CHAPTER 7
RELATIONS: THE SECOND TIME AROUND

Section 7.1

1. (a) {(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)}
(b) {(1,1),(2,2),(3,3),(4,4),(1,2)}
(c) {(1,1),(2,2),(1,2),(2,1)}

2. $-9, -2, 5, 12, 19$

3. (a) Let $f_1, f_2, f_3 \in F$ with $f_1(n) = n + 1$, $f_2(n) = 5n$, and $f_3(n) = 4n + 1/n$.
(b) Let $g_1, g_2, g_3 \in F$ with $g_1(n) = 3$, $g_2(n) = 1/n$, and $f_3(n) = \sin n$.

4. (a) The relation $\mathcal{R}$ on the set A is
(i) reflexive if $\forall x \in A\ (x, x) \in \mathcal{R}$
(ii) symmetric if $\forall x, y \in A\ [(x, y) \in \mathcal{R} \Longrightarrow (y, x) \in \mathcal{R}]$
(iii) transitive if $\forall x, y, z \in A\ [(x, y), (y, z) \in \mathcal{R} \Longrightarrow (x, z) \in \mathcal{R}]$
(iv) antisymmetric if $\forall x, y \in A\ [(x, y), (y, x) \in \mathcal{R} \Longrightarrow x = y]$.

(b) The relation $\mathcal{R}$ on the set A is
(i) not reflexive if $\exists x \in A\ (x, x) \notin \mathcal{R}$
(ii) not symmetric if $\exists x, y \in A\ [(x, y) \in \mathcal{R} \wedge (y, x) \notin \mathcal{R}]$
(iii) not transitive if $\exists x, y, z \in A\ [(x, y), (y, z) \in \mathcal{R} \wedge (x, z) \notin \mathcal{R}]$
(iv) not antisymmetric if $\exists x, y \in A\ [(x, y), (y, x) \in \mathcal{R} \wedge x \neq y]$.

5. (a) reflexive, antisymmetric, transitive
(b) transitive
(c) reflexive, symmetric, transitive
(d) symmetric
(e) (odd): symmetric
(f) (even): reflexive, symmetric, transitive
(g) reflexive, symmetric
(h) reflexive, transitive

6. The relation in part (a) is a partial order. The relations in parts (c) and (f) are equivalence relations.

7. (a) For all $x \in A, (x, x) \in \mathcal{R}_1, \mathcal{R}_2$, so $(x, x) \in \mathcal{R}_1 \cap \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$ is reflexive.

(b) All of these results are true. For example if $\mathcal{R}_1, \mathcal{R}_2$ are both transitive and $(x,y),(y,z) \in \mathcal{R}_1 \cap \mathcal{R}_2$ then $(x,y),(y,z) \in \mathcal{R}_1, \mathcal{R}_2$, so $(x,z) \in \mathcal{R}_1, \mathcal{R}_2$ (transitive property) and $(x,z) \in \mathcal{R}_1 \cap \mathcal{R}_2$. [The proofs for the symmetric and antisymmetric properties are similar.]

8. (a) For all $x \in A, (x,x) \in \mathcal{R}_1, \mathcal{R}_2 \subseteq \mathcal{R}_1 \cup \mathcal{R}_2$, so if either $\mathcal{R}_1$ or $\mathcal{R}_2$ is reflexive, then $\mathcal{R}_1 \cup \mathcal{R}_2$ is reflexive.

(b) (i) If $x, y \in A$ and $(x,y) \in \mathcal{R}_1 \cup \mathcal{R}_2$, assume without loss of generality, that $(x,y) \in \mathcal{R}_1$. $(x,y) \in \mathcal{R}_1$ and $\mathcal{R}_1$ symmetric $\Longrightarrow (y,x) \in \mathcal{R}_1 \Longrightarrow (y,x) \in \mathcal{R}_1 \cup \mathcal{R}_2$, so $\mathcal{R}_1 \cup \mathcal{R}_2$ is symmetric.

(ii) False: Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1),(1,2)\}, \mathcal{R}_2 = \{(2,1)\}$. Then $(1,2),(2,1) \in \mathcal{R}_1 \cup \mathcal{R}_2$, and $1 \neq 2$, so $\mathcal{R}_1 \cup \mathcal{R}_2$ is not antisymmetric.

(iii) False: Let $A = \{1,2,3\}, \mathcal{R}_1 = \{(1,1),(1,2)\}, \mathcal{R}_2 = \{(2,3)\}$. Then $(1,2),(2,3) \in \mathcal{R}_1 \cup \mathcal{R}_2$, but $(1,3) \notin \mathcal{R}_1 \cup \mathcal{R}_2$, so $\mathcal{R}_1 \cup \mathcal{R}_2$ is not transitive.

9.

(a) False: Let $A = \{1,2\}$ and $\mathcal{R} = \{(1,2),(2,1)\}$.

(b)
- (i) Reflexive: True
- (ii) Symmetric: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(1,1),(1,2)\}$.
- (iii) Antisymmetric & Transitive: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}$, $\mathcal{R}_2 = \{(1,2),(2,1)\}$.

(c)
- (i) Reflexive: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(1,1),(2,2)\}$.
- (ii) Symmetric: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}, \mathcal{R}_2 = \{(1,2),(2,1)\}$.
- (iii) Antisymmetric: True
- (iv) Transitive: False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2),(2,1)\}, \mathcal{R}_2 = \{(1,1),(1,2),(2,1),(2,2)\}$.

(d) True

10.

(a) 2^{12}	(b) $(2^4)(2^6) = 2^{10}$	(c) 2^6
(d) 2^{11}	(e) $(2^4)(2^5) = 2^9$	(f) $2^4 \cdot 3^6$
(g) $2^4 \cdot 3^5$	(h) (2^4)	(i) 1

11. (a) $\binom{2+2-1}{2}\binom{2+2-1}{2} = \binom{3}{2}\binom{3}{2} = 9$

(b) $\binom{3+2-1}{2}\binom{2+2-1}{2} = \binom{4}{2}\binom{3}{2} = 18$

(c) $\binom{4+2-1}{2}\binom{2+2-1}{2} = \binom{5}{2}\binom{3}{2} = 30$

(d) $\binom{4+2-1}{2}\binom{3+2-1}{2} = \binom{5}{2}\binom{4}{2} = 60$

(e) $\binom{2+2-1}{2}^4 = \binom{3}{2}^4 = 3^4 = 81$

(f) Since $13,860 = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, it follows that $\mathcal{R}$ contains $\binom{3+2-1}{2}^2\binom{2+2-1}{2}^3 = \binom{4}{2}^2\binom{3}{2}^3 = (36)(27) = 972$ ordered pairs.

12.
$$\begin{aligned}\text{Since } 5880 &= \binom{6+2-1}{2}\binom{4+2-1}{2}\binom{(k+1)+2-1}{2}\\ &= \binom{7}{2}\binom{5}{2}\binom{k+2}{2} = (21)(10)(\tfrac{1}{2})(k+2)(k+1),\\ \text{we find that } 56 &= (k+2)(k+1) \text{ and } k=6.\end{aligned}$$

For $n = p_1^5 p_2^3 p_3^6$ there are $(5+1)(3+1)(6+1) = (6)(4)(7) = 168$ positive integer divisors, so $|A| = 168$.

13. There may exist an element $a \in A$ such that for all $b \in B$ neither (a, b) nor $(b, a) \in \mathcal{R}$.

14. There are n ordered pairs of the form $(x, x), x \in A$. For each of the $(n^2 - n)/2$ sets $\{(x, y), (y, x)\}$ of ordered pairs where $x, y \in A, x \neq y$, one element is chosen. This results in a maximum value of $n + (n^2 - n)/2 = (n^2 + n)/2$.

The number of antisymmetric relations that can have this size is $2^{(n^2-n)/2}$.

15. $r - n$ counts the elements in $\mathcal{R}$ of the form $(a, b), a \neq b$. Since $\mathcal{R}$ is symmetric, $r - n$ is even.

16. (a) $x\mathcal{R}y$ if $x < y$.

(b) For example, suppose that $\mathcal{R}$ satisfies conditions (ii) and (iii). Since $\mathcal{R} \neq \emptyset$, let $(x, y) \in \mathcal{R}$, for $x, y \in A$. Since $\mathcal{R}$ is symmetric, it follows that $(y, x) \in \mathcal{R}$. Then by the transitive property we have $(x, x) \in \mathcal{R}$ (and $(y, y) \in \mathcal{R}$). But if $(x, x) \in \mathcal{R}$ the relation $\mathcal{R}$ is *not* irreflexive.

(c) $2^{(n^2-n)}$; $2^{n^2} - 2(2^{(n^2-n)})$

17. (a) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{0}\binom{21}{2}$

(b) $\binom{7}{5}\binom{21}{0} + \binom{7}{3}\binom{21}{1} + \binom{7}{1}\binom{21}{2}$

18. (a) Let $A_1 = f^{-1}(x)$, $A_2 = f^{-1}(y)$, and $A_3 = f^{-1}(z)$. Then $\mathcal{R} = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3)$, so $|\mathcal{R}| = 10^2 + 10^2 + 5^2 = 225$.

(b) $n_1^2 + n_2^2 + n_3^2 + n_4^2$

Section 7.2

1. $\mathcal{R} \circ \mathcal{S} = \{(1,3), (1,4)\}$; $\mathcal{S} \circ \mathcal{R} = \{(1,2), (1,3), (1,4), (2,4)\}$;
$\mathcal{R}^2 = \mathcal{R}^3 = \{(1,4), (2,4), (4,4)\}$;
$\mathcal{S}^2 = \mathcal{S}^3 = \{(1,1), (1,2), (1,3), (1,4)\}$.

2. Let $x \in A$. $\mathcal{R}$ reflexive $\Longrightarrow (x, x) \in \mathcal{R}$. $(x, x) \in \mathcal{R}, (x, x) \in \mathcal{R} \Longrightarrow (x, x) \in \mathcal{R} \circ \mathcal{R} = \mathcal{R}^2$.

3. $(a, d) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \Longrightarrow (a, c) \in \mathcal{R}_1 \circ \mathcal{R}_2, (c, d) \in \mathcal{R}_3$ for some $c \in C \Longrightarrow (a, b) \in \mathcal{R}_1, (b, c) \in \mathcal{R}_2, (c, d) \in \mathcal{R}_3$ for some $b \in B, c \in C \Longrightarrow (a, b) \in \mathcal{R}_1, (b, d) \in \mathcal{R}_2 \circ \mathcal{R}_3 \Longrightarrow (a, d) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$, and $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ \mathcal{R}_3 \subseteq \mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3)$.

4. (a) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) = \mathcal{R}_1 \circ \{(w,4),(w,5),(x,6),(y,4),(y,5),(y,6)\}$
$= \{(1,4),(1,5),(3,4),(3,5),(2,6),(1,6)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$
$= \{(1,5),(3,5),(2,6),(1,4),(1,6)\} \cup \{(1,4),(1,5),(3,4),(3,5)\}$
$= \{(1,4),(1,5),(1,6),(2,6),(3,4),(3,5)\}$

(b) $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = \mathcal{R}_1 \circ \{(w,5)\} = \{(1,5),(3,5)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,5),(3,5),(2,6),(1,4),(1,6)\} \cap \{(1,4),(1,5),(3,4),(3,5)\} = \{(1,4),(1,5),(3,5)\}$.

5. $\mathcal{R}_1 \circ (\mathcal{R}_2 \cap \mathcal{R}_3) = \mathcal{R}_2 \circ \{(m,3),(m,4)\} = \{(1,3),(1,4)\}$
$(\mathcal{R}_1 \circ \mathcal{R}_2) \cap (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,3),(1,4)\} \cap \{(1,3),(1,4)\} = \{(1,3),(1,4)\}$.

6. (a) $(x,z) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3) \iff$ for some $y \in B, (x,y) \in \mathcal{R}_1, (y,z) \in \mathcal{R}_2 \cup \mathcal{R}_3 \iff$ for some $y \in B, ((x,y) \in \mathcal{R}_1, (y,z) \in \mathcal{R}_2)$ or $((x,y) \in \mathcal{R}_1, (y,z) \in \mathcal{R}_3) \Longrightarrow (x,z) \in \mathcal{R}_1 \circ \mathcal{R}_2$ or $(x,z) \in \mathcal{R}_1 \circ \mathcal{R}_3 \iff (x,z) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$, so $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) \subseteq (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3)$. For the opposite inclusion, $(x,z) \in (\mathcal{R}_1 \circ \mathcal{R}_2) \cup (\mathcal{R}_1 \circ \mathcal{R}_3) \Longrightarrow (x,z) \in \mathcal{R}_1 \circ \mathcal{R}_2$ or $(x,z) \in \mathcal{R}_1 \circ \mathcal{R}_3$. Assume without loss of generality that $(x,z) \in \mathcal{R}_1 \circ \mathcal{R}_2$. Then there exists an element $y \in B$ so that $(x,y) \in \mathcal{R}_1$ and $(y,z) \in \mathcal{R}_2$. But $(y,z) \in \mathcal{R}_2 \Longrightarrow (y,z) \in \mathcal{R}_2 \cup \mathcal{R}_3$, so $(x,z) \in \mathcal{R}_1 \circ (\mathcal{R}_2 \cup \mathcal{R}_3)$, and the result follows.

(b) The proof here is similar to that in part (a). To show that the inclusion can be proper, let $A = B = C = \{1,2,3\}$ with $\mathcal{R}_1 = \{(1,2),(1,1)\}, \mathcal{R}_2 = \{(2,3)\}, \mathcal{R}_3 = \{(1,3)\}$. Then $\mathcal{R}_1 \circ (\mathcal{R}_2 \circ \mathcal{R}_3) = \mathcal{R}_1 \circ \emptyset = \emptyset$, but $(\mathcal{R}_1 \circ \mathcal{R}_2) \circ (\mathcal{R}_1 \circ \mathcal{R}_3) = \{(1,3)\}$.

7. This follows by the Pigeonhole Principle. Here the pigeons are the $2^{n^2}+1$ integers between 0 and 2^{n^2}, inclusive, and the pigeonholes are the 2^{n^2} relations on A.

8. Let $S = \{(1,1),(1,2),(1,4)\}$ and $T = \{(2,1),(2,2),(1,4)\}$.

9. Here there are two choices for each $a_{ii}, 1 \leq i < 6$. For each pair $a_{ij}, a_{ji}, 1 \leq i < j \leq 6$, there are two choices, and there are $(36-6)/2 = 15$ such pairs. Consequently there are $(2^6)(2^{15}) = 2^{21}$ such matrices.

10. For each 0 in E the matrix F can have either 0 or 1 (the other entries in F are 1). Since there are seven 0's in E there are 2^7 possible matrices F. There are 2^5 possible matrices G.

11. Consider the entry in the i-th row and j-th column of $M(\mathcal{R}_1 \circ \mathcal{R}_2)$. If this entry is a 1 then there exists $b_k \in B$ where $1 \leq k \leq n$ and $(a_i, b_k) \in \mathcal{R}_1, (b_k, c_j) \in \mathcal{R}_2$. Consequently, the entry in the i-th row and k-th column of $M(\mathcal{R}_1)$ is 1 and the entry in the k-th row and j-th column of $M(\mathcal{R}_2)$ is 1. This results in a 1 in the i-th row and j-th column in the product $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$.

Should the entry in row i and column j of $M(\mathcal{R}_1 \circ \mathcal{R}_2)$ be 0, then for each $b_k, 1 \leq k \leq n$, either $(a_i, b_k) \notin \mathcal{R}_1$ or $(b_k, c_j) \notin \mathcal{R}_2$. This means that in the matrices $M(\mathcal{R}_1), M(\mathcal{R}_2)$, if the entry in the i-th row and k-th column of $M(\mathcal{R}_1)$ is 1 then the entry in the k-th row and j-th column of $M(\mathcal{R}_2)$ is 0. Hence the entry in the i-th row and j-th column of $M(\mathcal{R}_1) \cdot M(\mathcal{R}_2)$ is 0.

12. (a) If $M(\mathcal{R}) = \mathbf{0}$, then $\forall x, y \in A \ \ (x,y) \notin \mathcal{R}$. Hence $\mathcal{R} = \emptyset$. Conversely, if $M(\mathcal{R}) \neq \mathbf{0}$, then $\exists x, y \in A$ where $x\mathcal{R}y$. Hence $(x,y) \in \mathcal{R}$ and $\mathcal{R} \neq \emptyset$.

(c) For $m = 1$, we have $M(\mathcal{R}^1) = M(\mathcal{R}) = [M(\mathcal{R})]^1$, so the result is true in this case. Assuming the truth of the statement for $m = k$ we have $M(\mathcal{R}^k) = [M(\mathcal{R})]^k$. Now consider $m = k+1$. $M(\mathcal{R}^{k+1}) = M(\mathcal{R} \circ \mathcal{R}^k) = M(\mathcal{R}) \cdot M(\mathcal{R}^k)$ (from Exercise 11) $= M(\mathcal{R}) \cdot [M(\mathcal{R})]^k = [M(\mathcal{R})]^{k+1}$. Consequently this result is true for all $m \geq 1$ by the Principle of Mathematical Induction.

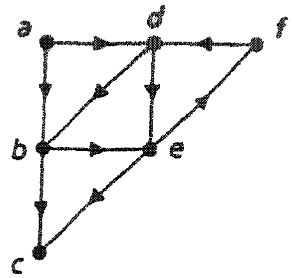
13. (a) $\mathcal{R}$ reflexive $\iff (x,x) \in \mathcal{R}$, for all $x \in A \iff m_{xx} = 1$ in $M = (m_{ij})_{n \times n}$, for all $x \in A \iff I_n \leq M$.

(b) $\mathcal{R}$ symmetric $\iff [\forall x, y \in A \ \ (x,y) \in \mathcal{R} \Longrightarrow (y,x) \in \mathcal{R}] \iff [\forall x, y \in A \ \ m_{xy} = 1$ in $M \Longrightarrow m_{yx} = 1$ in $M] \iff M = M^{tr}$.

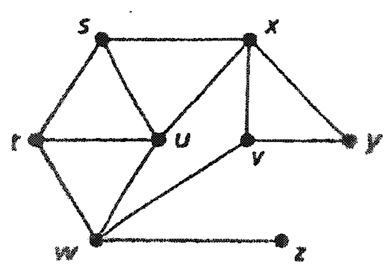
14.

```
10!   THIS PROGRAM MAY BE USED TO DETERMINE IF A RELATION
20!   ON A SET OF SIZE N, WHERE N <= 20, IS AN
30!   EQUIVALENCE RELATION. WE ASSUME WITHOUT LOSS OF
40!   GENERALITY THAT THE ELEMENTS ARE 1,2,3,...,N.
50!
60    INPUT "N ="; N
70    PRINT "   INPUT THE RELATION MATRIX FOR THE RELATION"
80    PRINT "BEING EXAMINED BY TYPING A(I,J) = 1 FOR EACH"
90    PRINT "1 <=  I <= N, 1 <=  J <= N, WHERE (I,J) IS IN"
100   PRINT "THE RELATION. WHEN ALL THE ORDERED PAIRS HAVE"
110   PRINT "BEEN ENTERED TYPE 'CONT' "
120   STOP
130   DIM A(20,20), C(20,20), D(20,20)
140   FOR K = 1 TO N
150       T = T + A(K,K)
160   NEXT K
170   IF T = N THEN &
              PRINT "R IS REFLEXIVE"; X = 1: GO TO 190
180   PRINT "R IS NOT REFLEXIVE"
190   FOR I = 1 TO N
200       FOR J = I + 1 TO N
210           IF A(I,J) <> A(J,I) THEN GO TO 260
220       NEXT J
230   NEXT I
240   PRINT "R IS SYMMETRIC": Y = 1
250   GO TO 270
260   PRINT "R IS NOT SYMMETRIC"
270   MAT C = A
280   MAT D = A*C
290   FOR I = 1 TO N
300       FOR J = 1 TO N
310           IF D(I,J) > 0 AND A(I,J) = 0 THEN GO TO 360
320       NEXT J
330   NEXT I
340   PRINT "R IS TRANSITIVE"; Z = 1
350   GO TO 370
360   PRINT "R IS NOT TRANSITIVE"
370   IF X + Y + Z = 3 THEN &
                  PRINT "R IS AN EQUIVALENCE RELATION" &
      ELSE PRINT "R IS NOT AN EQUIVALENCE RELATION"
380   END
```

15. (a)



(b)



16. (a) True (b) True (c) True (d) False

17. (i) $\mathcal{R} = \{(a,b),(b,a),(a,e),(e,a),(b,c),(c,b),(b,d),(d,b),(b,e),(e,b),(d,e),(e,d),(d,f),(f,d)\}$

$$M(\mathcal{R}) = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \end{array} \begin{array}{c} (a)\,(b)\,(c)\,(d)\,(e)\,(f) \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

For parts (ii), (iii), and (iv), the rows and columns of the relation matrix are indexed as in part (i).

(ii) $\mathcal{R} = \{(a,b),(b,e),(d,b),(d,c),(e,f)\}$

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

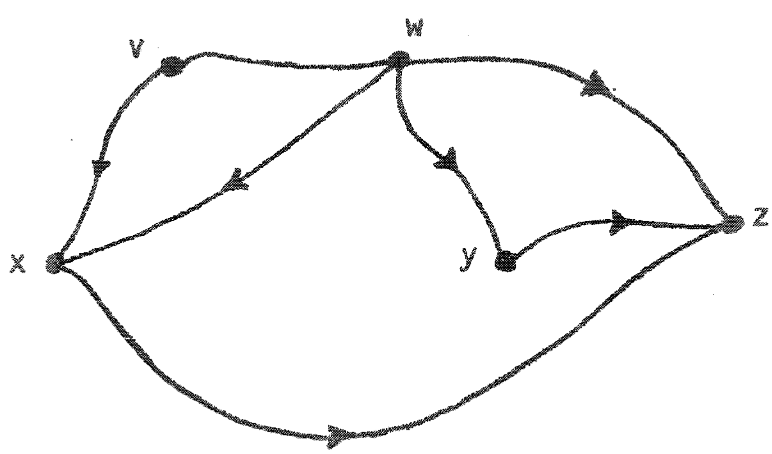
(iii) $\mathcal{R} = \{(a,a),(a,b),(b,a),(c,d),(d,c),(d,e),(e,d),(d,f),(f,d),(e,f),(f,e)\}$

$$M(\mathcal{R}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

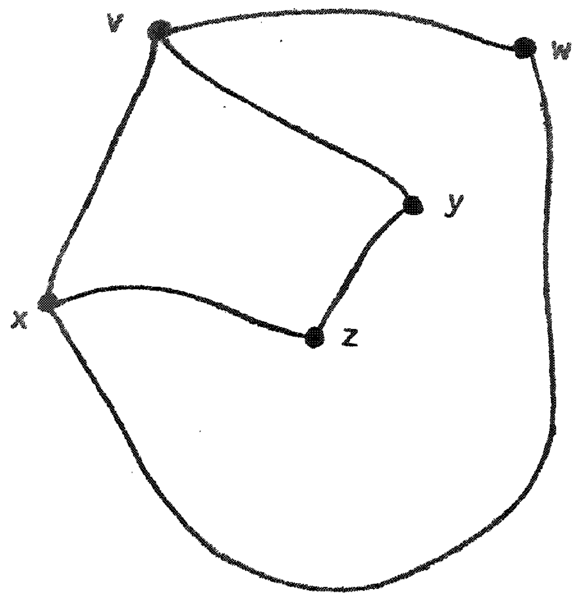
(iv) $\mathcal{R} = \{(b,a),(b,c),(c,b),(b,e),(c,d),(e,d)\}$

$$M(\mathcal{R}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

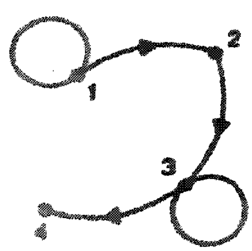
18. (a) $\mathcal{R} = \{(v,w),(v,x),(w,v),(w,x),(w,y),(w,z),(x,z),(y,z)\}$



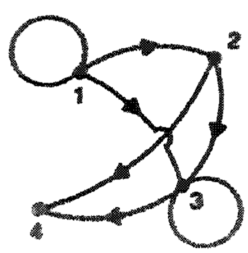
(b) $\mathcal{R} = \{(v,w),(v,x),(v,y),(w,v),(w,x),(x,v),(x,w),(x,z),(y,v),(y,z),(z,x),(z,y)\}$



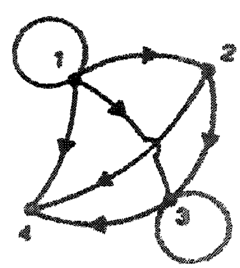
19. $\mathcal{R}$:



$\mathcal{R}^2$:



$\mathcal{R}^3$ and $\mathcal{R}^4$:



20. (a) (i) $\binom{7}{2}$

(ii) Each directed path corresponds to a subset of $\{2,3,4,5,6\}$. There are 2^5 subsets of $\{2,3,4,5,6\}$ and, consequently, 2^5 directed paths in G from 1 to 7.

(b) (i) $\binom{n}{2} = |E|$.

(ii) There are 2^{n-2} directed paths in G from 1 to n.

(iii) There are $2^{[((b-a)+1)-2]} = 2^{b-a-1}$ directed paths in G from a to b.

21. $2^{25}; (2^5)(2^{10}) = 2^{15}$

22. $2^{25}; (2^5)(2^{10}) = 2^{15}$

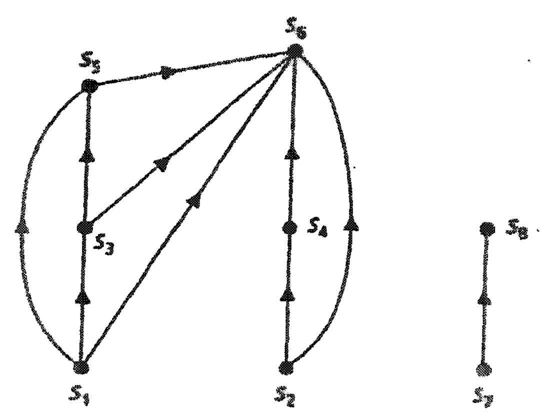
23. (a) $\mathcal{R}_1$: $\mathcal{R}_2$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

(b) Given an equivalence relation $\mathcal{R}$ on a finite set A, list the elements of A so that elements in the same cell of the partition (See Section 7.4.) are adjacent. The resulting relation matrix will then have square blocks of 1's along the diagonal (from upper left to lower right).

24. $\binom{6}{2}; \binom{7}{2}; \binom{n}{2}$

25.



26. (a) Let $k \in \mathbf{Z}^+$. Then $\mathcal{R}^{12k} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}$ and $\mathcal{R}^{12k+1} = \mathcal{R}$. The smallest value of $n > 1$ such that $\mathcal{R}^n = \mathcal{R}$ is $n = 13$. For all multiples of 12 the graph consists of all loops. When $n = 3, (5,5), (6,6), (7,7) \in \mathcal{R}^3$, and this is the smallest power of $\mathcal{R}$ that contains at least one loop.

(b) When $n = 2$, we find $(1,1),(2,2)$ in $\mathcal{R}$. For all $k \in \mathbf{Z}^+, \mathcal{R}^{30k} = \{(x,x) | x \in \mathbf{Z}^+, 1 \leq x \leq 10\}$ and $\mathcal{R}^{30k+1} = \mathcal{R}$. Hence $\mathcal{R}^{31}$ is the smallest power of $\mathcal{R}$ (for $n > 1$) where $\mathcal{R}^n = \mathcal{R}$.

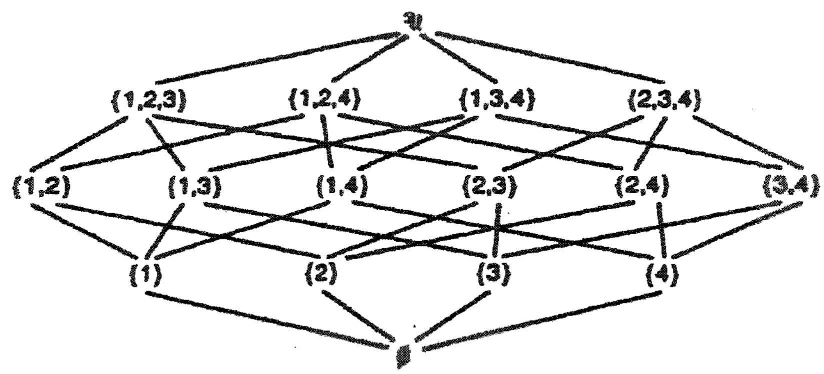
(c) Let $\mathcal{R}$ be a relation on set A where $|A| = m$. Let G be the directed graph associated with $\mathcal{R}$ – each component of G is a directed cycle C_i on m_i vertices, with $1 \leq i \leq k$. (Thus $m_1 + m_2 + \ldots + m_k = m$.) The smallest power of $\mathcal{R}$ where loops appear is $\mathcal{R}^t$, for $t = min\{m_i | 1 \leq i \leq k\}$.

Let $s = \mathrm{lcm}(m_1, m_2, \ldots, m_k)$. Then $\mathcal{R}^{rs}$ = the identity (equality) relation on A and $\mathcal{R}^{rs+1} = \mathcal{R}$, for all $r \in \mathbf{Z}^+$. The smallest power of $\mathcal{R}$ that reproduces $\mathcal{R}$ is $s + 1$.

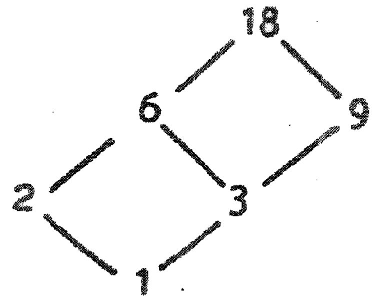
27. $\binom{n}{2} = 703 \Rightarrow n = 38$

Section 7.3

1.



2.



3. For all $a \in A, b \in B, a\mathcal{R}_1 a$ and $b\mathcal{R}_2 b$ so $(a,b)\mathcal{R}(a,b)$, and $\mathcal{R}$ is reflexive. Next $(a,b)\mathcal{R}(c,d), (c,d)\mathcal{R}(a,b) \Longrightarrow a\mathcal{R}_1 c, c\mathcal{R}_1 a$ and $b\mathcal{R}_2 d, d\mathcal{R}_2 b \Longrightarrow a = c, b = d \Longrightarrow (a,b) = (c,d)$, so $\mathcal{R}$ is antisymmetric. Finally, $(a,b)\mathcal{R}(c,d), (c,d)\mathcal{R}(e,f) \Longrightarrow a\mathcal{R}_1 c, c\mathcal{R}_1 e$ and $b\mathcal{R}_2 d, d\mathcal{R}_2 f \Longrightarrow a\mathcal{R}_1 e, b\mathcal{R}_2 f \Longrightarrow (a,b)\mathcal{R}(e,f)$, and $\mathcal{R}$ is transitive. Consequently, $\mathcal{R}$ is a partial order.

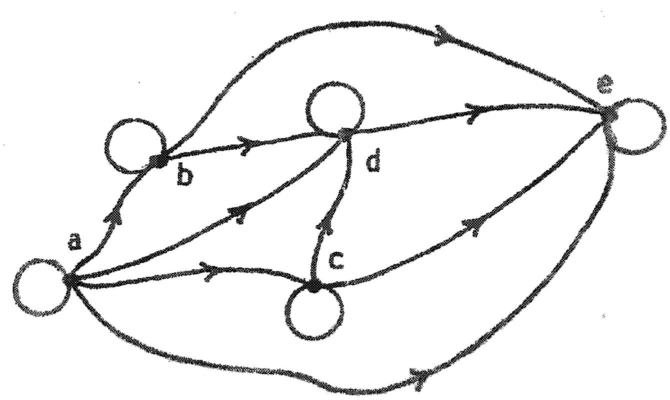
4. No. Let $A = B = \{1,2\}$ with each of $\mathcal{R}_1, \mathcal{R}_2$ the usual "is less than or equal to" relation. Then $\mathcal{R}$ is a partial order but it is not a total order for we cannot compare (1,2) and (2,1).

5. $\emptyset < \{1\} < \{2\} < \{3\} < \{1,2\} < \{1,3\} < \{2,3\} < \{1,2,3\}$. (There are other possibilities.)

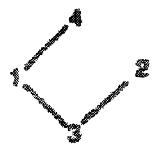
6. (a)

$$M(\mathcal{R}) = \begin{array}{c} \\ (a) \\ (b) \\ (c) \\ (d) \\ (e) \end{array} \begin{array}{c} \begin{array}{ccccc} (a) & (b) & (c) & (d) & (e) \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

(b)



(c) $a < b < c < d < e$ or $a < c < b < d < e$

7. (a) 

(b) $3 < 2 < 1 < 4$ or $3 < 1 < 2 < 4$.

(c) 2

8. Suppose that $x, y \in A$ and that both are least elements. Then $x\mathcal{R}y$ since x is a least element, and $y\mathcal{R}x$ since y is a least element. With $\mathcal{R}$ antisymmetric we have $x = y$.

9. Let x, y both be greatest lower bounds. Then $x\mathcal{R}y$ since x is a lower bound and y is a greatest lower bound. By similar reasoning $y\mathcal{R}x$. Since $\mathcal{R}$ is antisymmetric, $x = y$. [The proof for the *lub* is similar.]

10. Let $\mathcal{U} = \{1, 2, 3, 4\}$. Let A be the collection of all proper subsets of $\mathcal{U}$, partially ordered under set inclusion. Then {1,2,3}, {1,2,4}, {1,3,4}, and {2,3,4} are all maximal elements.

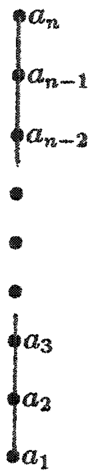
11. Let $\mathcal{U} = \{1, 2\}, A = \mathcal{P}(\mathcal{U})$, and $\mathcal{R}$ the inclusion relation. Then $(A, \mathcal{R})$ is a poset but not a total order. Let $B = \{\emptyset, \{1\}\}$. Then $(B \times B) \cap \mathcal{R}$ is a total order.

12. For all vertices $x, y \in A, x \neq y$, there is either an edge (x, y) or an edge (y, x), but not both. In addition, if $(x, y), (y, z)$ are edges in G then (x, z) is an edge in G. Finally, at every vertex of the graph there is a loop.

13. $n + \binom{n}{2}$

14. $n + \binom{n}{2}$

15. (a) The n elements of A are arranged along a vertical line. For if $A = \{a_1, a_2, \ldots a_n\}$, where $a_1\mathcal{R}a_2\mathcal{R}a_3\mathcal{R}\ldots\mathcal{R}a_n$, then the diagram can be drawn as



(b) $n!$

16. (a) Let $a \in A$ with a minimal. Then for $x \in A$, $x\mathcal{R}a \Longrightarrow x = a$. So if $M(\mathcal{R})$ is the relation matrix for $\mathcal{R}$, the column under 'a' has all 0's except for the one 1 for the ordered pair (a, a).

(b) Let $b \in A$, with b a greatest element. Then the column under 'b' in $M(\mathcal{R})$ has all 1's. If $c \in A$ and c is a least element, then the row of $M(\mathcal{R})$ determined by 'c' has all 1's.

17.

	lub	*glb*		*lub*	*glb*		*lub*	*glb*
(a)	{1,2}	$\emptyset$	(c)	{1,2}	$\emptyset$	(e)	{1,2,3}	$\emptyset$
(b)	{1,2,3}	$\emptyset$	(d)	{1,2,3}	{1}			

18. (a) (i) Only one such upper bound – {1,2,3}. (ii) Here the upper bound has the form $\{1, 2, 3, x\}$ where $x \in \mathcal{U}$ and $4 \leq x \leq 7$. Hence there are four such upper bounds. (iii) There are $\binom{4}{2}$ upper bounds of B that contain five elements from $\mathcal{U}$.

(b) $\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 2^4 = 16$

(c) $lub\, B = \{1, 2, 3\}$

(d) One – namely $\emptyset$

(e) $glb\, B = \emptyset$

19. For each $a \in \mathbf{Z}$ it follows that $a\mathcal{R}a$ because $a - a = 0$, an even nonnegative integer. Hence $\mathcal{R}$ is *reflexive*. If $a, b, c \in \mathbf{Z}$ with $a\mathcal{R}b$ and $b\mathcal{R}c$ then

$$a - b = 2m, \text{ for some } m \in \mathbf{N}$$
$$b - c = 2n, \text{ for some } n \in \mathbf{N},$$

and $a - c = (a - b) + (b - c) = 2(m + n)$, where $m + n \in \mathbf{N}$. Therefore, $a\mathcal{R}c$ and $\mathcal{R}$ is *transitive*. Finally, suppose that $a\mathcal{R}b$ and $b\mathcal{R}a$ for some $a, b \in \mathbf{Z}$. Then $a - b$ and $b - a$ are both nonnegative integers. Since this can only occur for $a - b = b - a$, we find that $[a\mathcal{R}b \wedge b\mathcal{R}a] \Rightarrow a = b$, so $\mathcal{R}$ is *antisymmetric*.

Consequently, the relation $\mathcal{R}$ is a partial order for $\mathbf{Z}$. But it is *not* a total order. For example, $2, 3 \in \mathbf{Z}$ and we have neither $2\mathcal{R}3$ nor $3\mathcal{R}2$, because neither -1 nor 1, respectively, is a nonnegative even integer.

20. (a) For all $(a,b) \in A$, $a = a$ and $b \leq b$, so $(a,b)\mathcal{R}(a,b)$ and the relation is reflexive. If $(a,b),(c,d) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(a,b)$, then if $a \neq c$ we find that
$(a,b)\mathcal{R}(c,d) \Rightarrow a < c$, and
$(c,d)\mathcal{R}(a,b) \Rightarrow c < a$,
and we obtain $a < a$. Hence we have $a = c$.
And now we find that
$(a,b)\mathcal{R}(c,d) \Rightarrow b \leq d$, and
$(c,d)\mathcal{R}(a,b) \Rightarrow d \leq b$,
so $b = d$. Therefore, $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(a,b) \Rightarrow (a,b) = (c,d)$, so the relation is antisymmetric. Finally, consider $(a,b),(c,d),(e,f) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(e,f)$. Then
(i) $a < c$, or (ii) $a = c$ and $b \leq d$; and
(i)$'$ $c < e$, or (ii)$'$ $c = e$ and $d \leq f$.
Consequently,
(i)$''$ $a < e$ or (ii)$''$ $a = e$ and $b \leq f$ — so, $(a,b)\mathcal{R}(e,f)$ and the relation is transitive.
The preceding shows that $\mathcal{R}$ is a partial order on A.

b) & c) There is only one minimal element — namely, (0,0). This is also the least element for this partial order.

The element (1,1) is the only maximal element for the partial order. It is also the greatest element.

d) This partial order is a total order. We find here that

$$(0,0)\mathcal{R}(0,1)\mathcal{R}(1,0)\mathcal{R}(1,1).$$

21. (a) The reflexive, antisymmetric, and transitive properties are established as in the previous exercise.
(b) & (c) Here the least element (and only minimal element) is (0,0). The element (2,2) is the greatest element (and the only maximal element).
(d) Once again we obtain a total order, for

$$(0,0)\mathcal{R}(0,1)\mathcal{R}(0,2)\mathcal{R}(1,0)\mathcal{R}(1,1)\mathcal{R}(1,2)\mathcal{R}(2,0)\mathcal{R}(2,1)\mathcal{R}(2,2).$$

22. Here $|X| = n+1$, $|A| = (n+1)^2$ and $|\mathcal{R}| = (n+1)^2 + \binom{(n+1)^2}{2}$.

23. (a) False. Let $\mathcal{U} = \{1,2\}, A = \mathcal{P}(\mathcal{U})$, and $\mathcal{R}$ be the inclusion relation. Then $(A, \mathcal{R})$ is a lattice where for all $S, T \in A$, $lub\{S,T\} = S \cup T$ and $glb\{S,T\} = S \cap T$. However, $\{1\}$ and $\{2\}$ are not related, so $(A,\mathcal{R})$ is not a total order.

(b) If $(A, \mathcal{R})$ is a total order, then for all $x, y \in A, x\mathcal{R}y$ or $y\mathcal{R}x$. For $x\mathcal{R}y, lub\{x,y\} = y$ and $glb\{x,y\} = x$. Consequently, $(A, \mathcal{R})$ is a lattice.

24. Since A is finite, A has a maximal element, by Theorem 7.3. If x, y $(x \neq y)$ are both maximal elements, since $x, y\mathcal{R}lub\{x,y\}$, then $lub\{x,y\}$ must equal either x or y. Assume $lub\{x,y\} = x$. Then $y\mathcal{R}x$, so y cannot be a maximal element. Hence A has a unique maximal element x. Now for each $a \in A, a \neq x$, if $lub\{a,x\} \neq x$, then we contradict x being a maximal element. Hence $a\mathcal{R}x$ for all $a \in A$, so x is the greatest element in A. [The proof for the least element is similar.]

25. (a) a (b) a (c) c (d) e (e) z (f) e (g) v
$(A, \mathcal{R})$ is a lattice with z the greatest (and only maximal) element and a the least (and only minimal) element.

26.

a)	5	b) and c)	$n+1$
d)	10	e) and f)	$n + (n-1) + \cdots + 2 + 1 = n(n+1)/2$.

27. Consider the vertex $p^a q^b r^c$, $0 \leq a < m$, $0 \leq b < n$, $0 \leq c < k$. There are mnk such vertices; each determines three edges — going to the vertices $p^{a+1}q^b r^c, p^a q^{b+1} r^c, p^a q^b r^{c+1}$. This accounts for $3mnk$ edges.

Now consider the vertex $p^m q^b r^c$, $0 \leq b < n$, $0 \leq c < k$. There are nk of these vertices; each detemines two edges — going to the vertices $p^m q^{b+1} r^c$, $p^m q^b r^{c+1}$. This accounts for $2nk$ edges. And similar arguments for the vertices $p^a q^n r^c (0 \leq a < m, 0 \leq c < k)$ and $p^a q^b r^k (0 \leq a < m, 0 \leq b < n)$ account for $2mk$ and $2mn$ edges, respectively.

Finally, each of the k vertices $p^m q^n r^c$, $0 \leq c < k$, determines one edge (going to $p^m q^n r^{c+1}$) and so these vertices account for k new edges. Likewise, each of the n vertices $p^m q^b r^k$, $0 \leq b < n$, determines one edge (going to $p^m q^{b+1} r^k$), and so these vertices account for n new edges. Lastly, each of the m vertices $p^a q^n r^k$, $0 \leq a < m$, determines one edge (going to $p^{a+1} q^n r^k$) and these vertices account for m new edges.

The preceding results give the total number of edges as $(m+n+k)+2(mn+mk+nk)+3mnk$.

28. a) $24 = 2^3 \cdot 3$. There are $4 \cdot 2 = 8$ divisors for this partial order and they can be totally ordered in $\frac{1}{5}\binom{8}{4} = 14$ ways.
b) $75 = 3 \cdot 5^2$. There are $2 \cdot 3 = 6$ divisors for this partial order and they can be totally ordered in $\frac{1}{4}\binom{6}{3} = 5$ ways.
c) $1701 = 3^5 \cdot 7$. Here the 12 divisors can be totally ordered in $\frac{1}{7}\binom{12}{6} = 132$ ways.

29. $429 = (\frac{1}{8})\binom{14}{7}$ so $k = 6$, and there are $2 \cdot 7 = 14$ positive integer divisors of $p^6 q$.

30. For the (0,1)-matrix $E = (e_{ij})_{m \times n}$ we have $e_{ij} = e_{ij}$, so $e_{ij} \leq e_{ij}$, for all $1 \leq i \leq m$, $1 \leq j \leq n$. Consequently, $E \leq E$ and the "precedes" relation is reflexive.

Now let $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$ be $(0,1)$-matrices, with $E \le F$ and $F \le E$. Then, for all $1 \le i \le m$, $1 \le j \le n$, $e_{ij} \le f_{ij}$ and $f_{ij} \le e_{ij} \Rightarrow e_{ij} = f_{ij}$, so $E = F$ – and the "precedes" relation is antisymmetric.

Finally, suppose that $E = (e_{ij})_{m\times n}$, $F = (f_{ij})_{m\times n}$, and $G = (g_{ij})_{m\times n}$ are $(0,1)$-matrices, with $E \le F$ and $F \le G$. Then, for all $1 \le i \le m$, $1 \le j \le n$, $e_{ij} \le f_{ij}$ and $f_{ij} \le g_{ij} \Rightarrow e_{ij} \le g_{ij}$, so $E \le G$ – and the "precedes" relation is transitive.

In so much as the "precedes" relation is reflexive, antisymmetric, and transitive, it follows that this relation is a partial order – making A into a poset.

Section 7.4

1. (a) Here the collection A_1, A_2, A_3 provides a partition of A.
(b) Although $A = A_1 \cup A_2 \cup A_3 \cup A_4$, we have $A_1 \cap A_2 \ne \emptyset$, so the collection A_1, A_2, A_3, A_4 does *not* provide a partition for A.

2. (a) There are three choices for placing 8 — in either A_1, A_2, or A_3. Hence there are three partitions of A for the conditions given.
(b) There are two possibilities with $7 \in A_1$, and two others with $8 \in A_1$. Hence there are four partitions of A under these conditions.
(c) If we place 7,8 in the same cell for a partition we obtain three of the possibilities. If not, there are three choices of cells for 7 and two choices of cells for 8 — and six more partitions that satisfy the stated restrictions. In total — by the rules of sum and product — there are $3 + (3)(2) = 3 + 6 = 9$ such partitions.

3. $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$.

4. (a) $[1] = \{1,2\} = [2]; [3] = \{3\}$
(b) $A = \{1,2\} \cup \{3\} \cup \{4,5\} \cup \{6\}$.

5. $\mathcal{R}$ is not transitive since $1\mathcal{R}2$, $2\mathcal{R}3$ but $1 \not\mathcal{R} 3$.

6. (a) For all $(x,y) \in A$, since $x = x$, it follows that $(x,y)\mathcal{R}(x,y)$, so $\mathcal{R}$ is reflexive. If $(x_1,y_1),(x_2,y_2) \in A$ and $(x_1,y_1)\mathcal{R}(x_2,y_2)$, then $x_1 = x_2$, so $x_2 = x_1$ and $(x_2,y_2)\mathcal{R}(x_1,y_1)$. Hence $\mathcal{R}$ is symmetric. Finally, let $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in A$ with $(x_1,y_1)\mathcal{R}(x_2,y_2)$ and $(x_2,y_2)\mathcal{R}(x_3,y_3)$. $(x_1,y_1)\mathcal{R}(x_2,y_2) \Longrightarrow x_1 = x_2; (x_2,y_2)\mathcal{R}(x_3,y_3) \Longrightarrow x_2 = x_3$. With $x_1 = x_2, x_2 = x_3$, it follows that $x_1 = x_3$, so $(x_1,y_1)\mathcal{R}(x_3,y_3)$ and $\mathcal{R}$ is transitive.

(b) Each equivalence class consists of the points on a vertical line. The collection of these vertical lines then provides a partition of the real plane.

7. (a) For all $(x,y) \in A, x + y = x + y \Longrightarrow (x,y)\mathcal{R}(x,y)$.
$(x_1,y_1)\mathcal{R}(x_2,y_2) \Longrightarrow x_1 + y_1 = x_2 + y_2 \Longrightarrow x_2 + y_2 = x_1 + y_1 \Longrightarrow$
$(x_2,y_2)\mathcal{R}(x_1,y_1)$. $(x_1,y_1)\mathcal{R}(x_2,y_2),(x_2,y_2)\mathcal{R}(x_3,y_3) \Longrightarrow$

$x_1 + y_1 = x_2 + y_2, x_2 + y_2 = x_3 + y_3$, so $x_1 + y_1 = x_3 + y_3$ and $(x_1, y_1)\mathcal{R}(x_3, y_3)$. Since $\mathcal{R}$ is reflexive, symmetric and transitive, it is an equivalence relation.

(b) $[(1,3)] = \{(1,3),(2,2),(3,1)\}$;
$[(2,4)] = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$; $[(1,1)] = \{(1,1)\}$.

(c) $A = \{(1,1)\} \cup \{(1,2),(2,1)\} \cup \{(1,3),(2,2),(3,1)\} \cup$
$\{(1,4),(2,3),(3,2),(4,1)\} \cup \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \cup$
$\{(2,5),(3,4),(4,3),(5,2)\} \cup \{(3,5),(4,4),(5,3)\} \cup \{(4,5),(5,4)\} \cup \{(5,5)\}$.

8. (a) For all $a \in A, a - a = 3 \cdot 0$, so $\mathcal{R}$ is reflexive. For $a, b \in A, a - b = 3c$, for some $c \in \mathbf{Z} \Longrightarrow b - a = 3(-c)$, for $-c \in \mathbf{Z}$, so $a\mathcal{R}b \Longrightarrow b\mathcal{R}a$ and $\mathcal{R}$ is symmetric. If $a, b, c \in A$ and $a\mathcal{R}b, b\mathcal{R}c$, then $a - b = 3m, b - c = 3n$, for some $m, n \in \mathbf{Z} \Longrightarrow (a - b) + (b - c) = 3m + 3n \Longrightarrow a - c = 3(m + n)$, so $a\mathcal{R}c$. Consequently, $\mathcal{R}$ is transitive.

(b) $[1] = [4] = [7] = \{1,4,7\}; [2] = [5] = \{2,5\}; [3] = [6] = \{3,6\}$.
$A = \{1,4,7\} \cup \{2,5\} \cup \{3,6\}$.

9. (a) For all $(a,b) \in A$ we have $ab = ab$, so $(a,b)\mathcal{R}(a,b)$ and $\mathcal{R}$ is reflexive. To see that $\mathcal{R}$ is symmetric, suppose that $(a,b),(c,d) \in A$ and that $(a,b)\mathcal{R}(c,d)$. Then $(a,b)\mathcal{R}(c,d) \Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c,d)\mathcal{R}(a,b)$, so $\mathcal{R}$ is symmetric. Finally, let $(a,b),(c,d),(e,f) \in A$ with $(a,b)\mathcal{R}(c,d)$ and $(c,d)\mathcal{R}(e,f)$. Then $(a,b)\mathcal{R}(c,d) \Rightarrow ad = bc$ and $(c,d)\mathcal{R}(e,f) \Rightarrow cf = de$, so $adf = bcf = bde$ and since $d \neq 0$, we have $af = be$. But $af = be \Rightarrow (a,b)\mathcal{R}(e,f)$, and consequently $\mathcal{R}$ is transitive.
It follows from the above that $\mathcal{R}$ is an equivalence relation on A.

(b) $[(2,14)] = \{(2,14)\}$
$[(-3,-9)] = \{(-3,-9),(-1,-3),(4,12)\}$
$[(4,8)] = \{(-2,-4),(1,2),(3,6),(4,8)\}$

(c) There are five cells in the partition — in fact,

$$A = [(-4,-20)] \cup [(-3,-9)] \cup [(-2,-4)] \cup [(-1,-11)] \cup [(2,14)].$$

10. (a) For all $X \subseteq A, B \cap X = B \cap X$, so $X\mathcal{R}X$ and $\mathcal{R}$ is reflexive. If $X, Y \subseteq A$, then $X\mathcal{R}Y \Longrightarrow X \cap B = Y \cap B \Longrightarrow Y \cap B = X \cap B \Longrightarrow Y\mathcal{R}X$, so $\mathcal{R}$ is symmetric. And finally, if $W, X, Y \subseteq A$ with $W\mathcal{R}X$ and $X\mathcal{R}Y$, then $W \cap B = X \cap B$ and $X \cap B = Y \cap B$. Hence $W \cap B = Y \cap B$, so $W\mathcal{R}Y$ and $\mathcal{R}$ is transitive. Consequently $\mathcal{R}$ is an equivalence relation on $\mathcal{P}(A)$.

(b) $\{\emptyset, \{3\}\} \cup \{\{1\},\{1,3\}\} \cup \{\{2\},\{2,3\}\} \cup \{\{1,2\},\{1,2,3\}\}$

(c) $[X] = \{\{1,3\},\{1,3,4\},\{1,3,5\},\{1,3,4,5\}\}$

(d) 8 - one for each subset of B.

11. (a) $(\frac{1}{2})\binom{6}{3}$ - The factor $(\frac{1}{2})$ is needed because each selection of size 3 should account for only one such equivalence relation, not two. For example, if $\{a,b,c\}$ is selected we get the partition $\{a, b, c\} \cup \{d, e, f\}$ that corresponds with an equivalence relation. But the selection $\{d, e, f\}$ gives us the same partition and corresponding equivalence relation.

(b) $\binom{6}{3}[1+3] = 4\binom{6}{3}$ – After selecting 3 of the elements we can partition the remaining 3 in

(i) 1 way into three equivalence classes of size 1; or

(ii) 3 ways into one equivalence class of size 1 and one of size 2.

(c) $\binom{6}{4}[1+1] = 2\binom{6}{4}$

(d) $(\frac{1}{2})\binom{6}{3} + 4\binom{6}{3} + 2\binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

12.

(a) $2^{10} = 1024$

(b) $\sum_{i=1}^{5} S(5, i) = 1 + 15 + 25 + 10 + 1 = 52$

(c) $1024 - 52 = 972$

(d) $S(5, 2) = 15$

(e) $\sum_{i=1}^{4} S(4, i) = 1 + 7 + 6 + 1 = 15$

(f) $\sum_{i=1}^{3} S(3, i) = 1 + 3 + 1 = 5$

(g) $\sum_{i=1}^{3} S(3, i) = 1 + 3 + 1 = 5$

(h) $(\sum_{i=1}^{3} S(3, i)) - (\sum_{i=1}^{2} S(2, i)) = 3$

13. 300

14. (a) Not possible. With $\mathcal{R}$ reflexive, $|\mathcal{R}| \geq 7$.

(b) $\mathcal{R} = \{(x, x)|x \in \mathbf{Z}, 1 \leq x \leq 7\}$.

(c) Not possible. With $\mathcal{R}$ symmetric, $|\mathcal{R}| - 7$ must be even.

(d) $\mathcal{R} = \{(x, x)|x \in \mathbf{Z}, 1 \leq x \leq 7\} \cup \{(1, 2), (2, 1)\}$.

(e) $\mathcal{R} = \{(x, x)|x \in \mathbf{Z}, 1 \leq x \leq 7\} \cup \{(1, 2), (2, 1)\} \cup \{(3, 4), (4, 3)\}$.

(f) and (h) Not possible with $r - 7$ odd.

(g) and (i) Not possible. See the remark at the end of Section 7.4.

15. Let $\{A_i\}_{i \in I}$ be a partition of a set A. Define $\mathcal{R}$ on A by $x\mathcal{R}y$ if for some $i \in I, x, y \in A_i$. For each $x \in A, x, x \in A_i$ for some $i \in I$, so $x\mathcal{R}x$ and $\mathcal{R}$ is reflexive. $x\mathcal{R}y \implies x, y \in A_i$, for some $i \in I \implies y, x \in A_i$, for some $i \in I \implies y\mathcal{R}x$, so $\mathcal{R}$ is symmetric. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x, y \in A_i$ and $y, z \in A_j$ for some $i, j \in I$. Since $A_i \cap A_j$ contains y and $\{A_i\}_{i \in I}$ is a partition, from $A_i \cap A_j = \emptyset$ it follows that $A_i = A_j$, so $i = j$. Hence $x, z \in A_i$, so $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

16. Let $P = \cup_{i \in I} A_i$ be a partition of A. Then $E = \cup_{i \in I}(A_i \times A_i)$ is an equivalence relation and $f(E) = P$, so f is *onto*.

Now let E_1, E_2 be two equivalence relations on A. If $E_1 \neq E_2$, then there exists $x, y \in A$ where $(x, y) \in E_1$ and $(x, y) \notin E_2$. Hence if $f(E_1) = P_1 = \cup_{i \in I} A_i$ and $f(E_2) = P_2 = \cup_{j \in J} A_j$, then $(x, y) \in E_1 \implies x, y \in A_i$, $\exists i \in I$, while $(x, y) \notin E_2 \implies$
$\forall j \in J\ (x \notin A_j \vee y \notin A_j)$. Consequently, $P_1 \neq P_2$ and f is *one-to-one*.

17. Proof: Since $\{B_1, B_2, B_3, \ldots, B_n\}$ is a partition of B, we have $B = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n$. Therefore $A = f^{-1}(B) = f^{-1}(B_1 \cup \ldots \cup B_n) = f^{-1}(B_1) \cup \ldots \cup f^{-1}(B_n)$ [by generalizing part (b) of Theorem 5.10]. For $1 \leq i < j \leq n$, $f^{-1}(B_i) \cap f^{-1}(B_j) = f^{-1}(B_i \cap B_j) = f^{-1}(\emptyset) = \emptyset$. Consequently, $\{f^{-1}(B_i)|1 \leq i \leq n, f^{-1}(B_i) \neq \emptyset\}$ is a partition of A.

Note: Part (b) of Example 7.55 is a special case of this result.

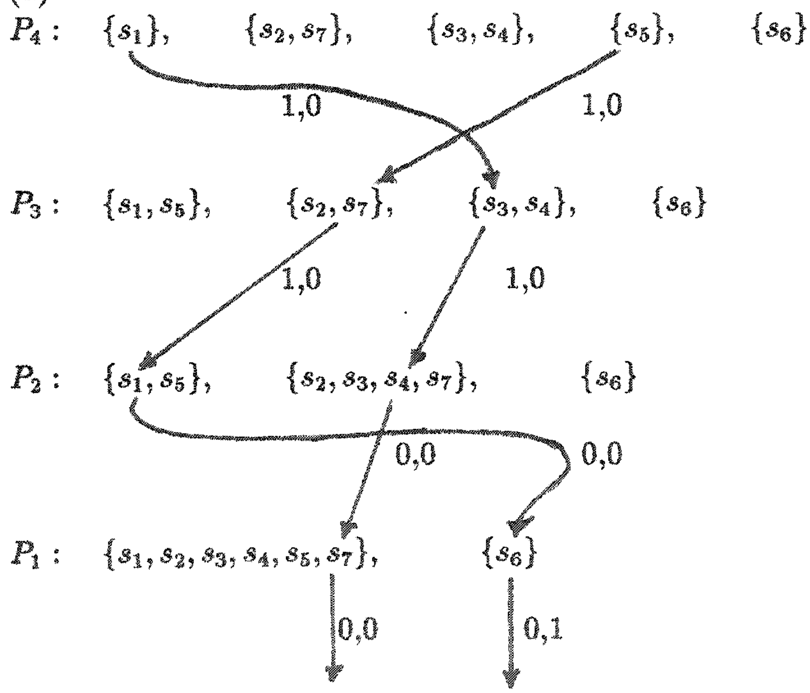
Section 7.5

1. (a) $P_1 : \{s_1, s_4\}, \{s_2, s_3, s_5\}$
$(\nu(s_1,0) = s_4)E_1(\nu(s_4,0) = s_1)$ but $(\nu(s_1,1) = s_1)\not{E}_1(\nu(s_4,1) = s_3)$, so $s_1\not{E}_2 s_4$.
$(\nu(s_2,1) = s_3)\not{E}_1(\nu(s_3,1) = s_4)$ so $s_2\not{E}_2 s_3$.
$(\nu(s_2,0) = s_3)E_1(\nu(s_5,0) = s_3)$ and $(\nu(s_2,1) = s_3)E_1(\nu(s_5,1) = s_3)$ so $s_2 E_2 s_5$.
Since $s_2\not{E}_2 S_3$ and $s_2 E_2 s_5$, it follows that $s_3\not{E}_2 s_5$.
Hence P_2 is given by $P_2 : \{s_1\}, \{s_2, s_5\}, \{s_3,\}, \{s_4\}$. $(\nu(s_2, x) = s_3)E_2(\nu(s_5, x) = s_3)$ for $x = 0, 1$. Hence $s_2 E_3 s_5$ and $P_2 = P_3$.
Consequently, states s_2 and s_5 are equivalent.

(b) States s_2 and s_5 are equivalent.

(c) States s_2 and s_7 are equivalent; s_3 and s_4 are equivalent.

2. (a)



Consequently, 1100 is a distinguishing sequence since $\omega(s_1, 1100) = 0000 \neq 0001 = \omega(s_5, 1100)$.

(b) 100 (c) 00

3. (a) s_1 and s_7 are equivalent; s_4 and s_5 are equivalent.

(b) (i) 0000 (ii) 0 (iii) 00

	ν		ω	
M:	0	1	0	1
s_1	s_4	s_1	1	0
s_2	s_1	s_2	1	0
s_3	s_6	s_1	1	0
s_4	s_3	s_4	0	0
s_6	s_2	s_1	1	0

Supplementary Exercises

1. (a) False. Let $A = \{1,2\}, I = \{1,2\}, \mathcal{R}_1 = \{(1,1)\}, \mathcal{R}_2 = \{(2,2)\}$. Then $\cup_{i\in I}\mathcal{R}_i$ is reflexive but neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is reflexive. Conversely, however, if $\mathcal{R}_i$ is reflexive for all (actually at least one) $i \in I$, then $\cup_{i\in I}\mathcal{R}_i$ is reflexive.

(b) True. $\cap_{i\in I}\mathcal{R}_i$ reflexive $\iff (a,a) \in \cap_{i\in I}\mathcal{R}_i$ for all $a \in A \iff (a,a) \in \mathcal{R}_i$ for all $a \in A$ and all $i \in I \iff \mathcal{R}_i$ is reflexive for all $i \in I$.

2. (i) (a) False. Let $A = \{1,2\}, \mathcal{R}_1 = \{(1,2)\}, \mathcal{R}_2 = \{(2,1)\}$. Then $\mathcal{R}_1 \cup \mathcal{R}_2$ is symmetric although neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is symmetric.

Conversely, however, if each $\mathcal{R}_i, i \in I$, is symmetric and $(x,y) \in \cup_{i\in I}\mathcal{R}_i$, then $(x,y) \in \mathcal{R}_i$ for some $i \in I$. Since $\mathcal{R}_i$ is symmetric, $(y,x) \in \mathcal{R}_i$, so $(y,x) \in \cup_{i\in I}\mathcal{R}_i$ and $\cup_{i\in I}\mathcal{R}_i$ is symmetric.

(b) If $(x,y) \in \cap_{i\in I}\mathcal{R}_i$, then $(x,y) \in \mathcal{R}_i$, for all $i \in I$. Since each $\mathcal{R}_i$ is symmetric, $(y,x) \in \mathcal{R}_i$, for all $i \in I$, so $(y,x) \in \cap_{i\in I}\mathcal{R}_i$ and $\cap_{i\in I}\mathcal{R}_i$ is symmetric.

The converse, however, is false. Let $A = \{1,2,3\}$, with $\mathcal{R}_1 = \{(1,2),(2,1),(1,3)\}$ and $\mathcal{R}_2 = \{(1,2),(2,1),(3,2)\}$. Then neither $\mathcal{R}_1$ nor $\mathcal{R}_2$ is symmetric, but $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1,2),(2,1)\}$ is symmetric.

(iii) (a) Let $A = \{1,2,3\}$ with $\mathcal{R}_1 = \{(1,2)\}$ and $\mathcal{R}_2 = \{(2,1)\}$. Then both $\mathcal{R}_1, \mathcal{R}_2$ are transitive but $\mathcal{R}_1 \cup \mathcal{R}_2$ is not transitive.

Conversely, for $A = \{1,2,3\}$ and $\mathcal{R}_1 = \{(1,3)\}, \mathcal{R}_2 = \{(1,2),(2,3)\}, \mathcal{R}_1 \cup \mathcal{R}_2 = \{(1,2),(2,3),(1,3)\}$ is transitive although $\mathcal{R}_2$ is not transitive.

(b) If $(x,y),(y,z) \in \cap_{i\in I}\mathcal{R}_i$, then $(x,y),(y,z) \in \mathcal{R}_i$ for all $i \in I$. With each $\mathcal{R}_i, i \in I$, transitive, it follows that $(x,z) \in \mathcal{R}_i$, so $(x,z) \in \cap_{i\in I}\mathcal{R}_i$ and $\cap_{i\in I}\mathcal{R}_i$ is transitive.

Conversely, however, $\{(1,2),(2,3)\} = \mathcal{R}_1$ and $\mathcal{R}_2 = \{(1,2)\}$ result in the transitive relation $\mathcal{R}_1 \cap \mathcal{R}_2 = \{(1,2)\}$ even though $\mathcal{R}_1$ is not transitive.

(ii) The results for part (ii) follow in a similar manner.

3. $(a,c) \in \mathcal{R}_2 \circ \mathcal{R}_1 \implies$ for some $b \in A, (a,b) \in \mathcal{R}_2, (b,c) \in \mathcal{R}_1$. With $\mathcal{R}_1, \mathcal{R}_2$ symmetric, $(b,a) \in \mathcal{R}_2, (c,b) \in \mathcal{R}_1$, so $(c,a) \in \mathcal{R}_1 \circ \mathcal{R}_2 \subseteq \mathcal{R}_2 \circ \mathcal{R}_1$. $(c,a) \in \mathcal{R}_2 \circ \mathcal{R}_1 \implies (c,d) \in \mathcal{R}_2, (d,a) \in \mathcal{R}_1$, for some $d \in A$. Then $(d,c) \in \mathcal{R}_2, (a,d) \in \mathcal{R}_1$ by symmetry, and $(a,c) \in$

$\mathcal{R}_1 \circ \mathcal{R}_2$, so $\mathcal{R}_2 \circ \mathcal{R}_1 \subseteq \mathcal{R}_1 \circ \mathcal{R}_2$ and the result follows.

4. (a) Reflexive, symmetric.

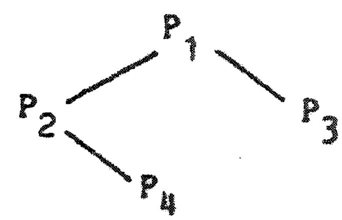
(b) Equivalence relation. Each equivalence class is of the form $A_r = \{t \in T|$ the area of $t = r, r \in \mathbf{R}^+\}$. Then $T = \cup_{r \in \mathbf{R}^+} A_r$.

(c) Reflexive, antisymmetric. (d) Symmetric.

(e) Equivalence relation. $[(1,1)] = \{(1,1),(2,2),(3,3),(4,4)\}$;
$[(1,2)] = \{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3)\}$;
$[(1,3)] = \{(1,3),(3,1),(2,4),(4,2)\}; [(1,4)] = \{(1,4),(4,1)\}$.
$A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)]$.

5. $(c,a) \in (\mathcal{R}_1 \circ \mathcal{R}_2)^c \iff (a,c) \in \mathcal{R}_1 \circ \mathcal{R}_2 \iff (a,b) \in \mathcal{R}_1, (b,c) \in \mathcal{R}_2$, for some $b \in B \iff (c,b) \in \mathcal{R}_2^c, (b,a) \in \mathcal{R}_1^c$, for some $b \in B \iff (c,a) \in \mathcal{R}_2^c \circ \mathcal{R}_1^c$.

6. (a) If P is a partition of A then $P \le P$, so $\mathcal{R}$ is reflexive. For partitions P_i, P_j of A if $P_i \le P_j$ and $P_j \le P_i$, then $P_i = P_j$ and $\mathcal{R}$ is antisymmetric. Finally, if P_i, P_j, P_k are partitions of A and $P_i \mathcal{R} P_j, P_j \mathcal{R} P_k$, then $P_i \le P_j$ and $P_j \le P_k$, so each cell of P_i is contained in a cell of P_k and $P_i \le P_k$. Hence $\mathcal{R}$ is transitive and is a partial order.
(b)



7. Let $\mathcal{U} = \{1,2,3,4,5\}, A = \mathcal{P}(\mathcal{U}) - \{\mathcal{U}, \emptyset\}$. Under the inclusion relation A is a poset with the five minimal elements $\{x\}, 1 \le x \le 5$, but no least element. Also, A has five maximal elements – the five subsets of $\mathcal{U}$ of size 4 – but no greatest element.

8. (b) $[(1,1)] = \{(1,1)\}$; $[(2,2)] = \{(1,4),(2,2),(4,1)\}$;
$[(3,2)] = \{(1,6),(2,3),(3,2),(6,1)\}$; $[(4,3)] = \{(2,6),(3,4),(4,3),(6,2)\}$.

9. $n = 10$

10. (a) For each $f \in \mathcal{F}, |f(n)| \le 1|f(n)|$ for all $n \ge 1$, so $f\mathcal{R}f$, and $\mathcal{R}$ is reflexive. Second, if $f, g \in \mathcal{F}$, then $f\mathcal{R}g \implies (f \in O(g)$ and $g \in O(f)) \implies (g \in O(f)$ and $f \in O(g)) \implies g\mathcal{R}f$, so $\mathcal{R}$ is symmetric. Finally, let $f, g, h \in \mathcal{F}$ with $f\mathcal{R}g, g\mathcal{R}f, g\mathcal{R}h$, and $h\mathcal{R}g$. Then there exist $m_1, m_2 \in \mathbf{R}^+$, and $k_1, k_2 \in \mathbf{Z}^+$ so that $|f(n)| \le m_1|g(n)|$ for all $n \ge k_1$, and $|g(n)| \le m_2|h(n)|$ for all $n \ge k_2$. Consequently, for all $n \ge max\{k_1, k_2\}$ we have $|f(n)| \le m_1|g(n)| \le m_1 m_2 |h(n)|$ so $f \in O(h)$. And in a similar manner $h \in O(f)$. So $f\mathcal{R}h$ and $\mathcal{R}$ is transitive.

(b) For each $f \in \mathcal{F}$, f is dominated by itself, so $[f]S[f]$ and S is reflexive. Second, if $[g],[h] \in \mathcal{F}'$ with $[g]S[h]$ and $[h]S[g]$, then $g\mathcal{R}h$ (as in part (a)), and $[g] = [h]$. Consequently, S is antisymmetric. Finally, if $[f],[g],[h] \in \mathcal{F}'$ with $[f]S[g]$ and $[g]S[h]$, then f is dominated by g and g is dominated by h. So, as in part (a), f is dominated by h and $[f]S[h]$, making S transitive.

(c) Let $f, f_1, f_2 \in \mathcal{F}$ with $f(n) = n$, $f_1(n) = n+3$, and $f_2(n) = 2-n$. Then $(f_1+f_2)(n) = 5$, and $f_1 + f_2 \notin [f]$, because f is not dominated by $f_1 + f_2$.

11.

(a)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	4	4	5
5	5	5	6
6	3	6	8
7	5		

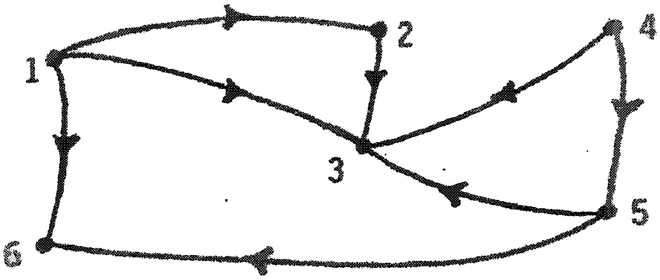
(b)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	5	4	4
5	4	5	5
		6	6

(c)

Adjacency List		Index List	
1	2	1	1
2	3	2	2
3	1	3	3
4	4	4	6
5	5	5	7
6	1	6	8
7	4		

12.



13. (a) For each $v \in V, v = v$ so $v\mathcal{R}v$. If $v\mathcal{R}w$ then there is a path from v to w. Since the graph G is undirected, the path from v to w is also a path from w to v, so $w\mathcal{R}v$ and $\mathcal{R}$ is symmetric. Finally, if $v\mathcal{R}w$ and $w\mathcal{R}x$, then a subset of the edges in the paths from v to w and w to x provide a path from v to x. Hence $\mathcal{R}$ is transitive and $\mathcal{R}$ is an equivalence relation.

(b) The cells of the partition are the (connected) components of G.

14. (a) $P_1 : \{s_1, s_3, s_7\}, \{s_2, s_4, s_5, s_6, s_8\}$
$P_2 : \{s_1, s_3, s_7\}, \{s_2, s_5, s_8\}, \{s_4, s_6\}$
$P_3 : \{s_1\}, \{s_3, s_7\}, \{s_2, s_5, s_8\}, \{s_4\}, \{s_6\}$
$P_4 = P_3$

	ν		ω	
M:	0	1	0	1
s_1	s_3	s_6	1	0
s_2	s_3	s_3	0	0
s_3	s_3	s_2	1	0
s_4	s_2	s_3	0	0
s_6	s_4	s_1	0	0

(b)

$P_3 : \quad \{s_1\}, \{s_3, s_7\}, \{s_2, s_5, s_8\}, \{s_4\}, \{s_6\}$

0,0 0,0

$P_2 : \quad \{s_1, s_3, s_7\}, \{s_2, s_5, s_8\}, \{s_4, s_6\}$

0,0 0,0

$P_1 : \quad \{s_1, s_3, s_7\}, \quad \{s_2, s_4, s_5, s_6, s_8\}$

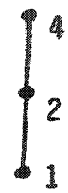
0,1 0,0

Hence $\omega(s_4, 000) = 001 \neq 000 = \omega(s_6, 000)$, so 000 is a distinguishing string for s_4 and s_6.

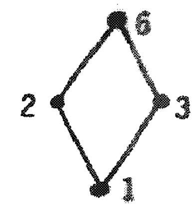
15. One possible order is 10, 3, 8, 6, 7, 9, 1, 4, 5, 2, where program 10 is run first and program 2 last.

16. (a) (i) $n = 2$:

(ii) $n = 4$:

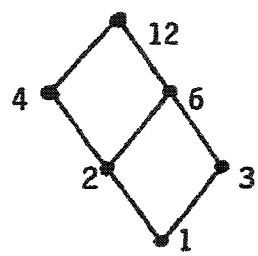


(iii) $n = 6$:

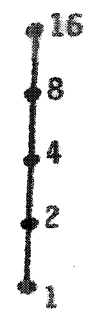


(iv) $n = 8$:

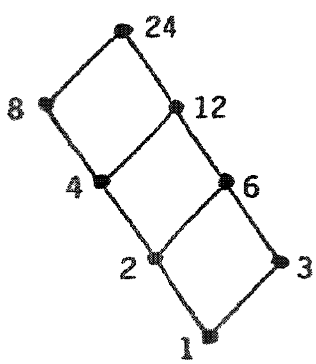
(v) $n = 12$:



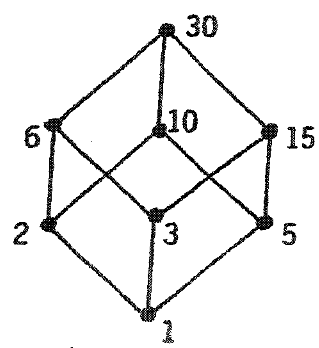
(vi) $n = 16$:



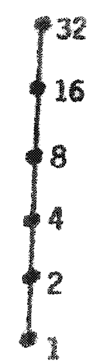
(viii) $n = 24$:



(viii) $n = 30$:



(ix) $n = 32$:



(b) For $2 \leq n \leq 35$, n can be written in one of the following nine forms: (i) p; (ii) p^2; (iii) pq; (iv) p^3; (v) p^2q; (vi) p^4; (vii) p^3q; (viii) pqr; (ix) p^5, where p, q, r denote distinct primes. The Hasse diagrams for these representations are given by the structures in part (a).

For $n = 36 = 2^2 \cdot 3^2$, we must introduce a new structure.

(c) The converse is false. $\tau(24) = 8 = \tau(30)$ but the Hasse diagrams in (vii) and (viii) of part (a) are not the same.

(d) This follows from the definitions of the gcd and lcm and the result of Example 4.45.

17. (b) $[(0.3, 0.7)] = \{(0.3, 0.7)\}$ $\qquad [(0.5, 0)] = \{(0.5, 0)\}$ $\qquad [(0.4, 1)] = \{(0.4, 1)\}$

$[(0,0.6)] = \{(0,0.6),(1,0.6)\}$ $\qquad$ $[(1,0.2)] = \{(0,0.2),(1,0.2)\}$
In general, if $0 < a < 1$, then $[(a,b)] = \{(a,b)\}$; otherwise, $[(0,b)] = \{(0,b),(1,b)\} = [(1,b)]$.
(c) The lateral surface of a cylinder of height 1 and base radius $1/2\pi$.

18. (a) If $C \subseteq \mathcal{U}$, then $0 \le |C| \le 3$. For $0 \le k \le 3$ there are $\binom{3}{k}$ subsets C of $\mathcal{U}$ where $|C| = k$; each such subset C determines 2^k subsets $B \subseteq C$. Hence the relation $\mathcal{R}$ contains $\binom{3}{0}2^0 + \binom{3}{1}2^1 + \binom{3}{2}2^2 + \binom{3}{3}2^3 = (1+2)^3 = 3^3 = 27$ ordered pairs.

(b) For $\mathcal{U} = \{1,2,3,4\}$ the number of ordered pairs in $\mathcal{R}$ is $\binom{4}{0}2^0 + \binom{4}{1}2^1 + \binom{4}{2}2^2 + \binom{4}{3}2^3 + \binom{4}{4}2^4 = (1+2)^4 = 3^4 = 81$.

(c) For $\mathcal{U} = \{1,2,3,\ldots,n\}$, where $n \ge 1$, there are 3^n ordered pairs in the relation $\mathcal{R}$.

19. Since $|\mathcal{U}| = n$, $|\mathcal{P}(\mathcal{U}| = 2^n$ and so there are $(2^n)(2^n) = 4^n$ ordered pairs of the form (A,B) where $A, B \subseteq \mathcal{U}$. From Exercise 18 (above) there are 3^n order pairs of the form (A,B) where $A \subseteq B$. [Note: If $(A,B) \in \mathcal{R}$, then so is (B,A).] Hence there are $3^n + 3^n - 2^n$ ordered pairs (A,B) where either $A \subseteq B$ or $B \subseteq A$, or both. We subtract 2^n because we have counted the 2^n ordered pairs (A,B), where $A = B$, twice. Therefore the number of ordered pairs in this relation is $4^n - (2 \cdot 3^n - 2^n) = 4^n - 2 \cdot 3^n + 2^n$.

20. (a) There are 2^m equivalence classes – one for each subset of B.

(b) 2^{n-m}

21. (a) (i) $B\mathcal{R}A\mathcal{R}C$; (ii) $B\mathcal{R}C\mathcal{R}F$
$B\mathcal{R}A\mathcal{R}C\mathcal{R}F$ is a maximal chain. There are six such maximal chains.

(b) Here $11 \;\mathcal{R}\; 385$ is a maximal chain of length 2, while $2 \;\mathcal{R}\; 6 \;\mathcal{R}\; 12$ is one of length 3. The length of a longest chain for this poset is 3.
(c) (i) $\emptyset \subseteq \{1\} \subseteq \{1,2\} \subseteq \{1,2,3\} \subseteq \mathcal{U}$;
(ii) $\emptyset \subseteq \{2\} \subseteq \{2,3\} \subseteq \{1,2,3\} \subseteq \mathcal{U}$.
There are $4! = 24$ such maximal chains.
(d) $n!$

22. If c_1 is not a minimal element of $(A,\mathcal{R})$, then there is an element $a \in A$ with $a\mathcal{R}c_1$. But then this contradicts the maximality of the chain $(C,\mathcal{R}')$

The proof for c_n maximal in $(A,\mathcal{R})$ is similar.

23. Let $a_1\mathcal{R}a_2\mathcal{R}\ldots\mathcal{R}a_{n-1}\mathcal{R}a_n$ be a longest (maximal) chain in $(A,\mathcal{R})$. Then a_n is a maximal element in $(A,\mathcal{R})$ and $a_1\mathcal{R}a_2\mathcal{R}\ldots\mathcal{R}a_{n-1}$ is a maximal chain in $(B,\mathcal{R}')$. Hence the length of a longest chain in $(B,\mathcal{R}')$ is at least $n-1$. If there is a chain $b_1\mathcal{R}'b_2\mathcal{R}'\ldots\mathcal{R}'b_n$ in $(B,\mathcal{R}')$ of length n, then this is also a chain of length n in $(A,\mathcal{R})$. But then b_n must be a maximal element of $(A,\mathcal{R})$, and this contradicts $b_n \in B$.

24. (a) {2,3,5}; {5,6,7,11}; {2,3,5,7,11}

(b) $\{\{1,2\},\{3,4\}\}$, $\{\{1,2,3\}, \{2,3,4\}\}$; 4

(c) Consider the set M of all maximal elements in $(A, \mathcal{R})$. If this set is not an antichain then there are two elements $a, b \in M$ where $a\mathcal{R}b$ or $b\mathcal{R}a$. Assume, without loss of generality, that $a\mathcal{R}b$. If this is so, then a is *not* a maximal element of $(A, \mathcal{R})$. Hence $(M, (M \times M) \cap \mathcal{R})$ is an antichain in $(A, \mathcal{R})$.

The proof for the set of all minimal elements is similar.

25. If $n = 1$, then for all $x, y \in A$, if $x \neq y$ then $x\mathcal{R}y$ and $y\mathcal{R}x$. Hence $(A, \mathcal{R})$ is an antichain, and the result follows.

Now assume the result true for $n = k \geq 1$, and let $(A, \mathcal{R})$ be a poset where the length of a longest chain is $k+1$. If M is the set of all maximal elements in $(A, \mathcal{R})$, then $M \neq \emptyset$ and M is an antichain in $(A, \mathcal{R})$. Also, by virtue of Exercise 23 above, $(A - M, \mathcal{R}')$, for $\mathcal{R}' = ((A - M) \times (A - M)) \cap \mathcal{R}$, is a poset with k the length of a longest chain. So by the induction hypothesis $A - M = C_1 \cup C_2 \cup \ldots \cup C_k$, a partition into k antichains. Consequently, $A = C_1 \cup C_2 \cup \ldots \cup C_k \cup M$, a partition into $k+1$ antichains.

26. (a) Since $96 = 2^5 \cdot 3$, there are $\frac{1}{7}\binom{12}{6} = 132$ ways to totally order the partial order of 12 positive integer divisors of 96.

(b) Here we have $96 > 32$ and must now totally order the partial order of 10 positive integer divisors of 48. This can be done in $\frac{1}{6}\binom{10}{5} = 42$ ways.

(c) Aside from 1 and 3 there are ten other positive integer divisors of 96. The Hasse diagram for the partial order of these ten integers – namely, 2,4,6,8,12,16,24,32,48,96 – is structurally the same as the Hasse diagram for the partial order of positive integer divisors of 48. So as in part (b) the answer is 42 ways.

(d) Here there are 14 such total orders.

27. (a) There are n edges – namely, $(0,1), (1,2), (2,3), \ldots, (n-1, n)$.

(b) The number of partitions, as described here, equals the number of compositions of n. So the answer is 2^{n-1}.

(c) The number of such partitions is $2^{3-1} \cdot 2^{5-1} = 64$, for there are 2^{3-1} compositions of 3 and 2^{5-1} compositions of $5 (= 12 - 7)$.