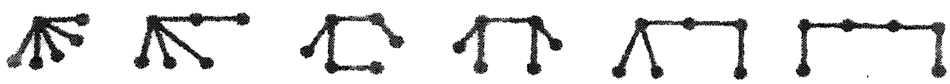
CHAPTER 12
TREES

Section 12.1

1. (a)



(b) 5

2. $|E_1| = 17 \Longrightarrow |V_2| = 18$. $|V_2| = 2|V_1| = 36 \Longrightarrow |E_2| = 35$.

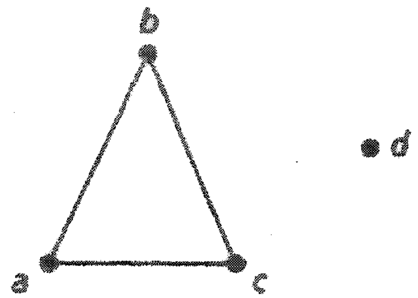
3. (a) Let $e_1, e_2, \ldots, e_7$ denote the numbers of edges for the seven trees, and let $v_1, v_2, \ldots v_7$, respectively, denote the numbers of vertices. Then $v_i = e_i + 1$, for all $1 \leq i \leq 7$, and $|V_1| = v_1 + v_2 + \ldots + v_7 = (e_1 + e_2 + \ldots + e_7) + 7 = 40 + 7 = 47$.

(b) Let n denote the number of trees in F_2. Then if e_i, v_i, $1 \leq i \leq n$, denote the numbers of edges and vertices, respectively, in these trees, it follows that $v_i = e_i + 1$, for all $1 \leq i \leq n$, and $62 = v_1+v_2+\ldots+v_n = (e_1+1)+(e_2+1)+\ldots+(e_n+1) = (e_1+e_2+\ldots+e_n)+n = 51+n$, so $n = 62 - 51 = 11$ trees in F_2.

4. $e = v - \kappa$

5. A path is a tree with only two pendant vertices.

6. (a) Since a tree contains no cycles it cannot have a subgraph homeomorphic to either K_5 or $K_{3,3}$.
(b) If $T = (V, E)$ is a tree then T is connected and, by part (a), T is planar. By Theorem 11.6, $|V| - |E| + 1 = 2$ or $|V| = |E| + 1$.

7. 

8. (a) Let x be the number of pendant vertices. Then $2|E| = \sum_{v\in V} \deg(v) = x + 4(2) + 1(3) + 2(4) + 1(5)$ and $|E| = |V| - 1 = x + 4 + 1 + 2 + 1 - 1 = x + 7$.
So $2(x + 7) = x + 24$ and $x = 10$.

(b) $2|E| = \sum_{v\in V} \deg(v) = v_1 + v_2(2) + v_3(3) + \ldots + v_m(m)$
$|E| = |V| - 1 = (v_1 + v_2 + \ldots v_m) - 1$
$2(v_1 + v_2 + \ldots + v_m - 1) = v_1 + 2v_2 + \ldots + mv_m$, so $v_1 = v_3 + 2v_4 + 3v_5 + \ldots + (m-2)v_m + 2$, and $|V| = v_1 + v_2 + \ldots + v_m = [v_3 + 2v_4 + \ldots + (m-2)v_m + 2] + v_2 + v_3 + \ldots + v_m = v_2 + 2v_3 + 3v_4 + \ldots + (m-1)v_m + 2$, $|E| = |V| - 1 = v_2 + 2v_3 + \ldots + (m-1)v_m + 1$.

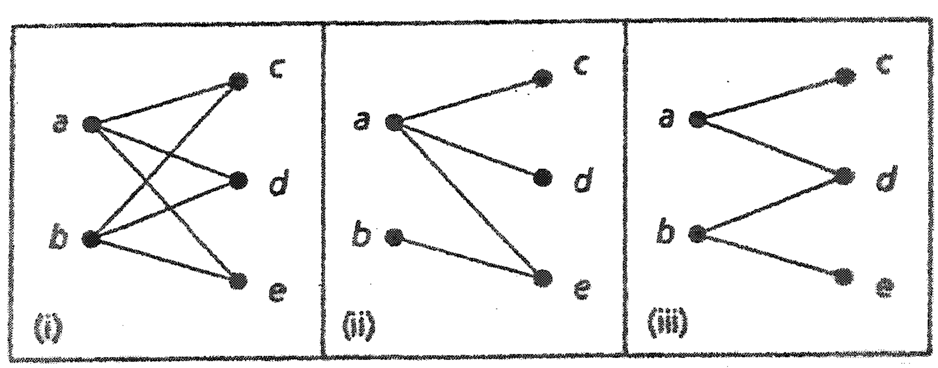
9. If there is a (unique) path between each pair of vertices in G then G is connected. If G contains a cycle then there is a pair of vertices x, y with two distinct paths connecting x and y. Hence, G is a loop-free connected undirected graph with no cycles, so G is a tree.

10. 31

11. Since T is a tree, there is a unique path connecting any two distinct vertices of T. Hence there are $\binom{n}{2}$ distinct paths in T.

12. If G contains no cycles then G is a tree. But then G must have at least two pendant vertices. This graph has only one pendant vertex.

13. (a) In part (i) of the given figure we find the complete bipartite graph $K_{2,3}$. Parts (ii) and (iii) of the figure provide two nonisomorphic spanning trees for $K_{2,3}$.
(b) Up to isomorphism these are the only spanning trees for $K_{2,3}$.



14. Let $V = \{x, y, w_1, w_2, \ldots, w_n\}$ be the vertices for $K_{2,n}$, where $V_1 = \{x, y\}$, $V_2 = \{w_1, w_2, \ldots, w_n\}$ and all edges have one vertex in V_1 and the other in V_2. If T is a spanning tree for $K_{2,n}$, then T has $n + 1$ edges and $\deg(x) + \deg(y) = n + 1$. So the number of nonisomorphic

spanning trees for $K_{2,n}$ is the number of partitions of n into two (nonzero) summands. This number is $\lfloor (n+1)/2 \rfloor$.

15. (a) 6: Any one of the six spanning trees for C_6 (the cycle on six vertices) together with the path connecting f to k.
(b) $6 \cdot 6 = 36$

16. (1) This graph has $9 = 3 \cdot 4 - 3 = 3 + 3(4-2)$ vertices, so any spanning tree for it will have eight edges. There are $12 = 3 \cdot 4$ edges (in total) so we shall remove four edges. Two edges must be removed from one 4-cycle (a cycle on four vertices) and one edge from each of the other two 4-cycles. When two edges from a 4-cycle are removed one must be from the 3-cycle (induced by a, b, and c) – otherwise, we get a disconnected subgraph. There are three ways to select the 4-cycle for removing two edges and three ways to select the edge not on the 3-cycle. We then select one edge from each of the remaining 4-cycles in $4 \cdot 4$ ways. So the number of nonidentical spanning trees for this graph is $3(4-1)(4^2) = 144$.

(2) Here the graph has $8 = 4 \cdot 3 - 4 = 4 + 4(3-2)$ vertices and $12 = 4 \cdot 3$ edges. There are $4(3-1)(3^3) = 216$ nonidentical spanning trees.

(3) This graph has $16 = 4 \cdot 5 - 4 = 4 + 4(5-2)$ vertices and $20 = 4 \cdot 5$ edges. There are $4(5-1)(5^3) = 2000$ nonidentical spanning trees.

17. (a) $n \geq m+1$
(b) Let k be the number of pendant vertices in T. From Theorem 11.2 and Theorem 12.3 we have

$$2(n-1) = 2|E| = \sum_{v \in V} \deg(v) \geq k + m(n-k).$$

Consequently, $[2(n-1) \geq k + m(n-k)] \Rightarrow [2n - 2 \geq k + mn - mk] \Rightarrow [k(m-1) \geq 2 - 2n + mn = 2 + (m-2)n \geq 2 + (m-2)(m+1) = 2 + m^2 - m - 2 = m^2 - m = m(m-1)]$, so $k \geq m$.

18. $\sum_{v \in V} \deg(v) = 2|E| = 2(|V| - 1) = 2(999) = 1998$.

19. (a) If the complement of T contains a cut set, then the removal of these edges disconnects G and there are vertices x, y with no path connecting them. Hence T is not a spanning tree for G.

(b) If the complement of C contains a spanning tree, then every pair of vertices in G has a path connecting them and this path includes no edges of C. Hence the removal of the edges in C from G does not disconnect G, so C is not a cut set for G.

20. $(d) \Longrightarrow (e)$: Let C be a cycle (in G) with r vertices and r edges. Since G is connected, the remaining vertices of G can each be connected to a vertex in C by a path (in G). Each such connection requires at least one new edge. Consequently, in G, $|E| \geq |V|$, contradicting $|V| = |E| + 1$. So G has no cycles and is connected, and G is a tree. Let G' be the graph obtained by adding edge $\{a, b\}$ to G. Since $\{a, b\} \notin E$, there is a unique path P, of length at least 2 in G, that connects a to b. In

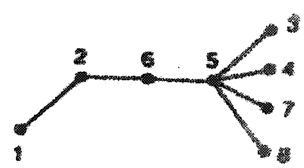
G', $P \cup \{\{a, b\}\}$ is a cycle. If G' contains a second cycle C_1, then C_1 must contain edge $\{a, b\}$. If not, then G would contain a cycle. This second cycle $C_1 = P_1 \cup \{\{a, b\}\}$, where P_1 is a path in G and $P_1 \neq P$. This contradicts Theorem 12.1.

$(e) \Longrightarrow (a)$: If G is not connected, let C_1, C_2 be components of G with $a \in C_1, b \in C_2$. Then adding the edge $\{a, b\}$ to G would not result in a cycle. Consequently, G is connected with no cycles, so G is a tree.

21. (a) (i) 3,4,6,3,8,4 (ii) 3,4,6,6,8,4

(b) No pendant vertex of the given tree appears in the sequence so the result is true for these vertices. When an edge $\{x, y\}$ is removed and y is a pendant vertex (of the tree or one of the resulting subtrees), then the $\deg(x)$ is decreased by 1 and x is placed in the sequence. As the process continues either (i) this vertex x becomes a pendant vertex in a subtree and is removed but not recorded in the sequence, or (ii) the vertex x is left as one of the last two vertices of an edge. In either case x has been listed in the sequence $(\deg(x) - 1)$ times.

(c)



(d) Input: The given Prüfer code $x_1, x_2, \ldots, x_{n-2}$.
Output: The unique tree T with n vertices labeled with $1, 2, \ldots, n$. (This tree T has the Prüfer code $x_1, x_2, \ldots, x_{n-2}$.)

$C := [x_1, x_2, \ldots, x_{n-2}]$ {Initializes C as a list (ordered set).}
$L := [1, 2, \ldots, n]$ {Initializes L as a list (ordered set).}
$T := \emptyset$
for $i := 1$ **to** $n - 2$ **do**
  $v :=$ smallest element in L not in C
  $w :=$ first entry in C
  $T := T \cup \{\{v, w\}\}$ {Add the new edge $\{v, w\}$ to the present forest.}
  delete v from L
  delete the first occurrence of w from C
$T := T \cup \{\{y, z\}\}$ {The vertices y, z are the last two remaining entries in L.}

22. Let V be the vertex set for K_n. From the previous exercise we know that there are $(n-1)^{n-3}$ spanning trees for the subgraph of K_n induced by $V - v$ (namely, the complete graph K_{n-1}). For v to be a pendant vertex it can be adjacent to only one of the $n - 1$ vertices in $V - v$. Consequently, there are $(n-1)[(n-1)^{n-3}]$ spanning trees of K_n where v is a pendant vertex.

23. (a) If the tree contains $n+1$ vertices then it is (isomorphic to) the complete bipartite graph $K_{1,n}$ – often called the *star* graph.
(b) If the tree contains n vertices then it is (isomorphic to) a path on n vertices.

24. Consider the Prüfer codes for the n^{n-2} labeled trees on n vertices. For a given labeled tree, the pendant vertices (of degree 1) have the labels which do *not* appear in the Prüfer code for that tree. If there are k pendant vertices, then there are k labels missing from the code and these can be selected in $\binom{n}{k}$ ways. That leaves $n-k$ labels that must *all* be placed in the $n-2$ positions of the Prüfer code. This can be counted as the number of *onto* functions from the set of $n-2$ positions to the set of $n-k$ labels – that is, $(n-k)!\ S(n-2, n-k)$.

The result then follows by the rule of product.

25. Let $E_1 = \{\{a,b\},\{b,c\},\{c,d\},\{d,e\},\{b,h\},\{d,i\},\{f,i\},\{g,i\}\}$ and
$E_2 = \{\{a,h\},\{b,i\},\{h,i\},\{g,h\},\{f,g\},\{c,i\},\{d,f\},\{e,f\}\}$.

Section 12.2

1.
(a) f,h,k,p,q,s,t (b) a (c) d
(d) e,f,j,q,s,t (e) q,t (f) 2
(g) k,p,q,s,t

2. (a)

Vertex	Level Number
p	35
s	36
t	36
v	37
w	38
x	38
y	39
z	39

(b) The vertex u has 37 ancestors.
(c) The vertex y has 39 ancestors.

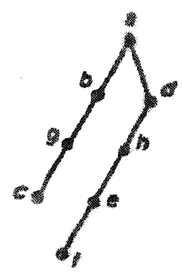
3. (a) $1 + w - x\ y * \pi \uparrow z\ 3$ (b) 0.4

4. (a) 5 (b) 2.1.3 (c) 4 (including the root)
(d) $2.1.3.x,\ 1 \le x \le 5$; 2.1.3, 2.1.2, 2.1.1, 2.1, 2, 1.

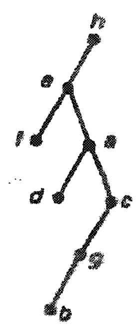
5. Preorder: r,j,h,g,e,d,b,a,c,f,i,k,m,p,s,n,q,t,v,w,u
Inorder: h,e,a,b,d,c,g,f,j,i,r,m,s,p,k,n,v,t,w,q,u
Postorder: a,b,c,d,e,f,g,h,i,j,s,p,m,v,w,t,u,q,n,k,r

6. Preorder: 1,2,5,9,14,15,10,16,17,3,6,4,7,8,11,12,13
Postorder: 14,15,9,16,17,10,5,2,6,3,7,11,12,13,8,4,1

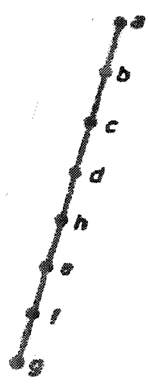
7. (a)
(i) & (iii)



(ii)



(b)
(i)



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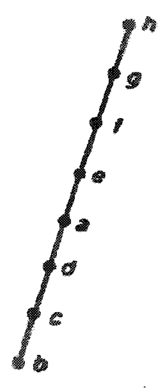
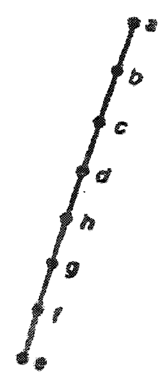
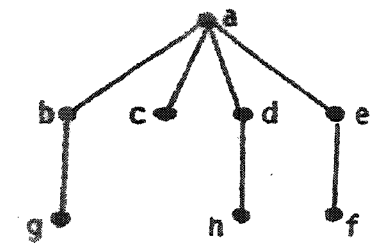


(iii)



8. (a) (i) & (iii)



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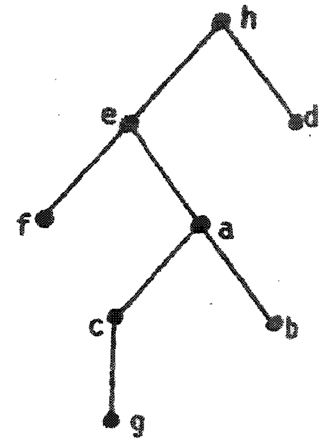
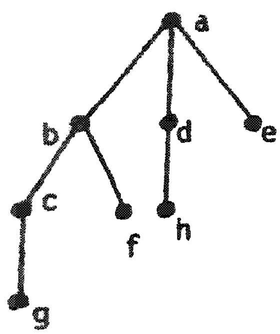
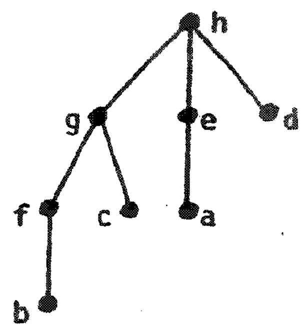


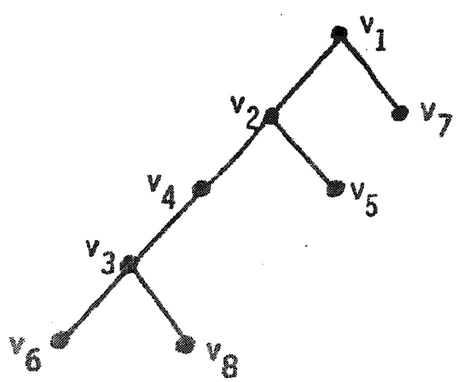
(b) (i) & (iii)



(ii)



9.



G is connected.

10. (a) Here the maximum height is $n-1$.
(b) In this case n must be odd and the maximum height is $(n-1)/2$.

11. Theorem 12.6
(a) Each internal vertex has m children so there are mi vertices that are the children of some other vertex. This accounts for all vertices in the tree except the root. Hence $n = mi + 1$
(b) $\ell + i = n = mi + 1 \Longrightarrow \ell = (m-1)i + 1$

(c) $\ell = (m-1)i + 1 \Longrightarrow i = (\ell - 1)/(m-1)$
$n = mi + 1 \Longrightarrow i = (n-1)/m$.

(Corollary 12.1)
Since the tree is balanced $m^{h-1} < \ell \leq m^h$ by Theorem 12.7.
$m^{h-1} < \ell \leq m^h \Longrightarrow \log_m(m^{h-1}) < \log_m(\ell) \leq \log_m(m^h) \Longrightarrow$
$(h-1) < \log_m \ell \leq h \Longrightarrow h = \lceil \log_m \ell \rceil$.

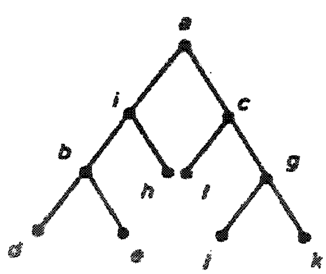
12. From Theorem 12.6 (c) we have
(a) $(\ell-1)/(m-1) = (n-1)/m \Longrightarrow (n-1)(m-1) = m(\ell-1) \Longrightarrow$
$n - 1 = (m\ell - m)/(m-1) \Longrightarrow n = [(m\ell - m)/(m-1)] + 1 =$
$[(m\ell - m) + (m-1)]/(m-1) = (m\ell - 1)/(m-1)$.

(b) $(\ell-1)/(m-1) = (n-1)/m \Longrightarrow \ell - 1 = (m-1)(n-1)/m \Longrightarrow$
$\ell = [(m-1)(n-1) + m]/m = [(m-1)n + 1]/m$.

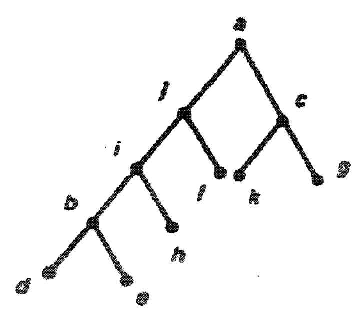
13. (a) From part (a) of Theorem 12.6 we have $|V|$ = number of vertices in $T = 3i + 1 = 3(34) + 1 = 103$. So T has $103 - 1 = 102$ edges. From part (b) of the same theorem we find that the number of leaves in T is $(3-1)(34) + 1 = 69$. [We can also obtain the number of leaves as $|V| - i = 103 - 34 = 69$.]

(b) It follows from part (c) of Theorem 12.6 that the given tree has $(817-1)/(5-1) = 816/4 = 204$ internal vertices.

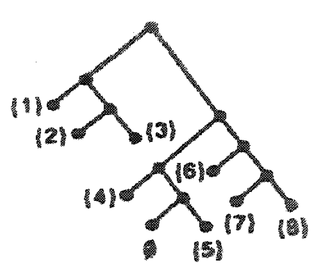
14. (a)



(b)



15. (a)



(b) 9; 5

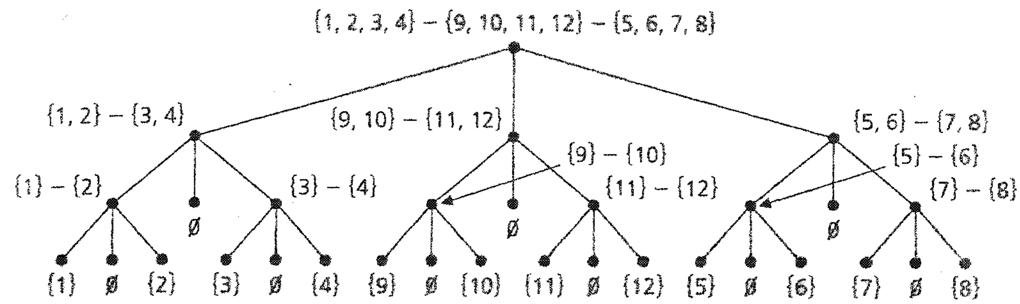
(c) $h(m-1)$; $(h-1) + (m-1)$

16. (a) From Theorem 12.6 (c), with $\ell = 25$, $m = 2$, it follows that $i = (25-1)/(2-1) = 24$. Hence 24 cans of tennis balls are opened and 24 matches are played.
(b) Either 4 or 5.

17. 21845; $1 + m + m^2 + \ldots + m^{h-1} = (m^h - 1)/(m-1)$.

18. $2[5 + 5^2 + 5^3 + 5^4 + 5^5 + 5^6 + 5^7]$; $2[5^5 + 5^6 + 5^7]$

19.



20. The number of vertices at level $h-1$ is m^{h-1}. Among these we find $m^{h-1} - b_{h-1}$ of the l leaves of T. Each of the b_{h-1} branch nodes account for m leaves (at level h). Therefore, $l = m^{h-1} - b_{h-1} + mb_{h-1} = m^{h-1} + (m-1)b_{h-1}$.

21. Let T be a complete binary tree with 31 vertices. The left and right subtrees of T are then *complete binary trees* on $2k+1$ and $30-(2k+1)$ vertices, respectively, with $0 \leq k \leq 14$.

The number of ways the left subtree can have $11(= 2\cdot 5+1)$ vertices is $(\frac{1}{6})\binom{10}{5}$. This leaves $19(= 2\cdot 9+1)$ vertices for the right subtree where there are $(\frac{1}{10})\binom{18}{9}$ possibilities. So by the rule of product there are $(\frac{1}{6})\binom{10}{5}(\frac{1}{10})\binom{18}{9} = 204,204$ complete binary trees on 31 vertices with 11 vertices in the left subtree of the root. A similar argument tells us that there are $(\frac{1}{11})\binom{20}{10}(\frac{1}{5})\binom{8}{4} = 235,144$ complete binary trees on 31 vertices with 21 vertices in the right subtree of the root.

22. $a_{n+1} = a_0a_n + a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1 + a_na_0$
[Compare with the equation for b_{n+1} in Section 10.5.]

23. (a) 1,2,5,11,12,13,14,3,6,7,4,8,9,10,15,16,17
(b) The preorder traversal of the rooted tree.

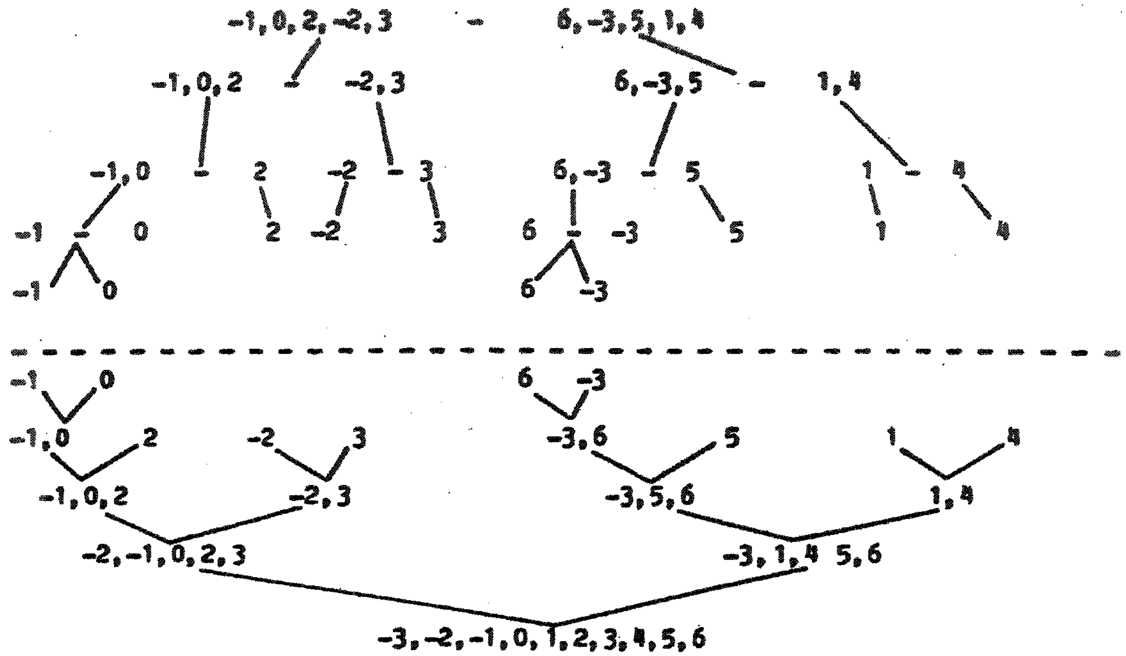
24. (a) 11,12,13,14,5,2,6,7,3,8,9,15,16,17,10,4,1
(b) The postorder traversal of the rooted tree.

Here the algorithm is iterative, while the one given in Definition 12.3 is recursive.

Section 12.3

1. (a) $L_1: 1,3,5,7,9 \qquad L_2: 2,4,6,8,10$
   (b) $L_1: \; 1,3,5,7,\ldots,2m-3,m+n$
   $L_2: \; 2,4,6,8,\ldots,2m-2,2m-1,2m,2m+1,\ldots,m+n-1$

2. (a)



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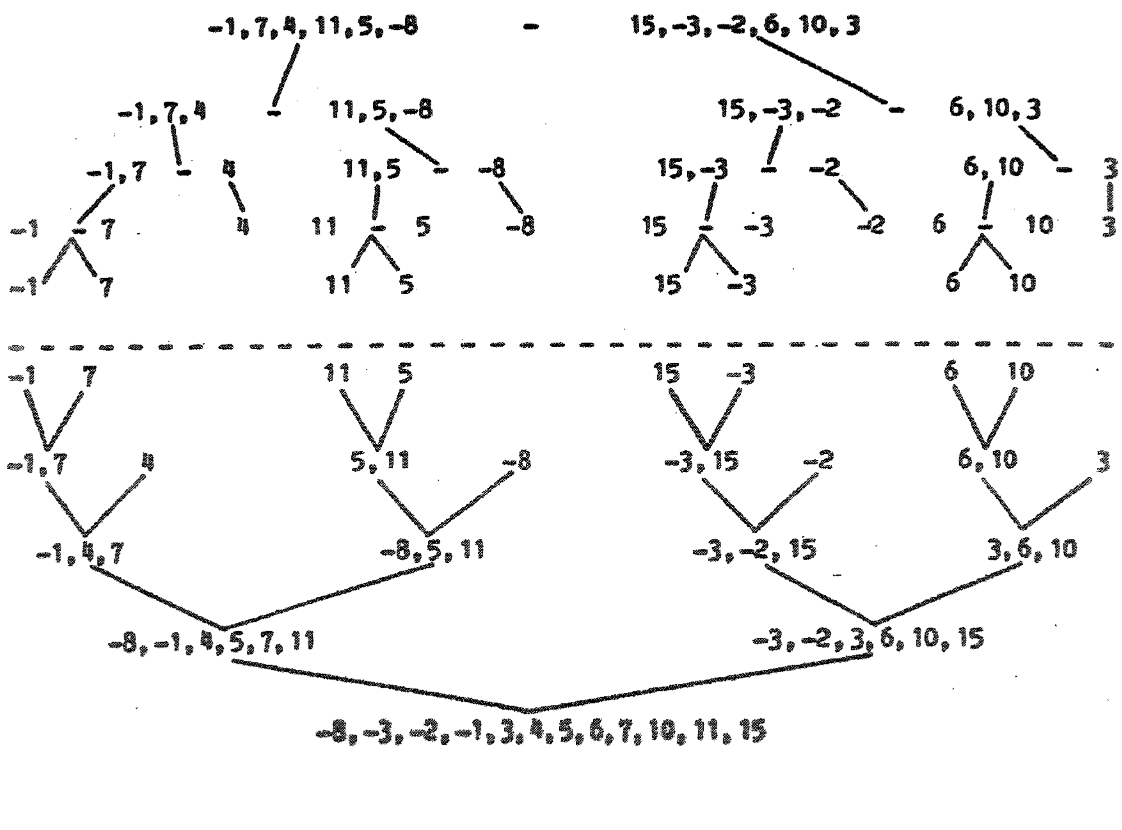
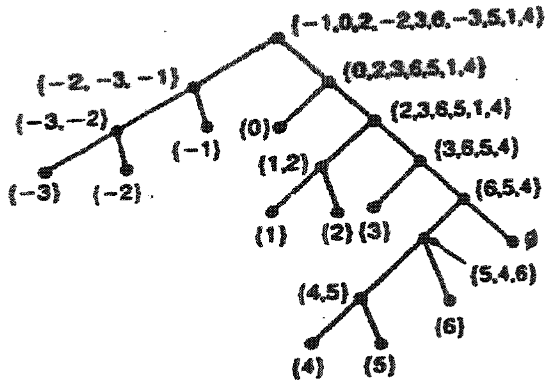


3. (a)



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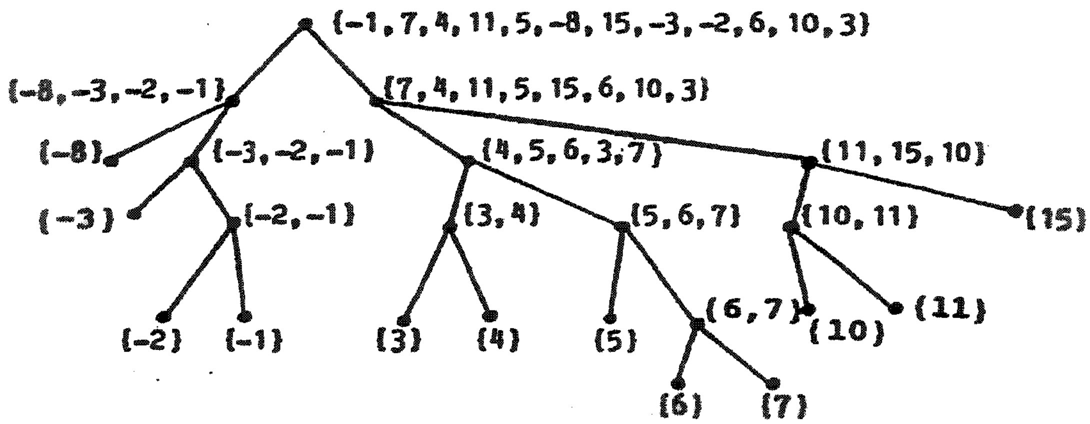


4. To establish this result we use mathematical induction (the alternative form). We know that $g(1) \leq g(2) \leq g(3) \leq g(4)$. So we assume that for all $i, j \in \{1, 2, 3, \ldots, n\}$, $i < j \Longrightarrow g(i) \leq g(j)$. Considering the case for $n + 1$ we have two results to examine.

(1) If $n + 1$ is odd then $n + 1 = 2k + 1$ for some $k \in \mathbf{Z}^+$. In the worst case, $g(n + 1) = g(2k+1) = g(k)+g(k+1)+[k+(k+1)-1] = g(k)+g(k+1)+2k \geq g(k)+g(k)+(2k-1) = g(2k) = g(n)$, since $g(k + 1) \geq g(k)$ by the induction hypothesis.

(2) If $n+1$ is even, then $n+1 = 2t$ for some $t \in \mathbf{Z}^+$. In the worst case, $g(n+1) = g(2t) = g(t)+g(t)+[t+t-1] = g(t)+g(t)+(2t-1) \geq g(t)+g(t-1)+(2t-2) = g(2t-1) = g(n)$, because $g(t) \geq g(t-1)$ by the induction hypothesis.

Consequently g is a monotone increasing function.

Section 12.4

1. (a) tear (b) tatener (c) rant

2. $x = y = z = 1$

3.

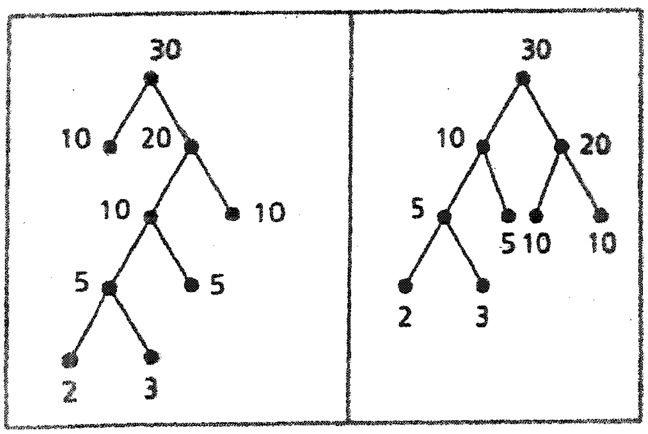
a:	111	e:	10	h:	010
b:	110101	f:	0111	i:	00
c:	0110	g:	11011	j:	110100
d:	0001				

4. (a) 2^3 (b) 2^7 (c) 2^{12} (d) 2^h

5. Since the tree has $m^7 = 279,936$ leaves, it follows that $m = 6$. From part(c) of Theorem 12.6 we find that there are $(m^7 - 1)/(m - 1) = (279,935)/5 = 55,987$ internal vertices.

6. $v = 1+m+m^2+\cdots+m^h = (1-m^{h+1})/(1-m) = (m^{h+1}-1)/(m-1)$, so $v(m-1)+1 = m^{h+1}$. Consequently, $h + 1 = \log_m[v(m-1)+1]$ and $h = \log_m[v(m-1)+1] - 1$.

7.



Amend part (a) of Step 2 for the Huffman tree algorithm as follows. If there are $n(> 2)$ such trees with smallest root weights w and w', then

(i) if $w < w'$ and $n - 1$ of these trees have root weight w', select a tree (of root weight w') with smallest height; and

(ii) if $w = w'$ (and all n trees have the same smallest root weight), select two trees (of root weight w) of smallest height.

8. (a) To merge lists L_1 and L_2 requires at most $75 + 40 - 1 = 114$ comparisons (from Lemma 12.1), for L_3 and L_4 at most $110 + 50 - 1 = 159$ comparisons. Merging the two resulting lists then requires at most $115 + 160 - 1 = 274$ comparisons for a total of at most $114 + 159 + 274 = 547$ comparisons.

(b) At most 114; at most 224 (338 at most, in total); at most 274 (in total, at most 612).

(c) Merge L_2 and L_4, then merge the resulting list (for L_2, L_4) with L_1, and finally merge the resulting list (for L_1, L_2, L_4) with L_3. This requires at most a total of $89+164+274 = 527$ comparisons.

(d) In order to minimize the number of comparisons in the sorting process construct an optimal tree with the weights w_i, $1 \leq i \leq n$, given by $w_i = |L_i|$.

Section 12.5

1. The articulation points are b, e, f, h, j, k. The biconnected components are
$B_1: \{\{a,b\}\}$; $B_2: \{\{d,e\}\}$;
$B_3: \{\{b,c\},\{c,f\},\{f,e\},\{e,b\}\}$; $B_4: \{\{f,g\},\{g,h\},\{h,f\}\}$;
$B_5: \{\{h,i\},\{i,j\},\{j,h\}\}$; $B_6: \{\{j,k\}\}$;
$B_7: \{\{k,p\},\{p,n\},\{n,m\},\{m,k\},\{p,m\}\}$.

2. If every path from x to y contains the vertex z, then splitting the vertex z will result in at least two components C_x, C_y where $x \in C_x$, $y \in C_y$. If not, there is a path that still connects x and y and this path does not include vertex z. Conversely, if z is an articulation point of G then the splitting of z results in at least two components C_1, C_2 for G. Select $x \in C_1$, $y \in C_2$. Since G is connected there is at least one path from x to y, but since x and y become separated upon the splitting of z, every path connecting x and y in G contains the vertex z.

3. (a) T can have as few as one or as many as $n-2$ articulation points. If T contains a vertex of degree $(n-1)$, then this vertex is the only articulation point. If T is a path with n vertices and $n-1$ edges, then the $n-2$ vertices of degree 2 are all articulation points.
(b) In all cases, a tree on n vertices has $n-1$ biconnected components. Each edge is a biconnected component.

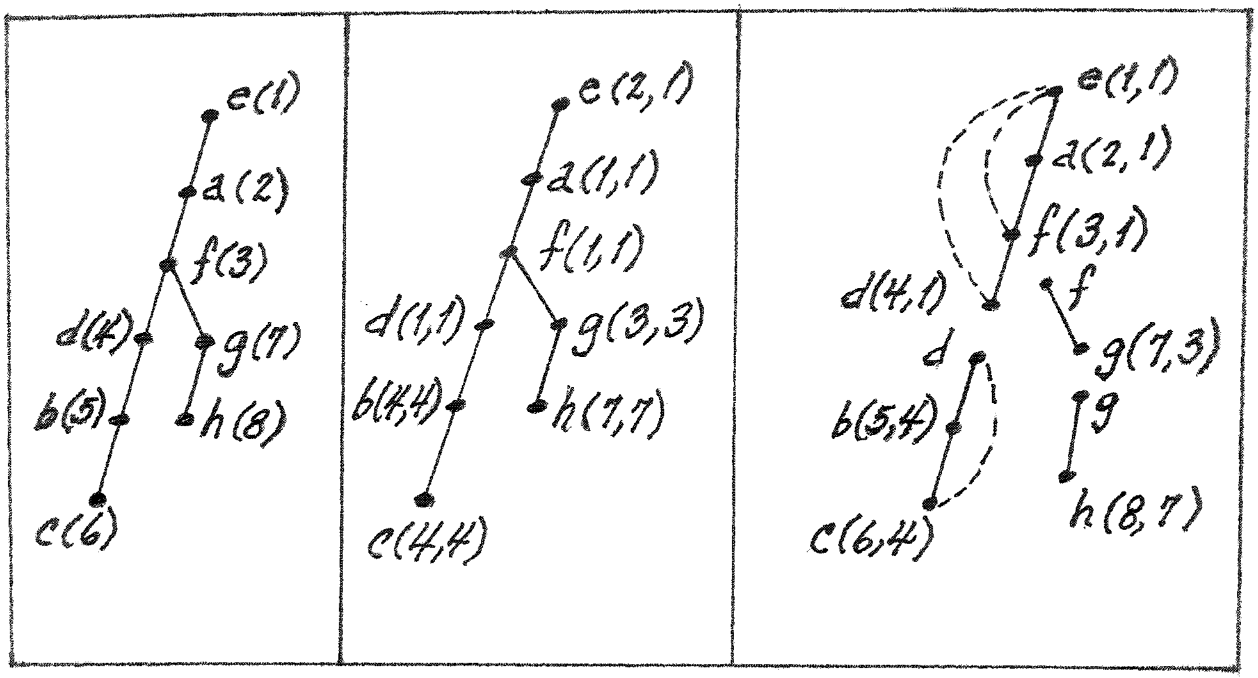
4. (a) From Exercise 2, if v is an articulation point in T then there are vertices x, y where every path from x to y includes vertex v. Hence $\deg(v) > 1$. Conversely, if $\deg(v) > 1$, let $a, b \in V$ such that $\{a,v\}$, $\{v,b\} \in E$. Then in splitting vertex v, the tree is separated into components C_a, C_b containing a, b, respectively. If not, there is another path from a to b that does not include v. This contradicts Theorem 12.1.

(b) Since G is connected, G has a spanning tree $T = (V, E')$. This tree has at least two pendant vertices. Let v be a pendant vertex in T. If v is an articulation point of G, then there are vertices x, y in G such that every path connecting x and y contains v. But then one of these paths must be in T. So $\deg_T(v) > 1$, contradicting v being a pendant vertex.

5. $\chi(G) = \max\{\chi(B_i) | 1 \leq i \leq k\}$.

6. The graph G has $n_1 \cdot n_2 \cdots n_8$ distinct spanning trees.

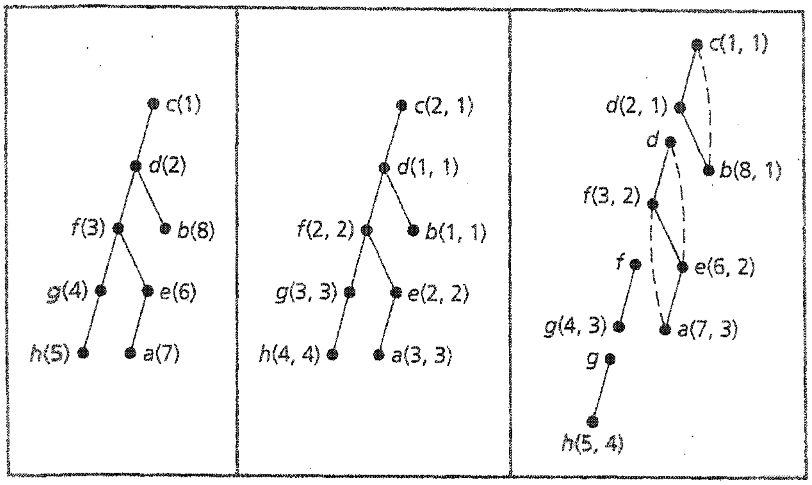
7. Proof: Suppose that G has a pendant vertex, say x, and that $\{w, x\}$ is the (unique) edge in E incident with x. Since $|V| \geq 3$ we know that $\deg(w) \geq 2$ and that $\kappa(G - w) \geq 2 > 1 = \kappa(G)$. Consequently, w is an articulation point of G.

8. 

(a) The first tree provides the depth-first spanning tree T for G with e as the root.
(b) The second tree provides $(\text{low}'(v), \text{low}(v))$ for each vertex v of G (and T). These results follow from step (2) of the algorithm.

For the third tree we find $(\text{dfi}(v), \text{low}(v))$ for each vertex v. Applying step (3) of the algorithm we find the articulation points d, f, and g, and the four biconnected components.

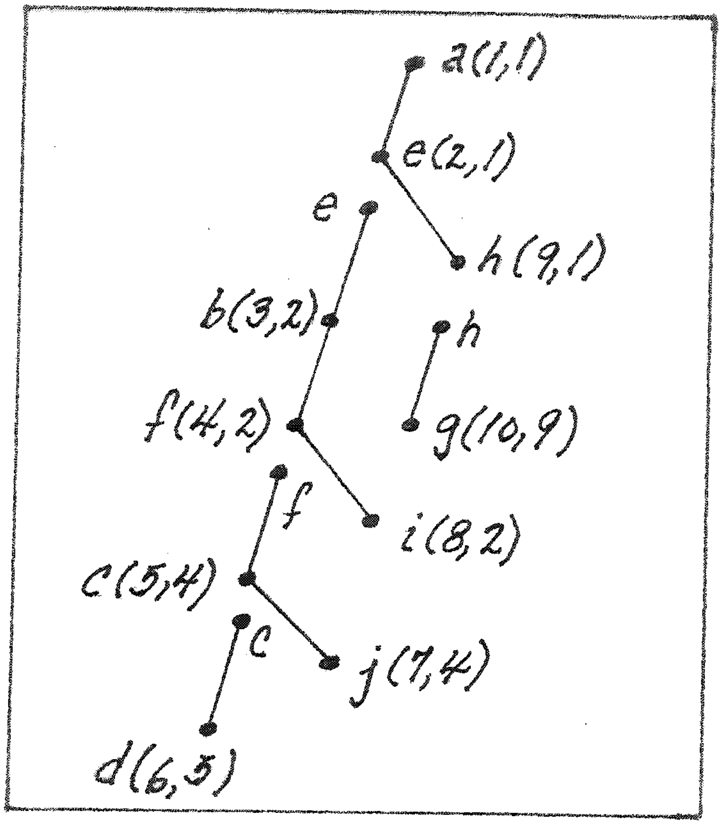
9.



(a) The first tree provides the depth-first spanning tree T for G where the order prescribed for the vertices is reverse alphabetical and the root is c.
(b) The second tree provides $(\text{low}'(v), \text{low}(v))$ for each vertex v of G (and T). These results follow from step (2) of the algorithm.

For the third tree we find $(\text{dfi}(v), \text{low}(v))$ for each vertex v. Applying step (3) of the algorithm we find the articulation points d, f, and g, and the four biconnected components.

10. The ordered pair next to each vertex v in the figure provides $(\text{dfi}(v), \text{low}(v))$. Following step (3) of the algorithm for determining the articulation points of G we see here that this graph has four articulation points – namely, c, e, f, and h. There are five biconnected components – the figure shows the spanning trees for these components.



11. No! For any loop-free connected undirected graph $G = (V, E)$ where $|V| \geq 2$, we have $\text{low}(x_1) = \text{low}(x_2) = 1$. (Note: Vertices x_1 and x_1 are always on the same biconnected component.)

12. (a) The vertex set for each graph is $V - \{v\}$. If $e = \{x, y\}$ is an edge in $\overline{G - v}$ then e is not in $G - v$, and since $x, y \neq v$, e is an edge in $\overline{G} - v$. For the opposite inclusion if $e = \{x, y\}$ is an edge in $\overline{G} - v$, then $x, y \neq v$ and e is not an edge in G, nor the subgraph $G - v$. Here e is an edge in $\overline{G - v}$.

Since $\overline{G - v}$ and $\overline{G} - v$ have the same vertex and edge sets, these graphs are equal.

(b) If v is an articulation point of G, then $\kappa(G - v) > \kappa(G)$, so $G - v$ is not connected. But then $\overline{G - v}$ is connected. So $\kappa(\overline{G} - v) = \kappa(\overline{G - v}) = 1 \leq \kappa(\overline{G})$, and consequently v cannot be an articulation point of $\overline{G}$.

13. Proof: If not, let $v \in V$ where v is an articulation point of G. Then $\kappa(G - v) > \kappa(G) = 1$. (From Exercise 19 of Section 11.6 we know that G is connected.) Now $G - v$ is disconnected with components $H_1, H_2, \ldots, H_t$, for $t \geq 2$. For $1 \leq i \leq t$, let $v_i \in H_i$. Then $H_i + v$ is a subgraph of $G - v_{i+1}$, and $\chi(H_i + v) \leq \chi(G - v_{i+1}) < \chi(G)$. (Here $v_{t+1} = v_1$.) Now let $\chi(G) = n$ and let $\{c_1, c_2, \ldots, c_n\}$ be a set of n colors. For each subgraph $H_i + v$, $1 \leq i \leq t$, we can properly color the vertices of $H_i + v$ with at most $n - 1$ colors — and can use c_1 to color vertex v for all of these t subgraphs. Then we can join these t subgraphs together at vertex v and obtain a proper coloring for the vertices of G where we use less than $n(= \chi(G))$ colors.

14. No! Consider the graph and breadth-first spanning tree shown in the figure. Here $\{c,d\} \in E$ and $\{c,d\} \notin E'$, but c is neither an ancestor nor a descendant of d in the tree T.

Supplementary Exercises

1. If G is a tree, consider G as a rooted tree. Then there are λ choices for coloring the root of G and $(\lambda - 1)$ choices for coloring each of its descendants. The result then follows by the rule of product.

   Conversely, if $P(G,\lambda) = \lambda(\lambda-1)^{n-1}$, then since the factor λ only occurs once, the graph G is connected. $P(G,\lambda) = \lambda(\lambda-1)^{n-1} = \lambda^n - (n-1)\lambda^{n-1} + \ldots + (-1)^{n-1}\lambda \Longrightarrow G$ has n vertices and $(n-1)$ edges. Therefore by part (d) of Theorem 12.5, G is a tree.

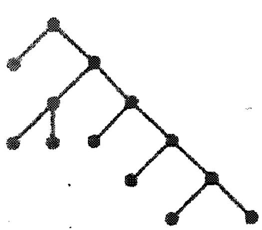
2. Model the problem with a complete quaternary tree rooted at the president.

   (a) Since there are 125 executives (vertices) there are 124 edges (phone calls).

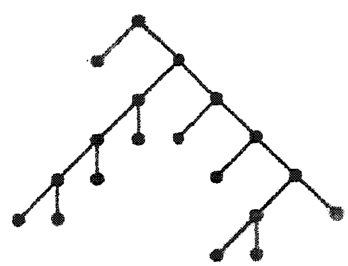
   (b) The total number of executives making calls is the number of internal vertices. From Theorem 12.6 (c), $i = (125-1)/4 = 61$. So 60 executives, in addition to the president, make calls.

3. (a) 1011001010100

   (b) (i)



   (ii)



   (c) Since the last two vertices visited in a preorder traversal are leaves, the last two symbols in the characteristic sequence of every complete binary tree are 00.

4. (a) {1,11} {3,23} {4,9} {6,15} {-5,18} {2,7} {-10,35} {-2,5}
   {-5,1,11,18} {2,3,7,23} {-10,4,9,35} {-2,5,6,15}
   {-10,-5,1,4,9,11,18,35} {-2,2,3,5,6,7,15,23}
   {-10,-5,-2,1,2,3,4,5,6,7,9,11,15,18,23,35}

   (b) $\sum_{i=1}^{k}(2^i - 1)2^{k-i}$

5. We assume that $G = (V,E)$ is connected – otherwise we work with a component of G. Since G is connected, and $\deg(v) \geq 2$ for all $v \in V$, it follows from Theorem 12.4 that G is not a tree. But every connected graph that is not a tree must contain a cycle.

6. From the first part of the definition of $\mathcal{R}$ the relation is reflexive. To establish the antisym-

metric property let $x\mathcal{R}y$ and $y\mathcal{R}x$ for $x, y \in V$. $x\mathcal{R}y \Longrightarrow x$ is on the path from r to y. If $x \neq y$ then x is encountered before y as we traverse the (unique) path from r to y. Hence by the uniqueness of such a path we cannot have $y\mathcal{R}x$. Hence $(x\mathcal{R}y \wedge y\mathcal{R}x) \Longrightarrow x = y$. Lastly, let $x, y, z \in V$ with $x\mathcal{R}y$ and $y\mathcal{R}z$. Then x is on the unique path from r to y and y is on the unique path from r to z. Since these paths are unique the path from r to z must include x so $x\mathcal{R}z$ and $\mathcal{R}$ is transitive.

7. For $1 \leq i (< n)$, let $x_i =$ the number of vertices v where $\deg(v) = i$. Then $x_1 + x_2 + \ldots + x_{n-1} = |V| = |E| + 1$, so $2|E| = 2(-1 + x_1 + x_2 + \ldots + x_{n-1})$. But $2|E| = \sum_{v \in V} \deg(v) = (x_1 + 2x_2 + 3x_3 + \ldots + (n-1)x_{n-1})$. Solving $2(-1 + x_1 + x_2 + \ldots + x_{n-1}) = x_1 + 2x_2 + \ldots + (n-1)x_{n-1}$ for x_1, we find that $x_1 = 2 + x_3 + 2x_4 + 3x_5 + \ldots + (n-3)x_{n-1} = 2 + \sum_{\deg(v_i) \geq 3} [\deg(v_i) - 2]$.

8. (a) For all $e \in E$, $e = e$, so $e\mathcal{R}e$ and $\mathcal{R}$ is reflexive.
If $e_1, e_2 \in E$ with $e_1 \neq e_2$ and $e_1\mathcal{R}e_2$, then e_1 and e_2 are edges of a cycle C of G. Hence e_2 and e_1 are edges of the cycle C, so $e_2\mathcal{R}e_1$ and $\mathcal{R}$ is symmetric.

Let e_1, e_2, e_3 be three distinct edges with $e_1\mathcal{R}e_2$ and $e_2\mathcal{R}e_3$. Let C_1 be a cycle of G containing e_1, e_2 and let C_2 be a cycle of G containing e_2, e_3. If $C_1 \neq C_2$, let C be the cycle of G made up from the edges of C_1, C_2, where common edges are removed. (In terms of edges, $C = C_1 \Delta C_2$.) Since e_1, e_3 are on C we have $e_1\mathcal{R}e_3$, and $\mathcal{R}$ is transitive.

(b) The partition of E induced by $\mathcal{R}$ provides the biconnected components of G.

9. (a) G^2 is isomorphic to K_5.
(b) G^2 is isomorphic to K_4.
(c) G^2 is isomorphic to K_{n+1}, so the number of new edges is $\binom{n+1}{2} - n = \binom{n}{2}$.

(d) If G^2 has an articulation point x, then there exists $u, v \in V$ such that every path (in G^2) from u to v passes through x. (This follows from Exercise 2 of Section 12.5.) Since G is connected, there exists a path P (in G) from u to v. If x is not on this path (which is also a path in G^2), then we contradict x being an articulation point in G^2. Hence the path P (in G) passes through x, and we can write $P: u \rightarrow u_1 \rightarrow \ldots \rightarrow u_{n-1} \rightarrow u_n \rightarrow x \rightarrow v_m \rightarrow v_{m-1} \rightarrow \ldots \rightarrow v_1 \rightarrow v$. But then in G^2 we add the edge $\{u_n, v_m\}$, and the path P' (in G^2) given by $P': u \rightarrow u_1 \rightarrow \ldots \rightarrow u_{n-1} \rightarrow u_n \rightarrow v_m \rightarrow v_{m-1} \rightarrow \ldots \rightarrow v_1 \rightarrow v$ does not pass through x. So x is not an articulation point of G^2, and G^2 has no articulation points.

10. (a) For the minimum value of $|V|$ we have six leaves at level 8 and the other $6^7 - 1$ leaves are at level 7. Since there are $6^7 + 5$ leaves, it follows from part (c) of Theorem 12.6 that $|V| = (6/5)[(6^7 + 5) - 1] + 1 = 335,929$.
For the maximum value of $|V|$ we have one leaf at level 7 and the other $(6^7 - 1)(6)$ leaves are at level 8. So there are $(6^8 - 6) + 1 = 6^8 - 5$ leaves in total. Once again we use part (c) of Theorem 12.6 to find that $|V| = (6/5)[(6^8 - 5) - 1] + 1 = 2,015,533$.

(b) Let ℓ denote the number of leaves in T. For the minimum case $\ell = (m^{h-1} - 1) + m =$

$m^{h-1}+(m-1)$ and $|V|=[m/(m-1)][m^{h-1}+(m-1)-1]+1$. For the maximum case we have $\ell=m(m^{h-1}-1)+1=m^h-m+1$ and $|V|=[m/(m-1)][m^h-m]+1$.

11. (a) $\ell_n=\ell_{n-1}+\ell_{n-2}$, for $n\geq 3$ and $\ell_1=\ell_2=1$. Since this is precisely the Fibonacci recurrence relation, we have $l_n=F_n$, the nth Fibonacci number, for $n\geq 1$.
(b) $i_n=i_{n-1}+i_{n-2}+1$, $n\geq 3$, $i_1=i_2=0$. The summand "+1" arises when we count the root, an internal vertex.
(Homogeneous part of solution):

$$i_n^{(h)}=i_{n-1}^{(h)}+i_{n-2}^{(h)},\ n\geq 3$$

$$i_n^{(h)}=A\alpha^n+B\beta^n,\text{ where }\alpha=(1+\sqrt{5})/2\text{ and }\beta=(1-\sqrt{5})/2.$$

(Particular part of solution):

$$i_n^{(p)}=C,\text{ a constant}$$

Upon substitution into the recurrence relation $i_n=i_{n-1}+i_{n-2}+1$, $n\geq 3$, we find that

$$C=C+C+1,$$

so $C=-1$,
and $i_n=A\alpha^n+B\beta^n-1$.
With $i_1=i_2=0$ we have

$$0=i_1=A\alpha+B\beta-1$$
$$0=i_2=A\alpha^2+B\beta^2-1,$$

and consequently,
$B=(\alpha-1)/[\beta(\alpha-\beta)]=[((1+\sqrt{5})/2)-1]/[((1-\sqrt{5})/2)(\sqrt{5})]=$
$[1+\sqrt{5}-2]/[(1-\sqrt{5})(\sqrt{5})]=-1/\sqrt{5}$, and $A=[1-B\beta]/\alpha=1/\sqrt{5}$. Therefore,

$$i_n=(1/\sqrt{5})\alpha^n-(1/\sqrt{5})\beta^n-1=F_n-1,$$

where F_n denotes the nth Fibonacci number, for $n\geq 1$.
(c) $v_n=\ell_n+i_n$, for all $n\in\mathbf{Z}^+$. Consequently, $v_n=F_n+F_n-1=2F_n-1$, where, as in parts (a) and (b), F_n denotes the nth Fibonacci number.

12. (a) For the graph G_3 in Fig. 12.48 (d) there are 12 nonidentical spanning trees in total.
(b) Consider the graph G_{n+1}. Here the nonidentical spanning trees arise from the following three cases any two of which are mutually exclusive.

(1) The edge $\{a,n+1\}$ is used: Here we can then use any of the t_n nonidentical spanning trees for G_n, and the result is a spanning tree for G_{n+1}.

(2) The edge $\{n+1,b\}$ is used: Here we have a situation similar to that in (1) and we get t_n additional nonidentical spanning trees for G_{n+1}.

(3) The edges $\{a,n+1\}$, $\{n+1,b\}$ are both used: Now for each vertex i, where $1\leq i\leq n$, we have two choices — include the edge $\{a,i\}$ or the edge $\{i,b\}$ (but *not* both). In this way we obtain the final 2^n nonidentical spanning trees for G_{n+1}.

The results in (1), (2), and (3) lead us to the following recurrence relation:

$$(*)\qquad t_{n+1}=2t_n+2^n,\qquad t_1=1,\qquad n\geq 1.$$

(Homogeneous Solution): $t_{n+1} = 2t_n$

$$t_n^{(h)} = A(2^n),\ A \text{ a constant.}$$

(Particular Solution): $t_n^{(p)} = Bn(2^n)$, B a constant.

Substituting $t_n^{(p)}$ into equation $(*)$ we find that

$B(n+1)(2^{n+1}) = 2Bn(2^n) + 2^n$

$Bn(2^{n+1}) + B(2^{n+1}) = Bn(2^{n+1}) + 2^n$

Consequently, $2B(2^n) = 2^n$ and $2B = 1$, or $B = 1/2$. Therefore, $t_n = A(2^n)+(1/2)n(2^n) = A(2^n) + n2^{n-1}$.

Since $t_1 = 1 = A(2) + 1$, $A = 0$ and $t_n = n2^{n-1}$, $n \geq 1$.

13. (a) For the spanning trees of G there are two mutually exclusive and exhaustive cases:

(i) The edge $\{x_1, y_1\}$ is in the spanning tree: These spanning trees are counted in b_n.

(ii) The edge $\{x_1, y_1\}$ is not in the spanning tree: In this case the edges $\{x_1, x_2\}$, $\{y_1, y_2\}$ are both in the spanning tree. Upon removing the edges $\{x_1, x_2\}, \{y_1, y_2\}$, and $\{x_1, y_1\}$, from the original ladder graph, we now need a spanning tree for the resulting smaller ladder graph with $n-1$ rungs. There are a_{n-1} spanning trees in this case.

(b) Here there are three mutually exclusive and exhaustive cases:

(i) The edges $\{x_1, x_2\}$ and $\{y_1, y_2\}$ are both in the spanning tree: Delete $\{x_1, x_2\}$, $\{y_1, y_2\}$, and $\{x_1, y_1\}$ from the graph. Then b_{n-1} counts those spanning trees for ladders with $n-1$ rungs where $\{x_2, y_2\}$ is included. For each of these delete $\{x_2, y_2\}$ and add $\{x_1, x_2\}$, $\{y_1, y_2\}$ and $\{x_1, y_1\}$.

(ii) The edge $\{x_1, x_2\}$ is in the spanning tree but the edge $\{y_1, y_2\}$ is not: Now the removal of the edges $\{x_1, y_1\}, \{x_1, x_2\}$, and $\{y_1, y_2\}$ from G results in a subgraph that is a ladder graph on $n-1$ rungs. This subgraph has a_{n-1} spanning trees.

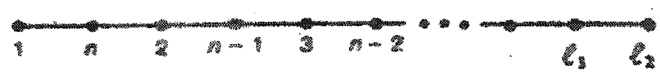
(iii) Here the edge $\{y_1, y_2\}$ is in the spanning tree but the edge $\{x_1, x_2\}$ is not: As in case (ii) there are a_{n-1} spanning trees.

On the basis of the preceding argument we have $b_n = b_{n-1} + 2a_{n-1}$, $n \geq 2$.

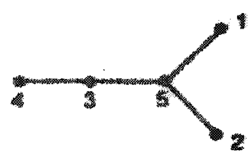
(c) $a_n = a_{n-1} + b_n$
$b_n = b_{n-1} + 2a_{n-1}$
$a_n = a_{n-1} + b_{n-1} + 2a_{n-1} = 3a_{n-1} + b_{n-1}$
$b_n = a_n - a_{n-1}$, so $b_{n-1} = a_{n-1} - a_{n-2}$
$a_n = 3a_{n-1} + a_{n-1} - a_{n-2} = 4a_{n-1} - a_{n-2}$, $n \geq 3$, $a_1 = 1$, $a_2 = 4$
$a_n - 4a_{n-1} + a_{n-2} = 0$
$r^2 - 4r + 1 = 0$
$r = (1/2)(4 \pm \sqrt{16-4}) = 2 \pm \sqrt{3}$
So $a_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$
$a_0 = 0 \Longrightarrow A + B = 0 \Longrightarrow B = -A$.

$a_1 = 1 = A(2+\sqrt{3}) - A(2-\sqrt{3}) = 2A\sqrt{3} \Longrightarrow A = 1/2\sqrt{3}$ and $B = -1/2\sqrt{3}$.
Therefore $a_n = (1/(2\sqrt{3}))[(2+\sqrt{3})^n - (2-\sqrt{3})^n], n \geq 0$.

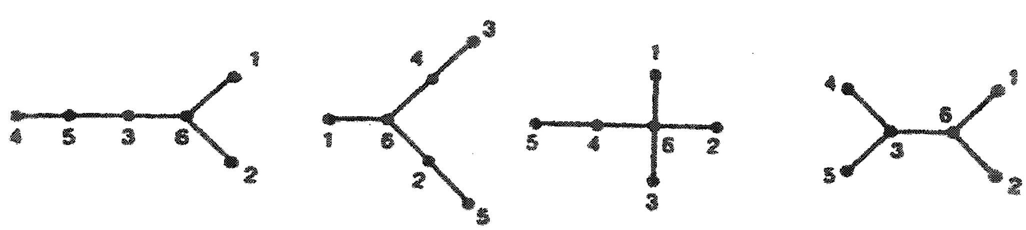
14. (a)



For n even, $\ell_1 = n/2$ and $\ell_2 = \ell_1 + 1$
For n odd, $\ell_1 = \ell_2 + 1$ and $\ell_2 = \lceil n/2 \rceil$
(b) Label the vertex of degree 1 with the label 1. Label the other n vertices (one vertex per label) with the labels $2, 3, \ldots, n, n+1$.
(c) For $|V| = 4$ the only trees are a path of length 3 and $K_{1,3}$. These are handled by parts (a) and (b), respectively.
For $|V| = 5$ there are three trees: (1) A path of length 4; (2) $K_{1,4}$; and (3) The tree with a vertex of degree 3. Trees (1) and (2) are handled by parts (a) and (b), respectively. The third tree may be labeled as follows.



For $|V| = 6$, there are six trees. The path of length 5 and $K_{1,5}$ are dealt with by parts (a) and (b), respectively. The other four trees may be labeled as follows.

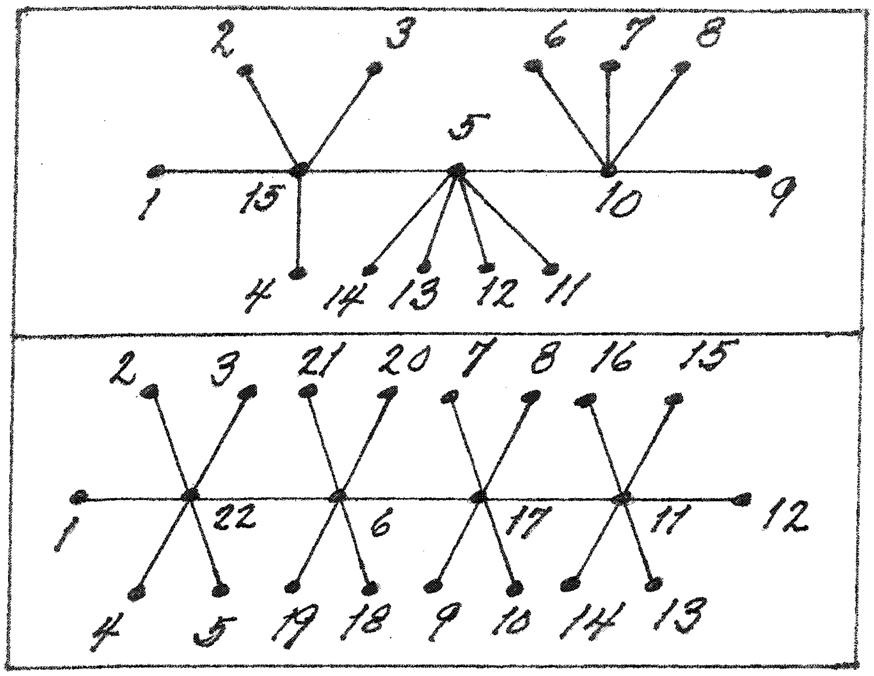


15. (a) (i) 3 (ii) 5

(b) $a_n = a_{n-1} + a_{n-2}$, $n \geq 5$, $a_3 = 2$, $a_4 = 3$.

$a_n = F_{n+1}$, the $(n+1)$st Fibonacci number.

16.



17. Here the input consists of

(a) the k (≥ 3) vertices of the spine – ordered from left to right as $v_1, v_2, \ldots, v_k$;

(b) $\deg(v_i)$, in the caterpillar, for all $1 \leq i \leq k$; and

(c) n, the number of vertices in the caterpillar, with $n \geq 3$.

If $k = 3$, the caterpillar is the complete bipartite graph (or, star) $K_{1,n-1}$, for some $n \geq 3$. We label v_1 with 1 and the remaining vertices with $2, 3, \ldots, n$. This provides the edge labels (the absolute value of the difference of the vertex labels) $1, 2, 3, \ldots, n-1$ – a graceful labeling.

For $k > 3$ we consider the following.

$\ell := 2$ $\{\ell$ is the largest low label$\}$

$h := n - 1$ $\{h$ is the smallest high label$\}$

label v_1 with 1

label v_2 with n

for $i := 2$ **to** $k - 1$ **do**

  if $2\lfloor i/2 \rfloor = i$ **then** $\{i$ is even$\}$

    begin

      if v_i has unlabeled leaves that are not on the spine **then**

        assign the $\deg(v_i) - 2$ labels from ℓ

        to $\ell + \deg(v_i) - 3$ to these leaves of v_i

      assign the label $\ell + \deg(v_i) - 2$ to v_{i+1}

```
            ℓ := ℓ + deg(v_i) − 1
        end
    else
        begin
            if  v_i has unlabeled leaves that are not on the spine then
                assign the deg(v_i) − 2 labels from h − [deg(v_i) − 3]
                to h to these leaves of v_i
            assign the label h − deg(v_i) + 2 to v_{i+1}
            h := h − deg(v_i) + 1
        end
```

18. (a) Fig. 12.50 10001000010001

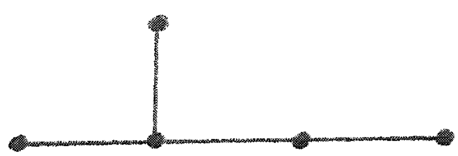
Fig. 12.51 1000010000100001000001

(b) Yes, when the caterpillar is a path.

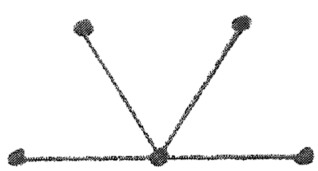
(c) Yes, when the caterpillar is the complete bipartite graph (or, star) $K_{1,n-1}$, where $n \geq 3$.

(d)

1111



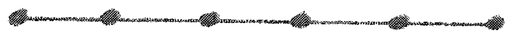
1011



1001

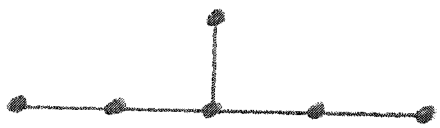
There are three nonisomorphic caterpillars on five vertices. Two of the corresponding binary strings are palindromes.

(e)

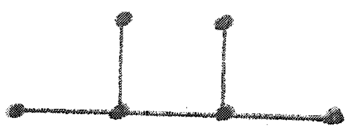


11111

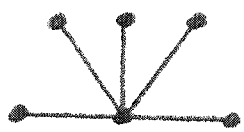
10111



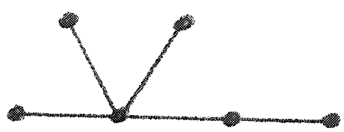
11011



10101



10001



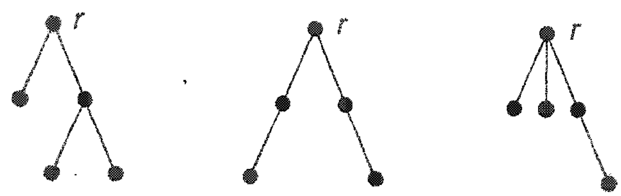
10011

There are six nonisomorphic caterpillars on six vertices. Four of the corresponding binary strings are palindromes.

(f) Since the caterpillar has n vertices it has $n-1$ edges, and its binary string has $n-1$ bits, where the first and last bits are 1s. For each of the remaining $n-3$ bits there are two choices – 0 or 1. This gives us 2^{n-3} binary strings. However, for each binary string s that is not a palindrome, the reversal of that string – namely, s^R – corresponds with a caterpillar that is isomorphic to the caterpillar determined by s. So each pair of these strings – s and s^R – determines only one (nonisomorphic) caterpillar. Further, each palindrome also determines a unique caterpillar. For the palindromes we have two choices for each of the first $\lceil (n-3)/2 \rceil$ positions (after the first 1). So there are $2^{\lceil (n-3)/2 \rceil}$ binary strings that are palindromes. Consequently, $2^{n-3} + 2^{\lceil (n-3)/2 \rceil}$ counts each of the nonisomorphic caterpillars on n vertices twice. Therefore, the number of nonisomorphic caterpillars on n vertices, for $n \geq 3$, is $(1/2)(2^{n-3} + 2^{\lceil (n-3)/2 \rceil})$.

19. (a) 1, −1, 1, 1, −1, −1   1, 1, −1, 1, −1, −1   1, −1, 1, −1, 1, −1

(b)



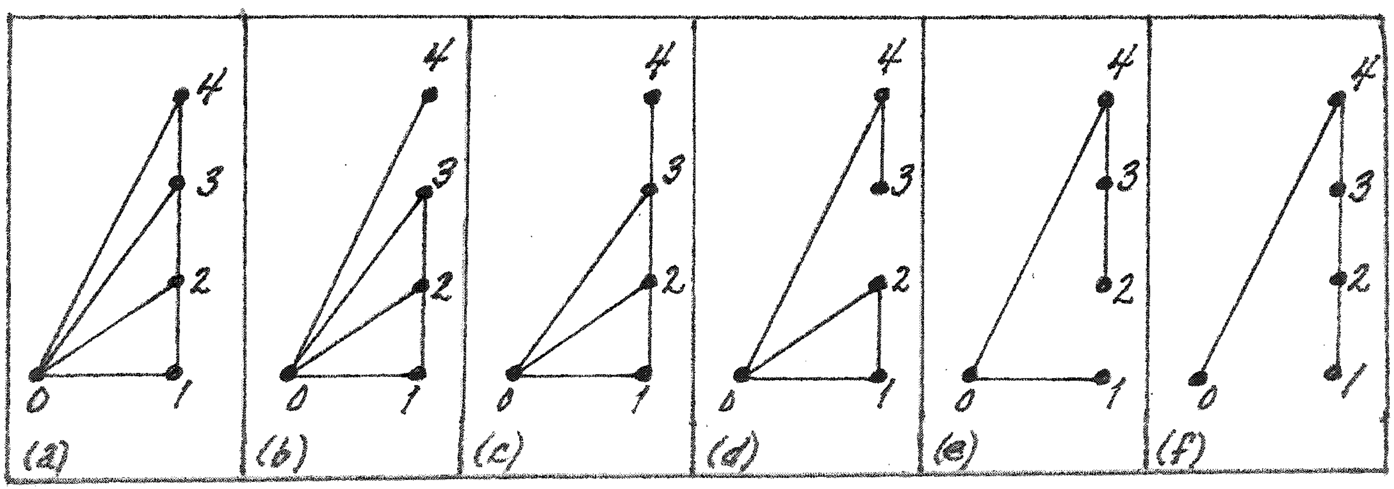
In total there are 14 ordered rooted trees on five vertices.

(c) This is another example where the Catalan numbers arise. There are $(\frac{1}{n+1})\binom{2n}{n}$ ordered rooted trees on $n+1$ vertices.

20. (a) Consider the case for $n = 4$, shown in part (a) of the figure. The five spanning subgraphs in parts (b)–(f) of the figure provide pairwise mutually exclusive situations that account for all the spanning trees of the graph given in part (a). As we scan the figure from left to right we find that

$$t_4 = t_3 + t_3 + t_2 + t_1 + t_0 = t_3 + \sum_{i=0}^{3} t_i.$$

This result generalizes to provide $t_{n+1} = t_n + \sum_{i=0}^{n} t_i$.



(b)

$$\begin{aligned} t_{n+1} &= t_n + \textstyle\sum_{i=0}^{n} t_i \\ &= 2t_n + \textstyle\sum_{i=0}^{n-1} t_i \\ &= 2t_n + [t_{n-1} + \textstyle\sum_{i=0}^{n-1} t_i] - t_{n-1} \\ &= 3t_n - t_{n-1}, n \geq 2 \end{aligned}$$

(c) $t_{n+1} = 3t_n - t_{n-1}, n \geq 2, t_2 = 3, t_1 = 1.$
Let $t_n = Ar^n$, $A \neq 0, r \neq 0.$
$r^2 - 3r + 1 = 0$
$r = (3 \pm \sqrt{5})/2$
So $t_n = B[(3+\sqrt{5})/2]^n + C[(3-\sqrt{5})/2]^n$.
Since $1 = t_1 = B[(3+\sqrt{5})/2] + C[(3-\sqrt{5})/2]$ and
$3 = t_2 = B[(3+\sqrt{5})/2)]^2 + C[(3-\sqrt{5})/2]^2$,
we find that
$B = 1/\sqrt{5}, C = -1/\sqrt{5}.$
Consequently, $t_n = (1/\sqrt{5})[(3+\sqrt{5})/2]^n - (1/\sqrt{5})[(3-\sqrt{5})/2]^n$, $n \geq 1$, $t_0 = 1$.

Recall that the nth Fibonacci number F_n is given by

$$F_n = (1/\sqrt{5})[(1+\sqrt{5})/2]^n - (1/\sqrt{5})[(1-\sqrt{5})/2]^n, n \geq 0.$$

For $n \geq 1$, $F_{2n} = (1/\sqrt{5})[(1+\sqrt{5})/2]^{2n} - (1/\sqrt{5})[(1-\sqrt{5})/2]^{2n} = (1/\sqrt{5})[(1+\sqrt{5})^2/4]^n - (1/\sqrt{5})[(1-\sqrt{5})^2/4]^n = (1/\sqrt{5})[(3+\sqrt{5})/2]^n - (1/\sqrt{5})[(3-\sqrt{5})/2]^n = t_n$.

21. (a) There are $\binom{5}{3} - 2 = 8$ nonidentical (though some are isomorphic) spanning trees for the kite induced by a, b, c, d. Since there are four vertices, a spanning tree has three edges and the only selections of three edges that do not provide a spanning tree are $\{a,c\}, \{b,c\}, \{a,b\}$ and $\{a,b\}, \{a,d\}, \{b,d\}$.

(b) There are $8 \cdot 1 \cdot 8 \cdot 1 \cdot 8 \cdot 1 \cdot 8 = 8^4$ nonidentical (though some are isomorphic) spanning trees of G that do not contain edge $\{c,h\}$. These spanning trees must include the edges $\{g,k\}, \{l,p\}$, and $\{d,o\}$, and there are eight nonidentical (though some are isomorphic) spanning trees for each of the four subgraphs that are kites.

(c) Consider the kite induced by a, b, c, d. There are eight two-tree forests for this kite that have no path between c and d. These forests can be obtained from the five edges of the kite by removing three edges at a time, as follows:

(i)	$\{a,b\}, \{a,c\}, \{b,c\}$	(ii)	$\{a,c\}, \{b,c\}, \{b,d\}$
(iii)	$\{a,c\}, \{a,d\}, \{b,c\}$	(iv)	$\{a,b\}, \{a,d\}, \{b,d\}$
(v)	$\{a,d\}, \{b,c\}, \{b,d\}$	(vi)	$\{a,c\}, \{a,d\}, \{b,d\}$
(vii)	$\{a,b\}, \{a,d\}, \{b,c\}$	(viii)	$\{a,b\}, \{a,c\}, \{b,d\}$

Vertex c is isolated for (i), (ii), (iii). For (iv), (v), (vi), vertex d is isolated. The forests for (vii), (viii) each contain two disconnected edges: $\{a,c\}, \{b,d\}$ for (vii) and $\{a,d\}, \{b,c\}$ for (viii).

Consequently, there are $4 \cdot 8 \cdot 1 \cdot 8 \cdot 1 \cdot 8 \cdot 1 \cdot 8 \cdot 1 = 4 \cdot 8^4$ nonidentical (though some are isomorphic) spanning trees for G that contain each of the four edges $\{c,h\}, \{g,k\}, \{l,p\}$, and $\{d,o\}$.

(d) In total there are $4 \cdot 8^4 + 4 \cdot 8^4 = 2(4 \cdot 8^4)$ nonidentical (though some are isomorphic) spanning trees for G.

(e) $2n8^n$