

## RV College of Engineering DEPARTMENT OF MATHEMATICS

Academic year 2023-2024 (Odd Semester 2023)

## DEPARTMENT OF MATHEMATICS

Improvement Test - Scheme and Solution

Course Title: Mathematics for Artificial Intelligence & Machine Learning, Course Code: MAT231ET

No.		Course Title: Mathematics for Artificial Intelligence & Machine Learning, Course Code: MAT231ET			
$ \begin{array}{c} \text{Ia} & \text{Given } V = \{(x,y) x,y \in \mathbb{R}\} \\ \text{Let } \alpha = (x_1,y_1), \beta = (x_2,y_2) \in V \text{ and } c,c' \in \mathbb{Q} \\ \text{(i)}(\alpha + \beta) + \gamma = ((x_1+x_2) + x_3, (y_1+y_2) + y_3) = (x_1+(x_2+x_3), y_1+(y_2+y_3)) = \alpha + (\beta+\gamma), \\ \text{(ii)} \alpha + 0 = (x_1,y_1) + (0,0) = \alpha = 0 + \alpha, \cdot (0,0) \text{ is the zero vector,} \\ \text{(iii)} \alpha + (-\alpha) = (x_1,y_1) + (-x_1,-y_1) = (0,0) = (-\alpha) + \alpha, \cdot -\alpha \text{ is the inverse of } \alpha, \\ \text{(iv)} \alpha + \beta = (x_1+x_2,y_1+y_2) = (x_2+x_1,y_2+y_1) = \beta + \alpha, \\ \text{(v)} c \cdot (\alpha+\beta) = (cx_1+cx_2,cy_1+cy_2) = c \cdot (x_1,y_1) + c \cdot (x_2,y_2) = c \cdot \alpha + c \cdot \beta, \\ \text{(vi)} (c+c') \cdot \alpha = (cx_1+c'x_1,cy_1+c'y_1) = c \cdot (x_1,y_1) + c' \cdot (x_1,y_1) = c \cdot \alpha + c' \cdot \alpha, \\ \text{(vii)} (c+c') \cdot \alpha = (cc'x_1,cc'y_1) = c \cdot (c'x_1,c'y_1) = c \cdot (c'x_1,c'y_1) = c \cdot \alpha + c' \cdot \alpha, \\ \text{(vii)} (c+c') \cdot \alpha = (cc'x_1,cc'y_1) = c \cdot (c'x_1,c'y_1) + c' \cdot (x_1,y_1) = c \cdot \alpha + c' \cdot \alpha, \\ \text{(vii)} (c+c') \cdot \alpha = (cc'x_1+c'x_1,cy_1+c'y_1) = \alpha, \text{ where } 1 \text{ is the unit element.} \\ \cdot V \text{ is a vector space over the field } \mathbb{Q}. \\ 1b  W = \left\{ \begin{bmatrix} a & b \\ b & d \\ b & d \\ b & d \end{bmatrix} \right  \alpha, b, d \in \mathbb{C} \right\}. \text{ Let } \alpha = \begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \\ b, d_1 \end{bmatrix}, \beta = \begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \\ b, d_2 \end{bmatrix} \in W, c \in \mathbb{C} \\ \text{(i)} \alpha + \beta = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ a_1+b_2 \end{bmatrix} \in W, \text{(ii)} c \cdot \alpha = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & cd_1 \end{bmatrix} \in W \\ \text{(ii)} T(A+B) = (A+B)C - C(A+B) = (AC-CA) \\ \text{Let } A, B \in M(\mathbb{R}) \text{ and } C \text{ be a fixed matrix } \in M(\mathbb{R}) \\ \text{(ii)} T(A+B) = (A+B)C - C(A+B) = (AC-CA) + (BC-CB) = T(A) + T(B) \\ \text{(ii)} T(c+A) = (c+A)C - C(c+A) = c \cdot (AC-CA) = c \cdot T(A) \\ \cdot T \text{ is a Linear Transformation.} \\ 2b  Given \{1+2t+t^2,2+2t-t^2,1+3t+2t^2\} = t = L.1, \text{ and they form a basis, dimension of the subspace is 3.} \\ 3  A = \begin{bmatrix} 2 & 2 & 2 \\ A & -2 & 1 \\ 1 & 3 & 2 \\ A & -3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -3 \\ 3 & -3 & -6 \end{bmatrix}, \begin{bmatrix} 2 & A & -2 & 1 \\ 2 & 2 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix} \\ Basis of C(A) = \{(2,-2,1), (4,-5,3), (1,3,5)\} \\ Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_1 = 0 - x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0, \\ n = 4, r = 3, \infty \text{ choose } 1 \text{ free variable. Let } x_3 \text{ be the f.} v. \\ A = 0, x_2 = 5x_3, x_1 = $		Solution	M		
(i) $(\alpha + \beta) + \gamma = ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) = \alpha + (\beta + \gamma),$ (ii) $\alpha + 0 = (x_1, y_1) + (0, 0) = \alpha = 0 + \alpha, \vdots (0, 0)$ is the zero vector, (iii) $\alpha + (-\alpha) = (x_1, y_1) + (-x_1, -y_1) = (0, 0) = (-\alpha) + \alpha, \vdots - \alpha$ is the inverse of $\alpha$ , (iv) $\alpha + \beta = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = \beta + \alpha$ , (v) $(\cdot (\alpha + \beta) = (cx_1 + cx_2, cy_1 + cy_2) = c \cdot (x_1, y_1) + c' \cdot (x_2, y_2) = c \cdot \alpha + c' \cdot \beta,$ (vi) $(c + c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) + c' \cdot (x_1, y_1) = c \cdot \alpha + c' \cdot \alpha,$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1 + c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha),$ (viii) $(c \cdot c') \cdot \alpha = (ccx_1 + c'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c$		Given $V = \{(x, y)   x, y \in \mathbb{R}\}$			
(ii) $\alpha + 0 = (x_1, y_1) + (0, 0) = \alpha = 0 + \alpha, \because (0, 0)$ is the zero vector, (iii) $\alpha + (-\alpha) = (x_1, y_1) + (-x_1, -y_2) = (0, 0) = (-\alpha) + \alpha, \because -\alpha$ is the inverse of $\alpha$ , (iv) $\alpha + \beta = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = \beta + \alpha$ , (v) $(\cdot (\alpha + \beta) = (cx_1 + cx_2, cy_1 + cy_2) = c \cdot (x_1, y_1) + c \cdot (x_2, y_2) = c \cdot \alpha + c \cdot \beta$ , (vi) $(c + c') \cdot \alpha = (cc'x_1, cc'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha)$ , (viii) $(c \cdot c') \cdot \alpha = (cc'x_1, cc'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha)$ , (viii) $(-\alpha - 1) \cdot (x_1, 1, y_1) = (x_1, y_1) = \alpha$ , where 1 is the unit element. $\therefore V$ is a vector space over the field $\mathbb{Q}$ .  1b $W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix}   \alpha, b, d \in \mathbb{C} \right\}$ . Let $\alpha = \begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \end{bmatrix}$ , $\beta = \begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \end{bmatrix} \in W$ , $c \in \mathbb{C}$ (i) $\alpha + \beta = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ b_1 + b_2 & d_1 + d_2 \end{bmatrix} \in W$ , (ii) $c \cdot \alpha = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & cd_1 \end{bmatrix} \in W$ 2 $\frac{a_1}{V} = \frac{a_1}{V} = \frac{a_1}{V} = \frac{a_1}{V} = \frac{a_2}{V} = $		Let $\alpha = (x_1, y_1), \beta = (x_2, y_2) \in V$ and $c, c' \in \mathbb{Q}$	1		
(iii) $a + (-a) = (x_1, y_1) + (-x_1, -y_1) = (0,0) = (-a) + a, \therefore -a$ is the inverse of $a$ , (iv) $a + \beta = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = \beta + a$ , (v) $c \cdot (a + \beta) = (cx_1 + cx_2, cy_1 + cy_2) = c \cdot (x_1, y_1) + c \cdot (x_2, y_2) = c \cdot a + c \cdot \beta$ , (vi) $(c + c') \cdot a = (cx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (x_1, y_1) + c \cdot (x_2, y_2) = c \cdot a + c' \cdot a$ , (viii) $(c \cdot c') \cdot a = (cx_1, c'x_1, cy_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot a)$ , (viii) $1 \cdot a = (1 \cdot x_1, 1 \cdot y_1) = (x_1, y_1) = a$ , where 1 is the unit element. $\therefore V$ is a vector space over the field $\mathbb{Q}$ .  1b $W = \{ \begin{bmatrix} a & b \\ b & d \end{bmatrix}   a, b, d \in \mathbb{C} \}$ . Let $a = \begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \end{bmatrix}, \beta = \begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \end{bmatrix} \in W, c \in \mathbb{C}$ (i) $a + \beta = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ a_1 + b_2 & d_1 + d_2 \end{bmatrix} \in W$ , (ii) $c \cdot a = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & cd_1 \end{bmatrix} \in W$ $\therefore W \text{ is a subspace of } M_{2\times 2}.$ 2a $T \cdot M(\mathbb{R}) \rightarrow M(\mathbb{R})$ defined by $T(A) = AC - CA$ Let $A, B \in M(\mathbb{R})$ and $C$ be a fixed matrix $\in M(\mathbb{R})$ (ii) $T(A + B) = (A + B)C - C(A + B) = (AC - CA) + (BC - CB) = T(A) + T(B)$ (iii) $T(c \cdot A) = (c \cdot A)C - C(c \cdot A) = c \cdot (AC - CA) = c \cdot T(A)$ $\therefore T \text{ is a Linear Transformation.}$ 2b Given $\{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2, 1 - t - 3t^2\}$ Considering the constants, the coefficients of $t$ and $t^2$ in matrix format implies $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} a_2 - 2R_1 & 1 \\ -2 & -2 & 3 \\ -2 & 7 & -4 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 3 \\ -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $\therefore \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2\} \text{ are L.I.}$ , and they form a basis, dimension of the subspace is 3.  3 $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & -6 & 5 \end{bmatrix} \begin{bmatrix} R_2 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_2 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_2 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_2 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_3 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_3 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_3 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} \begin{bmatrix} R_3 + R_1 & 0 & -1 & 5 \\ -2 & -5 & 7 & 3 \end{bmatrix} $		$(i)(\alpha + \beta) + \gamma = ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)) = \alpha + (\beta + \gamma),$			
$ \begin{aligned} & \text{(iii)} \ a + (-a) &= (x_1, y_1) + (-x_1, -y_1) &= (0.0) = (-a) + a, \dots - a \text{ is the inverse of } a, \\ & \text{(iv)} \ a + \beta &= (x_1 + x_2, y_1 + y_2) &= (x_2 + x_1, y_2 + y_1) &= \beta + a, \\ & \text{(v)} \ c \cdot (a + \beta) &= (cx_1 + cx_2, cy_1 + cy_2) &= c \cdot (x_1, y_1) + c \cdot (x_2, y_2) &= c \cdot a + c \cdot \beta, \\ & \text{(vi)} \ (c + c') \cdot a &= (cx_1 + c'x_1, cy_1 + c'y_1) &= c \cdot (x_1, y_1) + c' \cdot (x_1, y_1) &= c \cdot a + c' \cdot a, \\ & \text{(vii)} \ (c \cdot c') \cdot a &= (cc'x_1, cc'y_1) &= c \cdot (c'x_1, c'y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc'x_1, cc'y_1) &= c \cdot (c'x_1, c'y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc'x_1, cc'y_1) &= c \cdot (c'x_1, c'y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y_1) &= c \cdot (c' \cdot a), \\ & \text{(viii)} \ (c \cdot c') \cdot a &= (cc^x_1 + c^x_1, c^y_1) &= c \cdot (c^x_1, c^y$		(ii) $\alpha + 0 = (x_1, y_1) + (0, 0) = \alpha = 0 + \alpha, : (0, 0)$ is the zero vector,	1		
$ \begin{aligned} & \forall v : c \cdot (\alpha + \beta) = (cx_1 + cx_2, cy_1 + cy_2) = c \cdot (x_1, y_1) + c \cdot (x_2, y_2) = c \cdot \alpha + c \cdot \beta, \\ & \forall v : (c + c') \cdot \alpha = (cx_1 + c'x_1, cy_1 + c'y_1) = c \cdot (x_1, y_1) + c' \cdot (x_1, y_1) = c \cdot \alpha + c' \cdot \alpha, \\ & \forall v : (c \cdot c') \cdot \alpha = (cc'x_1, cc'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (v : (c \cdot c') \cdot \alpha) = (cc'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (v : (c \cdot c') \cdot \alpha) = (cc'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (v : (c \cdot c') \cdot \alpha) = (c'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (v : (c \cdot c') \cdot \alpha) = (c'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (v : (c \cdot c') \cdot \alpha) = (c'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (c : (c \cdot c') \cdot \alpha) = (c'x_1, c'y_1) = c \cdot (c'x_1, c'y_1) = c \cdot (c' \cdot \alpha), \\ & \forall v : (c : (c : (c \cdot c') \cdot \alpha) = (c'x_1, c'y_1) = c \cdot (c'x_1, c'x_1) = c \cdot ($		(iii) $\alpha + (-\alpha) = (x_1, y_1) + (-x_1, -y_1) = (0,0) = (-\alpha) + \alpha, : -\alpha$ is the inverse of $\alpha$ ,	1		
		(iv) $\alpha + \beta = (x_1 + x_2, y_1 + y_2) = (x_2 + x_1, y_2 + y_1) = \beta + \alpha$ ,	1		
$ \begin{aligned} & \text{(vi)} & (c + c') \cdot \alpha = (cc' x_1, cc' y_1) = c \cdot (c' x_1, y_1) = c \cdot (c' \cdot \alpha), \\ & \text{(vii)} & 1 \cdot \alpha = (1 \cdot x_1, 1 \cdot y_1) = (x_1, y_1) = a, \text{ where } 1 \text{ is the unit element.} \\ & \therefore V \text{ is a vector space over the field } \mathbb{Q}. \end{aligned} $ $ \begin{aligned} & \text{(vii)} & 1 \cdot \alpha = (1 \cdot x_1, 1 \cdot y_1) = (x_1, y_1) = a, \text{ where } 1 \text{ is the unit element.} \\ & \therefore V \text{ is a vector space over the field } \mathbb{Q}. \end{aligned} $ $ \begin{aligned} & \text{(wii)} & 1 \cdot \alpha = (1 \cdot x_1, 1 \cdot y_1) = (x_1, y_1) = (x_1, y_1) = a, \text{ where } 1 \text{ is the unit element.} \end{aligned} $ $ \begin{aligned} & \cdot V \text{ is a vector space over the field } \mathbb{Q}. \end{aligned} $ $ \begin{aligned} & \text{(wii)} & 1 \cdot \alpha = \left[ \left( \frac{a_1}{a} - \frac{b_1}{b_1} \right) + \frac{b_1}{a_1} \right], \beta = \left[ \frac{a_2}{b_2} - \frac{b_2}{b_2} \right] \in W, c \in \mathbb{C} \end{aligned} $ $ \end{aligned} \end{aligned} $ $ \begin{aligned} & \text{(i)} & \alpha + \beta = \left[ \frac{a_1 + a_2}{b_1 + b_2} - \frac{b_1 + b_2}{d_1 + d_2} \right] \in W, \text{ (ii)} & c \cdot \alpha = \left[ \frac{ca_1}{ca_1} - \frac{cb_1}{cb_1} \right] \in W \end{aligned} $ $ \end{aligned} \end{aligned} \end{aligned} $ $ \end{aligned} \end{aligned} \end{aligned} \end{aligned} \end{aligned} \end{aligned} $ $ \end{aligned} \end{aligned}$					
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$ \begin{array}{l} \text{(viii) } 1 \cdot \alpha = (1 \cdot x_1, 1 \cdot y_1) = (x_1, y_1) = \alpha, \text{ where } 1 \text{ is the unit element.} \\                                   $			1		
$ \begin{array}{ c c c c c } \hline 1b & W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \middle  a, b, d \in \mathbb{C} \right\}. \text{ Let } \alpha = \begin{bmatrix} a_1 & b_1 \\ b_1 & d_1 \end{bmatrix}, \beta = \begin{bmatrix} a_2 & b_2 \\ b_2 & d_2 \end{bmatrix} \in W, c \in \mathbb{C} \\ \hline (i) \alpha + \beta = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ b_1 + b_2 & d_1 + d_2 \end{bmatrix} \in W, \text{ (ii) } c \cdot \alpha = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & cd_1 \end{bmatrix} \in W \\ & \therefore W \text{ is a subspace of } M_{2\times 2}. \\ \hline 2a & T: M(\mathbb{R}) \to M(\mathbb{R}) \text{ defined by } T(A) = AC - CA \\ & \text{Let } A, B \in M(\mathbb{R}) \text{ and } C \text{ be a fixed matrix } \in M(\mathbb{R}) \\ & \text{ (ii) } T(c \cdot A) = (c \cdot A)C - C(A + B) = (AC - CA) + (BC - CB) = T(A) + T(B) \\ & \text{ (iii) } T(c \cdot A) = (c \cdot A)C - C(c \cdot A) = c \cdot (AC - CA) = c \cdot T(A) \\ & \therefore T \text{ is a Linear Transformation.} \\ \hline 2b & \text{Given } \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2, 1 - t - 3t^2 \} \\ & \text{Considering the constants, the coefficients of } t \text{ and } t^2 \text{ in matrix format implies} \\ \hline \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} & R_2 - 2R_1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} & \frac{2R_3 + R_2}{2R_4 - 3R_2} & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \frac{R_4 + R_3}{000000000000000000000000000000000000$			1		
$ \begin{array}{c} \vdots \ W \ \text{is a subspace of } M_{2\times 2} . \\ \hline 2a  T: M(\mathbb{R}) \to M(\mathbb{R}) \ \text{defined by } T(A) = AC - CA \\ \hline \text{Let } A, B \in M(\mathbb{R}) \ \text{and } C \ \text{be a fixed matrix} \in M(\mathbb{R}) \\ \hline \text{(i) } T(A+B) = (A+B)C - C(A+B) = (AC-CA) + (BC-CB) = T(A) + T(B) \\ \hline \text{(ii) } T(c \cdot A) = (c \cdot A)C - C(c \cdot A) = c \cdot (AC-CA) = c \cdot T(A) \\ \hline \vdots \ T \ \text{is a Linear Transformation.} \\ \hline 2b  \text{Given } \{1+2t+t^2,2+2t-t^2,1+3t+2t^2,1-t-3t^2\} \\ \hline \text{Considering the constants, the coefficients of } t \ \text{and } t^2 \ \text{in matrix format implies} \\ \hline \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} & R_2 - 2R_1 & 0 & -2 & -3 \\ R_4 - R_1 & 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} & 2R_3 + R_2 & 0 & 0 & -1 \\ 2R_4 - 3R_2 & 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} & R_4 + R_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ \hline \end{bmatrix} & \frac{1}{2} \\ 1 & -1 & -3 \end{bmatrix} & R_2 - 2R_1 & 0 & -2 & -3 \\ R_4 - R_1 & 0 & -3 & -4 \end{bmatrix} & 2R_3 + R_2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline \end{bmatrix} & \frac{1}{2} & \frac$	1b				
2a $T: M(\mathbb{R}) \to M(\mathbb{R})$ defined by $T(A) = AC - CA$ Let $A, B \in M(\mathbb{R})$ and $C$ be a fixed matrix $\in M(\mathbb{R})$ (i) $T(A + B) = (A + B)C - C(A + B) = (AC - CA) + (BC - CB) = T(A) + T(B)$ (ii) $T(c \cdot A) = (c \cdot A)C - C(c \cdot A) = c \cdot (AC - CA) = c \cdot T(A)$ $\therefore T$ is a Linear Transformation. 2b Considering the constants, the coefficients of $t$ and $t^2$ in matrix format implies $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ R_4 - R_1 & 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{2R_3 + R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 2R_4 - 3R_2 & 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2\} \text{ are L.I, and they form a basis, dimension of the subspace is 3.}$ $3$ $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ Basis of $R(A) = \{(2, 4, -2, 1), (-2, -5, 7, 3), (1, 3, -6, 5)\}$ Basis of $C(A) = \{(2, -2, 1), (4, -5, 3), (1, 3, 5)\}$ $Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0,$ $n = 4, r = 3, \therefore \text{ choose } 1 \text{ free variable. Let } x_3 \text{ be the f.v.}$ $x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9, 5, 1, 0)\}$ $1$ $A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{bmatrix} \xrightarrow{R_3 + 5R_2} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix}$ $A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0$		(i) $\alpha + \beta = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ b_1 + b_2 & d_1 + d_2 \end{bmatrix} \in W$ , (ii) $c \cdot \alpha = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & cd_1 \end{bmatrix} \in W$	$\begin{vmatrix} 2\\2 \end{vmatrix}$		
Let $A, B \in M(\mathbb{R})$ and $C$ be a fixed matrix $\in M(\mathbb{R})$ (i) $T(A+B) = (A+B)C - C(A+B) = (AC-CA) + (BC-CB) = T(A) + T(B)$ (ii) $T(c \cdot A) = (c \cdot A)C - C(c \cdot A) = c \cdot (AC-CA) = c \cdot T(A)$ $\therefore T$ is a Linear Transformation.  2b Given $\{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2, 1 - t - 3t^2\}$ Considering the constants, the coefficients of $t$ and $t^2$ in matrix format implies $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} = \begin{bmatrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 - R_1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 - 2 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} = \begin{bmatrix} 2R_3 + R_2 \\ 2R_4 - 3R_2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ Basis of $R(A) = \{(2, 4, -2, 1), (-2, -5, 7, 3), (1, 3, -6, 5)\}$ Basis of $R(A) = \{(2, 4, -2, 1), (4, -5, 3), (1, 3, 5)\}$ $Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0,$ $R = \{1, 1, 2, 3, 3, 3, 4, 4, 4, 5, 3, 4, 4, 4, 5, 3, 4, 4, 4, 5, 3, 4, 4, 4, 5, 3, 4, 4, 4, 5, 4, 5, 4, 5, 4, 5, 4, 5, 4, 5, 4, 5, 4, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5,$		$\therefore W$ is a subspace of $M_{2\times 2}$ .			
(i) $T(A+B) = (A+B)C - C(A+B) = (AC-CA) + (BC-CB) = T(A) + T(B)$ (ii) $T(c \cdot A) = (c \cdot A)C - C(c \cdot A) = c \cdot (AC-CA) = c \cdot T(A)$ $\therefore T$ is a Linear Transformation. 2b Given $\{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2, 1 - t - 3t^2\}$ Considering the constants, the coefficients of $t$ and $t^2$ in matrix format implies $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -4 \end{bmatrix} \xrightarrow{R_4 - 3R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_4 + R_3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2\} \text{ are L.I, and they form a basis, dimension of the subspace is 3.}$ $3 = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \xrightarrow{R_3 + 2R_2} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ Basis of $R(A) = \{(2, 4, -2, 1), (-2, -5, 7, 3), (1, 3, -6, 5)\}$ Basis of $C(A) = \{(2, -2, 1), (4, -5, 3), (1, 3, 5)\}$ $Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0,$ $n = 4, r = 3, \therefore \text{ choose 1 free variable. Let } x_3 \text{ be the f.v.}$ $x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9, 5, 1, 0)\}$ $A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} \xrightarrow{R_2 + R_1} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \end{bmatrix} \xrightarrow{R_3 + 5R_2} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0$	2a				
$\begin{array}{c} \text{(ii) } T(c \cdot A) = (c \cdot A)L - L(c \cdot A) = c \cdot (AL - LA) = c \cdot T(A) \\ \vdots T \text{ is a Linear Transformation.} \\ \text{2b}  \text{Given } \{1 + 2t + t^2, 2 + 2t - t^2, \ 1 + 3t + 2t^2, 1 - t - 3t^2 \} \\ \text{Considering the constants, the coefficients of } t \text{ and } t^2 \text{ in matrix format implies} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} & R_2 - 2R_1 & 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} & 2R_3 + R_2 & 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} & 2R_4 + R_3 & 1 & 2 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \cdot \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2\} \text{ are L.I, and they form a basis, dimension of the subspace is 3.} \\ \\ 3 & A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} & R_2 + R_1 & 2 & 4 & -2 & 1 \\ 2R_3 - R_1 & 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} & 2R_3 + 2R_2 & 2 & 4 & -2 & 1 \\ 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix} \\ & \text{Basis of } R(A) = \{(2, 4, -2, 1), (-2, -5, 7, 3), (1, 3, -6, 5)\} \\ & \text{Basis of } C(A) = \{(2, -2, 1), (4, -5, 3), (1, 3, 5)\} \\ & Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0, \\ & n = 4, r = 3, \text{ $c$ choose 1 free variable. Let $x_3$ be the f.v.} \\ & x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9, 5, 1, 0)\} \\ & A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} R_2 - 2R_1 & 2C - 2 & 1 \\ 0 & -1 & 1 & 3 \\ 2R_4 - R_1 & 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} R_3 + 5R_2 & 2C - 2 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 & 17$			$\frac{2}{2}$		
2b Given $\{1+2t+t^2,2+2t-t^2,1+3t+2t^2,1-t-3t^2\}$ Considering the constants, the coefficients of $t$ and $t^2$ in matrix format implies $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2R_3 + R_2 \\ 2R_4 - 3R_2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 1 \end{bmatrix} \sim R_4 + R_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore \{1+2t+t^2,2+2t-t^2,1+3t+2t^2\}$ are L.I, and they form a basis, dimension of the subspace is 3. $\begin{bmatrix} 3 & A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} R_2 + R_1 \\ 2R_3 - R_1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \sim R_3 + 2R_2 \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ Basis of $R(A) = \{(2,4,-2,1),(-2,-5,7,3),(1,3,-6,5)\}$ Basis of $C(A) = \{(2,-2,1),(4,-5,3),(1,3,5)\}$ Ax = 0 \(\Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0,17x_4 = 0, n = 4, r = 3, \(\cdot \cdot chose 1 \) free variable. Let $x_3$ be the f.v. $x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9,5,1,0)\}$ $A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} R_2 - 2R_1 R_3 + R_1 \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & 5 \\ 0 & 8 & 9 \end{bmatrix} R_3 + 5R_2 \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix}$ 2			-		
Considering the constants, the coefficients of $t$ and $t^2$ in matrix format implies $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \sim \begin{bmatrix} 2R_3 + R_2 \\ 2R_4 - 3R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} R_4 + R_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} $ $\therefore \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2\}$ are L.I, and they form a basis, dimension of the subspace is 3. $\begin{bmatrix} 3 & A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} R_2 + R_1 \\ 2R_3 - R_1 \\ 0 & 2 & -10 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ Basis of $R(A) = \{(2, 4, -2, 1), (-2, -5, 7, 3), (1, 3, -6, 5)\}$ Basis of $C(A) = \{(2, -2, 1), (4, -5, 3), (1, 3, 5)\}$ $Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0,$ $n = 4, r = 3, \therefore \text{ choose } 1 \text{ free variable. Let } x_3 \text{ be the f.v.}$ $x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9, 5, 1, 0)\}$ $A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} R_3 + 5R_2 \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix}$ $A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0$	01				
$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & -3 \end{bmatrix} \sim R_2 - 2R_1 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \end{bmatrix} \sim 2R_3 + R_2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 2R_4 - 3R_2 \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim R_4 + R_3 \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\therefore \{1 + 2t + t^2, 2 + 2t - t^2, 1 + 3t + 2t^2\} \text{ are L.I, and they form a basis, dimension of the subspace is 3.}$ $3  A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \sim R_2 + R_1 \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \sim R_3 + 2R_2 \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix}$ $Basis of R(A) = \{(2, 4, -2, 1), (-2, -5, 7, 3), (1, 3, -6, 5)\}$ $Basis of C(A) = \{(2, -2, 1), (4, -5, 3), (1, 3, 5)\}$ $Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0,$ $n = 4, r = 3, \therefore \text{ choose 1 free variable. Let } x_3 \text{ be the f.v.}$ $x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9, 5, 1, 0)\}$ $A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} R_3 + 5R_2 \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix}$ $A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0$	2b				
$ \begin{array}{c} \therefore \{1+2t+t^2,2+2t-t^2,1+3t+2t^2\} \text{ are L.I, and they form a basis, dimension of the} \\ \text{subspace is 3.} \\ \hline \\ 3 \\ A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} R_2 + R_1 \\ 2R_3 - R_1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \sim R_3 + 2R_2 \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix} \\ \text{Basis of } R(A) = \{(2,4,-2,1),(-2,-5,7,3),(1,3,-6,5)\} \\ \text{Basis of } C(A) = \{(2,-2,1),(4,-5,3),(1,3,5)\} \\ Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0, \\ n = 4, r = 3, \therefore \text{ choose 1 free variable. Let } x_3 \text{ be the f.v.} \\ x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9,5,1,0)\} \\ A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} R_2 - 2R_1 \\ R_3 + R_1 \\ 2R_4 - R_1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} \begin{bmatrix} R_3 + 5R_2 \\ R_4 + 8R_2 \\ R_4 + 8R_2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix} \\ A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0 \\ \end{array}$		<u> </u>			
$ \begin{array}{c} \therefore \{1+2t+t^2,2+2t-t^2,1+3t+2t^2\} \text{ are L.I, and they form a basis, dimension of the} \\ \text{subspace is 3.} \\ \hline \\ 3 \\ A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} R_2 + R_1 \\ 2R_3 - R_1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \sim R_3 + 2R_2 \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix} \\ \text{Basis of } R(A) = \{(2,4,-2,1),(-2,-5,7,3),(1,3,-6,5)\} \\ \text{Basis of } C(A) = \{(2,-2,1),(4,-5,3),(1,3,5)\} \\ Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0, \\ n = 4, r = 3, \therefore \text{ choose 1 free variable. Let } x_3 \text{ be the f.v.} \\ x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9,5,1,0)\} \\ A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} R_2 - 2R_1 \\ R_3 + R_1 \\ 2R_4 - R_1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} \begin{bmatrix} R_3 + 5R_2 \\ R_4 + 8R_2 \\ R_4 + 8R_2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix} \\ A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0 \\ \end{array}$		$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \end{bmatrix}$ $\begin{bmatrix} R_2 - 2R_1 \\ R_2 - 2R_1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \end{bmatrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \end{bmatrix}$			
$ \begin{array}{c} \therefore \{1+2t+t^2,2+2t-t^2,1+3t+2t^2\} \text{ are L.I, and they form a basis, dimension of the} \\ \text{subspace is 3.} \\ \hline \\ 3 \\ A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 1 & 3 & -6 & 5 \end{bmatrix} \sim \begin{bmatrix} R_2 + R_1 \\ 2R_3 - R_1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 2 & -10 & 9 \end{bmatrix} \sim R_3 + 2R_2 \begin{bmatrix} 2 & 4 & -2 & 1 \\ 0 & -1 & 5 & 4 \\ 0 & 0 & 0 & 17 \end{bmatrix} \\ \text{Basis of } R(A) = \{(2,4,-2,1),(-2,-5,7,3),(1,3,-6,5)\} \\ \text{Basis of } C(A) = \{(2,-2,1),(4,-5,3),(1,3,5)\} \\ Ax = 0 \Rightarrow 2x_1 + 4x_2 - 2x_3 + x_4 = 0, -x_2 + 5x_3 + 4x_4 = 0, 17x_4 = 0, \\ n = 4, r = 3, \therefore \text{ choose 1 free variable. Let } x_3 \text{ be the f.v.} \\ x_4 = 0, x_2 = 5x_3, x_1 = -9x_3, \text{ Hence basis of } N(A) = \{(-9,5,1,0)\} \\ A^T = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} R_2 - 2R_1 \\ R_3 + R_1 \\ 2R_4 - R_1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} \begin{bmatrix} R_3 + 5R_2 \\ R_4 + 8R_2 \\ R_4 + 8R_2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix} \\ A^T y = 0 \Rightarrow 2y_1 - 2y_2 + y_3 = 0, -y_2 + y_3 = 0, 17y_3 = 0 \Rightarrow y_3 = 0, y_2 = 0, y_1 = 0 \\ \end{array}$		$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \sim \frac{R_3 - R_1}{R_2 - R_1} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \sim 2R_4 - 3R_2 \begin{bmatrix} 0 & 0 & -1 \end{bmatrix} \sim R_4 + R_3 \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$	3		
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		3			
$A^{T} = \begin{bmatrix} 2 & -2 & 1 \\ 4 & -5 & 3 \\ -2 & 7 & -6 \\ 1 & 3 & 5 \end{bmatrix} R_{2} - 2R_{1} \\ R_{3} + R_{1} \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 5 & -5 \\ 0 & 8 & 9 \end{bmatrix} R_{3} + 5R_{2} \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \end{bmatrix}$ $A^{T}y = 0 \Rightarrow 2y_{1} - 2y_{2} + y_{3} = 0, -y_{2} + y_{3} = 0, 17y_{3} = 0 \Rightarrow y_{3} = 0, y_{2} = 0, y_{1} = 0$			1		
$A^{T}y = 0 \Rightarrow 2y_{1} - 2y_{2} + y_{3} = 0, -y_{2} + y_{3} = 0, 17y_{3} = 0 \Rightarrow y_{3} = 0, y_{2} = 0, y_{1} = 0$		$\begin{bmatrix} 2 & -2 & 1 \\ 4 & 5 & 2 \end{bmatrix} R_2 - 2R_1 \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_2 + 5R_2 \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$			
$A^{T}y = 0 \Rightarrow 2y_{1} - 2y_{2} + y_{3} = 0, -y_{2} + y_{3} = 0, 17y_{3} = 0 \Rightarrow y_{3} = 0, y_{2} = 0, y_{1} = 0$		$\begin{vmatrix} A^T = \begin{vmatrix} 4 & -5 & 3 \\ -2 & 7 & -6 \end{vmatrix} R_3 + R_1 \sim \begin{vmatrix} 0 & -1 & 1 \\ 0 & 5 & -5 \end{vmatrix} R_4 + 8R_2 \sim \begin{vmatrix} 0 & -1 & 1 \\ 0 & 0 & 17 \end{vmatrix}$			
		$\begin{bmatrix} 2 & 7 & 6 \\ 1 & 3 & 5 \end{bmatrix} 2R_4 - R_1 \begin{bmatrix} 6 & 3 & 3 \\ 0 & 8 & 9 \end{bmatrix} \begin{bmatrix} R_4 + 6R_2 \\ 0 & 0 & 0 \end{bmatrix}$	2		
		$A^{T}y = 0 \Rightarrow 2y_{1} - 2y_{2} + y_{3} = 0, -y_{2} + y_{3} = 0, 17y_{3} = 0 \Rightarrow y_{3} = 0, y_{2} = 0, y_{1} = 0$	1		
i i i i i i i i i i i i i i i i i i i			1		

## Academic year 2023-2024 (Odd Semester 2023)

4a	Given $T(1,2,1) = (7,0,3,5)$ , $T(2,1,1) = (5,2,2,3)T(-1,2,1) = (5,-4,3,5)$ ,	
	$T = \begin{bmatrix} 7 & 5 & 5 \\ 0 & 2 & -4 \\ 3 & 2 & 3 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 5 & 5 \\ 0 & 2 & -4 \\ 3 & 2 & 3 \\ 5 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1/2 & 3/2 & -5/2 \\ 0 & -1 & 2 \\ -1/2 & -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$	1 1 1
	$\therefore T(x, y, z) = (x + 3y, 2x - 2z, y + z, 2y + z)$	1
	$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} R_2 - 2R_1 \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & -2 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} 6R_3 + R_2 \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} 4R_4 - R_3 \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & -6 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ Basis of $R(T) = \{(1, 2, 0, 0), (3, 0, 1, 2), (0, -2, 1, 1)\}$	1 1
4b	$u_1 = (2, 5, -1), u_2 = (-2, 1, 1) \text{ and } y = (1, 2, 3)$	
40	$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 = \frac{9}{30} (2, 5, -1) + \frac{3}{6} (-2, 1, 1) = \left(\frac{-4}{10}, \frac{20}{10}, \frac{2}{10}\right) = \left(\frac{-2}{5}, \frac{10}{5}, \frac{1}{5}\right)$ $z = y - \hat{y} = \left(\frac{7}{5}, 0, \frac{14}{5}\right), \text{ distance } = \sqrt{9.8} = 3.13$	2 1 1
5	$\begin{bmatrix} 1 & -2 & -2 \end{bmatrix} \qquad u_1 = x_1 = (1, 2, -2, 1, 2)$	2
	$\begin{bmatrix} 1 & -2 & -2 \\ 2 & -4 & 1 \\ -2 & 0 & 4 \\ 1 & 0 & -2 \\ 2 & -2 & -2 \end{bmatrix}, x_1 = (1, 2, -2, 1, 2)  x_2 = (-2, -4, 0, 0, -2), u_2 = x_2 - \frac{x_2 \cdot u_1}{u_1 \cdot u_1} u_1 = (-2, -4, 0, 0, -2) - \frac{(-14)}{14} (1, 2, -2, 1, 2)  u_2 = (-1, -2, -2, 1, 0)$	2
	$u_3 = x_3 - \frac{x_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{x_3 \cdot u_2}{u_2 \cdot u_2} u_2 = (-2,1,4,-2,-2) - \frac{(-14)}{14} (1,2,-2,1,2) - \frac{(-10)}{10} (-1,-2,-2,1,0)$	2
	$u_3 = (-2, 1, 2, -1, 0)$	1
	$\begin{bmatrix} 1/\sqrt{14} & -1/\sqrt{10} & -2/\sqrt{5} \end{bmatrix}$	1
	$Q = \begin{bmatrix} 2/\sqrt{14} & -2/\sqrt{10} & 1/\sqrt{5} \\ -2/\sqrt{14} & -2/\sqrt{10} & 0 \\ 1/\sqrt{14} & 1/\sqrt{10} & 0 \\ 0 & 0 & \sqrt{5} \end{bmatrix}, R = Q^T A = \begin{bmatrix} \sqrt{14} & -\sqrt{14} & -\sqrt{14} \\ 0 & \sqrt{10} & -\sqrt{10} \\ 0 & 0 & \sqrt{5} \end{bmatrix}$	1
	$\begin{bmatrix} 2 - \begin{bmatrix} -2/\sqrt{14} & -2/\sqrt{10} & 0 & 0 & 0 & \sqrt{5} \\ 1/\sqrt{14} & 1/\sqrt{10} & 0 & 0 & 0 & \sqrt{5} \end{bmatrix}$	2
	$\begin{bmatrix} 1/\sqrt{14} & 1/\sqrt{10} & 0 \\ 2/\sqrt{14} & 0 & 0 \end{bmatrix}$	