Counting Techniques

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Overview

Combinations

Let us consider a set of n objects. An r-combination from the set of n objects is any selection of r objects, where the order of the objects does not matter.
Consider a set of letters $\{a, b, c\}$. Then abc,acb,bac,bca,cab,cba represent different permutations but they all represent the same combination
The number of r combinations from a set of n objects is denoted by C(n,r) or n_{C_r}
$n_{C_r} = \frac{n!}{r!(n-r)!}$

Examples

In how many ways can 3 cards be selected from a pack of cards?
Solution
There are 52 cards in a pack of cards.
Therefore the number of ways to select three cards is
52_{C_3}=52!/(3!49!)=22100

Examples

In how many ways can 3 diamonds and 2 clubs be selected from a pack of cards?

Solution

There are 13 diamonds and 13 clubs in a pack of cards.

Therefore the number of ways to select are

$13_{C_3}.13_{C_2}$=22308

A bag contains 4 different History books and 6 different English books. In how many ways can 3 books be selected so that there is at least 1 book on each subject?

Solution

The selection of three books with atleast 1 book of each subject can be done in the following ways:

A) There can be 1 book of history and 2 books of english.

B) There can be two books of history and 1 book of english.

Number of ways for A) $4_{C_1}.6_{C_2}$=60

Number of ways for B) $4_{C_2}.6_{C_1} = 36$

Total number of ways=60+36=96

Combination with repetition

Let us consider a set of 3 numbers $\{1, 2, 3\}$. Suppose we have to form a subset of two numbers and repetition is allowed. then we will have the following possibilities:
$\{\{1,1\},\{1,2\},\{1,3\},\{2,2\},\{2,3\},\{3,3\}\}$
However without repitition we will have only three possibilities.
$\{\{1,2\},\{2,3\},\{1,3\}\}$

Theorem

Let there be n elements in a set. If repetition of elements is allowed, then the number of r-combinations is given by
$C(n+r-1,r)$ or $C(n+r-1,n-1)$

Examples

1. Find the number of ways to select three balls from a bag that contains balls of 4 different colors, if repetition is allowed.

Solution

Here n=4,r=3

Therefore number of combinations=C(4+3-1,3)=C(6,3)=20

2. There are some cans of coke, Pepsi and sprite in a refrigerator. In how many ways can 5 cans be selected if repetition is allowed?

Solution

Here n=3,r=5

Total number of combinations=C(3+5-1,5)

=C(7,5)=21

3. How many solutions does the equation $x1+x2+x3=10$ have if x1,x2 and x3 are non negative integers?

Solution

The problem is similar to that of finding the number of ways to select 10 objects from a set of 3 elements, where each element can be repeated any number of times.

Here $n=3, r=10$

Therefore, total number of combinations$=C(3+10-1,10)$

$=66$

4. How many solutions does the equation x1+x2+x3+x4=30 have if $x1 \geq 2, x2 \geq 4, x3 \geq 5$ and $x4 \geq 6$ and all are integers.
Solution
x1=y1+2
x2=y2+4
x3=y3+5
x4=y4+6 where $(yi \geq 0, 1 \leq i \leq 4)$
Now the given equation is equivalent to
y1+2+y2+4+y3+5+y4+6=30
y1+y2+y3+y4=13
n=4,r=13
Therefore total number of combinations=C(4+13-1,13)=560

5. How many solutions does the equation x1+x2+x3=20 have if $0 \leq x1 \leq 6, 0 \leq x2 \leq 8$ and $0 \leq x3 \leq 9$?

Solution

10

Binomial Coefficients

Let n and r be positive integers such that r¡=n. We have already mentioned that an r-combination from a set of n elements is denoted by n_{C_r}
This number is also called the binomial coefficient because the number occurs as coefficient in the expansion of powers of binomial expression such as $(x + y)^n$

Binomial Theorem

Let x and y be variables and n be a non-negative integer, then

$(x + y)^n =$

$= n_{C_0} x^n y^0 + n_{C_1} x^{n-1} y^1 + ... + n_{C_{(n-1)}} x^1 y^{n-1} + n_{C_n} x^0 y^n$

Pascal's Identity

The binomial coefficient satisfy many different identities. One of the most important of these identities is the pascal's identity given as Theorem

Let n and k be positive integers with $n \geq k$, then

$n_{C_{k-1}} + n_{C_k} = n + 1_{C_k}$

Examples

1. Find the coefficient of x^8y^5 in the expansion of $(x+y)^{13}$
Solution
n=13,i=5
Therefore the coefficient of x^8y^5 is $13_{C_5} = 13!/(8!5!) = 1287$

Examples

2. Find the coefficient of x^7y^8 in the expansion of $(x+y)^{13}$
Solution
The coefficient of x^7y^8 is $15_{C_8}2^7 3^8$

Pigeonhole principle

Suppose we have 10 boxes and 11 balls. Each box will be 1 ball each up to 10 balls but for the 11th ball we have to chose a box from 1 to 10. In this way,at least 1 box will have more than 1 ball.
The principle can be states as
Given n pigeonhole, the minimum number of pigeons required to be sure that at-least one pigeonhole is occupied by two pigeons is n+1.

Examples

1. Find the number of students in a class so that 2 students were born in the same month.

Solution

pigeons=students

pigeonholes(n)=months

Thus the minimum number of students so that 2 students were born in the same month=n+1=13

2. How many people must be there in a group to guarantee that at-least two people have the same birthday?

Solution

Number of pigeonholes(n)=365

Therefore minimum number of people required=(n+1)=366

Try Yourself

3. Students are awarded marks in a subject. The maximum marks a student can obtain is 50. How many students must be in the class to be sure that atleast 2 students get the same marks?

Generalized Pigeonhole Principle

The principle can be stated as:
Given n pigeonholes, the minimum number of pigeons required to be sure that atleast one pigeonhole is occupied by k+1 pigeons is kn+1

Examples

1. Find the minimum number of students in a class to be sure that 4 of them were born in the same month.

Solution

n=12, K+1=4, Thus k=3

The minimum number of students are kn+1

$=3 \times 12 + 1$

$=37$

Try Yourself

2. Students are awarded grades A, B ,C and D. How many students must be there in a group so that atleast 6 students get the same grade.