JOINT DISTRIBUTION OF RANDOM VARIABLES

Topic Learning Objectives:

- To apply the knowledge of the statistical analysis and theory of probability in the study of uncertainties.
- Define the degree of dependence between two random variables and measuring the same .

Two or more random variables:

So far, only single random variables were considered. If one chooses a person at random and measures his or her height and weight, each measurement is a random variable – but taller people also tend to be heavier than shorter people, so the outcomes will be related. In order to deal with such probabilities, joint probability distribution of two random variables are studied in detail.

Joint Probability distribution for discrete random variables

Joint Probability Mass Function:

Let X and Y be random variables on the same sample space S with respective range spaces $\mathrm{R_X} = \{\mathrm{x_1, x_2, \dots, x_n}\}$ and $\mathrm{R_Y} = \{\mathrm{y_1, y_2, \dots, y_m}\}$. The joint distribution or joint probability function of X and Y is the function h on the product space $\mathrm{R_X} \times \mathrm{R_Y}$ defined by

$$h(x_i, y_j) \equiv P(X = x_i, Y = y_j) \equiv P(\{s \in S : X(s) = x_i,\ Y(s) = y_j\})$$

The function h has the following properties: (i) $h(x_i,\ y_j) \geq 0$, (ii) $\sum_i \sum_j h(x_i,\ y_j) = 1$

Thus, h defines a probability space on the product space $\mathrm{R_X} \times \mathrm{R_Y}$.

X \ Y	y_1	y_1	…	y_j	…	y_m	$\sum_i y_i$
x_1	$h(x_1, y_1)$	$h(x_1, y_2)$	…	$h(x_1, y_j)$	…	$h(x_1, y_m)$	$f(x_1)$
x_2	$h(x_2, y_1)$	$h(x_2, y_2)$	…	$h(x_2, y_j)$	…	$h(x_2, y_m)$	$f(x_2)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_i	$h(x_i, y_1)$	$h(x_i, y_2)$	…	$h(x_i, y_j)$	…	$h(x_i, y_m)$	$f(x_i)$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_m	$h(x_n, y_1)$	$h(x_n, y_2)$	…	$h(x_n, y_j)$	…	$h(x_n, y_m)$	$f(x_n)$
$\sum_i x_i$	$g(y_1)$	$g(y_2)$	…	$g(y_j)$	…	$g(y_m)$	

The functions f and g on the right side and the bottom side, respectively, of the joint distribution table are defined by

$f(x_i) = \sum_j h(x_i, y_j)$ and $g(y_j) = \sum_j h(x_i, y_j)$.

That is, $f(x_i)$ is the sum of the entries in the i^{th} row and $g(y_j)$ is the sum of the entries in the j^{th} column. They are called the marginal distributions of X and Y, respectively.

Expectation: Consider a function $\varphi(X, Y)$ of X and Y. Then the function

$E\{\varphi(X, Y)\} = \sum_i \sum_j h(x_i, \ y_j)\, \varphi(x_i, \ y_j)$

is called the mathematical expectation of $\varphi(X, Y)$ in the joint distribution of X and Y.

Co-variance and Correlation: Let X and Y be random variables with the joint distribution $h(x, y)$, and respective means μ_X and μ_Y. The covariance of X and Y, is denoted by cov(X, Y) and is defined as

$$cov(X, Y) = \sum_{i,j} (x_i - \mu_X)\,(y_j - \mu_Y)\, h(x_i, y_j)$$

$$cov(X, Y) = \sum_{i,j} x_i\, y_j h(x_i, y_j) - \mu_X \mu_Y$$

The correlation of X and Y is defined by $\rho(X, \ Y) = \frac{cov(X,Y)}{\sigma_X\, \sigma_Y}$

The correlation ρ is dimensionless and has the following properties:

(i) $\rho(X, \ Y) = \rho(Y, \ X)$,

(ii) $-1 \leq \rho \leq 1$,

(iii) $\rho(X, \ X) = 1, \ \rho(X, -X) = -1$,

(iv) $\rho(aX + b, \ cY + d) = \rho(X, \ Y)$ if $a, c \neq 0$.

Conditional probability distribution:

We know that the value x of the random variable X represents an event that is a subset of the sample space. If we use the definition of conditional probability,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0,$$

where A and B are now the events defined by $X = x$ and $Y = y$, respectively, then

$$P(Y = y|X = x) = \frac{P(X=x, Y=y)}{P(X=x)} = \frac{h(x,y)}{f(x)}, \text{ provided } f(x) > 0,$$

Where X and Y are discrete random variables.

Problems

1. A coin is tossed three times. Let X be equal to 0 or 1 according as a head or a tail occurs on the first toss. Let Y be equal to the total number of heads which occurs. Determine (i) the marginal distributions of X and Y, and (ii) the joint distribution of X and Y, (iii) expected values of $X, Y, X+Y$ and XY, (iv) σ_X and σ_Y, (v) $Cov(X,Y)$ and $\rho(X,Y)$.

Solution: Here the sample space is given by

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(i) The distribution of the random variable X is given by the following table

X (First toss Head or Tail)	0 (First toss Head)	1 (First toss Tail)
P(X) (Probability of random variable X)	$\frac{4}{8}$	$\frac{4}{8}$

which is the marginal distribution of the random variable X.

The distribution of the random variable Y is given by the following table

Y (Total number of Heads)	0 (zero Heads)	1 (one Head)	2 (two Head)	3 (three Head)
P(Y) (Probability of random variable Y)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

which is the marginal distribution of the random variable Y.

(ii) The joint distribution of the random variables $X$and Y is given by the following table

X \ Y	0 (zero Heads)	1 (one Head)	2 (two Head)	3 (three Head)
0 (First toss Head	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
1 (First toss Tail)	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0

(iii) $\mathrm{E}[\mathrm{X}] = \mu_\mathrm{X} = \sum \mathrm{x_i P(x_i)} = \mathrm{x_1 P(x_1) + x_2 P(x_2)} = 0 \times \frac{4}{8} + 1 \times \frac{4}{8} = \frac{4}{8}$

$$E[Y] = \mu_Y = \sum y_j P(y_j) = y_1P(y_1) + y_2P(y_2) + y_3P(y_3)$$

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{12}{8}$$

$$E[X+Y] = \sum\sum P_{ij}(x_i + y_j)$$

$$= P_{11}(x_1 + y_1) + P_{12}(x_1 + y_2) + P_{13}(x_1 + y_3) + P_{14}(x_1 + y_4) + P_{21}(x_2 + y_1) + P_{22}(x_2 + y_2) + P_{23}(x_2 + y_3) + P_{24}(x_2 + y_4)$$

$$= 0(0+0) + \frac{1}{8}(0+1) + \frac{2}{8}(0+2) + \frac{2}{8}(1+1) + \frac{1}{8}(1+2) + 0(1+3) = \frac{16}{8}$$

$$= 2.$$

$$E[XY] = \sum\sum P_{ij}(x_iy_j)$$

$$= P_{11}(x_1y_1) + P_{12}(x_1y_2) + P_{13}(x_1y_3) + P_{14}(x_1y_4) + P_{21}(x_2y_1) + P_{22}(x_2y_2) + P_{23}(x_2y_3) + P_{24}(x_2y_4)$$

$$= 0(0\times 0) + \frac{1}{8}(0\times 1) + \frac{2}{8}(0\times 2) + \frac{2}{8}(1\times 1) + \frac{1}{8}(1\times 2) + 0(1\times 3) = 2.$$

(iv) $\sigma_X^2 = E[X^2] - \mu_X^2 = \sum x_i^2 P(x_i) - [E(X)]^2 = x_1^2P(x_1) + x_2^2P(x_2) = 0^2 \times \frac{4}{8} + 1^2 \times \frac{4}{8} - \left(\frac{4}{8}\right)^2 = \frac{1}{4}$

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2$$

$$= \sum y_i^2 P(y_i) - [E(Y)]^2 = y_1^2P(y_1) + y_2^2P(y_2) + y_3^2P(y_3) + +y_4^2P(y_4)$$

$$= 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} - \left(\frac{2}{8}\right)^2 = \frac{3}{4}$$

(v) $Cov(X,Y) = E(XY) - \mu_X\mu_Y = \frac{1}{2} - \frac{1}{2} \times \frac{3}{2} = -\frac{1}{4}$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{-1/4}{(1/2)(\sqrt{3}/2)} = -\frac{1}{\sqrt{3}}.$$

2. The joint distribution of two random variables X and Y is given by the following table:

X \ Y	2	3	4
1	0.06	0.15	0.09
2	0.14	0.35	0.21

Determine the individual distributions of X and Y. Also, verify that X and Y are stochastically independent.

Solution: X takes values 1, 2 and Y takes the values 2, 3, 4. Also, $h_{11} = 0.06, h_{12} = 0.15$

$h_{13} = 0.09, h_{21} = 0.14, h_{22} = 0.35, \; h_{23} = 0.21$

Therefore, $f_1 = h_{11} + h_{12} + h_{13} = 0.3, \; f_2 = h_{21} + h_{22} + h_{23} = 0.7,$

$g_1 = h_{11} + h_{21} = 0.2, g_2 = h_{12} + h_{22} = 0.5, g_3 = h_{13} + h_{23} = 0.3.$

The distribution of X is given by

x_i	1	2
f_i	0.3	0.7

The distribution of Y is given by

y_j	2	3	4
g_j	0.2	0.5	0.3

$f_1 g_1 = 0.06 = h_{11}, \quad f_1 g_2 = 0.15 = h_{12,} \; f_1 g_3 = 0.09 = h_{13,}$

$f_2 g_1 = 0.14 = h_{21}, \quad f_2 g_2 = 0.35 = h_{22,} \; f_2 g_3 = 0.21 = h_{23,}$

Thus, $f_i g_j = h_{ij}$ for all values of i and j so, X and Y are stochastically independent.

3.The joint distribution of two random variables X and Y is given by the following table:

X \ Y	0	1
0	0.1	0.2
1	0.4	0.2
2	0.1	0

(a) Find $P(X + Y > 1)$

(b) Determine the individual (marginal) probability distributions of X and Y and verify that X and Y are not independent.

(c) Find $P(X = 2|Y = 0)$.

(d) Find the conditional distribution of X given $Y = 1$.

Solution: Note that X takes the values 0, 1, 2 and Y takes the values 0, 1

$h_{11} = 0.1, \; h_{12} = 0.2, \; h_{21} = 0.4, \; h_{22} = 0.2, h_{31} = 0.1, \; h_{32} = 0,$

(a) The event $X + Y > 1$ occurs only when the pair (X, Y) takes the values $(1,1), (2,0)$ and $(2,1)$. The probability that this event occurs is therefore

$P(X + Y > 1) = h_{22} + h_{31} + h_{32} = 0.2 + 0.1 + 0 = 0.3.$

(b) $f_1 = h_{11} + h_{12} = 0.1 + 0.2 = 0.3.$

$f_2 = h_{21} + h_{22} = 0.4 + 0.2 = 0.6.$

$f_3 = h_{31} + h_{32} = 0.1 + 0 = 0.1.$

$g_1 = h_{11} + h_{21} + h_{31} = 0.6$

$g_2 = h_{12} + h_{22} + h_{32} = 0.4.$

The distribution of X is given by

x_i	0	1	2
f_i	0.6	0.6	0.1

The distribution of Y is given by

y_j	0	1
g_j	0.6	0.4

It is verified that $f_1 \; g_1 = 0.18 \neq h_{11}$.

Therefore, X and Y are not stochastically independent.

(c) $P(X = 2|Y = 0) = \frac{h(2,0)}{g(0)} = \frac{h_{31}}{g_1} = \frac{0.1}{0.6} = \frac{1}{6}$

(d) Conditional distribution of X given $Y = 1$ is

$$P(X = x|Y = 1) = \frac{h(x,1)}{g(1)} = \frac{h_{i2}}{g_2}$$

$$P(X = 0|Y = 1) = \frac{h(0,1)}{g(1)} = \frac{h_{12}}{g_2} = \frac{0.2}{0.4} = 0.5$$

$$P(X = 1|Y = 1) = \frac{h(1,1)}{g(1)} = \frac{h_{22}}{g_2} = \frac{0.2}{0.4} = 0.5$$

$$P(X = 2|Y = 1) = \frac{h(2,1)}{g(1)} = \frac{h_{32}}{g_2} = \frac{0}{0.4} = 0$$

4. The joint distribution of two random variables X and Y is given by $p_{ij} = k(i + j),\ i = 1, 2, 3, 4; j = 1, 2, 3.$ Find (i) k and (ii) the marginal distributions of X and Y. Show that X and Y are not independent.

Solution: For the given p_{ij} ,

$$\sum_i \sum_j h_{ij} = \sum_{i=1}^{4} \sum_{j=1}^{3} h = k \sum_{i=1}^{4} \sum_{j=1}^{3} (i + j)$$

$$= k \sum_{i=1}^{4} \{(i + 1) + (i + 2) + (i + 3)\} = k \sum_{i=1}^{4} (3\, i + 6)$$

$$= k\, \{(3 + 6) + (3 \times 2 + 6) + (3 \times 3 + 6) + (3 \times 4 + 6)\} = 54k$$

Since

$$\sum_i \sum_j h_{ij} = 1, i.e., 54k = 1 \;\; or \;\; k = 1/54$$

$$f_i = \sum_j h_{ij} = \sum_{j=1}^{3} h_{ij} = k \sum_{j=1}^{3}(i+j) = \frac{i+2}{18}$$

$$g_j = \sum_i h_{ij} = \sum_{j=1}^{4} h_{ij} = k \sum_{j=1}^{3}(i+j) = \frac{2j+5}{27}$$

Therefore, the marginal distributions of X and Y are

$\{f_i\} = \left\{\frac{i+2}{18}\right\}$, i=1,2,3,4 and $\{g_j\} = \left\{\frac{2j+5}{27}\right\}$, j=1,2,3.

Finally note that $f_i \; g_j \neq h_{11}$, so X and Y are not independent.

5. The joint probability distribution of two random variables X and Y is given by the following table.

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Find the marginal distribution of X and Y, and evaluate $cov(X,Y)$. Find $P(X=4|Y=3)$ and $P(Y=3|X=4)$

Solution: From the table, note that

$$f_1 = \frac{1}{8} + \frac{1}{24} + \frac{1}{12} = \frac{1}{4}$$

$$f_2 = \frac{1}{4} + \frac{1}{4} + 0 = \frac{1}{2}$$

$$f_3 = \frac{1}{8} + \frac{1}{24} + \frac{1}{12} = \frac{1}{4}$$

$$g_1 = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

$$g_2 = \frac{1}{24} + \frac{1}{4} + \frac{1}{24} = \frac{1}{3}$$

$$g_3 = \frac{1}{12} + 0 + \frac{1}{12} = \frac{1}{6}$$

The marginal distribution of X is given by the table:

x_i	2	4	6
f_i	1/4	1/2	1/4

And the marginal distribution of Y is given by the table:

y_j	1	3	9
g_j	1/2	1/3	1/6

Therefore, the means of these distributions are respectively,

$$\mu_X = \sum x_i\, P(x_i) = \left(2 \times \frac{1}{4}\right) + \left(4 \times \frac{1}{2}\right) + \left(6 \times \frac{1}{4}\right) = 4$$

$$\mu_Y = \sum y_j\, P(y_j) = \left(1 \times \frac{1}{2}\right) + \left(3 \times \frac{1}{3}\right) + \left(9 \times \frac{1}{6}\right) = 3$$

$$\mathrm{E}[XY] = \sum_i \sum_j h_{ij}\ x_i x_j$$

$$= (h_{11}x_1y_1 + h_{12}x_1y_2 + h_{13}x_1y_3) + (h_{21}x_2y_1 + h_{22}x_2y_2 + h_{23}x_2y_3)$$
$$+ (h_{31}x_3y_1 + h_{32}x_3y_2 + h_{31}x_3y_3)$$

$$= \left(2 \times \tfrac{1}{8}\right) + \left(6 \times \tfrac{1}{24}\right) + \left(18 \times \tfrac{1}{12}\right) + \left(4 \times \tfrac{1}{4}\right) + \left(12 \times \tfrac{1}{4}\right) + 36 \times 0 + \left(6 \times \tfrac{1}{8}\right) +$$
$$\left(18 \times \tfrac{1}{24}\right) + \left(54 \times \tfrac{1}{12}\right)$$

$= 2 + 4 + 6 = 12$

$Cov\,(X, Y) = E[XY] - \mu_X\mu_Y = 12 - 12 = 0$

$\rho(X, Y) = 0.$

$$P(X = 4|Y = 3) = \frac{h(4,3)}{g(3)} = \frac{h_{22}}{g_2} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

$$P(Y = 3|X = 4) = \frac{h(4,3)}{f(4)} = \frac{h_{22}}{f_2} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{2}{4} = 0.5$$

Exercise:

1) The joint probability distribution of two random variables X and Y is given by the following table.

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

(a) Find the marginal distribution of X and Y, and evaluate $cov(X, Y)$.

(b) Also determine whether μ_X and μ_Y.

(c) Find $P(Y = -1|X = 1)$ and $P(X = 2|Y = 4)$

2) Two textbooks are selected at random from a shelf containing three statistics texts, two mathematics texts and three engineering texts. Denoting the number of books selected in each subject by S, M and E respectively, find (a) the joint distribution of S and M, (b) the marginal distributions of S, M and E, and (c) Find the correlation of the random variables S and M.
3) Consider an experiment that consists of 2 throws of a fair die. Let X be the number of 4s and Y be the number of 5s obtained in the two throws. Find the joint probability distribution of X and Y. Also evaluate $P(2X + Y < 3)$.

Joint Probability distribution for continuous random variables

Joint probability density function

Let x and y be two continuous random variables. Suppose there exists a real valued function $h(x, y)$ of x and y such that the following conditions hold:

(i) $\mathrm{h(x, y)} \geq 0$ for all x, y

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{h(x, y)}\, dx\, dy$ exists and is equal to 1.

Then, $\mathrm{h(x, y)}$ is called joint probability density function.

If $[a, b]$ and $[c, d]$ are any two intervals, then the probability that $x \in [a, b]$ and $y \in [c, d]$, is denoted by $P(a \leq x \leq b, \quad c \leq y \leq d)$ is defined by the formula

$$P(a \leq x \leq b, \quad c \leq y \leq d) = \int_a^b \int_c^d h(x, y)\, dy\, dx.$$

For any specified real numbers u, v, the function

F(u, v) = $\int_{-\infty}^{u}\int_{-\infty}^{v} h(x, y)\, dy\, dx$

is called the joint or the compound cumulative distribution function.

Where $F(u, v) = P(-\infty < x \leq u, \quad -\infty < y \leq v)$ and $\frac{\partial^2 F}{\partial u \partial v} = p(u, v)$.

Further, the function $h_1(x) = \int_{-\infty}^{\infty} h(x, y)\, dy$ is called marginal density function of x, and the function $h_2(y) = \int_{-\infty}^{\infty} h(x, y)\, dx$ is called marginal density function of y. $h_1(x)$ is the density function of x and $h_2(y)$ is the density function of y.

The variables x and y are said to stochastically independent if $h_1(x)h_2(y) = h(x, y)$.

If $\varphi(x, y)$ is a function of x and y, then the expectation of $\varphi(x, y)$ is defined by

E$\{\varphi(x, y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x, y)\, h(x, y)\, dx\, dy.$

The covariance between x and y is defined as

$Cov(x, y) = E\{xy\} - E\{x\}\, E\{y\}.$

Conditional Probability:

The idea of conditional probability function of discrete random variables of extended to the case of continuous random variables.

If X and Y are continuous random variables, then the conditional probability distribution Y given X is $f(y|x) = \frac{h(x,y)}{h_1(x)}$

where $h(x, y)$ is the joint density function of X and Y, $h_1(x)$ is the marginal density function of X.

$$P(c < Y < d|a < x < b) = \frac{P(a < X < b, c < Y < d)}{P(a < x < b)}$$

Problems

1. Find the constant 'k' so that

$$\mathrm{h(x,y)} = \begin{cases} k(x+1)e^{-y}, & 0 < x < 1,\ y > 0 \\ 0, & elsewhere \end{cases}$$

is a joint probability density function. Are x and y independent?

Solution: Observe that $h(x, y) \geq 0$ for x, y, if $k \geq 0$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,dx\,dy = \int_{y=0}^{\infty}\int_{x=0}^{1} h(x,y)\,dx\,dy$$

$$= k\left\{\int_0^1 (x+1)dx\right\}\left\{\int_0^{\infty} e^{-y}\,dy\right\}$$

$$= k\left\{\frac{2^2 - 1^2}{2}\right\}\{0+1\} = \frac{3}{2}k.$$

Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,dx\,dy = 1$ if $\mathrm{k} = \frac{2}{3}$.

Therefore, $h(x,y)$ is a joint probability density function if $\mathrm{k} = \frac{2}{3}$.

With $\mathrm{k} = \frac{2}{3}$, the marginal density functions are

$$h_1(x) = \int_{-\infty}^{\infty} h(x,y)\,dy, \quad 0 < x < 1$$

$$= \frac{2}{3}(x+1)\int_0^{\infty} e^{-y}\,dy$$

$$= \frac{2}{3}(x+1)(0+1).$$

$$= \frac{2}{3}(x+1), \quad 0 < x < 1$$

Next,

$$h_2(y) = \int_{-\infty}^{\infty} h(x,y)\,dx, \;\; y > 0$$
$$= \frac{2}{3}e^{-y}\int_0^1 (x+1)\,dx = \frac{2}{3}e^{-y}\left\{\frac{2^2}{2} - \frac{1}{2}\right\}$$
$$= e^{-y}, \;\; y > 0.$$

Therefore, $h_1(x)h_2(y) = h(x,y)$ and hence x and y are stochastically independent.

2. The life time x and brightness y of a light bulb are modeled as continuous random variables with joint density function

$$h(x,y) = \alpha\beta e^{-(\alpha x+\beta y)}, \qquad 0 < x < \infty, 0 < y < \infty.$$

Where α and β are appropriate constants. Find (i) the marginal density functions of x and y, and (ii) the compound cumulative distributive function.

Solution: For the given distribution, the marginal density function of x is

$$h_1(x) = \int_{-\infty}^{\infty} h(x,y)\,dy = h_1(x) = \int_0^{\infty} \alpha\beta e^{-(\alpha x+\beta y)}\,dy$$
$$= \alpha\beta\, e^{-\alpha x}\int_0^{\infty} e^{-\beta y}dy = \alpha\, e^{-\alpha x}, \;\; 0 < x < \infty$$

the marginal density function of y is

$$h_2(y) = \int_{-\infty}^{\infty} h(x,y)\,dx = \beta\, e^{-\beta y}, \quad 0 < y < \infty.$$

Further, the compound cumulative distribution function is

$$\mathrm{F}\,(\mathrm{u},\mathrm{v}) = \int_{-\infty}^{u}\int_{-\infty}^{v} h(x,y)\,dy\,dx = \int_0^u\int_0^v \alpha\beta e^{-(\alpha x+\beta y)}dy\,dx$$
$$= \alpha\beta\left\{\int_0^u e^{-\alpha x}dx\right\}\left\{\int_0^v e^{-\beta y}dy\right\}$$
$$= \alpha\beta\left\{\frac{1}{\alpha}(1 - e^{-\alpha u})\right\}\left\{\frac{1}{\beta}(1 - e^{-\beta v})\right\}$$
$$= (1 - e^{-\alpha u})(1 - e^{-\beta v}), \qquad 0 < u < \infty, \quad 0 < v < \infty.$$

3. The joint probability density function of two random variables x and y is given by

$$\mathrm{h(x,y)} = \begin{cases} 2, & 0 < x < \; y < 1 \\ 0, & \quad elsewhere \end{cases}$$

(i) Find the covariance between x and y.

Solution: The marginal density function of x is

$$h_1(x) = \int_{-\infty}^{\infty} h(x,y)\,dy = \begin{cases} \int_x^1 2\,dy = 2(1-x), & 0 < x < 1 \\ \quad 0 & elsewhere \end{cases}$$

The marginal density function of y is

$$h_2(y) = \int_{-\infty}^{\infty} h(x,y)\,dx = \begin{cases} \int_0^y 2\,dx = 2y, & 0 < y < 1 \\ 0 & elsewhere \end{cases}$$

$$E[x] = \int_{-\infty}^{\infty} x\ h_1(x)dx = \int_0^1 x\{2\,(1-x)\}\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx = 2\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{3},$$

$$E[y] = \int_{-\infty}^{\infty} y\ h_2(y)dy = \int_0^1 y(2y)dy = \frac{2}{3},$$

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,h(x,y)\,dx\,dy = \int_0^1 2y\left\{\int_0^y x\,dx\right\}\,dy = \int_0^1 y^3\,dy = \frac{1}{4}.$$

Therefore,

$$Cov\,(x,y) = E[xy] - E[x]E[y] = \frac{1}{4} - \frac{1}{3}\cdot\frac{2}{3} = \frac{1}{36}.$$

4. Verify that $f(x,y) = \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0 \\ 0, & elsewhere \end{cases}$ is a density function of a joint probability distribution. Then evaluate the following:

(i) $P\left(\frac{1}{2} < x < 2, 0 < y < 4\right)$ (ii) $P(x < 1)$ (iii) $P(x > y)$ (iv) $P(x + y \leq 1)$,

(v) $P(0 < x < 1 | y = 2)$.

Solution: Given $f(x,y) \geq 0$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x+y)}dx\,dy = \int_{-\infty}^{\infty} e^{-x}\,dx \int_0^{\infty} e^{-y}\,dy$$

$$= (0+1)(0+1) = 1.$$

Therefore, $f(x,y)$ is a density function.

(i) $P\left(\frac{1}{2} < x < 2,\ 0 < y < 4\right) = \int_{1/2}^2\int_0^4 f(x,y)\,dy\,dx = \int_{1/2}^2\int_0^4 e^{-(x+y)}dy\,dx$

$$= \int_{1/2}^2 e^{-x}\,dx \int_0^4 e^{-y}dy = \left(e^{-1/2} - e^{-2}\right)(1 - e^{-4}).$$

(ii) The marginal density function of x is

$$h_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-(x+y)}\,dy = e^{-x}\int_0^{\infty} e^{-y}\,dy = e^{-x}$$

Therefore, $P(x < 1) = \int_0^1 h_1(x)\,dx = \int_0^1 e^{-x}\,dx = 1 - \frac{1}{e}$.

(iii) $P(x \leq y) = \int_0^{\infty}\left\{\int_0^y f(x,y)\,dx\right\}dy = \int_0^{\infty}\left\{\int_0^y e^{-(x+y)}\,dx\right\}dy$

$$= \int_0^\infty e^{-y}\left(\int_0^y e^{-x}\,dx\right)dy = \int_0^\infty e^{-y}(1-e^{-y})\,dy$$

$$= \int_0^\infty (e^{-y} - e^{-2y})\,dy = 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, $P(x > y) = 1 - P(x \le y) = 1 - \frac{1}{2} = \frac{1}{2}$.

(iv) $P(x + y \le 1) = \iint_A f(x,y)dA$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)\,dy\,dx = \int_0^1\left\{\int_0^{1-x} e^{-(x+y)}\,dy\right\}dx$$

$$= \int_0^1 e^{-x}\left\{\int_0^{1-x} e^{-y}\,dy\right\}dx = \int_0^1 e^{-x}\{1 - e^{-(1-x)}\}\,dx$$

$$= \int_0^1 (e^{-x} - e^{-1})\,dx = 1 - \frac{2}{e}.$$

(v) $P(0 < x < 1 | y = 2) = \frac{P(0 < x < 1 | y = 2)}{P(y=2)}$

[putting $y = 2$] $P(0 < x < 1 | y = 2) = \frac{\int_0^1 e^{-(x+2)}dx}{\int_0^\infty e^{-(x+2)}dx} = 1 - \frac{1}{e} = 0.63$

Exercise:

1) If the joint probability function for $f(x,y)$ is

$$f(x,y) = \begin{cases} c(x^2+y^2), & 0 \le x \le 1, 0 \le y \le 1, c \ge 0 \\ 0, & elsewhere \end{cases}$$ is a density function of a joint probability distribution. Then evaluate the following:

(i) the value of the constant c. (ii) the marginal density functions of x and y.

(iii) $P\left(x < \frac{1}{2}, y > \frac{1}{2}\right)$ (iv) $P\left(\frac{1}{4} < x < \frac{3}{4}\right)$ (v) $P\left(y < \frac{1}{2}\right)$.

2) For the distribution given by the density function

$$f(x,y) = \begin{cases} \frac{1}{96}xy, & 0 < x < 4, 1 < y < 5 \\ 0, & elsewhere \end{cases}$$

evaluate (i) $P(1 < x < 2, 2 < y < 3)$, (ii) $P(x > 3, y \le 2)$ (iii) $P(y \le x)$, (iv) $(x + y \le 3)$

3) For the distribution defined by the density function

$$f(x,y) = \begin{cases} 3xy(x+y), & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & elsewhere \end{cases}$$

find the covariance between x and y.

4) For the distribution defined by the density function

$$f(x, y) = \begin{cases} \frac{1}{8}(6 - x - y), & 0 < x < 2, \quad 0 < y < 4 \\ 0, & elsewhere \end{cases}$$

Evaluate (i) $P(x < 1, y < 3)$, (ii)$P(x + y < 3)$, (iii) the covariance between x and y and (iv) $P(x < 1|y < 3)$

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Video Links:

https://www.youtube.com/watch?v=82Ad1orN-NA

https://www.youtube.com/watch?v=eYthpvmqcf0

https://www.youtube.com/watch?v=L0zWnBrjhng

https://www.youtube.com/watch?v=Om68Hkd7pfw

https://www.youtube.com/watch?v=RYIb1u3C13I