RV College of Engineering®

HANDBOOK OF MATHEMATICS

FOR

SECOND YEAR B.E. PROGRAM

TRIGONOMETRY

Basic Functions

- $sin\,\theta = \frac{Opposite\ Side}{Hypotenuse}$
- $cos\,\theta = \frac{Adjacent\ Side}{Hypotenuse}$
- $tan\,\theta = \frac{Opposite\ Side}{Adjacent\ Side} = \frac{\sin\theta}{\cos\theta}$
- $sec\,\theta = \frac{Hypotenuse}{Adjacent\ Side} = \frac{1}{\cos\theta}$
- $cosec\,\theta = \frac{Hypotenuse}{Opposite\ Side} = \frac{1}{\sin\theta}$
- $cot\,\theta = \frac{Adjacent\ Side}{Opposite\ Side} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

Identities

- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\tan(-x) = -\tan x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\tan\left(\frac{\pi}{2} - x\right) = \cot x$
- $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
- $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$
- $\sin(\pi - x) = \sin x$
- $\cos(\pi - x) = -\cos x$
- $\tan(\pi - x) = -\tan x$
- $\sin(\pi + x) = -\sin x$
- $\cos(\pi + x) = -\cos x$
- $\tan(\pi + x) = \tan x$
- $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
- $\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$
- $\tan\left(\frac{3\pi}{2} - x\right) = \cot x$
- $\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$
- $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$
- $\tan\left(\frac{3\pi}{2} + x\right) = -\cot x$
- $\sin(x + y) = \sin x \cos y + \cos x \sin y$
- $\cos(x + y) = \cos x \cos y - \sin x \sin y$
- $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
- $\sin(2x) = 2 \sin x \cos x$
- $\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$
  $= 2\cos^2 x - 1$
- $\tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$
- $\cos^2 x + \sin^2 x = 1$
- $\sec^2 x - \tan^2 x = 1$
- $\mathrm{cosec}^2\, x - \cot^2 x = 1$
- $\sin 3x = 3\sin x - 4\sin^3 x$
- $\cos 3x = 4\cos^3 x - 3\cos x$
- $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
- $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$
- $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

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BASIC CALCULUS

Differentiation

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
a	0	a^x	$a^x \log_e a$
$x^n, n \neq -1$	nx^{n-1}	e^{ax}	ae^{ax}
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	$\log_e x$	$\frac{1}{x}$
$\sin x$	$\cos x$	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos x$	$-\sin x$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan x$	$\sec^2 x$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\operatorname{cosec} x$	$-\cot x \operatorname{cosec} x$	$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\sec x$	$\tan x \sec x$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x$	$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\sinh x$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh x$	$\operatorname{sech}^2 x$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\operatorname{cosech} x$	$-\coth x \operatorname{cosech} x$	$\operatorname{cosech}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2+1}}$
$\operatorname{sech} x$	$-\tanh x \operatorname{sech} x$	$\operatorname{sech}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{1-x^2}}$
$\coth x$	$-\operatorname{cosech}^2 x$	$\coth^{-1} x$	$\frac{1}{1-x^2}$

Rules of differentiation

- $\frac{d}{dx}(fg) = gf' + fg'$
- $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$
- $\frac{d}{dx}\big(f(t)\big) = \frac{d}{dt}\big(f(t)\big)\frac{dt}{dx}$

Integration

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}$	$\frac{1}{x}$	$\log_e x$
e^{ax}	$\frac{e^{ax}}{a}$	$\log_e x$	$x(\log_e x - 1)$
a^x	$\frac{a^x}{\log_e a}$	$\operatorname{cosec} x$	$\log_e(\operatorname{cosec} x - \cot x)$
$\sin x$	$-\cos x$	$\sec x$	$\log_e(\sec x + \tan x)$
$\cos x$	$\sin x$	$\cot x$	$\log_e \sin x$
$\tan x$	$\log_e \sec x$	$\sec^2 x$	$\tan x$
$\sinh x$	$\cosh x$	$\operatorname{cosec}^2 x$	$-\cot x$
$\cosh x$	$\sinh x$	$\tanh x$	$\log_e \cosh x$
$\frac{1}{\sqrt{a^2-x^2}}$	$sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2+x^2}}$	$sinh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{x^2-a^2}}$	$cosh^{-1}\left(\frac{x}{a}\right)$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log_e\left(\frac{a+x}{a-x}\right)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)$
$\sqrt{a^2-x^2}$	$\frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2 sin^{-1}\left(\frac{x}{a}\right)\right]$	$e^{ax}\sin bx$	$\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$
$u(x)v(x)$	$u\int v\,dx - \int\left[\frac{du}{dx}\left[\int v\,dx\right]dx\right]$	$e^{ax}\cos bx$	$\frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

COMPLEX ANALYSIS

Algebra of complex numbers:

- If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$ is non-negative real number.
- $z = x + iy$ is represented by a point $P(x, y)$ in the XY plane, $x - axis$ is real axis, $y - axis$ is imaginary axis, plane is complex plane.
- $z = re^{i\theta}$ is the polar form of complex number z where $r = \sqrt{x^2 + y^2}$ and $\theta = tan^{-1}\left(\frac{y}{x}\right)$.
- $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ represents the distance between the points z_1 and z_2 in complex plane.
- $|z - z_0| = R$ represents complex equation of circle with centre z_0 and radius R.
- $|z - z_0| < R$ represents the region with in, but not on, a circle of radius R centred at the point z_0, the point z_0 is said to be interior point.
- $|z - z_0| \leq R$ represents the region with in, and on, a circle of radius R centred at the point z_0.
- $|z - z_0| > R$ represents the region outside the circle with centre z_0 and radius R.

Cauchy-Riemann (C-R) equations:

- In Cartesian form: If $f(z) = u(x, y) + i\, v(x, y)$ is differentiable at $z = x + iy$, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
- In polar form: If $f(z) = u(r, \theta) + i\, v(r, \theta)$, then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Harmonic Functions: A function ϕ is said to be a harmonic function if it satisfies Laplace equation $\nabla^2\phi = 0$.

- In Cartesian form $\phi(x, y)$ is harmonic if $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.
- In polar form $\phi(r, \theta)$ is harmonic if $\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$.

Taylor theorem: Taylor series expansion for the function $f(z)$ about the point $z = a$ is

$$f(z) = f(a) + (z - a)f'(a) + (z - a)^2\frac{f''(a)}{2!} + \cdots$$

Maclaurin theorem: Maclaurin series for the function $f(z)$ about the point $z = 0$ is

$$f(z) = f(0) + zf'(0) + z^2\frac{f''(0)}{2!} + \cdots$$

Binomial Expansion: If $|x| < 1$, then

- $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
- $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$
- $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

Laurent's theorem: Let $c_1$and c_2 are concentric circles cantered at "a", then Laurent's series of $f(z)$ about the point $z = a$,

$$f(z) = \sum_{n=0}^{\infty} a_n\,(z - a)^n + \sum_{n=1}^{\infty}\frac{b_n}{(z - a)^n},$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{n+1}}\, dw \quad b_n = \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-a)^{-n+1}}\, dw$ for $n = 0, 1, 2 \ldots$

Determination of poles: If $f(z) = \frac{\phi(z)}{(z-a)^m}$ where $\phi(z)$ is analytic and not zero at the point 'a', then 'a' is a pole of order m of $f(z)$. The poles of $f(z)$ may be obtained by solving the equation $\frac{1}{f(z)} = 0$.

Residue: The coefficient of $\frac{1}{z-a}$ in the Laurent's expansion of $f(z)$ is the Residue of $f(z)$ at the pole $z = a$.

Determination of a residue: If 'a' is a pole of order m ≥ 1 of $f(z)$, then residue of $f(z)$ at 'a' is $\frac{1}{(m-1)!}\left[\lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\right]$.

Cauchy's residue theorem: Let C be a simple closed curve and $f(z)$ be analytic within and on C except at a finite number of poles $a_1, a_2, \dots, a_n$ which lie inside C , then

$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \cdots + R_n)$$

where $R_1, R_2 \dots R_n$ are the residues of $f(z)$ at $a_1, a_2, \dots, a_n$ respectively.

PARTIAL DIFFERENTIAL EQUATIONS

Lagrange's linear equation: The first order linear partial differential equation of the form $P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = R$, where P, Q and R are functions of x, y, z is known as Lagrange's Linear equation.

Subsidiary/Auxiliary equation: The equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is known as the subsidiary/ auxiliary equation of as Lagrange's Linear equation $P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = R$.

One-dimensional wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{T}{\rho}$ the phase speed, T is the tension, and ρ density of the string.

One-dimensional heat equation: $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$, where $c^2 = \frac{\kappa}{s\rho}$ the thermal diffusivity, κ thermal conductivity, s specific heat and ρ density of the material of the body.

Two-dimensional Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

Laplace equation: $u_{xx} + u_{yy} = 0$

- Standard 5- point formula: $u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$
- Diagonal 5- point formula: $u_{i,j} = \frac{1}{4}\left[u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j-1}\right]$.

One-dimensional heat equation: $u_t = c^2 u_{xx}$

- Schmidt formula: $u_{i,j+1} = \alpha\left(u_{i-1,j} + u_{i+1,j}\right) + (1-2\alpha)u_{i,j}$ where $\alpha = \frac{kc^2}{h^2}$.
- Bendre-Schmidt relation: $u_{i,j+1} = \frac{1}{2}\left[u_{i-1,j} + u_{i+1,j}\right]$ when $\alpha = \frac{1}{2}$.

One-dimensional wave equation: $u_{tt} = c^2 u_{xx}$

- Explicit formula: $u_{i,j+1} = \beta^2\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1-\beta^2)\, u_{i,j} - u_{i,j-1}$, where $\beta^2 = \frac{c^2k^2}{h^2}$.
- For $\beta = 1$ and $k = \frac{h}{c}$, $u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$.

The above scheme is used with standard initial and boundary conditions.

CALCULUS OF VARIATION

Euler's equation: A necessary condition for the functional $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$ to be an extremum is that $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$.

Alternate forms:

- $\frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}$
- $\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y'\,\frac{\partial f}{\partial y'}\right) = 0$
- $\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'}y' - \frac{\partial^2 f}{\partial (y')^2}y'' = 0$

Cases of Euler's Equation:

- If f is only the function of y', then Euler's equation will be: $\frac{df}{dy'} = c$ where c is an arbitrary constant.
- If f is independent of y, then Euler's equation will be: $\frac{\partial f}{\partial y'} = c$ where c is an arbitrary constant.
- If f is independent of y' then Euler's equation will be: $\frac{\partial f}{\partial y} = 0$.
- If f is independent of x and y then Euler's equation will be: $y''\frac{\partial^2 f}{\partial y'^2} = 0$.
- If f does not contain x then Euler's equation will be: $f - y'\frac{\partial f}{\partial y'} = c$ where c is an arbitrary constant.

Cartesian coordinate system: $u_1 = x, u_2 = y, u_3 = z$

Element of arc length $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

Cylindrical coordinate system: $u_1 = r, u_2 = \theta\,, u_3 = z$

Element of arc length $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + (dz)^2}$

Spherical coordinate system: $u_1 = r\,, u_2 = \theta$ and $u_3 = \phi$.

Element of arc length $ds = \sqrt{(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2}$.

LAPLACE TRANSFORM

Gamma function:

- $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1} dx, (n > 0)$
- $\Gamma(1) = 1$
- $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$
- $\Gamma(n+1) = \begin{cases} n\Gamma(n), & n > 0 \\ n!, & n \text{ postive integer} \end{cases}$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\Gamma(n) = \frac{\Gamma(n+1)}{n}, (n < 0, \neq -1, -2, \dots)$

Beta Function:

- $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$
- $\beta(m,n) = \beta(n,m)$
- $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$
- $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
- $\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}} dt$

Laplace transform of $f(t)$: $L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$

Transform of elementary functions:

- $L(e^{at}) = \frac{1}{s-a}, \quad s > a$
- $L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}, s > |a|$
- $L(\sin at) = \frac{a}{s^2+a^2}, \quad s > 0$
- $L(\cosh at) = \frac{s}{s^2-a^2}, \quad s > |a|$
- $L(\cos at) = \frac{s}{s^2+a^2}, \quad s > 0$
- $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$
- $L[H(t-a)] = \frac{e^{-as}}{s}$, where H is Heaviside unit step function

Properties of Laplace transform:

- $L[a f(t) + b g(t)] = a L[f(t)] + b L[g(t)]$.
- If $L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$, where a is a positive constant.
- Let a be any real constant then $L[e^{at} f(t)] = F(s-a)$
- If $L[f(t)] = F(s)$, then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s), n = 1, 2, 3,\dots$
- If $L[f(t)] = F(s)$, then $L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s) ds$.
- If $L[f(t)] = F(s)$, then $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$
- If $L[f(t)] = F(s)$, then $L\int_0^t f(t) dt = \frac{1}{s} F(s)$
- If $f(t)$ is a periodic function of period T, then $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$.
- If $L\{f(t)\} = F(s)$, then $L[f(t-a) H(t-a)] = e^{-as} F(s)$
- If $f(t)$ is a continuous function at $t = a$, then $\int_0^\infty f(t)\delta(t-a) dt = f(a)$, where $\delta(t-a)$ is unit impulse function.

Inverse Laplace transform of $F(s)$ using Convolution theorem: If $L^{-1}[F(s)] = f(t)$ and $L^{-1}[G(s)] = g(t)$, then $L^{-1}[F(s)G(s)] = \int_0^t f(u) g(t-u) du = f(t) * g(t)$.

FOURIER SERIES

Fourier series of $f(x)$ in the interval $(a, a+2l)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right),$$

where Fourier coefficients a_0, a_n, b_n are given by

$a_0 = \frac{1}{l}\int_a^{a+2l} f(x)dx,$ $\quad a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,$ $\quad n = 1, 2, 3, \ldots$ and
$b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx,$ $\quad n = 1, 2, 3, \ldots$

Complex Fourier Series of $f(x)$ in the interval $(a, a+2l)$:

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$, where $c_n = \frac{1}{2l}\int_a^{a+2l} f(x)\, e^{\frac{-in\pi x}{l}}\, dx$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

Relation between Fourier and complex Fourier coefficients:

$a_0 = 2c_0, a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n})$

Half-range Fourier series:

- Sine series of $f(x)$ in $(0, l)$: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where
  $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$
- Cosine series of $f(x)$ in $(0, l)$: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$, where
  $a_0 = \frac{2}{l}\int_0^l f(x)dx$ and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

Harmonic Analysis: Let the periodic function $y = f(x)$ takes values $y_0, y_1, y_2, \ldots, y_m$ corresponding to a given set of equi-spaced values $x_0, x_1, x_2, \ldots, x_m$ in $(a, a + 2l)$, Fourier series of $f(x)$ is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$, where a_0, a_n and b_n are computed using the formulae:

$$a_0 = 2\frac{\sum y}{m}, \quad a_n = 2\frac{\sum y\cos\left(\frac{n\pi x}{l}\right)}{m}, \quad b_n = 2\frac{\sum y\sin\left(\frac{n\pi x}{l}\right)}{m}.$$

- Let the periodic function $y = f(x)$ takes values $y_0, y_1, y_2, \ldots, y_m$ corresponding to a given set of equi spaced values $x_0, x_1, x_2, \ldots, x_m$ in $(0, l)$, Half range cosine Fourier series and Half range sine Fourier series of $f(x)$ are respectively:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}a_0 + a_1 \cos\left(\frac{\pi x}{l}\right) + a_2 \cos\left(\frac{2\pi x}{l}\right) + \cdots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= b_1 \sin\left(\frac{\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right) + \cdots,$$

where a_0, a_n and b_n are computed from the table by using the formulae:

$$a_0 = 2\frac{\Sigma y}{m}, \quad a_n = 2\frac{\Sigma y\cos\left(\frac{n\pi x}{l}\right)}{m}, \quad b_n = 2\frac{\Sigma y\sin\left(\frac{n\pi x}{l}\right)}{m}$$

FOURIER TRANSFORMS

Complex Fourier transform or Fourier transform of $f(x)$:

- Fourier transform of $f(x)$: $\hat{f}(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$, provided the integral exists.
- Inverse Fourier transform of $\hat{f}(\alpha)$: $F^{-1}\left[\hat{f}(\alpha)\right] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-i\alpha x}d\alpha$.

Fourier sine and cosine transforms:

- Fourier sine transform of $f(x)$: $\hat{f}_s(\alpha) = F_s[f(x)] = \int_0^{\infty} f(x) \sin \alpha x \; dx$
- Inverse Fourier sine transform $\hat{f}_s(\alpha)$: $F_s^{-1}\left[\hat{f}_s(\alpha)\right] = \frac{2}{\pi}\int_{-\infty}^{\infty} \hat{f}_s(\alpha) \sin \alpha x \; d\alpha$
- Fourier cosine transform of $f(x)$: $\hat{f}_c(\alpha) = F_c[f(x)] = \int_0^{\infty} f(x) \cos \alpha x \, dx$
- Inverse Fourier cosine transform of $\hat{f}_s(\alpha)$**:** $F_c^{-1}\left[\hat{f}_s(\alpha)\right] = \frac{2}{\pi}\int_{-\infty}^{\infty} \hat{f}_c(\alpha) \cos \alpha x \, d\alpha$

Relation between Fourier sine and cosine transform:

- $F_s[xf(x)] = -\frac{d}{d\alpha}F_c[f(x)]$ and $F_c[xf(x)] = \frac{d}{d\alpha}F_s[f(x)]$

Properties of Fourier transforms:

- For any two functions $f(x)$ and $\phi(x)$ (whose Fourier transforms exist) and any two constants a and b, $F[af(x) + b\phi(x)] = aF[f(x)] + bF[\phi(x)]$
- If $F[f(x)] = \hat{f}(\alpha)$, then for any non-zero constant a, $F[f(ax)] = \frac{1}{|a|}\hat{f}\left(\frac{\alpha}{a}\right)$
- If $F[f(x)] = \hat{f}(\alpha)$, then for any non-zero constant a,
  (i) $F[f(x-a)] = e^{i\alpha a}\hat{f}(\alpha)$
  (ii) $F\left[e^{iax}f(x)\right] = \hat{f}(\alpha + a)$
- $F[f'(x)] = -isF[f(x)]$ and $F[f''(x)] = -s^2F[f(x)]$
- If $F[f(x)] = \hat{f}(\alpha)$, then
  (i) $F[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}(\alpha + a) + \hat{f}(\alpha - a)\right]$ and
  (ii) $F[f(x)\sin ax] = \frac{1}{2}\left[\hat{f}(\alpha + a) - \hat{f}(\alpha - a)\right]$, where $'a'$ is a real constant.
- If $\hat{f}_s(\alpha)$ and $\hat{f}_c(\alpha)$ are Fourier sine and cosine transforms of $f(x)$ respectively, then
  (i) $F_s[f(x)\cos ax] = \frac{1}{2}\left[\hat{f}_s(\alpha + a) + \hat{f}_s(\alpha - a)\right]$,
  (ii) $F_c[f(x)\sin ax] = \frac{1}{2}\left[\hat{f}_s(\alpha + a) - \hat{f}_s(\alpha - a)\right]$,
  (iii) $F_s[f(x)\sin ax] = \frac{1}{2}\left[\hat{f}_c(\alpha - a) + \hat{f}_c(\alpha + a)\right]$.

Convolution theorem: *If* $F[f(x)] = \hat{f}(\alpha)$ and $F[g(x)] = \hat{g}(\alpha)$, then

$$F^{-1}\left[\hat{f}(\alpha)\hat{g}(\alpha)\right] = f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

Parseval's identity:

- If the Fourier transform of $f(x)$ and $g(x)$ are $F(\alpha)$ and $G(\alpha)$, respectively, then
  (i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\bar{G}(\alpha)d\alpha \;=\; \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$
  (ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\bar{G}(\alpha)d\alpha \;=\; \int_{-\infty}^{\infty} |f(x)|^2dx$

- If Fourier cosine and sine transform of $f(x)$ and $g(x)$ are $F_c(\alpha)$, $G_c(\alpha)$ and $F_s(\alpha)$, $G_s(\alpha)$ respectively, then
  (i) $\frac{2}{\pi}\int_0^\infty F_c(\alpha)G_c(\alpha)d\alpha = \int_0^\infty f(x)g(x)dx$
  (ii) $\frac{2}{\pi}\int_0^\infty F_s(\alpha)G_s(\alpha)d\alpha = \int_0^\infty f(x)g(x)dx$
  (iii) $\frac{2}{\pi}\int_0^\infty [F_c(\alpha)]^2 d\alpha = \int_0^\infty [f(x)]^2 dx$
  (iv) $\frac{2}{\pi}\int_0^\infty [F_s(\alpha)]^2 d\alpha = \int_0^\infty [f(x)]^2 dx$

LINEAR ALGEBRA

Basic transformation matrices:

- Stretch matrix/dilation matrix: $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, where a is real constant.
- Rotation matrix: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- Projection matrix: $\begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}$
- Reflection matrix: $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

Gram-Schmidt process: If $\{x_1, x_2, \ldots, x_p\}$ is a basis for a subspace W of vector space R^n, then corresponding orthogonal basis $\{v_1, v_2, \ldots, v_p\}$ for W, where

$$v_1 = x_1 \text{ and } v_i = x_i - \left(\frac{x_i \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_i \cdot v_2}{v_2 \cdot v_2}\right) v_2 - \cdots - \left(\frac{x_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}}\right) v_{i-1} \text{ for } i = 2,3,4,\ldots,p.$$

STATISTICS

Moments for ungrouped data:

- The r^{th} moment about origin: $\mu_r' = \frac{1}{n}\sum_{i=1}^n x_i^r$, where $r = 1, 2, 3, \ldots$, $x_1, x_2, \cdots x_n$ are n observations
- The r^{th} central moment: $\mu_r = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^r$, where $r = 1, 2, 3, \cdots$ and $\bar{x}$ is mean

Moments for grouped data:

- The r^{th} moment about origin: $\mu_r' = \frac{1}{N}\sum_{i=1}^n f_i x_i^r$, $r = 1, 2, 3, \cdots$, where observations $x_1, x_2, \ldots, x_n$ are the mid points of the class-intervals and $f_1, f_2, \ldots, f_n$ are their corresponding frequencies and $N = \sum_{i=1}^n f_i$
- The r^{th} central moment: $\mu_r = \frac{1}{N}\sum_{i=1}^n f_i(x_i - \bar{x})^r$, $r = 1, 2, 3 \cdots$ & $\bar{x}$ is mean
- The r^{th} moment about any point A: $\mu_r' = \frac{1}{N}\sum_{i=1}^n f_i(x_i - A)^r$, $r = 1, 2, 3 \cdots$

Relation between raw moments (about origin or any point) and central moments:

- $\mu_r = \mu_r' - {}^rC_1\, \mu_{r-1}'\, \mu_1' + {}^rC_2\, \mu_{r-2}'\, {\mu'}_1^2 + \ldots + (-1)^r {\mu_1'}^r$, r = 1, 2, 3 …
- $\mu_r' = \mu_r + {}^rC_1 \mu_{r-1}\, \mu_1' + {}^rC_2\, \mu_{r-2} {\mu_1'}^2 + \ldots + {\mu_1'}^r$

Measures of kurtosis: $\beta_2 = \frac{\mu_4}{\mu_2^2}$

Measures of Skewness (Karl Pearson's coefficient): $S_k = \frac{\sqrt{\beta_1}\,(\beta_2+3)}{2(5\beta_2-6\beta_1-9)}$, where $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$

Fitting of a straight line $y = a + bx$: The normal equations for estimating the values of a and b are

$$\sum y = n\mathrm{a} + b\sum x,\ \sum xy = a\sum x + b\sum x^2$$

Fitting of a second-degree equation (quadratic) $y = a + bx + cx^2$: The normal equations for estimating the values of a , b , c are

$$\sum y = n\mathrm{a} + b\sum x + c\sum x^2,$$
$$\sum xy = \mathrm{a}\sum x + b\sum x^2 + c\sum x^3,$$
$$\sum x^2 y = \mathrm{a}\sum x^2 + b\sum x^3 + c\sum x^4.$$

Correlation coefficient (Karl Pearson correlation coefficient):

- $r = \frac{\sum(x-\bar{x})(y-\bar{y})}{n\sigma_x\sigma_y}$, where $\sigma_x^2 = \frac{\sum(x-\bar{x})^2}{n}$ variance of the x series, $\sigma_y^2 = \frac{\sum(y-\bar{y})^2}{n}$ variance of the y series,
- $\overline{x} = \frac{\sum x}{n} \rightarrow$ Mean of the x series $\overline{y} = \frac{\sum(x-\overline{x})^2}{n} \rightarrow$ mean of the y series.
- $r = \frac{n\sum xy - (\sum x)(\sum y)}{\sqrt{\{n\sum x^2 - (\sum x)^2\}\{n\sum y^2 - (\sum y)^2\}}}$.

Rank correlation coefficient r_s (Spearman's rank correlation coefficient):

- If $x_1, x_2, \dots, x_n$ and $\mathrm{y}_1, \mathrm{y}_2, \dots, \mathrm{y}_\mathrm{n}$ be the ranks of n individuals in characteristics A and B respectively, then $r_s = 1 - \frac{6\sum_{i=1}^{n} {d_i}^2}{n(n^2-1)}$, where d_i is difference between ranks assigned in characteristics A and B. and n is number of pairs of data.
- Rank correlation coefficient for tied ranks: $r_s = 1 - \frac{6\left[\sum_{i=1}^{n} {d_i}^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \cdots\right]}{n(n^2-1)}$, where $m_1, m_2, \dots$ are number of repetitions of ranks.

Linear regression:

- Regression line of y on x: $y - \bar{y} = b_{yx}(x - \overline{x})$, where
$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(x-\overline{x})^2} = \frac{n\sum \mathrm{xy} - \sum x\sum y}{n\sum x^2 - (\sum x)^2}$$
- Regression line of x on y: $x - \overline{x} = b_{xy}(y - \overline{y})$, where
$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{\sum(x-\overline{x})(y-\overline{y})}{\sum(y-\overline{y})^2} = \frac{n\sum xy - \sum x\sum y}{n\sum y^2 - (\sum y)^2}$$
- Angle between two lines of regression: $\tan\theta = \frac{1-r^2}{r}\ \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$.

Multiple linear regression: Fitting of a multiple linear regression model: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k$ for n sets of data (y_i, x_{ij}), where $i = 1, \dots, n$ and $j = 1, \dots, k$. The normal equations for estimating the values of $\beta_0, \beta_1, \beta_2, \dots, \beta_k$ are

$$n\beta_0 + \beta_1\sum_{i=1}^{n} x_{i1} + \beta_2\sum_{i=1}^{n} x_{i2} + \cdots \beta_k\sum_{i=1}^{n} x_{ik} = \sum_{i=1}^{n} y_i,$$

$$\beta_0\sum_{i=1}^{n} x_{i1} + \beta_1\sum_{i=1}^{n} x_{i1}^2 + \beta_2\sum_{i=1}^{n} x_{i1}x_{i2} + \cdots \beta_k\sum_{i=1}^{n} x_{i1}x_{ik} = \sum_{i=1}^{n} x_{i1}y_i,$$

$$\vdots$$

$$\beta_0\sum_{i=1}^{n} x_{ik} + \beta_1\sum_{i=1}^{n} x_{ik}x_{i1} + \beta_2\sum_{i=1}^{n} x_{ik}x_{i2} + \cdots \beta_k\sum_{i=1}^{n} x_{ik}^2 = \sum_{i=1}^{n} x_{ik}y_i.$$

Multivariate data (Sample):

Dimensions	Sample	Sample Mean	Sample variance (unbiased) / Sample Covariance
1-dimensional	$x_1 \ldots . x_n$	$\bar{x} = \sum_{i=1}^{n} x_i$	$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
p-dimensional	p-random variable vector $X = \begin{bmatrix} X_1 \\ . \\ . \\ . \\ Xp \end{bmatrix}$ where $x_1, x_2, \ldots, x_n \in R^p$ n-dimensional data matrix $X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ & . & & \\ & . & & \\ & . & & \\ x_{n1} & x_{n2} & \ldots & x_{np} \end{bmatrix}_{n \times p}$	Sample Mean: $\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$, j = 1, 2, ...,p. and Sample Mean Vector: $\bar{x} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_{i1} \\ . \\ . \\ . \\ \frac{1}{n}\sum_{i=1}^{n} x_{ip} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ . \\ . \\ . \\ \bar{x}_p \end{bmatrix}$	Sample Variance: $s_{jj} = s_j^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ij} - \bar{x}_j\right)^2$, j = 1, 2, ...,p and i = j. and Sample Covariance: $s_{jk} = s_{kj} = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_{ij} - \bar{x}_j\right)\left(x_{ik} - \bar{x}_k\right)$, $1 \le k, j \le p$ and $j \neq k$. Sample Covariance Matrix: $S = \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1p} \\ s_{21} & s_{22} & \ldots & s_{2p} \\ & . & & \\ & . & & \\ & . & & \\ s_{p1} & s_{p2} & \ldots & s_{pp} \end{bmatrix}_{p \times p}$ $= \frac{1}{n-1}\sum_{i=1}^{n}\begin{bmatrix} (x_{i1} - \bar{x}_1)^2 \ldots (x_{i1} - \bar{x}_1)(x_{ip} - \bar{x}_p) \\ . \\ . \\ . \\ (x_{ip} - \bar{x}_p)(x_{i1} - \bar{x}_1) \ldots (x_{ip} - \bar{x}_p)^2 \end{bmatrix}_{p \times p}$

Multivariate data (Population):

Dimensions	Population	Population Mean	Population variance / Population Covariance
1-dimensional	$x_1 \ldots . x_n$	$\mu = \sum_{i=1}^{n} x_i$	$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$

p-dimensional	p-random variable vector $$X = \begin{bmatrix} X_1 \\ . \\ . \\ . \\ X_p \end{bmatrix}$$ where $x_1 \ldots . x_n \in R^p$ n-dimensional data matrix $$X = \begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1p} \\ x_{21} & x_{22} & \ldots & x_{2p} \\ & . & & \\ & . & & \\ & . & & \\ x_{n1} & x_{n2} & \ldots & x_{np} \end{bmatrix}_{n \times p}$$ Here each column in the data matrix corresponds to a random variable X_j.	Population Mean: $$\mu_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij},$$ j = 1, 2, ...,p. and Population Mean Vector: $$\mu = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_{i1} \\ . \\ . \\ . \\ \frac{1}{n}\sum_{i=1}^{n} x_{ip} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ . \\ . \\ . \\ \mu_p \end{bmatrix}$$	Population Variance: $$\sigma_{jj} = \sigma_j^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_{ij} - \mu_j\right)^2,$$ j = 1, 2, ...,p and i = j. and Population Covariance: $$\sigma_{jk} = \sigma_{kj} = \frac{1}{n}\sum_{i=1}^{n}\left(x_{ij} - \mu_j\right)\left(x_{ik} - \mu_k\right),$$ $1 \leq k, j \leq p$ and $j \neq k$. Population Covariance Matrix: $$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2p} \\ & . & & \\ & . & & \\ & . & & \\ \sigma_{p1} & \sigma_{p2} & \ldots & \sigma_{pp} \end{bmatrix}_{p \times p}$$ $$= \frac{1}{n}\sum_{i=1}^{n} \begin{bmatrix} \left(x_{i1} - \mu_1\right)^2 \ldots \left(x_{i1} - \mu_1\right)\left(x_{ip} - \mu_p\right) \\ . \\ . \\ . \\ \left(x_{ip} - \mu_p\right)\left(x_{i1} - \mu_1\right) \ldots \left(x_{ip} - \mu_p\right)^2 \end{bmatrix}_{p \times p}$$

Rank Correlation, Partial Correlation, Multiple Correlation	
In the case of tri variate data X_1, X_2 and X_3	i) Coefficient of partial correlation of X_1 and X_2, keeping X_3 constant: $$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}}$$ ii) Coefficient of partial correlation of X_1 and X_3, keeping X_2 constant: $$r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{\sqrt{1 - r_{12}^2}\sqrt{1 - r_{23}^2}}$$ iii) Coefficient of partial correlation of X_2 and X_3, keeping X_1 constant: $$r_{23.1} = \frac{r_{23} - r_{12}r_{13}}{\sqrt{1 - r_{12}^2}\sqrt{1 - r_{13}^2}}$$ **Note**: In all the above three cases $r_{ij} = r_{ji}$ for all $i \neq j$.

<table>
<tr>
<td>In the case of tri variate data: X_1, X_2 and X_3</td>
<td>
i. Multiple correlation coefficient of X_1 on X_2 and X_3: $R_{1.23} = \sqrt{\frac{r_{12}^2+r_{13}^2-2r_{12}r_{13}r_{23}}{1-r_{23}^2}}$
ii. Multiple correlation coefficient of X_2 on X_1 and X_3: $R_{2.13} = \sqrt{\frac{r_{21}^2+r_{23}^2-2r_{21}r_{23}r_{13}}{1-r_{13}^2}}$
iii. Multiple correlation coefficient of X_2 on X_1 and X_3: $R_{3.12} = \sqrt{\frac{r_{31}^2+r_{32}^2-2r_{31}r_{32}r_{12}}{1-r_{21}^2}}$
Given variables x, y and z, we define the multiple correlation coefficient.
$R_{z.xy} = \sqrt{\frac{r_{zx}^2+r_{zy}^2-2r_{zx}r_{zy}r_{yx}}{1-r_{yx}^2}}$ and $r_{xy} = \frac{n\sum xy-\sum x\sum y}{\sqrt{\{n(\sum x^2)-(\sum x)^2\}\{n(\sum y^2)-(\sum y)^2\}}}$
Note: In all the above three cases $r_{ij} = r_{ji}$ for all $i \neq j$.
</td>
</tr>
<tr>
<td colspan="2" align="center">Analysis of Variance (ANOVA)</td>
</tr>
</table>

One-way ANOVA Table

Sources of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F_{cal}
Between classes	$k-1$	SSB	MSSB	$F_{cal} = \frac{MSSB}{MSSW}$
Within classes	$N-k$	SSW	MSSW	
Total	$N-1$	SST		

k: number of classes, $N = pq$ is number of rows × number pf columns, n: number of entries in each X_i

T: $Grand\ sum$: ΣX_i , Correction factor $(C.F.) = \frac{T^2}{N}$

SSB: Sum of squares between classes = $\frac{(\Sigma X_i)^2}{n_i} - C.F.$

SST: Total sum of squares = $\Sigma X_i^2 - C.F.$

SSW: Sum of squares within classes = $SST - SSB$

MSSB: Mean sum of squares between classes = $\frac{SSB}{k-1}$

MSSW: Mean sum of squares within classes = $\frac{SSW}{N-k}$

Conclusion:

If $F_{cal} < F_{tab}$, there is NO significant difference, Null Hypothesis H_0 is accepted.

If $F_{cal} > F_{tab}$, there is a significant difference, Null Hypothesis H_0 is rejected and Alternative Hypothesis, H_1 is accepted.

Note: In one-way ANOVA, F_{cal} can also be computed using columns.

Two-way ANOVA table

Source of variation	Degrees of freedom	Sum of squares	Mean sum of squares	F_{cal}
Between rows	$r-1$	SSR	MSSR	$F_{cal\,(rows)} = \frac{MSSR}{MSSE}$
Between columns	$c-1$	SSC	MSSC	$F_{cal\,(columns)} = \frac{MSSC}{MSSE}$
Error (residual)	$(r-1)(c-1)$	SSE	MSSE	
Total	$rc-1$	SST		

Number of levels of row factor = r, Number of levels of column factor = c

Total number of observations = rc = (rows × columns)

Observations in ij^{th} cell of table = x_{ij} , Sum of c observations in i^{th} row = $T_{Ri} = \Sigma_j x_{ij}$

Sum of r observations in j^{th} column = $T_{Cj} = \Sigma_i x_{ij}$

Sum of all observations = $T = \Sigma_i T_{Ri} = \Sigma_j T_{Cj}$

Correction factor = $C.F. = \frac{T^2}{rc}$

Sum of squares between rows: SSR = $\sum \frac{T_{Ri}^2}{c} - C.F.$

Sum of squares between columns: SSC = $\sum \frac{T_{Cj}^2}{r} - C.F.$

Total sum of squares: SST = $\Sigma_i \Sigma_j x_{ij}^2 - C.F.$

Error (residual) sum of squares: SSE = SST-SSR-SSC

MSSR = mean sum of squares between rows = $\frac{SSR}{r-1}$

MSSC = mean sum of squares between columns = $\frac{SSC}{c-1}$

MSSE = mean sum of squares of error = $\frac{SSE}{(r-1)(c-1)}$

Conclusion:

If $F_{cal} < F$ tab, there is no significant difference, null hypothesis H_0 is accepted.

If $F_{cal} > F$ tab, there is a significant difference, null hypothesis H_0 is rejected and alternative hypothesis, H_1 is accepted.

PROBABILITY AND RANDOM VARIABLES

Probability of an event: If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E then the probability $'p'$ of happening of E is given by $p = P(E) = \frac{m}{n}$

Addition theorem:

For two events: If A and B are any two events with respective probabilities $P(A)$ and $P(B)$, then the probability of occurrence of at least one of the events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

For three events: If A, B and C are any three events with respective probabilities $P(A), P(B)$ and $P(C)$, then the probability of occurrence of at least one of the events is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

Conditional probability: Conditional probability of B, given A, denoted by $P(B|A)$, is defined by $P(A|B) = \frac{P(A \cap B)}{P(A)}$, provided $P(A) > 0$.

Multiplication rule: Suppose A and B are events in a sample space S with $P(A) > 0$ multiplication rule is: $P(A \cap B) = P(A)\, P(B|A)$

Bayes' theorem (rule): If $B_1, B_2, \ldots, B_n$ are mutually disjoint events with $P(B_i) \neq 0$ $(i = 1,2, \ldots, n)$ then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} B_i$ such that $P(A) > 0$, then $\mathrm{P(B_i|A)} = \frac{\mathrm{P(B_i)P(A|B_i)}}{\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{P(B_i)P(A|B_i)}}$.

Discrete random variable: Let X be a discrete random variable. A function $p(x)$ is a probability mass function of the discrete random variable X if $p(x) \geq 0, \forall x \in X$ and $\sum_x p(x) = 1$.

- Expectation, $E(X) = \sum_x x p(x)$
- If $Y = g(X)$, then $E(Y) = \sum_x g(x) p(x)$
- Variance, $Var(X) = E\left[\left(X - E(X)\right)^2\right] = E(X^2) - [E(X)]^2$
- Standard deviation, $\sigma_X = \sqrt{Var(X)}$
- The cumulative distribution function, $F(t) = P(X \leq t) = \sum_{x \leq t} p(x)$

Continuous random variable: Suppose X is a continuous random variable. A function $f(x)$ is called a probability density function of the continuous random variable X if $f(x) \geq 0$ $\forall x \in X$ and $\int_{-\infty}^{\infty} f(x) dx = 1$.

- Expectation, $E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$
- If $Y = g(X)$, then $E(Y) = \int_{-\infty}^{\infty} g(x)\, f(x) dx$
- Variance, $Var(X) = E\left[\left(X - E(X)\right)^2\right] = E(X^2) - [E(X)]^2$
- Standard deviation, $\sigma_X = \sqrt{Var(X)}$
- Cumulative distribution function, $F(t) = P(X \leq t) = \int_{-\infty}^{t} f(x) dx$
- $P(a \leq X \leq b) = P(a \leq X < b) = P(a < X \leq b) = P(a < X < b) = \int_a^b f(x) dx$

Markov's inequality: For any random variable X with finite $E[X]$ and any $k > 0$, the probability that X is at least k times its expected value is at-most $\frac{1}{k}$. That is, $P\big[X \geq kE[X]\big] \leq \frac{1}{k}$ or $P[X \geq k] \leq \frac{E[X]}{k}$.

Chebyshev's inequality: Any random variable X with expectation $\mu = E[X]$ and variance $\sigma^2 = Var[X]$ belongs to the interval $\mu \pm k = [\mu - k, \mu + k]$ with probability of at least $1 - \left(\frac{\sigma}{k}\right)^2$. That is, $P\{|X - \mu| \geq k\} \leq \left(\frac{\sigma}{k}\right)^2 = \frac{\sigma^2}{k^2}$

Joint probability mass function: Suppose X and Y are two discrete random variables. A function $p(x, y)$ is called a joint probability mass function of X and Y if $p(x, y) \geq 0, \forall x \in X, y \in Y$ and $\sum_x \sum_y p(x, y) = 1$.

- Let $Z = g(X, Y)$. Expectation, $E[Z] = \sum_x \sum_y g(x, y) p(x, y)$
- The marginal distributions of X alone and of Y alone are: $g(x) = \sum_y p(x, y)$ and $h(y) = \sum_x p(x, y)$
- Covariance, $Cov(X, Y) = E(XY) - E(X)E(Y)$
- Correlation of X and Y, $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
- For discrete random variables X and Y with joint pmf $p(x, y)$ and x, y such that $g(x) > 0, h(y) > 0$, then the conditional probability mass functions are
  (i) $P(X = x|Y = y) = \frac{p(x,y)}{h(y)}$, (ii) $P(Y = y|X = x) = \frac{p(x,y)}{g(x)}$.
- If X and Y are independent, then $E(XY) = E(X)E(Y)$

Joint probability density function: Suppose X and Y are two continuous random variables. A function $f(x, y)$ is called a joint probability density function of X and Y if $f(x, y) \geq 0, \forall x \in X, y \in Y$ and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dxdy = 1$.

- Let $Z = g(X, Y)$. Expectation, $E[Z] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dxdy$
- The marginal distributions of X alone and of Y alone are $g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$ and $h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$
- Covariance, $Cov(X, Y) = E(XY) - E(X)E(Y)$
- Correlation of X and Y, $\rho(X, Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
- If X and Y are independent, then $E(XY) = E(X)E(Y)$
- The conditional distribution of the random variable Y given that $X = x$ is $f(y|x) = \frac{f(x,y)}{g(x)}$, provided $g(x) > 0$
- The conditional distribution of the random variable X given that $Y = y$ is $f(x|y) = \frac{f(x,y)}{h(y)}$, provided $h(y) > 0$.

PROBABILITY DISTRIBUTIONS

Bernoulli distribution:

- The probability mass function is given by $f(x, p) = p^x (1 - p)^{1-x}$; $x \in (0,1)$
- Mean, $\mu = p$
- Variance, $\sigma^2 = pq$
- Standard deviation, $\sigma = \sqrt{pq}$

Binomial distribution:

- The probability function of the binomial distribution is given by $b(x;n,p) = n_{C_x}\, p^x q^{n-x}$, where $x = 1,2,3,\cdots$, p is the probability of success and $q = 1 - p$ is the probability of failure.
- Mean, $\mu = np$
- Variance, $\sigma^2 = npq$
- Standard deviation, $\sigma = \sqrt{npq}$

Poisson distribution:

- The probability function of the Poisson distribution is given by $p(x;\lambda) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, where λ is the parameter of the Poisson distribution.
- Mean, $\mu = \lambda$
- Variance, $\sigma^2 = \lambda$
- Standard deviation, $\sigma = \sqrt{\lambda}$

Geometric distribution:

- Probability mass function is given by $P(X = x) = (1-p)^{x-1}p$, where $0 < p < 1$
- Mean, $\mu = 1/p$
- Variance, $\sigma^2 = \frac{1-p}{p^2}$
- Standard deviation $\sigma = \frac{\sqrt{1-p}}{p}$

Exponential distribution: A continuous random variable X assuming non-negative values is said to have an exponential distribution with parameter $\lambda > 0$, if its probability density function is given by

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & otherwise \end{cases}$$

- Mean $\mu = \frac{1}{\lambda}$
- Variance $\sigma^2 = \left(\frac{1}{\lambda}\right)^2$
- Standard deviation $\sigma = \frac{1}{\lambda}$

Uniform distribution: The probability density function is $f(x) = \frac{1}{b-a}\;\; for\ a \leq x \leq b$

- Mean $\mu = \frac{a+b}{2}$
- Variance $\sigma^2 = \frac{(b-a)^2}{12}$
- Standard deviation $\sigma = \sqrt{\frac{(b-a)^2}{12}}$

Normal distribution: A random variable X is said to have a normal distribution with parameters μ (called "mean") and σ^2 (called "variance") if its density function is given by the probability law:

$$f(x;\ \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\, exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \text{ for } -\infty < x < \infty, -\infty < \mu < \infty \text{ and } 0 < \sigma < \infty\,.$$

- Standard normal distribution or $z -$ distribution is given by

$$f(z;\ 0,1) = \frac{1}{\sqrt{2\pi}}\, exp\left[\frac{-z^2}{2}\right], \qquad -\infty < z < \infty,$$

- Cumulative standard normal distribution is
$$\phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}u^2}\ du.$$
Note: The value of the integral can be calculated by the Normal distribution table.

SAMPLING THEORY

Numerical summation of data: If the n observations in a sample are denoted by $x_1, x_2, \dots, x_n$, then

- Sample mean: $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
- Sample variance: $s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$
- Standard deviation is the positive square root of the sample variance.
- If the population is finite with size N, then for the sampling distribution of $\bar{x}$:

  Mean: $\mu_{\bar{x}} = \mu$
  Variance: $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n} \times \frac{N-n}{N-1}$, if the sampling is without replacement
  $\sigma_{\bar{x}}^2 = \frac{\sigma^2}{n}$, if the sampling is with replacement or population is infinite.

Sampling distributions: Let X be a random variable of a population with mean μ and variance σ^2. If the random variables $X_1, X_2, \dots, X_n$ are a random sample on size n, then sample mean $\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$

Sampling distribution of means

- Mean: $\mu_{\bar{X}} = \mu$
- Variance: $\sigma_{\bar{X}}^2 = \frac{\sigma^2}{n}$
- If the population is normal or sample size is sufficiently large, then the distribution of $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ is approximately standard normal.

Sampling distribution of a difference in sample means

- Mean: $\mu_{(\bar{X}_1 - \bar{X}_2)} = \mu_{\bar{X}_1} - \mu_{\bar{X}_2} = \mu_1 - \mu_2$
- Variance: $\sigma^2_{(\bar{X}_1 - \bar{X}_2)} = \sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$
- If the populations are normal or sample sizes are sufficiently large, then the distribution of $Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$ is approximately standard normal.

Sampling distribution of proportions:

- Mean $\mu_{\hat{P}} = p$
- Standard deviation $\sigma_{\hat{P}} = \sqrt{\frac{pq}{n}} = \left[\frac{p(1-p)}{n}\right]^{\frac{1}{2}}$
- The distribution of $Z = \frac{\hat{P} - p}{\sqrt{\frac{p(1-p)}{n}}}$ is approximately standard normal if n is large or $np \ge 5$ and $np(1-p) \ge 5$.

Statistical decisions:

- Levels of significance for two – tailed test

Level of significance	Critical value	Acceptance region
0.05	$z_c = 1.96$	$(-1.96, 1.96)$
0.01	$z_c = 2.58$	$(-2.58, 2.58)$

- Levels of significance for one– tailed test

Level of significance	Critical value		Critical region	
	Right-tailed test	Left-tailed test	Right-tailed test	Left-tailed test
0.05	$z_c = 1.645$	$-z_c = -1.645$	$(1.645, \infty)$	$(-\infty, -1.645)$
0.01	$z_c = 2.33$	$-z_c = -2.33$	$(2.33, \infty)$	$(-\infty, -2.33)$

- For the standard normal variate Z, and the test value of Z is z_0, then the P-value is determined as follows:
  (i) For left-tail test, the P-value is $P(Z \le z_0)$
  (ii) For right-tail test, the P-value is $P(Z \ge z_0)$
  (iii) For two tail test, the P-value is $P(Z \le -|z_0|) + P(Z \ge |z_0|) = 2\big(1 - \phi(|z_0|)\big)$

Test of significance:

z- test:

- Test statistics:
  (i) $z = \frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$, if σ is known.
  (ii) $z = \frac{\bar{X}-\mu_0}{S/\sqrt{n}}$, if σ is unknown and n is large.

t-test:

- Test statistic:
  $T = \frac{\bar{X}-\mu_0}{S/\sqrt{n}}$, if σ is unknown.

Chi – square (χ^2) test:

- If s^2 is the variance of a random sample of size n taken from a normal population having variance σ^2, then the statistic

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2}$$

  is a chi-squared distribution with $\nu = n - 1$ degrees of freedom.

- If $f_1, f_2, f_3, \dots \dots f_n$ are the observed frequencies and $e_1, e_2, e_3 \dots \dots e_n$ are the expected or theoretical frequencies. The statistic $\chi^2 = \sum_{i=1}^{n} \frac{(f_i - e_i)^2}{e_i}$

F- test:

- Test statistic: $F_0 = \frac{S_1^2}{S_2^2}$