7

Relations: The Second Time Around

In Chapter 5 we introduced the concept of a (binary) relation. Returning to relations in this chapter, we shall emphasize the study of relations on a set A — that is, subsets of $A \times A$. Within the theory of languages and finite state machines from Chapter 6, we find many examples of relations on a set A, where A represents a set of strings from a given alphabet or a set of internal states from a finite state machine. Various properties of relations are developed, along with ways to represent finite relations for computer manipulation. Directed graphs reappear as a way to represent such relations. Finally, two types of relations on a set A are especially important: equivalence relations and partial orders. Equivalence relations, in particular, arise in many areas of mathematics. For the present we shall use an equivalence relation on the set of internal states in a finite state machine M in order to find a machine M_1, with as few internal states as possible, that performs whatever tasks M is capable of performing. The procedure is known as the minimization process.

7.1 Relations Revisited: Properties of Relations

We start by recalling some fundamental ideas considered earlier.

Definition 7.1 For sets A, B, any subset of $A \times B$ is called a *(binary) relation* from A to B. Any subset of $A \times A$ is called a *(binary) relation* on A.

As mentioned in the sentence following Definition 5.2, our primary concern is with binary relations. Consequently, for us the word "relation" will once again mean binary relation, unless something otherwise is specified.

EXAMPLE 7.1

a) Define the relation $\mathcal{R}$ on the set $\mathbf{Z}$ by $a \,\mathcal{R}\, b$, or $(a, b) \in \mathcal{R}$, if $a \leq b$. This subset of $\mathbf{Z} \times \mathbf{Z}$ is the ordinary "less than or equal to" relation on the set $\mathbf{Z}$, and it can also be defined on $\mathbf{Q}$ or $\mathbf{R}$, but not on $\mathbf{C}$.

b) Let $n \in \mathbf{Z}^+$. For $x, y \in \mathbf{Z}$, the *modulo n relation* $\mathcal{R}$ is defined by $x \,\mathcal{R}\, y$ if $x - y$ is a multiple of n. With $n = 7$, we find, for instance, that $9 \,\mathcal{R}\, 2$, $-3 \,\mathcal{R}\, 11$, $(14, 0) \in \mathcal{R}$, but $3 \not\!\mathcal{R}\, 7$ (that is, 3 is *not* related to 7).

c) For the universe $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$ consider the (fixed) set $C \subseteq \mathcal{U}$ where $C = \{1, 2, 3, 6\}$. Define the relation $\mathcal{R}$ on $\mathcal{P}(\mathcal{U})$ by $A \mathcal{R} B$ when $A \cap C = B \cap C$. Then the sets $\{1, 2, 4, 5\}$ and $\{1, 2, 5, 7\}$ are related since $\{1, 2, 4, 5\} \cap C = \{1, 2\} = \{1, 2, 5, 7\} \cap C$. Likewise we find that $X = \{4, 5\}$ and $Y = \{7\}$ are so related because $X \cap C = \emptyset = Y \cap C$. However, the sets $S = \{1, 2, 3, 4, 5\}$ and $T = \{1, 2, 3, 6, 7\}$ are *not* related — that is, $S \not\mathcal{R} T$ — since $S \cap C = \{1, 2, 3\} \neq \{1, 2, 3, 6\} = T \cap C$.

EXAMPLE 7.2

Let Σ be an alphabet, with language $A \subseteq \Sigma^*$. For $x, y \in A$, define $x \mathcal{R} y$ if x is a prefix of y. Other relations can be defined on A by replacing "prefix" with either "suffix" or "substring."

EXAMPLE 7.3

Consider a finite state machine $M = (S, \mathcal{I}, \mathcal{O}, \nu, \omega)$.

a) For $s_1, s_2 \in S$, define $s_1 \mathcal{R} s_2$ if $\nu(s_1, x) = s_2$, for some $x \in \mathcal{I}$. Relation $\mathcal{R}$ establishes the *first level of reachability*.

b) The relation for the *second level of reachability* can also be given for S. Here $s_1 \mathcal{R} s_2$ if $\nu(s_1, x_1x_2) = s_2$, for some $x_1x_2 \in \mathcal{I}^2$. This can be extended to higher levels if the need arises. For the general *reachability* relation we have $\nu(s_1, y) = s_2$, for some $y \in \mathcal{I}^*$.

c) Given $s_1, s_2 \in S$ the relation of *1-equivalence*, which is denoted by $s_1 \text{E}_1 s_2$ and is read "s_1 is 1-equivalent to s_2", is defined when $\omega(s_1, x) = \omega(s_2, x)$ for all $x \in \mathcal{I}$. Consequently, $s_1 \text{E}_1 s_2$ indicates that if machine M starts in either state s_1 or s_2, the output is the same for each element of $\mathcal{I}$. This idea can be extended to states being *k-equivalent*, where we write $s_1 \text{E}_k s_2$ if $\omega(s_1, y) = \omega(s_2, y)$, for all $y \in \mathcal{I}^k$. Here the same output string is obtained for each input string in $\mathcal{I}^k$ if we start at either s_1 or s_2.
If two states are k-equivalent for all $k \in \mathbf{Z}^+$, then they are called *equivalent*. We shall look further into this idea later in the chapter.

We now start to examine some of the properties a relation can satisfy.

Definition 7.2

A relation $\mathcal{R}$ on a set A is called *reflexive* if for all $x \in A$, $(x, x) \in \mathcal{R}$.

To say that a relation $\mathcal{R}$ is reflexive simply means that each element x of A is related to itself. All the relations in Examples 7.1 and 7.2 are reflexive. The general reachability relation in Example 7.3(b) and all of the relations mentioned in part (c) of that example are also reflexive. [What goes wrong with the relations for the first and second levels of reachability given in parts (a) and (b) of Example 7.3?]

EXAMPLE 7.4

For $A = \{1, 2, 3, 4\}$, a relation $\mathcal{R} \subseteq A \times A$ will be reflexive if and only if $\mathcal{R} \supseteq \{(1, 1), (2, 2), (3, 3), (4, 4)\}$. Consequently, $\mathcal{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$ is not a reflexive relation on A, whereas $\mathcal{R}_2 = \{(x, y) | x, y \in A, x \leq y\}$ is reflexive on A.

EXAMPLE 7.5

Given a finite set A with $|A| = n$, we have $|A \times A| = n^2$, so there are 2^{n^2} relations on A. How many of these are reflexive?

If $A = \{a_1, a_2, \ldots, a_n\}$, a relation $\mathcal{R}$ on A is reflexive if and only if $\{(a_i, a_i) | 1 \leq i \leq n\} \subseteq \mathcal{R}$. Considering the other $n^2 - n$ ordered pairs in $A \times A$ [those of the form (a_i, a_j), where $i \neq j$ for $1 \leq i, j \leq n$] as we construct a reflexive relation $\mathcal{R}$ on A, we either include or exclude each of these ordered pairs, so by the rule of product there are $2^{(n^2-n)}$ reflexive relations on A.

Definition 7.3

Relation $\mathcal{R}$ on set A is called *symmetric* if $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$, for all $x, y \in A$.

EXAMPLE 7.6

With $A = \{1, 2, 3\}$, we have:

a) $\mathcal{R}_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, a symmetric, but not reflexive, relation on A;

b) $\mathcal{R}_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$, a reflexive, but not symmetric, relation on A;

c) $\mathcal{R}_3 = \{(1, 1), (2, 2), (3, 3)\}$ and $\mathcal{R}_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, two relations on A that are both reflexive and symmetric; and

d) $\mathcal{R}_5 = \{(1, 1), (2, 3), (3, 3)\}$, a relation on A that is neither reflexive nor symmetric.

To count the symmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$, we write $A \times A$ as $A_1 \cup A_2$, where $A_1 = \{(a_i, a_i) | 1 \leq i \leq n\}$ and $A_2 = \{(a_i, a_j) | 1 \leq i, j \leq n, i \neq j\}$, so that every ordered pair in $A \times A$ is in exactly one of A_1, A_2. For A_2, $|A_2| = |A \times A| - |A_1| = n^2 - n = n(n-1)$, an even integer. The set A_2 contains $(1/2)(n^2 - n)$ subsets S_{ij} of the form $\{(a_i, a_j), (a_j, a_i)\}$ where $1 \leq i < j \leq n$. In constructing a symmetric relation $\mathcal{R}$ on A, for each ordered pair in A_1 we have our usual choice of exclusion or inclusion. For each of the $(1/2)(n^2 - n)$ subsets $S_{ij} (1 \leq i < j \leq n)$ taken from A_2 we have the same two choices. So by the rule of product there are $2^n \cdot 2^{(1/2)(n^2-n)} = 2^{(1/2)(n^2+n)}$ symmetric relations on A.

In counting those relations on A that are both reflexive and symmetric, we have only one choice for each ordered pair in A_1. So we have $2^{(1/2)(n^2-n)}$ relations on A that are both reflexive and symmetric.

Definition 7.4

For a set A, a relation $\mathcal{R}$ on A is called *transitive* if, for all $x, y, z \in A$, $(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$. (So if x "is related to" y, and y "is related to" z, we want x "related to" z, with y playing the role of "intermediary.")

EXAMPLE 7.7

All the relations in Examples 7.1 and 7.2 are transitive, as are the relations in Example 7.3(c).

EXAMPLE 7.8

Define the relation $\mathcal{R}$ on the set $\mathbf{Z}^+$ by $a \mathcal{R} b$ if a (exactly) divides b — that is, $b = ca$ for some $c \in \mathbf{Z}^+$. Now if $x \mathcal{R} y$ and $y \mathcal{R} z$, do we have $x \mathcal{R} z$? We know that $x \mathcal{R} y \Rightarrow y = sx$ for some $s \in \mathbf{Z}^+$ and $y \mathcal{R} z \Rightarrow z = ty$ where $t \in \mathbf{Z}^+$. Consequently, $z = ty = t(sx) = (ts)x$ for $ts \in \mathbf{Z}^+$, so $x \mathcal{R} z$ and $\mathcal{R}$ is transitive. In addition, $\mathcal{R}$ is reflexive, but not symmetric, because, for example, $2 \mathcal{R} 6$ but $6 \not\mathcal{R} 2$.

EXAMPLE 7.9

Consider the relation $\mathcal{R}$ on the set $\mathbf{Z}$ where we define $a \mathcal{R} b$ when $ab \geq 0$. For all integers x we have $xx = x^2 \geq 0$, so $x \mathcal{R} x$ and $\mathcal{R}$ is reflexive. Also, if $x, y \in \mathbf{Z}$ and $x \mathcal{R} y$, then

$$x \mathcal{R} y \Rightarrow xy \geq 0 \Rightarrow yx \geq 0 \Rightarrow y \mathcal{R} x,$$

so the relation $\mathcal{R}$ is symmetric as well. However, here we find that $(3, 0), (0, -7) \in \mathcal{R}$—since $(3)(0) \geq 0$ and $(0)(-7) \geq 0$—but $(3, -7) \notin \mathcal{R}$ because $(3)(-7) < 0$. Consequently, this relation is *not* transitive.

EXAMPLE 7.10

If $A = \{1, 2, 3, 4\}$, then $\mathcal{R}_1 = \{(1, 1), (2, 3), (3, 4), (2, 4)\}$ is a transitive relation on A, whereas $\mathcal{R}_2 = \{(1, 3), (3, 2)\}$ is not transitive because $(1, 3), (3, 2) \in \mathcal{R}_2$ but $(1, 2) \notin \mathcal{R}_2$.

At this point the reader is probably ready to start counting the number of transitive relations on a finite set. But this is not possible here. For unlike the cases dealing with the reflexive and symmetric properties, there is no known general formula for the total number of transitive relations on a finite set. However, at a later point in this chapter we shall have the necessary ideas to count the relations $\mathcal{R}$ on a finite set, where $\mathcal{R}$ is (simultaneously) reflexive, symmetric, and transitive.

For now we consider one last property for relations.

Definition 7.5

Given a relation $\mathcal{R}$ on a set A, $\mathcal{R}$ is called *antisymmetric* if for all $a, b \in A$, $(a \,\mathcal{R}\, b$ and $b \,\mathcal{R}\, a) \Rightarrow a = b$. (Here the only way we can have both a "related to" b and b "related to" a is if a and b are one and the same element from A.)

EXAMPLE 7.11

For a given universe $\mathcal{U}$, define the relation $\mathcal{R}$ on $\mathcal{P}(\mathcal{U})$ by $(A, B) \in \mathcal{R}$ if $A \subseteq B$, for $A, B \subseteq \mathcal{U}$. So $\mathcal{R}$ is the subset relation of Chapter 3 and if $A \,\mathcal{R}\, B$ and $B \,\mathcal{R}\, A$, then we have $A \subseteq B$ and $B \subseteq A$, which gives us $A = B$. Consequently, this relation is antisymmetric, as well as reflexive and transitive, but it is not symmetric.

Before we are led astray into thinking that "not symmetric" is synonymous with "antisymmetric", let us consider the following.

EXAMPLE 7.12

For $A = \{1, 2, 3\}$, the relation $\mathcal{R}$ on A given by $\mathcal{R} = \{(1, 2), (2, 1), (2, 3)\}$ is not symmetric because $(3, 2) \notin \mathcal{R}$, and it is not antisymmetric because $(1, 2), (2, 1) \in \mathcal{R}$ but $1 \neq 2$. The relation $\mathcal{R}_1 = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

How many relations on A are antisymmetric? Writing

$$A \times A = \{(1, 1), (2, 2), (3, 3)\} \cup \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\},$$

we make two observations as we try to construct an antisymmetric relation $\mathcal{R}$ on A.

1) Each element $(x, x) \in A \times A$ can be either included or excluded with no concern about whether or not $\mathcal{R}$ is antisymmetric.
2) For an element of the form (x, y), $x \neq y$, we must consider both (x, y) and (y, x) and we note that for $\mathcal{R}$ to remain antisymmetric we have three alternatives: (a) place (x, y) in $\mathcal{R}$; (b) place (y, x) in $\mathcal{R}$; or (c) place neither (x, y) nor (y, x) in $\mathcal{R}$. [What happens if we place both (x, y) and (y, x) in $\mathcal{R}$?]

So by the rule of product, the number of antisymmetric relations on A is $(2^3)(3^3) = (2^3)(3^{(3^2-3)/2})$. If $|A| = n > 0$, then there are $(2^n)(3^{(n^2-n)/2})$ antisymmetric relations on A.

For our next example we return to the concept of function dominance, which we first defined in Section 5.7.

EXAMPLE 7.13

Let $\mathscr{F}$ denote the set of all functions with domain $\mathbf{Z}^+$ and codomain $\mathbf{R}$; that is, $\mathscr{F} = \{f \mid f: \mathbf{Z}^+ \to \mathbf{R}\}$. For $f, g \in \mathscr{F}$, define the relation $\mathscr{R}$ on $\mathscr{F}$ by $f \mathscr{R} g$ if f is dominated by g (or $f \in O(g)$). Then $\mathscr{R}$ is reflexive and transitive.

If $f, g: \mathbf{Z}^+ \to \mathbf{R}$ are defined by $f(n) = n$ and $g(n) = n + 5$, then $f \mathscr{R} g$ and $g \mathscr{R} f$ but $f \neq g$, so $\mathscr{R}$ is *not antisymmetric*. In addition, if $h: \mathbf{Z}^+ \to \mathbf{R}$ is given by $h(n) = n^2$, then $(f, h), (g, h) \in \mathscr{R}$, but neither (h, f) nor (h, g) is in $\mathscr{R}$. Consequently, the relation $\mathscr{R}$ is also *not symmetric*.

At this point we have seen the four major properties that arise in the study of relations. Before closing this section we define two more notions, each of which involves three of these four properties.

Definition 7.6

A relation $\mathscr{R}$ on a set A is called a *partial order*, or a *partial ordering relation*, if $\mathscr{R}$ is reflexive, antisymmetric, and transitive.

EXAMPLE 7.14

The relation in Example 7.1(a) is a partial order, but the relation in part (b) of that example is not because it is not antisymmetric. All the relations of Example 7.2 are partial orders, as is the subset relation of Example 7.11.

Our next example provides us with the opportunity to relate this new idea of a partial order with results we studied in Chapters 1 and 4.

EXAMPLE 7.15

We start with the set $A = \{1, 2, 3, 4, 6, 12\}$—the set of positive integer divisors of 12—and define the relation $\mathscr{R}$ on A by $x \mathscr{R} y$ if x (exactly) divides y. As in Example 7.8 we find that $\mathscr{R}$ is reflexive and transitive. In addition, if $x, y \in A$ and we have both $x \mathscr{R} y$ and $y \mathscr{R} x$, then

$$x \mathscr{R} y \Rightarrow y = ax, \text{ for some } a \in \mathbf{Z}^+, \quad \text{and}$$
$$y \mathscr{R} x \Rightarrow x = by, \text{ for some } b \in \mathbf{Z}^+.$$

Consequently, it follows that $y = ax = a(by) = (ab)y$, and since $y \neq 0$, we have $ab = 1$. Because $a, b \in \mathbf{Z}^+$, $ab = 1 \Rightarrow a = b = 1$, so $y = x$ and $\mathscr{R}$ is antisymmetric—hence it defines a partial order for the set A.

Now suppose we wish to know how many ordered pairs occur in this relation $\mathscr{R}$. We may simply list the ordered pairs from $A \times A$ that comprise $\mathscr{R}$:

$$\mathscr{R} = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 12), (2, 2), (2, 4), (2, 6), (2, 12), (3, 3), (3, 6), (3, 12), (4, 4), (4, 12), (6, 6), (6, 12), (12, 12)\}$$

In this way we learn that there are 18 ordered pairs in the relation. But if we then wanted to consider the same type of partial order for the set of positive integer divisors of 1800, we should definitely be discouraged by this method of simply *listing* all the ordered pairs. So let us examine the relation $\mathcal{R}$ a little closer. By the Fundamental Theorem of Arithmetic we may write $12 = 2^2 \cdot 3$ and then realize that if $(c, d) \in \mathcal{R}$, then

$$c = 2^m \cdot 3^n \quad \text{and} \quad d = 2^p \cdot 3^q,$$

where $m, n, p, q \in \mathbf{N}$ with $0 \le m \le p \le 2$ and $0 \le n \le q \le 1$.

When we consider the fact that $0 \le m \le p \le 2$, we find that each possibility for m, p is simply a selection of size 2 from a set of size 3 — namely, the set $\{0, 1, 2\}$ — where repetitions are allowed. (In any such selection, if there is a smaller nonnegative integer, then it is assigned to m.) In Chapter 1 we learned that such a selection can be made in $\binom{3+2-1}{2} = \binom{4}{2} = 6$ ways. And, in like manner, n and q can be selected in $\binom{2+2-1}{2} = \binom{3}{2} = 3$ ways. So by the rule of product there should be $(6)(3) = 18$ ordered pairs in $\mathcal{R}$ — as we found earlier by actually listing all of them.

Now suppose we examine a similar situation, the set of positive integer divisors of $1800 = 2^3 \cdot 3^2 \cdot 5^2$. Here we are dealing with $(3+1)(2+1)(2+1) = (4)(3)(3) = 36$ divisors, and a typical ordered pair for this partial order (given by division) looks like $(2^r \cdot 3^s \cdot 5^t, 2^u \cdot 3^v \cdot 5^w)$, where $r, s, t, u, v, w \in \mathbf{N}$ with $0 \le r \le u \le 3$, $0 \le s \le v \le 2$, and $0 \le t \le w \le 2$. So the number of ordered pairs in the relation is

$$\binom{4+2-1}{2}\binom{3+2-1}{2}\binom{3+2-1}{2} = \binom{5}{2}\binom{4}{2}\binom{4}{2} = (10)(6)(6) = 360,$$

and we definitely should *not* want to have to list all of the ordered pairs in the relation in order to obtain this result.

In general, for $n \in \mathbf{Z}^+$ with $n > 1$, use the Fundamental Theorem of Arithmetic to write $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$, where $k \in \mathbf{Z}^+$, $p_1 < p_2 < p_3 < \cdots < p_k$, and p_i is prime and $e_i \in \mathbf{Z}^+$ for each $1 \le i \le k$. Then n has $\prod_{i=1}^{k}(e_i + 1)$ positive integer divisors. And when we consider the same type of partial order for this set (of positive integer divisors of n), we find that the number of ordered pairs in the relation is

$$\prod_{i=1}^{k} \binom{(e_i+1)+2-1}{2} = \prod_{i=1}^{k} \binom{e_i+2}{2}.$$

In closing this section we introduce the equivalence relation — a concept that is very important in the study of mathematics.

Definition 7.7 An *equivalence relation* $\mathcal{R}$ on a set A is a relation that is reflexive, symmetric, and transitive.

EXAMPLE 7.16

a) The relation in Example 7.1(b) and all the relations in Example 7.3(c) are equivalence relations.

b) If $A = \{1, 2, 3\}$, then
$\mathcal{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$,
$\mathcal{R}_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$,
$\mathcal{R}_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$, and
$\mathcal{R}_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\} = A \times A$
are all equivalence relations on A.

c) For a given finite set A, $A \times A$ is the largest equivalence relation on A, and if $A = \{a_1, a_2, \ldots, a_n\}$, then the equality relation $\mathcal{R} = \{(a_i, a_i) | 1 \le i \le n\}$ is the smallest equivalence relation on A.

d) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{x, y, z\}$, and $f: A \to B$ be the onto function

$$f = \{(1, x), (2, z), (3, x), (4, y), (5, z), (6, y), (7, x)\}.$$

Define the relation $\mathcal{R}$ on A by $a \mathcal{R} b$ if $f(a) = f(b)$. Then, for instance, we find here that $1 \mathcal{R} 1$, $1 \mathcal{R} 3$, $2 \mathcal{R} 5$, $3 \mathcal{R} 1$, and $4 \mathcal{R} 6$.

For each $a \in A$, $f(a) = f(a)$ because f is a function—so $a \mathcal{R} a$, and $\mathcal{R}$ is reflexive. Now suppose that $a, b \in A$ and $a \mathcal{R} b$. Then $a \mathcal{R} b \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow b \mathcal{R} a$, so $\mathcal{R}$ is symmetric. Finally, if $a, b, c \in A$ with $a \mathcal{R} b$ and $b \mathcal{R} c$, then $f(a) = f(b)$ and $f(b) = f(c)$. Consequently, $f(a) = f(c)$, and we see that $(a \mathcal{R} b \wedge b \mathcal{R} c) \Rightarrow a \mathcal{R} c$. So $\mathcal{R}$ is transitive. Since $\mathcal{R}$ is reflexive, symmetric, and transitive, it is an equivalence relation.

Here $\mathcal{R} = \{(1, 1), (1, 3), (1, 7), (2, 2), (2, 5), (3, 1), (3, 3), (3, 7), (4, 4), (4, 6), (5, 2), (5, 5), (6, 4), (6, 6), (7, 1), (7, 3), (7, 7)\}$.

e) If $\mathcal{R}$ is a relation on a set A, then $\mathcal{R}$ is both an equivalence relation and a partial order on A if and only if $\mathcal{R}$ is the equality relation on A.

EXERCISES 7.1

1. If $A = \{1, 2, 3, 4\}$, give an example of a relation $\mathcal{R}$ on A that is

a) reflexive and symmetric, but not transitive

b) reflexive and transitive, but not symmetric

c) symmetric and transitive, but not reflexive

2. For relation (b) in Example 7.1, determine five values of x for which $(x, 5) \in \mathcal{R}$.

3. For the relation $\mathcal{R}$ in Example 7.13, let $f: \mathbf{Z}^+ \to \mathbf{R}$ where $f(n) = n$.

a) Find three elements $f_1, f_2, f_3 \in \mathcal{F}$ such that $f_i \mathcal{R} f$ and $f \mathcal{R} f_i$, for all $1 \le i \le 3$.

b) Find three elements $g_1, g_2, g_3 \in \mathcal{F}$ such that $g_i \mathcal{R} f$ but $f \not\mathcal{R} g_i$, for all $1 \le i \le 3$.

4. a) Rephrase the definitions for the reflexive, symmetric, transitive, and antisymmetric properties of a relation $\mathcal{R}$ (on a set A), using quantifiers.

b) Use the results of part (a) to specify when a relation $\mathcal{R}$ (on a set A) is (i) *not* reflexive; (ii) *not* symmetric; (iii) *not* transitive; and (iv) *not* antisymmetric.

5. For each of the following relations, determine whether the relation is reflexive, symmetric, antisymmetric, or transitive.

a) $\mathcal{R} \subseteq \mathbf{Z}^+ \times \mathbf{Z}^+$ where $a \mathcal{R} b$ if $a|b$ (read "a divides b," as defined in Section 4.3).

b) $\mathcal{R}$ is the relation on $\mathbf{Z}$ where $a \mathcal{R} b$ if $a|b$.

c) For a given universe $\mathcal{U}$ and a fixed subset C of $\mathcal{U}$, define $\mathcal{R}$ on $\mathcal{P}(\mathcal{U})$ as follows: For $A, B \subseteq \mathcal{U}$ we have $A \mathcal{R} B$ if $A \cap C = B \cap C$.

d) On the set A of all lines in $\mathbf{R}^2$, define the relation $\mathcal{R}$ for two lines ℓ_1, ℓ_2 by $\ell_1 \mathcal{R} \ell_2$ if ℓ_1 is perpendicular to ℓ_2.

e) $\mathcal{R}$ is the relation on $\mathbf{Z}$ where $x \mathcal{R} y$ if $x + y$ is odd.

f) $\mathcal{R}$ is the relation on $\mathbf{Z}$ where $x \mathcal{R} y$ if $x - y$ is even.

g) Let T be the set of all triangles in $\mathbf{R}^2$. Define $\mathcal{R}$ on T by $t_1 \mathcal{R} t_2$ if t_1 and t_2 have an angle of the same measure.

h) $\mathcal{R}$ is the relation on $\mathbf{Z} \times \mathbf{Z}$ where $(a, b)\mathcal{R}(c, d)$ if $a \le c$. [*Note*: $\mathcal{R} \subseteq (\mathbf{Z} \times \mathbf{Z}) \times (\mathbf{Z} \times \mathbf{Z})$.]

6. Which relations in Exercise 5 are partial orders? Which are equivalence relations?

7. Let $\mathcal{R}_1, \mathcal{R}_2$ be relations on a set A. (a) Prove or disprove that $\mathcal{R}_1, \mathcal{R}_2$ reflexive $\Rightarrow \mathcal{R}_1 \cap \mathcal{R}_2$ reflexive. (b) Answer part (a) when each occurrence of "reflexive" is replaced by (i) symmetric; (ii) antisymmetric; and (iii) transitive.

8. Answer Exercise 7, replacing each occurrence of $\cap$ by $\cup$.

9. For each of the following statements about relations on a set A, where $|A| = n$, determine whether the statement is true or false. If it is false, give a counterexample.

a) If $\mathcal{R}$ is a relation on A and $|\mathcal{R}| \ge n$, then $\mathcal{R}$ is reflexive.

b) If $\mathcal{R}_1, \mathcal{R}_2$ are relations on A and $\mathcal{R}_2 \supseteq \mathcal{R}_1$, then $\mathcal{R}_1$ reflexive (symmetric, antisymmetric, transitive) $\Rightarrow \mathcal{R}_2$ reflexive (symmetric, antisymmetric, transitive).

c) If $\mathcal{R}_1, \mathcal{R}_2$ are relations on A and $\mathcal{R}_2 \supseteq \mathcal{R}_1$, then $\mathcal{R}_2$ reflexive (symmetric, antisymmetric, transitive) $\Rightarrow \mathcal{R}_1$ reflexive (symmetric, antisymmetric, transitive).

d) If $\mathcal{R}$ is an equivalence relation on A, then $n \le |\mathcal{R}| \le n^2$.

10. If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive; (b) symmetric; (c) reflexive and symmetric; (d) reflexive and contain (x, y); (e) symmetric and contain (x, y); (f) antisymmetric; (g) antisymmetric and contain (x, y); (h) symmetric and antisymmetric; and (i) reflexive, symmetric, and antisymmetric.

11. Let $n \in \mathbf{Z}^+$ with $n > 1$, and let A be the set of positive integer divisors of n. Define the relation $\mathcal{R}$ on A by $x \mathcal{R} y$ if x

(exactly) divides y. Determine how many ordered pairs are in the relation $\mathcal{R}$ when n is (a) 10; (b) 20; (c) 40; (d) 200; (e) 210; and (f) 13860.

12. Suppose that p_1, p_2, p_3 are distinct primes and that $n, k \in \mathbf{Z}^+$ with $n = p_1^5 p_2^3 p_3^k$. Let A be the set of positive integer divisors of n and define the relation $\mathcal{R}$ on A by $x \mathcal{R} y$ if x (exactly) divides y. If there are 5880 ordered pairs in $\mathcal{R}$, determine k and $|A|$.

13. What is wrong with the following argument?

Let A be a set with $\mathcal{R}$ a relation on A. If $\mathcal{R}$ is symmetric and transitive, then $\mathcal{R}$ is reflexive.

Proof: Let $(x, y) \in \mathcal{R}$. By the symmetric property, $(y, x) \in \mathcal{R}$. Then with $(x, y), (y, x) \in \mathcal{R}$, it follows by the transitive property that $(x, x) \in \mathcal{R}$. Consequently, $\mathcal{R}$ is reflexive.

14. Let A be a set with $|A| = n$, and let $\mathcal{R}$ be a relation on A that is antisymmetric. What is the maximum value for $|\mathcal{R}|$? How many antisymmetric relations can have this size?

15. Let A be a set with $|A| = n$, and let $\mathcal{R}$ be an equivalence relation on A with $|\mathcal{R}| = r$. Why is $r - n$ always even?

16. A relation $\mathcal{R}$ on a set A is called *irreflexive* if for all $a \in A$, $(a, a) \notin \mathcal{R}$.

a) Give an example of a relation $\mathcal{R}$ on $\mathbf{Z}$ where $\mathcal{R}$ is irreflexive and transitive but not symmetric.

b) Let $\mathcal{R}$ be a nonempty relation on a set A. Prove that if $\mathcal{R}$ satisfies any two of the following properties — irreflexive, symmetric, and transitive — then it cannot satisfy the third.

c) If $|A| = n \geq 1$, how many different relations on A are irreflexive? How many are neither reflexive nor irreflexive?

17. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. How many symmetric relations on A contain exactly (a) four ordered pairs? (b) five ordered pairs? (c) seven ordered pairs? (d) eight ordered pairs?

18. a) Let $f: A \to B$, where $|A| = 25$, $B = \{x, y, z\}$, and $|f^{-1}(x)| = 10$, $|f^{-1}(y)| = 10$, $|f^{-1}(z)| = 5$. If we define the relation $\mathcal{R}$ on A by $a \mathcal{R} b$ if $a, b \in A$ and $f(a) = f(b)$, how many ordered pairs are there in the relation $\mathcal{R}$?

b) For $n, n_1, n_2, n_3, n_4 \in \mathbf{Z}^+$, let $f: A \to B$, where $|A| = n$, $B = \{w, x, y, z\}$, $|f^{-1}(w)| = n_1$, $|f^{-1}(x)| = n_2$, $|f^{-1}(y)| = n_3$, $|f^{-1}(z)| = n_4$, and $n_1 + n_2 + n_3 + n_4 = n$. If we define the relation $\mathcal{R}$ on A by $a \mathcal{R} b$ if $a, b \in A$ and $f(a) = f(b)$, how many ordered pairs are there in the relation $\mathcal{R}$?

7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

Since our interest in relations is focused on those for finite sets, we are concerned with ways of representing such relations so that the properties of Section 7.1 can be easily verified. For this reason we now develop the necessary tools: relation composition, zero-one matrices, and directed graphs.

In a manner analogous to the composition of functions, relations can be combined in the following circumstances.

Definition 7.8 If A, B, and C are sets with $\mathcal{R}_1 \subseteq A \times B$ and $\mathcal{R}_2 \subseteq B \times C$, then the *composite relation* $\mathcal{R}_1 \circ \mathcal{R}_2$ is a relation from A to C defined by $\mathcal{R}_1 \circ \mathcal{R}_2 = \{(x, z) | x \in A, z \in C$, and there exists $y \in B$ with $(x, y) \in \mathcal{R}_1, (y, z) \in \mathcal{R}_2\}$.

Beware! The composition of two relations is written in an order opposite to that for function composition. We shall see why in Example 7.21.

EXAMPLE 7.17 Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, and $C = \{5, 6, 7\}$. Consider $\mathcal{R}_1 = \{(1, x), (2, x), (3, y), (3, z)\}$, a relation from A to B, and $\mathcal{R}_2 = \{(w, 5), (x, 6)\}$, a relation from B to C. Then $\mathcal{R}_1 \circ \mathcal{R}_2 = \{(1, 6), (2, 6)\}$ is a relation from A to C. If $\mathcal{R}_3 = \{(w, 5), (w, 6)\}$ is another relation from B to C, then $\mathcal{R}_1 \circ \mathcal{R}_3 = \emptyset$.

EXAMPLE 7.18

Let A be the set of employees at a computing center, while B denotes a set of high-level programming languages, and C is a set of projects $\{p_1, p_2, \ldots, p_8\}$ for which managers must make work assignments using the people in A. Consider $\mathcal{R}_1 \subseteq A \times B$, where an ordered pair of the form (L. Alldredge, Java) indicates that employee L. Alldredge is proficient in Java (and perhaps other programming languages). The relation $\mathcal{R}_2 \subseteq B \times C$ consists of ordered pairs such as (Java, p_2), indicating that Java is considered an essential language needed by anyone who works on project p_2. In the composite relation $\mathcal{R}_1 \circ \mathcal{R}_2$ we find (L. Alldredge, p_2). If no other ordered pair in $\mathcal{R}_2$ has p_2 as its second component, we know that if L. Alldredge was assigned to p_2 it was solely on the basis of his proficiency in Java. (Here $\mathcal{R}_1 \circ \mathcal{R}_2$ has been used to set up a matching process between employees and projects on the basis of employee knowledge of specific programming languages.)

Comparable to the associative law for function composition, the following result holds for relations.

THEOREM 7.1

Let A, B, C, and D be sets with $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, and $\mathcal{R}_3 \subseteq C \times D$. Then $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$.

Proof: Since both $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$ and $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$ are relations from A to D, there is some reason to believe they are equal. If $(a, d) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$, then there is an element $b \in B$ with $(a, b) \in \mathcal{R}_1$ and $(b, d) \in (\mathcal{R}_2 \circ \mathcal{R}_3)$. Also, $(b, d) \in (\mathcal{R}_2 \circ \mathcal{R}_3) \Rightarrow$ $(b, c) \in \mathcal{R}_2$ and $(c, d) \in \mathcal{R}_3$ for some $c \in C$. Then $(a, b) \in \mathcal{R}_1$ and $(b, c) \in \mathcal{R}_2 \Rightarrow$ $(a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2$. Finally, $(a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2$ and $(c, d) \in \mathcal{R}_3 \Rightarrow (a, d) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$, and $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3$. The opposite inclusion follows by similar reasoning.

As a result of this theorem no ambiguity arises when we write $\mathcal{R}_1 \circ \mathcal{R}_2 \circ \mathcal{R}_3$ for either of the relations in Theorem 7.1. In addition, we can now define the powers of a relation $\mathcal{R}$ on a set.

Definition 7.9

Given a set A and a relation $\mathcal{R}$ on A, we define the *powers of* $\mathcal{R}$ recursively by (a) $\mathcal{R}^1 = \mathcal{R}$; and (b) for $n \in \mathbf{Z}^+$, $\mathcal{R}^{n+1} = \mathcal{R} \circ \mathcal{R}^n$.

Note that for $n \in \mathbf{Z}^+$, $\mathcal{R}^n$ is a relation on A.

EXAMPLE 7.19

If $A = \{1, 2, 3, 4\}$ and $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, then $\mathcal{R}^2 = \{(1, 4), (1, 2), (3, 4)\}$, $\mathcal{R}^3 = \{(1, 4)\}$, and for $n \geq 4$, $\mathcal{R}^n = \emptyset$.

As the set A and the relation $\mathcal{R}$ on A grow larger, calculations such as those in Example 7.19 become tedious. To avoid this tedium, the tool we need is the computer, once a way can be found to tell the machine about the set A and the relation $\mathcal{R}$ on A.

Definition 7.10

An $m \times n$ *zero-one matrix* $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in m rows and n columns, where each e_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq n$, denotes the entry in the ith row and jth column of E, and each such entry is 0 or 1. [We can also write (0, 1)-matrix for this type of matrix.]

EXAMPLE 7.20

The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is a 3×4 (0, 1)-matrix where, for example, $e_{11} = 1$, $e_{23} = 0$, and $e_{31} = 1$.

In working with these matrices, we use the standard operations of matrix addition and multiplication *with the stipulation that* $1 + 1 = 1$. (Hence the addition is called Boolean.)

EXAMPLE 7.21

Consider the sets A, B, and C and the relations $\mathcal{R}_1$, $\mathcal{R}_2$ of Example 7.17. With the orders of the elements in A, B, and C fixed as in that example, we define the *relation matrices* for $\mathcal{R}_1$, $\mathcal{R}_2$ as follows:

$$M(\mathcal{R}_1) = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \!\!\begin{array}{c} \begin{array}{cccc} (w) & (x) & (y) & (z) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \qquad M(\mathcal{R}_2) = \begin{array}{c} \\ (w) \\ (x) \\ (y) \\ (z) \end{array} \!\!\begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

In constructing $M(\mathcal{R}_1)$, we are dealing with a relation from A to B, so the elements of A are used to mark the rows of $M(\mathcal{R}_1)$ and the elements of B designate the columns. Then to denote, for example, that $(2, x) \in \mathcal{R}_1$, we place a 1 in the row marked (2) and the column marked (x). Each 0 in this matrix indicates an ordered pair in $A \times B$ that is missing from $\mathcal{R}_1$. For example, since $(3, w) \notin \mathcal{R}_1$, there is a 0 for the entry in row (3) and column (w) of the matrix $M(\mathcal{R}_1)$. The same process is used to obtain $M(\mathcal{R}_2)$.

Multiplying these matrices,† we find that

$$M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (4) \end{array} \!\!\begin{array}{c} \begin{array}{ccc} (5) & (6) & (7) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = M(\mathcal{R}_1 \circ \mathcal{R}_2),$$

where the rows of the 4×3 matrix $M(\mathcal{R}_1 \circ \mathcal{R}_2)$ are marked by the elements of A while its columns are marked by the elements of C. In general we have: If $\mathcal{R}_1$ is a relation from A to B and $\mathcal{R}_2$ is a relation from B to C, then $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = M(\mathcal{R}_1 \circ \mathcal{R}_2)$. That is, the product of the relation matrices for $\mathcal{R}_1$, $\mathcal{R}_2$, in that order, equals the relation matrix of the composite relation $\mathcal{R}_1 \circ \mathcal{R}_2$. (This is why the composition of two relations was written in the order specified in Definition 7.8.)

The reader will be asked to prove the general result of Example 7.21, along with some results from our next example, in Exercises 11 and 12 at the end of this section.

Further properties of relation matrices are exhibited in the following example.

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†The reader who is not familiar with matrix multiplication or simply wishes a brief review should consult Appendix 2.

EXAMPLE 7.22

Let $A = \{1, 2, 3, 4\}$ and $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$, as in Example 7.19. Keeping the order of the elements in A fixed, we define the *relation matrix* for $\mathcal{R}$ as follows: $M(\mathcal{R})$ is the 4×4 (0, 1)-matrix whose entries m_{ij}, for $1 \leq i, j \leq 4$, are given by

$$m_{ij} = \begin{cases} 1, & \text{if } (i, j) \in \mathcal{R}, \\ 0, & \text{otherwise.} \end{cases}$$

In this case we find that

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now how can this be of any use? If we compute $(M(\mathcal{R}))^2$ using the convention that $1 + 1 = 1$, then we find that

$$(M(\mathcal{R}))^2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which happens to be the relation matrix for $\mathcal{R} \circ \mathcal{R} = \mathcal{R}^2$. (Check Example 7.19.) Furthermore,

$$(M(\mathcal{R}))^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which is also the relation matrix for the relation $\mathcal{R}^4$—that is, $(M(\mathcal{R}))^4 = M(\mathcal{R}^4)$. Also, recall that $\mathcal{R}^4 = \emptyset$, as we learned in Example 7.19.

What has happened here carries over to the general situation. We now state some results about relation matrices and their use in studying relations.

Let A be a set with $|A| = n$ and $\mathcal{R}$ a relation on A. If $M(\mathcal{R})$ is the relation matrix for $\mathcal{R}$, then

a) $M(\mathcal{R}) = \mathbf{0}$ (the matrix of all 0's) if and only if $\mathcal{R} = \emptyset$

b) $M(\mathcal{R}) = \mathbf{1}$ (the matrix of all 1's) if and only if $\mathcal{R} = A \times A$

c) $M(\mathcal{R}^m) = [M(\mathcal{R})]^m$, for $m \in \mathbf{Z}^+$

Using the (0, 1)-matrix for a relation, we now turn to the recognition of the reflexive, symmetric, antisymmetric, and transitive properties. To accomplish this we need the concepts introduced in the following three definitions.

Definition 7.11

Let $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$ be two $m \times n$ (0, 1)-matrices. We say that *E precedes*, or *is less than, F*, and we write $E \leq F$, if $e_{ij} \leq f_{ij}$, for all $1 \leq i \leq m$, $1 \leq j \leq n$.

EXAMPLE 7.23

With $E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, we have $E \leq F$. In fact, there are eight (0, 1)-matrices G for which $E \leq G$.

Definition 7.12

For $n \in \mathbf{Z}^+$, $I_n = (\delta_{ij})_{n \times n}$ is the $n \times n$ (0, 1)-matrix where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Definition 7.13

Let $A = (a_{ij})_{m \times n}$ be a (0, 1)-matrix. The *transpose* of A, written A^{tr}, is the matrix $(a^*_{ji})_{n \times m}$ where $a^*_{ji} = a_{ij}$, for all $1 \leq j \leq n$, $1 \leq i \leq m$.

EXAMPLE 7.24

For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$, we find that $A^{\text{tr}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

As this example demonstrates, the ith row (column) of A equals the ith column (row) of A^{tr}. This indicates a method we can use in order to obtain the matrix A^{tr} from the matrix A.

THEOREM 7.2

Given a set A with $|A| = n$ and a relation $\mathcal{R}$ on A, let M denote the relation matrix for $\mathcal{R}$. Then

a) $\mathcal{R}$ is reflexive if and only if $I_n \leq M$.

b) $\mathcal{R}$ is symmetric if and only if $M = M^{\text{tr}}$.

c) $\mathcal{R}$ is transitive if and only if $M \cdot M = M^2 \leq M$.

d) $\mathcal{R}$ is antisymmetric if and only if $M \cap M^{\text{tr}} \leq I_n$. (The matrix $M \cap M^{\text{tr}}$ is formed by operating on corresponding entries in M and M^{tr} according to the rules $0 \cap 0 = 0 \cap 1 = 1 \cap 0 = 0$ and $1 \cap 1 = 1$—that is, the usual multiplication for 0's and/or 1's.)

Proof: The results follow from the definitions of the relation properties and the (0, 1)-matrix. We demonstrate this for part (c), using the elements of A to designate the rows and columns in M, as in Examples 7.21 and 7.22.

Let $M^2 \leq M$. If $(x, y), (y, z) \in \mathcal{R}$, then there are 1's in row (x), column (y) and in row (y), column (z) of M. Consequently, in row (x), column (z) of M^2 there is a 1. This 1 must also occur in row (x), column (z) of M because $M^2 \leq M$. Hence $(x, z) \in \mathcal{R}$ and $\mathcal{R}$ is transitive.

Conversely, if $\mathcal{R}$ is transitive and M is the relation matrix for $\mathcal{R}$, let s_{xz} be the entry in row (x) and column (z) of M^2, with $s_{xz} = 1$. For s_{xz} to equal 1 in M^2, there must exist at least one $y \in A$ where $m_{xy} = m_{yz} = 1$ in M. This happens only if $x \mathcal{R} y$ and $y \mathcal{R} z$. With $\mathcal{R}$ transitive, it then follows that $x \mathcal{R} z$. So $m_{xz} = 1$ and $M^2 \leq M$.

The proofs of the remaining parts are left to the reader.

The relation matrix is a useful tool for the computer recognition of certain properties of relations. Storing information as described here, this matrix is an example of a *data* *structure*. Also of interest is how the relation matrix is used in the study of graph theory† and how graph theory is used in the recognition of certain properties of relations.

At this point we shall introduce some fundamental concepts in graph theory. Often these concepts will be given within examples and not in terms of formal definitions. In Chapter 11, however, the presentation will not assume what is given here and will be more rigorous and comprehensive.

Definition 7.14

Let V be a finite nonempty set. A *directed graph* (or *digraph*) G *on* V is made up of the elements of V, called the *vertices* or *nodes* of G, and a subset E, of $V \times V$, that contains the *(directed) edges*, or *arcs*, of G. The set V is called the *vertex set* of G, and the set E is called the *edge set*. We then write $G = (V, E)$ to denote the graph.

If $a, b \in V$ and $(a, b) \in E$‡, then there is an edge *from* a *to* b. Vertex a is called the *origin* or *source* of the edge, with b the *terminus*, or *terminating vertex*, and we say that b is *adjacent from* a and that a is *adjacent to* b. In addition, if $a \neq b$, then $(a, b) \neq (b, a)$. An edge of the form (a, a) is called a *loop* (at a).

EXAMPLE 7.25

For $V = \{1, 2, 3, 4, 5\}$, the diagram in Fig. 7.1 is a directed graph G on V with edge set $\{(1, 1), (1, 2), (1, 4), (3, 2)\}$. Vertex 5 is a part of this graph even though it is not the origin or terminus of an edge. It is referred to as an *isolated* vertex. As we see here, edges need not be straight line segments, and there is no concern about the length of an edge.

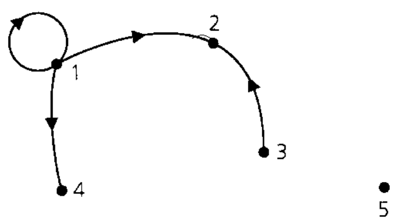


Figure 7.1

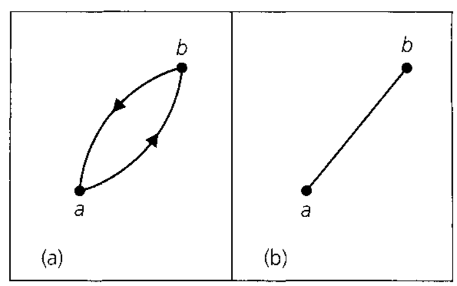


Figure 7.2

When we develop a *flowchart* to study a computer program or algorithm, we deal with a special type of directed graph where the shapes of the vertices may be important in the analysis of the algorithm. Road maps are directed graphs, where the cities and towns are represented by vertices and the highways linking any two localities are given by edges. In road maps, an edge is often directed in both directions. Consequently, if G is a directed graph and $a, b \in V$, with $a \neq b$, and both $(a, b), (b, a) \in E$, then the single undirected edge $\{a, b\} = \{b, a\}$ in Fig. 7.2(b) is used to represent the two directed edges shown in Fig. 7.2(a). In this case, a and b are called *adjacent* vertices. (Directions may also be disregarded for loops.)

†Since the terminology of graph theory is not standardized, the reader may find some differences between definitions given here and in other texts.

‡In this chapter we allow only one edge from a to b. Situations where multiple edges occur are called *multigraphs*. These are discussed in Chapter 11.

Directed graphs play an important role in many situations in computer science. The following example demonstrates one of these.

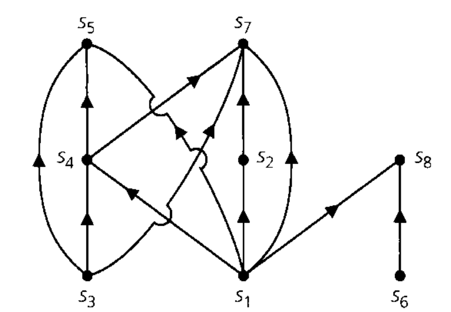
EXAMPLE 7.26

Computer programs can be processed more rapidly when certain statements in the program are executed concurrently. But in order to accomplish this we must be aware of the dependence of some statements on earlier statements in the program. For we cannot execute a statement that needs results from other statements — statements that have not yet been executed.

In Fig. 7.3(a) we have eight assignment statements that constitute the beginning of a computer program. We represent these statements by the eight corresponding vertices $s_1, s_2, s_3, \ldots, s_8$ in part (b) of the figure, where a directed edge such as (s_1, s_5) indicates that statement s_5 cannot be executed until statement s_1 has been executed. The resulting directed graph is called the *precedence graph* for the given lines of the computer program. Note how this graph indicates, for example, that statement s_7 cannot be executed until after each of the statements s_1, s_2, s_3, and s_4 has been executed. Also, we see how a statement such as s_1 must be executed before it is possible to execute any of the statements s_2, s_4, s_5, s_7, or s_8. In general, if a vertex (statement) s is adjacent *from m* other vertices (and no others), then the corresponding statements for these m vertices must be executed before statement s can be executed. Similarly, should a vertex (statement) s be adjacent *to n* other vertices, then each of the corresponding statements for these vertices requires the execution of statement s before it can be executed. Finally, from the precedence graph we see that the statements s_1, s_3, and s_6 can be processed concurrently. Following this, the statements s_2, s_4, and s_8 can be executed at the same time, and then the statements s_5 and s_7. (Or we could process statements s_2 and s_4 concurrently, and then the statements s_5, s_7, and s_8.)

(s_1)	b	$:= 3$
(s_2)	c	$:= b + 2$
(s_3)	a	$:= 1$
(s_4)	d	$:= a * b + 5$
(s_5)	e	$:= d - 1$
(s_6)	f	$:= 7$
(s_7)	e	$:= c + d$
(s_8)	g	$:= b * f$

(a)



(b)

Figure 7.3

Now we want to consider how relations and directed graphs are interrelated. For a start, given a set A and a relation $\mathcal{R}$ on A, we can construct a directed graph G with vertex set A and edge set $E \subseteq A \times A$, where $(a, b) \in E$ if $a, b \in A$ and $a \mathcal{R} b$. This is demonstrated in the following example.

EXAMPLE 7.27

For $A = \{1, 2, 3, 4\}$, let $\mathcal{R} = \{(1, 1), (1, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2)\}$ be a relation on A. The directed graph associated with $\mathcal{R}$ is shown in Fig. 7.4(a), where the undirected edge $\{2, 3\}(= \{3, 2\})$ is used in place of the pair of distinct directed edges $(2, 3)$ and $(3, 2)$. If the directions in Fig. 7.4(a) are ignored, we get the *associated undirected graph* shown in part (b) of the figure. Here we see that the graph is *connected* in the sense that for any two vertices x, y, with $x \neq y$, there is a *path* starting at x and ending at y. Such a path consists of a *finite sequence of undirected edges*, so the edges $\{1, 2\}$, $\{2, 4\}$ provide a path from 1 to 4, and the edges $\{3, 4\}$, $\{4, 2\}$, and $\{2, 1\}$ provide a path from 3 to 1. The sequence of edges $\{3, 4\}$, $\{4, 2\}$, and $\{2, 3\}$ provides a path from 3 to 3. Such a *closed* path is called a *cycle*. This is an example of an undirected cycle of *length* 3, because it has three edges in it.

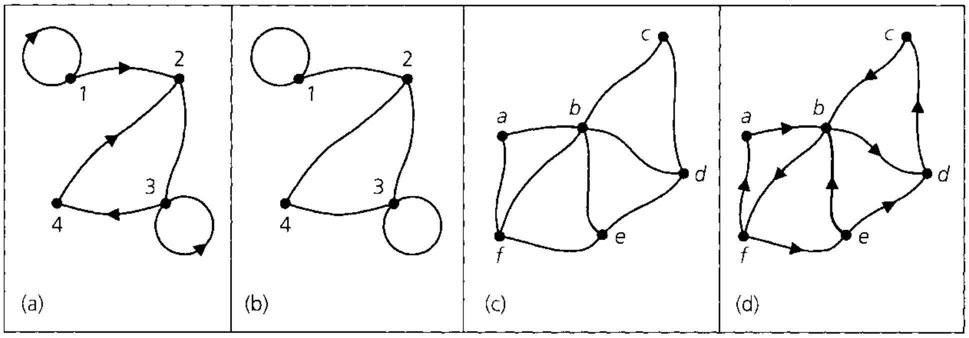


Figure 7.4

When we are dealing with paths (in both directed and undirected graphs), no vertex may be repeated. Therefore, the sequence of edges $\{a, b\}$, $\{b, e\}$, $\{e, f\}$, $\{f, b\}$, $\{b, d\}$ in Fig. 7.4(c) is *not* considered to be a path (from a to d) because we pass through the vertex b more than once. In the case of cycles, the path starts and terminates at the same vertex and has *at least three edges*. In Fig. 7.4(d) the sequence of edges (b, f), (f, e), (e, d), (d, c), (c, b) provides a *directed cycle* of length 5. The six edges (b, f), (f, e), (e, b), (b, d), (d, c), (c, b) do *not* yield a directed cycle in the figure because of the repetition of vertex b. If their directions are ignored, the corresponding six edges, in part (c) of the figure, likewise pass through vertex b more than once. Consequently, these edges are not considered to form a cycle for the undirected graph in Fig. 7.4(c).

Now since we require a cycle to have *length* at least 3, we shall not consider loops to be cycles. We also note that loops have no bearing on graph connectivity.

We choose to define the next idea formally because of its relevance to what we did earlier in Section 6.3.

Definition 7.15 A directed graph G on V is called *strongly connected* if for all $x, y \in V$, where $x \neq y$, there is a path (in G) of directed edges from x to y — that is, either the directed edge (x, y) is in G or, for some $n \in \mathbf{Z}^+$ and distinct vertices $v_1, v_2, \ldots, v_n \in V$, the directed edges $(x, v_1), (v_1, v_2), \ldots, (v_n, y)$ are in G.

It is in this sense that we talked about strongly connected machines in Chapter 6. The graph in Fig. 7.4(a) is connected but not strongly connected. For example, there is no directed path from 3 to 1. In Fig. 7.5 the directed graph on $V = \{1, 2, 3, 4\}$ is strongly connected and *loop-free*. This is also true of the directed graph in Fig. 7.4(d).

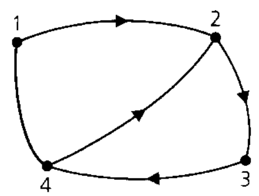


Figure 7.5

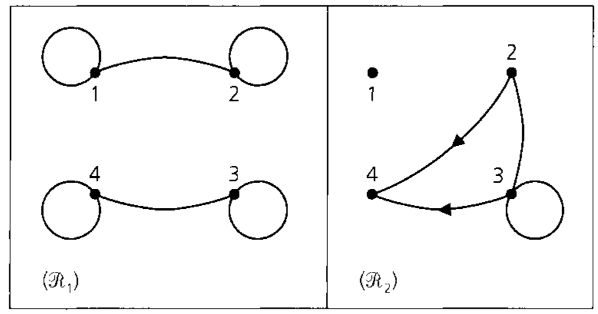


Figure 7.6

EXAMPLE 7.28

For $A = \{1, 2, 3, 4\}$, consider the relations $\mathcal{R}_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$ and $\mathcal{R}_2 = \{(2, 4), (2, 3), (3, 2), (3, 3), (3, 4)\}$. As Fig. 7.6 illustrates, the graphs of these relations are *disconnected*. However, each graph is the union of two connected pieces called the *components* of the graph. For $\mathcal{R}_1$ the graph is made up of two strongly connected components. For $\mathcal{R}_2$, one component consists of an isolated vertex, and the other component is connected but not strongly connected.

EXAMPLE 7.29

The graphs in Fig. 7.7 are examples of undirected graphs that are loop-free and have an edge for every pair of distinct vertices. These graphs illustrate the *complete graphs* on n vertices which are denoted by K_n. In Fig. 7.7 we have examples of the complete graphs on three, four, and five vertices, respectively. The complete graph K_2 consists of two vertices x, y and an edge connecting them, whereas the complete graph K_1 consists of one vertex and no edges because loops are not allowed.

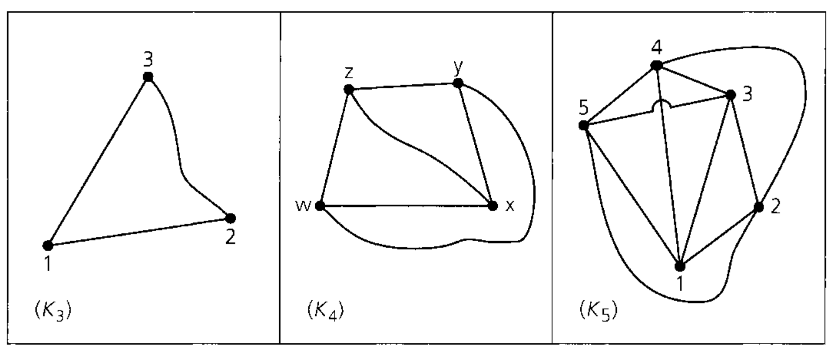


Figure 7.7

In this drawing of K_5 two edges cross, namely, $\{3, 5\}$ and $\{1, 4\}$. However, there is no point of intersection creating a new vertex. If we try to avoid the crossing of edges by drawing the graph differently, we run into the same problem all over again. This difficulty will be examined in Chapter 11 when we deal with the planarity of graphs.

A digraph G on a vertex set V gives rise to a relation $\mathcal{R}$ on V where $x \mathcal{R} y$ if (x, y) is an edge in G. Consequently, there is a (0, 1)-matrix for G, and since this relation matrix comes about from the adjacencies of pairs of vertices, it is referred to as the *adjacency matrix* for G as well as the relation matrix for $\mathcal{R}$.

At this point we tie together the properties of relations and the structure of directed graphs.

EXAMPLE 7.30

If $A = \{1, 2, 3\}$ and $\mathcal{R} = \{(1, 1), (1, 2), (2, 2), (3, 3), (3, 1)\}$, then $\mathcal{R}$ is a reflexive antisymmetric relation on A, but it is neither symmetric nor transitive. The directed graph associated with $\mathcal{R}$ consists of five edges. Three of these edges are loops that result from the reflexive property of $\mathcal{R}$. (See Fig. 7.8.) In general, if $\mathcal{R}$ is a relation on a finite set A, then $\mathcal{R}$ is reflexive if and only if its directed graph contains a loop at each vertex (element of A).

EXAMPLE 7.31

The relation $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 3), (3, 2)\}$ is symmetric on $A = \{1, 2, 3\}$, but it is not reflexive, antisymmetric, or transitive. The directed graph for $\mathcal{R}$ is found in Fig. 7.9. In general, a relation $\mathcal{R}$ on a finite set A is symmetric if and only if its directed graph may be drawn so that it contains only loops and undirected edges.

EXAMPLE 7.32

For $A = \{1, 2, 3\}$, consider $\mathcal{R} = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$. The directed graph for $\mathcal{R}$ is shown in Fig. 7.10. Here $\mathcal{R}$ is transitive and antisymmetric but not reflexive or symmetric. The directed graph indicates that a relation on a set A is transitive if and only if it satisfies the following: For all $x, y \in A$, if there is a (directed) path from x to y in the associated graph, then there is an edge (x, y) also. [Here $(1, 2)$, $(2, 3)$ is a (directed) path from 1 to 3, and we also have the edge $(1, 3)$ for transitivity.] Notice that the directed graph in Fig. 7.3 of Example 7.26 also has this property.

The relation $\mathcal{R}$ is antisymmetric because there are no ordered pairs in $\mathcal{R}$ of the form (x, y) and (y, x) with $x \neq y$. To use the directed graph of Fig. 7.10 to characterize antisymmetry, we observe that for any two vertices x, y, with $x \neq y$, the graph contains at most one of the edges (x, y) or (y, x). Hence there are no undirected edges aside from loops.

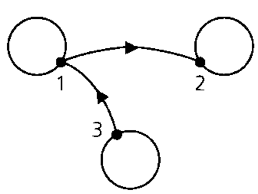


Figure 7.8

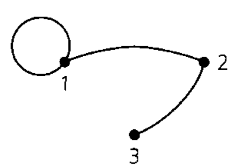


Figure 7.9

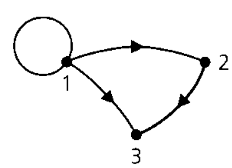


Figure 7.10

Our final example deals with equivalence relations.

EXAMPLE 7.33

For $A = \{1, 2, 3, 4, 5\}$, the following are equivalence relations on A:

$$\mathcal{R}_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\},$$
$$\mathcal{R}_2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}.$$

Their associated graphs are shown in Fig. 7.11. If we ignore the loops in each graph, we find the graph decomposed into components such as K_1, K_2, and K_3. In general, a relation on a finite set A is an equivalence relation if and only if its associated graph is one complete graph augmented by loops at every vertex or consists of the disjoint union of complete graphs augmented by loops at every vertex.

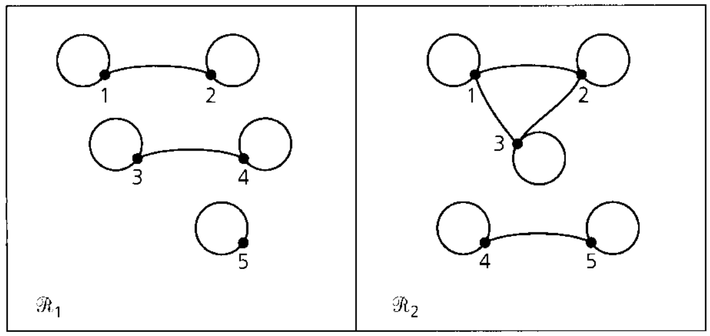


Figure 7.11

EXERCISES 7.2

1. For $A = \{1, 2, 3, 4\}$, let $\mathcal{R}$ and $\mathcal{S}$ be the relations on A defined by $\mathcal{R} = \{(1, 2), (1, 3), (2, 4), (4, 4)\}$ and $\mathcal{S} = \{(1, 1), (1, 2), (1, 3), (2, 3), (2, 4)\}$. Find $\mathcal{R} \circ \mathcal{S}$, $\mathcal{S} \circ \mathcal{R}$, $\mathcal{R}^2$, $\mathcal{R}^3$, $\mathcal{S}^2$, and $\mathcal{S}^3$.

2. If $\mathcal{R}$ is a reflexive relation on a set A, prove that $\mathcal{R}^2$ is also reflexive on A.

3. Provide a proof for the opposite inclusion in Theorem 7.1.

4. Let $A = \{1, 2, 3\}$, $B = \{w, x, y, z\}$, and $C = \{4, 5, 6\}$. Define the relations $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, and $\mathcal{R}_3 \subseteq B \times C$, where $\mathcal{R}_1 = \{(1, w), (3, w), (2, x), (1, y)\}$, $\mathcal{R}_2 = \{(w, 5), (x, 6), (y, 4), (y, 6)\}$, and $\mathcal{R}_3 = \{(w, 4), (w, 5), (y, 5)\}$. (a) Determine $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3)$ and $(\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$. (b) Determine $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3)$ and $(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$.

5. Let $A = \{1, 2\}$, $B = \{m, n, p\}$, and $C = \{3, 4\}$. Define the relations $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, and $\mathcal{R}_3 \subseteq B \times C$ by $\mathcal{R}_1 = \{(1, m), (1, n), (1, p)\}$, $\mathcal{R}_2 = \{(m, 3), (m, 4), (p, 4)\}$, and $\mathcal{R}_3 = \{(m, 3), (m, 4), (p, 3)\}$. Determine $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3)$ and $(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$.

6. For sets A, B, and C, consider relations $\mathcal{R}_1 \subseteq A \times B$, $\mathcal{R}_2 \subseteq B \times C$, and $\mathcal{R}_3 \subseteq B \times C$. Prove that (a) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$; and (b) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3)$.

7. For a relation $\mathcal{R}$ on a set A, define $\mathcal{R}^0 = \{(a, a) | a \in A\}$. If $|A| = n$, prove that there exist $s, t \in \mathbf{N}$ with $0 \leq s < t \leq 2^{n^2}$ such that $\mathcal{R}^s = \mathcal{R}^t$.

8. With $A = \{1, 2, 3, 4\}$, let $\mathcal{R} = \{(1, 1), (1, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$ be a relation on A. Find two relations $\mathcal{S}$, $\mathcal{T}$ on A where $\mathcal{S} \neq \mathcal{T}$ but $\mathcal{R} \circ \mathcal{S} = \mathcal{R} \circ \mathcal{T} = \{(1, 1), (1, 2), (1, 4)\}$.

9. How many 6×6 (0, 1)-matrices A are there with $A = A^{\text{tr}}$?

10. If $E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, how many (0, 1)-matrices F satisfy $E \leq F$? How many (0, 1)-matrices G satisfy $G \leq E$?

11. Consider the sets $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$, and $C = \{c_1, c_2, \ldots, c_p\}$, where the elements in each set remain fixed in the order given here. Let $\mathcal{R}_1$ be a relation from A to B, and let $\mathcal{R}_2$ be a relation from B to C. The relation matrix for $\mathcal{R}_i$ is $M(\mathcal{R}_i)$, where $i = 1, 2$. The rows and columns of these matrices are indexed by the elements from the appropriate sets A, B, and C according to the orders already prescribed. The matrix for $\mathcal{R}_1 \circ \mathcal{R}_2$ is the $m \times p$ matrix $M(\mathcal{R}_1 \circ \mathcal{R}_2)$, where the elements of A (in the order given) index the rows and the elements of C (also in the order given) index the columns.

Show that for all $1 \leq i \leq m$ and $1 \leq j \leq p$, the entries in the ith row and jth column of $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$ and $M(\mathcal{R}_1 \circ \mathcal{R}_2)$ are equal. [Hence $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2) = M(\mathcal{R}_1 \circ \mathcal{R}_2)$.]

12. Let A be a set with $|A| = n$, and consider the order for the listing of its elements as fixed. For $\mathcal{R} \subseteq A \times A$, let $M(\mathcal{R})$ denote the corresponding relation matrix.

a) Prove that $M(\mathcal{R}) = \mathbf{0}$ (the $n \times n$ matrix of all 0's) if and only if $\mathcal{R} = \emptyset$.

b) Prove that $M(\mathcal{R}) = \mathbf{1}$ (the $n \times n$ matrix of all 1's) if and only if $\mathcal{R} = A \times A$.

c) Use the result of Exercise 11, along with the Principle of Mathematical Induction, to prove that $M(\mathcal{R}^m) = [M(\mathcal{R})]^m$, for all $m \in \mathbf{Z}^+$.

13. Provide the proofs for Theorem 7.2(a), (b), and (d).

14. Use Theorem 7.2 to write a computer program (or to develop an algorithm) for the recognition of equivalence relations on a finite set.

15. a) Draw the digraph $G_1 = (V_1, E_1)$ where $V_1 = \{a, b, c, d, e, f\}$ and $E_1 = \{(a, b), (a, d), (b, c), (b, e), (d, b), (d, e), (e, c), (e, f), (f, d)\}$.

b) Draw the undirected graph $G_2 = (V_2, E_2)$ where $V_2 = \{s, t, u, v, w, x, y, z\}$ and $E_2 = \{\{s, t\}, \{s, u\}, \{s, x\}, \{t, u\}, \{t, w\}, \{u, w\}, \{u, x\}, \{v, w\}, \{v, x\}, \{v, y\}, \{w, z\}, \{x, y\}\}$.

16. For the directed graph $G = (V, E)$ in Fig. 7.12, classify each of the following statements as true or false.

a) Vertex c is the origin of two edges in G.

b) Vertex g is adjacent to vertex h.

c) There is a directed path in G from d to b.

d) There are two directed cycles in G.

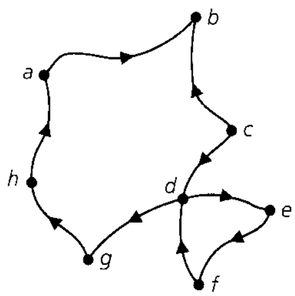


Figure 7.12

17. For $A = \{a, b, c, d, e, f\}$, each graph, or digraph, in Fig. 7.13 represents a relation $\mathcal{R}$ on A. Determine the relation $\mathcal{R} \subseteq A \times A$ in each case, as well as its associated relation matrix $M(\mathcal{R})$.

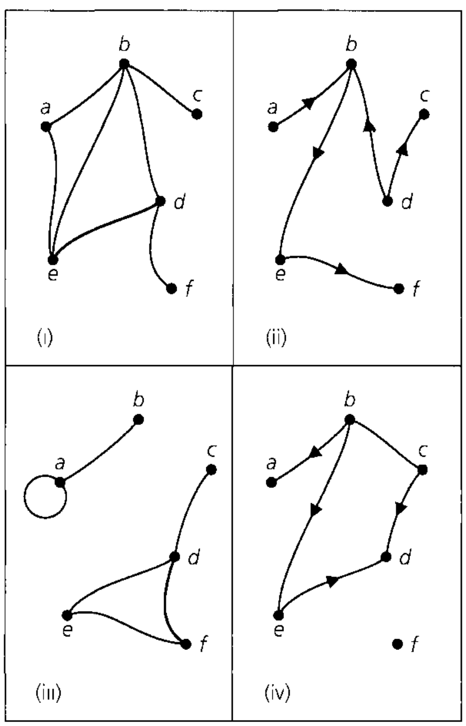


Figure 7.13

18. For $A = \{v, w, x, y, z\}$, each of the following is the (0, 1)-matrix for a relation $\mathcal{R}$ on A. Here the rows (from top to bottom) and the columns (from left to right) are indexed in the order v, w, x, y, z. Determine the relation $\mathcal{R} \subseteq A \times A$ in each case, and draw the directed graph G associated with $\mathcal{R}$.

a) $M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b) $M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

19. For $A = \{1, 2, 3, 4\}$, let $\mathcal{R} = \{(1, 1), (1, 2), (2, 3), (3, 3), (3, 4)\}$ be a relation on A. Draw the directed graph G on A that is associated with $\mathcal{R}$. Do likewise for $\mathcal{R}^2$, $\mathcal{R}^3$, and $\mathcal{R}^4$.

20. a) Let $G = (V, E)$ be the directed graph where $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $E = \{(i, j) | 1 \leq i < j \leq 7\}$.

i) How many edges are there for this graph?

ii) Four of the directed paths in G from 1 to 7 may be given as:

1) (1, 7);

2) (1, 3), (3, 5), (5, 6), (6, 7);

3) (1, 2), (2, 3), (3, 7); and

4) (1, 4), (4, 7).

How many directed paths (in total) exist in G from 1 to 7?

b) Now let $n \in \mathbf{Z}^+$ where $n \geq 2$, and consider the directed graph $G = (V, E)$ with $V = \{1, 2, 3, \ldots, n\}$ and $E = \{(i, j) | 1 \leq i < j \leq n\}$.

i) Determine $|E|$.

ii) How many directed paths exist in G from 1 to n?

iii) If $a, b \in \mathbf{Z}^+$ with $1 \leq a < b \leq n$, how many directed paths exist in G from a to b?

(The reader may wish to refer back to Exercise 20 in Section 3.1.)

21. Let $|A| = 5$. (a) How many directed graphs can one construct on A? (b) How many of the graphs in part (a) are actually undirected?

22. For $|A| = 5$, how many relations $\mathcal{R}$ on A are there? How many of these relations are symmetric?

23. a) Keeping the order of the elements fixed as 1, 2, 3, 4, 5, determine the (0, 1) relation matrix for each of the equivalence relations in Example 7.33.

b) Do the results of part (a) lead to any generalization?

24. How many (undirected) edges are there in the complete graphs K_6, K_7, and K_n, where $n \in \mathbf{Z}^+$?

25. Draw a precedence graph for the following segment found at the start of a computer program:

```
(s1) a := 1
(s2) b := 2
(s3) a := a + 3
(s4) c := b
(s5) a := 2 * a - 1
(s6) b := a * c
(s7) c := 7
(s8) d := c + 2
```

26. a) Let $\mathcal{R}$ be the relation on $A = \{1, 2, 3, 4, 5, 6, 7\}$, where the directed graph associated with $\mathcal{R}$ consists of the two components, each a directed cycle, shown in Fig. 7.14. Find the smallest integer $n > 1$, such that $\mathcal{R}^n = \mathcal{R}$. What is the smallest value of $n > 1$ for which the graph of $\mathcal{R}^n$ contains some loops? Does it ever happen that the graph of $\mathcal{R}^n$ consists of only loops?

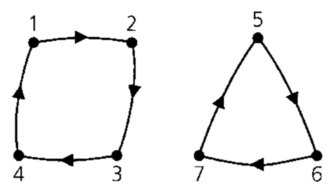


Figure 7.14

b) Answer the same questions from part (a) for the relation $\mathcal{R}$ on $A = \{1, 2, 3, \ldots, 9, 10\}$, if the directed graph associated with $\mathcal{R}$ is as shown in Fig. 7.15.

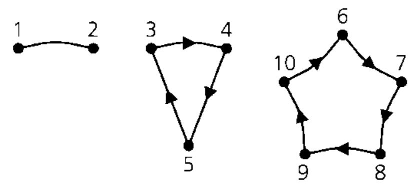


Figure 7.15

c) Do the results in parts (a) and (b) indicate anything in general?

27. If the complete graph K_n has 703 edges, how many vertices does it have?

7.3 Partial Orders: Hasse Diagrams

If you ask children to recite the numbers they know, you'll hear a uniform response of "1, 2, 3, ..." Without paying attention to it, they list these numbers in increasing order. In this section we take a closer look at this idea of order, something we may have taken for granted. We start with some observations about the sets **N**, **Z**, **Q**, **R**, and **C**.

The set **N** is closed under the binary operations of (ordinary) addition and multiplication, but if we seek an answer to the equation $x + 5 = 2$, we find that no element of **N** provides a solution. So we enlarge **N** to **Z**, where we can perform subtraction as well as addition and multiplication. However, we soon run into trouble trying to solve the equation $2x + 3 = 4$. Enlarging to **Q**, we can perform nonzero division in addition to the other operations. Yet this soon proves to be inadequate; the equation $x^2 - 2 = 0$ necessitates the introduction of the real but irrational numbers $\pm\sqrt{2}$. Even after we expand from **Q** to **R**, more trouble arises when we try to solve $x^2 + 1 = 0$. Finally we arrive at **C**, the complex numbers, where any polynomial equation of the form $c_n x^n + c_{n-1}x^{n-1} + \cdots + c_2 x^2 + c_1 x + c_0 = 0$, where $c_i \in \mathbf{C}$ for $0 \leq i \leq n$, $n > 0$ and $c_n \neq 0$, can be solved. (This result is known as the Fundamental Theorem of Algebra. Its proof requires material on functions of a complex variable, so no proof is given here.) As we kept building up from **N** to **C**, gaining more ability to solve polynomial equations, something was lost when we went from **R** to **C**. In **R**, given numbers r_1, r_2, with $r_1 \neq r_2$, we know that either $r_1 < r_2$ or $r_2 < r_1$. However, in **C** we have $(2 + i) \neq (1 + 2i)$, but what meaning can we attach to a statement such as "$(2+i) < (1+2i)$"? We have lost the ability to "order" the elements in this number system!

As we start to take a closer look at the notion of order we proceed as in Section 7.1 and let A be a set with $\mathcal{R}$ a relation on A. The pair $(A, \mathcal{R})$ is called a *partially ordered set*, or *poset*, if relation $\mathcal{R}$ on A is a partial order, or a partial ordering relation (as given in Definition 7.6). If A is called a poset, we understand that there is a partial order $\mathcal{R}$ on A that makes A into this poset. Examples 7.1(a), 7.2, 7.11, and 7.15 are posets.

EXAMPLE 7.34

Let A be the set of courses offered at a college. Define the relation $\mathcal{R}$ on A by $x \mathcal{R} y$ if x, y are the same course or if x is a prerequisite for y. Then $\mathcal{R}$ makes A into a poset.

EXAMPLE 7.35

Define $\mathcal{R}$ on $A = \{1, 2, 3, 4\}$ by $x \mathcal{R} y$ if $x | y$ — that is, x (exactly) divides y. Then $\mathcal{R} = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 3), (1, 4), (2, 4)\}$ is a partial order, and $(A, \mathcal{R})$ is a poset. (This is similar to what we learned in Example 7.15.)

EXAMPLE 7.36

In the construction of a house certain jobs, such as digging the foundation, must be performed before other phases of the construction can be undertaken. If A is a set of tasks that must be performed in building a house, we can define a relation $\mathcal{R}$ on A by $x \mathcal{R} y$ if x, y denote the same task or if task x must be performed before the start of task y. In this way we place an order on the elements of A, making it into a poset that is sometimes referred to as a PERT (Program Evaluation and Review Technique) network. (Such networks came into play during the 1950s in order to handle the complexities that arose in organizing the many individual activities required for the completion of projects on a very large scale. This technique was actually developed and first used by the U.S. Navy in order to coordinate the many projects that were necessary for the building of the Polaris submarine.)

Consider the diagrams given in Fig. 7.16. If part (a) were part of the directed graph associated with a relation $\mathcal{R}$, then because $(1, 2), (2, 1) \in \mathcal{R}$ with $1 \neq 2$, $\mathcal{R}$ could not be antisymmetric. For part (b), if the diagram were part of the graph of a transitive relation $\mathcal{R}$, then $(1, 2), (2, 3) \in \mathcal{R} \Rightarrow (1, 3) \in \mathcal{R}$. Since $(3, 1) \in \mathcal{R}$ and $1 \neq 3$, $\mathcal{R}$ is not antisymmetric, so it cannot be a partial order.

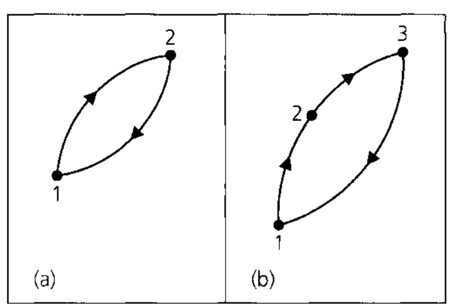


Figure 7.16

From these observations, if we are given a relation $\mathcal{R}$ on a set A, and we let G be the directed graph associated with $\mathcal{R}$, then we find that:

i) If G contains a pair of edges of the form (a, b), (b, a), for $a, b \in A$ with $a \neq b$, or

ii) If $\mathcal{R}$ is transitive and G contains a directed cycle (of length greater than or equal to three),

then the relation $\mathcal{R}$ cannot be antisymmetric, so $(A, \mathcal{R})$ fails to be a partial order.

EXAMPLE 7.37

Consider the directed graph for the partial order in Example 7.35. Figure 7.17(a) is the graphical representation of $\mathcal{R}$. In part (b) of the figure, we have a somewhat simpler diagram, which is called the *Hasse diagram* for $\mathcal{R}$.

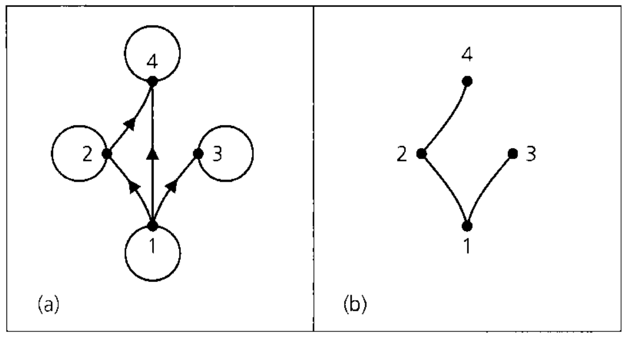


Figure 7.17

When we know that a relation $\mathcal{R}$ is a partial order on a set A, we can eliminate the loops at the vertices of its directed graph. Since $\mathcal{R}$ is also transitive, having the edges $(1, 2)$ and $(2, 4)$ is enough to insure the existence of edge $(1, 4)$, so we need not include that edge. In this way we obtain the diagram in Fig. 7.17(b), where we have not lost the directions on the edges — the directions are assumed to go from the bottom to the top.

In general, if $\mathcal{R}$ is a partial order on a finite set A, we construct a Hasse diagram for $\mathcal{R}$ on A by drawing a line segment from x *up to* y, if $x, y \in A$ with $x \mathcal{R} y$ and, most important, if there is no other element $z \in A$ such that $x \mathcal{R} z$ and $z \mathcal{R} y$. (So there is nothing "in between" x and y.) If we adopt the convention of reading the diagram from bottom to top, then it is not necessary to direct any edges.

EXAMPLE 7.38

In Fig. 7.18 we have the Hasse diagrams for the following four posets. (a) With $\mathcal{U} = \{1, 2, 3\}$ and $A = \mathcal{P}(\mathcal{U})$, $\mathcal{R}$ is the subset relation on A. (b) Here $\mathcal{R}$ is the "(exactly) divides" relation

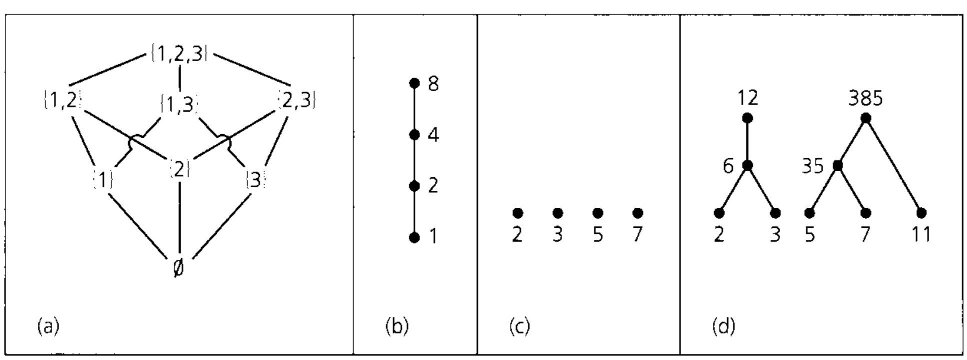


Figure 7.18

applied to $A = \{1, 2, 4, 8\}$. (c) and (d) Here the same relation as in part (b) is applied to $\{2, 3, 5, 7\}$ in part (c) and to $\{2, 3, 5, 6, 7, 11, 12, 35, 385\}$ in part (d). In part (c) we note that a Hasse diagram can have all isolated vertices; it can also have two (or more) connected pieces, as shown in part (d).

EXAMPLE 7.39

Let $A = \{1, 2, 3, 4, 5\}$. The relation $\mathcal{R}$ on A, defined by $x \mathcal{R} y$ if $x \leq y$, is a partial order. This makes A into a poset that we can denote by $(A, \leq)$. If $B = \{1, 2, 4\} \subset A$, then the set $(B \times B) \cap \mathcal{R} = \{(1, 1), (2, 2), (4, 4), (1, 2), (1, 4), (2, 4)\}$ is a partial order on B.

In general if $\mathcal{R}$ is a partial order on A, then for each subset B of A, $(B \times B) \cap \mathcal{R}$ makes B into a poset where the partial order on B is induced from $\mathcal{R}$.

We turn now to a special type of partial order.

Definition 7.16

If $(A, \mathcal{R})$ is a poset, we say that A is *totally ordered* (or, *linearly ordered*) if for all $x, y \in A$ either $x \mathcal{R} y$ or $y \mathcal{R} x$. In this case $\mathcal{R}$ is called a *total order* (or, a *linear order*).

EXAMPLE 7.40

a) On the set **N**, the relation $\mathcal{R}$ defined by $x \mathcal{R} y$ if $x \leq y$ is a total order.

b) The subset relation applied to $A = \mathcal{P}(\mathcal{U})$, where $\mathcal{U} = \{1, 2, 3\}$, is a partial, but not total, order: $\{1, 2\}, \{1, 3\} \in A$ but we have neither $\{1, 2\} \subseteq \{1, 3\}$ nor $\{1, 3\} \subseteq \{1, 2\}$.

c) The Hasse diagram in part (b) of Fig. 7.18 shows a total order. In Fig. 7.19(a) we have the directed graph for this total order—alongside its Hasse diagram in part (b).

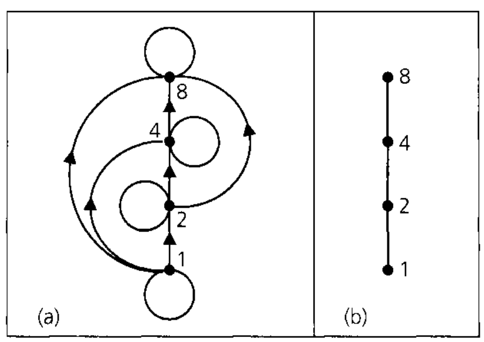


Figure 7.19

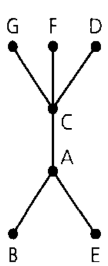


Figure 7.20

Could these notions of partial and total order ever arise in an industrial problem?

Say a toy manufacturer is about to market a new product and must include a set of instructions for its assembly. In order to assemble the new toy, there are seven tasks, denoted A, B, C, . . . , G, that one must perform in the partial order given by the Hasse diagram of Fig. 7.20. Here we see, for example, that all of the tasks B, A, and E must be completed before we can work on task C. Since the set of instructions is to consist of a listing of these tasks, numbered 1, 2, 3, . . . , 7, how can the manufacturer write the listing and make sure that the partial order of the Hasse diagram is maintained?

What we are really asking for here is whether we can take the partial order $\mathcal{R}$, given by the Hasse diagram, and find a total order $\mathcal{T}$ on these tasks for which $\mathcal{R} \subseteq \mathcal{T}$. The answer is yes, and the technique that we need is known as *topological sorting*.

Topological Sorting Algorithm

(for a partial order $\mathcal{R}$ on a set A with $|A| = n$)

Step 1: **Set $k = 1$. Let H_1 be the Hasse diagram of the partial order.**

Step 2: **Select a vertex v_k in H_k such that no (implicitly directed) edge in H_k starts at v_k.**

Step 3: **If $k = n$, the process is completed and we have a total order**

$$\mathcal{T}: v_n < v_{n-1} < \cdots < v_2 < v_1$$

that contains $\mathcal{R}$.

If $k < n$, then remove from H_k the vertex v_k and all (implicitly directed) edges of H_k that terminate at v_k. Call the result H_{k+1}. Increase k by 1 and return to step (2).

Here we have presented our algorithm as a precise list of instructions, with no concern about the particulars of the pseudocode used in earlier chapters and with no reference to its implementation in a particular computer language.

Before we apply this algorithm[†] to the problem at hand, we should observe the deliberate use of "a" before the word "vertex" in step (2). This implies that the selection need not be unique and that we can get several different total orders $\mathcal{T}$ containing $\mathcal{R}$. Also, in step (3), for vertices v_{i-1} where $2 \leq i \leq n$, the notation $v_i < v_{i-1}$ is used because it is more suggestive of "v_i before v_{i-1}" than is the notation $v_i \mathcal{T} v_{i-1}$.

In Fig. 7.21, we show the Hasse diagrams that evolve as we apply the topological sorting algorithm to the partial order in Fig. 7.20. Below each diagram, the total order is listed as it evolves.

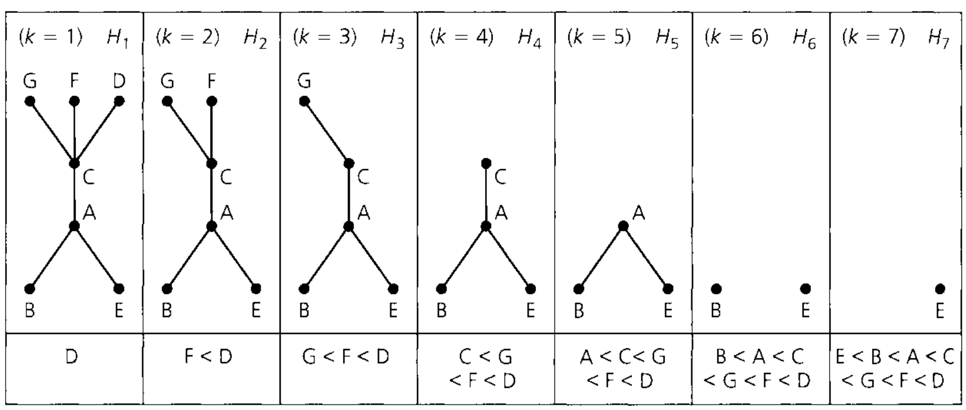


Figure 7.21

If the toy manufacturer writes the instructions in a list as 1-E, 2-B, 3-A, 4-C, 5-G, 6-F, 7-D, he or she will have a total order that preserves the partial order needed for correct assembly. This total order is one of 12 possible answers.

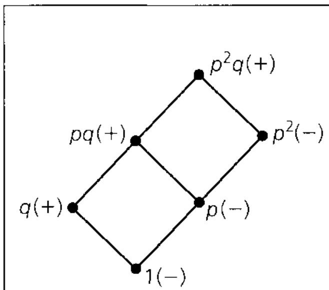
[†]Here we are only concerned with applying this algorithm. Hence we are assuming that it works and we shall not present a proof of that fact. Furthermore, we may operate similarly with other algorithms we encounter.

As is typical in discrete and combinatorial mathematics, this algorithm provides a procedure that reduces the size of the problem with each successive application.

The next example provides a situation where the number of distinct total orders for a particular partial order is determined.

EXAMPLE 7.41†

Let p, q be distinct primes. In part (a) of Fig. 7.22 we have the Hasse diagram for the partial order $\mathscr{R}$ of all positive-integer divisors of p^2q. Applying the topological sorting algorithm to this Hasse diagram, we find in Fig. 7.22(b) the five total orders $\mathscr{T}_i$, where $\mathscr{R} \subseteq \mathscr{T}_i$, for $1 \leq i \leq 5$.



(a)

$\mathscr{T}_1$: $p^2q > pq > q > p^2 > p > 1$
+, +, +, −, −, −
$\mathscr{T}_2$: $p^2q > pq > p^2 > p > q > 1$
+, +, −, −, +, −
$\mathscr{T}_3$: $p^2q > p^2 > pq > q > p > 1$
+, −, +, +, −, −
$\mathscr{T}_4$: $p^2q > pq > p^2 > q > p > 1$
+, +, −, +, −, −
$\mathscr{T}_5$: $p^2q > p^2 > pq > p > q > 1$
+, −, +, −, +, −

(b)

Figure 7.22

Now look at Fig. 7.22 again. This time focus on the three plus signs and three minus signs in part (a) of the figure and in the list below each total order in part (b). When we apply the topological sorting algorithm to the given partial order $\mathscr{R}$, step (2) of the algorithm implies that the first divisor selected is always p^2q. This accounts for the first plus sign in each $\mathscr{T}_i$, $1 \leq i \leq 5$. Continuing to apply the algorithm we get two more plus signs and the three minus signs.

Could there ever be more minus signs than plus signs in our corresponding list, as a total order is developed? For example, could we start with +, −, −,? If so, we have failed to correctly apply step (2) of the topological sorting algorithm — we should have recognized pq as the unique candidate to select after p^2q and p^2. In fact, for $0 \leq k \leq 2$, p^kq must be selected before p^k can be. Consequently, for each list of three plus signs and three minus signs, there is always at least as many plus signs as minus signs, as the list is read from left to right. Comparing now with the result in part (a) of Example 1.43, we see that the number of total orders for the given partial order is $5 = \frac{1}{3+1}\binom{2\cdot 3}{3}$. Further, for $n \geq 1$, the topological sorting algorithm can be applied to the partial order of all positive divisors of $p^{n-1}q$ to yield $\frac{1}{n+1}\binom{2n}{n}$ total orders, another instance where the Catalan numbers arise.

In the topological sorting algorithm, we saw how the Hasse diagram was used in determining a total order containing a given poset $(A, \mathscr{R})$. This algorithm now prompts us to examine further properties of a partial order. At the start, particular emphasis will be given

† This example refers back to the optional material on Catalan numbers in Section 1.5. It may be skipped with no loss of continuity.

to a vertex like the vertex v_k in step (2) of the algorithm. The special property exhibited by such a vertex is now considered in the following.

Definition 7.17 If $(A, \mathcal{R})$ is a poset, then an element $x \in A$ is called a *maximal* element of A if for all $a \in A$, $a \neq x \Rightarrow x \not\mathcal{R}\, a$. An element $y \in A$ is called a *minimal* element of A if whenever $b \in A$ and $b \neq y$, then $b \not\mathcal{R}\, y$.

If we use the contrapositive of the first statement in Definition 7.17, then we can state that $x (\in A)$ is a maximal element if for each $a \in A$, $x \mathcal{R} a \Rightarrow x = a$. In a similar manner, $y \in A$ is a minimal element if for each $b \in A$, $b \mathcal{R} y \Rightarrow b = y$.

EXAMPLE 7.42

Let $\mathcal{U} = \{1, 2, 3\}$ and $A = \mathcal{P}(\mathcal{U})$.

a) Let $\mathcal{R}$ be the subset relation on A. Then $\mathcal{U}$ is maximal and $\emptyset$ is minimal for the poset $(A, \subseteq)$.

b) For B, the collection of proper subsets of $\{1, 2, 3\}$, let $\mathcal{R}$ be the subset relation on B. In the poset $(B, \subseteq)$, the sets $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$ are all maximal elements; $\emptyset$ is still the only minimal element.

EXAMPLE 7.43

With $\mathcal{R}$ the "less than or equal to" relation on the set $\mathbf{Z}$, we find that $(\mathbf{Z}, \leq)$ is a poset with neither a maximal nor a minimal element. The poset $(\mathbf{N}, \leq)$, however, has minimal element 0 but no maximal element.

EXAMPLE 7.44

When we look back at the partial orders in parts (b), (c), and (d) of Example 7.38, the following observations come to light.

1) The partial order in part (b) has the unique maximal element 8 and the unique minimal element 1.
2) Each of the four elements — 2, 3, 5, and 7 — is both a maximal element and a minimal element for the poset in part (c) of Example 7.38.
3) In part (d) the elements 12 and 385 are both maximal. Each of the elements 2, 3, 5, 7, and 11 is a minimal element for this partial order.

Are there any conditions indicating when a poset must have a maximal or minimal element?

THEOREM 7.3 If $(A, \mathcal{R})$ is a poset and A is finite, then A has both a maximal and a minimal element.

Proof: Let $a_1 \in A$. If there is no element $a \in A$ where $a \neq a_1$ and $a_1 \mathcal{R} a$, then a_1 is maximal. Otherwise there is an element $a_2 \in A$ with $a_2 \neq a_1$ and $a_1 \mathcal{R} a_2$. If no element $a \in A$, $a \neq a_2$, satisfies $a_2 \mathcal{R} a$, then a_2 is maximal. Otherwise we can find $a_3 \in A$ so that $a_3 \neq a_2$, $a_3 \neq a_1$ (Why?) while $a_1 \mathcal{R} a_2$ and $a_2 \mathcal{R} a_3$. Continuing in this manner, since A is finite, we get to an element $a_n \in A$ with $a_n \not\mathcal{R}\, a$ for all $a \in A$ where $a \neq a_n$, so a_n is maximal.

The proof for a minimal element follows in a similar way.

Returning now to the topological sorting algorithm, we see that in each iteration of step (2) of the algorithm, we are selecting a maximal element from the original poset $(A, \mathcal{R})$, or a poset of the form $(B, \mathcal{R}')$ where $\emptyset \neq B \subset A$ and $\mathcal{R}' = (B \times B) \cap \mathcal{R}$. At least one such element exists (in each iteration) by virtue of Theorem 7.3. Then in the second part of step (3), if x is the maximal element selected [in step (2)], we remove from the present poset all elements of the form (a, x). This results in a smaller poset.

We turn now to the study of some additional concepts involving posets.

Definition 7.18 If $(A, \mathcal{R})$ is a poset, then an element $x \in A$ is called a *least* element if $x \mathcal{R} a$ for all $a \in A$. Element $y \in A$ is called a *greatest* element if $a \mathcal{R} y$ for all $a \in A$.

EXAMPLE 7.45 Let $\mathcal{U} = \{1, 2, 3\}$, and let $\mathcal{R}$ be the subset relation.

a) With $A = \mathcal{P}(\mathcal{U})$, the poset $(A, \subseteq)$ has $\emptyset$ as a least element and $\mathcal{U}$ as a greatest element.

b) For B = the collection of nonempty subsets of $\mathcal{U}$, the poset $(B, \subseteq)$ has $\mathcal{U}$ as a greatest element. There is no least element here, but there are three minimal elements.

EXAMPLE 7.46 For the partial orders in Example 7.38, we find that

1) The partial order in part (b) has a greatest element 8 and a least element 1.

2) There is no greatest element or least element for the poset in part (c).

3) No greatest element or least element exists for the partial order in part (d).

We have seen that it is possible for a poset to have several maximal and minimal elements. What about least and greatest elements?

THEOREM 7.4 If the poset $(A, \mathcal{R})$ has a greatest (least) element, then that element is unique.

Proof: Suppose that $x, y \in A$ and that both are greatest elements. Since x is a greatest element, $y \mathcal{R} x$. Likewise, $x \mathcal{R} y$ because y is a greatest element. As $\mathcal{R}$ is antisymmetric, it follows that $x = y$.

The proof for the least element is similar.

Definition 7.19 Let $(A, \mathcal{R})$ be a poset with $B \subseteq A$. An element $x \in A$ is called a *lower bound* of B if $x \mathcal{R} b$ for all $b \in B$. Likewise, an element $y \in A$ is called an *upper bound* of B if $b \mathcal{R} y$ for all $b \in B$.

An element $x' \in A$ is called a *greatest lower bound* (glb) of B if it is a lower bound of B and if for all other lower bounds x'' of B we have $x'' \mathcal{R} x'$. Similarly $y' \in A$ is a *least upper bound* (lub) of B if it is an upper bound of B and if $y' \mathcal{R} y''$ for all other upper bounds y'' of B.

EXAMPLE 7.47 Let $\mathcal{U} = \{1, 2, 3, 4\}$, with $A = \mathcal{P}(\mathcal{U})$, and let $\mathcal{R}$ be the subset relation on A. If $B = \{\{1\}, \{2\}, \{1, 2\}\}$, then $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 2, 3, 4\}$ are all upper bounds for

B (in $(A, \mathcal{R})$), whereas $\{1, 2\}$ is a least upper bound (and is in B). Meanwhile, a greatest lower bound for B is $\emptyset$, which is not in B.

EXAMPLE 7.48

Let $\mathcal{R}$ be the "less than or equal to" relation for the poset $(A, \mathcal{R})$.

a) If $A = \mathbf{R}$ and $B = [0, 1]$, then B has glb 0 and lub 1. Note that $0, 1 \in B$. For $C = (0, 1]$, C has glb 0 and lub 1, and $1 \in C$ but $0 \notin C$.

b) Keeping $A = \mathbf{R}$, let $B = \{q \in \mathbf{Q} | q^2 < 2\}$. Then B has $\sqrt{2}$ as a lub and $-\sqrt{2}$ as a glb, and neither of these real numbers is in B.

c) Now let $A = \mathbf{Q}$, with B as in part (b). Here B has no lub or glb.

These examples lead us to the following result.

THEOREM 7.5

If $(A, \mathcal{R})$ is a poset and $B \subseteq A$, then B has at most one lub (glb).

Proof: We leave the proof to the reader.

We close this section with one last ordered structure.

Definition 7.20

The poset $(A, \mathcal{R})$ is called a *lattice* if for all $x, y \in A$ the elements lub$\{x, y\}$ and glb$\{x, y\}$ both exist in A.

EXAMPLE 7.49

For $A = \mathbf{N}$ and $x, y \in \mathbf{N}$, define $x \mathcal{R} y$ by $x \leq y$. Then lub$\{x, y\} = \max\{x, y\}$, glb$\{x, y\} = \min\{x, y\}$, and $(\mathbf{N}, \leq)$ is a lattice.

EXAMPLE 7.50

For the poset in Example 7.45(a), if $S, T \subseteq \mathcal{U}$, with lub$\{S, T\} = S \cup T$ and glb$\{S, T\} = S \cap T$, then $(\mathcal{P}(\mathcal{U}), \subseteq)$ is a lattice.

EXAMPLE 7.51

Consider the poset in Example 7.38(d). Here we find, for example, that

$$\text{lub}\{2, 3\} = 6,\ \ \text{lub}\{3, 6\} = 6,\ \ \text{lub}\{5, 7\} = 35,\ \ \text{lub}\{7, 11\} = 385,\ \ \text{lub}\{11, 35\} = 385,$$

and

$$\text{glb}\{3, 6\} = 3,\ \ \text{glb}\{2, 12\} = 2,\ \ \text{glb}\{35, 385\} = 35.$$

However, even though lub$\{2, 3\}$ exists, there is no glb for the elements 2 and 3. In addition, we are also lacking (among other considerations) glb$\{5, 7\}$, glb$\{11, 35\}$, glb$\{3, 35\}$, and lub$\{3, 35\}$. Consequently, this partial order is not a lattice.

EXERCISES 7.3

1. Draw the Hasse diagram for the poset $(\mathcal{P}(\mathcal{U}), \subseteq)$, where $\mathcal{U} = \{1, 2, 3, 4\}$.

2. Let $A = \{1, 2, 3, 6, 9, 18\}$, and define $\mathcal{R}$ on A by $x \mathcal{R} y$ if $x | y$. Draw the Hasse diagram for the poset $(A, \mathcal{R})$.

3. Let $(A, \mathcal{R}_1)$, $(B, \mathcal{R}_2)$ be two posets. On $A \times B$, define relation $\mathcal{R}$ by $(a, b) \mathcal{R} (x, y)$ if $a \mathcal{R}_1 x$ and $b \mathcal{R}_2 y$. Prove that $\mathcal{R}$ is a partial order.

4. If $\mathcal{R}_1$, $\mathcal{R}_2$ in Exercise 3 are total orders, is $\mathcal{R}$ a total order?

5. Topologically sort the Hasse diagram in part (a) of Example 7.38.

6. For $A = \{a, b, c, d, e\}$, the Hasse diagram for the poset $(A, \mathcal{R})$ is shown in Fig. 7.23. (a) Determine the relation matrix for $\mathcal{R}$. (b) Construct the directed graph G (on A) that is associated with $\mathcal{R}$. (c) Topologically sort the poset $(A, \mathcal{R})$.

7. The directed graph G for a relation $\mathcal{R}$ on set $A = \{1, 2, 3, 4\}$ is shown in Fig. 7.24. (a) Verify that $(A, \mathcal{R})$ is a poset and find its Hasse diagram. (b) Topologically sort $(A, \mathcal{R})$. (c) How many more directed edges are needed in Fig. 7.24 to extend $(A, \mathcal{R})$ to a total order?

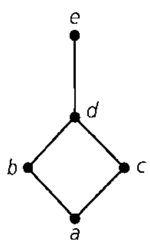


Figure 7.23

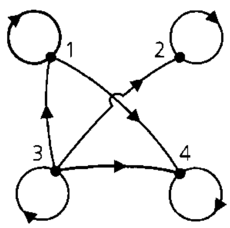


Figure 7.24

8. Prove that if a poset $(A, \mathcal{R})$ has a least element, it is unique.

9. Prove Theorem 7.5.

10. Give an example of a poset with four maximal elements but no greatest element.

11. If $(A, \mathcal{R})$ is a poset but not a total order, and $\emptyset \neq B \subset A$, does it follow that $(B \times B) \cap \mathcal{R}$ makes B into a poset but not a total order?

12. If $\mathcal{R}$ is a relation on A, and G is the associated directed graph, how can one recognize from G that $(A, \mathcal{R})$ is a total order?

13. If G is the directed graph for a relation $\mathcal{R}$ on A, with $|A| = n$, and $(A, \mathcal{R})$ is a total order, how many edges (including loops) are there in G?

14. Let $M(\mathcal{R})$ be the relation matrix for relation $\mathcal{R}$ on A, with $|A| = n$. If $(A, \mathcal{R})$ is a total order, how many 1's appear in $M(\mathcal{R})$?

15. **a)** Describe the structure of the Hasse diagram for a totally ordered poset $(A, \mathcal{R})$, where $|A| = n \geq 1$.

b) For a set A where $|A| = n \geq 1$, how many relations on A are total orders?

16. **a)** For $A = \{a_1, a_2, \ldots, a_n\}$, let $(A, \mathcal{R})$ be a poset. If $M(\mathcal{R})$ is the corresponding relation matrix, how can we recognize a maximal or minimal element of the poset from $M(\mathcal{R})$?

b) How can one recognize the existence of a greatest or least element in $(A, \mathcal{R})$ from the relation matrix $M(\mathcal{R})$?

17. Let $\mathcal{U} = \{1, 2, 3, 4\}$, with $A = \mathcal{P}(\mathcal{U})$, and let $\mathcal{R}$ be the subset relation on A. For each of the following subsets B (of A), determine the lub and glb of B.

a) $B = \{\{1\}, \{2\}\}$

b) $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$

c) $B = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

d) $B = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$

e) $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$

18. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7\}$, with $A = \mathcal{P}(\mathcal{U})$, and let $\mathcal{R}$ be the subset relation on A. For $B = \{\{1\}, \{2\}, \{2, 3\}\} \subseteq A$, determine each of the following.

a) The number of upper bounds of B that contain (i) three elements of $\mathcal{U}$; (ii) four elements of $\mathcal{U}$; (iii) five elements of $\mathcal{U}$

b) The number of upper bounds that exist for B

c) The lub for B

d) The number of lower bounds that exist for B

e) The glb for B

19. Define the relation $\mathcal{R}$ on the set $\mathbf{Z}$ by $a \mathcal{R} b$ if $a - b$ is a nonnegative even integer. Verify that $\mathcal{R}$ defines a partial order for $\mathbf{Z}$. Is this partial order a total order?

20. For $X = \{0, 1\}$, let $A = X \times X$. Define the relation $\mathcal{R}$ on A by $(a, b) \mathcal{R} (c, d)$ if (i) $a < c$; or (ii) $a = c$ and $b \leq d$. (a) Prove that $\mathcal{R}$ is a partial order for A. (b) Determine all minimal and maximal elements for this partial order. (c) Is there a least element? Is there a greatest element? (d) Is this partial order a total order?

21. Let $X = \{0, 1, 2\}$ and $A = X \times X$. Define the relation $\mathcal{R}$ on A as in Exercise 20. Answer the same questions posed in Exercise 20 for this relation $\mathcal{R}$ and set A.

22. For $n \in \mathbf{Z}^+$, let $X = \{0, 1, 2, \ldots, n-1, n\}$ and $A = X \times X$. Define the relation $\mathcal{R}$ on A as in Exercise 20. Remember that each element in this total order $\mathcal{R}$ is an ordered pair whose components are themselves ordered pairs. How many such elements are there in $\mathcal{R}$?

23. Let $(A, \mathcal{R})$ be a poset. Prove or disprove each of the following statements.

a) If $(A, \mathcal{R})$ is a lattice, then it is a total order.

b) If $(A, \mathcal{R})$ is a total order, then it is a lattice.

24. If $(A, \mathcal{R})$ is a lattice, with A finite, prove that $(A, \mathcal{R})$ has a greatest element and a least element.

25. For $A = \{a, b, c, d, e, v, w, x, y, z\}$, consider the poset $(A, \mathcal{R})$ whose Hasse diagram is shown in Fig. 7.25. Find

a) glb$\{b, c\}$

b) glb$\{b, w\}$

c) glb$\{e, x\}$

d) lub$\{c, b\}$

e) lub$\{d, x\}$

f) lub$\{c, e\}$

g) lub$\{a, v\}$

Is $(A, \mathcal{R})$ a lattice? Is there a maximal element? a minimal element? a greatest element? a least element?

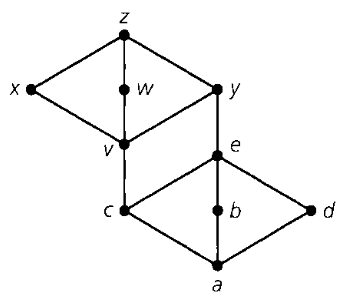


Figure 7.25

26. Given partial orders $(A, \mathcal{R})$ and $(B, \mathcal{S})$, a function $f: A \to B$ is called *order-preserving* if for all $x, y \in A$, $x \mathcal{R} y \Rightarrow f(x) \mathcal{S} f(y)$. How many such order-preserving functions are there for each of the following, where $\mathcal{R}$, $\mathcal{S}$ both denote $\leq$ (the usual "less than or equal to" relation)?

a) $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$;

b) $A = \{1, \ldots, n\}$, $n \geq 1$, $B = \{1, 2\}$;

c) $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbf{Z}^+$, $n \geq 1$, $a_1 < a_2 < \cdots < a_n$, $B = \{1, 2\}$;

d) $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$;

e) $A = \{1, 2\}$, $B = \{1, \ldots, n\}$, $n \geq 1$; and

f) $A = \{1, 2\}$, $B = \{b_1, b_2, \ldots, b_n\} \subset \mathbf{Z}^+$, $n \geq 1$, $b_1 < b_2 < \cdots < b_n$.

27. Let p, q, r, s be four distinct primes and $m, n, k, \ell \in \mathbf{Z}^+$. How many edges are there in the Hasse diagram of all positive divisors of (a) p^3; (b) p^m; (c) p^3q^2; (d) p^mq^n; (e) $p^3q^2r^4$; (f) $p^mq^nr^k$; (g) $p^3q^2r^4s^7$; and (h) $p^mq^nr^ks^\ell$?

28. Find the number of ways to totally order the partial order of all positive-integer divisors of (a) 24; (b) 75; and (c) 1701.

29. Let p, q be distinct primes and $k \in \mathbf{Z}^+$. If there are 429 ways to totally order the partial order of positive-integer divisors of p^kq, how many positive-integer divisors are there for this partial order?

30. For $m, n \in \mathbf{Z}^+$, let A be the set of all $m \times n$ (0, 1)-matrices. Prove that the "precedes" relation of Definition 7.11 makes A into a poset.

7.4 Equivalence Relations and Partitions

As we noted earlier in Definition 7.7, a relation $\mathcal{R}$ on a set A is an equivalence relation if it is reflexive, symmetric, and transitive. For any set $A \neq \emptyset$, the relation of equality is an equivalence relation on A, where two elements of A are related if they are identical; equality thus establishes the property of "sameness" among the elements of A.

If we consider the relation $\mathcal{R}$ on $\mathbf{Z}$ defined by $x \mathcal{R} y$ if $x - y$ is a multiple of 2, then $\mathcal{R}$ is an equivalence relation on $\mathbf{Z}$ where all even integers are related, as are all odd integers. Here, for example, we do not have $4 = 8$, but we do have $4 \mathcal{R} 8$, for we no longer care about the size of a number but are concerned with only two properties: "evenness" and "oddness." This relation splits $\mathbf{Z}$ into two subsets consisting of the odd and even integers: $\mathbf{Z} = \{\ldots, -3, -1, 1, 3, \ldots\} \cup \{\ldots, -4, -2, 0, 2, 4, \ldots\}$. This splitting up of $\mathbf{Z}$ is an example of a partition, a concept closely related to the equivalence relation. In this section we investigate this relationship and see how it helps us count the number of equivalence relations on a finite set.

Definition 7.21 Given a set A and index set I, let $\emptyset \neq A_i \subseteq A$ for each $i \in I$. Then $\{A_i\}_{i \in I}$ is a *partition* of A if

a) $A = \bigcup_{i \in I} A_i$ and **b)** $A_i \cap A_j = \emptyset$, for all $i, j \in I$ where $i \neq j$.

Each subset A_i is called a *cell* or *block* of the partition.

EXAMPLE 7.52 If $A = \{1, 2, 3, \ldots, 10\}$, then each of the following determines a partition of A:

a) $A_1 = \{1, 2, 3, 4, 5\}$, $A_2 = \{6, 7, 8, 9, 10\}$

b) $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 6, 7, 9\}$, $A_3 = \{5, 8, 10\}$

c) $A_i = \{i, i+5\}$, $1 \leq i \leq 5$

In these three examples we note how each element of A belongs to *exactly one* cell in each partition.

EXAMPLE 7.53

Let $A = \mathbf{R}$ and, for each $i \in \mathbf{Z}$, let $A_i = [i, i+1)$. Then $\{A_i\}_{i \in \mathbf{Z}}$ is a partition of $\mathbf{R}$.

Now just how do partitions come into play with equivalence relations?

Definition 7.22

Let $\mathcal{R}$ be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x, denoted $[x]$, is defined by $[x] = \{y \in A | y \, \mathcal{R} \, x\}$.

EXAMPLE 7.54

Define the relation $\mathcal{R}$ on $\mathbf{Z}$ by $x \, \mathcal{R} \, y$ if $4|(x-y)$. Since $\mathcal{R}$ is reflexive, symmetric, and transitive, it is an equivalence relation and we find that

$$[0] = \{\ldots, -8, -4, 0, 4, 8, 12, \ldots\} = \{4k | k \in \mathbf{Z}\}$$
$$[1] = \{\ldots, -7, -3, 1, 5, 9, 13, \ldots\} = \{4k+1 | k \in \mathbf{Z}\}$$
$$[2] = \{\ldots, -6, -2, 2, 6, 10, 14, \ldots\} = \{4k+2 | k \in \mathbf{Z}\}$$
$$[3] = \{\ldots, -5, -1, 3, 7, 11, 15, \ldots\} = \{4k+3 | k \in \mathbf{Z}\}.$$

But what about $[n]$, where n is an integer other than 0, 1, 2, or 3? For example, what is $[6]$? We claim that $[6] = [2]$ and to prove this we use Definition 3.2 (for the equality of sets) as follows. If $x \in [6]$, then from Definition 7.22 we know that $x \, \mathcal{R} \, 6$. Here this means that 4 divides $(x-6)$, so $x - 6 = 4k$ for some $k \in \mathbf{Z}$. But then $x - 6 = 4k \Rightarrow x - 2 = 4(k+1) \Rightarrow$ 4 divides $(x-2) \Rightarrow x \, \mathcal{R} \, 2 \Rightarrow x \in [2]$, so $[6] \subseteq [2]$. For the opposite inclusion start with an element y in $[2]$. Then $y \in [2] \Rightarrow y \, \mathcal{R} \, 2 \Rightarrow$ 4 divides $(y-2) \Rightarrow y - 2 = 4l$ for some $l \in \mathbf{Z} \Rightarrow y - 6 = 4(l-1)$, where $l - 1 \in \mathbf{Z} \Rightarrow$ 4 divides $y - 6 \Rightarrow y \, \mathcal{R} \, 6 \Rightarrow y \in [6]$, so $[2] \subseteq [6]$. From the two inclusions it now follows that $[6] = [2]$, as claimed.

Further, we also find, for example, that $[2] = [-2] = [-6]$, $[51] = [3]$, and $[17] = [1]$. Most important, $\{[0], [1], [2], [3]\}$ provides a partition of $\mathbf{Z}$.

[*Note*: Here the index set for the partition is implicit. If, for instance, we let $A_0 = [0]$, $A_1 = [1]$, $A_2 = [2]$, and $A_3 = [3]$, then one possible index set I (as in Definition 7.21) is $\{0, 1, 2, 3\}$. When a collection of sets is called a partition (of a given set) but no index set is specified, the reader should realize that the situation is like the one given here — where the index set is implicit.]

EXAMPLE 7.55

Define the relation $\mathcal{R}$ on the set $\mathbf{Z}$ by $a \, \mathcal{R} \, b$ if $a^2 = b^2$ (or, $a = \pm b$). For all $a \in \mathbf{Z}$, we have $a^2 = a^2$ — so $a \, \mathcal{R} \, a$ and $\mathcal{R}$ is reflexive. Should $a, b \in \mathbf{Z}$ with $a \, \mathcal{R} \, b$, then $a^2 = b^2$ and it follows that $b^2 = a^2$, or $b \, \mathcal{R} \, a$. Consequently, relation $\mathcal{R}$ is symmetric. Finally, suppose that $a, b, c \in \mathbf{Z}$ with $a \, \mathcal{R} \, b$ and $b \, \mathcal{R} \, c$. Then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$ and $a \, \mathcal{R} \, c$. This makes the given relation transitive. Having established the three needed properties, we now know that $\mathcal{R}$ is an equivalence relation.

What can we say about the corresponding partition of $\mathbf{Z}$?

Here one finds that $[0] = \{0\}$, $[1] = [-1] = \{-1, 1\}$, $[2] = [-2] = \{-2, 2\}$, and, in general, for each $n \in \mathbf{Z}^+$, $[n] = [-n] = \{-n, n\}$. Furthermore, we have the *partition*

$$\mathbf{Z} = \bigcup_{n=0}^{\infty}[n] = \bigcup_{n\in\mathbf{N}}[n] = \{0\} \cup \left(\bigcup_{n=1}^{\infty}\{-n, n\}\right) = \{0\} \cup \left(\bigcup_{n\in\mathbf{Z}^+}\{-n, n\}\right).$$

These examples lead us to the following general situation.

THEOREM 7.6

If $\mathcal{R}$ is an equivalence relation on a set A, and $x, y \in A$, then (a) $x \in [x]$; (b) $x \,\mathcal{R}\, y$ if and only if $[x] = [y]$; and (c) $[x] = [y]$ or $[x] \cap [y] = \emptyset$.

Proof:

a) This result follows from the reflexive property of $\mathcal{R}$.

b) The proof here is somewhat reminiscent of what was done in Example 7.54.

If $x \,\mathcal{R}\, y$, let $w \in [x]$. Then $w \,\mathcal{R}\, x$ and because $\mathcal{R}$ is transitive, $w \,\mathcal{R}\, y$. Hence $w \in [y]$ and $[x] \subseteq [y]$. With $\mathcal{R}$ symmetric, $x \,\mathcal{R}\, y \Rightarrow y \,\mathcal{R}\, x$. So if $t \in [y]$, then $t \,\mathcal{R}\, y$ and by the transitive property, $t \,\mathcal{R}\, x$. Hence $t \in [x]$ and $[y] \subseteq [x]$. Consequently, $[x] = [y]$.

Conversely, let $[x] = [y]$. Since $x \in [x]$ by part (a), then $x \in [y]$ or $x \,\mathcal{R}\, y$.

c) This property tells us that two equivalence classes can be related in only one of two possible ways. Either they are identical or they are disjoint.

We assume that $[x] \neq [y]$ and show how it then follows that $[x] \cap [y] = \emptyset$. If $[x] \cap [y] \neq \emptyset$, then let $v \in A$ with $v \in [x]$ and $v \in [y]$. Then $v \,\mathcal{R}\, x$, $v \,\mathcal{R}\, y$, and, since $\mathcal{R}$ is symmetric, $x \,\mathcal{R}\, v$. Now $(x \,\mathcal{R}\, v$ and $v \,\mathcal{R}\, y) \Rightarrow x \,\mathcal{R}\, y$, by the transitive property. Also $x \,\mathcal{R}\, y \Rightarrow [x] = [y]$ by part (b). This contradicts the assumption that $[x] \neq [y]$, so we reject the supposition that $[x] \cap [y] \neq \emptyset$, and the result follows.

Note that if $\mathcal{R}$ is an equivalence relation on A, then by parts (a) and (c) of Theorem 7.6 the distinct equivalence classes determined by $\mathcal{R}$ provide us with a partition of A.

EXAMPLE 7.56

a) If $A = \{1, 2, 3, 4, 5\}$ and $\mathcal{R} = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$, then $\mathcal{R}$ is an equivalence relation on A. Here $[1] = \{1\}$, $[2] = \{2, 3\} = [3]$, $[4] = \{4, 5\} = [5]$, and $A = [1] \cup [2] \cup [4]$ with $[1] \cap [2] = \emptyset$, $[1] \cap [4] = \emptyset$, and $[2] \cap [4] = \emptyset$. So $\{[1], [2], [4]\}$ determines a partition of A.

b) Consider part (d) of Example 7.16 once again. We have $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{x, y, z\}$, and $f\colon A \to B$ is the onto function

$$f = \{(1, x), (2, z), (3, x), (4, y), (5, z), (6, y), (7, x)\}.$$

The relation $\mathcal{R}$ defined on A by $a \,\mathcal{R}\, b$ if $f(a) = f(b)$ was shown to be an equivalence relation. Here

$$f^{-1}(x) = \{1, 3, 7\} = [1]\ (= [3] = [7]),$$
$$f^{-1}(y) = \{4, 6\} = [4]\ (= [6]), \quad \text{and}$$
$$f^{-1}(z) = \{2, 5\} = [2]\ (= [5]).$$

With $A = [1] \cup [4] \cup [2] = f^{-1}(x) \cup f^{-1}(y) \cup f^{-1}(z)$, we see that $\{f^{-1}(x), f^{-1}(y), f^{-1}(z)\}$ determines a partition of A.

In fact, for any nonempty sets A, B, if $f\colon A \to B$ is an onto function, then $A = \bigcup_{b\in B} f^{-1}(b)$ and $\{f^{-1}(b) | b \in B\}$ provides us with a partition of A.

EXAMPLE 7.57

In the programming language C++ a nonexecutable specification statement called the *union construct* allows two or more variables in a given program to refer to the same memory location.

For example, within a program the statements

```
union
{
 int a;
 int c;
 int p;
};
union
{
 int up;
 int down;
};
```

inform the C++ compiler that the integer variables a, c, and p will share one memory location while the integer variables *up* and *down* will share another. Here the set of all program variables is partitioned by the equivalence relation $\mathcal{R}$, where $v_1 \mathcal{R} v_2$ if v_1 and v_2 are program variables that share the same memory location.

EXAMPLE 7.58

Having seen examples of how an equivalence relation induces a partition of a set, we now go backward. If an equivalence relation $\mathcal{R}$ on $A = \{1, 2, 3, 4, 5, 6, 7\}$ induces the partition $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$, what is $\mathcal{R}$?

Consider the cell $\{1, 2\}$ of the partition. This subset implies that $[1] = \{1, 2\} = [2]$, and so $(1, 1), (2, 2), (1, 2), (2, 1) \in \mathcal{R}$. (The first two ordered pairs are necessary for the reflexive property of $\mathcal{R}$; the others preserve symmetry.)

In like manner, the cell $\{4, 5, 7\}$ implies that under $\mathcal{R}$, $[4] = [5] = [7] = \{4, 5, 7\}$ and that, as an equivalence relation, $\mathcal{R}$ must contain $\{4, 5, 7\} \times \{4, 5, 7\}$. In fact,

$$\mathcal{R} = (\{1, 2\} \times \{1, 2\}) \cup (\{3\} \times \{3\}) \cup (\{4, 5, 7\} \times \{4, 5, 7\}) \cup (\{6\} \times \{6\}),$$

and

$$|\mathcal{R}| = 2^2 + 1^2 + 3^2 + 1^2 = 15.$$

The results in Examples 7.54, 7.55, 7.56, and 7.58 lead us to the following.

THEOREM 7.7

If A is a set, then

a) any equivalence relation $\mathcal{R}$ on A induces a partition of A, and

b) any partition of A gives rise to an equivalence relation $\mathcal{R}$ on A.

Proof: Part (a) follows from parts (a) and (c) of Theorem 7.6. For part (b), given a partition $\{A_i\}_{i \in I}$ of A, define relation $\mathcal{R}$ on A by $x \mathcal{R} y$, if x and y are in the same cell of the partition. We leave to the reader the details of verifying that $\mathcal{R}$ is an equivalence relation.

On the basis of this theorem and the examples we have examined, we state the next result. A proof for it is outlined in Exercise 16 at the end of the section.

THEOREM 7.8 For any set A, there is a one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.

We are primarily concerned with using this result for finite sets.

EXAMPLE 7.59

a) If $A = \{1, 2, 3, 4, 5, 6\}$, how many relations on A are equivalence relations?

We solve this problem by counting the partitions of A, realizing that a partition of A is a distribution of the (distinct) elements of A into identical containers, with no container left empty. From Section 5.3 we know, for example, that there are $S(6, 2)$ partitions of A into two identical nonempty containers. Using the Stirling numbers of the second kind, as the number of containers varies from 1 to 6, we have $\sum_{i=1}^{6} S(6, i) = 203$ different partitions of A. Consequently, there are 203 equivalence relations on A.

b) How many of the equivalence relations in part (a) satisfy $1, 2 \in [4]$?

Identifying 1, 2, and 4 as the "same" element under these equivalence relations, we count as in part (a) for the set $B = \{1, 3, 5, 6\}$ and find that there are $\sum_{i=1}^{4} S(4, i) = 15$ equivalence relations on A for which $[1] = [2] = [4]$.

We close by noting that if A is a finite set with $|A| = n$, then for all $n \leq r \leq n^2$, there is an equivalence relation $\mathcal{R}$ on A with $|\mathcal{R}| = r$ if and only if there exist $n_1, n_2, \ldots, n_k \in \mathbf{Z}^+$ with $\sum_{i=1}^{k} n_i = n$ and $\sum_{i=1}^{k} n_i^2 = r$.

EXERCISES 7.4

1. Determine whether each of the following collections of sets is a partition for the given set A. If the collection is not a partition, explain why it fails to be.

a) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $A_1 = \{4, 5, 6\}$, $A_2 = \{1, 8\}$, $A_3 = \{2, 3, 7\}$.

b) $A = \{a, b, c, d, e, f, g, h\}$; $A_1 = \{d, e\}$, $A_2 = \{a, c, d\}$, $A_3 = \{f, h\}$, $A_4 = \{b, g\}$.

2. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. In how many ways can we partition A as $A_1 \cup A_2 \cup A_3$ with

a) $1, 2 \in A_1$, $3, 4 \in A_2$, and $5, 6, 7 \in A_3$?

b) $1, 2 \in A_1$, $3, 4 \in A_2$, $5, 6 \in A_3$, and $|A_1| = 3$?

c) $1, 2 \in A_1$, $3, 4 \in A_2$, and $5, 6 \in A_3$?

3. If $A = \{1, 2, 3, 4, 5\}$ and $\mathcal{R}$ is the equivalence relation on A that induces the partition $A = \{1, 2\} \cup \{3, 4\} \cup \{5\}$, what is $\mathcal{R}$?

4. For $A = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5), (6, 6)\}$ is an equivalence relation on A. (a) What are [1], [2], and [3] under this equivalence relation? (b) What partition of A does $\mathcal{R}$ induce?

5. If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1, 2\}$, $A_2 = \{2, 3, 4\}$, and $A_3 = \{5\}$, define relation $\mathcal{R}$ on A by $x \mathcal{R} y$ if x and y are in the same subset A_i, for $1 \leq i \leq 3$. Is $\mathcal{R}$ an equivalence relation?

6. For $A = \mathbf{R}^2$, define $\mathcal{R}$ on A by $(x_1, y_1) \mathcal{R} (x_2, y_2)$ if $x_1 = x_2$.

a) Verify that $\mathcal{R}$ is an equivalence relation on A.

b) Describe geometrically the equivalence classes and partition of A induced by $\mathcal{R}$.

7. Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$, and define $\mathcal{R}$ on A by $(x_1, y_1) \mathcal{R} (x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$.

a) Verify that $\mathcal{R}$ is an equivalence relation on A.

b) Determine the equivalence classes $[(1, 3)]$, $[(2, 4)]$, and $[(1, 1)]$.

c) Determine the partition of A induced by $\mathcal{R}$.

8. If $A = \{1, 2, 3, 4, 5, 6, 7\}$, define $\mathcal{R}$ on A by $(x, y) \in \mathcal{R}$ if $x - y$ is a multiple of 3.

a) Show that $\mathcal{R}$ is an equivalence relation on A.

b) Determine the equivalence classes and partition of A induced by $\mathcal{R}$.

9. For $A = \{(-4, -20), (-3, -9), (-2, -4), (-1, -11), (-1, -3), (1, 2), (1, 5), (2, 10), (2, 14), (3, 6), (4, 8), (4, 12)$ define the relation $\mathcal{R}$ on A by $(a, b) \mathcal{R} (c, d)$ if $ad = bc$.

a) Verify that $\mathcal{R}$ is an equivalence relation on A.

b) Find the equivalence classes $[(2, 14)]$, $[(-3, -9)]$, and $[(4, 8)]$.

c) How many cells are there in the partition of A induced by $\mathcal{R}$?

10. Let A be a nonempty set and fix the set B, where $B \subseteq A$. Define the relation $\mathcal{R}$ on $\mathcal{P}(A)$ by $X \mathcal{R} Y$, for $X, Y \subseteq A$, if $B \cap X = B \cap Y$.

a) Verify that $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(A)$.

b) If $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, find the partition of $\mathcal{P}(A)$ induced by $\mathcal{R}$.

c) If $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$, find $[X]$ if $X = \{1, 3, 5\}$.

d) For $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$, how many equivalence classes are in the partition induced by $\mathcal{R}$?

11. How many of the equivalence relations on $A = \{a, b, c, d, e, f\}$ have (a) exactly two equivalence classes of size 3? (b) exactly one equivalence class of size 3? (c) one equivalence class of size 4? (d) at least one equivalence class with three or more elements?

12. Let $A = \{v, w, x, y, z\}$. Determine the number of relations on A that are (a) reflexive and symmetric; (b) equivalence relations; (c) reflexive and symmetric but not transitive; (d) equivalence relations that determine exactly two equivalence classes; (e) equivalence relations where $w \in [x]$; (f) equivalence relations where $v, w \in [x]$; (g) equivalence relations where $w \in [x]$ and $y \in [z]$; and (h) equivalence relations where $w \in [x]$, $y \in [z]$, and $[x] \neq [z]$.

13. If $|A| = 30$ and the equivalence relation $\mathcal{R}$ on A partitions A into (disjoint) equivalence classes A_1, A_2, and A_3, where $|A_1| = |A_2| = |A_3|$, what is $|\mathcal{R}|$?

14. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. For each of the following values of r, determine an equivalence relation $\mathcal{R}$ on A with $|\mathcal{R}| = r$, or explain why no such relation exists. (a) $r = 6$; (b) $r = 7$; (c) $r = 8$; (d) $r = 9$; (e) $r = 11$; (f) $r = 22$; (g) $r = 23$; (h) $r = 30$; (i) $r = 31$.

15. Provide the details for the proof of part (b) of Theorem 7.7.

16. For any set $A \neq \emptyset$, let $P(A)$ denote the set of all partitions of A, and let $E(A)$ denote the set of all equivalence relations on A. Define the function $f\colon E(A) \to P(A)$ as follows: If $\mathcal{R}$ is an equivalence relation on A, then $f(\mathcal{R})$ is the partition of A induced by $\mathcal{R}$. Prove that f is one-to-one and onto, thus establishing Theorem 7.8.

17. Let $f\colon A \to B$. If $\{B_1, B_2, B_3, \ldots, B_n\}$ is a partition of B, prove that $\{f^{-1}(B_i) | 1 \leq i \leq n,\ f^{-1}(B_i) \neq \emptyset\}$ is a partition of A.

7.5 Finite State Machines: The Minimization Process

In Section 6.3 we encountered two finite state machines that performed the same task but had different numbers of internal states. (See Figs. 6.9 and 6.10.) The machine with the larger number of internal states contains *redundant* states — states that can be eliminated because other states will perform their functions. Since minimization of the number of states in a machine reduces its complexity and cost, we seek a process for transforming a given machine into one that has no redundant internal states. This process is known as the *minimization process*, and its development relies on the concepts of equivalence relation and partition.

Starting with a given finite state machine $M = (S, \mathcal{I}, \mathcal{O}, \nu, \omega)$, we define the relation E_1 on S by $s_1 \mathrel{E_1} s_2$ if $\omega(s_1, x) = \omega(s_2, x)$, for all $x \in \mathcal{I}$. This relation E_1 is an equivalence relation on S, and it partitions S into subsets such that two states are in the same subset if they produce the same output for each $x \in \mathcal{I}$. Here the states s_1, s_2 are called *1-equivalent*.

For each $k \in \mathbf{Z}^+$, we say that the states s_1, s_2 are *k-equivalent* if $\omega(s_1, x) = \omega(s_2, x)$ for all $x \in \mathcal{I}^k$. Here ω is the extension of the given output function to $S \times \mathcal{I}^*$. The relation of k-equivalence is also an equivalence relation on S; it partitions S into subsets of k-equivalent states. We write $s_1 \mathrel{E_k} s_2$ to denote that s_1 and s_2 are k-equivalent.

Finally, if $s_1, s_2 \in S$ and s_1, s_2 are k-equivalent for all $k \geq 1$, then we call s_1 and s_2 *equivalent* and write $s_1 \mathrel{E} s_2$. When this happens, we find that if we keep s_1 in our machine, then s_2 will be redundant and can be removed. Hence our objective is to determine the partition of S induced by E and to select one state for each equivalence class. Then we shall have a minimal realization of the given machine.

To accomplish this, let us start with the following observations.

a) If two states in a machine are not 2-equivalent, could they possibly be 3-equivalent? (or k-equivalent, for $k \geq 4$?)

The answer is no. If $s_1, s_2 \in S$ and $s_1 \not\mathrm{E}_2\, s_2$ (that is, s_1 and s_2 are not 2-equivalent), then there is at least one string $xy \in \mathscr{I}^2$ such that $\omega(s_1, xy) = v_1v_2 \neq w_1w_2 = \omega(s_2, xy)$, where $v_1, v_2, w_1, w_2 \in \mathscr{O}$. So with regard to E_3, we find that $s_1 \not\mathrm{E}_3\, s_2$ because for any $z \in \mathscr{I}$, $\omega(s_1, xyz) = v_1v_2v_3 \neq w_1w_2w_3 = \omega(s_2, xyz)$.

In general, to find states that are $(k+1)$-equivalent, we look at states that are k-equivalent.

b) Now suppose that $s_1, s_2 \in S$ and $s_1 \mathrm{E}_2\, s_2$. We wish to determine whether $s_1 \mathrm{E}_3\, s_2$. That is, does $\omega(s_1, x_1x_2x_3) = \omega(s_2, x_1x_2x_3)$ for all strings $x_1x_2x_3 \in \mathscr{I}^3$? Consider what happens. First we get $\omega(s_1, x_1) = \omega(s_2, x_1)$, because $s_1 \mathrm{E}_2\, s_2 \Rightarrow s_1 \mathrm{E}_1\, s_2$. Then there is a transition to the states $\nu(s_1, x_1)$ and $\nu(s_2, x_1)$. Consequently, $\omega(s_1, x_1x_2x_3) = \omega(s_2, x_1x_2x_3)$ if $\omega(\nu(s_1, x_1), x_2x_3) = \omega(\nu(s_2, x_1), x_2x_3)$ [that is, if $\nu(s_1, x_1) \mathrm{E}_2\, \nu(s_2, x_1)$].

In general, for $s_1, s_2, \in S$, where $s_1 \mathrm{E}_k\, s_2$, we find that $s_1 \mathrm{E}_{k+1}\, s_2$ if (and only if) $\nu(s_1, x) \mathrm{E}_k\, \nu(s_2, x)$ for all $x \in \mathscr{I}$.

With these observations to guide us, we now present an algorithm for the minimization of a finite state machine M.

Step 1: Set $k = 1$. We determine the states that are 1-equivalent by examining the rows in the state table for M. For $s_1, s_2 \in S$ it follows that $s_1 \mathrm{E}_1\, s_2$ when s_1, s_2 have the same output rows.

Let P_1 be the partition of S induced by E_1.

Step 2: Having determined P_k, we obtain P_{k+1} by noting that if $s_1 \mathrm{E}_k\, s_2$, then $s_1 \mathrm{E}_{k+1}\, s_2$ when $\nu(s_1, x) \mathrm{E}_k\, \nu(s_2, x)$ for all $x \in \mathscr{I}$. We have $s_1 \mathrm{E}_k\, s_2$ if s_1, s_2 are in the same cell of the partition P_k. Likewise, $\nu(s_1, x) \mathrm{E}_k\, \nu(s_2, x)$ for each $x \in \mathscr{I}$, if $\nu(s_1, x)$ and $\nu(s_2, x)$ are in the same cell of the partition P_k. In this way P_{k+1} is obtained from P_k.

Step 3: If $P_{k+1} = P_k$, the process is complete. We select one state from each equivalence class and these states yield a minimal realization of M.

If $P_{k+1} \neq P_k$, we increase k by 1 and return to step (2).

We illustrate the algorithm in the following example.

EXAMPLE 7.60

With $\mathscr{I} = \mathscr{O} = \{0, 1\}$, let M be given by the state table shown in Table 7.1. Looking at the output rows, we see that s_3 and s_4 are 1-equivalent, as are s_2, s_5, and s_6. Here E_1 partitions S as follows:

$$P_1: \{s_1\}, \{s_2, s_5, s_6\}, \{s_3, s_4\}.$$

For each $s \in S$ and each $k \in \mathbf{Z}^+$, $s \mathrm{E}_k\, s$, so as we continue this process to determine P_2, we shall not concern ourselves with equivalence classes of only one state.

Since $s_3 \mathrm{E}_1\, s_4$, there is a chance that we could have $s_3 \mathrm{E}_2\, s_4$. Here $\nu(s_3, 0) = s_2$, $\nu(s_4, 0) = s_5$ with $s_2 \mathrm{E}_1\, s_5$, and $\nu(s_3, 1) = s_4$, $\nu(s_4, 1) = s_3$ with $s_4 \mathrm{E}_1\, s_3$. Hence $\nu(s_3, x) \mathrm{E}_1\, \nu(s_4, x)$, for all $x \in \mathscr{I}$, and $s_3 \mathrm{E}_2\, s_4$. Similarly, $\nu(s_2, 0) = s_5$, $\nu(s_5, 0) = s_2$ with $s_5 \mathrm{E}_1\, s_2$, and $\nu(s_2, 1) = s_2$, $\nu(s_5, 1) = s_5$ with $s_2 \mathrm{E}_1\, s_5$. Thus $s_2 \mathrm{E}_2\, s_5$. Finally, $\nu(s_5, 0) = s_2$ and

$\nu(s_6, 0) = s_1$, but $s_2 \not\!E_1\, s_1$, so $s_5 \not\!E_2\, s_6$. (Why don't we investigate the possibility of $s_2\, E_2\, s_6$?) Equivalence relation E_2 partitions S as follows:

$$P_2\colon \{s_1\}, \{s_2, s_5\}, \{s_3, s_4\}, \{s_6\}.$$

Since $P_2 \neq P_1$, we continue the process to get P_3. In determining whether $s_2\, E_3\, s_5$, we see that $\nu(s_2, 0) = s_5$, $\nu(s_5, 0) = s_2$, and $s_5\, E_2\, s_2$. Also, $\nu(s_2, 1) = s_2$, $\nu(s_5, 1) = s_5$, and $s_2\, E_2\, s_5$. With $\nu(s_2, x)\, E_2\, \nu(s_5, x)$ for all $x \in \mathcal{I}$, we have $s_2\, E_3\, s_5$. For s_3, s_4, $(\nu(s_3, 0) = s_2)$ E_2 $(s_5 = \nu(s_4, 0))$ and $(\nu(s_3, 1) = s_4)\, E_2\, (s_3 = \nu(s_4, 1))$, so $s_3\, E_3\, s_4$ and E_3 induces the partition P_3: $\{s_1\}, \{s_2, s_5\}, \{s_3, s_4\}, \{s_6\}$.

Table 7.1

	ν		ω	
	0	1	0	1
s_1	s_4	s_3	0	1
s_2	s_5	s_2	1	0
s_3	s_2	s_4	0	0
s_4	s_5	s_3	0	0
s_5	s_2	s_5	1	0
s_6	s_1	s_6	1	0

Table 7.2

	ν		ω	
	0	1	0	1
s_1	s_3	s_3	0	1
s_2	s_2	s_2	1	0
s_3	s_2	s_3	0	0
s_6	s_1	s_6	1	0

Now $P_3 = P_2$ so the process is completed, as indicated in step (3) of the algorithm. We find that s_5 and s_4 may be regarded as redundant states. Removing them from the table, and replacing all further occurrences of them by s_2 and s_3, respectively, we arrive at Table 7.2. This is a minimal machine that performs the same tasks as the machine given in Table 7.1.

If we do not want states that skip a subscript, we can always relabel the states in this minimal machine. Here we would have $s_1, s_2, s_3, s_4\ (= s_6)$, but this s_4 is not the same s_4 we started with in Table 7.1.

You may be wondering how we knew that we could stop the process when $P_3 = P_2$. For after all, couldn't it happen that perhaps $P_4 \neq P_3$, or that $P_4 = P_3$ but $P_5 \neq P_4$? To prove that this never occurs, we define the following idea.

Definition 7.23 If P_1, P_2 are partitions of a set A, then P_2 is called a *refinement* of P_1, and we write $P_2 \leq P_1$, if every cell of P_2 is contained in a cell of P_1. When $P_2 \leq P_1$ and $P_2 \neq P_1$ we write $P_2 < P_1$. This occurs when at least one cell in P_2 is properly contained in a cell in P_1.

In the minimization process of Example 7.60, we had $P_3 = P_2 < P_1$. Whenever we apply the algorithm, as we get P_{k+1} from P_k, we always find that $P_{k+1} \leq P_k$, because $(k+1)$-equivalence implies k-equivalence. So each successive partition refines the preceding partition.

THEOREM 7.9 In applying the minimization process, if $k \geq 1$ and P_k and P_{k+1} are partitions with $P_{k+1} = P_k$, then $P_{r+1} = P_r$ for all $r \geq k + 1$.

Proof: If not, let $r\ (\geq k+1)$ be the smallest subscript such that $P_{r+1} \neq P_r$. Then $P_{r+1} < P_r$, so there exist $s_1, s_2 \in S$ with $s_1\, E_r\, s_2$ but $s_1 \not\!E_{r+1}\, s_2$. But $s_1\, E_r\, s_2 \Rightarrow \nu(s_1, x)\, E_{r-1}\, \nu(s_2, x)$, for all $x \in \mathscr{I}$, and with $P_r = P_{r-1}$, we then find that $\nu(s_1, x)\ \mathrm{E}_r\ \nu(s_2, x)$, for all $x \in \mathscr{I}$, so $s_1\ \mathrm{E}_{r+1}\ s_2$. Consequently, $P_{r+1} = P_r$.

We close this section with the following related idea. Let M be a finite state machine with $s_1, s_2 \in S$, and s_1, s_2 not equivalent. If $s_1\ \not{\mathrm{E}}_1\ s_2$, then these states produce different output rows in the state table for M. In this case it is easy to find an $x \in \mathscr{I}$ such that $\omega(s_1, x) \neq \omega(s_2, x)$, and this distinguishes these nonequivalent states. Otherwise, s_1 and s_2 produce the same output rows in the table but there is a smallest integer $k \geq 1$ such that $s_1\ \mathrm{E}_k\ s_2$ but $s_1\ \not{\mathrm{E}}_{k+1}\ s_2$. Now if we are to distinguish these states, we need to find a string $x = x_1x_2 \cdots x_kx_{k+1} \in \mathscr{I}^{k+1}$ such that $\omega(s_1, x) \neq \omega(s_2, x)$, even though $\omega(s_1, x_1x_2 \cdots x_k) = \omega(s_2, x_1x_2 \cdots x_k)$. Such a string x is called a *distinguishing string* for the states s_1 and s_2. There may be more than one such string, but each has the same (minimal) length $k + 1$.

Before we try to find a distinguishing string for two nonequivalent states in a specific finite state machine, let us examine the major idea at play here. So suppose that $s_1, s_2 \in S$ and that for some (fixed) $k \in \mathbf{Z}^+$ we have $s_1\ \mathrm{E}_k\ s_2$ but $s_1\ \not{\mathrm{E}}_{k+1}\ s_2$. What can we conclude?

We find that

$$
\begin{aligned}
s_1\ \not{\mathrm{E}}_{k+1}\ s_2 &\Rightarrow \exists x_1 \in \mathscr{I}\ [\nu(s_1, x_1)\ \not{\mathrm{E}}_k\ \nu(s_2, x_1)] \\
&\Rightarrow \exists x_1 \in \mathscr{I}\ \exists x_2 \in \mathscr{I}\ [\nu(\nu(s_1, x_1), x_2)\ \not{\mathrm{E}}_{k-1}\ \nu(\nu(s_2, x_1), x_2)], \\
\text{or} &\quad\ \exists x_1 \in \mathscr{I}\ \exists x_2 \in \mathscr{I}\ [\nu(s_1, x_1x_2)\ \not{\mathrm{E}}_{k-1}\ \nu(s_2, x_1x_2)] \\
&\Rightarrow \exists x_1, x_2, x_3 \in \mathscr{I}\ [\nu(s_1, x_1x_2x_3)\ \not{\mathrm{E}}_{k-2}\ \nu(s_2, x_1x_2x_3)] \\
&\Rightarrow \ldots \\
&\Rightarrow \exists x_1, x_2, \ldots, x_i \in \mathscr{I}\ [\nu(s_1, x_1x_2 \cdots x_i)\ \not{\mathrm{E}}_{k+1-i}\ \nu(s_2, x_1x_2 \cdots x_i)] \\
&\Rightarrow \ldots \\
&\Rightarrow \exists x_1, x_2, \ldots, x_k \in \mathscr{I}\ [\nu(s_1, x_1x_2 \cdots x_k)\ \not{\mathrm{E}}_1\ \nu(s_2, x_1x_2 \cdots x_k)].
\end{aligned}
$$

This last statement about the states $\nu(s_1, x_1x_2 \cdots x_k)$, $\nu(s_2, x_1x_2 \cdots x_k)$ not being 1-equivalent implies that we can find $x_{k+1} \in \mathscr{I}$ where

$$\omega(\nu(s_1, x_1x_2 \cdots x_k), x_{k+1}) \neq \omega(\nu(s_2, x_1x_2 \cdots x_k), x_{k+1}). \tag{1}$$

That is, these *single* output symbols from $\mathscr{O}$ are different.

The result denoted by Eq. (1) also implies that

$$\omega(s_1, x) = \omega(s_1, x_1x_2 \cdots x_kx_{k+1}) \neq \omega(s_2, x_1x_2 \cdots x_kx_{k+1}) = \omega(s_2, x).$$

In this case we have two output strings of length $k + 1$ that agree for the first k symbols and differ in the $(k + 1)$st symbol.

We shall use the preceding observations, together with the partitions $P_1, P_2, \ldots, P_k, P_{k+1}$ of the minimization process, in order to deal with the following example.

EXAMPLE 7.61

From Example 7.60 we have the partitions shown below. Here $s_2\ \mathrm{E}_1\ s_6$, but $s_2\ \not{\mathrm{E}}_2\ s_6$. So we seek an input string x of length 2 such that $\omega(s_2, x) \neq \omega(s_6, x)$.

1) We start at P_2, where for s_2, s_6, we find that $\nu(s_2, 0) = s_5$ and $\nu(s_6, 0) = s_1$ are in different cells of P_1 — that is,

$$s_5 = \nu(s_2, 0)\ \not{\mathrm{E}}_1\ \nu(s_6, 0) = s_1.$$

[The input 0 and output 1 (for $\omega(s_2, 0) = 1 = \omega(s_6, 0)$) provide the labels for the arrows going from the cells of P_2 to those of P_1.]

$$
\begin{array}{l}
P_2\colon \{s_1\}, \{s_2, s_5\}, \{s_3, s_4\}, \{s_6\} \\
\qquad\quad 0, 1 \qquad\qquad 0, 1 \\
P_1\colon \{s_1\}, \{s_2, s_5, s_6\}, \{s_3, s_4\} \\
\qquad 0, 0 \downarrow \quad 0, 1 \downarrow
\end{array}
$$

2) Working with s_1 and s_5 in the partition P_1 we see that

$$\omega(\nu(s_2, 0), 0) = \omega(s_5, 0) = 1 \neq 0 = \omega(s_1, 0) = \omega(\nu(s_6, 0), 0).$$

3) Hence $x = 00$ is a minimal distinguishing string for s_2 and s_6 because $\omega(s_2, 00) = 11 \neq 10 = \omega(s_6, 00)$.

EXAMPLE 7.62

Applying the minimization process to the machine given by the state table in part (a) of Table 7.3, we obtain the partitions in part (b) of the table. (Here $P_4 = P_3$.) We find that the states s_1 and s_4 are 2-equivalent but not 3-equivalent. To construct a minimal distinguishing string for these two states, we proceed as follows:

1) Since $s_1 \not E_3 s_4$, we use partitions P_3 and P_2 to find $x_1 \in \mathscr{I}$ (namely, $x_1 = 1$) so that

$$(\nu(s_1, 1) = s_2) \not E_2 (s_5 = \nu(s_4, 1)).$$

2) Then $\nu(s_1, 1) \not E_2 \nu(s_4, 1) \Rightarrow \exists x_2 \in \mathscr{I}$ (here $x_2 = 1$) with $(\nu(s_1, 1), 1) \not E_1 (\nu(s_4, 1), 1)$, or $\nu(s_1, 11) \not E_1 \nu(s_4, 11)$. We used the partitions P_2 and P_1 to obtain $x_2 = 1$.
3) Now we use the partition P_1 where we find that for $x_3 = 1 \in \mathscr{I}$,

$$\omega(\nu(s_1, 11), 1) = 0 \neq 1 = \omega(\nu(s_4, 11), 1) \quad \text{or}$$
$$\omega(s_1, 111) = 100 \neq 101 = \omega(s_4, 111).$$

In part (b) of Table 7.3, we see how we arrived at the minimal distinguishing string $x = 111$ for these states. (Also note how this part of the table indicates that 11 is a minimal distinguishing string for the states s_2 and s_5, which are 1-equivalent but not 2-equivalent.)

Table 7.3

	ν		ω	
	0	1	0	1
s_1	s_4	s_2	0	1
s_2	s_5	s_2	0	0
s_3	s_4	s_2	0	1
s_4	s_3	s_5	0	1
s_5	s_2	s_3	0	0

(a)

$$
\begin{array}{l}
P_3\colon \{s_1, s_3\}, \{s_2\}, \{s_4\}, \{s_5\} \\
\qquad 1, 1 \qquad\qquad\quad 1, 1 \\
P_2\colon \{s_1, s_3, s_4\}, \{s_2\}, \{s_5\} \\
\qquad\qquad 1, 0 \qquad 1, 0 \\
P_1\colon \{s_1, s_3, s_4\}, \{s_2, s_5\} \\
\qquad 1, 1 \downarrow \qquad\quad \downarrow 1, 0
\end{array}
$$

(b)

A great deal more can be done with finite state machines. Among other omissions, we have avoided offering any rigorous explanation or proof of why the minimization process works. The interested reader should consult the chapter references for more on this topic.

EXERCISES 7.5

1. Apply the minimization process to each machine in Table 7.4.

Table 7.4

	ν		ω	
	0	1	0	1
s_1	s_4	s_1	0	1
s_2	s_3	s_3	1	0
s_3	s_1	s_4	1	0
s_4	s_1	s_3	0	1
s_5	s_3	s_3	1	0

(a)

	ν		ω	
	0	1	0	1
s_1	s_6	s_3	0	0
s_2	s_5	s_4	0	1
s_3	s_6	s_2	1	1
s_4	s_4	s_3	1	0
s_5	s_2	s_4	0	1
s_6	s_4	s_6	0	0

(b)

	ν		ω	
	0	1	0	1
s_1	s_6	s_3	0	0
s_2	s_3	s_1	0	0
s_3	s_2	s_4	0	0
s_4	s_7	s_4	0	0
s_5	s_6	s_7	0	0
s_6	s_5	s_2	1	0
s_7	s_4	s_1	0	0

(c)

2. For the machine in Table 7.4(c), find a (minimal) distinguishing string for each given pair of states: (a) s_1, s_5; (b) s_2, s_3; (c) s_5, s_7.

3. Let M be the finite state machine given in the state diagram shown in Fig. 7.26.

a) Minimize machine M.

b) Find a (minimal) distinguishing string for each given pair of states: (i) s_3, s_6; (ii) s_3, s_4; and (iii) s_1, s_2.

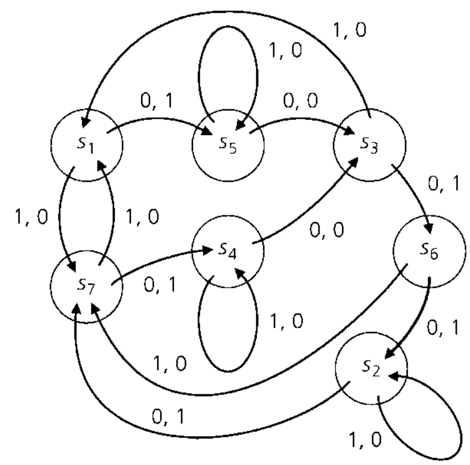


Figure 7.26

7.6 Summary and Historical Review

Once again the relation concept surfaces. In Chapter 5 this idea was introduced as a generalization of the function. Here in Chapter 7 we concentrated on relations and the special properties: reflexive, symmetric, antisymmetric, and transitive. As a result we focused on two special kinds of relations: partial orders and equivalence relations.

A relation $\mathcal{R}$ on a set A is a partial order, making A into a poset, if $\mathcal{R}$ is reflexive, antisymmetric, and transitive. Such a relation generalizes the familiar "less than or equal to" relation on the real numbers. Try to imagine calculus, or even elementary algebra, without it! Or take a simple computer program and see what happens if the program is entered into the computer haphazardly, permuting the order of the statements. Order is with us wherever we turn. We have grown so accustomed to it that we sometimes take it for granted. The origins of the subject of partially ordered sets (and lattices) came about during the nineteenth century in the work of George Boole (1815–1864), Richard Dedekind (1831–1916), Charles Sanders Peirce (1839–1914), and Ernst Schröder (1841–1902). The work of Garrett Birkhoff (1911–1996) in the 1930s, however, is where the initial work on partially ordered sets and lattices was developed to the point where these areas emerged as subjects in their own right.

For a finite poset, the Hasse diagram, a special type of directed graph, provides a pictorial representation of the order defined by the poset; it also proves useful when a total order, including the given partial order, is needed. These diagrams are named for the German number theorist Helmut Hasse (1898–1979). He introduced them in his textbook *Höhere Algebra* (published in 1926) as an aid in the study of the solutions of polynomial equations. The method we employed to derive a total order from a partial order is called topological sorting and it is used in the solution of PERT (Program Evaluation and Review Technique) networks. As mentioned earlier, this method was developed and first used by the U.S. Navy.

Although the equivalence relation differs from the partial order in only one property, it is quite different in structure and application. We make no attempt to trace the origin of the equivalence relation, but the ideas behind the reflexive, symmetric, and transitive properties can be found in *I Principii di Geometria* (1889), the work of the Italian mathematician Giuseppe Peano (1858–1932). The work of Carl Friedrich Gauss (1777–1855) on *congruence*, which he developed in the 1790s, also utilizes these ideas in spirit, if not in name.

Giuseppe Peano (1858–1932)

Carl Friedrich Gauss (1777–1855)

Basically, an equivalence relation $\mathcal{R}$ on a set A generalizes equality; it induces a characteristic of "sameness" among the elements of A. This "sameness" notion then causes the set A to be partitioned into subsets called *equivalence classes*. Conversely, we find that a partition of a set A induces an equivalence relation on A. The partition of a set arises in many places in mathematics and computer science. In computer science many searching algorithms rely on a technique that successively reduces the size of a given set A that is being searched. By partitioning A into smaller and smaller subsets, we apply the searching procedure in a more efficient manner. Each successive partition refines its predecessor, the key needed, for example, in the minimization process for finite state machines.

Throughout the chapter we emphasized the interplay between relations, directed graphs, and (0, 1)-matrices. These matrices provide a rectangular array of information about a relation, or graph, and prove useful in certain calculations. Storing information like this, in rectangular arrays and in consecutive memory locations, has been practiced in computer science since the late 1940s and early 1950s. For more on the historical background of such considerations, consult pages 456–462 of D. E. Knuth [3]. Another way to store information about a graph is the *adjacency list representation.* (See Supplementary Exercise 11.) In the study of data structures, *linked lists* and *doubly linked lists* are prominent in implementing such a representation. For more on this, consult the text by A. V. Aho, J. E. Hopcroft, and J. D. Ullman [1].

With regard to graph theory, we are in an area of mathematics that dates back to 1736 when the Swiss mathematician Leonhard Euler (1707–1783) solved the problem of the seven bridges of Königsberg. Since then, much more has evolved in this area, especially in conjunction with data structures in computer science.

For similar coverage of some of the topics in this chapter, see Chapter 3 of D. F. Stanat and D. F. McAllister [6]. An interesting presentation of the "Equivalence Problem" can be found on pages 353–355 of D. E. Knuth [3] for those wanting more information on the role of the computer in conjunction with the concept of the equivalence relation.

The early work on the development of the minimization process can be found in the paper by E. F. Moore [5], which builds upon prior ideas of D. A. Huffman [2]. Chapter 10 of Z. Kohavi [4] covers the minimization process for different types of finite state machines and includes some hardware considerations in their design.

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SUPPLEMENTARY EXERCISES

1. Let A be a set and I an index set where, for each $i \in I$, $\mathcal{R}_i$ is a relation on A. Prove or disprove each of the following.

a) $\bigcup_{i \in I} \mathcal{R}_i$ is reflexive on A if and only if each $\mathcal{R}_i$ is reflexive on A.

b) $\bigcap_{i \in I} \mathcal{R}_i$ is reflexive on A if and only if each $\mathcal{R}_i$ is reflexive on A.

2. Repeat Exercise 1 with "reflexive" replaced by (i) symmetric; (ii) antisymmetric; (iii) transitive.

3. For a set A, let $\mathcal{R}_1$ and $\mathcal{R}_2$ be symmetric relations on A. If $\mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{R}_2 \circ \mathcal{R}_1$, prove that $\mathcal{R}_1 \circ \mathcal{R}_2 = \mathcal{R}_2 \circ \mathcal{R}_1$.

4. For each of the following relations on the set specified, determine whether the relation is reflexive, symmetric, antisymmetric, or transitive. Also determine whether it is a partial order or an equivalence relation, and, if the latter, describe the partition induced by the relation.

a) $\mathcal{R}$ is the relation on $\mathbf{Q}$ where $a \mathcal{R} b$ if $|a - b| < 1$.

b) Let T be the set of all triangles in the plane. For $t_1, t_2 \in T$, define $t_1 \mathcal{R} t_2$ if t_1, t_2 have the same area.

c) For T as in part (b), define $\mathcal{R}$ by $t_1 \mathcal{R} t_2$ if at least two sides of t_1 are contained within the perimeter of t_2.

d) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Define $\mathcal{R}$ on A by $x \mathcal{R} y$ if $xy \geq 10$.

5. For sets A, B, and C with relations $\mathcal{R}_1 \subseteq A \times B$ and $\mathcal{R}_2 \subseteq B \times C$, prove or disprove that $(\mathcal{R}_1 \circ \mathcal{R}_2)^c = \mathcal{R}_2^c \circ \mathcal{R}_1^c$.

6. For a set A, let $C = \{P_i | P_i \text{ is a partition of } A\}$. Define relation $\mathcal{R}$ on C by $P_i \mathcal{R} P_j$ if $P_i \leq P_j$ — that is, P_i is a refinement of P_j.

a) Verify that $\mathcal{R}$ is a partial order on C.

b) For $A = \{1, 2, 3, 4, 5\}$, let P_i, $1 \leq i \leq 4$, be the following partitions: P_1: $\{1, 2\}, \{3, 4, 5\}$; P_2: $\{1, 2\}, \{3, 4\}, \{5\}$; P_3: $\{1\}, \{2\}, \{3, 4, 5\}$; P_4: $\{1, 2\}, \{3\}, \{4\}, \{5\}$. Draw the Hasse diagram for $C = \{P_i | 1 \leq i \leq 4\}$, where C is partially ordered by refinement.

7. Give an example of a poset with 5 minimal (maximal) elements but no least (greatest) element.

8. Let $A = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$. Define $\mathcal{R}$ on A by $(x_1, y_1) \mathcal{R} (x_2, y_2)$, if $x_1 y_1 = x_2 y_2$.

a) Verify that $\mathcal{R}$ is an equivalence relation on A.

b) Determine the equivalence classes $[(1, 1)]$, $[(2, 2)]$, $[(3, 2)]$, and $[(4, 3)]$.

9. If the complete graph K_n has 45 edges, what is n?

10. Let $\mathcal{F} = \{f\colon \mathbf{Z}^+ \to \mathbf{R}\}$ — that is, $\mathcal{F}$ is the set of all functions with domain $\mathbf{Z}^+$ and codomain $\mathbf{R}$.

a) Define the relation $\mathcal{R}$ on $\mathcal{F}$ by $g \mathcal{R} h$, for $g, h \in \mathcal{F}$, if g is dominated by h and h is dominated by g — that is, $g \in \Theta(h)$. (See Exercises 14, 15 for Section 5.7.) Prove that $\mathcal{R}$ is an equivalence relation on $\mathcal{F}$.

b) For $f \in \mathcal{F}$, let $[f]$ denote the equivalence class of f for the relation $\mathcal{R}$ of part (a). Let $\mathcal{F}'$ be the set of equivalence classes induced by $\mathcal{R}$. Define the relation $\mathcal{S}$ on $\mathcal{F}'$ by $[g] \mathcal{S} [h]$, for $[g], [h] \in \mathcal{F}'$, if g is dominated by h. Verify that $\mathcal{S}$ is a partial order.

c) For $\mathcal{R}$ in part (a), let $f, f_1, f_2 \in \mathcal{F}$ with $f_1, f_2 \in [f]$. If $f_1 + f_2\colon \mathbf{Z}^+ \to \mathbf{R}$ is defined by $(f_1 + f_2)(n) = f_1(n) + f_2(n)$, for $n \in \mathbf{Z}^+$, prove or disprove that $f_1 + f_2 \in [f]$.

11. We have seen that the adjacency matrix can be used to represent a graph. However, this method proves to be rather inefficient when there are many 0's (that is, few edges) present. A better method uses the *adjacency list representation*, which is made up of an *adjacency list* for each vertex v and an *index list*. For the graph shown in Fig. 7.27, the representation is given by the two lists in Table 7.5.

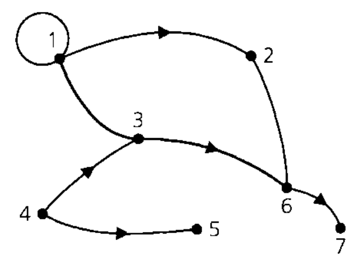


Figure 7.27

Table 7.5

Adjacency List		Index List	
1	1	1	1
2	2	2	4
3	3	3	5
4	6	4	7
5	1	5	9
6	6	6	9
7	3	7	11
8	5	8	11
9	2		
10	7		

For each vertex v in the graph, we list, preferably in numerical order, each vertex w that is adjacent from v. Hence for 1, we list 1, 2, 3 as the first three adjacencies in our adjacency list. Next to 2 in the index list we place a 4, which tells us where to start looking in the adjacency list for the adjacencies from 2. Since there is a 5 to the right of 3 in the index list, we know that the only adjacency from 2 is 6. Likewise, the 7 to the right of 4 in the index list directs us to the seventh entry in the adjacency list — namely, 3 — and we find that vertex 4 is adjacent to vertices 3 (the seventh vertex in the adjacency list) and 5 (the eighth vertex in the adjacency list). We stop at vertex 5 because of the 9 to the right of vertex 5 in the index list. The 9's in the index list next to 5 and 6 indicate that no vertex is adjacent from vertex 5. In a similar way, the 11's next to 7 and 8 in the index list tell us that vertex 7 is not adjacent to any vertex in the given directed graph.

In general, this method provides an easy way to determine the vertices adjacent from a vertex v. They are listed in the positions $\text{index}(v)$, $\text{index}(v) + 1, \ldots, \text{index}(v + 1) - 1$ of the adjacency list.

Finally, the last pair of entries in the index list — namely, 8 and 11 — is a "phantom" that indicates where the adjacency list would pick up from *if* there were an eighth vertex in the graph.

Represent each of the graphs in Fig. 7.28 in this manner.

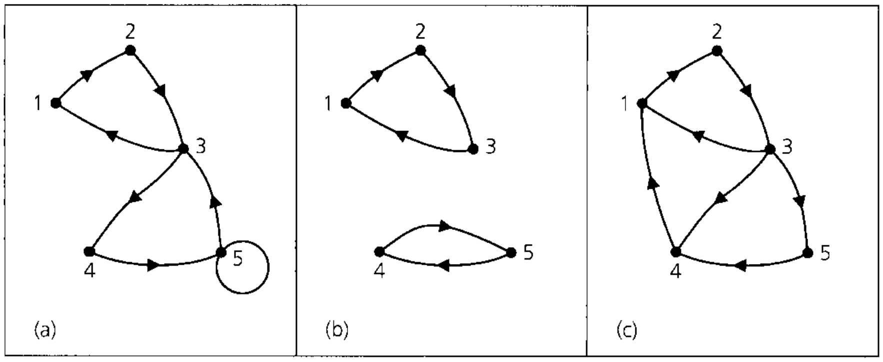


Figure 7.28

12. The adjacency list representation of a directed graph G is given by the lists in Table 7.6. Construct G from this representation.

Table 7.6

Adjacency List		Index List	
1	2	1	1
2	3	2	4
3	6	3	5
4	3	4	5
5	3	5	8
6	4	6	10
7	5	7	10
8	3	8	10
9	6		

13. Let G be an undirected graph with vertex set V. Define the relation $\mathcal{R}$ on V by $v \mathcal{R} w$ if $v = w$ or if there is a path from v to w (or from w to v since G is undirected). (a) Prove that $\mathcal{R}$ is an equivalence relation on V. (b) What can we say about the associated partition?

14. a) For the finite state machine given in Table 7.7, determine a minimal machine that is equivalent to it.

b) Find a minimal string that distinguishes states s_4 and s_6.

15. At the computer center Maria is faced with running 10 computer programs which, because of priorities, are restricted by the following conditions: (a) $10 > 8, 3$; (b) $8 > 7$; (c) $7 > 5$; (d) $3 > 9, 6$; (e) $6 > 4, 1$; (f) $9 > 4, 5$; (g) $4, 5, 1 > 2$; where, for example, $10 > 8, 3$ means that program number 10 must be run before programs 8 and 3. Determine an order for running these programs so that the priorities are satisfied.

16. a) Draw the Hasse diagram for the set of positive integer divisors of (i) 2; (ii) 4; (iii) 6; (iv) 8; (v) 12; (vi) 16; (vii) 24; (viii) 30; (ix) 32.

Table 7.7

	ν		ω	
	0	1	0	1
s_1	s_7	s_6	1	0
s_2	s_7	s_7	0	0
s_3	s_7	s_2	1	0
s_4	s_2	s_3	0	0
s_5	s_3	s_7	0	0
s_6	s_4	s_1	0	0
s_7	s_3	s_5	1	0
s_8	s_7	s_3	0	0

b) For all $2 \leq n \leq 35$, show that the Hasse diagram for the set of positive-integer divisors of n looks like one of the nine diagrams in part (a). (Ignore the numbers at the vertices and concentrate on the structure given by the vertices and edges.) What happens for $n = 36$?

c) For $n \in \mathbf{Z}^+$, $\tau(n) =$ the number of positive-integer divisors of n. (See Supplementary Exercise 32 in Chapter 5.) Let $m, n \in \mathbf{Z}^+$ and S, T be the sets of all positive-integer divisors of m, n, respectively. The results of parts (a) and (b) imply that if the Hasse diagrams of S, T are structurally the same, then $\tau(m) = \tau(n)$. But is the converse true?

d) Show that each Hasse diagram in part (a) is a lattice if we define $\text{glb}\{x, y\} = \gcd(x, y)$ and $\text{lub}\{x, y\} = \text{lcm}(x, y)$.

17. Let U denote the set of all points in and on the unit square shown in Fig. 7.29. That is, $U = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$. Define the relation $\mathcal{R}$ on U by $(a, b) \mathcal{R} (c, d)$ if (1) $(a, b) = (c, d)$, or (2) $b = d$ and $a = 0$ and $c = 1$, or (3) $b = d$ and $a = 1$ and $c = 0$.

a) Verify that $\mathcal{R}$ is an equivalence relation on U.

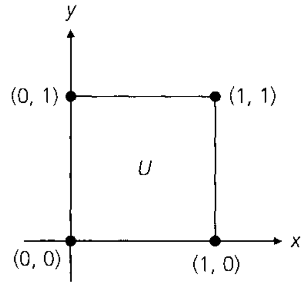


Figure 7.29

b) List the ordered pairs in the equivalence classes $[(0.3, 0.7)]$, $[(0.5, 0)]$, $[(0.4, 1)]$, $[(0, 0.6)]$, $[(1, 0.2)]$. For $0 \leq a \leq 1, 0 \leq b \leq 1$, how many ordered pairs are in $[(a, b)]$?

c) If we "glue together" the ordered pairs in each equivalence class, what type of surface comes about?

18. a) For $\mathcal{U} = \{1, 2, 3\}$, let $A = \mathcal{P}(\mathcal{U})$. Define the relation $\mathcal{R}$ on A by $B \mathcal{R} C$ if $B \subseteq C$. How many ordered pairs are there in the relation $\mathcal{R}$?

b) Answer part (a) for $\mathcal{U} = \{1, 2, 3, 4\}$.

c) Generalize the results of parts (a) and (b).

19. For $n \in \mathbf{Z}^+$, let $\mathcal{U} = \{1, 2, 3, \ldots, n\}$. Define the relation $\mathcal{R}$ on $\mathcal{P}(\mathcal{U})$ by $A \mathcal{R} B$ if $A \nsubseteq B$ and $B \nsubseteq A$. How many ordered pairs are there in this relation?

20. Let A be a finite nonempty set with $B \subseteq A$ (B fixed), and $|A| = n$, $|B| = m$. Define the relation $\mathcal{R}$ on $\mathcal{P}(A)$ by $X \mathcal{R} Y$, for $X, Y \subseteq A$, if $X \cap B = Y \cap B$. Then $\mathcal{R}$ is an equivalence relation, as verified in Exercise 10 of Section 7.4. (a) How many equivalence classes are in the partition of $\mathcal{P}(A)$ induced by $\mathcal{R}$? (b) How many subsets of A are in each equivalence class of the partition induced by $\mathcal{R}$?

21. For $A \neq \emptyset$, let $(A, \mathcal{R})$ be a poset, and let $\emptyset \neq B \subseteq A$ such that $\mathcal{R}' = (B \times B) \cap \mathcal{R}$. If $(B, \mathcal{R}')$ is totally ordered, we call $(B, \mathcal{R}')$ a *chain* in $(A, \mathcal{R})$. In the case where B is finite, we may order the elements of B by $b_1 \mathcal{R}' b_2 \mathcal{R}' b_3 \mathcal{R}' \cdots \mathcal{R}' b_{n-1} \mathcal{R}' b_n$ and say that the chain has *length n*. A chain (of length n) is called *maximal* if there is no element $a \in A$ where $a \notin \{b_1, b_2, b_3, \ldots, b_n\}$ and $a \mathcal{R} b_1$, $b_n \mathcal{R} a$, or $b_i \mathcal{R} a \mathcal{R} b_{i+1}$, for some $1 \leq i \leq n - 1$.

a) Find two chains of length 3 for the poset given by the Hasse diagram in Fig. 7.20. Find a maximal chain for this poset. How many such maximal chains does it have?

b) For the poset given by the Hasse diagram in Fig. 7.18(d), find two maximal chains of different lengths. What is the length of a longest (maximal) chain for this poset?

c) Let $\mathcal{U} = \{1, 2, 3, 4\}$ and $A = \mathcal{P}(\mathcal{U})$. For the poset $(A, \subseteq)$, find two maximal chains. How many such maximal chains are there for this poset?

d) If $\mathcal{U} = \{1, 2, 3, \ldots, n\}$, how many maximal chains are there in the poset $(\mathcal{P}(\mathcal{U}), \subseteq)$?

22. For $\emptyset \neq C \subseteq A$, let $(C, \mathcal{R}')$ be a maximal chain in the poset $(A, \mathcal{R})$, where $\mathcal{R}' = (C \times C) \cap \mathcal{R}$. If the elements of C are ordered as $c_1 \mathcal{R}' c_2 \mathcal{R}' \cdots \mathcal{R}' c_n$, prove that c_1 is a minimal element in $(A, \mathcal{R})$ and that c_n is maximal in $(A, \mathcal{R})$.

23. Let $(A, \mathcal{R})$ be a poset in which the length of a longest (maximal) chain is $n \geq 2$. Let M be the set of all maximal elements in $(A, \mathcal{R})$, and let $B = A - M$. If $\mathcal{R}' = (B \times B) \cap \mathcal{R}$, prove that the length of a longest chain in $(B, \mathcal{R}')$ is $n - 1$.

24. Let $(A, \mathcal{R})$ be a poset, and let $\emptyset \neq C \subseteq A$. If $(C \times C) \cap \mathcal{R} = \emptyset$, then for all distinct $x, y \in C$ we have $x \not\mathcal{R}\, y$ and $y \not\mathcal{R}\, x$. The elements of C are said to form an *antichain* in the poset $(A, \mathcal{R})$.

a) Find an antichain with three elements for the poset given in the Hasse diagram of Fig. 7.18(d). Determine a largest antichain containing the element 6. Determine a largest antichain for this poset.

b) If $\mathcal{U} = \{1, 2, 3, 4\}$, let $A = \mathcal{P}(\mathcal{U})$. Find two different antichains for the poset $(A, \subseteq)$. How many elements occur in a largest antichain for this poset?

c) Prove that in any poset $(A, \mathcal{R})$, the set of all maximal elements and the set of all minimal elements are antichains.

25. Let $(A, \mathcal{R})$ be a poset in which the length of a longest chain is n. Use mathematical induction to prove that the elements of A can be partitioned into n antichains $C_1, C_2, \ldots, C_n$ (where $C_i \cap C_j = \emptyset$, for $1 \leq i < j \leq n$).

26. a) In how many ways can one totally order the partial order of positive-integer divisors of 96?

b) How many of the total orders in part (a) start with $96 > 32$?

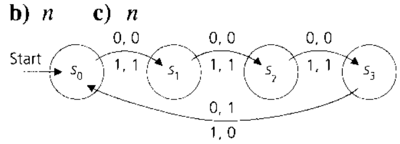
c) How many of the total orders in part (a) end with $3 > 1$?

d) How many of the total orders in part (a) start with $96 > 32$ and end with $3 > 1$?

e) How many of the total orders in part (a) start with $96 > 48 > 32 > 16$?

27. Let n be a fixed positive integer and let $A_n = \{0, 1, \ldots, n\} \subseteq \mathbf{N}$. (a) How many edges are there in the Hasse diagram for the total order $(A_n, \leq)$, where "$\leq$" is the ordinary "less than or equal to" relation? (b) In how many ways can the edges in the Hasse diagram of part (a) be partitioned so that the edges in each cell (of the partition) provide a path (of one or more edges)? (c) In how many ways can the edges in the Hasse diagram for $(A_{12}, \leq)$ be partitioned so that the edges in each cell (of the partition) provide a path (of one or more edges) and one of the cells is $\{(3, 4), (4, 5), (5, 6), (6, 7)\}$?

Supplementary Exercises – p. 334

1. a) True b) False c) True d) True e) True f) True
3. Let $x \in \Sigma$ and $A = \{x\}$. Then $A^2 = \{xx\}$ and $(A^2)^* = \{\lambda, x^2, x^4, \ldots\}$. $A^* = \{\lambda, x, x^2, x^3, \ldots\}$ and $(A^*)^2 = A^*$, so $(A^2)^* \neq (A^*)^2$.
5. $\mathcal{C}_{02} = \{1, 00\}^*\{0\}$ $\mathcal{C}_{22} = \{0\}\{1, 00\}^*\{0\}$ $\mathcal{C}_{11} = \emptyset$
   $\mathcal{C}_{00} = \{1, 00\}^* - \{\lambda\}$ $\mathcal{C}_{10} = \{1\}\{1, 00\}^* \cup \{10\}\{1, 00\}^*$
7. a) By the pigeonhole principle there is a first state s that is encountered twice. Let y be the output string that resulted since s was first encountered, until we reach this state a second time. Then from that point on the output is $yyy\ldots$.
   b) n c) n
9. 

11. a)

	ν		ω	
	0	1	0	1
(s_0, s_3)	(s_0, s_4)	(s_1, s_3)	1	1
(s_0, s_4)	(s_0, s_3)	(s_1, s_4)	0	1
(s_1, s_3)	(s_1, s_3)	(s_2, s_3)	1	1
(s_1, s_4)	(s_1, s_4)	(s_2, s_4)	1	1
(s_2, s_3)	(s_2, s_3)	(s_0, s_4)	1	1
(s_2, s_4)	(s_2, s_4)	(s_0, s_3)	1	0

   b) $\omega((s_0, s_3), 1101) = 1111$; M_1 is in state s_0, and M_2 is in state s_4.

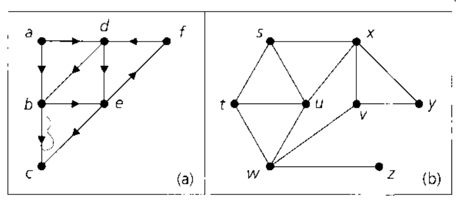
Chapter 7
Relations: The Second Time Around

Section 7.1 – p. 343

1. a) $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (2, 3), (3, 2)\}$
   b) $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$ c) $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$
3. a) Let $f_1, f_2, f_3 \in \mathcal{F}$ with $f_1(n) = n + 1$, $f_2(n) = 5n$, and $f_3(n) = 4n + 1/n$.
   b) Let $g_1, g_2, g_3 \in \mathcal{F}$ with $g_1(n) = 3$, $g_2(n) = 1/n$, and $g_3(n) = \sin n$.
5. a) Reflexive, antisymmetric, transitive b) Transitive
   c) Reflexive, symmetric, transitive d) Symmetric e) Symmetric
   f) Reflexive, symmetric, transitive g) Reflexive, symmetric h) Reflexive, transitive
7. a) For all $x \in A$, $(x, x) \in \mathcal{R}_1, \mathcal{R}_2$, so $(x, x) \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive.
   b) (i) $(x, y) \in \mathcal{R}_1 \cap \mathcal{R}_2 \Rightarrow (x, y) \in \mathcal{R}_1, \mathcal{R}_2 \Rightarrow (y, x) \in \mathcal{R}_1, \mathcal{R}_2 \Rightarrow (y, x) \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ is symmetric.
   (ii) $(x, y), (y, x) \in \mathcal{R}_1 \cap \mathcal{R}_2 \Rightarrow (x, y), (y, x) \in \mathcal{R}_1, \mathcal{R}_2$. By the antisymmetry of $\mathcal{R}_1$ (or $\mathcal{R}_2$), $x = y$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ is antisymmetric.
   (iii) $(x, y), (y, z) \in \mathcal{R}_1 \cap \mathcal{R}_2 \Rightarrow (x, y), (y, z) \in \mathcal{R}_1, \mathcal{R}_2 \Rightarrow (x, z) \in \mathcal{R}_1, \mathcal{R}_2$ (transitive property) $\Rightarrow (x, z) \in \mathcal{R}_1 \cap \mathcal{R}_2$, so $\mathcal{R}_1 \cap \mathcal{R}_2$ is transitive.
9. a) False: let $A = \{1, 2\}$ and $\mathcal{R} = \{(1, 2), (2, 1)\}$.
   b) (i) Reflexive: true
   (ii) Symmetric: false. Let $A = \{1, 2\}$, $\mathcal{R}_1 = \{(1, 1)\}$, and $\mathcal{R}_2 = \{(1, 1), (1, 2)\}$.
   (iii) Antisymmetric and transitive: false. Let $A = \{1, 2\}$, $\mathcal{R}_1 = \{(1, 2)\}$, and $\mathcal{R}_2 = \{(1, 2), (2, 1)\}$.
   d) True.
11. a) $\binom{2+2-1}{2}\binom{2+2-1}{2} = \binom{3}{2}\binom{3}{2} = 9$ b) 18 c) $\binom{4+2-1}{2}\binom{2+2-1}{2} = \binom{5}{2}\binom{3}{2} = 30$
   d) 60 e) 81 f) 972

13. There may exist an element $a \in A$ such that for all $b \in A$, neither (a, b) nor (b, a) is in $\mathcal{R}$.
15. $r - n$ counts the elements in $\mathcal{R}$ of the form (a, b), $a \neq b$. Since $\mathcal{R}$ is symmetric, $r - n$ is even.
17. **a)** $\binom{7}{4}\binom{21}{0} + \binom{7}{2}\binom{21}{1} + \binom{7}{0}\binom{21}{2}$ **b)** $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{1}\binom{21}{2}$
d) $\binom{7}{6}\binom{21}{1} + \binom{7}{4}\binom{21}{2} + \binom{7}{2}\binom{21}{3} + \binom{7}{0}\binom{21}{4}$

Section 7.2—p. 354

1. $\mathcal{R} \circ \mathcal{S} = \{(1, 3), (1, 4)\}$; $\mathcal{S} \circ \mathcal{R} = \{(1, 2), (1, 3), (1, 4), (2, 4)\}$;
$\mathcal{R}^2 = \mathcal{R}^3 = \{(1, 4), (2, 4), (4, 4)\}$; $\mathcal{S}^2 = \mathcal{S}^3 = \{(1, 1), (1, 2), (1, 3), (1, 4)\}$
3. $(a, d) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \Rightarrow (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2$, $(c, d) \in \mathcal{R}_3$ for some $c \in C \Rightarrow (a, b) \in \mathcal{R}_1$, $(b, c) \in \mathcal{R}_2$, $(c, d) \in \mathcal{R}_3$ for some $b \in B$, $c \in C \Rightarrow (a, b) \in \mathcal{R}_1$, $(b, d) \in \mathcal{R}_2 \circ \mathcal{R}_3 \Rightarrow$ $(a, d) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$, and $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$
5. $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = \mathcal{R}_1 \circ \{(m, 3), (m, 4)\} = \{(1, 3), (1, 4)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1, 3), (1, 4)\} \cap \{(1, 3), (1, 4)\} = \{(1, 3), (1, 4)\}$
7. This follows by the pigeonhole principle. Here the pigeons are the $2^{n^2} + 1$ integers between 0 and 2^{n^2}, inclusive, and the pigeonholes are the 2^{n^2} relations on A.
9. 2^{21}
11. Consider the entry in the ith row and jth column of $M(\mathcal{R}_1 \circ \mathcal{R}_2)$. If this entry is a 1, then there exists $b_k \in B$ where $1 \leq k \leq n$ and $(a_i, b_k) \in \mathcal{R}_1$, $(b_k, c_j) \in \mathcal{R}_2$. Consequently, the entry in the ith row and kth column of $M(\mathcal{R}_1)$ is 1, and the entry in the kth row and jth column of $M(\mathcal{R}_2)$ is 1. This results in a 1 in the ith row and jth column in the product $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$.
Should the entry in row i and column j of $M(\mathcal{R}_1 \circ \mathcal{R}_2)$ be 0, then for each b_k, where $1 \leq k \leq n$, either $(a_i, b_k) \notin \mathcal{R}_1$ or $(b_k, c_j) \notin \mathcal{R}_2$. This means that in the matrices $M(\mathcal{R}_1)$, $M(\mathcal{R}_2)$, if the entry in the ith row and kth column of $M(\mathcal{R}_1)$ is 1, then the entry in the kth row and jth column of $M(\mathcal{R}_2)$ is 0. Hence the entry in the ith row and jth column of $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$ is 0.
13. **d)** Let s_{xy} be the entry in row (x) and column (y) of M. Then s_{yx} appears in row (x) and column (y) of M^{tr}. $\mathcal{R}$ is antisymmetric $\Longleftrightarrow (s_{xy} = s_{yx} = 1 \Rightarrow x = y) \Longleftrightarrow M \cap M^{tr} \leq I_n$.
15. 

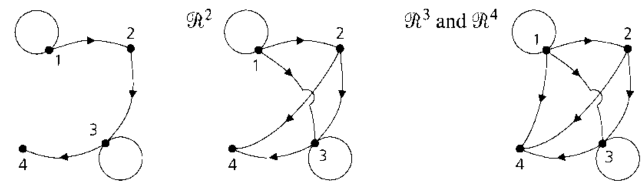
17. (i) $\mathcal{R} = \{(a, b), (b, a), (a, e), (e, a), (b, c), (c, b), (b, d), (d, b), (b, e), (e, b), (d, e), (e, d), (d, f), (f, d)\}$:

$$M(\mathcal{R}) = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \end{array} \begin{array}{c} \begin{array}{cccccc} (a) & (b) & (c) & (d) & (e) & (f) \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

For part (ii) the rows and columns of the relation matrix are indexed as in part (i).
(ii) $\mathcal{R} = \{(a, b), (b, e), (d, b), (d, c), (e, f)\}$:

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

19. $\mathcal{R}$ $\mathcal{R}^2$ $\mathcal{R}^3$ and $\mathcal{R}^4$

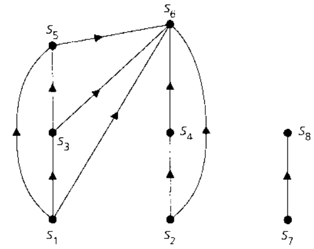


21. **a)** 2^{25} **b)** 2^{15}

23. **a)** $\mathcal{R}_1$: $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ $\mathcal{R}_2$: $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

b) Given an equivalence relation $\mathcal{R}$ on a finite set A, list the elements of A so that elements in the same cell of the partition (see Section 7.4) are adjacent. The resulting relation matrix will then have square blocks of 1's along the diagonal (from upper left to lower right).

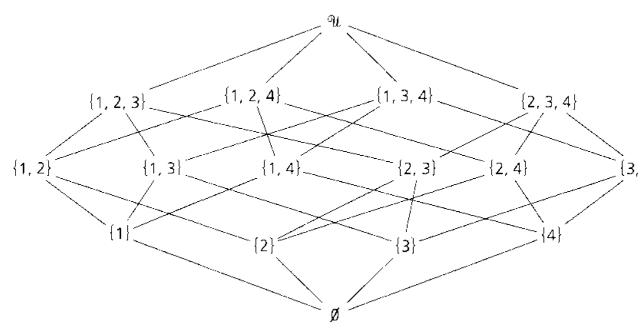
25.
(s_1) a := 1
(s_2) b := 2
(s_3) a := a + 3
(s_4) c := b
(s_5) a := 2 * a − 1
(s_6) b := a * c
(s_7) c := 7
(s_8) d := c + 2



27. $n = 38$

Section 7.3 – p. 364

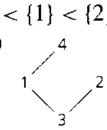
1.



3. For all $a \in A$, $b \in B$, we have $a\,\mathcal{R}_1\,a$ and $b\,\mathcal{R}_2\,b$, so $(a, b)\,\mathcal{R}\,(a, b)$ and $\mathcal{R}$ is reflexive. $(a, b)\,\mathcal{R}\,(c, d)$, $(c, d)\,\mathcal{R}\,(a, b) \Rightarrow a\,\mathcal{R}_1\,c$, $c\,\mathcal{R}_1\,a$ and $b\,\mathcal{R}_2\,d$, $d\,\mathcal{R}_2\,b \Rightarrow a = c$, $b = d \Rightarrow (a, b) = (c, d)$, so $\mathcal{R}$ is antisymmetric. $(a, b)\,\mathcal{R}\,(c, d)$, $(c, d)\,\mathcal{R}\,(e, f) \Rightarrow a\,\mathcal{R}_1\,c$, $c\,\mathcal{R}_1\,e$ and $b\,\mathcal{R}_2\,d$, $d\,\mathcal{R}_2\,f \Rightarrow a\,\mathcal{R}_1\,e$, $b\,\mathcal{R}_2\,f \Rightarrow (a, b)\,\mathcal{R}\,(e, f)$, and this implies that $\mathcal{R}$ is transitive.

5. $\emptyset < \{1\} < \{2\} < \{3\} < \{1, 2\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}$ (There are other possibilities.)

7. **a)**



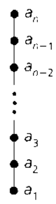
b) $3 < 2 < 1 < 4$ or $3 < 1 < 2 < 4$ **c)** 2

9. Let x, y both be least upper bounds. Then $x\,\mathcal{R}\,y$, since y is an upper bound and x is a least upper bound. Likewise, $y\,\mathcal{R}\,x$. $\mathcal{R}$ antisymmetric $\Rightarrow x = y$. (The proof for the glb is similar.)

11. Let $\mathcal{U} = \{1, 2\}$, $A = \mathcal{P}(\mathcal{U})$, and let $\mathcal{R}$ be the inclusion relation. Then $(A, \mathcal{R})$ is a poset but not a total order. Let $B = \{\emptyset, \{1\}\}$. Then $(B \times B) \cap \mathcal{R}$ is a total order.

13. $n + \binom{n}{2}$

15. a) The n elements of A are arranged along a vertical line. For if $A = \{a_1, a_2, \ldots, a_n\}$ where $a_1 \mathcal{R}\, a_2 \mathcal{R}\, a_3 \mathcal{R} \cdots \mathcal{R}\, a_n$, then the diagram can be drawn as follows:



b) $n!$

17.

	lub	glb
a)	$\{1, 2\}$	$\emptyset$
b)	$\{1, 2, 3\}$	$\emptyset$
c)	$\{1, 2\}$	$\emptyset$
d)	$\{1, 2, 3\}$	$\{1\}$
e)	$\{1, 2, 3\}$	$\emptyset$

19. For each $a \in \mathbf{Z}$ it follows that $a \mathcal{R}\, a$ because $a - a = 0$, an even nonnegative integer. Hence $\mathcal{R}$ is *reflexive*. If $a, b, c \in \mathbf{Z}$ with $a \mathcal{R}\, b$ and $b \mathcal{R}\, c$, then

$$a - b = 2m, \quad \text{for some } m \in \mathbf{N}$$
$$b - c = 2n, \quad \text{for some } n \in \mathbf{N},$$

and $a - c = (a - b) + (b - c) = 2(m + n)$, where $m + n \in \mathbf{N}$. Therefore, $a \mathcal{R}\, c$ and $\mathcal{R}$ is *transitive*. Finally, suppose that $a \mathcal{R}\, b$ and $b \mathcal{R}\, a$ for some $a, b \in \mathbf{Z}$. Then $a - b$ and $b - a$ are both nonnegative integers. Since this can only occur for $a - b = b - a = 0$, we find that $[a \mathcal{R}\, b \wedge b \mathcal{R}\, a] \Rightarrow a = b$, so $\mathcal{R}$ is *antisymmetric*.

Consequently, the relation $\mathcal{R}$ is a partial order for $\mathbf{Z}$. But it is *not* a total order. For example, $2, 3 \in \mathbf{Z}$ and we have neither $2 \mathcal{R}\, 3$ nor $3 \mathcal{R}\, 2$, because neither -1 nor 1, respectively, is a nonnegative even integer.

21. b) & c) Here the least element (and only minimal element) is $(0, 0)$. The element $(2, 2)$ is the greatest element (and the only maximal element).
d) $(0, 0) \mathcal{R}\, (0, 1) \mathcal{R}\, (0, 2) \mathcal{R}\, (1, 0) \mathcal{R}\, (1, 1) \mathcal{R}\, (1, 2) \mathcal{R}\, (2, 0) \mathcal{R}\, (2, 1) \mathcal{R}\, (2, 2)$

23. a) False. Let $\mathcal{U} = \{1, 2\}$, $A = \mathcal{P}(\mathcal{U})$, and let $\mathcal{R}$ be the inclusion relation. Then $(A, \mathcal{R})$ is a lattice where for all $S, T \in A$, $\text{lub}\{S, T\} = S \cup T$ and $\text{glb}\{S, T\} = S \cap T$. However, $\{1\}$ and $\{2\}$ are not related, so $(A, \mathcal{R})$ is not a total order.

25. a) a **b)** a **c)** c **d)** e **e)** z **f)** e **g)** v
$(A, \mathcal{R})$ is a lattice with z the greatest (and only maximal) element and a the least (and only minimal) element.

27. a) 3 **b)** m **c)** 17 **d)** $m + n + 2mn$ **e)** 133
f) $m + n + k + 2(mn + mk + nk) + 3mnk$ **g)** 1484
h) $m + n + k + \ell + 2(mn + mk + m\ell + nk + n\ell + k\ell) + 3(mnk + mn\ell + mk\ell + nk\ell) + 4mnk\ell$

29. $429 = \left(\frac{1}{8}\right)\binom{14}{7}$ so $k = 6$, and there are $2 \cdot 7 = 14$ positive integer divisors of $p^6 q$.

Section 7.4 – p. 370

1. a) Here the collection A_1, A_2, A_3 provides a partition of A.
b) Although $A = A_1 \cup A_2 \cup A_3 \cup A_4$, we have $A_1 \cap A_2 \neq \emptyset$, so the collection A_1, A_2, A_3, A_4 does *not* provide a partition for A.

3. $\mathcal{R} = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$

5. $\mathcal{R}$ is not transitive since $1 \mathcal{R}\, 2$ and $2 \mathcal{R}\, 3$ but $1 \not\mathcal{R}\, 3$.

7. a) For all $(x, y) \in A$, $x + y = x + y \Rightarrow (x, y) \mathcal{R}\, (x, y)$.
$(x_1, y_1) \mathcal{R}\, (x_2, y_2) \Rightarrow x_1 + y_1 = x_2 + y_2 \Rightarrow x_2 + y_2 = x_1 + y_1 \Rightarrow (x_2, y_2) \mathcal{R}\, (x_1, y_1)$.
$(x_1, y_1) \mathcal{R}\, (x_2, y_2)$, $(x_2, y_2) \mathcal{R}\, (x_3, y_3) \Rightarrow x_1 + y_1 = x_2 + y_2$, $x_2 + y_2 = x_3 + y_3$, so $x_1 + y_1 = x_3 + y_3$ and $(x_1, y_1) \mathcal{R}\, (x_3, y_3)$. Since $\mathcal{R}$ is reflexive, symmetric, and transitive, it is an equivalence relation.
b) $[(1, 3)] = \{(1, 3), (2, 2), (3, 1)\}$; $[(2, 4)] = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$
$[(1, 1)] = \{(1, 1)\}$

c) $A = \{(1,1)\} \cup \{(1,2),(2,1)\} \cup \{(1,3),(2,2),(3,1)\} \cup \{(1,4),(2,3),(3,2),(4,1)\} \cup \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \cup \{(2,5),(3,4),(4,3),(5,2)\} \cup \{(3,5),(4,4),(5,3)\} \cup \{(4,5),(5,4)\} \cup \{(5,5)\}$

9. **a)** For all $(a, b) \in A$ we have $ab = ab$, so $(a, b) \mathcal{R} (a, b)$ and $\mathcal{R}$ is reflexive. To see that $\mathcal{R}$ is symmetric suppose that $(a, b), (c, d) \in A$ and that $(a, b) \mathcal{R} (c, d)$. Then $(a, b) \mathcal{R} (c, d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c, d) \mathcal{R} (a, b)$, so $\mathcal{R}$ is symmetric. Finally, let $(a, b), (c, d), (e, f) \in A$ with $(a, b) \mathcal{R} (c, d)$ and $(c, d) \mathcal{R} (e, f)$. Then $(a, b) \mathcal{R} (c, d) \Rightarrow ad = bc$ and $(c, d) \mathcal{R} (e, f) \Rightarrow cf = de$, so $adf = bcf = bde$ and since $d \neq 0$, we have $af = be$. But $af = be \Rightarrow (a, b) \mathcal{R} (e, f)$, and consequently $\mathcal{R}$ is transitive. It follows from the above that $\mathcal{R}$ is an equivalence relation on A.

b) $[(2, 14)] = \{(2, 14)\}$ $[(-3, -9)] = \{(-3, -9), (-1, -3), (4, 12)\}$
$[(4, 8)] = \{(-2, -4), (1, 2), (3, 6), (4, 8)\}$

c) There are five cells in the partition — in fact,

$$A = [(-4, -20)] \cup [(-3, -9)] \cup [(-2, -4)] \cup [(-1, -11)] \cup [(2, 14)].$$

11. **a)** $\left(\frac{1}{2}\right)\binom{6}{3}$ **b)** $4\binom{6}{3}$ **c)** $2\binom{6}{4}$ **d)** $\left(\frac{1}{2}\right)\binom{6}{3} + 4\binom{6}{3} + 2\binom{6}{4} + \binom{6}{5} + \binom{6}{6}$ **13.** 300

15. Let $\{A_i\}_{i \in I}$ be a partition of a set A. Define $\mathcal{R}$ on A by $x \mathcal{R} y$ if for some $i \in I$, we have $x, y \in A_i$. For each $x \in A$, $x, x \in A_i$ for some $i \in I$, so $x \mathcal{R} x$ and $\mathcal{R}$ is reflexive. $x \mathcal{R} y \Rightarrow x, y \in A_i$, for some $i \in I \Rightarrow y, x \in A_i$ for some $i \in I \Rightarrow y \mathcal{R} x$, so $\mathcal{R}$ is symmetric. If $x \mathcal{R} y$ and $y \mathcal{R} z$, then $x, y \in A_i$ and $y, z \in A_j$ for some $i, j \in I$. Since $A_i \cap A_j$ contains y and $\{A_i\}_{i \in I}$ is a partition, from $A_i \cap A_j \neq \emptyset$ it follows that $A_i = A_j$, so $i = j$. Hence $x, z \in A_i$, so $x \mathcal{R} z$ and $\mathcal{R}$ is transitive.

17. *Proof*: Since $\{B_1, B_2, B_3, \ldots, B_n\}$ is a partition of B, we have $B = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$. Therefore $A = f^{-1}(B) = f^{-1}(B_1 \cup \cdots \cup B_n) = f^{-1}(B_1) \cup \cdots \cup f^{-1}(B_n)$ [by generalizing part (b) of Theorem 5.10]. For $1 \leq i < j \leq n$, $f^{-1}(B_i) \cap f^{-1}(B_j) = f^{-1}(B_i \cap B_j) = f^{-1}(\emptyset) = \emptyset$. Consequently, $\{f^{-1}(B_i) | 1 \leq i \leq n, f^{-1}(B_i) \neq \emptyset\}$ is a partition of A.
Note: Part (b) of Example 7.56 is a special case of this result.

Section 7.5 – p. 376

1. **a)** s_2 and s_5 are equivalent. **b)** s_2 and s_5 are equivalent.
c) s_2 and s_7 are equivalent; s_3 and s_4 are equivalent.

3. **a)** s_1 and s_7 are equivalent; s_4 and s_5 are equivalent.
b) (i) 0 0 0 0
(ii) 0
(iii) 0 0

	ν		ω	
M:	0	1	0	1
s_1	s_4	s_1	1	0
s_2	s_1	s_2	1	0
s_3	s_6	s_1	1	0
s_4	s_3	s_4	0	0
s_6	s_2	s_1	1	0

Supplementary Exercises – p. 378

1. **a)** False. Let $A = \{1, 2\}$, $I = \{1, 2\}$, $\mathcal{R}_1 = \{(1, 1)\}$, and $\mathcal{R}_2 = \{(2, 2)\}$. Then $\bigcup_{i \in I} \mathcal{R}_i$ is reflexive, but neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is reflexive. Conversely, however, if $\mathcal{R}_i$ is reflexive for all (actually, at least one) $i \in I$, then $\bigcup_{i \in I} \mathcal{R}_i$ is reflexive.

3. $(a, c) \in \mathcal{R}_2 \circ \mathcal{R}_1 \Rightarrow$ for some $b \in A$, $(a, b) \in \mathcal{R}_2$, $(b, c) \in \mathcal{R}_1$. With $\mathcal{R}_1, \mathcal{R}_2$ symmetric, $(b, a) \in \mathcal{R}_2$, $(c, b) \in \mathcal{R}_1$, so $(c, a) \in \mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{R}_2 \circ \mathcal{R}_1$. $(c, a) \in \mathcal{R}_2 \circ \mathcal{R}_1 \Rightarrow (c, d) \in \mathcal{R}_2$, $(d, a) \in \mathcal{R}_1$, for some $d \in A$. Then $(d, c) \in \mathcal{R}_2$, $(a, d) \in \mathcal{R}_1$ by symmetry, and $(a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2$, so $\mathcal{R}_2 \circ \mathcal{R}_1 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2$ and the result follows.

5. $(c, a) \in (\mathcal{R}_1 \circ \mathcal{R}_2)^c \Longleftrightarrow (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2 \Longleftrightarrow (a, b) \in \mathcal{R}_1, (b, c) \in \mathcal{R}_2$, for some $b \in B \Longleftrightarrow (b, a) \in \mathcal{R}_1^c, (c, b) \in \mathcal{R}_2^c$, for some $b \in B \Longleftrightarrow (c, a) \in \mathcal{R}_2^c \circ \mathcal{R}_1^c$.

7. Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $A = \mathcal{P}(\mathcal{U}) - \{\mathcal{U}, \emptyset\}$. Under the inclusion relation, A is a poset with the five minimal elements $\{x\}$, $1 \leq x \leq 5$, but no least element. Also, A has five maximal elements — the five subsets of $\mathcal{U}$ of size 4 — but no greatest element.

9. $n = 10$

11. a)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	4	4	5
5	5	5	6
6	3	6	8
7	5		

b)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	5	4	4
5	4	5	5
		6	6

c)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	4	4	6
5	5	5	7
6	1	6	8
7	4		

13. b) The cells of the partition are the connected components of G.

15. One possible order is 10, 3, 8, 6, 7, 9, 1, 4, 5, 2, where program 10 is run first and program 2 last.

17. b) $[(0.3, 0.7)] = \{(0.3, 0.7)\}$ $\quad [(0.5, 0)] = \{(0.5, 0)\}$ $\quad [(0.4, 1)] = \{(0.4, 1)\}$
$[(0, 0.6)] = \{(0, 0.6), (1, 0.6)\}$ $\quad [(1, 0.2)] = \{(0, 0.2), (1, 0.2)\}$
In general, if $0 < a < 1$, then $[(a, b)] = \{(a, b)\}$; otherwise, $[(0, b)] = \{(0, b), (1, b)\} = [(1, b)]$.

c) The lateral surface of a cylinder of height 1 and base radius $1/2\pi$.

19. $4^n - 2(3^n) + 2^n$

21. a) (i) $B \mathcal{R} A \mathcal{R} C$; (ii) $B \mathcal{R} C \mathcal{R} F$
$B \mathcal{R} A \mathcal{R} C \mathcal{R} F$ is a maximal chain. There are six such maximal chains.

b) Here $11 \mathcal{R} 385$ is a maximal chain of length 2, while $2 \mathcal{R} 6 \mathcal{R} 12$ is one of length 3. The length of a longest chain for this poset is 3.

c) (i) $\emptyset \subseteq \{1\} \subseteq \{1, 2\} \subseteq \{1, 2, 3\} \subseteq \mathcal{U}$; (ii) $\emptyset \subseteq \{2\} \subseteq \{2, 3\} \subseteq \{1, 2, 3\} \subseteq \mathcal{U}$
There are $4! = 24$ such maximal chains.

d) $n!$

23. Let $a_1 \mathcal{R} a_2 \mathcal{R} \cdots \mathcal{R} a_{n-1} \mathcal{R} a_n$ be a longest (maximal) chain in $(A, \mathcal{R})$. Then a_n is a maximal element in $(A, \mathcal{R})$ and $a_1 \mathcal{R} a_2 \mathcal{R} \cdots \mathcal{R} a_{n-1}$ is a maximal chain in $(B, \mathcal{R}')$. Hence the length of a longest chain in $(B, \mathcal{R}')$ is at least $n - 1$. If there is a chain $b_1 \mathcal{R}' b_2 \mathcal{R}' \cdots \mathcal{R}' b_n$ in $(B, \mathcal{R}')$ of length n, then this is also a chain of length n in $(A, \mathcal{R})$. But then b_n must be a maximal element of $(A, \mathcal{R})$, and this contradicts $b_n \in B$.

25. If $n = 1$, then for all $x, y \in A$, if $x \neq y$ then $x \not\mathcal{R} y$ and $y \not\mathcal{R} x$. Hence $(A, \mathcal{R})$ is an antichain, and the result follows. Now assume the result true for $n = k \geq 1$, and let $(A, \mathcal{R})$ be a poset where the length of a longest chain is $k + 1$. If M is the set of all maximal elements in $(A, \mathcal{R})$, then $M \neq \emptyset$ and M is an antichain in $(A, \mathcal{R})$. Also, by virtue of Exercise 23, $(A - M, \mathcal{R}')$, for $\mathcal{R}' = ((A - M) \times (A - M)) \cap \mathcal{R}$, is a poset with k the length of a longest chain. So by the induction hypothesis, $A - M = C_1 \cup C_2 \cup \cdots \cup C_k$, a partition into k antichains. Consequently, $A = C_1 \cup C_2 \cup \cdots \cup C_k \cup M$, a partition into $k + 1$ antichains.

27. a) n **b)** 2^{n-1} **c)** 64

Chapter 8
The Principle of Inclusion and Exclusion

Section 8.1 – p. 396

1. Let $x \in S$ and let n be the number of conditions (from among c_1, c_2, c_3, c_4) satisfied by x.

($n = 0$): Here x is counted once in $N(\overline{c}_2\overline{c}_3\overline{c}_4)$ and once in $N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4)$.

($n = 1$): If x satisfies c_1 (and not c_2, c_3, c_4), then x is counted once in $N(\overline{c}_2\overline{c}_3\overline{c}_4)$ and once in $N(c_1\overline{c}_2\overline{c}_3\overline{c}_4)$.