Conditional Probability

When an experiment produces a pair of random variables X and Y, observing a sample value of one of them provides partial information about the other. The partial knowledge consists of the value of one of the random variables: either $X = x$ or $Y = y$.

Learning $Y=y$ changes our knowledge of Y and modified knowledge of X. The new model is either a conditional PMF of X given Y, ~~or~~ or a conditional PDF of X given Y.

Conditional PMF

For discrete random variables X and Y with joint PMF $P(x,y)$ and x and y such that $P(x)>0$ and $P(y)>0$,

$$P(x|y) = \frac{P(x,y)}{P(y)}, \quad P(y|x) = \frac{P(x,y)}{P(x)}$$

Conditional PDF

For discrete random variables X and Y with joint PDF $f(x,y)$ and x and y such that $P_1(x)>0$ and $P_2(y)>0$,

$$f(x|y) = \frac{f(x,y)}{P_2(y)}, \quad f(y|x) = \frac{f(x,y)}{P_1(x)}$$

Note: If X and Y are independent, then

$P(x|y) = P(x)$, $P(y|x) = P(y)$ in discrete case

$f(x|y) = P_1(x)$, $f(y|x) = P_2(y)$ in continuous case

* Random variables X and Y have the joint PMF given by

X \ Y	1	2	3	4
1	1/4	0	0	0
2	1/8	1/8	0	0
3	1/12	1/12	1/12	0
4	1/16	1/16	1/16	1/16

, find the conditional PMF of Y given $X=x$ for each x.

sol^n

X	1	2	3	4
P(x)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$p(y|x=1) =$

Y	1	2	3	4
P(y|x=1)	$\frac{1/4}{1/4}$	$\frac{0}{1/4}$	$\frac{0}{1/4}$	$\frac{0}{1/4}$

$=$

Y	1	2	3	4
P(y|x=1)	1	0	0	0

$P(y|x=2) =$

Y	1	2	3	4
P(y|x=2)	$\frac{1/8}{1/4}$	$\frac{1/8}{1/4}$	$\frac{0}{1/4}$	$\frac{0}{1/4}$

$=$

Y	1	2	3	4
P(y|x=2)	$\frac{1}{2}$	$\frac{1}{2}$	0	0

$P(y|x=3) =$

Y	1	2	3	4
P(y|x=3)	$\frac{1/12}{1/4}$	$\frac{1/12}{1/4}$	$\frac{1/12}{1/4}$	$\frac{1/12}{1/4}$

$=$

Y	1	2	3	4
P(y|x=3)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

$p(y|x=4) =$

Y	1	2	3	4
P(y|x=4)	$\frac{1/16}{1/4}$	$\frac{1/16}{1/4}$	$\frac{1/16}{1/4}$	$\frac{1/16}{1/4}$

$=$

Y	1	2	3	4
P(y|x=4)	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

* Random variables X and Y have the joint PDF

$$f(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$, Find (i) $f(y|x)$ for $0 \le x \le 1$ (ii) $f(x|y)$ for $0 \le y \le 1$

sol^n $P_1(x) = \int_0^x 2\,dy = 2y\Big]_0^x = 2x$; $P_2(y) = \int_y^1 2\,dx = 2x\Big]_y^1 = 2-2y$

$$f(y|x) = \frac{f(x,y)}{P_1(x)} = \begin{cases} \frac{2}{2x} = \frac{1}{x}, & 0 \le y \le x \le 1 \\ \frac{0}{2x} = 0, & \text{otherwise} \end{cases}$$

$$f(x|y) = \frac{f(x,y)}{P_2(y)} = \begin{cases} \frac{2}{2(1-y)} = \frac{1}{1-y}, & 0 \le y \le x \le 1 \\ \frac{0}{2(1-y)} = 0, & \text{otherwise} \end{cases}$$

Conditional mean and Conditional variance

For two random variables X and Y, the expected value of X, given $Y=y$ is

Discrete case: $E[X|Y=y] = \sum_x x\, P(x|y)$

Continuous case: $E[X|Y=y] = \int_{-\infty}^{\infty} x\, f(x|y)\, dx$

the variance of X, given $Y=y$ is

Discrete case: $Var[X|Y=y] = \sum_x x^2 P(x|y) - \left\{\sum_x x P(x|y)\right\}^2$

Continuous case: $Var[X|Y=y] = \int_{-\infty}^{\infty} x^2 f(x|y)\, dx - \left\{\int_{-\infty}^{\infty} x f(x|y)\, dx\right\}^2$

* The conditional PMFs of

X \ Y	1	2	3	4
1	1/4	0	0	0
2	1/8	1/8	0	0
3	1/12	1/12	1/12	0
4	1/16	1/16	1/16	1/16

are given by $p(y|x=1) = \begin{cases} 1, & y=1 \\ 0, & \text{otherwise} \end{cases}$, $P(y|x=2) = \begin{cases} 1/2, & y \in \{1,2\} \\ 0, & \text{otherwise} \end{cases}$

$P(y|x=3) = \begin{cases} 1/3, & y \in \{1,2,3\} \\ 0, & \text{otherwise} \end{cases}$, $p(y|x=4) = \begin{cases} 1/4, & y \in \{1,2,3,4\} \\ 0, & \text{otherwise.} \end{cases}$

Find $E[Y|X=x]$ for $x = 1, 2, 3, 4$.

Sol^n $E[Y|X=1] = \Sigma y P(y|x=1) = 1 \times 1 = 1$

$E[Y|X=2] = \Sigma y P(y|x=2) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = \frac{3}{2}$

$E[Y|X=3] = \Sigma y P(y|x=3) = 1 \times \frac{1}{3} + 2 \times \frac{1}{3} + 3 \times \frac{1}{3} = \frac{6}{3} = 2$

$E[Y|X=4] = \Sigma y P(y|x=4) = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$

* The random variables X and Y having the joint PDF, $f(x,y) = \begin{cases} 2, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise,} \end{cases}$ the conditional PDF of X given Y is given by $f(x|y) = \begin{cases} \frac{1}{1-y}, & 0 \le y \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$

Find the conditional expected values $E[X|Y=y]$ -

Sol^n $E[X|Y=y] = \int_{-\infty}^{\infty} x f(x|y)\, dx$

$$= \int_{y}^{1} \frac{x}{1-y}\, dx$$

$$= \frac{1}{(1-y)} \times \frac{x^2}{2}\Big]_y^1$$

$$= \frac{1}{2(1-y)}(1-y^2)$$

$$= \frac{(1+y)(1-y)}{2(1-y)}$$

$$E[X|Y=y] = \frac{1+y}{2}$$

The joint PMF of two random variables X and Y is given by

$X \backslash Y$	-1	0	1
-1	$1/12$	$1/6$	$1/4$
0	$1/18$	$1/9$	$1/6$
1	$1/36$	$1/18$	$1/12$

Find the (i) Conditional PMF of Y given $X = -1$

(ii) conditional PMF of X given $Y = 1$

Also find (i) $E[Y|X=x]$ for $x=-1$

(ii) $E[X|y=y]$ for each $y=1$

(iii) $Var[Y|X=x]$ for $x=-1$

$Var[X|Y=y]$ for $y=1$

Sol^n

X	-1	0	1
$P(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

,

Y	-1	0	1
$P(Y)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$P(Y/X=-1)$

Y	-1	0	1
$P(y/x=-1)$	$\frac{1/12}{1/6}$	$\frac{1/18}{1/6}$	$\frac{1/36}{1/6}$

$=$

Y	-1	0	1
$p(y/x=-1)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$P(X/Y=1)$

X	-1	0	1
$P(x/y=1)$	$\frac{1/36}{1/6}$	$\frac{1/18}{1/6}$	$\frac{1/12}{1/6}$

$=$

X	-1	0	1
$P(x/y=1)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

$$E[Y|X=-1] = \sum y\,P(y/x=-1) = -1\times\frac{1}{2} + 0\times\frac{1}{3} + 1\times\frac{1}{6} = -\frac{2}{6} = -\frac{1}{3}$$

$$E[X|y=1] = \sum x\,P(x/y=1) = -1\times\frac{1}{6} + 0\times\frac{1}{3} + 1\times\frac{1}{2} = \frac{2}{6} = \frac{1}{3}$$

$$Var[Y|X=-1] = \sum y^2 P(y/x=-1) - \{\sum y\,P(y/x=-1)\}^2$$

$$= 1\times\frac{1}{2} + 0\times\frac{1}{3} + 1\times\frac{1}{6} - \left(-\frac{1}{3}\right)^2$$

$$= \frac{5}{9}$$

$$Var[X|Y=1] = \sum x^2 P(x|y=1) - \{\sum x\,P(x|y=1)\}^2$$

$$= 1\times\frac{1}{6} + 0\times\frac{1}{3} + 1\times\frac{1}{2} - \left(\frac{1}{3}\right)^2$$

$$= \frac{5}{9}$$

* Let R be the uniform $(0,1)$ random variable.
Given $R = r$, X is the uniform$(0,r)$ random variable.
Find the conditional PDF of R given X.

Sol^n Given
$$f(r) = \begin{cases} 1, & 0 \le r \le 1 \\ 0, & \text{otherwise} \end{cases}, \quad f(x|r) = \begin{cases} \frac{1}{r}, & 0 \le x < r \\ 0, & \text{otherwise} \end{cases}$$

joint PDF of R and X is
$$f(r, x) = f(x|r)f(r) = \begin{cases} \frac{1}{r}, & 0 < x < r < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$f(x) = \int_{-\infty}^{\infty} f(r,x)\,dr = \int_{x}^{1} \frac{1}{r}\,dr = \ln r\Big]_{x}^{1} = \ln 1 - \ln x = -\ln x$$

$$f(r|x) = \frac{f(r,x)}{f(x)} = \begin{cases} \frac{1/r}{-\ln x} = -\frac{1}{r\ln x}, & x < r < 1 \\ 0, & \text{otherwise.} \end{cases}$$

* Given the joint density function $f(x,y) = \begin{cases} \frac{6-x-y}{8}, & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise,} \end{cases}$

find $P(1<y<3 \mid x=1)$

Sol^n. $P_1(x) = \int_{y=2}^{4} \frac{6-x-y}{8}\,dy = \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4 = \frac{1}{8}\left[(24-4x-8)-(12-2x-2)\right] = \frac{1}{8}(6-2x)$

$$P(y|x) = \begin{cases} \dfrac{\frac{6-x-y}{8}}{\frac{6-2x}{8}}, & \cancel{2<y<4} = \dfrac{6-x-y}{6-2x}, \quad 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

$$P(y|x=1) = \begin{cases} \frac{5-y}{4}, & 2<y<4 \\ 0, & \text{otherwise} \end{cases}$$

$$P(1<y<3 \mid x=1) = \int_1^3 P(y|x=1)\,dy = \int_1^2 0\,dy + \int_2^3 \frac{5-y}{4}\,dy + \int_3^4 0\,dy = \frac{1}{4}\left[5y - \frac{y^2}{2}\right]_2^3$$
$$= \frac{1}{4}\left[\left(15-\frac{9}{2}\right) - \left(10 - \frac{4}{2}\right)\right] = \frac{1}{4}\times\frac{5}{2} = \frac{5}{8}$$