CHAPTER 5
RELATIONS AND FUNCTIONS

Section 5.1

1. $A \times B = \{(1,2),(2,2),(3,2),(4,2),(1,5),(2,5),(3,5),(4,5)\}$
$B \times A = \{(2,1),(2,2),(2,3),(2,4),(5,1),(5,2),(5,3),(5,4)\}$
$A \cup (B \times C) = \{1,2,3,4,(2,3),(2,4),(2,7),(5,3),(5,4),(5,7)\}$
$(A \cup B) \times C = (A \times C) \cup (B \times C) = \{(1,3),(2,3),(3,3),(4,3),(5,3),(1,4),(2,4),(3,4),(4,4),(5,4),(1,7),(2,7),(3,7),(4,7),(5,7)\}$

2. (a) $\{(1,2)\}; \{(1,2),(1,4),(1,5),(2,2),(2,4)\}; A \times B$
(b) $\{(1,1),(2,2),(3,3)\}; \{(1,1),(1,2),(1,3),(2,2),(3,3)\}; \{(1,2),(2,1),(3,3)\}$.

3. (a) $|A \times B| = |A||B| = 9$
(b) Since a relation from A to B is a subset of $A \times B$, there are 2^9 relations from A to B.
(c) Since $|A \times A| = 9$, there are 2^9 relations on A.
(d) For the other seven ordered pairs in $A \times B$ there are two choices: include it in the relation or leave it out. Hence there are 2^7 relations from A to B that contain (1,2) and (1,5).
(e) $\binom{9}{5}$ (f) $\binom{9}{7} + \binom{9}{8} + \binom{9}{9}$

4. If either A or B is $\emptyset$ and when $A = B$.

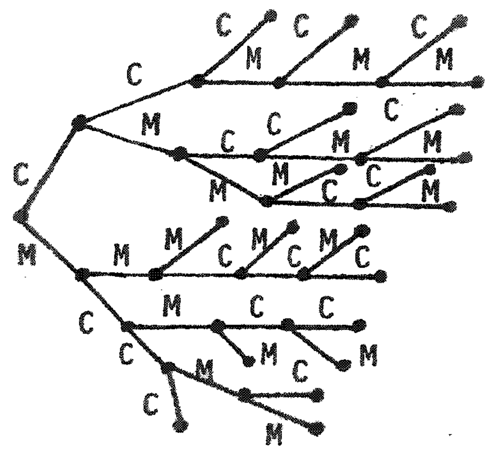
5. (a) Assume that $A \times B \subseteq C \times D$ and let $a \in A$ and $b \in B$. Then $(a,b) \in A \times B$, and since $A \times B \subseteq C \times D$ we have $(a,b) \in C \times D$. But $(a,b) \in C \times D \Rightarrow a \in C$ and $b \in D$. Hence, $a \in A \Rightarrow a \in C$, so $A \subseteq C$, and $b \in B \Rightarrow b \in D$, so $B \subseteq D$.
Conversely, suppose that $A \subseteq C$ and $B \subseteq D$, and that $(x,y) \in A \times B$. Then $(x,y) \in A \times B \Rightarrow x \in A$ and $y \in B \Rightarrow x \in C$ (since $A \subseteq C$) and $y \in D$ (since $B \subseteq D$) $\Rightarrow (x,y) \in C \times D$. Consequently, $A \times B \subseteq C \times D$.

(b) If any of the sets A, B, C, D is empty we still find that

$$[(A \subseteq C) \wedge (B \subseteq D)] \Rightarrow [A \times B \subseteq C \times D].$$

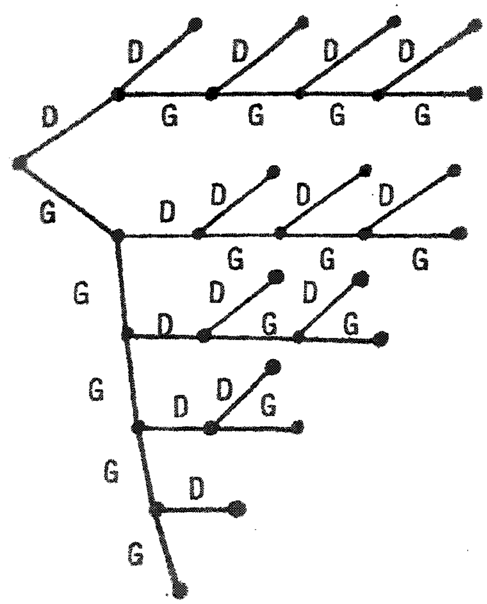
However, the converse need not hold. For example, let $A = \emptyset$, $B = \{1,2\}$, $C = \{1,2\}$ and $D = \{1\}$. Then $A \times B = \emptyset$ — if not, there exists an ordered pair (x,y) in $A \times B$, and this means that the empty set A contains an element x. And so $A \times B = \emptyset \subseteq C \times D$ — but $B = \{1,2\} \nsubseteq \{1\} = D$.

6.



7. (a) Since $|A| = 5$ and $|B| = 4$ we have $|A \times B| = |A||B| = 5 \cdot 4 = 20$. Consequently, $A \times B$ has 2^{20} subsets, so $|\mathcal{P}(A \times B)| = 2^{20}$.
(b) If $|A| = m$ and $|B| = n$, for $m, n \in \mathbf{N}$, then $|A \times B| = mn$. Consequently, $|\mathcal{P}(A \times B)| = 2^{mn}$.

8.



9. (b) $A \times (B \cup C) = \{(x,y)|x \in A$ and $y \in (B \cup C)\} = \{(x,y)|x \in A$ and $(y \in B$ or $y \in C)\} = \{(x,y)|(x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)\} = \{(x,y)|x \in A$ and $y \in B\} \cup \{(x,y)|x \in A$ and $y \in C\} = (A \times B) \cup (A \times C)$.
(c) & (d) The proofs here are similar to that given in part (b).

10. $1 + 2 + 2(3) + 2(3)(5) = 39;\ 38$

11. $(x,y) \in A \times (B - C) \Longleftrightarrow x \in A$ and $y \in B - C \Longleftrightarrow x \in A$ and $(y \in B$ and $y \notin C) \Longleftrightarrow (x \in A$ and $y \in B)$ and $(x \in A$ and $y \notin C) \Longleftrightarrow (x,y) \in A \times B$ and $(x,y) \notin A \times C \Longleftrightarrow (x,y) \in (A \times B) - (A \times C)$..

12. $2^{(3|B|)} = 4096 \Longrightarrow 3|B| = 12 \Longrightarrow |B| = 4$.

13. (a) (1) $(0,2) \in \mathcal{R}$; and
(2) If $(a,b) \in \mathcal{R}$, then $(a+1, b+5) \in \mathcal{R}$.
(b) From part (1) of the definition we have $(0,2) \in \mathcal{R}$. By part (2) of the definition we then find that
(i) $(0,2) \in \mathcal{R} \Rightarrow (0+1, 2+5) = (1,7) \in \mathcal{R}$;
(ii) $(1,7) \in \mathcal{R} \Rightarrow (1+1, 7+5) = (2,12) \in \mathcal{R}$;
(iii) $(2,12) \in \mathcal{R} \Rightarrow (2+1, 12+5) = (3,17) \in \mathcal{R}$; and
(iv) $(3,17) \in \mathcal{R} \Rightarrow (3+1, 17+5) = (4,22) \in \mathcal{R}$.

14. (a) (1) $(1,1), (2,1) \in \mathcal{R}$; and
(2) If $(a,b) \in \mathcal{R}$, then $(a+1, b+1)$ and $(a+1, b)$ are in $\mathcal{R}$.
(b) Start with (2,1) in $\mathcal{R}$ — from part (1) of the definition. Then by part (2) we get
(i) $(2,1) \in \mathcal{R} \Rightarrow (2+1, 1+1) = (3,2) \in \mathcal{R}$;
(ii) $(3,2) \in \mathcal{R} \Rightarrow (3+1, 2) = (4,2) \in \mathcal{R}$; and
(iii) $(4,2) \in \mathcal{R} \Rightarrow (4+1, 2) = (5,2) \in \mathcal{R}$.

Start with (1,1) in $\mathcal{R}$ — from part (1) of the definition. Then we find from part (2) that
(i) $(1,1) \in \mathcal{R} \Rightarrow (1+1, 1+1) = (2,2) \in \mathcal{R}$;
(ii) $(2,2) \in \mathcal{R} \Rightarrow (2+1, 2+1) = (3,3) \in \mathcal{R}$; and
(iii) $(3,3) \in \mathcal{R} \Rightarrow (3+1, 3+1) = (4,4) \in \mathcal{R}$.

Section 5.2

1. (a) Function: Range $= \{7, 8, 11, 16, 23, \ldots\}$
(b) Relation, not a function. For example, both (4,2) and $(4,-2)$ are in the relation.
(c) Function: Range = the set of all real numbers.
(d) Relation, not a function. Both (0,1) and $(0,-1)$ are in the relation.
(e) Since $|R| > 5$, R cannot be a function.

2. The formula cannot be used for the domain of real numbers since $f(\sqrt{2})$, $f(-\sqrt{2})$ are undefined. Since $\sqrt{2}, -\sqrt{2} \notin \mathbf{Z}$ the formula does define a real valued function on the domain $\mathbf{Z}$.

3. (a) $\{(1,x),(2,x),(3,x),(4,x)\}$, $\{(1,y),(2,y),(3,y),(4,y)\}$, $\{(1,z),(2,z),(3,z),(4,z)\}$
$\{(1,x),(2,y),(3,x),(4,y)\}$, $\{(1,x),(2,y),(3,z),(4,x)\}$
(b) 3^4 (c) 0 (d) 4^3 (e) 24 (f) 3^3 (g) 3^2 (h) 3^2

4. $3^{|A|} = 2187 \Longrightarrow |A| = 7$

5. (a) $A \cap B = \{(x,y)|y = 2x+1 \text{ and } y = 3x\}$
$2x + 1 = 3x \Rightarrow x = 1$
So $A \cap B = \{(1,3)\}$.
(b) $B \cap C = \{(x,y)|y = 3x \text{ and } y = x - 7\}$
$3x = x - 7 \Rightarrow 2x = -7$, so $x = -7/2$.
Consequently, $B \cap C = \{(-7/2, 3(-7/2)\} = \{(-7/2, -21/2)\}$.
(c) $\overline{\overline{A} \cup \overline{C}} = \overline{\overline{A}} \cap \overline{\overline{C}} = A \cap C = \{(x,y)|y = 2x+1 \text{ and } y = x - 7\}$
Now $2x + 1 = x - 7 \Rightarrow x = -8$, and so $A \cap C = \{(-8,-15)\}$.
(d) We know that $\overline{B} \cup \overline{C} = \overline{B \cap C}$, and since $B \cap C = \{(-7/2, -21/2)\}$ we have $\overline{B} \cup \overline{C} = \mathbf{R}^2 - \{(-7/2,-21/2)\} = \{(x,y)|x \neq -7/2 \text{ or } y \neq -21/2\}$.

6.
(a) (i) $A \cap B = \{(1,3)\}$ (ii) $B \cap C = \{\} = \emptyset$
(iii) $\overline{\overline{A} \cup \overline{C}} = \{(-8,-15)\}$ (iv) $\overline{B} \cup \overline{C} = \mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$

(b) (i) $A \cap B = \{(1,3)\}$ (ii) $B \cap C = \{\} = \emptyset$
(iii) $\overline{\overline{A} \cup \overline{C}} = \emptyset$ (iv) $\overline{B} \cup \overline{C} = \mathbf{Z}^+ \times \mathbf{Z}^+$

7.
(a) $\lfloor 2.3 - 1.6 \rfloor = \lfloor 0.7 \rfloor = 0$ (b) $\lfloor 2.3 \rfloor - \lfloor 1.6 \rfloor = 2 - 1 = 1$
(c) $\lceil 3.4 \rceil \lfloor 6.2 \rfloor = 4 \cdot 6 = 24$ (d) $\lfloor 3.4 \rfloor \lceil 6.2 \rceil = 3 \cdot 7 = 21$
(e) $\lfloor 2\pi \rfloor = 6$ (f) $2\lceil \pi \rceil = 8$

8. (a) True (b) False: Let $a = 1.5$. Then $\lfloor 1.5 \rfloor = 1 \neq 2 = \lceil 1.5 \rceil$
(c) True (d) False: Let $a = 1.5$. Then $-\lceil a \rceil = -2 \neq -1 = \lceil -a \rceil$.

9. (a) $\ldots [-1, -6/7) \cup [0, 1/7) \cup [1, 8/7) \cup [2, 15/7) \cup \ldots$
(b) $[1, 8/7)$ (c) $\mathbf{Z}$ (d) $\mathbf{R}$

10. $\mathbf{R}$

11. (a) $\ldots \cup (-7/3, -2] \cup (-4/3, -1] \cup (-1/3, 0] \cup (2/3, 1] \cup (5/3, 2] \cup \ldots = \bigcup_{m \in \mathbf{Z}^+} (m - 1/3, m]$

(b) $\ldots \cup ((-2n-1)/n, -2] \cup ((-n-1)/n, -1] \cup (-1/n, 0] \cup ((n-1)/n, 1] \cup ((2n-1)/n, 2] \cup \ldots$

$$= \bigcup_{m \in \mathbf{Z}^+} (m - 1/n, m]$$

12. Proof: (Case 1: $k|n$) Here $n = qk$ for $q \in \mathbf{Z}^+$, and $(n-1)/k = (qk-1)/k = q - (1/k)$ with $q - 1 \leq q - (1/k) < q$. Therefore $\lceil n/k \rceil = \lceil q \rceil = q = (q-1)+1 = \lfloor (n-1)/k \rfloor + 1$. (Case 2: $k \not| n$) Now we have $n = qk + r$, where $q, r \in \mathbf{Z}^+$ with $r < k$, and $n/k = q + (r/k)$ with $0 < (r/k) < 1$. So $n - 1 = qk + (r-1)$ and $(n-1)/k = q + [(r-1)/k]$ with $0 \leq [(r-1)/k] < 1$. Consequently, $\lceil n/k \rceil = \lceil q + (r/k) \rceil = q + 1 = \lfloor (n-1)/k \rfloor + 1$.

13. a) Proof (i): If $a \in \mathbf{Z}^+$, then $\lceil a \rceil = a$ and $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1 \rfloor = 1$. If $a \notin \mathbf{Z}^+$, write $a = n + c$, where $n \in \mathbf{Z}^+$ and $0 < c < 1$. Then $\lceil a \rceil / a = (n+1)/(n+c) = 1 + (1-c)/(n+c)$, where $0 < (1-c)/(n+c) < 1$. Hence $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1 + (1-c)/(n+c) \rfloor = 1$.

Proof (ii): For $a \in \mathbf{Z}^+$, $\lfloor a \rfloor = a$ and $\lceil \lfloor a \rfloor / a \rceil = \lceil 1 \rceil = 1$. When $a \notin \mathbf{Z}^+$, let $a = n + c$, where $n \in \mathbf{Z}^+$ and $0 < c < 1$. Then $\lfloor a \rfloor / a = n/(n+c) = 1 - [c/(n+c)]$, where $0 < c/(n+c) < 1$. Consequently $\lceil \lfloor a \rfloor / a \rceil = \lceil 1 - (c/(n+c)) \rceil = 1$.

b) Consider $a = 0.1$. Then
(i) $\lfloor \lceil a \rceil / a \rfloor = \lfloor 1/0.1 \rfloor = \lfloor 10 \rfloor = 10 \neq 1$; and
(ii) $\lceil \lfloor a \rfloor / a \rceil = \lceil 0/0.1 \rceil = 0 \neq 1$.

In fact (ii) is false for all $0 < a < 1$, since $\lceil \lfloor a \rfloor / a \rceil = 0$ for all such values of a. In the case of (i), when $0 < a \leq 0.5$, it follows that $\lceil a \rceil / a \geq 2$ and $\lfloor \lceil a \rceil / a \rfloor \geq 2 \neq 1$. However, for $0.5 < a < 1$, $\lceil a \rceil / a = 1/a$ where $1 < 1/a < 2$, and so $\lfloor \lceil a \rceil / a \rfloor = 1$ for $0.5 < a < 1$.

14.

(a)
$$
\begin{aligned}
a_2 &= 2a_{\lfloor 2/2 \rfloor} = 2a_1 = 2\\
a_3 &= 2a_{\lfloor 3/2 \rfloor} = 2a_1 = 2\\
a_4 &= 2a_{\lfloor 4/2 \rfloor} = 2a_2 = 4\\
a_5 &= 2a_{\lfloor 5/2 \rfloor} = 2a_2 = 4\\
a_6 &= 2a_{\lfloor 6/2 \rfloor} = 2a_3 = 4\\
a_7 &= 2a_{\lfloor 7/2 \rfloor} = 2a_3 = 4\\
a_8 &= 2a_{\lfloor 8/2 \rfloor} = 2a_4 = 8
\end{aligned}
$$

(b) Proof: (By the Alternative Form of the Principle of Mathematical Induction)
For $n = 1$ we have $a_1 = 1 \leq 1$, so the result is true in this first case. (This provides the basis step for the proof.)
Now assume the result true for some $k \geq 1$ and all $n = 1, 2, 3, \ldots, k-1, k$. For $n = k+1$ we have $a_{k+1} = 2a_{\lfloor (k+1)/2 \rfloor} \leq 2\lfloor (k+1)/2 \rfloor$, where the inequality follows from the assumption of the induction hypothesis.
When k is odd, then, $\lfloor (k+1)/2 \rfloor = (k+1)/2$ and we have $a_{k+1} \leq 2[(k+1)/2] = k+1$.
When k is even, then $\lfloor (k+1)/2 \rfloor = \lfloor (k/2) + (1/2) \rfloor = (k/2)$, and here we find that $a_{k+1} \leq 2(k/2) = k \leq k+1$.
In either case it follows from $a_{\lfloor (k+1)/2 \rfloor} \leq \lfloor (k+1)/2 \rfloor$ that $a_{k+1} \leq k+1$. So we have established the inductive step of the proof.
Therefore, it follows from the Alternative Form of the Principle of Mathematical Induction that

$$\forall n \in \mathbf{Z}^+ \quad a_n \leq n.$$

15. (a) One-to-one. The range is the set of all odd integers.
(b) One-to-one. Range $= \mathbf{Q}$
(c) Since $f(1) = f(0), f$ is not one-to-one. The range of $f = \{0, \pm 6, \pm 24, \pm 60, \ldots\} = \{n^3 - n | n \in \mathbf{Z}\}$.
(d) One-to-one. Range $= (0, +\infty) = \mathbf{R}^+$
(e) One-to-one. Range $= [-1, 1]$
(f) Since $f(\pi/4) = f(3\pi/4), f$ is not one-to-one. The range of $f = [0, 1]$.

16.

(a) {4,9}	(b) {4,9}	(c) [0,9)
(d) [0,9)	(e) [0,49]	(f) $[9, 16) \cup [25, 36]$

17. The extension must include $f(1)$ and $f(4)$. Since $|B| = 4$ there are four choices for each of 1 and 4, so there are $4^2 = 16$ ways to extend the given function g.

18. Let $A = \{1, 2\}, B = \{3, 4\}$ and $f = \{(1, 3), (2, 3)\}$. For $A_1 = \{1\}, A_2 = \{2\}$, $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$ while $f(A_1) \cap f(A_2) = \{3\} \cap \{3\} = \{3\}$.

19. (a) $f(A_1 \cup A_2) = \{y \in B | y = f(x),\ x \in A_1 \cup A_2\} = \{y \in B | y = f(x),\ x \in A_1$ or $x \in A_2\} = \{y \in B | y = f(x),\ x \in A_1\} \cup \{y \in B | y = f(x),\ x \in A_2\} = f(A_1) \cup f(A_2)$.
(c) $y \in f(A_1) \cap f(A_2) \Longrightarrow y = f(x_1) = f(x_2),\ x_1 \in A_1,\ x_2 \in A_2 \Longrightarrow y = f(x_1)$ with $x_1 = x_2$, since f is one-to-one $\Longrightarrow y \in f(A_1 \cap A_2)$.

20. The number of injective (or, one-to-one) functions from A to B is $(|B|!)/(|B| - 5)! = 6720$, and $|B| = 8$.

21. No. Let $A = \{1, 2\}, X = \{1\}, Y = \{2\}, B = \{3\}$. For $f = \{(1, 3), (2, 3)\}$ we have $f|_X$, $f|_Y$ one-to-one, but f is not one-to-one.

22. (a) A monotone increasing function $f : X_7 \to X_5$ determines a selection, with repetitions allowed, of size 7 from {1,2,3,4,5}, and vice versa. For example, the selection 1,1,2,2,3,5,5 corresponds to the monotone increasing function $g : X_7 \to X_5$, where $g = \{(1, 1), (2, 1), (3, 2), (4, 2), (5, 3), (6, 5), (7, 5)\}$. (Note the second components.) Consequently, the number of monotone increasing functions $f : X_7 \to X_5$ is $\binom{5+7-1}{7} = \binom{11}{7} = 330$.

(b) $\binom{9+6-1}{6} = \binom{14}{6} = 3003$.

(c) For $m, n \in \mathbf{Z}^+$, the number of monotone increasing functions $f : X_m \to X_n$ is $\binom{n+m-1}{m}$.

(d) Since $f(4) = 4$, it follows that $f(\{1, 2, 3\}) \subseteq \{1, 2, 3, 4\}$ and $f(\{5, 6, 7, 8, 9, 10\}) \subseteq \{4, 5, 6, 7, 8\}$ because f is monotone increasing. The number of these functions is $\binom{4+3-1}{3}\binom{5+6-1}{6} = \binom{6}{3}\binom{10}{6} = (20)(210) = 4200$.

(e) $\binom{12}{4}\binom{5}{2} = 4950$.

(f) Let $m, n, k, \ell \in \mathbf{Z}^+$ with $1 \leq k \leq m$ and $1 \leq \ell \leq n$. If $f : X_m \to X_n$ is monotone increasing and $f(k)=\ell$, then $f(\{1,2,\ldots,k-1\}) \subseteq \{1,2,\ldots,\ell\}$ and $f(\{k+1,\ldots,m\}) \subseteq \{\ell,\ell+1,\ldots,n\}$. So there are $\binom{\ell+(k-1)-1}{(k-1)}\binom{(n-\ell+1)+(m-(k+1)+1)-1}{(m-(k+1)+1)} = \binom{\ell+k-2}{k-1}\binom{n+m-\ell-k}{m-k}$ such functions.

23. (a) $f(a_{ij}) = 12(i-1)+j$ (b) $f(a_{ij}) = 10(i-1)+j$ (c) $f(a_{ij}) = 7(i-1)+j$

24. $g(a_{ij}) = m(j-1)+i$

25. (a) (i) $f(a_{ij}) = n(i-1)+(k-1)+j$
(ii) $g(a_{ij}) = m(j-1)+(k-1)+i$
(b) $k+(mn-1) \le r$

26. (a) There is only one function in S_1, namely $f: A \to B$ where $f(a)=f(b)=1$ and $f(c)=2$. Hence $|S_1|=1$.
(b) Since $f(c)=3$ we have two choices — namely 1,2 — for each of $f(a)$ and $f(b)$. Consequently, $|S_2| = 2^2$.
(c) With $f(c)=i+1$ there are i choices — namely $1,2,3,\ldots,i-1,i$ — for each of $f(a)$ and $f(b)$, so $|S_i|=i^2$.
(d) Any function f in T_1 is determined by two elements x,y in B, where $1 \le x < y \le n+1$ and $f(a)=f(b)=x$, $f(c)=y$. We can select these two elements from B in $\binom{n+1}{2}$ ways, so $|T_1| = \binom{n+1}{2}$.
(e) For T_2 we have $f(a)<f(b)<f(c)$, so we need three distinct elements from B, and these can be chosen in $\binom{n+1}{3}$ ways. The argument for T_3 is similar.
(f) $S = S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_n$, where $S_i \cap S_j = \emptyset$ for all $1 \le i < j \le n$, and $S = T_1 \cup T_2 \cup T_3$ with $T_1 \cap T_2 = T_1 \cap T_3 = T_2 \cap T_3 = \emptyset$.
(g) From part (f) we have $|S| = \sum_{i=1}^{n}|S_i| = \sum_{i=1}^{n} i^2 = \sum_{j=1}^{3}|T_j| = \binom{n+1}{2} + 2\binom{n+1}{3}$. Hence $\sum_{i=1}^{n} i^2 = (n+1)(n)/2 + 2(n+1)(n)(n-1)/6 = (n+1)(n)[(1/2)+(n-1)/3] = (n+1)(n)[(3+2n-2)/6] = n(n+1)(2n+1)/6$.

27. (a) $A(1,3) = A(0,A(1,2)) = A(1,2)+1 = A(0,A(1,1))+1 = [A(1,1)+1]+1 = A(1,1)+2 = A(0,A(1,0))+2 = [A(1,0)+1]+2 = A(1,0)+3 = A(0,1)+3 = (1+1)+3 = 5$

$A(2,3) = A(1,A(2,2))$
$A(2,2) = A(1,A(2,1))$
$A(2,1) = A(1,A(2,0)) = A(1,A(1,1))$
$A(1,1) = A(0,A(1,0)) = A(1,0)+1 = A(0,1)+1 = (1+1)+1 = 3$
$A(2,1) = A(1,3) = A(0,A(1,2)) = A(1,2)+1 = A(0,A(1,1)) = [A(1,1)+1]+1 = 5$
$A(2,2) = A(1,5) = A(0,A(1,4)) = A(1,4)+1 = A(0,A(1,3))+1 = A(1,3)+2 = A(0,A(1,2))+2 = A(1,2)+3 = A(0,A(1,1))+3 = A(1,1)+4 = 7$
$A(2,3) = A(1,7) = A(0,A(1,6)) = A(1,6)+1 = A(0,A(1,5))+1 = A(0,7)+1 = (7+1)+1 = 9$

(b) Since $A(1,0) = A(0,1) = 2 = 0+2$, the result holds for the case where $n = 0$. Assuming the truth of the (open) statement for some k (≥ 0) we have $A(1,k) = k+2$. Then we find that $A(1,k+1) = A(0, A(1,k)) = A(1,k)+1 = (k+2)+1 = (k+1)+2$, so the truth at $n = k$ implies the truth at $n = k+1$. Consequently, $A(1,n) = n+2$ for all $n \in \mathbf{N}$ by the Principle of Mathematical Induction.

(c) Here we find that $A(2,0) = A(1,1) = 1+2 = 3$ (by the result in part(b)). So $A(2,0) = 3 + 2 \cdot 0$ and the given (open) statement is true in this first case.
Next we assume the result true for some k (≥ 0) — that is, we assume that $A(2,k) = 3+2k$. For $k+1$ we then find that $A(2,k+1) = A(1, A(2,k)) = A(2,k)+2$ (by part (b)) $=$ $(3+2k)+2$ (by the induction hypothesis) $= 3+2(k+1)$. Consequently, for all $n \in \mathbf{N}$, $A(2,n) = 3+2n$ — by the Principle of Mathematical Induction.

(d) Once again we consider what happens for $n = 0$. Since $A(3,0) = A(2,1) = 3+2(1)$ (by part (c)) $= 5 = 2^{0+3} - 3$, the result holds in this first case.
So now we assume the given (open) statement is true for some k (≥ 0) and this gives us the induction hypothesis: $A(3,k) = 2^{k+3} - 3$. For $n = k+1$ it then follows that $A(3,k+1) = A(2, A(3,k)) = 3 + 2A(3,k)$ (by part (c)) $= 3 + 2(2^{k+3} - 3)$ (by the induction hypothesis) $= 2^{(k+1)+3} - 3$, so the result holds for $n = k+1$ whenever it does for $n = k$. Therefore, $A(3,n) = 2^{n+3} - 3$, for all $n \in \mathbf{N}$ — by the Principle of Mathematical Induction.

28. (a) $\binom{5}{4}4^4 + \binom{5}{3}4^3 + \binom{5}{2}4^2 + \binom{5}{1}4^1 = (4+1)^5 - \binom{5}{5}4^5 - \binom{5}{0}4^0 = 5^5 - 4^5 - 1$

(b) $\binom{m}{m-1}n^{m-1} + \binom{m}{m-2}n^{m-2} + \cdots + \binom{m}{1}n^1 = (n+1)^m - n^m - 1.$

Section 5.3

1. Let $A = \{1,2,3,4\}, B = \{v,w,x,y,z\}$: (a) $f = \{(1,v),(2,v),(3,w),(4,x)\}$
(b) $f = \{(1,v),(2,x),(3,y),(4,z)\}$
(c) Let $A = \{1,2,3,4,5\}, B = \{w,x,y,z\}, f = \{(1,w),(2,w),(3,x),(4,y),(5,z)\}$.
(d) Let $A = \{1,2,3,4\}, B = \{w,x,y,z\}, f = \{(1,w),(2,x),(3,y),(4,z)\}$.

2. (a) One-to-one and onto.
(b) One-to-one but not onto. The range consists of all the odd integers.
(c) One-to-one and onto.
(d) Since $f(-1) = f(1)$, f is not one-to-one. Also f is not onto. The range of $f = \{0,1,4,9,16,\ldots\}$.
(e) Since $f(0) = f(-1)$, f is not one-to-one. Also f is not onto. The range of $f = \{0,2,6,12,20,\ldots\}$.
(f) One-to-one but not onto. The range of $f = \{\ldots,-64,-27,-8,-1,0,1,8,27,\ldots\}$.

3. (a), (b), (c), (f) One-to-one and onto.
(d) Neither one-to-one nor onto. Range $= [0,+\infty)$
(e) Neither one-to-one nor onto. Range $= [-1/4,+\infty)$

4. (a) 6^4; $6!/2!$; 0 (b) 4^6; $(4!)S(6,4)$; 0

5. For $n=5, m=3, \sum_{k=0}^{5}(-1)^k\binom{5}{5-k}(5-k)^3 = (-1)^0\binom{5}{5}(5)^3+(-1)^1\binom{5}{4}(4)^3+(-1)^2\binom{5}{3}(3)^3+(-1)^3\binom{5}{2}(2)^3+(-1)^4\binom{5}{1}(1)^3+(-1)^5\binom{5}{0}(0)^3 = 125-320+70-80+5=0$

6. (a) $\sum_{i=1}^{5}\binom{5}{i}(i!)S(7,i) = \binom{5}{1}(1!)S(7,1)+\binom{5}{2}(2!)S(7,2)+\binom{5}{3}(3!)S(7,3)+\binom{5}{4}(4!)S(7,4)+\binom{5}{5}(5!)S(7,5) = (5)(1)(1)+(10)(2)(63)+(10)(6)(301)+(5)(24)(350)+(1)(120)(14) = 78,125 = 5^7$.

(b) The expression m^n counts the number of ways to distribute n distinct objects among m distinct containers.

For $1 \leq i \leq m$, let i count the number of distinct containers that we actually use — that is, those that are *not* empty after the n distinct objects are distributed. This number of distinct containers can be chosen in $\binom{m}{i}$ ways. Once we have the i distinct containers we can distribute the n distinct objects among these i distinct containers, with no container left empty, in $(i!)S(n,i)$ ways — where $S(n,i)=0$ when $n<i$. Then $\sum_{i=1}^{m}\binom{m}{i}(i!)S(n,i)$ also counts the number of ways to distribute n distinct objects among m distinct containers.

Hence $m^n = \sum_{i=1}^{m}\binom{m}{i}(i!)S(n,i)$.

7. (a) (i) $2!S(7,2)$ (ii) $\binom{5}{2}[2!S(7,2)]$ (iii) $3!S(7,3)$
(iv) $\binom{5}{3}[3!S(7,3)]$ (v) $4!S(7,4)$ (vi) $\binom{5}{4}[4!S(7,4)]$

(b) $\binom{n}{k}[k!S(m,k)]$

8. Let A be the set of compounds and B the set of assistants. Then the number of assignments with no idle assistants is the number of onto functions from set A to set B. There are $5!S(9,5)$ such functions.

9. For each $r \in \mathbf{R}$ there is at least one $a \in \mathbf{R}$ such that $a^5-2a^2+a-r=0$ because the polynomial x^5-2x^2+x-r has odd degree and real coefficients. Consequently, f is onto. However, $f(0)=0=f(1)$, so f is not one-to-one.

10. (a) $(4!)S(7,4)$
(b) $(3!)S(6,3)$ (Here container II contains only the blue ball) $+$ $(4!)S(6,4)$ (Here container II contains more than just the blue ball).
(c) $S(7,4)+S(7,3)+S(7,2)+S(7,1)$.

11.

m \ n	1	2	3	4	5	6	7	8	9	10
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

12. (a) Since $31,100,905 = 5 \times 11 \times 17 \times 29 \times 31 \times 37$, we find that there are $S(6,3) = 90$ unordered factorizations of 31,100,905 into three factors — each greater than 1.
(b) If the order of the factors in part (a) is considered relevant then there are $(3!)S(6,3) = 540$ such factorizations.
(c) $\sum_{i=2}^{6} S(6,i) = S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6) = 31 + 90 + 65 + 15 + 1 = 202$
(d) $\sum_{i=2}^{6} (i!)S(6,i) = (2!)S(6,2) + (3!)S(6,3) + (4!)S(6,4) + (5!)S(6,5) + (6!)S(6,6) =$

$$(2)(31) + (6)(90) + (24)(65) + (120)(15) + (720)(1) = 4682.$$

13. (a) Since $156,009 = 3 \times 7 \times 17 \times 19 \times 23$, it follows that there are $S(5,2) = 15$ two-factor unordered factorizations of 156,009, where each factor is greater than 1.
(b) $\sum_{i=2}^{5} S(5,i) = S(5,2) + S(5,3) + S(5,4) + S(5,5) = 15 + 25 + 10 + 1 = 51.$
(c) $\sum_{i=2}^{n} S(n,i).$

14.

```
10    Dim S(12, 12)
20    For I = 1 To 12
30          S(I,I) = 1
40    Next I
50    Print "M = :  1"
60    For M = 2 To 12
70          Print "M ="; M; ":  1, ";
80          For N = 2 To M-1
90                S(M,N) = S(M-1,N-1) + N*S(M-1,N)
100               Print S(M,N); ", ";
110         Next N
120         Print "  1"
130   Next M
140   End
```

15. a) $n = 4$: $\sum_{i=1}^{4} i!S(4,i)$; $n = 5$: $\sum_{i=1}^{5} i!S(5,i)$

In general, the answer is $\sum_{i=1}^{n} i!S(n,i)$.

b) $\binom{15}{12}\sum_{i=1}^{12} i!S(12,i)$.

16. a) (i) $10!$

(ii) The given outcome — namely, $\{C_2, C_3, C_7\}$, $\{C_1, C_4, C_9, C_{10}\}$, $\{C_5\}$, $\{C_6, C_8\}$ — is an example of a distribution of ten distinct objects among four distinct containers, with no container left empty. [Or it is an example of an onto function $f : A \to B$ where $A = \{C_1, C_2, \ldots, C_{10}\}$ and $B = \{1,2,3,4\}$.] There are $4!S(10,4)$ such distributions [or functions].

The answer to the question is $\sum_{i=1}^{10} i!S(10,i)$.

(iii) $\binom{10}{3}\sum_{i=1}^{7} i!S(7,i)$.

b) $\binom{9}{2}\sum_{i=1}^{7} i!S(7,i)$

c) For $0 \le k \le 9$, the number of outcomes where C_3 is tied for first place with k other candidates is $\binom{9}{k}\sum_{i=1}^{9-k} i!S(9-k,i)$. [Part (b) above is the special case where $k = 3-1 = 2$.]

Summing over the possible values of k we have the answer $\sum_{k=0}^{9}\binom{9}{k}\sum_{i=1}^{9-k} i!S(9-k,i)$.

17. Let $a_1, a_2, \ldots, a_m, x$ denote the $m+1$ distinct objects. Then $S_r(m+1,n)$ counts the number of ways these objects can be distributed among n identical containers so that each container receives at least r of the objects.

Each of these distributions falls into exactly one of two categories:

1) The element x is in a container with r or more other objects: Here we start with $S_r(m,n)$ distributions of $a_1, a_2, \ldots, a_m$ into n identical containers – each container receiving at least r of the objects. Now we have n *distinct* containers – distinguished by their contents. Consequently, there are n choices for locating the object x. As a result, this category provides $nS_r(m,n)$ of the distributions.

2) The element x is in a container with $r-1$ of the other objects: These other $r-1$ objects can be chosen in $\binom{m}{r-1}$ ways, and then these objects – along with x – can be placed in one of the n containers. The remaining $m+1-r$ distinct objects can then be distributed among the $n-1$ identical containers – where each container receives at least r of the objects – in $S_r(m+1-r, n-1)$ ways. Hence this category provides the remaining $\binom{m}{r-1}S_r(m+1-r, n-1)$ distributions.

18. (a) For $n > m$ we have $s(m,n) = 0$, because there are more tables than people.

(b) For $m \geq 1$, (i) $s(m,m) = 1$ because the ordering of the m tables is not taken into account; and, (ii) $s(m,1) = (m-1)!$, as in Example 1.16.
(c) Here there are two people at one table and one at each of the other $m-1$ tables. There are $\binom{m}{2}$ such arrangements.
(d) When m people are seated around $m-2$ tables there are two cases to consider: (1) One table with three occupants and $m-3$ tables, each with one occupant — there are $\binom{m}{3}(2!)$ such arrangements; and, (2) Two tables, each with two occupants, and $m-4$ tables each with a single occupant — there are $(1/2)\binom{m}{2}\binom{m-2}{2}$ of these arrangements. We then find that $\binom{m}{3}(2!)+(1/2)\binom{m}{2}\binom{m-2}{2} = (1/3)(m)(m-1)(m-2)+(1/2)[(1/2)(m)(m-1)][(1/2)(m-2)(m-3)] = (m)(m-1)(m-2)[(1/3)+(1/8)(m-3)] = (1/24)(m)(m-1)(m-2)(3m-1)$.

19. (a) We know that $s(m,n)$ counts the number of ways we can place m people — call them $p_1, p_2, \ldots, p_m$ — around n circular tables, with at least one occupant at each table. These arrangements fall into two disjoint sets: (1) The arrangements where p_1 is alone: There are $s(m-1,n-1)$ such arrangements; and, (2) The arrangements where p_1 shares a table with at least one of the other $m-1$ people: There are $s(m-1,n)$ ways where $p_2, p_3, \ldots, p_m$ can be seated around the n tables so that every table is occupied. Each such arrangement determines a total of $m-1$ locations (at all the n tables) where p_1 can now be seated — this for a total of $(m-1)s(m-1,n)$ arrangements. Consequently, $s(m,n) = (m-1)s(m-1,n) + s(m-1,n-1)$, for $m \geq n > 1$.

(b) For $m = 2$, we have $s(m,2) = 1 = 1!(1/1) = (m-1)!\sum_{i=1}^{m-1}\frac{1}{i}$. So the result is true in this case; this establishes the basis step for a proof by mathematical induction. Assuming the result for $m = k(\geq 2)$ we have $s(k,2) = (k-1)!\sum_{i=1}^{k-1}\frac{1}{i}$. Using the result from part (a) we now find that $s(k+1,2) = ks(k,2)+s(k,1) = k(k-1)!\sum_{i=1}^{k-1}\frac{1}{i}+(k-1)! = k!\sum_{i=1}^{k-1}\frac{1}{i}+(1/k)k! = k!\sum_{i=1}^{k}\frac{1}{i}$. The result now follows for all $m \geq 2$ by the Principle of Mathematical Induction.

Section 5.4

1. Here we find, for example, that
$f(f(a,b),c) = f(a,c) = c$, while
$f(a,f(b,c)) = f(a,b) = a$, so f is *not* associative.

2. (a) For all $a,b \in \mathbf{R}$, $f(a,b) = \lceil a+b \rceil = \lceil b+a \rceil = f(b,a)$, because the real numbers are commutative under addition. Hence f is a commutative (closed) binary operation.
(b) This binary operation is *not* associative. For example,

$$f(f(3.2,4.7),6.4) = f(\lceil 3.2{+}4.7 \rceil,6.4) = f(\lceil 7.9 \rceil,6.4) = f(8,6.4) = \lceil 8{+}6.4 \rceil = \lceil 14.4 \rceil = 15,$$

while,

$$f(3.2, f(4.7, 6.4)) = f(3.2, \lceil 4.7+6.4 \rceil) = f(3.2, \lceil 11.1 \rceil) = f(3.2, 12) = \lceil 3.2+12 \rceil = \lceil 15.2 \rceil = 16.$$

(c) There is no identity element. If $a \in \mathbf{R} - \mathbf{Z}$ then for any $b \in \mathbf{R}$, $\lceil a+b \rceil \in \mathbf{Z}$. So if x were the identity element we would have $a = f(a, x) = \lceil a+x \rceil$ with $a \in \mathbf{R} - \mathbf{Z}$ and $\lceil a+x \rceil \in \mathbf{Z}$.

3. (a) $f(x, y) = x + y - xy = y + x - yx = f(y, x)$, so the binary operation is commutative.
$f(f(w, x), y) = f(w, x) + y - f(w, x)y = (w + x - wx) + y - (w + x - wx)y = w + x + y - wx - wy - xy + wxy.$
$f(w, f(x, y)) = w + f(x, y) - w \cdot f(x, y) = w + (x + y - xy) - w(x + y - xy) = w + x + y - wx - wy - xy + wxy.$
Since $f(f(w, x), y) = f(w, f(x, y))$, the (closed) binary operation is associative.
(b), (d) Commutative and associative
(c) Neither commutative nor associative.

4. (a) The identity is $z = 0$.
(d) The identity is $z = 3$.
(b), (c) Neither of these (closed) binary operations has an identity.

5. (a) 25 (b) 5^{25} (c) 5^{25} (d) 5^{10}

6. (a) 5^{24} (b) 5^{15}
(c) $3 \cdot 5^{15}$, because neither a nor b can be an identity.
(d) $3 \cdot 5^{9}$

7. (a) Yes (b) Yes (c) No

8. Each element in A is of the form 2^i for some $1 \leq i \leq 5$, and $\gcd(2^i, 2^5) = 2^i = \gcd(2^5, 2^i)$, so $2^5 = 32$ is the identity element for f.

9. (a) $|A| = (32)(38) = 1216$.
(b) The identity element for f is $p^{31}q^{37}$.

10. For $n \in \mathbf{Z}^+$ let $p_1, p_2, \ldots, p_n$ be distinct primes and for each $1 \leq i \leq n$ let M_i be a fixed positive integer. If $A = \{ \prod_{1 \leq i \leq n} p_i^{e_i} \mid e_i \in \mathbf{N},\ 0 \leq e_i \leq M_i \}$ define the closed binary operation $f : A \times A \to A$ by $f(a, b) = \gcd(a, b)$.
Then $|A| = \prod_{i=1}^{n} (M_i + 1)$ and the identity element for f is $\prod_{i=1}^{n} p_i^{M_i}$.

11. By the Well-Ordering Principle A has a least element and this same element is the identity for g. If A is finite then A will have a largest element and this same element will be the identity for f. If A is infinite then f cannot have an identity.

12.
(a) $\pi_A(D) = [0, +\infty)$ $\pi_B(D) = \mathbf{R}$
(b) $\pi_A(D) = \mathbf{R}$ $\pi_B(D) = [-1, 1]$
(c) $\pi_A(D) = [-1, 1]$ $\pi_B(D) = [-1, 1]$

13. (a) 5 (b) {(25,25,6), (25,2,4), (60,40,20), (25,40,10)}
(c) A_1, A_2

14. (a) 5
(b) $\{(1, A), (1, D), (1, E), (2, A), (2, D), (2, E)\}$;
$\{(10000, 1, 100), (400, 1, 100), (30, 1, 100), (4000, 1, 250), (400, 1, 250), (15, 1, 250)\}$
(c) $A_1 \times A_2$; $A_2 \times A_5$; $A_3 \times A_5$

Section 5.5

1. Here the socks are the pigeons and the colors are the pigeonholes.

2. The result follows by the Pigeonhole Principle where the eight people are the pigeons and the pigeonholes are the seven days of the week.

3. $26^2 + 1 = 677$

4. Subdivide the set S into the 14 subsets: $\{3\}, \{7, 103\}, \{11, 99\}, \{15, 95\}, \ldots, \{43, 67\}, \{47, 63\}, \{51, 59\}, \{55\}$. By the Pigeonhole Principle if we select at least 15 elements of S then we must have the elements in one of the two-element subsets and these sum to 110.

5. (a) For each $x \in \{1, 2, 3, \ldots, 300\}$ wrote $x = 2^n \cdot m$, where $n \geq 0$ and $\gcd(2, m) = 1$. There are 150 possibilities for m: namely, $1, 3, 5, \ldots, 299$. In selecting 151 numbers from $\{1, 2, 3, \ldots, 300\}$ there must be two numbers of the form $x = 2^s \cdot m$, $y = 2^t \cdot m$. If $x < y$ then $x|y$; otherwise $y < x$ and $y|x$.
(b) If $n + 1$ integers are selected from the set $\{1, 2, 3, \ldots, 2n\}$, then there must be two integers x, y in the selection where $x|y$ or $y|x$.

6. Any selection of size 101 from S must contain two consecutive integers n, $n + 1$ and $\gcd(n, n + 1) = 1$.

7. (a) Here the pigeons are the integers $1, 2, 3, \ldots, 25$ and the pigeonholes are the 13 sets: $\{1, 25\}, \{2, 24\}, \ldots, \{11, 15\}, \{12, 14\}, \{13\}$. In selecting 14 integers we get the elements in at least one two-element subset, and these sum to 26.
(b) If $S = \{1, 2, 3, \ldots, 2n + 1\}$, for n a positive integer, then any subset of size $n + 2$ from S must contain two elements that sum to $2n + 2$.

8. (a) Since $|S| \geq 3$, $\exists x, y \in S$ where x, y are both even or both odd. In either case $x + y$ is even.
(b) $5 (= 2^2 + 1)$ (c) $9 (= 2^3 + 1)$

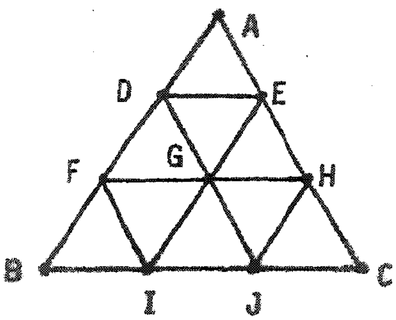
(d) For $n \in \mathbf{Z}^+$ let $S = \{(a_1, a_2, \ldots, a_n) | a_i \in \mathbf{Z}^+, 1 \leq i \leq n\}$. If $|S| \geq 2^n + 1$, then S contains two ordered n-tuples $(x_1, x_2, \ldots x_n)$, $(y_1, y_2, \ldots, y_n)$ such that $x_i + y_i$ is even $\forall\, 1 \leq i \leq n$.

(e) 5 – as in part (b).

9. (a) For any $t \in \{1, 2, 3, \ldots, 100\}, 1 \leq \sqrt{t} \leq 10$. Selecting 11 elements from $\{1, 2, 3, \ldots, 100\}$ there must be two, say x and y, where $\lfloor \sqrt{x} \rfloor = \lfloor \sqrt{y} \rfloor$, so that $0 < |\sqrt{x} - \sqrt{y}| < 1$.

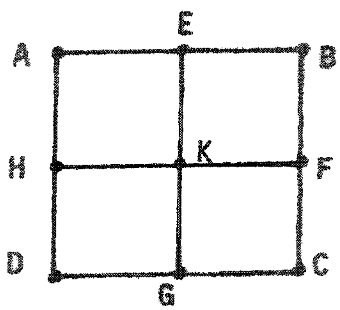
(b) Let $n \in \mathbf{Z}^+$. If $n + 1$ elements are selected from $\{1, 2, 3, \ldots, n^2\}$, then there exist two, say x and y, where $0 < |\sqrt{x} - \sqrt{y}| < 1$.

10.



In triangle ABC, divide each side into three equal parts and form the nine congruent triangles shown in the figure. Let R_1 be the interior of triangle ADE together with the points on segment DE, excluding D,E. Region R_2 is the interior of triangle DFG together with the points on segments DG, FG, excluding D,F. Regions $R_3, \ldots, R_9$ are defined similarly so that the interior of $\triangle$ ABC is the union of these nine regions and $R_i \cap R_j = \emptyset$, for $i \neq j$. Then if 10 points are chosen in the interior of $\triangle$ ABC, at least two of these points are in R_i for some $1 \leq i \leq 9$, and these two points are at a distance less than $1/3$ from each other.

11.



Divide the interior of the square into four smaller congruent squares as shown in the figure. Each smaller square has diagonal length $1/\sqrt{2}$. Let region R_1 be the interior of square AEKH together with the points on segment EK, excluding point E. Region R_2 is the interior of square EBFK together with the points on segment FK, excluding points F,K. Regions R_3, R_4 are defined in a similar way. Then if five points are chosen in the interior of square ABCD, at least two are in R_i for some $1 \leq i \leq 4$ and these points are within $1/\sqrt{2}$ (units) of each other.

12. For any five-element subset E of A we find that $1+2+3+4+5 = 15 \leq s_E \leq 115 \leq 21+22+23+24+25$, so there are 116 possible values for such a sum s_E. Since $|A| = 9$, there are $\binom{9}{5} = 126$ five-element subsets of A.

The result now follows by the Pigeonhole Principle where the 126 five-element subsets of A are the pigeons and the 116 possible sums are the pigeonholes.

13. Consider the subsets A of S where $1 \leq |A| \leq 3$. Since $|S| = 5$, there are $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} = 25$ such subsets A. Let s_A denote the sum of the elements in A. Then $1 \leq s_A \leq 7+8+9 = 24$. So by the Pigeonhole Principle, there are two subsets of S whose elements yield the same sum.

14. For $1 \leq i \leq 42$, let x_i count the total number of resumés Brace has sent out from the start of his senior year to the end of the i-th day. Then $1 \leq x_1 < x_2 < \ldots < x_{42} \leq 60$, and $x_1 + 23 < x_2 + 23 < \ldots < x_{42} + 23 \leq 83$. We have 42 distinct numbers $x_1, x_2, \ldots, x_{42}$, and 42 other distinct numbers $x_1 + 23, x_2 + 23, \ldots, x_{42} + 23$, all between 1 and 83 inclusive. By the Pigeonhole Principle $x_i = x_j + 23$ for some $1 \leq j < i \leq 42$; $x_i - x_j = 23$.

15. For $(\emptyset \neq) T \subseteq S$, we have $1 \leq s_T \leq m + (m-1) + \cdots + (m-6) = 7m - 21$. The set S has $2^7 - 1 = 128 - 1 = 127$ nonempty subsets. So by the Pigeonhole Principle we need to have $127 > 7m - 21$ or $148 > 7m$. Hence $7 \leq m \leq 21$.

16. Proof: Consider the $k+1$ integers: (1) 3; (2) 33; (3) 333; ...; and $(k+1)$ $333\ldots3$, where for all $1 \leq i \leq k+1$, the i-th integer has i digits – each of which is a 3. Since there are $k+1$ integers, it follows from the Division Algorithm and the Pigeonhole Principle that two of these integers, say a and b, have the same remainder when divided by k. Suppose that $a = q_1k + r$, $b = q_2k + r$, and that $a > b$. Then $a - b = (q_1 - q_2)k$, so $k|(a-b)$ and the only digits in $a - b$ are 0's and 3's. [Note: The integer 3 is not special. The result is also true if we replace 3 by any of the digits 1, 2, 4, 5, 6, 7, 8, 9. However, we cannot obtain the result without using the digit 0.]

17. (a) 2,4,1,3
(b) 3,6,9,2,5,8,1,4,7
(c) For $n \geq 2$, there exists a sequence of n^2 distinct real numbers with no decreasing or increasing subsequence of length $n+1$. For example, consider $n, 2n, 3n, \ldots, (n-1)n, n^2, (n-1), (2n-1), \ldots, (n^2-1), (n-2), (2n-2), \ldots, (n^2-2), \ldots, 1, (n+1), (2n+1), \ldots, (n-1)n+1$.
(d) The result in Example 5.49 (for $n \geq 2$) is best possible – in the sense that we cannot reduce the length of the sequence from n^2+1 to n^2 and still obtain the desired subsequence of length $n+1$.

18. This follows from the result due to Paul Erdős and George Szekeres: A sequence of $50(= 7^2+1)$ distinct real numbers contains a decreasing or increasing subsequence of length $8(= 7+1)$.

19. Proof: If not each pigeonhole contains at most k pigeons – for a total of at most kn pigeons. But we have $kn+1$ pigeons. So we have a contradiction and the result then follows.

20. (a) 7 (b) 13 (c) $6(n-1)+1$

21. (a) 1001 (b) 2001
(c) Let $k, n \in \mathbf{Z}^+$. The smallest value for $|S|$ (where $S \subset \mathbf{Z}^+$) so that there exist n elements $x_1, x_2, \ldots, x_n \in S$ where all n of these integers have the same remainder upon division by k is $k(n-1)+1$.

22. Proof: If not, each pigeonhole contains at most $\lfloor (m-1)/n \rfloor$ pigeons – for a total of $n\lfloor (m-1)/n \rfloor \leq m-1$ pigeons. But this contradicts the fact that we have m pigeons. The result then follows.
[Note: This result is true even if $m \leq n$.]

23. Proof: If not, then the number of pigeons roosting in the first pigeonhole is $x_1 \leq p_1 - 1$, the number of pigeons roosting in the second pigeonhole is $x_2 \leq p_2 - 1, \ldots$, and the number roosting in the n-th pigeonhole is $x_n \leq p_n - 1$. Hence the total number of pigeons is $x_1 + x_2 + \cdots + x_n = (p_1 - 1) + (p_2 - 1) + \cdots + (p_n - 1) = p_1 + p_2 + \cdots + p_n - n < p_1 + p_2 + \cdots + p_n - n + 1$, the number of pigeons we started with. The result now follows because of this contradiction.

24. 42

Section 5.6

1. (a) There are 7! bijective functions on A – of these, 6! satisfy $f(1) = 1$. Hence there are $7! - 6! = 6(6!)$ bijective functions $f : A \to A$ where $f(1) \neq 1$.
(b) $n! - (n-1)! = (n-1)(n-1)!$

2. (a) Here f, g have the same domain A and some codomain $\mathbf{R}$, and for all $x \in A$ we find that

$$g(x) = \frac{2x^2 - 8}{x+2} = \frac{2(x^2-4)}{x+2} = \frac{2(x-2)(x+2)}{(x+2)} = 2(x-2) = 2x - 4 = f(x).$$

Consequently, $f = g$.
(b) Here there is a problem and $f \neq g$. In fact for any nonempty subset A of $\mathbf{R}$, if $-2 \in A$ then g is not defined for A because $g(-2) = 0/0$. [We note that $\frac{x^2-4}{x+2} = x - 2$, for $x \neq -2$.]

3. $9x^2 - 9x + 3 = g(f(x)) = 1 - (ax+b) + (ax+b)^2 = a^2x^2 + (2ab - a)x + (b^2 - b + 1)$. By comparing coefficients on like powers of x, $a = 3, b = -1$ or $a = -3, b = 2$.

4. $g \circ f = \{(1,4), (2,6), (3,10), (4,14)\}$

5. $g^2(A) = g(T \cap (S \cup A)) = T \cap (S \cup [T \cap (S \cup A)]) =$
$T \cap [(S \cup T) \cap (S \cup (S \cup A))] = T \cap [(S \cup T) \cap (S \cup A)] =$
$[T \cap (S \cup T)] \cap (S \cup A) = T \cap (S \cup A) = g(A)$.

6. $(f \circ g)(x) = f(cx + d) = a(cx + d) + b$
$(g \circ f)(x) = g(ax + b) = c(ax + b) + d$
$(f \circ g)(x) = (g \circ f)(x) \iff acx + ad + b = acx + bc + d \iff ad + b = bc + d$

7. (a) $(f \circ g)(x) = 3x - 1$; $(g \circ f)(x) = 3(x-1)$;

$$(g \circ h)(x) = \begin{cases} 0, & x \text{ even;} \\ 3, & x \text{ odd} \end{cases} \qquad (h \circ g)(x) = \begin{cases} 0, & x \text{ even;} \\ 1, & x \text{ odd} \end{cases}$$

$$(f \circ (g \circ h))(x) = f((g \circ h)(x)) = \begin{cases} -1, & x \text{ even;} \\ 2, & x \text{ odd} \end{cases}$$

$$((f \circ g) \circ h)(x) = \begin{cases} (f \circ g)(0), & x \text{ even} \\ (f \circ g)(1), & x \text{ odd} \end{cases} = \begin{cases} -1, & x \text{ even} \\ 2, & x \text{ odd} \end{cases}$$

(b) $f^2(x) = f(f(x)) = x - 2$; $f^3(x) = x - 3$; $g^2(x) = 9x$; $g^3(x) = 27x$; $h^2 = h^3 = h^{500} = h$.

8. (a) If $c \in C$, there is an element $a \in A$ such that $(g \circ f)(a) = c$. Then $g(f(a)) = c$ with $f(a) \in B$, so g is onto.

(b) Let $x, y \in A$. $f(x) = f(y) \Longrightarrow g(f(x)) = g(f(y)) \Longrightarrow (g \circ f)(x) = (g \circ f)(y) \Longrightarrow x = y$, since $g \circ f$ is one-to-one.

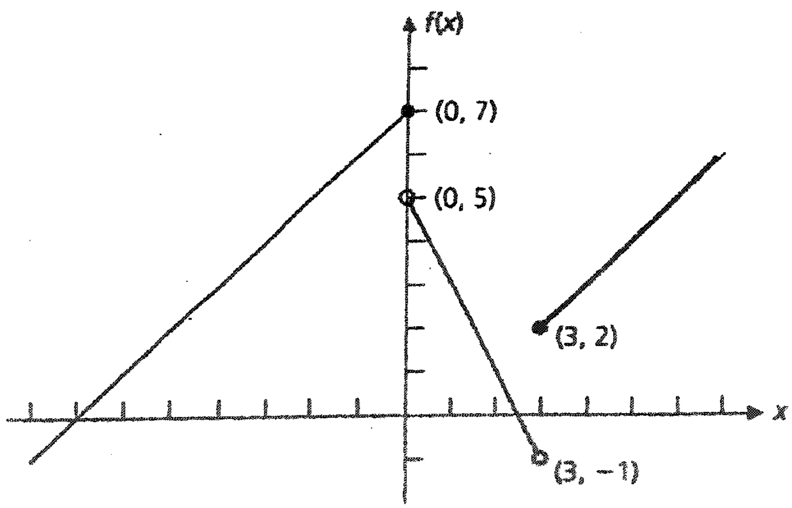
9. (a) $f^{-1}(x) = \frac{1}{2}(\ell n x - 5)$
(b) For $x \in \mathbf{R}^+, (f \circ f^{-1})(x) = f(\frac{1}{2}(\ell n x - 5)) = e^{2((1/2)(\ell n x - 5)) + 5} = e^{\ell n x - 5 + 5} = e^{\ell n x} = x$; for $x \in \mathbf{R}$, $(f^{-1} \circ f)(x) = f^{-1}(e^{2x+5}) = \frac{1}{2}[\ell n(e^{2x+5}) - 5] = \frac{1}{2}[2x + 5 - 5] = x$.

10. (a) $f^{-1} = \{(x, y) | 2y + 3x = 7\}$ (b) $f^{-1} = \{(x, y) | ay + bx = c,\ b \neq 0,\ a \neq 0\}$
(c) $f^{-1} = \{(x, y) | y = x^{1/3}\} = \{(x, y) | x = y^3\}$
(d) Here $f(0) = f(-1) = 0$, so f is not one-to-one, and consequently f is not invertible.

11. f, g invertible $\Longrightarrow$ each of f, g is both one-to-one and onto $\Longrightarrow g \circ f$ is one-to-one and onto $\Longrightarrow g \circ f$ invertible. Since $(g \circ f) \circ (f^{-1} \circ g^{-1}) = 1_C$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = 1_A$, $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$. By uniqueness of inverses $f^{-1} \circ g^{-1} = (g \circ f)^{-1}$.

12. (a) $f^{-1}(\{2\}) = \{a \in A | f(a) \in \{2\}\} = \{a \in A | f(a) = 2\} = \{1\}$
(b) $f^{-1}(\{6\}) = \{a \in A | f(a) \in \{6\}\} = \{a \in A | f(a) = 6\} = \{2, 3, 5\}$
(c) $f^{-1}(\{6, 8\}) = \{a \in A | f(a) \in \{6, 8\}\} = \{a \in A | f(a) = 6 \text{ or } f(a) = 8\} = \{2, 3, 4, 5, 6\}$, because $f(2) = f(3) = f(5) = 6$ and $f(4) = f(6) = 8$.
(d) $f^{-1}(\{6, 8, 10\}) = \{2, 3, 4, 5, 6\} = f^{-1}(\{6, 8\})$ since $f^{-1}(\{10\}) = \emptyset$.
(e) $f^{-1}(\{6, 8, 10, 12\}) = \{2, 3, 4, 5, 6, 7\}$
(f) $f^{-1}(\{10, 12\}) = \{7\}$

13.



(a) $f^{-1}(-10) = \{x \in \mathbf{R} \mid x \leq 10 \text{ and } x + 7 = -10\} = \{-17\}$
$f^{-1}(0) = \{-7, 5/2\}$
$f^{-1}(4) = \{-3, 1/2, 5\}$
$f^{-1}(6) = \{-1, 7\}$
$f^{-1}(7) = \{0, 8\}$
$f^{-1}(8) = \{9\}$

(b) (i) $f^{-1}([-5,-1]) = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -5 \leq x+7 \leq -1\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } -5 \leq -2x+5 \leq -1\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -5 \leq x-1 \leq -1\} = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -12 \leq x \leq -8\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } 3 \leq x \leq 5\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -4 \leq x \leq 0\} = [-12,-8] \cup \emptyset \cup \emptyset = [-12,-8]$

(ii) $f^{-1}([-5,0]) = [-12,-7] \cup [5/2,3)$

(iii) $f^{-1}([-2,4]) = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -2 \leq x+7 \leq 4\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } -2 \leq -2x+5 \leq 4\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -2 \leq x-1 \leq 4\} = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } -9 \leq x \leq -3\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } 1/2 \leq x \leq 7/2\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } -1 \leq x \leq 5\} = [-9,-3] \cup [1/2,3) \cup [3,5] = [-9,-3] \cup [1/2,5]$

(iv) $f^{-1}((5,10)) = (-2,0] \cup (6,11)$

(v) $f^{-1}([11,17)) = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } 11 \leq x+7 < 17\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } 11 \leq -2x+5 < 17\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } 11 \leq x-1 < 17\} = \{x \in \mathbf{R} \mid x \leq 0 \text{ and } 4 \leq x < 10\} \cup \{x \in \mathbf{R} \mid 0 < x < 3 \text{ and } -6 < x \leq -3\} \cup \{x \in \mathbf{R} \mid 3 \leq x \text{ and } 12 \leq x < 18\} = \emptyset \cup \emptyset \cup [12,18) = [12,18)$

14. (a) $\{-1,0,1\}$ (b) $\{-1,0,1\}$ (c) $[-1,1]$ (d) $(-1,1)$
(e) $[-2,2]$ (f) $(-3,-2) \cup [-1,0) \cup (0,1] \cup (2,3)$

15. Since $f^{-1}(\{6,7,8\}) = \{1,2\}$ there are three choices for each of $f(1)$ and $f(2)$ – namely, 6, 7 or 8. Furthermore $3,4,5 \notin f^{-1}(\{6,7,8\})$ so $3,4,5 \in f^{-1}(\{9,10,11,12\})$ and we have four choices for each of $f(3)$, $f(4)$, and $f(5)$. Therefore, it follows by the rule of product that there are $3^2 \cdot 4^3 = 576$ functions $f: A \to B$ where $f^{-1}(\{6,7,8\}) = \{1,2\}$.

16.
(a) $[0,2)$ (b) $[-1,2)$ (c) $[0,1)$ (d) $[0,2)$
(e) $[-1,3)$ (f) $[-1,0) \cup [2,4)$

17. (a) The range of $f = \{2,3,4,\ldots\} = \mathbf{Z}^+ - \{1\}$.
(b) Since 1 is not in the range of f the function is not onto.
(c) For all $x, y \in \mathbf{Z}^+$, $f(x) = f(y) \Rightarrow x+1 = y+1 \Rightarrow x = y$, so f is one-to-one.
(d) The range of g is $\mathbf{Z}^+$.
(e) Since $g(\mathbf{Z}^+) = \mathbf{Z}^+$, the codomain of g, this function is onto.
(f) Here $g(1) = 1 = g(2)$, and $1 \neq 2$, so g is not one-to-one.
(g) For all $x \in \mathbf{Z}^+$, $(g \circ f)(x) = g(f(x)) = g(x+1) = \max\{1, (x+1)-1\} = \max\{1, x\} = x$, since $x \in \mathbf{Z}^+$. Hence $g \circ f = 1_{\mathbf{Z}^+}$.
(h) $(f \circ g)(2) = f(\max\{1,1\}) = f(1) = 1+1 = 2$
$(f \circ g)(3) = f(\max\{1,2\}) = f(2) = 2+1 = 3$
$(f \circ g)(4) = f(\max\{1,3\}) = f(3) = 3+1 = 4$
$(f \circ g)(7) = f(\max\{1,6\}) = f(6) = 6+1 = 7$
$(f \circ g)(12) = f(\max\{1,11\}) = f(11) = 11+1 = 12$
$(f \circ g)(25) = f(\max\{1,24\}) = f(24) = 24+1 = 25$
(i) No, because the functions f, g are *not* inverses of each other. The calculations in part (h) may suggest that $f \circ g = 1_{\mathbf{Z}^+}$ since $(f \circ g)(x) = x$ for $x \geq 2$. But we also find that $(f \circ g)(1) = f(\max\{1,0\}) = f(1) = 2$, so $(f \circ g)(1) \neq 1$, and, consequently, $f \circ g \neq 1_{\mathbf{Z}^+}$.

18. (a) $f(\emptyset, \emptyset) = \emptyset = f(\emptyset, \{1\})$ and $(\emptyset, \emptyset) \neq (\emptyset, \{1\})$, so f is not one-to-one.

$g(\{1\}, \{2\}) = \{1,2\} = g(\{1,2\}, \{2\})$ and $(\{1\}, \{2\}) \neq (\{1,2\}, \{2\})$, so g is not one-to-one.

$h(\{1\}, \{2\}) = \{1,2\} = h(\{2\}, \{1\})$ and $(\{1\}, \{2\}) \neq (\{2\}, \{1\})$, so h is not one-to-one.

(b) For each subset A of $\mathbf{Z}^+$, $f(A, A) = g(A, A) = h(A, \emptyset) = A$, so each of the three functions f, g, and h, is an onto function.

(c) From the results in part (a) it follows that none of these functions is invertible.

(d) The sets $f^{-1}(\emptyset)$, $h^{-1}(\emptyset)$, $f^{-1}(\{1\})$, $h^{-1}(\{3\})$, $f^{-1}(\{4,7\})$, and $h^{-1}(\{5,9\})$, are all infinite.

(e) $|g^{-1}(\emptyset) = \{(\emptyset, \emptyset)\}$, so $|g^{-1}(\emptyset)| = 1$.

$g^{-1}(\{2\}) = \{(\emptyset, \{2\}), (\{2\}, \emptyset), (\{2\}, \{2\})\}$, so $|g^{-1}(\{2\})| = 3$

$|g^{-1}(\{8, 12\})| = 9$.

19. (a) $a \in f^{-1}(B_1 \cap B_2) \iff f(a) \in B_1 \cap B_2 \iff f(a) \in B_1$ and $f(a) \in B_2 \iff a \in f^{-1}(B_1)$ and $a \in f^{-1}(B_2) \iff a \in f^{-1}(B_1) \cap f^{-1}(B_2)$

(c) $a \in f^{-1}(\overline{B_1}) \iff f(a) \in \overline{B_1} \iff f(a) \notin B_1 \iff a \notin f^{-1}(B_1) \iff a \in \overline{f^{-1}(B_1)}$

20. (a) (i) $f(x) = 2x$; (ii) $f(x) = \lfloor x/2 \rfloor$

(b) No. The set $\mathbf{Z}$ is not finite.

21. (a) Suppose that $x_1, x_2 \in \mathbf{Z}$ and $f(x_1) = f(x_2)$. Then either $f(x_1)$, $f(x_2)$ are both even or they are both odd. If they are both even, then $f(x_1) = f(x_2) \Rightarrow -2x_1 = -2x_2 \Rightarrow x_1 = x_2$. Otherwise, $f(x_1)$, $f(x_2)$ are both odd and $f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. Consequently, the function f is one-to-one.

In order to prove that f is an onto function let $n \in \mathbf{N}$. If n is even, then $(-n/2) \in \mathbf{Z}$ and $(-n/2) < 0$, and $f(-n/2) = -2(-n/2) = n$. For the case where n is odd we find that $(n+1)/2 \in \mathbf{Z}$ and $(n+1)/2 > 0$, and $f((n+1)/2) = 2[(n+1)/2] - 1 = (n+1) - 1 = n$. Hence f is onto.

(b) $f^{-1} : \mathbf{N} \to \mathbf{Z}$, where

$$f^{-1}(x) = \begin{cases} (\frac{1}{2})(x+1), & x = 1, 3, 5, 7 \ldots \\ -x/2 \quad , & x = 0, 2, 4, 6, \ldots \end{cases}$$

22. It follows from Theorem 5.11 that there are 5! invertible functions $f : A \longrightarrow B$.

23. (a) For all $n \in \mathbf{N}$, $(g \circ f)(n) = (h \circ f)(n) = (k \circ f)(n) = n$.

(b) The results in part (a) do not contradict Theorem 5.7. For although $g \circ f = h \circ f = k \circ f = 1_{\mathbf{N}}$, we note that

(i) $(f \circ g)(1) = f(\lfloor 1/3 \rfloor) = f(0) = 3 \cdot 0 = 0 \neq 1$, so $f \circ g \neq 1_{\mathbf{N}}$;

(ii) $(f \circ h)(1) = f(\lfloor 2/3 \rfloor) = f(0) = 3 \cdot 0 = 0 \neq 1$, so $f \circ h \neq 1_{\mathbf{N}}$; and

(iii) $(f \circ k)(1) = f(\lfloor 3/3 \rfloor) = f(1) = 3 \cdot 1 = 3 \neq 1$, so $f \circ k \neq 1_{\mathbf{N}}$.

Consequently, none of g, h, and k, is the inverse of f. (After all, since f is *not* onto it is *not* invertible.)

Section 5.7

1.

(a) $f \in O(n)$	(b) $f \in O(1)$	(c) $f \in O(n^3)$
(d) $f \in O(n^2)$	(e) $f \in O(n^3)$	(f) $f \in O(n^2)$
(g) $f \in O(n^2)$		

2. Let $m = 1$ and $k = 1$ in Definition 5.23. Then $\forall n \geq k\ |f(n)| = n < n + (1/n) = |g(n)|$, so $f \in O(g)$.
Now let $m = 2$ and $k = 1$. Then $\forall n \geq k\ |g(n)| = n + (1/n) \leq n + n = 2n = 2|f(n)|$, and $g \in O(f)$.

3. (a) For all $n \in \mathbf{Z}^+, 0 \leq \log_2 n < n$. So let $k = 1$ and $m = 200$ in Definition 5.23. Then $|f(n)| = 100 \log_2 n = 100((1/2)\ \log_2 n) < 200((1/2)n) = 200|g(n)|$, so $f \in O(g)$.
(b) For $n = 6$, $2^n = 64 < 3096 = 4096 - 1000 = 2^{12} - 1000 = 2^{2n} - 1000$. Assuming that $2^k < 2^{2k} - 1000$ for $n = k \geq 6$, we find that $2 < 2^2 \Longrightarrow 2(2^k) < 2^2(2^{2k} - 1000) < 2^2 2^{2k} - 1000$, or $2^{k+1} < 2^{2(k+1)} - 1000$, so $f(n) < g(n)$ for all $n \geq 6$. Therefore, with $k = 6$ and $m = 1$ in Definition 5.23 we find that for $n \geq k\ |f(n)| \leq m|g(n)|$ and $f \in O(g)$.
(c) For all $n \geq 4$, $n^2 \leq 2^n$ (A formal proof of this can be given by mathematical induction.) So let $k = 4$ and $m = 3$ in Definition 5.23. Then for $n \geq k$, $|f(n)| = 3n^2 \leq 3(2^n) < 3(2^n + 2n) = m|g(n)|$ and $f \in O(g)$.

4. Let $m = 11$ and $k = 1$. Then $\forall n \geq k\ |f(n)| = n + 100 \leq 11n^2 = m|g(n)|$, so $f \in O(g)$. However, $\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+$ choose $n > max\{k, 100 + m\}$. Then $n^2 > (100 + m)n = 100n + mn > 100m + mn = m(100 + n) = m|f(n)|$, so $g \notin O(f)$.

5. To show that $f \in O(g)$, let $k = 1$ and $m = 4$ in Definition 5.23. Then for all $n \geq k$, $|f(n)| = n^2 + n \leq n^2 + n^2 = 2n^2 \leq 2n^3 = 4((1/2)(n^3)) = 4|g(n)|$, and f is dominated by g.
To show that $g \notin O(f)$, we follow the idea given in Example 5.66 – namely that

$$\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+\ \exists n \in \mathbf{Z}^+\ [(n \geq k) \wedge (|g(n)| > m|f(n)|)].$$

So not matter what the values of m and k are, choose $n > max\{4m, k\}$. Then $|g(n)| = (1/2)n^3 > (1/2)(4m)n^2 = m(2n^2) \geq m(n^2 + n) = m|f(n)|$, so $g \notin O(f)$.

6. $\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+$ choose $n > max\{k, m\}$ with n odd. Then $n = |f(n)| > m = m \cdot 1 = m|g(n)|$, so $f \notin O(g)$. In a similar way, $\forall m \in \mathbf{R}^+\ \forall k \in \mathbf{Z}^+$ now choose $n > max\{k, m\}$ with n even. Then $n = |g(n)| > m = m \cdot 1 = m|f(n)|$, and $g \notin O(f)$.

7. For all $n \geq 1, \log_2 n \leq n$, so with $k = 1$ and $m = 1$ in Definition 5.23 we have $|g(n)| = \log_2 n \leq n = m \cdot n = m|f(n)|$. Hence $g \in O(f)$.
To show that $f \in O(g)$ we first observe that $\lim_{n \to \infty} \frac{n}{\log_2 n} = +\infty$. (This can be established by using L'Hospital's Rule from the Calculus.) Since $\lim_{n \to \infty} \frac{n}{\log_2 n} = +\infty$ we

find that for every $m \in \mathbf{R}^+$ and $k \in \mathbf{Z}^+$ there is an $n \in \mathbf{Z}^+$ such that $\frac{n}{\log_2 n} > m$, or $|f(n)| = n > m \log_2 n = m|g(n)|$. Hence $f \notin O(g)$.

8. $f \in O(g) \Longrightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k_1 \in \mathbf{Z}^+$ so that $\forall n \geq k_1 \; |f(n)| \leq m_1|g(n)|$. $g \in O(h) \Longrightarrow \exists m_2 \in \mathbf{R}^+ \; \exists k_2 \in \mathbf{Z}^+$ so that $\forall n \geq k_2 \; |g(n)| \leq m_2|h(n)|$. Therefore, $\forall n \geq max\{k_1, k_2\}$ we have $|f(n)| \leq m_1|g(n)| \leq m_1 m_2 |h(n)|$ and $f \in O(h)$.

9. Since $f \in O(g)$, there exists $m \in \mathbf{R}^+, k \in \mathbf{Z}^+$ so that $|f(n)| \leq m|g(n)|$ for all $n \geq k$. But then $|f(n)| \leq [m/|c|]|cg(n)|$ for all $n \geq k$, so $f \in O(cg)$.

10. (a) Let $k = 1$ and $m = 1$ in Definition 5.23.
(b) If $h \in O(f)$ and $f \in O(g)$, then $h \in O(g)$ by Exercise 8. Likewise, if $h \in O(g)$ and $g \in O(f)$ then $h \in O(f)$ – again by Exercise 8.
(c) This follows from parts (a) and (b).

11. (a) For all $n \geq 1$, $f(n) = 5n^2 + 3n > n^2 = g(n)$. So with $M = 1$ and $k = 1$, we have $|f(n)| \geq M|g(n)|$ for all $n \geq k$ and it follows that $f \in \Omega(g)$.

(b) For all $n \geq 1$, $g(n) = n^2 = (1/10)(5n^2 + 5n^2) > (1/10)(5n^2 + 3n) = (1/10)f(n)$. So with $M = (1/10)$ and $k = 1$, we find that $|g(n)| \geq M|f(n)|$ for all $n \geq k$ and it follows that $g \in \Omega(f)$.

(c) For all $n \geq 1$, $f(n) = 5n^2 + 3n > n = h(n)$. With $M = 1$ and $k = 1$, we have $|f(n)| \geq M|h(n)|$ for all $n \geq k$ and so $f \in \Omega(h)$.

(d) Suppose that $h \in \Omega(f)$. If so, there exist $M \in \mathbf{R}^+$ and $k \in \mathbf{Z}^+$ with $n = |h(n)| \geq M|f(n)| = M(5n^2 + 3n)$ for all $n \geq k$. Then $0 < M \leq n/(5n^2+3n) = 1/(5n+3)$. But how can M be a positive *constant* while $1/(5n+3)$ approaches 0 as n (a *variable*) gets larger? From this contradiction it follows that $h \notin \Omega(f)$.

12. Proof: Suppose that $f \in \Omega(g)$. Then there exist $M \in \mathbf{R}^+$ and $k \in \mathbf{Z}^+$ such that $|f(n)| > M|g(n)|$ for all $n \geq k$. Consequently, $|g(n)| \leq (1/M)|f(n)|$ for all $n \geq k$, so $g \in O(f)$.

Conversely, $g \in O(f) \Rightarrow \exists m \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; (|g(n)| \leq m|f(n)|) \Rightarrow \exists m \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; (|f(n)| \geq (1/m)|g(n)|) \Rightarrow \exists M \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; (|f(n)| \geq M|g(n)|) \Rightarrow f \in \Omega(g)$. [Here $M = 1/m$.] [Note: Upon replacing each occurrence of $\Rightarrow$ by $\Leftrightarrow$ we can establish this "if and only if" proof without the first (separate) part in the first paragraph.]

13. (a) For $n \geq 1$, $f(n) = \sum_{i=1}^{n} i = n(n+1)/2 = (n^2/2) + (n/2) > (n^2/2)$. With $k = 1$ and $M = 1/2$, we have $|f(n)| \geq M|n^2|$ for all $n \geq k$. Hence $f \in \Omega(n^2)$.
(b) $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 > \lceil n/2 \rceil^2 + \cdots + n^2 > \lceil n/2 \rceil^2 + \cdots + \lceil n/2 \rceil^2 = \lceil (n+1)/2 \rceil \lceil n/2 \rceil^2 > n^3/8$. With $k = 1$ and $M = 1/8$, we have $|g(n)| \geq M|n^3|$ for all $n \geq k$. Hence $g \in \Omega(n^3)$.

Alternately, for $n \geq 1$, $g(n) = \sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6 = (2n^3 + 3n^2 + n)/6 > n^3/6$.

With $k=1$ and $M=1/6$, we find that $|g(n)| \geq M|n^3|$ for all $n \geq k$ – so $g \in \Omega(n^3)$.
(c) $\sum_{i=1}^{n} i^t = 1^t+2^t+\cdots+n^t > \lceil n/2 \rceil^t+\cdots+n^t > \lceil n/2 \rceil^t+\cdots+\lceil n/2 \rceil^t = \lceil (n+1)/2 \rceil \lceil n/2 \rceil^t > (n/2)^{t+1}$. With $k=1$ and $M=(1/2)^{t+1}$, we have $|h(n)| \geq M|n^{t+1}|$ for all $n \geq k$. Hence $h \in \Omega(n^{t+1})$.

14. Proof: $f \in \Theta(g) \Rightarrow \exists m_1, m_2 \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; m_1|g(n)| \leq |f(n)| \leq m_2|g(n)| \Rightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \leq k \; m_1|g(n)| \leq |f(n)|$ and $\exists m_2 \in \mathbf{R}^+ \; \exists k \in \mathbf{Z}^+ \; \forall n \geq k \; |f(n)| \leq m_2|g(n)| \Rightarrow f \in \Omega(g)$ and $f \in O(g)$.

Conversely, $f \in \Omega(g) \Rightarrow \exists m_1 \in \mathbf{R}^+ \; \exists k_1 \in \mathbf{Z}^+ \; \forall n \geq k_1 \; m_1|g(n)| \leq |f(n)|$. Likewise, $f \in O(g) \Rightarrow \exists m_2 \in \mathbf{R}^+ \; \exists k_2 \in \mathbf{Z}^+ \; \forall n \geq k_2 \; |f(n)| \leq m_2|g(n)|$. Let $k = \max\{k_1, k_2\}$. Then for all $n \geq k$, $m_1|g(n)| \leq |f(n)| \leq m_2|g(n)|$, so $f \in \Theta(g)$.

15. Proof: $f \in \Theta(g) \Rightarrow f \in \Omega(g)$ and $f \in O(g)$ (from Exercise 14 of this section) $\Rightarrow g \in O(f)$ and $g \in \Omega(f)$ (from Exercise 12 of this section) $\Rightarrow g \in \Theta(f)$.

16. Proof: Part (a) follows from Exercises 14 and 13(a) of this section and part (a) of Example 5.68.

The situation is similar for parts (b) and (c).

Section 5.8

1.
(a) $f \in O(n^2)$ (b) $f \in O(n^3)$ (c) $f \in O(n^2)$
(d) $f \in O(\log_2 n)$ (e) $f \in O(n \log_2 n)$

2. (a) $f \in O(n)$ (b) $f \in O(n)$

3. (a) For the following program segment the value of the integer n, and the values of the array entries $A[1], A[2], A[3], \ldots, A[n]$ are supplied beforehand. Also, the variables i, Max, and Location that are used here are integer variables.

```
Begin
    Max := A[1];
    Location := 1;
    If n = 1 then
        Begin
            Writeln ('The first occurrence of the maximum ');
            Write ('entry in the array is at position 1.')
        End;
    If n > 1 then
        Begin
            For i := 2 to n do
                If Max < A[i] then
                    Begin
```

```
                Max := A[i];
                Location := i
            End;
        Writeln (' The first occurrence of the maximum ');
        Write (' entry in the array is at position ', i:0, '.')
    End
End;
```

(b) If, as in Exercise 2, we define the worst-case complexity function $f(n)$ as the number of times the comparison Max $< A[i]$ is executed, then $f(n) = n - 1$ for all $n \in \mathbf{Z}^+$, and $f \in O(n)$.

4. (a) For the following program segment the value of the integer n, and the values of the array entries $A[1], A[2], A[3], \ldots, A[n]$ are supplied earlier in the program. Also the variables i, Max, and Min that are used here are integer variables.

```
Begin
    Min := A[1];
    Max := A[1];
    For i := 2 to n do
        Begin
            If A[i] < Min then
                Min := A[i];
            If A[i] > Max then
                Max := A[i];
        End;
    Writeln (' The minimum value in the array is ', Min :0);
    Write (' and the maximum value is ', Max:0, '.')
End;
```

(b) Here we define the worst-case time-complexity function $f(n)$ as the number of comparisons that are executed in the For loop. Consequently, $f(n) = 2(n-1)$ for all $n \in \mathbf{Z}^+$ and $f \in O(n)$.

5. (a) Here there are five additions and ten multiplications.
(b) For the general case there are n additions and $2n$ multiplications.

6. (a) For each iteration of the for loop there is one addition and one multiplication. Therefore, in total, there are five additions and five multiplications.
(b) For the general case there are n additions and n multiplications.

7. Proof: For $n = 1$, we find that $a_1 = 0 = \lfloor 0 \rfloor = \lfloor \log_2 1 \rfloor$, so the result is true in this first case.
Now assume the result true for all $n = 1, 2, 3, \ldots, k$, where $k \geq 1$, and consider the cases for $n = k + 1$.

(i) $n = k+1 = 2^m$, where $m \in \mathbf{Z}^+$: Here $a_n = 1 + a_{\lfloor n/2 \rfloor} = 1 + a_{2^{m-1}} = 1 + \lfloor \log_2 2^{m-1} \rfloor = 1 + (m-1) = m = \lfloor \log_2 2^m \rfloor = \lfloor \log_2 n \rfloor$; and

(ii) $n = k+1 = 2^m + r$, where $m \in \mathbf{Z}^+$ and $0 < r < 2^m$: Here $2^m < n < 2^{m+1}$, so we have

(1) $2^{m-1} < (n/2) < 2^m$;

(2) $2^{m-1} = \lfloor 2^{m-1} \rfloor \leq \lfloor n/2 \rfloor < \lfloor 2^m \rfloor = 2^m$; and

(3) $m - 1 = \log_2 2^{m-1} \leq \log_2 \lfloor n/2 \rfloor < \log_2 2^m = m$.

Consequently, $\lfloor \log_2 \lfloor n/2 \rfloor \rfloor = m - 1$ and $a_n = 1 + a_{\lfloor n/2 \rfloor} = 1 + \lfloor \log_2 \lfloor n/2 \rfloor \rfloor = 1 + (m-1) = m = \lfloor \log_2 n \rfloor$.

Therefore it follows from the Alternative Form of the Principle of Mathematical Induction that $a_n = \lfloor \log_2 n \rfloor$ for all $n \in \mathbf{Z}^+$.

8. We claim that $a_n = \lceil \log_2 n \rceil$ for all $n \in \mathbf{Z}^+$.
Proof: When $n = 1$ we have $a_1 = 0 = \lceil 0 \rceil = \lceil \log_2 1 \rceil$, and this establishes our basis step. For the inductive step we assume the result true for all $n = 1, 2, 3, \ldots, k$ (≥ 1) and consider what happens at $n = k + 1$.

(i) $n = k+1 = 2^m$, where $m \in \mathbf{Z}^+$: Here $a_n = 1 + a_{\lceil n/2 \rceil} = 1 + a_{2^{m-1}} = 1 + \lceil \log_2 2^{m-1} \rceil = 1 + (m-1) = m = \lceil \log_2 2^m \rceil = \lceil \log_2 n \rceil$.

(ii) $n = k+1 = 2^m + r$, where $m \in \mathbf{Z}^+$ and $0 < r < 2^m$: Here $2^m < n < 2^{m+1}$ and we find that

(1) $2^{m-1} < n/2 < 2^m$;

(2) $2^{m-1} = \lceil 2^{m-1} \rceil < \lceil n/2 \rceil \leq \lceil 2^m \rceil = 2^m$; and

(3) $m - 1 = \log_2 2^{m-1} < \log_2 \lceil n/2 \rceil \leq \log_2 2^m = m$.

Therefore, $\lceil \log_2 \lceil n/2 \rceil \rceil = m$ and $a_n = 1 + a_{\lceil n/2 \rceil} = 1 + \lceil \log_2 \lceil n/2 \rceil \rceil = 1 + m = \lceil \log_2 n \rceil$, since $2^m < n < 2^{m+1} \Rightarrow \log_2 2^m = m < \log_2 n < m + 1 = \log_2 2^{m+1} \Rightarrow m < \lceil \log_2 n \rceil = m + 1$.

Consequently, it follows from the Alternative Form of the Principle of Mathematical Induction that $a_n = \lceil \log_2 n \rceil$ for all $n \in \mathbf{Z}^+$.

9. Here $np = 3/4$ and $q = 1 - np = 1/4$, so $E(X) = np(n+1)/2 + nq = (3/4)[(n+1)/2] + (1/4)n = (3/8)n + (3/8) + (1/4)n = (5/8)n + (3/8)$.

10. $Pr(X = i) = i/[n(n+1)]$, so $\sum_{i=1}^{n} Pr(X = i) = \sum_{i=1}^{n} i/[n(n+1)] = (1/[n(n+1)]) \sum_{i=1}^{n} i = (1/[n(n+1)])[n(n+1)/2] = 1/2$ and $q = 1 - (1/2) = 1/2$.
$E(X) = \sum_{i=1}^{n} i^2/[n(n+1)] + (1/2)n = [1/[n(n+1)]] \sum_{i=1}^{n} i^2 + (1/2)n = \frac{1}{n(n+1)} \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n}{2} = \frac{2n+1}{6} + \frac{n}{2} = \frac{5n}{6} + \frac{1}{6}$.

11.

a)
```
procedure LocateRepeat (n: positive integer; a1, a2, a3, ..., an: integers)
begin
    location := 0
    i := 2
    while i ≤ n and location = 0 do
        begin
            j := 1
            while j < i and location = 0 do
```

```
                if a_j = a_i then location := i
                else j := j + 1
            i := i + 1
        end
end {location is the subscript of the first array entry that repeats a previous
    array entry; location is 0 if the array contains n distinct integers}
```

b) For $n \geq 2$, let $f(n)$ count the maximum number of times the second **while** loop is executed. The second **while** loop is executed at most $n-1$ times for each value of i, where $2 \leq i \leq n$. Consequently, $f(n) = 1+2+3+\cdots+(n-1) = (n-1)(n)/2$, which occurs when the array consists of n distinct integers or when the only repeat is a_{n-1} and a_n. Since $(n-1)(n)/2 = (1/2)(n^2 - n)$ we have $f \in O(n^2)$.

12.

a)

```
procedure FirstDecrease (n: positive integer; a_1, a_2, a_3, ..., a_n: integers)
begin
    location := 0
    i := 2
    while i ≤ n and location = 0 do
            if a_i < a_{i-1} then location := i
            else i := i + 1
end {location is the subscript of the first array entry that is smaller than its
    immediate predecessor; location is 0 if the n integers in the array
    are in increasing order}
```

b) For $n \geq 2$, let $f(n)$ count the maximum number of comparisons made in the **while** loop. This is $n-1$, which occurs if the integers in the array are in ascending order or if $a_1 < a_2 < a_3 < \ldots < a_{n-1}$ and $a_n < a_{n-1}$. Consequently, $f \in O(n)$.

Supplementary Exercises

1. (a) If either A or B is $\emptyset$ then $A \times B = \emptyset = A \cap B$ and the result is true.
For A, B nonempty we find that:
$(x,y) \in (A \times B) \cap (B \times A) \Rightarrow (x,y) \in A \times B$ and $(x,y) \in B \times A \Rightarrow (x \in A$ and $y \in B)$ and $(x \in B$ and $y \in A) \Rightarrow x \in A \cap B$ and $y \in A \cap B \Rightarrow (x,y) \in (A \cap B) \times (A \cap B)$; and
$(x,y) \in (A \cap B) \times (A \cap B) \Rightarrow (x \in A$ and $x \in B)$ and $(y \in A$ and $y \in B) \Rightarrow (x,y) \in A \times B$ and $(x,y) \in B \times A \Rightarrow (x,y) \in (A \times B) \cap (B \times A)$.
Consequently, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$.

(b) If either A or B is $\emptyset$ then $A \times B = \emptyset = B \times A$ and the result follows.
If not, let $(x,y) \in (A \times B) \cup (B \times A)$. Then

$(x,y) \in (A \times B) \cup (B \times A) \Rightarrow (x,y) \in A \times B$ or $(x,y) \in (B \times A) \Rightarrow (x \in A$ and $y \in B)$ or $(x \in B$ and $y \in A) \Rightarrow (x \in A$ or $x \in B)$ and $(y \in A$ or $y \in B) \Rightarrow x, y \in A \cup B \Rightarrow (x,y) \in (A \cup B) \times (A \cup B)$.

2. (a) True (b) False: Let $A = \{1,2\}, B = \{x,y\}, f = \{(1,x),(2,y)\}$.
(c) False: Let $f : \mathbf{Z} \to \mathbf{Z}, f(x) = 2x$. (d) True.
(e) False: Let $A = \{1,2\}, B = \{1,2,3\}, C = \{1,2,3,4\}, f = \{(1,1),(2,2)\}$,
$g = \{(1,1),(2,2),(3,3)\}, h = \{(1,1),(2,2),(3,4)\}$.
(f) False. Let $A = \{1,2,3,4\}, B = \{5,6\}, A_1 = \{1,2\}, A_2 = \{2,3,4\}$,
$f = \{(1,5),(2,6),(3,5),(4,5)\}$. Then $f(A_1 \cap A_2) = f(2) = \{6\}$, but $f(A_1) \cap f(A_2) = \{5,6\}$.
(g) True

3. (a) $f(1) = f(1 \cdot 1) = 1 \cdot f(1) + 1 \cdot f(1)$, so $f(1) = 0$.
(b) $f(0) = 0$
(c) Proof (by Mathematical Induction): When $a = 0$ the result is true, so consider $a \neq 0$. For $n = 1$, $f(a^n) = f(a) = 1 \cdot a^0 \cdot f(a) = na^{n-1}f(a)$, so the result follows in this first case, and this establishes our basis step. Assume the result true for $n = k (\geq 1)$ – that is, $f(a^k) = ka^{k-1}f(a)$. For $n = k+1$ we have $f(a^{k+1}) = f(a \cdot a^k) = af(a^k) + a^k f(a) = aka^{k-1}f(a) + a^k f(a) = ka^k f(a) + a^k f(a) = (k+1)a^k f(a)$. Consequently, the truth of the result for $n = k+1$ follows from the truth of the result for $n = k$. So by the Principle of Mathematical Induction the result is true for all $n \in \mathbf{Z}^+$.

4. $2^{|A \times B|} = 262,144 \Longrightarrow |A \times B| = 18 \Longrightarrow |A| = 2, |B| = 9$ or $|A| = 3, |B| = 6$.

5. $(x,y) \in (A \cap B) \times (C \cap D) \Longleftrightarrow x \in A \cap B, y \in C \cap D \Longleftrightarrow (x \in A, y \in C)$ and $(x \in B, y \in D) \Longrightarrow (x,y) \in A \times C$ and $(x,y) \in B \times D \Longleftrightarrow (x,y) \in (A \times C) \cap (B \times D)$

6. (a) $5!$ (b) $4!$

7. If $0 \leq x < 1$, then $\lfloor x \rfloor = 0$ and $x^2 = 1/2$. So $x = 1/\sqrt{2}$.
If $1 \leq x < 2$, then $\lfloor x \rfloor = 1$ and $x^2 = 3/2$. So $x = \sqrt{3/2}$.
For $k \in \mathbf{Z}^+$ and $k \geq 2$, if $k \leq x < k+1$, then $\lfloor x \rfloor = k$ and if x satisfies the given equation we have $x^2 = k + (1/2)$. But for $k \geq 2$ we find that $k(k-1) > 0$, so $k(k-1) \geq 1 > 1/2$, and $k^2 - k > 1/2$. Now $k^2 > k + (1/2) \Rightarrow k > \sqrt{k + (1/2)} = x$ and we do *not* have $k \leq x < k+1$.
Finally, let $k \in \mathbf{Z}^+$ and consider $-k \leq x < -k+1$. Then $x^2 - \lfloor x \rfloor = x^2 - (-k) = x^2 + k$, and $x^2 - \lfloor x \rfloor = 1/2 \Rightarrow x^2 = -k + 1/2 < 0$, so x cannot be a real number.
Consequently, there are only two real numbers that satisfy the equation $x^2 - \lfloor x \rfloor = 1/2$ — namely, $x = 1/\sqrt{2}$ and $x = \sqrt{3/2}$.

8. Proof: First we show that the result holds for the first part of the recursive definition. Since $2 \cdot 1 = 2 \geq 1$ we find the result true in part (1). In order to complete the proof we need to verify that every ordered pair (s,t) in $\mathcal{R}$ that results from part (2) of the definition satisfies the condition $2s \geq t$. We consider three cases:

(i) $(a+1, b)$ with $(a, b) \in \mathcal{R}$: Here we have $2a \geq b$, and since $a+1 \geq a$ it follows that $2(a+1) \geq 2a \geq b$;
(ii) $(a+1, b+1)$ with $(a, b) \in \mathcal{R}$: Now we find that $2a \geq b \Rightarrow 2a+2 \geq b+1 \Rightarrow 2(a+1) \geq b+1$; and
(iii) $(a+1, b+2)$ with $(a, b) \in \mathcal{R}$: In this last case it follows that $2a \geq b \Rightarrow 2a+2 \geq b+2 \Rightarrow 2(a+1) \geq b+2$.
Consequently, for all $(a, b) \in \mathcal{R}$ we have $2a \geq b$.

9. (a) $f^2(x) = f(f(x)) = a(f(x)+b) - b = a[(a(x+b)-b)+b] - b = a^2(x+b) - b$
$f^3(x) = f(f^2(x)) = f(a^2(x+b)-b) = a[(a^2(x+b)-b)+b] - b = a^3(x+b) - b$

(b) Conjecture: For $n \in \mathbf{Z}^+$, $f^n(x) = a^n(x+b) - b$. Proof (by Mathematical Induction): The formula is true for $n = 1$ – by the definition of $f(x)$. Hence we have our basis step. Assume the formula true for $n = k(\geq 1)$ – that is, $f^k(x) = a^k(x+b) - b$. Now consider $n = k+1$. We find that $f^{k+1}(x) = f(f^k(x)) = f(a^k(x+b)-b) = a[(a^k(x+b)-b)+b] - b = a^{k+1}(x+b) - b$. Since the truth of the formula at $n = k$ implies the truth of the formula at $n = k+1$, it follows that the formula is valid for all $n \in \mathbf{Z}^+$ – by the Principle of Mathematical Induction.

10. Let $n = |A| - |A_1|$. Since $|B|^n$ is the number of ways to extend f to A and $|B|^n = 6^n = 216$, then $n = 3$ and $|A| = 8$.

11. (a) $(7 \times 6 \times 5 \times 4 \times 3)/(7^5) \doteq 0.15$.
(b) For the computer program the elements of B are replaced by $\{1,2,3,4,5,6,7\}$.

```
10   Random
20   Dim F(5)
30   For I = 1 To 5
40        F(I) = Int(Rnd*7 + 1)
50   Next I
60   For J = 2 To 5
70        For K = 1 To J - 1
80             If F(J) = F(K) then GOTO 120
90        Next K
100  Next J
110  GOTO 140
120  C = C + 1
130  GOTO 10
140  C = C + 1
150  Print "After "; C; " generations the resulting"
160  Print "function is one-to-one."
170  Print "The one-to-one function is given as:"
180  For I = 1 To 5
190       Print "("; I; ","; F(I); ")"
```

```
200 Next I
210 End
```

12. For each subset A of S, let s_A denote the sum of the elements of A. Consider only those nonempty subsets A of S where $|A| \leq 5$. There are $2^7 - 1 - 1 - 7 = 119$ such subsets and here $1 \leq s_A \leq 20+21+22+23+24 = 110$. The result follows by the Pigeonhole Principle for there are 119 subsets (pigeons) and 110 possible sums (pigeonholes).

13. For $1 \leq i \leq 10$, let x_i be the number of letters typed on day i. Then $x_1 + x_2 + x_3 + \ldots + x_8 + x_9 + x_{10} = 84$, or $x_3 + \ldots + x_8 = 54$. Suppose that $x_1 + x_2 + x_3 < 25, x_2 + x_3 + x_4 < 25, \ldots, x_8 + x_9 + x_{10} < 25$. Then $x_1 + 2x_2 + 3(x_3 + \ldots + x_8) + 2x_9 + x_{10} < 8(25) = 200$, or $3(x_3 + \ldots + x_8) < 160$. Consequently, $54 = x_3 + \ldots + x_8 < (160)/3 = 53\ 1/3$.

14. If two elements in $\{x_1, x_2, \ldots, x_7\}$ have the same units digit then their difference is divisible by 10. If this does not happen consider the ten possible units digits as follows: $\{0\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}$ – these are the pigeonholes for the problem. When the seven pigeons $\{x_1, x_2, \ldots, x_7\}$ go to the pigeonholes where their units digits are located, at least one two-element subset is filled and those two numbers (pigeons) will sum to a multiple of 10.

15. For $\prod_{k=1}^{n}(k - i_k)$ to be odd, $(k - i_k)$ must be odd for all $1 \leq k \leq n$, i.e., one of k, i_k must be even and the other odd. Since n is odd, $n = 2m+1$ and in the list $1, 2, \ldots, n$, there are m even integers and $m+1$ odd integers. Let $1, 3, 5, \ldots, n$ be the pigeons and $i_1, i_3, i_5, \ldots, i_n$ the pigeonholes. At most m of the pigeonholes can be even integers, so $(k - i_k)$ must be even for at least one $k = 1, 3, 5, \ldots, n$. Consequently, $\prod_{k=1}^{n}(k - i_k)$ is even.

16. (a) The answer is the number of onto functions $f : A \longrightarrow B$ where $|A| = 10$ (weekly chores) and $|B| = 3$ (for the three young men). There are $3!S(10,3)$ such functions.

(b) $2!S(9,2)$ (Thomas only mows the lawn) $+3!S(9,3)$ (Thomas does more than just mow the lawn).

17. Let the n distinct objects be $x_1, x_2, \ldots, x_n$. Place x_n in a container. Now there are two *distinct* containers. For each of $x_1, x_2, \ldots, x_{n-1}$ there are two choices and this gives 2^{n-1} distributions. Among these there is one where $x_1, x_2, \ldots, x_{n-1}$ are in the container with x_n, so we remove this distribution and find $S(n,2) = 2^{n-1} - 1$.

18. (a) $\binom{12}{9}$ (b) $5!S(9,5)$

(c) $4!S(7,4)$ (Donald gets only the two books on basketball) $+ 5!S(7,5)$ (Donald gets the two books on basketball and at least one other book.)

19. (a) and (b) $m!S(n,m)$

20. $S(n, n-2)$ is the number of ways to place n distinct objects into $n-2$ identical containers with no container left empty. There are two cases. One container contains three objects and the others one. This can happen in $\binom{n}{3}$ ways. The other possibility is that two containers each contain two objects and the others one. This happens in $(1/2)\binom{n}{2}\binom{n-2}{2} = (n!)/[2!2!2!(n-4)!] = 3\binom{n}{4}$ ways.

21. Fix $m = 1$. For $n = 1$ the result is true. Assume $f \circ f^k = f^k \circ f$ and consider $f \circ f^{k+1}$. $f \circ f^{k+1} = f \circ (f \circ f^k) = f \circ (f^k \circ f) = (f \circ f^k) \circ f = f^{k+1} \circ f$. Hence $f \circ f^n = f^n \circ f$ for all $n \in \mathbf{Z}^+$. Now assume that for $t \geq 1$, $f^t \circ f^n = f^n \circ f^t$. Then $f^{t+1} \circ f^n = (f \circ f^t) \circ f^n = f \circ (f^t \circ f^n) = f \circ (f^n \circ f^t) = (f \circ f^n) \circ f^t = (f^n \circ f) \circ f^t = f^n \circ (f \circ f^t) = f^n \circ f^{t+1}$, so $f^m \circ f^n = f^n \circ f^m$ for all $m, n \in \mathbf{Z}^+$.

22. (b) $y \in f(\bigcap_{i \in I} A_i) \iff y = f(x)$, for some $x \in \bigcap_{i \in I} A_i \implies$ $y \in f(A_i)$, for all $i \in I \iff y \in \bigcap_{i \in I} f(A_i)$.

(c) From part (b), $f(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} f(A_i)$. For the opposite inclusion let $y \in \bigcap_{i \in I} f(A_i)$. Then $y \in f(A_i)$ for all $i \in I$, so $y = f(x_i), x_i \in A_i$, for each $i \in I$. Since f is one-to-one, all of these x_i's, $i \in I$, yield only one element $x \in \bigcap_{i \in I} A_i$. Hence $y = f(x) \in f(\bigcap_{i \in I} A_i)$, so $\bigcap_{i \in I} f(A_i) \subseteq f(\bigcap_{i \in I} A_i)$ and the equality follows.

The proof for part (a) is done in a similar way.

23. Proof: Let $a \in A$. Then

$$f(a) = g(f(f(a))) = f(g(f(f(f(a))))) = f(g \circ f^3(a)).$$

From $f(a) = g(f(f(a)))$ we have $f^2(a) = (f \circ f)(a) = f(g(f(f(a))))$. So $f(a) = f(g \circ f^3(a)) = f(g(f(f(f(a))))) = f^2(f(a)) = f^2(g(f^2(a))) = f(f(g(f(f(a))))) = f(g(f(a))) = g(a)$.

Consequently, $f = g$.

24. (a) $n^{(n \times n)} = n^{(n^2)}$ (b) $n^{(n^3)}$ (c) $n^{(n^k)}$

(d) Since $|A| = n$, there are n choices for each selection of size k, with repetitions allowed, from the set A of size n. There are $r = \binom{n+k-1}{k}$ possible selections and n^r commutative k-ary operations on A.

25. a) Note that $2 = 2^1$, $16 = 2^4$, $128 = 2^7$, $1024 = 2^{10}$, $8192 = 2^{13}$, and $65536 = 2^{16}$. Consider the exponents on 2. If four numbers are selected from $\{1, 4, 7, 10, 13, 16\}$, there is at least one pair whose sum is 17. Hence if four numbers are selected from S, there are two numbers whose product is $2^{17} = 131072$.

b) Let $a, b, c, d, n \in \mathbf{Z}^+$. Let $S = \{b^a, b^{a+d}, b^{a+2d}, \ldots, b^{a+nd}\}$. If $\lceil \frac{n}{2} \rceil + 1$ numbers are selected from S then there are at least two of them whose product is b^{2a+nd}.

26. (a) $\chi_{A \cap B}, \chi_A \cdot \chi_B$ both have domain $\mathcal{U}$ and codomain $\{0,1\}$. For each $x \in \mathcal{U}$, $\chi_{A \cap B}(x) = 1$ iff $x \in A \cap B$ iff $x \in A$ and $x \in B$ iff $\chi_A(x) = 1$ and $\chi_B(x) = 1$. Also, $\chi_{A \cap B}(x) =$

0 iff $x \notin A \cap B$ iff $x \notin A$ or $x \notin B$ iff $\chi_A(x) = 0$ or $\chi_B(x) = 0$ iff $\chi_A \cdot \chi_B(x) = 0$. Hence $\chi_{A\cap B} = \chi_A \cdot \chi_B$.

(b) The proof here is similar to that of part (a).

(c) $\chi_{\bar{A}}(x) = 1$ iff $x \in \bar{A}$ iff $x \notin A$ iff $\chi_A(x) = 0$ iff $(1 - \chi_A)(x) = 1$. $\chi_{\bar{A}}(x) = 0$ iff $x \notin \bar{A}$ iff $x \in A$ iff $\chi_A(x) = 1$ iff $(1 - \chi_A)(x) = 0$. Hence $\chi_{\bar{A}} = 1 - \chi_A$.

27. $f \circ g = \{(x, z), (y, y), (z, x)\}; g \circ f = \{(x, x), (y, z), (z, y)\};$
$f^{-1} = \{(x, z), (y, x), (z, y)\}; g^{-1} = \{(x, y), (y, x), (z, z)\};$
$(g \circ f)^{-1} = \{(x, x), (y, z), (z, y)\} = f^{-1} \circ g^{-1}; g^{-1} \circ f^{-1} = \{(x, z), (y, y), (z, x)\}.$

28. (a) $f^{-1}(8) = \{x|5x + 3 = 8\} = \{1\}$.
(b) $|x^2 + 3x + 1| = 1 \Longrightarrow x^2 + 3x + 1 = 1$ or $x^2 + 3x + 1 = -1 \Longrightarrow x^2 + 3x = 0$ or $x^2 + 3x + 2 = 0 \Longrightarrow (x)(x + 3) = 0$ or $(x + 2)(x + 1) = 0 \Longrightarrow x = 0, -3$ or $x = -1, -2$. Hence $g^{-1}(1) = \{-3, -2, -1, 0\}$.
(c) $\{-8/5, -8/3\}$

29. Under these conditions we know that $f^{-1}(\{6, 7, 9\}) = \{2, 4, 5, 6, 9\}$. Consequently we have
(i) two choices for each of $f(1)$, $f(3)$, and $f(7)$ – namely, 4 or 5;
(ii) two choices for each of $f(8)$ and $f(10)$ – namely, 8 or 10; and
(iii) three choices for each of $f(2)$, $f(4)$, $f(5)$, $f(6)$, and $f(9)$ – namely, 6, 7, or 9.
Therefore, by the rule of product, it follows that the number of functions satisfying these conditions is $2^3 \cdot 2^2 \cdot 3^5 = 7776$.

30. Since $f^1 = f$ and $(f^{-1})^1 = f^{-1}$, the result is true for $n = 1$. Assume the result for $n = k : (f^k)^{-1} = (f^{-1})^k$. For $n = k + 1, (f^{k+1})^{-1} = (f \circ f^k)^{-1} = (f^k)^{-1} \circ (f^{-1}) = (f^{-1})^k \circ (f^{-1})^1 = (f^{-1})^1 \circ (f^{-1})^k$ (by Exercise 21) $= (f^{-1})^{k+1}$. Therefore, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbf{Z}^+$.

31. (a) $(\pi \circ \sigma)(x) = (\sigma \circ \pi)(x) = x$
(b) $\pi^n(x) = x - n; \sigma^n(x) = x + n (n \geq 2)$.
(c) $\pi^{-n}(x) = x + n; \sigma^{-n}(x) = x - n (n \geq 2)$.

32. (a) $\tau(n) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$
(b) $k = 2 : \tau(2) = \tau(3) = \tau(5) = 2$
$k = 3 : \tau(2^2) = \tau(3^2) = \tau(5^2) = 3$
$k = 4 : \tau(6) = \tau(8) = \tau(10) = 4$
$k = 5 : \tau(2^4) = \tau(3^4) = \tau(5^4) = 5$
$k = 6 : \tau(12) = \tau(18) = \tau(20) = 6$
(c) For all $k > 1$ and any prime p, $\tau(p^{k-1}) = k$.
(d) Let $a = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ and $b = q_1^{f_1} q_2^{f_2} \cdots q_t^{f_t}$, where $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_t$ are $k + t$ distinct primes, and $e_1, e_2, \ldots, e_k, f_1, f_2, \ldots f_t \in \mathbf{Z}^+$. Then
$\tau(ab) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1)(f_1 + 1)(f_2 + 1) \cdots (f_t + 1)$

$= [(e_1+1)(e_2+1)\cdots(e_k+1)][(f_1+1)(f_2+1)\cdots(f_t+1)] = \tau(a)\tau(b).$

33. (a) Here there are eight distinct primes and each subset A satisfying the stated property determines a distribution of the eight distinct objects in $X = \{2,3,5,7,11,13,17,19\}$ into four identical containers with no container left empty. There are $S(8,4)$ such distributions.

(b) $S(n,m)$

34. Define $f : \mathbf{Z}^+ \to \mathbf{R}$ by $f(n) = 1/n$.

35. (a) Let $m = 1$ and $k = 1$. Then for all $n \geq k, |f(n)| \leq 2 < 3 \leq |g(n)| = m|g(n)|$, so $f \in O(g)$.

(b) Let $m = 4$ and $k = 1$. Then for all $n \geq k, |g(n)| \leq 4 = 4 \cdot 1 \leq 4|f(n)| = m|f(n)|$, so $g \in O(f)$.

36. (a)$f \in O(f_1) \Longrightarrow \exists m_1 \in \mathbf{R}^+\ \exists k_1 \in \mathbf{Z}^+$ such that $|f(n)| \leq m_1|f_1(n)|\ \forall n \geq k_1$.
$g \in O(g_1) \Longrightarrow \exists m_2 \in \mathbf{R}^+\ \exists k_2 \in \mathbf{Z}^+$ such that $|g(n)| \leq m_2|g_1(n)|\ \forall n \geq k_2$.
Let $m = max\{m_1, m_2\}$. Then for all $n \geq max\{k_1, k_2\}, |(f+g)(n)| = |f(n)+g(n)| = |f(n)| + |g(n)| \leq m_1|f_1(n)| + m_2|g_1(n)| \leq m(|f_1(n)| + |g_1(n)|) = m|f_1(n) + g_1(n)| = m|(f_1+g_1)(n)|$, so $(f+g) \in O(f_1+g_1)$.

(b) Let $f, f_1, g, g_1 : \mathbf{Z}^+ \longrightarrow \mathbf{R}$ be defined by $f(n) = n, f_1(n) = 1-n, g(n) = 1, g_1(n) = n$.

37. First note that if $\log_a n = r$, then $n = a^r$ and $\log_b n = \log_b(a^r) = r\log_b a = (\log_b a)(\log_a n)$.
Now let $m = (\log_b a)$ and $k = 1$. Then for all $n \geq k, |g(n)| = \log_b n = (\log_b a)(\log_a n) = m|f(n)|$, so $g \in O(f)$.
Finally, with $m = (\log_b a)^{-1} = \log_a b$ and $k = 1$, we find that for all $n \geq k, |f(n)| = \log_a n = (\log_a b)(\log_b n) = m|g(n)|$. Hence $f \in O(g)$.