UNIT-I

PROBABILITY AND RANDOM VARIABLES

Topic Learning Objectives:

- To apply the knowledge of the statistical analysis and theory of probability in the study of uncertainties.
- To use probability theory to solve random physical phenomena and implement appropriate distribution models.

Prerequisites:

If an experiment is repeated under essentially homogeneous and similar conditions we generally come across two types of situations:

(i) The result or what is usually known as the 'outcome' is unique or certain.

(ii) The result is not unique but may be one of the several possible outcomes.

The phenomena covered by (i) are known as 'deterministic' or 'predictable' phenomena. By a deterministic phenomenon we mean one in which the result can be predicted with certainty.

For example:

(a) The velocity 'v' of a particle after time 't ' is given by v = u + at where ‘u’ is the initial velocity and ‘a’ is the acceleration. This equation uniquely determines ‘v’ if the right-hand quantities are known.

(b) Ohm’s Law , viz., C = E/R where C is the flow of current, E the potential difference between the two ends of the conductor and R the resistance, uniquely determines the value C as soon as E and R are given.

A deterministic model is defined as a model which stipulates that the conditions under which an experiment is performed to determine the outcome of the experiment. For a number of situations the deterministic model suffices. However, there are phenomena (as covered by (ii) above) which do not lend themselves to deterministic approach and are known as 'unpredictable' or 'probabilistic' phenomena. For example:

(a) In tossing of a coin one is not sure if a head or tail will be obtained.

(b) If a light tube has lasted for t hours, nothing can be said about its further life. It may fail to function any moment.

In such cases we talk of chance or probability which is taken to be a quantitative measure of certainty.

Some basic definitions:

Trial and Event: Consider an experiment which, though repeated under essentially identical conditions, does not give unique results but may result in any one of the several possible outcomes. The experiment is known as a trial and the outcomes are known as events or casts. For example:

(i) Throwing of a die is a trial and getting l (or 2 or 3, ... or 6) is an event.

(ii) Tossing of a coin is a trial and getting head (H) or tail (T) is an event.

(iii) Drawing two cards from a pack of well-shuffled cards is a trial and getting a king and a queen are events.

Exhaustive Events: The total number of possible outcomes in any trial is known as exhaustive events or exhaustive cases. For example:

(i) In tossing of a coin there are two exhaustive cases, *viz.*, head and. tai1.

(ii) In throwing of a die, there are six, exhaustive cases since anyone of the 6 faces 1, 2, ... , 6 may come uppermost.

(iii) In drawing two cards from a pack of cards the exhaustive number of cases is 52_{C_2}, since 2 cards can be drawn out of 52 cards in 52_{C_2} ways.

Favourable Events or Cases: The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event. For example:

(i) In drawing a card from a pack of cards the number of cases favourable to drawing of an ace is 4, for drawing a spade 13 and for drawing a red card is 26.

(ii) In throwing of two dice, the number of cases favourable to getting the sum 5 is : (1,4) (4,1) (2,3) (3,2), *i.e.*, 4.

Mutually exclusive events: Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes the happening of all the others, i.e., if no two or more of them can happen simultaneously in the same trial. For example:

(i) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of others, in the same trial, is ruled out.

(ii) Similarly in tossing a coin the events head and tail are mutually exclusive.

Equally likely events: Outcomes of a trial are set to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others. For example:

(i) In tossing an unbiased or uniform coin, head or tail are likely events.

(ii) In throwing an unbiased die, all the six faces are equally likely to come.

Independent events: Several events are said to be independent if the happening (or non-happening) of an event is not affected by the supplementary knowledge concerning the occurrence of any number of the remaining events. For example:

(i) In tossing an unbiased coin the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(ii) If we draw a card from a pack of well-shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second draw is dependent on the first draw.

There are three systematic approaches to the study of probability as mentioned below.

Mathematical or Classical or 'a priori' Probability: If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E then the probability $'p'$ of happening of E is given by

$$p = P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}$$

Since the number of cases favourable to the 'non-happening' of the event E are (n - m) ,the probability $'q'$ that E will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p \text{ gives } p + q = 1$$

Obviously p as well as q are non-negative and cannot exceed unity, *i.e*,

$0 \leq p \leq 1\,, 0 \leq q \leq 1$.

Statistical or Empirical Probability: If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event happens to the number of trials, as the number of trials become indefinitely large, is called the probability of happening of the event. (It is assumed that the limit is finite and unique).

Symbolically, if in n trials an event E happens m times, then the probability 'p' of the happening of E is given by $p = P(E) = \lim_{n\to\infty} \frac{m}{n}$.

Axiomatic Probability: Let A be any event in the sample space S, then P(A) is called the probability of event A, if the following axioms are satisfied.

Axiom 1: $\mathrm{P(A)} \geq 0$

Axiom 2: $\mathrm{P}(S) = 1$, S being the sure event

Axiom 3: For two mutually exclusive events A & B, $\mathrm{P(A \cup B) = P(A) + P(B)}$

Some important results:

1. The probability of an event always lies between 0 and i.e., $0 \leq P(A) \leq 1$.
2. If A and A^{1} are complementary events to each other defined on a random experiment then $P(A) + P(A^{1}) = 1$.
3. Addition Theorem: If A and B are any two events with respective probabilities P(A) and P(B), then the probability of occurrence of at least one of the events is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
4. The probability of null event is zero i.e., $P(\emptyset) = 0$.
5. For any two events A and B of a sample space S
   (i) $P(A - B) = P(A) - P(A \cap B)$
   (ii) $P(B - A) = P(B) - P(A \cap B)$
   (iii) $P(\overline{A} \cap B) = P(B) - P(A \cap B) = P(A \cup B) - P(A)$
   (iv) $P[(A - B) \cup (B - A)] = P(A) + P(B) - 2P(A \cap B)$
6. Addition Theorem for three events: If A, B and C are any three events with respective probabilities P(A), P(B) and P(C), then the probability of occurrence of at least one of the events is given by
   $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

Conditional probability: The probability of an event B occurring when it is known that some event A has already occurred is called a **conditional probability** and is denoted by P(B|A). The symbol P(B|A) is usually read as “the probability that B occurs given that A occurs” or simply “the probability of B, given A.”

i.e., The conditional probability of B, given A, denoted by P(B|A), is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0.$$

Two events A and B are **independent** if and only if $P(B \mid A) = P(B)$ or $P(A \mid B) = P(A)$, assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**. The condition $P(B \mid A) = P(B)$ implies that $P(A \mid B) = P(A)$, and conversely.

Multiplication theorem for Conditional Probability: Suppose A and B are events in a sample space S with $P(A) > 0$. By definition of conditional probability, multiplying the conditional probability formula by P(A), we obtain the following important **multiplicative rule** (or **product rule**), which enables us to calculate the probability that two events will both occur.

$P(A \cap B) = P(A)\, P(B|A)$, provided $P(A) > 0$.

Thus, the probability that both *A* and *B* occur is equal to the probability that *A* occurs multiplied by the conditional probability that *B* occurs, given that *A* occurs.

Since the events A ∩ B and B ∩ A are equivalent, thus we can also write

$P(A \cap B) = P(B \cap A) = P(B)\, P(A \mid B)$.

Theorem on Total Probability: If the events $B_1, B_2, \dots B_k$ constitute a partition of the sample space S such that $P(Bi) \neq 0$ for $i = 1, 2, \dots, k$, then for any event A of S,

$P(A) = \sum_{i=i}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i)$.

Baye's theorem (rule):

If $B_1, B_2, \dots, B_n$ are mutually disjoint events with $P(B_i) \neq 0$ $(i = 1,2,\dots,n)$ then for any arbitrary event A which is a subset of $\cup_{i=1}^{n} B_i$ such that $P(A) > 0$, then

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}.$$

Problems:

1. In a school 25% of the students failed in first language, 15% of the students failed in second language and 10% of the students failed in both. If a student is selected at random find the probability that
   (i) He failed in first language if he had failed in the second language.
   (ii) He failed in second language if he had failed in the first language.
   (iii) He failed in either of the two languages.

   Solution: Let A be set of students failing in the first language and B be the set of students failing in the second language. We have by data

   $P(A) = 25/100 = 1/4$, $P(B) = 15/100 = 3/20$, $P(A \cap B) = \frac{10}{100} = 1/10$.

   $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{3}{20}} = 2/3$$

   $$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{\frac{1}{10}}{\frac{1}{4}} = 2/5$$

   $P(A \cup B) = P(A) + P(B) - P(A\cap B) = 1/4 + 3/20 - 1/10 = 3/10$

2. Three machines A, B, C produces 50%, 30% and 20% of the items in factory. The percentage of defective outputs are respectively 3%, 4% and 5%. If an item is selected at random. What is the probability that it is defective? What is the probability that it is from A?

   Solution: Let D denotes the defective item.

   Given $P(A) = 0.5$ and $P(D \mid A) = 0.03$, $P(B) = 0.3$ and $P(D \mid B) = 0.04$,

   $P(C) = 0.2$ and $P(D \mid C) = 0.05$.

   Now $P(D) = P(A)\,P(D \mid A) + P(B)\,P(D \mid B) + P(C)\,P(D \mid C) = 0.037$

   By Baye's theorem,

   Probability that the defective item is from A = $P(A \mid D) = P(A)P(D \mid A)/P(D)$

= (0.5) (0.03)/0.037

= 0.4054

3. In a college where boys and girls are equal proportion, it was found that 10 out of 100 boy and 25 out of 100 girls were using the same brand of a two wheeler. If a student using that was selected at random what is the probability of being a boy?

Solution:

P(Boy) = P(B) = 1/2 = P(Girl)= P(G)

Let E be the event of choosing a student using that brand of vehicle.

Therefore, $P(E|B) = 10/100 = 0.1$ and $P(E|G) = 25/100 = 0.25$

Now, $P(E) = P(B)\,P(E|B) + P(G)\,P(E|G) = 0.175$.

We have to find $P(B|E)$ and by Baye's theorem

$P(B|E) = \frac{P(B)\,P(E|B)}{P(E)} = \frac{(0.5)(0.1)}{0.175} = 2/5$ = 0.2857.

4. In a recent survey in a statistics class, it was determined that only 60% of the students attend class on thursday. From past data it was noted that 98% of those who went to class on thursday pass the course, while only 20% of those who did not go to class on thursday passed the course.

a) What percentage of students is expected to pass the course?

b) Given that a student passes the course, what is the probability that he/she attended classes on thursday.

Solution:

A_1: the students attend class on thursday

A_2: the students do not attend class on thursday

B_1: the students pass the course

B_2: the students do not pass the course

a) $P(A_1) = 0.6, P(A_2) = 1 - P(A_1) = 0.4, P(B_1|A_1) = 0.98, P(B_1|A_2) = 0.2$

$$P(B_1) = P(B_1 \cap A_1) + P(B_1 \cap A_2)$$
$$= P(A_1) * P(B_1|A_1) + P(A_2) * P(B_1|A_2)$$
$$= 0.6 * 0.98 + 0.4 * 0.2 = 0.668$$

Therefore, percentage of students who pass the course = 66%

b) By Bayes' theorem,

$$P(A_1|B_1) = \frac{P(A_1 \cap B_1)}{P(B_1)} = \frac{P(A_1) * P(B_1|A_1)}{P(A_1) * P(B_1|A_1) + P(A_2) * P(B_1|A_2)}$$
$$= \frac{0.6 * 0.98}{0.6 * 0.98 + 0.4 * 0.2}$$

= 0.854

5. In an electronics laboratory, there are identically looking capacitors of three makes A_1, A_2 and A_3 in the ratio 2:3:4. It is known that 1% of A_1, 1.5% of A_2 and 2% of A_3 are defective. What percentage of capacitors in the laboratory is defective? If a capacitor picked at defective is found to be defective, what is the probability it is of make A_3?

Solution: Let D be the event that the item is defective.

Here we have to find $P(D)$and $P(A_3|D)$.

Here $P(A_1) = \frac{2}{9}, P(A_2) = \frac{1}{3}$ and $P(A_3) = \frac{4}{9}$.

The conditional probabilities are $P(D|A_1) = 0.01, P(D|A_2) = 0.015$ and $P(D|A_3) = 0.02$

$$P(D) = P(A_1) * P(D|A_1) + P(A_2) * P(D|A_2) + P(A_3) * P(D|A_3)$$

$$= \frac{2}{9} * 0.01 + \frac{1}{3} * 0.015 + \frac{4}{9} * 0.02$$

$$= 0.0167$$

and $P(A_3|D) = \frac{P(A_3)*P(D|A_3)}{P(D)}$

$$= \frac{\frac{4}{9}*0.02}{0.0167} = 0.533 .$$

Exercise:

1. There are three bags; first containing 1 white, 2 red, 3 green balls; second 2 white, 3 red, 1 green balls and third 3 white, 1 red, 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and one red. Find the probability that the balls so drawn came from the second bag.

2. A factory uses three machines X, Y, Z to produce certain items.
   i. Machine X produces 50% of the items of which 3% are defective.
   ii. Machine Y produces 30% of the items of which 4% are defective.
   iii. Machine Z produces 20% of the items of which 5% are defective.

   Suppose a defective item is found among the output. Find the probability that it came from each of the machines.

3. In a certain college 25% of boys and 10% of girls are studying Mathematics. The girls constitute 60% of the student body.
   i. What is the probability that Mathematics is being studied? Ans: 0.16
   ii. If a student is selected at random and is found to be studying Mathematics, find the probability that the student is a (i) girl (ii) boy.

4. A large industrial firm uses three local motels to provide overnight accommodations for its clients. From past experience it is known that 20% of the clients are assigned rooms at the Ramada Inn, 50% at the Sheraton, and 30% at the Lakeview Motor Lodge. If the plumbing is

faulty in 5% of the rooms at the Ramada Inn, in 4% of the rooms at the Sheraton, and in 8% of the rooms at the Lakeview Motor Lodge, what is the probability that

i. a client will be assigned a room with faulty plumbing? Ans: 0.054

ii. a person with a room having faulty plumbing was assigned accommodations at the Lakeview Motor Lodge?

5. In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that he knows the answer and 1 – p the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability 1/5, where 5 is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he answered it correctly.

Answers: 1. 6/11 2. 40.5%, 32.5% and 27% 3. (i) 0.375 (ii) 0.625 4. 4/9 5. 5p/4p + 1

Random variable

Intuitively by a random variable we mean a real number X connected with the outcome of a random experiment E. For example, if E consists of three tosses of a coin, one can consider the random variable which is the number of heads (0, 1, 2 or 3).

Outcome	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
Value of X	3	2	2	2	1	1	1	0

Let S denote the sample space of a random experiment. A random variable means it is a rule which assigns a numerical value to each and every outcome of the experiment. Thus, random variable is a function X (ω) with domain S and range (-∞, ∞) such that for every real number *a*, the event [ω: X (ω) ≤ *a*] ϵ B the field of subsets in S. It is denoted as f: S → R.

Note that all the outcomes of the experiment are associated with a unique number. Therefore, f is an example of a random variable. Usually a random variable is denoted by letters such as X, Y, Z etc. The image set of the random variable may be written as f(S) = {0, 1, 2, 3}.

There are two types of random variables. They are;

1. Discrete Random Variable (DRV)
2. Continuous Random Variable (CRV).

Discrete Random Variable: A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3, … . Discrete random variables are usually (but not necessarily) counts. If a random variable takes at most a countable number of values, it is called a **discrete random variable**. In other words, a real valued function defined on a discrete sample space is called a discrete random variable.

Examples of Discrete Random Variable:

(i) In the experiment of throwing a die, define X as the number that is obtained. Then X takes any of the values 1 – 6. Thus, X(S) = {1, 2, 3…6} which is a finite set and hence X is a DRV.

(ii) If X be the random variable denoting the number of marks scored by a student in a subject of an examination, then X(S) = {0, 1, 2, 3,…100}. Then, X is a DRV.

(iii) The number of children in a family is a DRV.

(iv) The number of defective light bulbs in a box of ten is a DRV.

Probability Mass Function: Suppose X is a one-dimensional discrete random variable taking at most a countably infinite number of values x_1, x_2, …. With each possible outcome x_i, one can associate a number $p_i = P(X = x_i) = p(x_i)$, called the probability of x_i.

The numbers $p(x_i)$; $i = 1, 2, \ldots$ must satisfy the following conditions:

(i) $p(x_i) \geq 0 \; \forall \, i$,

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.

This function p is called the **probability mass function** of the random variable X and the set $\{x_i, p(x_i)\}$ is called the probability distribution of the random variable X.

Remarks:

1. The set of values which X takes is called the spectrum of the random variable.
2. For discrete random variable, knowledge of the probability mass function enables us to compute probabilities of arbitrary events. In fact, if E is a set of real numbers, we have $(X \in E) = \sum_{x \in E \cap S} p(x)$, where S is the sample space.

Discrete Distribution Function: In this case there is a countable number of points $x_1, x_2, x_3, \ldots$ and numbers $p_i \geq 0, \sum_{i=1}^{\infty} p_i = 1$ such that $(X) = \sum_{(i: x_i \leq x)} p_i$.

Mean/Expected Value, Variance and Standard Deviation of DRV:

The **mean or expected value** of a DRV X is defined as

$E(X) = \mu = \sum P(X = x_i) * x_i = \sum P_i X_i$.

The **variance** of a DRV X is defined as

$Var\,(X) = \sigma^2 = \sum P(X = x_i) * (x_i - \mu)^2 = \sum P_i (x_i - \mu)^2 = \sum P_i x_i^2 - \mu^2$.

The **standard deviation** of DRV X is defined as

$SD\,(X) = \sigma = \sqrt{\sigma^2} = \sqrt{Var\,(X)}$.

Continuous Random Variable: A continuous random variable is not defined at specific values. Instead, it is defined over an interval of values, and is represented by the area under a curve. Thus, a random variable X is said to be continuous if it can take all possible values between certain limits. In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers. Here, the probability of observing any single value is equal to zero, since the number of values which may be assumed by the random variable is infinite.

A continuous random variable is a random variable that (at least conceptually) can be measured to any desired degree of accuracy.

Examples of Continuous Random Variable:

(i) Rainfall in a particular area can be treated as CRV.

(ii) Age, height and weight related problems can be included under CRV.

(iii) The amount of sugar in an orange is a CRV.

(iv) The time required to run a mile is a CRV.

Important Remark: In case of DRV, the probability at a point i.e., P (x = c) is not zero for some fixed c. However, in case of CRV the probability at a point is always zero.

i.e., P(x = c) = 0 for all possible values of c.

Probability Density Function: The probability density function (p.d.f) of a random variable X usually denoted by $f_x(x)$ or simply by $f(x)$ has the following obvious properties:

i) $f(x) \geq 0, -\infty < x < \infty$

ii) $\int_{-\infty}^{\infty} f(x)dx = 1$

iii) The probability $P(E)$ given by $P(E) = \int f(x)dx$ is well defined for any event E.

If $f(x)$ is the p.d.f of x, then the probability that x belongs to A, where A is some interval (a, b) is given by the integral of $f(x)$ over that interval.

i.e., $P(X \in A) = \int_a^b f(x)dx$

Cumulative Density Function: Cumulative density function of a continuous random variable is defined as $F(x) = \int_{-\infty}^{x} f(t)dt$ for $-\infty < x < \infty$.

Mean/Expectation, Variance and Standard deviation of CRV:

The mean or expected value of a CRV X is defined as $\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$

The variance of a CRV X is defined as $Var(X) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

The standard deviation of a CRV X is given by = $\sqrt{Var(X)}$.

Examples:

1. The probability density function of a discrete random variable X is given below:

x	0	1	2	3	4	5	6
$P(X = x) = f(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k; (ii) F (4); (iii) $P(X \geq 5)$; (iv) $P(2 \leq X < 5)$; (v) E(X) and (vi) Var (X).

Solution: To find the value of k, consider the sum of all the probabilities which equals to $49k$. Equating this to 1, we obtain $k = 1/49$. Therefore, distribution of X may now be written as

x	0	1	2	3	4	5	6
$P(X = x) = f(x)$	1/49	3/49	5/49	7/49	9/49	11/49	13/49

Using this, we may solve the other problems in hand.

$F(4) = P[X \leq 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4] = \frac{25}{49}$.

$$P[X \geq 5] = P[X = 5] + P[X = 6] = \frac{24}{49}.$$

$$P[2 \leq X < 5] = P[X = 2] + P[X = 3] + P[X = 4] = \frac{21}{49}.$$

Next to find $E(X)$, consider

$$E(X) = \sum_i x_i * f(x_i) = \frac{203}{49}.$$

To obtain Variance, it is necessary to compute

$E(X^2) = \sum_i x_i^2 * f(x_i) = \frac{973}{49}$.

Thus, Variance of X is obtained by using the relation,

$Var(X) = E(X^2) - [E(X)]^2 = \frac{973}{49} - \left(\frac{203}{49}\right)^2$.

2. A random variable, X, has the following distribution function.

X	-2	-1	0	1	2	3
$f(x_i)$	0.1	k	0.2	$2k$	0.3	k

Find (i) k; (ii) F (2); (iii) $P(-2 < X < 2)$; (iv) $P(-1 < X \leq 2)$; (v) $E(X)$ and (vi) Variance.

Solution: Consider the result, namely, sum of all the probabilities equals 1,

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$ yields $k = 0.1$. In view of this, distribution function of X may be formulated as

X	-2	-1	0	1	2	3
$f(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1

Note that $F(2) = P[X \leq 2]$

$= P[X = -2] + P[X = -1] + P[X = 0] + P[X = 1] + P[X = 2]$

= 0.9. The same also be obtained using the result,

$\mathrm{F}(2) = \mathrm{P}[\mathrm{X} \leq 2] = 1 - \mathrm{P}[\mathrm{X} < 1] = 1 - \{\mathrm{P}[\mathrm{X} = -2] + \mathrm{P}[\mathrm{X} = -1] + \mathrm{P}[\mathrm{X} = 0]\} = 0.6$.

Next, $P(-2 < X < 2) = P[X = -1] + P[X = 0] + P[X = 1] = 0.5$.

Clearly, $P(-1 < X \leq 2) = 0.7$.

Now, consider $(X) = \sum_i x_i * f(x_i) = 0.8$.

Then $E(X^2) = \sum_i x_i^2 * f(x_i) = 2.8$. $Var(X) = E(X^2) - \{E(X)\}^2 = 2.8 - 0.64 = 2.16$.

3. A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution: Let *X* be a random variable whose values *x* are the possible numbers of defective computers purchased by the school. Then *x* can only take the numbers 0, 1, and 2. Now

$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95}$, $f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190}$

$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}$.

Thus, the probability distribution of *X* is

x	0	1	2
$f(x)$	68/95	51/190	3/190

4. If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags, find a formula for the probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency.

Solution: Since the probability of selling an automobile with side airbags is 0.5, the $2^4 = 16$ points in the sample space are equally likely to occur. Therefore, the denominator for all probabilities, and also for our function, is 16. To obtain the number of ways of selling 3 cars with side airbags, we need to consider the number of ways of partitioning 4 outcomes into two cells, with 3 cars with side airbags assigned to one cell and the model without side airbags assigned to the other. This can be done in $\binom{4}{3} = 4$ ways. In general, the event of selling *x* models with side airbags and 4 - x models without side airbags can occur in $\binom{4}{x}$ ways, where x can be 0, 1, 2, 3, or 4. Thus, the probability distribution $f(x) = P(X = x)$ is

$$f(x) = \left(\frac{1}{16}\right)\binom{4}{x}\ for\ \ x = 0,1,2,3,4.$$

5. The following table gives the pmf of tossing of 3 coins, and the random variable X as getting a head. Construct the cumulative distribution table.

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

Solution:

X	0	1	2	3
F(X)	1/8	4/8	7/8	1

6. The diameter of an electric cable, say X, is assumed to be a continuous random variable with p.d.f $f(x) = \begin{cases} 6x(1-x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

(i) Check that above is p.d.f.

(ii) Find $P\left(\frac{2}{3} < x < 1\right)$

(iii) Determine a number b such that $P(X < b) = P(X > b)$.

Solution: (i) $f(x) \ge 0$ in the given interval.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= 0 + \int_0^1 6x(1-x)\,dx + 0$$

$= \left\{\frac{6x^2}{2} - \frac{6x^3}{3}\right\}$ by putting limits $x = 0$ to 1 we get

$$= 1$$

(ii) $P\left(\frac{2}{3} < x < 1\right) = \int_{2/3}^{1} f(x)dx = \int_{2/3}^{1}(6x - 6x^2)dx = \frac{7}{27}$.

(iii) $P(X < b) = P(X > b)$

$$\int_0^b f(x)dx \ = \int_b^1 f(x)dx$$

$$6\int_0^b x(1-x)dx = 6\int_b^1 x(1-x)dx$$

$$\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)\right]$$

$$3b^2 - 2b^3 = [1 - 3b^2 + 2b^3]$$

$$4b^3 - 6b^2 + 1 = 0$$

$$(2b - 1)(2b^2 - 2b - 1) = 0$$

From this b = ½ is the only real value lying between 0 and 1 and satisfying the given condition.

7. Suppose that the error in the reaction temperature, in ◦C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, -1 < x < 2 \\ 0\,, \quad \text{elsewhere} \end{cases}$$

(i) Verify that $f(x)$ is a probability density function.

(ii) Find P(0 < X ≤ 1).

Solution: a) $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3} dx = 1$. Hence the given function is a p.d.f.

b) P(0 < X ≤ 1) = $\int_{0}^{1} \frac{x^2}{3} dx = \frac{1}{9}$.

8. The length of time (in minutes) that a certain lady speaks on telephone is found to be a random variable with probability function $f(x) = \begin{cases} Ae^{\frac{-x}{5}} & for\ x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(i) Find A

(ii) Find the probability that she will speak on the phone

(a) more than 10 min (b) less than 5 min (c) between 5 & 10 min.

Solution: (i) Given $f(x)$ is p.d.f. i.e., $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = 1$$

$$\xrightarrow{yields} 0 + \int_{0}^{\infty} Ae^{\frac{-x}{5}} dx = 1$$

$$\xrightarrow{yields} A = \frac{1}{5}$$

(ii) (a) $P(x > 10) = \int_{10}^{\infty} f(x)dx = \int_{10}^{\infty} \frac{1}{5} e^{\frac{-x}{5}}\, dx = e^{-2} = 0.1353$

(b) $P(x < 5) = \int_{-\infty}^{5} f(x)dx = \int_{0}^{5} \frac{1}{5} e^{\frac{-x}{5}}\, dx = -e^{-1} + 1 = 0.6322$

(c) $P(5 < x < 10) = \int_{5}^{10} f(x)dx = \int_{5}^{10} \frac{1}{5} e^{\frac{-x}{5}}\, dx = -e^{-2} + e^{-1} = 0.2325$.

9. Suppose X is a continuous random variable with the following probability density function $f(x) = 3x^2$ for $0 < x < 1$. Find the mean and variance of X.

Solution: Mean = $\mu = \int_{-\infty}^{\infty} xf(x)dx$

$$= \int_{-\infty}^{0} xf(x)dx + \int_{0}^{1} xf(x)dx + \int_{1}^{\infty} xf(x)dx$$

$$= 0 + \int_{0}^{1} x * 3x^2 dx + 0 = \int_{0}^{1} 3x^3 dx = \frac{3}{4}\ .$$

Variance = $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

$$= \int_0^1 x^2 f(x)dx - \mu^2$$

$$= \int_0^1 x^2 * 3x^2 dx - \left(\frac{3}{4}\right)^2$$

$$= \int_0^1 3x^4 dx - \left(\frac{3}{4}\right)^2 = \frac{3}{80}.$$

10. The lifetime in years, of some electronic component is a continuous random variable with the density function $f(x) = \begin{cases} \frac{k}{x^4}, x \geq 1 \\ 0, x < 1 \end{cases}$. Find (i) k, (ii) the cumulative distribution function and (iii) the probability for the lifetime to exceed 2 years.

Solution:

(i) $\int_1^\infty \frac{k}{x^4} dx = 1 \Rightarrow -\frac{k}{3x^3}\Big|_1^\infty = 1 \Rightarrow \left(-0 + \frac{k}{3}\right) = 1 \Rightarrow k = 3$

(ii) $F(t) = \int_1^t \frac{3}{x^4} dx = -\frac{3}{3x^3}\Big|_1^t = -\frac{1}{t^3} + 1$

(iii) $P(x > 2) = 1 - P(x \leq 2) = 1 - F(2) = 1 - \left(-\frac{1}{2^3} + 1\right) = \frac{1}{8}$

Exercise:

1. Two cards are drawn randomly, simultaneously from a well shuffled deck of 52 cards. Find the variance for the number of aces.
2. If X is a discrete random variable taking values 1,2,3,… with $P(x) = \frac{1}{2}\left(\frac{2}{3}\right)^x$. Find P(X being an odd number) by first establishing that P(x) is a probability function.
3. The probability mass function of a random variable X is zero except the points x = 0,1,2. At these points it has the values $p(0) = 3c^3, p(1) = 4c - 10c^2$ and $p(2) = 5c - 1$ for some $c > 0$.
   a) Determine the value of c.
   b) Compute the probabilities $P(X < 2)$and $P(1 < X \leq 2)$.
   c) Find the largest x such that $F(x) < \frac{1}{2}$.
   d) Find the smallest x such that $F(x) \geq \frac{1}{3}$.
4. If X is a random variable with $P(X = x) = \frac{1}{2^x}$, where $x = 1,2,3, \ldots \infty$.
   Find i) P(X) (ii) P(X = even) (iii) P(X = divisible by 3).

5. A continuous random variable has the density function $f(x) = \begin{cases} kx^2 & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

   Find k and hence find $P(x < 3)$, $P(x > 1)$.

6. Let X be a continuous random variable with p.d.f.

$$f(x)=\begin{cases} ax\,, & 0\le x\le 1\\ a\,, & 1\le x\le 2\\ -ax+3a\,, & 2\le x\le 3\\ 0\,, & \text{otherwise}\end{cases}$$

(i) Determine the constant. (ii) Compute $P(X\le 1.5)$.

7. Find the mean and variance of the probability density function $f(x)=\frac{1}{2}e^{-|x|}$

8. A continuous distribution of a variable X in the range (-3, 3) is defined by

$$f(x)=\begin{cases} \frac{1}{16}(3+x)^2\,, & -3\le x\le -1\\ \frac{1}{16}(6-2x^2), & -1\le x\le 1\\ \frac{1}{16}(3-x)^2\,, & 1\le x\le 3\end{cases}$$

(i) Verify that the area under the curve is unity.

(ii) Find the mean and variance of the above distribution.

9. Let X be a discrete random variable with $X=\{1,2,3,\ldots\}$ and $P(X)=\frac{1}{2^x}$ $for\ x=1,2,3,$ ….Find the CDF of X.

10. The probability density function of a random variable x is given by

$f(x)=\begin{cases} x\,, & 0\le x\le 1\\ 2-x\,, & 1<x\le 2\end{cases}$. What is the cdf?

Answers: 1. 0.1392 2. 3/5 3. 1/3, 1/3, 2/3, 1, 1 4. 1, 1/3, 1/7 5. 1/18, 1, 13/27

6. 1/2, 1/2 7. Mean = 0 and Variance = 2 8. Unit area and 0, 1. 9. $F(X)=\sum_1^{\infty}\frac{1}{2^x}=\frac{2^x-1}{2^x}$, 10. $F(t)=\begin{cases} 0, & -\infty<t<0\\ \frac{t^2}{2}, & 0\le t\le 1\\ 1-\frac{1}{2}(t-2)^2, & 1\le t\le 2\\ 1\,, & t>2\end{cases}$

Inequalities of Markov and Chebyshev:

Often, given a random variable Xwhose distribution is unknown but whose expected value $\mu$is known, we may want to ask how likely it is for $X$to be 'far' from μ, or how likely it is for this random variable to be 'very large.' This would give us some idea of the spread of the distribution, though perhaps not a complete picture.

Markov's Inequality:

For any random variable X with finite $E[X]$ and any $k > 0$, the probability that X is atleast k times its expected value is at-most $\frac{1}{k}$.

$$\text{i.e., } P\big[X \geq kE[X]\big] \leq \frac{1}{k} \quad \text{or} \quad P[X \geq k] \leq \frac{E[X]}{k}.$$

Examples:

1. A biased coin, with probability of tossing a head being $\frac{1}{5}$, is tossed 10 times. Estimate the probability of getting at least 8 heads in 10 tosses.

Solution:

$$E[X] = np = 10 \times \frac{1}{5} = 2$$

By Markov's inequality, $P[X \geq 8] \leq \frac{2}{8} = \frac{1}{5} = 0.25$

The probability of at-least 8 heads is:

$n = 10, p = \frac{1}{5}$,

then $P[X \geq 8] = P[X = 8] + P[X = 9] + P[X = 10]$

$$= {}^{10}C_8 \left(\frac{1}{5}\right)^8 \left(\frac{4}{5}\right)^2 + {}^{10}C_9 \left(\frac{1}{5}\right)^9 \left(\frac{4}{5}\right)^1 + {}^{10}C_8 \left(\frac{1}{5}\right)^{10} \left(\frac{4}{5}\right)^0 = 0.0000779 < 0.25$$

2. A random variable X takes the value 0 with probability 24/25 and the value 5 with probability 1/25. Estimate using Markov inequality a bound on the probability that X is at-least 5.

Solution:

$$E[X] = \sum xp(x) = 0 \times \frac{24}{25} + 5 \times \frac{1}{25} = 1/5$$

$P[X \geq 5] = \frac{1/5}{5} = \frac{1}{25}$ and this is exactly the probability of $X = 5$.

Chebyshev's Inequality:

Any random variable X with expectation $\mu = E[X]$ and variance $\sigma^2 = var[X]$ belongs to the interval $\mu \pm k = [\mu - k, \mu + k]$with probability of at-least $1 - \left(\frac{\sigma}{k}\right)^2$

$$\text{i.e., } P\{|X - \mu| \geq k\} \leq \left(\frac{\sigma}{k}\right)^2 = \frac{\sigma^2}{k^2}$$

Examples:

1. Let X be a random variable with mean 4 and variance 2. Use Chebyshev's inequality to obtain an upper bound on $P[|X-4| \geq 2]$.

Solution:

$P[|X-4| \geq 2] \leq \frac{2}{2^2} = \frac{1}{2}$.

2. Suppose a fair coin is flipped 100 times. Find a bound on the probability that the number of times the coin lands on heads is at least 60 or at most 40.

Solution:

Let X be the number of times the coin lands on heads.

$$n = 100, p = \frac{1}{2}, q = \frac{1}{2}$$

Then $E[X] = np = 100 \times \frac{1}{2} = 50, Var[X] = npq = 100 \times \frac{1}{2} \times \frac{1}{2} = 25$

By Chebyshev, we have $P(X < 40 \cup X > 60) = P(|X - \mu| \geq 10) \leq \frac{25}{10^2} = \frac{1}{4}$.

Exercise:

1. A biased coin lands heads with probability $\frac{1}{10}$. This coin is flipped 200 times. Use Markov's inequality to give an upper bound on the probability that the coin lands heads at least 120 times. Find a bound on the probability that the number of times the coin lands on heads is at least 25 or at most 15.
2. The average height of a raccoon is 10 inches. (i) Given an upper bound on the probability that a certain raccoon is at least 15 inches tall, (ii) The standard deviation of this height distribution is 2 inches. Find a lower bound on the probability that a certain raccoon is between 5 and 15 inches tall.

Answers:

1. Markov's bound $= \frac{1}{6}$, Chebyshev's bound $=\frac{18}{25}$, 2. (i) $\frac{2}{3}$, (ii) $\frac{4}{25}$

Video Links:

1. Bayes theorem
https://www.youtube.com/watch?v=bUI8ovd07uI
https://www.youtube.com/watch?v=OByl4RJxnKA

2. Conditional probability
https://www.youtube.com/watch?v=eHfhpAhGdvY

3. Random variables
https://www.youtube.com/watch?v=82Ad1orN-NA