CHAPTER 1
FUNDAMENTAL PRINCIPLES OF COUNTING

Sections 1.1 and 1.2

1. (a) By the rule of sum, there are $8 + 5 = 13$ possibilities for the eventual winner.
(b) Since there are eight Republicans and five Democrats, by the rule of product we have $8 \times 5 = 40$ possible pairs of opposing candidates.
(c) The rule of sum in part (a); the rule of product in part (b).

2. By the rule of product there are $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$ license plates where the first two symbols are vowels and the last four are even digits.

3. By the rule of product there are (a) $4 \times 12 \times 3 \times 2 = 288$ distinct Buicks that can be manufactured. Of these, (b) $4 \times 1 \times 3 \times 2 = 24$ are blue.

4. (a) From the rule of product there are $10 \times 9 \times 8 \times 7 = P(10,4) = 5040$ possible slates.
(b) (i) There are $3 \times 9 \times 8 \times 7 = 1512$ slates where a physician is nominated for president.
(ii) The number of slates with exactly one physician appearing is $4 \times [3 \times 7 \times 6 \times 5] = 2520$.
(iii) There are $7 \times 6 \times 5 \times 4 = 840$ slates where no physician is nominated for any of the four offices. Consequently, $5040 - 840 = 4200$ slates include at least one physician.

5. Based on the evidence supplied by Jennifer and Tiffany, from the rule of product we find that there are $2 \times 2 \times 1 \times 10 \times 10 \times 2 = 800$ different license plates.

6. (a) Here we are dealing with the permutations of 30 objects (the runners) taken 8 (the first eight finishing positions) at a time.. So the trophies can be awarded in $P(30,8) = 30!/22!$ ways.
(b) Roberta and Candice can finish among the top three runners in 6 ways. For each of these 6 ways, there are $P(28,6)$ ways for the other 6 finishers (in the top 8) to finish the race. By the rule of product there are $6 \cdot P(28,6)$ ways to award the trophies with these two runners among the top three.

7. By the rule of product there are 2^9 possibilities.

8. By the rule of product there are (a) $12!$ ways to process the programs if there are no restrictions; (b) $(4!)(8!)$ ways so that the four higher priority programs are processed first; and (c) $(4!)(5!)(3!)$ ways where the four top priority programs are processed first and the three programs of least priority are processed last.

9. (a) $(14)(12) = 168$
(b) $(14)(12)(6)(18) = 18{,}144$
(c) $(8)(18)(6)(3)(14)(12)(14)(12) = 73{,}156{,}608$

10. Consider one such arrangement – say we have three books on one shelf and 12 on the other. This can be accomplished in 15! ways. In fact for any subdivision (resulting in two nonempty shelves) of the 15 books we get 15! ways to arrange the books on the two shelves. Since there are 14 ways to subdivide the books so that each shelf has at least one book, the total number of ways in which Pamela can arrange her books in this manner is (14)(15!).

11. (a) There are four roads from town A to town B and three roads from town B to town C, so by the rule of product there are $4 \times 3 = 12$ roads from A to C that pass through B. Since there are two roads from A to C directly, there are $12 + 2 = 14$ ways in which Linda can make the trip from A to C.
(b) Using the result from part (a), together with the rule of product, we find that there are $14 \times 14 = 196$ different round trips (from A to C and back to A).
(c) Here there are $14 \times 13 = 182$ round trips.

12. (1) a,c,t (2) a,t,c (3) c,a,t (4) c,t,a (5) t,a,c (6) t,c,a

13. (a) $8! = P(8,8)$ (b) $7!$ $6!$

14. (a) $P(7,2) = 7!/(7-2)! = 7!/5! = (7)(6) = 42$
(b) $P(8,4) = 8!/(8-4)! = 8!/4! = (8)(7)(6)(5) = 1680$
(c) $P(10,7) = 10!/(10-7)! = 10!/3! = (10)(9)(8)(7)(6)(5)(4) = 604,800$
(d) $P(12,3) = 12!/(12-3)! = 12!/9! = (12)(11)(10) = 1320$

15. Here we must place a,b,c,d in the positions denoted by x: e <u>x</u> e <u>x</u> e <u>x</u> e <u>x</u> e. By the rule of product there are 4! ways to do this.

16. (a) With repetitions allowed there are 40^{25} distinct messages.
(b) By the rule of product there are $40 \times 30 \times 30 \times \ldots \times 30 \times 30 \times 40 = (40^2)(30^{23})$ messages.

17. Class A: $(2^7 - 2)(2^{24} - 2) = 2,113,928,964$
Class B: $2^{14}(2^{16} - 2) = 1,073,709,056$
Class C: $2^{21}(2^8 - 2) = 532,676,608$

18. From the rule of product we find that there are $(7)(4)(3)(6) = 504$ ways for Morgan to configure her low-end computer system.

19. (a) $7! = 5040$ (b) $4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = (4!)(3!) = 144$
(c) $(3!)(5)(4!) = 720$ (d) $(3!)(4!)(2) = 288$

20. (a) Since there are three A's, there are $8!/3! = 6720$ arrangements.

(b) Here we arrange the six symbols D,T,G,R,M, AAA in $6! = 720$ ways.

21. (a) $12!/(3!2!2!2!)$
(b) $[11!/(3!2!2!2!)]$ (for AG) + $[11!/(3!2!2!2!)]$ (for GA)
(c) Consider one case where all the vowels are adjacent: S,C,L,G,C,L, OIOOIA. These seven symbols can be arranged in $(7!)/(2!2!)$ ways. Since O,O,O,I,I,A can be arranged in $(6!)/(3!2!)$ ways, the number of arrangements with all the vowels adjacent is $[7!/(2!2!)][6!/(3!2!)]$.

22. (Case 1: The leading digit is 5) $(6!)/(2!)$
(Case 2: The leading digit is 6) $(6!)/(2!)^2$
(Case 3: The leading digit is 7) $(6!)/(2!)^2$
In total there are $[(6!)/(2!)][1 + (1/2) + (1/2)] = 6! = 720$ such positive integers n.

23. Here the solution is the number of ways we can arrange 12 objects — 4 of the first type, 3 of the second, 2 of the third, and 3 of the fourth. There are $12!/(4!3!2!3!) = 277,200$ ways.

24. $P(n+1,r) = (n+1)!/(n+1-r)! = [(n+1)/(n+1-r)] \cdot [n!/(n-r)!] = [(n+1)/(n+1-r)]P(n,r)$.

25. (a) $n = 10$ (b) $n = 5$
(c) $2n!/(n-2)! + 50 = (2n)!/(2n-2)! \Longrightarrow 2n(n-1) + 50 = (2n)(2n-1) \Longrightarrow n^2 = 25 \Longrightarrow n = 5$.

26. Any such path from (0,0) to (7,7) or from (2,7) to (9,14) is an arrangement of 7 R's and 7 U's. There are $(14!)/(7!7!)$ such arrangements.
In general, for m, n nonnegative integers, and any real numbers a, b, the number of such paths from (a, b) to $(a+m, b+n)$ is $(m+n)!/(m!n!)$.

27. (a) Each path consists of 2 H's, 1 V, and 7 A's. There are $10!/(2!1!7!)$ ways to arrange these 10 letters and this is the number of paths.
(b) $10!/(2!1!7!)$
(c) If a, b, and c are any real numbers and m, n, and p are nonnegative integers, then the number of paths from (a, b, c) to $(a+m, b+n, c+p)$ is $(m+n+p)!/(m!n!p!)$.

28. (a) The **for** loop for i is executed 12 times, while those for j and k are executed $10-5+1 = 6$ and $15-8+1 = 8$ times, respectively. Consequently, following the execution of the given program segment, the value of *counter* is

$$0 + 12(1) + 6(2) + 8(3) = 48.$$

(b) Here we have three tasks — T_1, T_2, and T_3. Task T_1 takes place each time we traverse the instructions in the i loop. Similarly, tasks T_2 and T_3 take place during each iteration of the j and k loops, respectively. The final value for the integer variable counter follows by the rule of sum.

29. (a) & (b) By the rule of product the **print** statement is executed $12 \times 6 \times 8 = 576$ times.

30. (a) For five letters there are $26 \times 26 \times 26 \times 1 \times 1 = 26^3$ palindromes. There are $26 \times 26 \times 26 \times 1 \times 1 \times 1 = 26^3$ palindromes for six letters.
(b) When letters may not appear more than two times, there are $26 \times 25 \times 24 = 15,600$ palindromes for either five or six letters.

31. By the rule of product there are (a) $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136,080$ six-digit integers with no leading zeros and no repeated digit. (b) When digits may be repeated there are 9×10^5 such six-digit integers.
(i) (a) $(9 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 0) + $(8 \times 8 \times 7 \times 6 \times 5 \times 4)$ (for the integers ending in 2,4,6, or 8) = 68,800. (b) When the digits may be repeated there are $9 \times 10 \times 10 \times 10 \times 10 \times 5 = 450,000$ six-digit even integers.
(ii) (a) $(9 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 0) + $(8 \times 8 \times 7 \times 6 \times 5 \times 1)$ (for the integers ending in 5) = 28,560. (b) $9 \times 10 \times 10 \times 10 \times 10 \times 2 = 180,000$.
(iii) We use the fact that an integer is divisible by 4 if and only if the integer formed by the last two digits is divisible by 4. (a) $(8 \times 7 \times 6 \times 5 \times 6)$ (last two digits are 04, 08, 20, 40, 60, or 80) + $(7 \times 7 \times 6 \times 5 \times 16)$ (last two digits are 12, 16, 24, 28, 32, 36, 48, 52, 56, 64, 68, 72, 76, 84, 92, or 96) = 33,600. (b) $9 \times 10 \times 10 \times 10 \times 25 = 225,000$.

32. (a) For positive integers n, k, where $n = 3k$, $n!/(3!)^k$ is the number of ways to arrange the n objects $x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_k, x_k, x_k$. This must be an integer.
(b) If n, k are positive integers with $n = mk$, then $n!/(m!)^k$ is an integer.

33. (a) With 2 choices per question there are $2^{10} = 1024$ ways to answer the examination.
(b) Now there are 3 choices per question and 3^{10} ways.

34. $(4!/2!)$ (No 7's) + $(4!)$ (One 7 and one 3) + $(2)(4!/2!)$ (One 7 and two 3's) + $(4!/2!)$ (Two 7's and no 3's) + $(2)(4!/2!)$ (Two 7's and one 3) + $(4!/(2!2!))$ (Two 7's and two 3's). The total gives us 102 such four-digit integers.

35. (a) $6!$ (b) Let A,B denote the two people who insist on sitting next to each other. Then there are $5!$ (A to the right of B) + $5!$ (B to the right of A) = $2(5!)$ seating arrangements.

36. (a) Locate A. There are two cases to consider. (1) There is a person to the left of A on the same side of the table. There are $7!$ such seating arrangements. (2) There is a person to the right of A on the same side of the table. This gives $7!$ more arrangements. So there are $2(7!)$ possibilities. (b) 7200

37. We can select the 10 people to be seated at the table for 10 in $\binom{16}{10}$ ways. For each such selection there are $9!$ ways of arranging the 10 people around the table. The remaining six people can be seated around the other table in $5!$ ways. Consequently, there are $\binom{16}{10}9!5!$ ways to seat the 16 people around the two given tables.

38. The nine women can be situated around the table in 8! ways. Each such arrangement provides nine spaces (between women) where a man can be placed. We can select six of these places and situate a man in each of them in $\binom{9}{6}6! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways. Consequently, the number of seating arrangements under the given conditions is $(8!)\binom{9}{6}6! = 2,438,553,600$.

39.

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procedure SumOfFact(i, sum: positive integers; j,k: nonnegative integers;
                        factorial: array [0..9] of ten positive integers)
begin
   factorial [0] := 1
   for i := 1 to 9 do
      factorial [i] := i * factorial [i - 1]

   for i := 1 to 9 do
      for j := 0 to 9 do
         for k := 0 to 9 do
            begin
               sum := factorial [i] + factorial [j] + factorial [k]
               if (100 * i + 10 * j + k) = sum then
                  print (100 * i + 10 * j + k)
            end
end
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The unique answer is 145 since $(1!) + (4!) + (5!) = 1 + 24 + 120 = 145$.

Section 1.3

1. $\binom{6}{2} = 6!/[2!(6\text{-}2)!] = 6!/(2!4!) = (6)(5)/2 = 15$

a	b	b	c	c	e
a	c	b	d	c	f
a	d	b	e	d	e
a	e	b	f	d	f
a	f	c	d	e	f

2. Order is not relevant here and Diane can make her selection in $\binom{12}{5} = 792$ ways.

3. (a) $C(10, 4) = 10!/(4!6!) = (10)(9)(8)(7)/(4)(3)(2)(1) = 210$
(b) $\binom{12}{7} = 12!/(7!5!) = (12)(11)(10)(9)(8)/(5)(4)(3)(2)(1) = 792$

(c) $C(14,12) = 14!/(12!2!) = (14)(13)/(2)(1) = 91$
(d) $\binom{15}{10} = 15!/(10!5!) = (15)(14)(13)(12)(11)/(5)(4)(3)(2)(1) = 3003$

4. (a) $2^6 - 1 = 63$ (b) $\binom{6}{3} = 20$ (c) $\binom{6}{2} + \binom{6}{4} + \binom{6}{6} = 31$

5. (a) There are $P(5,3) = 5!/(5-3)! = 5!/2! = (5)(4)(3) = 60$ permutations of size 3 for the five letters m, r, a, f, and t.
(b) There are $C(5,3) = 5!/[3!(5-3)!] = 5!/(3!2!) = 10$ combinations of size 3 for the five letters m, r, a, f, and t. They are

a,f,m	a,f,r	a,f,t	a,m,r	a,m,t
a,r,t	f,m,r	f,m,t	f,r,t	m,r,t

6.
$$\binom{n}{2} + \binom{n-1}{2} = (\frac{1}{2})(n)(n-1) + (\frac{1}{2})(n-1)(n-2) = (\frac{1}{2})(n-1)[n+(n-2)] =$$
$(\frac{1}{2})(n-1)(2n-2) = (n-1)^2$.

7. (a) $\binom{20}{12}$ (b) $\binom{10}{6}\binom{10}{6}$
(c) $\binom{10}{2}\binom{10}{10}$(2 women)$+\binom{10}{4}\binom{10}{8}$ (4 women)$+\ldots+\binom{10}{10}\binom{10}{2}$ (10 women) $= \sum_{i=1}^{5}\binom{10}{2i}\binom{10}{12-2i}$
(d) $\binom{10}{7}\binom{10}{5}$ (7 women) $+ \binom{10}{8}\binom{10}{4}$ (8 women) $+ \binom{10}{9}\binom{10}{3}$ (9 women) $+$
$\binom{10}{10}\binom{10}{2}$ (10 women) $= \sum_{i=7}^{10}\binom{10}{i}\binom{10}{12-i}$.
(e) $\sum_{i=8}^{10}\binom{10}{i}\binom{10}{12-i}$

8. (a) $\binom{4}{1}\binom{13}{5}$ (b) $\binom{4}{4}\binom{48}{1}$ (c) $\binom{13}{1}\binom{4}{4}\binom{48}{1}$ (d) $\binom{4}{3}\binom{4}{2}$
(e) $\binom{4}{3}\binom{12}{1}\binom{4}{2}$ (f) $\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3744$
(g) $\binom{13}{1}\binom{4}{3}\binom{48}{1}\binom{44}{1}/2$ (Division by 2 is needed since no distinction is made for the order in which the other two cards are drawn.) This result equals $54,912 = \binom{13}{1}\binom{4}{3}\binom{48}{2} - 3744 =$
$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$.
(h) $\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1}$.

9. (a) $\binom{8}{2}$ (b) $\binom{8}{4}$ (c) $\binom{8}{6}$ (d) $\binom{8}{6} + \binom{8}{7} + \binom{8}{8}$.

10. $\binom{12}{5}$; $\binom{10}{3}$.

11. (a) $\binom{10}{7} = 120$ (b) $\binom{8}{5} = 56$ (c) $\binom{6}{4}\binom{4}{3}$ (four of the first six) $+\binom{6}{5}\binom{4}{2}$ (five of the first six) $+\binom{6}{6}\binom{4}{1}$ (all of the first six) $= (15)(4) + (6)(6) + (1)(4) = 100$.

12. (a) The first three books can be selected in $\binom{12}{3}$ ways. The next three in $\binom{9}{3}$ ways. The third set of three in $\binom{6}{3}$ ways and the fourth set in $\binom{3}{3}$ ways. Consequently, the 12 books can be distributed in $\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = (12!)/[(3!)^4]$ ways.

(b) $\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2} = (12!)/[(4!)^2(2!)^2]$.

13. The letters M,I,I,I,P,P,I can be arranged in $[7!/(4!)(2!)]$ ways. Each arrangement provides eight locations (one at the start of the arrangement, one at the finish, and six between letters) for placing the four nonconsecutive S's. Four of these locations can be selected in $\binom{8}{4}$ ways. Hence, the total number of these arrangements is $\binom{8}{4}\ [7!/(4!)(2!)]$.

14. $\binom{n}{11} = 12,376$ when $n = 17$.

15. (a) Two distinct points determine a line. With 15 points, no three collinear, there are $\binom{15}{2}$ possible lines.

(b) There are $\binom{25}{3}$ possible triangles or planes, and $\binom{25}{4}$ possible tetrahedra.

16. (a) $\sum_{i=1}^{6}(i^2+1) = (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+(5^2+1)+(6^2+1) = 2+5+10+17+26+37$
$= 97$

(b) $\sum_{j=-2}^{2}(j^3-1) = [(-2)^3-1]+[(-1)^3-1]+(0^3-1)+(1^3-1)+(2^3-1) = -9-2-1+0+7$
$= -5$

(c) $\sum_{i=0}^{10}[1+(-1)^i] = 2+0+2+0+2+0+2+0+2+0+2 = 12$

(d) $\sum_{k=n}^{2n}(-1)^k = [(-1)^n+(-1)^{n+1}]+[(-1)^{n+2}+(-1)^{n+3}]+\ldots+[(-1)^{2n-1}+(-1)^{2n}]$
$= 0+0+\ldots+0 = 0$

(e) $\sum_{i=1}^{6}i(-1)^i = -1+2-3+4-5+6 = 3$

17. (a) $\sum_{k=2}^{n}\frac{1}{k!}$ (b) $\sum_{i=1}^{7}i^2$ (c) $\sum_{j=1}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{7}(-1)^{k+1}k^3$

(d) $\sum_{i=0}^{n}\frac{i+1}{n+i}$ (e) $\sum_{i=0}^{n}(-1)^i\left[\frac{n+i}{(2i)!}\right]$

18. (a) $10!/(4!3!3!)$ (b) $\binom{10}{8}2^2+\binom{10}{9}2+\binom{10}{10}$

(c) $\binom{10}{4}$ (four 1's, six 0's) + $\binom{10}{2}\binom{8}{1}$ (two 1's, one 2, seven 0's) +
$\binom{10}{2}$ (two 2's, eight 0's)

19. $\binom{10}{3}$ (three 1's, seven 0's) + $\binom{10}{1}\binom{9}{1}$ (one 1, one 2, eight 0's) +
$\binom{10}{1}$ (one 3, nine 0's) = 220
$\binom{10}{4}+\binom{10}{2}+\binom{10}{1}\binom{9}{2}+\binom{10}{1}\binom{9}{1} = 705$

$(2^{10})(\sum_{i=0}^{5}\binom{10}{2i})$ – Select an even number of locations for 0,2. This is done in $\binom{10}{2i}$ ways for $0 \leq i \leq 5$. Then for the $2i$ positions selected there are two choices; for the $10-2i$ remaining positions there are also two choices – namely, 1,3.

20. (a) We can select 3 vertices from A, B, C, D, E, F, G, H in $\binom{8}{3}$ ways, so there are $\binom{8}{3} = 56$ distinct inscribed triangles.
(b) $\binom{8}{4} = 70$ quadrilaterals.
(c) The total number of polygons is $\binom{8}{3}+\binom{8}{4}+\binom{8}{5}+\binom{8}{6}+\binom{8}{7}+\binom{8}{8} = 2^8-[\binom{8}{0}+\binom{8}{1}+\binom{8}{2}] = 256-[1+8+28] = 219$.

21. There are $\binom{n}{3}$ triangles if sides of the n-gon may be used. Of these $\binom{n}{3}$ triangles, when $n \geq 4$ there are n triangles that use two sides of the n-gon and $n(n-4)$ triangles that use only one side. So if the sides of the n-gon cannot be used, then there are $\binom{n}{3} - n - n(n-4)$, $n \geq 4$, triangles.

22. (a) From the rule of product it follows that there are $4 \times 4 \times 6 = 96$ terms in the complete expansion of $(a+b+c+d)(e+f+g+h)(u+v+w+x+y+z)$.
(b) The terms bvx and egu do not occur as summands in this expansion.

23. (a) $\binom{12}{9}$ (b) $\binom{12}{9}(2^3)$
(c) Let $a = 2x$ and $b = -3y$. By the binomial theorem the coefficient of a^9b^3 in the expansion of $(a+b)^{12}$ is $\binom{12}{9}$. But $\binom{12}{9}\, a^9b^3 = \binom{12}{9}(2x)^9(-3y)^3 = \binom{12}{9}(2^9)(-3)^3\, x^9y^3$, so the coefficient of x^9y^3 is $\binom{12}{9}(2^9)(-3)^3$.

24. $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{t-1}}{n_t} =$
$\left(\frac{n!}{n_1!(n-n_1)!}\right)\left(\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\right)\left(\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\right)\cdots\left(\frac{n_t!}{n_t!0!}\right)$
$= \frac{n!}{n_1!n_2!n_3!\ldots n_t!}$.

25. (a) $\binom{4}{1,1,2} = 12$ (b) $\binom{4}{0,1,1,2} = 12$
(c) $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$ (d) $\binom{4}{1,1,2}(-2)(3)^2 = -216$
(e) $\binom{8}{3,2,1,2}(2)^3(-1)^2(3)(-2)^2 = 161,280$

26. (a) $\binom{10}{2,2,2,2,2} = (10!)/(2!)^5 = 113,400$
(b) $\binom{12}{2,2,2,2,4}(2)^2(-1)^2(3)^2(1)^2(-2)^4 = [(12!)/[(2!)^4(4!)]](2)^2(3)^2(2)^4 = 718,502,400$
(c) $\binom{12}{0,2,2,2,2,4}(1)^2(-2)^2(1)^2(5)^2(3)^4 = [(12!)/(0!)(2!)^4(4!)]](2)^2(5)^2(3)^4 = 10,103,940,000$

27. In each of parts (a)–(e) replace the variables by 1 and evaluate the results.
(a) 2^3 (b) 2^{10} (c) 3^{10} (d) 4^5 (e) 4^{10}

28. a) $\sum_{i=0}^{n} \frac{1}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i} = 2^n/n!$

b) $\sum_{i=0}^{n} \frac{(-1)^i}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n} \frac{(-1)^i n!}{i!(n-i)!} = \frac{1}{n!}\sum_{i=0}^{n}(-1)^i\binom{n}{i} = \frac{1}{n!}(0) = 0.$

29. $n\binom{m+n}{m} = n\frac{(m+n)!}{m!n!} = \frac{(m+n)!}{m!(n-1)!} =$
$(m+1)\frac{(m+n)!}{(m+1)(m!)(n-1)!} = (m+1)\frac{(m+n)!}{(m+1)!(n-1)!} = (m+1)\binom{m+n}{m+1}$

30. The sum is the binomial expansion of $(1+2)^n = 3^n$.

31. (a) $1 = [(1+x)-x]^n = (1+x)^n - \binom{n}{1}x^1(1+x)^{n-1} + \binom{n}{2}x^2(1+x)^{n-2} - \ldots + (-1)^n\binom{n}{n}x^n$.
(b) $1 = [(2+x)-(x+1)]^n$ (c) $2^n = [(2+x)-x]^n$

32. $\sum_{i=0}^{50}\binom{50}{i}8^i = (1+8)^{50} = 9^{50} = [(\pm 3)^2)]^{50} = (\pm 3)^{100}$, so $x = \pm 3$.

33. (a) $\sum_{i=1}^{3}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) = a_3 - a_0$

(b) $\sum_{i=1}^{n}(a_i - a_{i-1}) = (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) =$
$a_n - a_0$

(c) $\sum_{i=1}^{100}(\frac{1}{i+2} - \frac{1}{i+1}) = (\frac{1}{3} - \frac{1}{2}) + (\frac{1}{4} - \frac{1}{3}) + (\frac{1}{5} - \frac{1}{4}) + \ldots + (\frac{1}{101} - \frac{1}{100}) + (\frac{1}{102} - \frac{1}{101}) =$
$\frac{1}{102} - \frac{1}{2} = \frac{1-51}{102} = \frac{-50}{102} = \frac{-25}{51}$.

34.

```
procedure Select2(i,j: positive integers)
begin
   for i := 1 to 5 do
      for j := i + 1 to 6 do
         print (i,j)
end
```

```
procedure Select3(i,j,k: positive integers)
begin
   for i := 1 to 4 do
      for j := i + 1 to 5 do
         for k := j + 1 to 6 do
            print (i,j,k)
end
```

Section 1.4

1. Let $x_i, 1 \le i \le 5$, denote the amounts given to the five children.
(a) The number of integer solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 10,\ 0 \le x_i,\ 1 \le i \le 5$, is $\binom{5+10-1}{10} = \binom{14}{10}$. Here $n = 5$, $r = 10$.
(b) Giving each child one dime results in the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 5,\ 0 \le x_i,\ 1 \le i \le 5$. There are $\binom{5+5-1}{5} = \binom{9}{5}$ ways to distribute the remaining five dimes.
(c) Let x_5 denote the amount for the oldest child. The number of solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 10,\ 0 \le x_i,\ 1 \le i \le 4,\ 2 \le x_5$ is the number of solutions to $y_1 + y_2 + y_3 + y_4 + y_5 = 8,\ 0 \le y_i,\ 1 \le i \le 5$, which is $\binom{5+8-1}{8} = \binom{12}{8}$.

2. Let $x_i,\ 1 \le i \le 5$, denote the number of candy bars for the five children with x_1 the number for the youngest. $(x_1 = 1):\ \ x_2 + x_3 + x_4 + x_5 = 14$. Here there are $\binom{4+14-1}{14} = \binom{17}{14}$ distributions. $(x_1 = 2):\ \ x_2 + x_3 + x_4 + x_5 = 13$. Here the number of distributions is $\binom{4+13-1}{13} = \binom{16}{13}$. The answer is $\binom{17}{14} + \binom{16}{13}$ by the rule of sum.

3. $\binom{4+20-1}{20} = \binom{23}{20}$

4. (a) $\binom{31}{12}$ (b) $\binom{31+12-1}{12} = \binom{42}{12}$
(c) There are 31 ways to have 12 cones with the same flavor. So there are $\binom{42}{12} - 31$ ways to order the 12 cones and have at least two flavors.

5. (a) 2^5
(b) For each of the n distinct objects there are two choices. If an object is not selected, then one of the n identical objects is used in the selection. This results in 2^n possible selections of size n.

6. $\binom{12}{4,4,4}\binom{22}{12}$

7. (a) $\binom{4+32-1}{32} = \binom{35}{32}$ (b) $\binom{4+28-1}{28} = \binom{31}{28}$
(c) $\binom{4+8-1}{8} = \binom{11}{8}$ (d) 1
(e) $x_1 + x_2 + x_3 + x_4 = 32,\ x_i \ge -2,\ 1 \le i \le 4$. Let $y_i = x_i + 2,\ 1 \le i \le 4$. The number of solutions to the given problem is then the same as the number of solutions to $y_1 + y_2 + y_3 + y_4 = 40,\ y_i \ge 0,\ 1 \le i \le 4$. This is $\binom{4+40-1}{40} = \binom{43}{40}$.
(f) $\binom{4+28-1}{28} - \binom{4+3-1}{3} = \binom{31}{28} - \binom{6}{3}$, where the term $\binom{6}{3}$ accounts for the solutions where $x_4 \ge 26$.

8. For the chocolate donuts there are $\binom{3+5-1}{5} = \binom{7}{5}$ distributions. There are $\binom{3+4-1}{4} = \binom{6}{4}$ ways to distribute the jelly donuts. By the rule of product there are $\binom{7}{5}\binom{6}{4}$ ways to distribute the donuts as specified.

9. $$230,230 = \binom{n+20-1}{20} = \binom{n+19}{20} \Longrightarrow n = 7$$

10. Here we want the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100$, $x_i \geq 3$, $1 \leq i \leq 6$. (For $1 \leq i \leq 6$, x_i counts the number of times the face with i dots is rolled.) This is equal to the number of nonnegative integer solutions there are to $y_1+y_2+y_3+y_4+y_5+y_6 = 82$, $y_i \geq 0$, $1 \leq i \leq 6$. Consequently the answer is $\binom{6+82-1}{82} = \binom{87}{82}$.

11. (a) $\binom{10+5-1}{5} = \binom{14}{5}$ (b) $\binom{7+5-1}{5} + 3\binom{7+4-1}{4} + 3\binom{7+3-1}{3} + \binom{7+2-1}{2} = \binom{11}{5} + 3\binom{10}{4} + 3\binom{9}{3} + \binom{8}{2}$, where the first summand accounts for the case where none of 1,3,7 appears, the second summand for when exactly one of 1,3,7 appears once, the third summand for the case of exactly two of these digits appearing once each, and the last summand for when all three appear.

12. (a) The number of solutions for $x_1 + x_2 + \ldots + x_5 < 40, x_i \geq 0$, $1 \leq i \leq 5$, is the same as the number for $x_1 + x_2 + \ldots + x_5 \leq 39, x_i \geq 0$, $1 \leq i \leq 5$, and this equals the number of solutions for $x_1 + x_2 + \ldots + x_5 + x_6 = 39, x_i \geq 0$, $1 \leq i \leq 6$. There are $\binom{6+39-1}{39} = \binom{44}{39}$ such solutions.
(b) Let $y_i = x_i + 3$, $1 \leq i \leq 5$, and consider the inequality $y_1 + y_2 + \ldots + y_5 \leq 54$, $y_i \geq 0$. There are [as in part (a)] $\binom{6+54-1}{54} = \binom{59}{54}$ solutions.

13. (a) $\binom{4+4-1}{4} = \binom{7}{4}$.
(b) $\binom{3+7-1}{7}$ (container 4 has one marble) $+\binom{3+5-1}{5}$ (container 4 has three marbles) $+\binom{3+3-1}{3}$ (container 4 has five marbles) $+\binom{3+1-1}{1}$ (container 4 has seven marbles) $= \sum_{i=0}^{3} \binom{9-2i}{7-2i}$.

14. (a) $\binom{8}{2,4,1,0,1}(3)^2(2)^4$
(b) The terms in the expansion have the form $v^a w^b x^c y^d z^e$ where a, b, c, d, e are nonnegative integers that sum to 8. There are $\binom{5+8-1}{8} = \binom{12}{8}$ terms.

15. Consider one such distribution – the one where there are six books on each of the four shelves. Here there are 24! ways for this to happen. And we see that there are also 24! ways to place the books for any other such distribution.

The number of distributions is the number of positive integer solutions to

$$x_1 + x_2 + x_3 + x_4 = 24.$$

This is the same as the number of nonnegative integer solutions for

$$y_1 + y_2 + y_3 + y_4 = 20.$$

[Here $y_i + 1 = x_i$ for all $1 \leq i \leq 4$.]
So there are $\binom{4+20-1}{20} = \binom{23}{20}$ such distributions of the books, and consequently, $\binom{23}{20}(24!)$ ways in which Beth can arrange the 24 books on the four shelves with at least one book on each shelf.

16. For equation (1) we need the number of nonnegative integer solutions for $w_1 + w_2 + w_3 + \ldots + w_{19} = n - 19$, where $w_i \geq 0$ for all $1 \leq i \leq 19$. This is $\binom{19+(n-19)-1}{(n-19)} = \binom{n-1}{n-19}$. The number of positive integer solutions for equation (2) is the number of nonnegative integer solutions for

$$z_1 + z_2 + z_3 + \ldots + z_{64} = n - 64,$$

and this is $\binom{64+(n-64)-1}{(n-64)} = \binom{n-1}{n-64}$.
So $\binom{n-1}{n-19} = \binom{n-1}{n-64} = \binom{n-1}{63}$ and $n - 19 = 63$. Hence $n = 82$.

17. (a) $\binom{5+12-1}{12} = \binom{16}{12}$ (b) 5^{12}

18. (a) There are $\binom{3+6-1}{6} = \binom{8}{6}$ solutions for $x_1+x_2+x_3 = 6$ and $\binom{4+31-1}{31} = \binom{34}{31}$ solutions for $x_4 + x_5 + x_6 + x_7 = 31$, where $x_i \geq 0$, $1 \leq i \leq 7$. By the rule of product the pair of equations has $\binom{8}{6}\binom{34}{31}$ solutions.
(b) $\binom{5}{3}\binom{34}{31}$

19. Here there are $r = 4$ nested **for** loops, so $1 \leq m \leq k \leq j \leq i \leq 20$. We are making selections, with repetition, of size $r = 4$ from a collection of size $n = 20$. Hence the **print** statement is executed $\binom{20+4-1}{4} = \binom{23}{4}$ times.

20. Here there are $r = 3$ nested **for** loops and $1 \leq i \leq j \leq k \leq 15$. So we are making selections, with repetition, of size $r = 3$ from a collection of size $n = 15$. Therefore the statement

$$counter := counter + 1$$

is executed $\binom{15+3-1}{3} = \binom{17}{3}$ times, and the final value of the variable *counter* is $10+\binom{17}{3} = 690$.

21. The **begin-end** segment is executed $\binom{10+3-1}{3} = \binom{12}{3} = 220$ times. After the execution of this segment the value of the variable *sum* is $\sum_{i=1}^{220} i = (220)(221)/2 = 24,310$.

22. $\binom{n+2}{3} = \sum_{i=1}^{n}\binom{i+1}{2} \Longrightarrow \frac{(n+2)(n+1)n}{6} = \frac{1}{2}\sum_{i=1}^{n}(i+1)i \Longrightarrow \frac{(n+2)(n+1)n}{6} = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i \Longrightarrow \frac{1}{2}\sum_{i=1}^{n} i^2 = \frac{(n+2)(n+1)n}{6} - \frac{(n+1)n}{4} \Longrightarrow \sum_{i=1}^{n} i^2 = n(n+1)[\frac{n+2}{3} - \frac{1}{2}] = n(n+1)[\frac{2n+4-3}{6}] = \frac{n(n+1)(2n+1)}{6}$.

23. (a) Put one object into each container. Then there are $m - n$ identical objects to place into n distinct containers. This yields $\binom{n+(m-n)-1}{m-n} = \binom{m-1}{m-n} = \binom{m-1}{n-1}$ distributions.
(b) Place r objects into each container. The remaining $m - rn$ objects can then be distributed among the n distinct containers in $\binom{n+(m-rn)-1}{m-rn} = \binom{m-1+(1-r)n}{m-rn} = \binom{m-1+(1-r)n}{n-1}$ ways.

24. (a)

procedure *Selections1(i,j*: nonnegative integers)

```
begin
    for i := 0 to 10 do
        for j := 0 to 10 − i do
            print (i,j, 10 − i − j)
end
```

(b) For all $1 \leq i \leq 4$ let $y_i = x_i + 2 \geq 0$. Then the number of integer solutions to $x_1 + x_2 + x_3 + x_4 = 4$, where $-2 \leq x_i$ for $1 \leq i \leq 4$, is the number of integer solutions to $y_1 + y_2 + y_3 + y_4 = 12$, where $y_i \geq 0$ for $1 \leq i \leq 4$. We use this observation in the following.

```
procedure Selections2(i,j,k: nonnegative integers)
begin
    for i := 0 to 12 do
        for j := 0 to 12 − i do
            for k := 0 to 12 − i − j do
                print (i,j,k, 12 − i − j − k)
end
```

25. If the summands must all be even, then consider one such composition – say,

$$20 = 10 + 4 + 2 + 4 = 2(5 + 2 + 1 + 2).$$

Here we notice that $5 + 2 + 1 + 2$ provides a composition of 10. Further, each composition of 10, when multiplied through by 2, provides a composition of 20, where each summand is even. Consequently, we see that the number of compositions of 20, where each summand is even, equals the number of compositions of 10 – namely, $2^{10-1} = 2^9$.

26. Each such composition can be factored as k times a composition of m. Consequently, there are 2^{m-1} compositions of n, where $n = mk$ and each summand in a composition is a multiple of k.

27. a) Here we want the number of integer solutions for $x_1 + x_2 + x_3 = 12$, $x_1, x_3 > 0$, $x_2 = 7$. The number of integer solutions for $x_1 + x_3 = 5$, with $x_1, x_3 > 0$, is the same as the number of integer solutions for $y_1 + y_3 = 3$, with $y_1, y_3 \geq 0$. This is $\binom{2+3-1}{3} = \binom{4}{3} = 4$.

b) Now we must also consider the integer solutions for $w_1 + w_2 + w_3 = 12$, $w_1, w_3 > 0$, $w_2 = 5$. The number here is $\binom{2+5-1}{5} = \binom{6}{5} = 6$.

Consequently, there are $4 + 6 = 10$ arrangements that result in three runs.

c) The number of arrangements for four runs requires two cases [as above in part (b)].

If the first run consists of heads, then we need the number of integer solutions for $x_1 + x_2 + x_3 + x_4 = 12$, where $x_1 + x_3 = 5$, $x_1, x_3 > 0$ and $x_2 + x_4 = 7$, $x_2, x_4 > 0$. This number is $\binom{2+3-1}{3}\binom{2+5-1}{5} = \binom{4}{3}\binom{6}{5} = 4 \cdot 6 = 24$. When the first run consists of tails we get $\binom{6}{5}\binom{4}{3} = 6 \cdot 4 = 24$ arrangements.

In all there are $2(24) = 48$ arrangements with four runs.

d) If the first run starts with an H, then we need the number of integer solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ where $x_1 + x_3 + x_5 = 5$, $x_1, x_3, x_5 > 0$ and $x_2 + x_4 = 7$, $x_2, x_4 > 0$. This is $\binom{3+2-1}{2}\binom{2+5-1}{5} = \binom{4}{2}\binom{6}{5} = 36$. For the case where the first run starts with a T, the number of arrangements is $\binom{3+4-1}{4}\binom{2+3-1}{3} = \binom{6}{4}\binom{4}{3} = 60$.

In total there are $36 + 60 = 96$ ways for these 12 tosses to determine five runs.

e) $\binom{3+4-1}{4}\binom{3+2-1}{2} = \binom{6}{4}\binom{4}{2} = 90$ – the number of arrangements which result in six runs, if the first run starts with an H. But this is also the number when the first run starts with a T. Consequently, six runs come about in $2 \cdot 90 = 180$ ways.

f) $2\binom{1+4-1}{4}\binom{1+6-1}{6} + 2\binom{2+3-1}{3}\binom{2+5-1}{5} + 2\binom{3+2-1}{2}\binom{3+4-1}{4} + 2\binom{4+1-1}{1}\binom{4+3-1}{3} + 2\binom{5+0-1}{0}\binom{5+2-1}{2} = 2\sum_{i=0}^{4}\binom{4}{4-i}\binom{6}{6-i} = 2[1 \cdot 1 + 4 \cdot 6 + 6 \cdot 15 + 4 \cdot 20 + 1 \cdot 15] = 420$.

28. (a) For $n \geq 4$, consider the strings made up of n bits – that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?
(b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?
(c) Provide a combinatorial proof for the following:

$$\text{For } n \geq 1, \quad 2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

(a) A string of this type consists of x_1 1's followed by x_2 0's followed by x_3 1's followed by x_4 0's followed by x_5 1's followed by x_6 0's, where,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = n, \quad x_1, x_6 \geq 0, \quad x_2, x_3, x_4, x_5 > 0.$$

The number of solutions to this equation equals the number of solutions to

$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = n - 4, \quad \text{where } y_i \geq 0 \text{ for } 1 \leq i \leq 6.$$

This number is $\binom{6+(n-4)-1}{n-4} = \binom{n+1}{n-4} = \binom{n+1}{5}$.

(b) For $n \geq 6$, a string with this structure has x_1 1's followed by x_2 0's followed by x_3 1's ... followed by x_8 0's, where

$$x_1 + x_2 + x_3 + \cdots + x_8 = n, \quad x_1, x_8 \geq 0, \quad x_2, x_3, \ldots, x_7 > 0.$$

The number of solutions to this equation equals the number of solutions to

$$y_1 + y_2 + y_3 + \cdots + y_8 = n - 6, \quad \text{where } y_i \geq 0 \text{ for } 1 \leq i \leq 8.$$

This number is $\binom{8+(n-6)-1}{n-6} = \binom{n+1}{n-6} = \binom{n+1}{7}$.

(c) There are 2^n strings in total and $n+1$ strings where there are k 1's followed by $n-k$ 0's, for $k = 0, 1, 2, \ldots, n$. These $n+1$ strings contain no occurrences of 01, so there are $2^n - (n+1) = 2^n - \binom{n+1}{1}$ strings that contain at least one occurrence of 01. There are $\binom{n+1}{3}$ strings that contain (exactly) one occurrence of 01, $\binom{n+1}{5}$ strings with (exactly) two occurrences, $\binom{n+1}{7}$ strings with (exactly) three occurrences, ... , and for

(i) n odd, we can have at most $\frac{n-1}{2}$ occurrences of 01. The number of strings with $\frac{n-1}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+1} = n,\ x_1, x_{n+1} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+1} = n - (n-1) = 1, \text{ where } y_1, y_2, \ldots, y_{n+1} \geq 0.$$

This number is $\binom{(n+1)+1-1}{1} = \binom{n+1}{1} = \binom{n+1}{n} = \binom{n+1}{2(\frac{n-1}{2})+1}$.

(ii) n even, we can have at most $\frac{n}{2}$ occurrences of 01. The number of strings with $\frac{n}{2}$ occurrences of 01 is the number of integer solutions for

$$x_1 + x_2 + \cdots + x_{n+2} = n,\ x_1, x_{n+2} \geq 0, \quad x_2, x_3, \ldots, x_n > 0.$$

This is the same as the number of integer solutions for

$$y_1 + y_2 + \cdots + y_{n+2} = n - n = 0, \text{ where } y_i \geq 0 \text{ for } 1 \leq i \leq n+2.$$

This number is $\binom{(n+2)+0-1}{0} = \binom{n+1}{0} = \binom{n+1}{n+1} = \binom{n+1}{2(\frac{n}{2})+1}$.

Consequently,

$$2^n - \binom{n+1}{1} = \binom{n+1}{3} + \binom{n+1}{5} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even,} \end{cases}$$

and the result follows.

Section 1.5

1.

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} =$$

$$\frac{(2n)!(n+1)}{(n+1)!n!} - \frac{(2n)!n}{n!(n+1)!} = \frac{(2n)![(n+1)-n]}{(n+1)!n!} = \frac{1}{(n+1)}\frac{(2n)!}{n!n!} =$$

$$(\frac{1}{n+1})\binom{2n}{n}$$

2. $b_7 = 429$ $b_8 = 1430$ $b_9 = 4862$ $b_{10} = 16796$

3. (a) $5(= b_3)$; $14(= b_4)$

(b) For $n \geq 0$ there are $b_n(= \frac{1}{(n+1)}\binom{2n}{n})$ such paths from (0,0) to (n, n).

(c) For $n \geq 0$ the first move is U and the last is H.

4. (a) $b_6 = 132$ (b) $b_5 = 42$ (c) $b_7 = 429$

5. Using the results in the third column of Table 1.10 we have:

111000	110010	101010
1 2 3 4 5 6	1 2 5 3 4 6	1 3 5 2 4 6

6.

(a) (i) 1 3 4 7 / 2 5 6 8 (ii) 1 2 5 7 / 3 4 6 8 (iii) 1 2 3 5 / 4 6 7 8

(b) (i) 10111000 (ii) 11100010 (iii) 11011000

7. There are $b_5(= 42)$ ways.

8. (a) (i) 1110001010 (ii) 1010101010 (iii) 1111001000

(b) (i) $(((ab(c(de \leftrightarrow (((ab)(c(de)))f)$
(ii) $((ab((cd(e \leftrightarrow ((ab)((cd)(ef)))$
(iii) $(a(((bc(de \leftrightarrow (a(((bc)(de))f))$

9. (i) When $n = 4$ there are $14(= b_4)$ such diagrams.

(ii) For any $n \geq 0$, there are b_n different drawings of n semicircles on and above a horizontal line, with no two semicircles intersecting. Consider, for instance, the diagram in part (f) of the figure. Going from left to right, write 1 the first time you encounter a semicircle and write 0 the second time that semicircle is encountered. Here we get the list 110100. The list 110010 corresponds with the drawing in part (g). This correspondence shows that the number of such drawings for n semicircles is the same as the number of lists of n 1's and n 0's where, as the list is read from left to right, the number of 0's never exceeds the number of 1's.

10. (a) In total there are $\binom{10}{7} = \binom{10}{3}$ paths from (0,0) to (7,3), each made up of seven R's and three U's. From these $\binom{10}{7}$ paths we remove those that violate the stated condition – namely, those paths where the number of U's exceeds the number R's (at some first position in the path). For example, consider one such path:

$$R\,U\,R\,U\,U\,R\,R\,R\,R\,R.$$

Here the condition is violated, for the first time, after the third U. Transform the given path as follows:

$$R\ U\ R\ U\ U \vdots R\ R\ R\ R\ R \leftrightarrow R\ U\ R\ U\ U\ U\ U\ U\ U\ U.$$

Here the entries up to and including the first violation remain unchanged, while those following the first violation are changed: R's become U's and U's become R's. This correspondence shows us that the number of paths that violate the given condition is the same as the number of paths made up of eight U's and two R's – and there are $\binom{10}{8} = \binom{10}{2}$ such paths.

Consequently, the answer is
$\binom{10}{7} - \binom{10}{8} = \frac{10!}{7!3!} - \frac{10!}{8!2!} = \frac{10!(8)}{8!3!} - \frac{10!(3)}{8!3!} = (\frac{5}{8})\frac{10!}{7!3!} = (\frac{7+1-3}{7+1})\binom{10}{7}$.

(b) $\binom{m+n}{n} - \binom{m+n}{n+1} = \frac{(m+n)!}{n!m!} - \frac{(m+n)!}{(n+1)!(m-1)!}$
$= \frac{(m+n)!(n+1)-(m+n)!m}{(n+1)!m!} = (\frac{n+1-m}{n+1})(\frac{(m+n)!}{n!m!}) = (\frac{n+1-m}{n+1})\binom{m+n}{n}$.
[Note that when $m = n$, this becomes $(\frac{1}{n+1})\binom{2n}{n}$, the formula for the nth Catalan number.]

11. Consider one of the $(\frac{1}{6+1})\binom{2\cdot 6}{6} = (\frac{1}{7})\binom{12}{6}$ ways in which the \$5 and \$10 bills can be arranged – say,

(*) \$5, \$5, \$10, \$5, \$5, \$10, \$10, \$10, \$5, \$5, \$10, \$10.

Here we consider the six \$5 bills as indistinguishable – likewise, for the six \$10 bills. However, we consider the patrons as distinct. Hence, there are 6! ways for the six patrons, each with a \$5 bill, to occupy positions 1, 2, 4, 5, 9, and 10, in the arrangement (*). Likewise, there are 6! ways to locate the other six patrons (each with a \$10 bill). Consequently, here the number of arrangements is

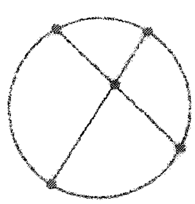
$$(\frac{1}{7})\binom{12}{6}(6!)(6!) = (\frac{1}{7})(12!) = 68,428,800.$$

Supplementary Exercises

1. $\binom{4}{1}\binom{7}{2} + \binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{6}$

2. (a) 5^9 (b) $5(4^8)$

3.



Select any four of these twelve points (on the circumference). As seen in the figure, these points determine a pair of chords that intersect. Consequently, the largest number of points of intersection for all possible chords is $\binom{12}{4} = 495$.

4. (a) $\binom{25}{2}^3$
(b) $3\binom{25}{1}^2\binom{25}{4}$ (four hymns from one book, one from each of the other two) $+ 6\binom{25}{1}\binom{25}{2}\binom{25}{3}$ (one hymn from one book, two hymns from a second book, and three from the third book) $+\binom{25}{2}^3$ (two hymns from each of the three books).

5. (a) 10^{25}
(b) There are 10 choices for the first flag. For the second flag there are 11 choices: The nine poles with no flag, and above or below the first flag on the pole where it is situated. There are 12 choices for the third flag, 13 choices for the fourth,..., and 34 choices for the last (25th). Hence there are $(34!)/(9!)$ possible arrangements.
(c) There are 25! ways to arrange the flags. For each arrangement consider the 24 spaces, one between each pair of flags. Selecting 9 of these spaces provides a distribution among the 10 flagpoles where every flagpole has at least one flag and order is relevant. Hence there are $(25!)\binom{24}{9}$ such arrangements.

6. Consider the 45 heads and the 46 positions they determine: (1) One position to the left of the first head; (2) One position between the i-th head and the $(i+1)$-st head, where $1 \leq i \leq 44$; and, (3) One position to the right of the 45-th (last) head. To answer the question posed we need to select 15 of the 46 positions. This we can do in $\binom{46}{15}$ ways.

In an alternate way, let x_i denote the number of heads to the left of the i-th tail, for $1 \leq i \leq 15$. Let x_{16} denote the number of heads to the right of the 15th tail. Then we want the number of integer solutions for

$$x_1 + x_2 + x_3 + \ldots + x_{15} + x_{16} = 45,$$

where $x_1 \geq 0$, $x_{16} \geq 0$, and $x_i > 0$ for $2 \leq i \leq 15$. This is the number of integer solutions for

$$y_1 + y_2 + y_3 + \ldots + y_{15} + y_{16} = 31,$$

with $y_i \geq 0$ for $1 \leq i \leq 16$. Consequently the answer is $\binom{16+31-1}{31} = \binom{46}{31} = \binom{46}{15}$.

7. (a) $C(12,8)$ (b) $P(12,8)$

8. There are $(7!/2!)$ ways to arrange the seven symbols OE,W,N,N,D,R,G. In each arrangement there are 6 locations for the I so that it is not adjacent to a vowel, so there are $(6)(7!/2!)$ arrangements. The three vowels can be divided up into a pair and a single vowel in six ways (order counts), so the total number of arrangements is $(6^2)(7!/2!)$.

9. (a) There are two blocks, for example, that differ only in size. There are four that differ only in color, one that differs only in the material used for construction, and five that differ only in shape. In total there are $2 + 4 + 1 + 5 = 12$ blocks that differ from the ***small red wooden square*** block in exactly one way.
(b) There are $\binom{4}{2} = 6$ ways of selecting the two differing properties. Each such pair must be considered separately.

(i) Material, size: Here there are $1 \times 2 = 2$ such blocks.
(ii) Material, color: This pair yields $1 \times 4 = 4$ such blocks.
(iii) Material, shape: For this pair we obtain $1 \times 5 = 5$ such blocks.
(iv) Size, color: Here we get $2 \times 4 = 8$ of the blocks.
(v) Size, shape: This pair gives us $2 \times 5 = 10$ such blocks.
(vi) Color, shape: For this pair we find $4 \times 5 = 20$ of the blocks we need to count.

In total there are $2 + 4 + 5 + 8 + 10 + 20 = 49$ of Dustin's blocks that differ from the *large blue plastic hexagonal* block in exactly two ways.

10. Since 'R' is the 18th letter of the alphabet, the first and middle initials can be chosen in $\binom{17}{2} = (17)(16)/2 = 136$ ways.

Alternately, since 'R' is the 18th letter of the alphabet, consider what happens when the middle initial is any letter between 'B' and 'Q'. For middle initial 'Q' there are 16 possible first initials. For middle initial 'P' there are 15 possible choices. Continuing back to 'B' where there is only one choice (namely 'A') for the first initial, we find that the total number of choices is $1 + 2 + 3 + \ldots + 15 + 16 = (16)(17)/2 = 136$.

11. The number of linear arrangements of the 11 horses is $11!/(5!3!3!)$. Each circular arrangement represents 11 linear arrangements, so there are $(1/11)[11!/(5!3!3!)]$ ways to arrange the horses on the carousel.

12. (a) $P(16,12)$ (b) $\binom{12}{2} P(15,10)$

13. (a) (i) $\binom{5}{4} + \binom{5}{2}\binom{4}{2} + \binom{4}{4}$ (ii) $\binom{5+4-1}{4} + \binom{5+2-1}{2}\binom{4+2-1}{2} + \binom{4+4-1}{4} = \binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4}$ (iii) $\binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4} - 9$
(b) (i) $\binom{5}{1}\binom{4}{3} + \binom{5}{3}\binom{4}{1}$ (ii) and (iii) $\binom{5}{1}\binom{4+3-1}{3} + \binom{5+3-1}{3}\binom{4}{1} = \binom{5}{1}\binom{6}{3} + \binom{7}{3}\binom{4}{1}$.

14. (a) If there are no restrictions Mr. Kelly can make the assignments in $12! = 479{,}001{,}600$ ways.
(b) Mr. DiRocco and Mr. Fairbanks can be assigned in $4 \times 3 = 12$ ways, and the other 10 assistants can then be assigned in $10!$ ways. Consequently, in this situation, Mr. Kelly can make one of $12(10!) = 43{,}545{,}600$ assignments.
(c) Suppose that Mr. Hyland is assigned to the first floor and Mr. Thornhill is assigned to the third floor. This can be accomplished in $4 \times 4 \times (10!) = 58,060,800$ ways. There are $3 \times 2 = 6$ ways to assign these two assistants to different floors, so in this case we have $(3 \times 2) \times [4 \times 4 \times (10!)] = 348,364,800$ possibilities.
Alternately, from part (b), there are $3 \times [12(10!)] = 130,636,800$ ways in which Mr. Hyland and Mr. Thornhill could be assigned to the same floor — and $(12!) - [(3)(12)(10!)] = 348{,}364{,}800$.

15. (a) For each increasing four-digit integer we have four distinct digits, which can only be arranged in one way. These four digits can be chosen in $\binom{9}{4} = 126$ ways. And these same four digits can also be arranged as a decreasing four-digit integer.

To complete the solution we must account for the decreasing four-digit integers where the units digit is 0. There are $\binom{9}{3} = 84$ of these.

Consequently there are $2\binom{9}{4} + \binom{9}{3} = 343$ such four-digit integers.

(b) For each nondecreasing four-digit integer we have four nonzero digits, with repetitions allowed. These four digits can be selected in $\binom{9+4-1}{4} = \binom{12}{4}$ ways. And these same four digits account for a nonincreasing four-digit integer. So at this point we have $2\binom{12}{4} - 9$ of the four-digit integers we want to count. (The reason we subtract 9 is because we have counted the nine integers 1111, 2222, 3333, ..., 9999 twice in $2\binom{12}{4}$.)

We have not accounted for those nonincreasing four-digit integers where the units digit is 0. There are $\binom{10+3-1}{3} - 1 = \binom{12}{3} - 1$ of these four-digit integers. (Here we subtracted 1 since we do not want to include 0000.)

Therefore there are $[2\binom{12}{4} - 9] + [\binom{12}{3} - 1] = [2\binom{12}{4} + \binom{12}{3}] - 10 = 1200$ such four-digit integers.

16. (a) $\binom{5}{2,1,2}(1/2)^2(-3)^2 = 135/2$
(b) Each term is of the form $x^{n_1}y^{n_2}z^{n_3}$ where each n_i, $1 \leq i \leq 3$, is a nonnegative integer and $n_1 + n_2 + n_3 = 5$. Consequently, there are $\binom{3+5-1}{5} = \binom{7}{5}$ terms.
(c) Replace x, y, and z by 1. Then the sum of all the coefficients in the expansion is $((1/2) + 1 - 3)^5 = (-3/2)^5$.

17. (a) First place person A at the table. There are five distinguishable places available for A (e.g., any of the positions occupied by A,B,C,D,E in Fig. 1.11(a)). Then position the other nine people relative to A. This can be done in 9! ways, so there are (5)(9!) seating arrangements.
(b) There are three distinct ways to position A,B so that they are seated on longer sides of the table across from each other. The other eight people can then be located in 8! different ways, so the total number of arrangements is (3)(8!).

18. (a) For $x_1 + x_2 + x_3 = 6$ there are $\binom{3+6-1}{6} = \binom{8}{6}$ nonnegative integer solutions. With $x_1 + x_2 + x_3 = 6$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 15$, the number of nonnegative integer solutions for $x_4 + x_5 = 9$ is $\binom{2+9-1}{9} = \binom{10}{9}$. The number of solutions for the pair of equations is $\binom{8}{6}\binom{10}{9}$.
(b) Let $0 \leq k \leq 6$. For $x_1 + x_2 + x_3 = k$ there are $\binom{3+k-1}{k} = \binom{k+2}{k}$ solutions. To solve $x_4 + x_5 \leq 15 - k$, consider $x_4 + x_5 + x_6 = 15 - k$, $x_4, x_5, x_6 \geq 0$. Here there are $\binom{3+15-k-1}{15-k} = \binom{17-k}{15-k}$ solutions. The total number of solutions is $\sum_{k=0}^{6}\binom{k+2}{k}\binom{17-k}{15-k}$.

19. (a) Here A must win set 5 and exactly two of the four earlier sets. This can be done in $\binom{4}{2}$ ways. With seven possible scores for each set there are $\binom{4}{2}7^5$ ways for the scores to be recorded.

(b) Here A can win in four sets in $\binom{3}{2}$ ways, and scores can be recorded in $\binom{3}{2}7^4$ ways. So if A wins in four or five sets, then the scores can be recorded in $[\binom{3}{2}7^4 + \binom{4}{2}7^5]$ ways. Since B may be the winner, the final answer is $2[\binom{3}{2}7^4 + \binom{4}{2}7^5]$.

20. We can choose r objects from n in $\binom{n}{r}$ ways. Once the r objects are selected they can be arranged in a circle in $(r-1)!$ ways. So there are $\binom{n}{r}(r-1)!$ circular arrangements of the n objects taken r at a time.

21. For every positive integer n, $0 = (1-1)^n = \binom{n}{0}(1)^0 - \binom{n}{1}(1)^1 + \binom{n}{2}(1)^2 - \binom{n}{3}(1)^3 + \ldots + (-1)^n\binom{n}{n}(1)^n$, and $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$

22. (a) $7!/3!$ (b) $5!$ (c) $\binom{5}{3}(4!)$

23. (a) There are $P(20,12) = \frac{20!}{8!} = (20)(19)(18)\cdots(11)(10)(9)$ ways in which Francesca can fill her bookshelf.
(b) There are $\binom{17}{9}$ ways in which Francesca can select nine other books. Then she can arrange those nine books and the three books on tennis on her bookshelf in $12!$ ways. Consequently, among the arrangements in part (a), there are $\binom{17}{9}(12!)$ arrangements that include Francesca's three books on tennis.

24. Following the execution of this program segment the value of *counter* is
$10+(12-1+1)(r-1+1)(2)+[3+4+\ldots+(s-3+1)](4)+(12-3+1)(6)+(t-7+1)(8) =$
$10+(12)(r)(2)+[(1/2)(s-3+1)(s-3+2)-2-1](4)+(10)(6)+(t-6)(8) =$
$22+24r+8t+2(s-2)(s-1)-12 = 14+24r+8t+2s(s-3)$.

25. (a) For 17 there must be an odd number, between 1 and 17 inclusive, of 1's.
For $2k+1$ 1's, where $0 \le k \le 8$, there are $2k+2$ locations to select, with repetitions allowed. The selection size is the number of 2's, which is $(1/2)[17-(2k+1)] = 8-k$. The selection can be made in $\binom{2k+2+(8-k)-1}{8-k} = \binom{9+k}{8-k}$ ways, and so the answer is $\sum_{k=0}^{8}\binom{9+k}{8-k} = 2584$.
(b) In the case of 18 the number of 1's must be even: $2k$, for $0 \le k \le 9$. If there are $2k$ 1's, there are $2k+1$ locations, with repetitions allowed, for the $(1/2)(18-2k) = 9-k$ 2's. The selection can be made in $\binom{2k+1+(9-k)-1}{9-k} = \binom{9+k}{9-k}$ ways, and the answer is $\sum_{k=0}^{9}\binom{9+k}{9-k} = 4181$.
(c) For n odd, let $n = 2k+1$ for $k \ge 0$. The number of ways to write n as an ordered sum of 1's and 2's is $\sum_{i=0}^{k}\binom{k+1+i}{k-i}$.
For n even, let $n = 2k$ for $k \ge 1$. Here the answer is $\sum_{i=1}^{k}\binom{k+i}{k-i}$.

26. (a) (i) 1 (one 3) + 1 (three 3's) + 1 (five 3's) = 3.
(ii) $\binom{8}{1}$ (one 3) + $\binom{7}{3}$ (three 3's) + $\binom{6}{5}$ (five 3's).
(b) (i) 1 (no 3's) + 1 (two 3's) + 1 (four 3's) + 1 (six 3's) = 4.
(ii) $\binom{9}{0}$ (no 3's) + $\binom{8}{2}$ (two 3's) + $\binom{7}{4}$ (four 3's) + $\binom{6}{6}$ (six 3's).

27. (a) The number of positive integer solutions to the given equation is the same as the number of nonnegative integer solutions for $y_1 + y_2 + \ldots + y_r = n - r$, where $y_i \geq 0$ for all $1 \leq i \leq r$. Here there are $\binom{r+(n-r)-1}{n-r} = \binom{n-1}{n-r} = \binom{n-1}{r-1}$ solutions.
(b) The total is $\sum_{r=1}^{n} \binom{n-1}{r-1} = \binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{n-1} = 2^{n-1}$.

28. (a) There are $5 - 1 = 4$ horizontal moves and $9 - 2 = 7$ vertical moves. One can arrange 4 R's and 7 U's in $11!/(4!7!)$ ways.
(b) Since a diagonal move takes the place of one horizontal move and one vertical move, the number of diagonal moves is between 0 and 4, inclusive. The resulting cases are as follows:

(0 D's):	4 R's, 7 U's:	$11!/(4!7!)$
(1 D's):	3 R's, 6 U's:	$10!/(1!3!6!)$
(2 D's):	2 R's, 5 U's:	$9!/(2!2!5!)$
(3 D's):	1 R, 4 U's:	$8!/(3!1!4!)$
(4 D's):	0 R's, 3 U's:	$7!/(4!0!3!)$

The answer is the sum of the results: $\sum_{i=0}^{4}[(11-i)!/(i!(4-i)!(7-i)!)]$.

29. (a) $11!/(7!4!)$
(b) $[11!/(7!4!)] - [4!/(2!2!)][4!/(3!1!)]$
(c) $[11!/(7!4!)] + [10!/(6!3!1!)] + [9!/(5!2!2!)] + [8!/(4!1!3!)] + [7!/(3!0!4!)]$ (for part (a))
$\{[11!/(7!4!)] + [10!/(6!3!1!)] + [9!/(5!2!2!)] + [8!/(4!1!3!)] + [7!/(3!0!4!)]\} -$
$[\{[4!/(2!2!)] + [3!/(1!1!1!)] + [2!/2!]\} \times \{[4!/(3!1!)] + [3!/(2!1!)]\}]$ (for part (b)).

30. Here we want certain paths from (1,1) to (14,4) where the moves are of the form:
$(m, n) \to (m+1, n+1)$, if the $(n+1)$-st ballot is for Katalin.
$(m, n) \to (m+1, n-1)$, if the $(n+1)$-st ballot is for Donna.

These paths are the ones that never touch or cross the horizontal (or x-) axis. In general, an ordered pair (m, n) here indicates that m ballots have been counted with Katalin leading by n votes. The number of ways to count the ballots according to the prescribed conditions is

$$\binom{13}{8} - \binom{13}{9} = 1287 - 715 = 572.$$

31. Each rectangle (contained within the 8×5 grid) is determined by four corners of the form $(a, b), (c, b), (c, d), (a, d)$, where a, b, c, d are integers with $0 \leq a < c \leq 8$ and $0 \leq b < d \leq 5$. We can select the pair a, c in $\binom{9}{2}$ ways and the pair b, d in $\binom{6}{2}$ ways. Consequently, the number of rectangles is $\binom{9}{2}\binom{6}{2} = 540$.

32. Here we consider the number of integer solutions for

$$x_1 + x_2 + x_3 = 6, \quad x_i > 0, \quad 1 \leq i \leq 3, \text{ and } w_1 + w_2 = 6, \quad w_i > 0, \quad 1 \leq i \leq 2.$$

This equals the number of integer solutions for

$$y_1 + y_2 + y_3 = 3, \quad y_i \geq 0, \quad 1 \leq i \leq 3, \text{ and } z_1 + z_2 = 3, \quad z_i \geq 0, \quad 1 \leq i \leq 2.$$

So the answer is $\binom{3+3-1}{3}\binom{2+3-1}{3} = \binom{5}{3}\binom{4}{3}$.

33. There are $\binom{6}{4} = 15$ ways to choose the four quarters when Hunter will take these electives. For each of these choices of four quarters, there are $12 \cdot 11 \cdot 10 \cdot 9$ ways to assign the electives. So, in total, there are $\binom{6}{4} \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 178,200$ ways for Hunter to select and schedule these four electives.

34. Consider the family as one unit. Then we are trying to arrange nine distinct objects – the family and the eight other people – around the table. This can be done in 8! ways. Since the family unit can be arranged in four ways, the total number of arrangements under the prescribed conditions is 4(8!).