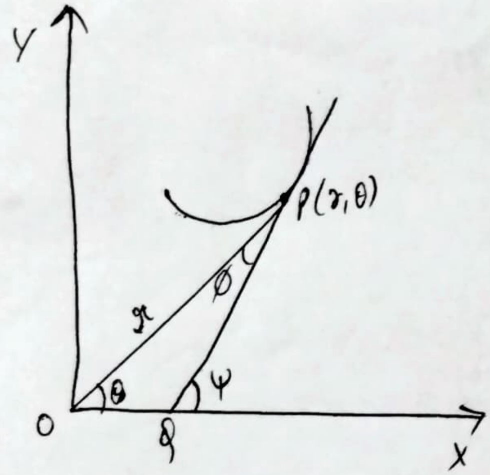
FULL Mathematics Model Question Paper - 1 2023

BMATS 101, Maths for CS stream

Q1. (a) Prove $\tan\phi = r\dfrac{d\theta}{dr}$



Clearly, $\boxed{\psi = \theta + \phi}$

$\Rightarrow \tan\psi = \tan(\theta+\phi)$ {take ~~tan~~ both sides}

$$\Rightarrow \tan\psi = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi}$$

$\left\{ \because \tan(x+y) = \dfrac{\tan x + \tan y}{1-\tan x \tan y} \right\}$

Also, $\tan\psi = \dfrac{dy}{dx}$, $x = r\cos\theta$
$y = r\sin\theta$
r is Not Constant

$\Rightarrow$ diff w.r.t. 'θ'

$$\Rightarrow \frac{dx}{d\theta} = -r\sin\theta + \cos\theta\frac{dr}{d\theta} \qquad \frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + r'\sin\theta}{-r\sin\theta + r'\cos\theta} \qquad r' = \frac{dr}{d\theta}$$

$\div$ by $r'\cos\theta$

$$\frac{dy}{dx} = \frac{\frac{r\cos\theta}{r'\cos\theta} + \frac{r'\sin\theta}{r'\cos\theta}}{\frac{-r\sin\theta}{r'\cos\theta} + \frac{r'\cos\theta}{r'\cos\theta}} \Rightarrow \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} = \frac{dy}{dx} = \tan\psi$$

$\because \quad \tan \psi = \dfrac{\tan\phi + \tan\theta}{1 - \tan\phi \tan\theta} = \dfrac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta}$

$\therefore$ from above we get,

$$\boxed{\tan\phi = \frac{r}{r'} = r\frac{d\theta}{dr}}$$

(b). Angle b/w $r_1 = a\log\theta \qquad r_2 = \dfrac{\theta}{\log\theta}$

$\hookrightarrow \quad |\phi_1 - \phi_2|$

$\Rightarrow \quad r_1 = a\log\theta \quad \Rightarrow \quad r_1' = \dfrac{a}{\theta} \quad \Rightarrow \tan\phi_1 = \dfrac{r_1}{r_1'} = \dfrac{\cancel{a}(\log\theta).\theta}{\cancel{a}}$

$\tan\phi_1 = \log\theta^{\theta} \qquad \Rightarrow \quad \phi_1 = \tan^{-1}(\log\theta^{\theta})$

$r_2' = \dfrac{\theta}{\log\theta} \quad \Rightarrow \quad 1 + \dfrac{1}{\log\theta} \quad \Rightarrow r_2' = \dfrac{1+\log\theta}{\log\theta}$

$\tan\phi_2 = \dfrac{r_2}{r_2'} = \dfrac{\theta}{\cancel{\log\theta}} \times \dfrac{\cancel{\log\theta}}{1+\log\theta} = \dfrac{\theta}{1+\log\theta}$

$\phi_2 = \tan^{-1}\left(\dfrac{\theta}{1+\log\theta}\right)$

$\Rightarrow$ Angle $\Rightarrow \boxed{|\phi_1 - \phi_2| = \left|\tan^{-1}(\log\theta^{\theta}) - \tan^{-1}\left(\dfrac{\theta}{1+\log\theta}\right)\right|}$

Angle b/w the Curves, $r_1 = a\log\theta$ & $r_2 = \dfrac{\theta}{\log\theta}$

(c). Show, that Radius of Curvature at any point of the Cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ is $4a\cos\left(\frac{\theta}{2}\right)$.

Now, $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ diff^n w.r.t. 'x'

$$\frac{dx}{d\theta} = a + a\cos\theta \quad , \quad \frac{dy}{d\theta} = a\sin\theta$$

$$\Rightarrow \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\cancel{2}\sin\theta/2\cos\theta/2}{\cancel{2}\cos^2\theta/2} = \tan\theta/2$$

$$\Rightarrow \frac{dy}{dx} = \boxed{y_1 = \tan\theta/2}$$

$$y_2\left(\frac{d^2y}{dx^2}\right) = \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{d\theta}{dx}$$

as, $\frac{d\theta}{dx}$ or $\frac{dx}{d\theta} = a + a\cos\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{a(1+\cos\theta)}$

$$\Rightarrow y_2 = \frac{1}{2}\sec^2\frac{\theta}{2} \times \frac{1}{a2\cos^2\theta/2} = \frac{\sec^4\theta/2}{4a}$$

$$\boxed{y_2 = \frac{\sec^4\theta/2}{4a}}$$

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}} \Rightarrow \rho = \frac{(1+\tan^2\theta/2)^{3/2}}{\sec^4\theta/2} \times 4a$$

$$\rho = \frac{(\sec^2\theta/2)^{3/2}}{\sec^4\theta/2} \times 4a \Rightarrow \frac{\sec^3\theta/2}{\sec^4\theta/2} \times 4a$$

$$\boxed{\rho = 4a\cos(\theta/2)}$$

hence proved

Q2. (a). Show $r_1 = a(1+\sin\theta)$ & $r_2 = a(1+\sin\theta)$ cut each other orthogonally.

Cut orthogonally $\Rightarrow$ $\boxed{\tan\phi_1 \times \tan\phi_2 = -1}$

$\Rightarrow r_1 = a(1+\sin\theta) \Rightarrow r_1' = a\cos\theta \Rightarrow \tan\phi_1 = \frac{r_1}{r_1'} = \frac{\cancel{a}(1+\sin\theta)}{\cancel{a}\cos\theta} = \frac{1+\sin\theta}{\cos\theta}$

*** The curve cannot cut orthogonally bcz both are same.

(b). Find pedal eqⁿ of $\frac{2a}{r} = (1+\cos\theta)$.

take log both sides

$\log\left(\frac{2a}{r}\right) = \log(1+\cos\theta) \Rightarrow \log 2a - \log r = \log(1+\cos\theta)$

diffⁿ w.r.t. θ

$\Rightarrow \cancel{-}\frac{r'}{r} = \frac{\cancel{-}\sin\theta}{1+\cos\theta} \Rightarrow \frac{r'}{r} = \frac{\sin\theta}{1+\cos\theta} \Rightarrow \frac{r}{r'} = \tan\phi = \frac{1+\cos\theta}{\sin\theta}$

$\Rightarrow \tan\phi = \frac{\cancel{2}\cos^2\theta/2}{\cancel{2}\sin\theta/2\cos\theta/2} \Rightarrow \tan\phi = \frac{\cos\theta/2}{\sin\theta/2} = \cot\theta/2$

$\tan\phi = \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$ $\boxed{\phi = \frac{\pi}{2} - \frac{\theta}{2}}$

$\therefore p = r\sin\phi \Rightarrow p = r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \Rightarrow p = r\cos\theta/2$

Now, we will find $\cos\theta/2$ from the given eqⁿ

$\Rightarrow \frac{2a}{r} = 1+\cos\theta \Rightarrow \frac{\cancel{2}a}{r} = \cancel{2}\cos^2\theta/2 \Rightarrow \cos(\theta/2) = \sqrt{\frac{a}{r}}$

$\Rightarrow p = r\sqrt{\frac{a}{r}} \Rightarrow p^2 = r^2\frac{a}{r} \Rightarrow p^2 = ar \Rightarrow 2p\frac{dp}{dr} = a$

diffⁿ w.r.t. r →

$\frac{dp}{dr} = \frac{a}{2p} \Rightarrow \frac{dr}{dp} = \frac{2p}{a} \Rightarrow \rho = r\frac{dr}{dp} \Rightarrow \boxed{\rho = \frac{2pr}{a}}$ ** Ans (c). Using Modern Mathematical tool write a Code to plot the Curve of

$r = 2|\cos 2\theta|$.

Using Python (Numpy & Matplotlib)

```
import numpy as np
import matplotlib.pyplot as plt
theta = np.linspace(-np.pi, np.pi, 1000)
r = 2 * np.abs(np.cos(2 * theta))
x = r * np.cos(theta)
y = r * np.sin(theta)
fig, ax = plt.subplots()
ax.plot(x, y)
ax.set_aspect('equal')
ax.set_title('r=2|cos(2*theta)|')
ax.grid(True)
plt.show()
```

Source: Chat gpt :)

Q3 (a) Expand $\log(\sec x)$ by Maclaurian Series upto x^4?

Maclaurian Series $\rightarrow f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$

$f(0) = \log \sec 0 = 0$

$f'(x) = \tan x \; \left\{\frac{1}{\sec x} \cdot \sec x \tan x\right\} \quad f'(0) = 0$

$f''(x) = \sec^2 x \quad f''(0) = 1$

$f'''(x) = 2\sec^2 x \tan x \quad f'''(0) = 0$

$f^{IV}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$

$f^{IV}(0) = 2$

$\therefore$

$$f(x) = f(\log \sec x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!}$$

$$\Rightarrow 0 + 0 . \frac{x}{1} + 1 . \frac{x^2}{2} + 0 . \frac{x^3}{3!} + 2 . \frac{x^4}{4!}$$

$$\Rightarrow \boxed{\frac{x^2}{2} + \frac{x^4}{12} + \cdots} \cdots$$

(b). If $u = e^{(ax+by)} . f(ax-by)$ prove $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu$?

$$\Rightarrow \frac{\partial u}{\partial x} = a\,e^{(ax+by)} . f(ax-by) + a\,e^{(ax+by)} f'(ax-by)$$

$$b\frac{\partial u}{\partial x} = ab\{e^{(ax+by)} . f(ax-by) + e^{(ax+by)} . f'(ax-by)\}.$$

$$\frac{\partial u}{\partial y} = b\,e^{(ax+by)} . f(ax-by) - b\,e^{(ax+by)} f'(ax-by) \Rightarrow a\frac{\partial u}{\partial y} = ab\{e^{(ax+by)} . f(ax-by) - e^{(ax+by)} . f'(ax-by)\}$$

$$\Rightarrow b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = ab\{e^{(ax+by)} . f(ax-by) + \cancel{e^{(ax+by)} . f'(ax-by)} + e^{(ax+by)} . f(ax-by) - \cancel{e^{(ax+by)} . f'(ax-by)}\}$$

$$b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = ab\{2\,e^{(ax+by)} . f(ax-by)\}$$

$$\Rightarrow \boxed{b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2abu}$$

(c). $f(x,y) = x^3 + 3xy^2 - 3y^2 - 3x^2 + 4$ Find Extreme Values.

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0 \qquad \frac{\partial f}{\partial y} = 6xy - 6y = 0 \Rightarrow 6y(x-1) = 0$$

$$\therefore \boxed{y = 0 \text{ or } x = 1}$$

$\therefore$ When, $y = 0 \Rightarrow 3x^2 + 3(0)^2 - 6x = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$y = 0 \Rightarrow x = 0$ or 2 $\Rightarrow$ Points $\rightarrow (0,0)\ (2,0)$

When $x = 1$

$3(1)^2 + 3y^2 - 6 = 0 \Rightarrow 3y^2 = 3,\ y^2 = 1 \Rightarrow \underline{y = \pm 1}$

Points $\underline{(1,1)}$, $\underline{(1,-1)}$

Now, $A = \frac{\partial^2 f}{\partial x^2} = 6x - 6$, $C = \frac{\partial^2 f}{\partial y^2} = 6x - 6$, $B = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$\Rightarrow \underline{AC - B^2 \Rightarrow (6x-6)^2 - 36y^2}$ & $\underline{A = (6x-6)}$

Now, at $(0,0)$ $AC - B^2 \Rightarrow (0-6)^2 - 0 > 0$ & $A < 0$ $\therefore$ Maximum Point

at $(2,0)$ $AC - B^2 \Rightarrow (12-6)^2 - 0 > 0$ & $A > 0$ $\therefore$ Minimum Point

at $(1,1)$ $AC - B^2 \Rightarrow (0)^2 - 36(1)^2 < 0$ $\therefore$ Saddle Point

at $(1,-1)$ $AC - B^2 \Rightarrow (0)^2 - 36(-1)^2 < 0$ $\therefore$ Saddle Point

$\therefore$ Maximum at $(0,0)$ $f(x,y) = \underline{4}$

Minimum at $(2,0)$ $f(x,y) = (2)^3 - 3(2)^2 + 4 = 8 - 12 + 4 = \underline{0}$

Q4. (a) Evaluate (i) $\lim_{x\to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$ (ii) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

(i) $K = \lim_{x\to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$, $\boxed{K = \lim_{x\to 0} (f(x))^{g(x)} \Rightarrow \log K = \lim_{x\to 0} g(x)\{f(x) - 1\}}$

$\log K = \lim_{x\to 0} \frac{1}{x}\left\{\frac{a^x + b^x}{2} - 1\right\} \Rightarrow \log K = \lim_{x\to 0} \frac{a^x + b^x - 2}{2x} = \frac{0}{0}$ form $\therefore$ Apply L-Hospital's Rule

$\Rightarrow \log K = \lim_{x\to 0} \frac{a^x \log a + b^x \log b}{2} \Rightarrow \log K = \frac{a^0 \log a + b^0 \log b}{2}$

$\log k = \log (ab)^{1/2} \Rightarrow \boxed{K = (ab)^{1/2}}$

(b). $x + y + z = u \quad y + z = uv \quad z = uvw \quad \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$?

$y = uv - uvw \implies x = u - (y+z) \implies x = u - \cancel{uvw} - uv + \cancel{uvw}$

$x = u - uv \quad y = uv - uvw \quad z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}$$

$x_u = 1 - v \quad x_v = -u \quad x_w = 0$

$y_u = v - vw \quad y_v = u - uw \quad y_w = -uv$

$z_u = vw \quad z_v = uw \quad z_w = uv$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{bmatrix} = 1-v(u^2v - \cancel{u^2vw} + \cancel{u^2vw}) + u(v^2u - \cancel{v^2uw} + \cancel{v^2uw})$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 1-v(u^2v) + u(v^2u) = u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$\implies \boxed{\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v}$$

(c.) Using Modern Mathematical Tool write a program/code to show that $u_{xx} + u_{yy} = 0$ given $u = e^x(x\cos(y) - y\sin(y))$.

```
from sympy import *
from sympy.abc import rho, alpha
x, y = symbols('x, y')
u = exp(x)*(x*Cos(y) - y*Sin(y))
display (u)
dux = diff(u, x)
display (dux)
```

```
duy = diff(u,y)
display(duy)
duxy = diff(dux,y)
duyx = diff(duy,x)
if duxy == duyx:
    print("equal")
else:
    print("Not equal").
```

Q.5. (a) Solve: $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$

$P = \frac{1}{x}, \quad Q = x^2$

divide by $y^6 \Rightarrow \quad y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$

let $y^{-5} = v \Rightarrow -5y^{-6}\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow y^{-6}\frac{dy}{dx} = \frac{dv}{dx}\frac{1}{-5}$

$\Rightarrow \frac{dv}{dx} \times \frac{1}{-5} + \frac{v}{x} = x^2 \Rightarrow$ Multiply by -5

$\Rightarrow \frac{dv}{dx} - 5\frac{v}{x} = -5x^2 \qquad P = -\frac{5}{x}, \; Q = -5x^2$

$\int P\,dx = -5\int \frac{1}{x}dx = \log|x|^{-5}$

$I.F. = e^{\log|x|^{-5}} \Rightarrow x^{-5} = I.F.$

Sol^n of diff^n eq^n $\Rightarrow V(I.F.) = \int Q(I.F.)dx + c$

$\Rightarrow \frac{1}{x^5 y^5} = \int -5x^2 \times x^{-5}dx \Rightarrow -5\int \frac{1}{x^3}dx$

$$\Rightarrow \frac{1}{x^5y^5} = -5\int x^{-3}dx \Rightarrow \frac{1}{x^5y^5} = -5\times\frac{x^{-2}}{-2}+c$$

$$\Rightarrow \frac{1}{x^5y^5} = \frac{5}{2x^2}+c \Rightarrow \boxed{\frac{1}{x^5y^5} - \frac{5}{2x^2} = c}$$

(b). Find Orthogonal trajectory of $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ where λ is Parameter

$$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1 \quad —①$$

diff' w.r.t. 'x'

$$\frac{2x}{a^2}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0 \Rightarrow \frac{x}{a^2}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{y}{b^2+\lambda}\frac{dy}{dx} = -\frac{x}{a^2} \Rightarrow \frac{1}{b^2+\lambda}\frac{dy}{dx} = \frac{-x}{a^2y} \Rightarrow \frac{1}{b^2+\lambda} = \frac{-x}{a^2y}\frac{dx}{dy}$$

Substitute it in ①

$$\Rightarrow \frac{x^2}{a^2}+y^{\cancel{2}}\left(\frac{-x}{a^2\cancel{y}}\frac{dx}{dy}\right)=1 \Rightarrow \text{Replace } \frac{dy}{dx} \text{ by } \frac{-dx}{dy}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{xy}{a^2}\frac{dy}{dx}=1 \Rightarrow x^2+xy\frac{dy}{dx}=a^2$$

$$\Rightarrow xy\frac{dy}{dx}=a^2-x^2 \Rightarrow y\,dy = \left(\frac{a^2}{x}-\frac{x^2}{x}\right)dx$$

$$\Rightarrow y\,dy = \left(\frac{a^2}{x^2}-x\right)dx \Rightarrow \int y\,dy = \int\left(\frac{a^2}{x}-x\right)dx$$

$$\Rightarrow \frac{y^2}{2} = a^2\log(x) - \frac{x^2}{2}+c$$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}-a^2\log x = c$$

$$\Rightarrow \boxed{x^2+y^2-2a^2\log x = c_1}$$

(C) Solve $xyp^2-(x^2+y^2)y+xy=0$

Given eq^n $xyp^2-(x^2+y^2)p+xy=0$ is of the form $ax^2+bx+c=0$

$a=xy \quad b=-(x^2+y^2) \quad \& \quad c=xy$

$$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow p=\frac{x^2+y^2\pm\sqrt{(x^2+y^2)^2-4x^2y^2}}{2xy}$$

$$=\frac{x^2+y^2\pm\sqrt{(x^2-y^2)^2}}{2xy}$$

$$=\frac{x^2+y^2\pm(x^2-y^2)}{2xy}$$

$$p=\frac{x^2+\cancel{y^2}+x^2-\cancel{y^2}}{2xy} \qquad p=\frac{x^2+y^2+y^2-x^2}{2xy}=\frac{2y^2}{2xy}=\frac{y}{x}$$

$$=\frac{2x^2}{2xy}=\frac{x}{y}$$

$$\frac{dy}{dx}=\frac{x}{y} \qquad \Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

$$\int y\,dy=\int x\,dx \qquad \int\frac{1}{y}dy=\int\frac{1}{x}dx$$

$$\frac{y^2}{2}=\frac{x^2}{2}+\frac{c}{2} \qquad \log y=\log x+\log c$$

$$\Rightarrow \boxed{y-xc=0}$$

$$\boxed{y^2=x^2+c}$$

Q6(a). Solve. $(x^2+y^2+x)dx+xy\,dy=0$

The given diff^n eq^n is of the form

$$Mdx+Ndy=0$$

where, $M=x^2+y^2+x \quad \& \quad N=xy$

$$\frac{\partial M}{\partial y}=2y \qquad \frac{\partial N}{\partial x}=y$$

Clearly, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Let $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = 2y - y = y$ Close to N

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{y}{xy} = \frac{1}{x} = f(x)$

$I.F. = e^{\int f(x).dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

$M = I.F.\ M$ $\qquad$ $N = I.F.\ N$

$x(x^2+y^2+x)$ $\qquad$ $= x \cdot xy$

$x^3 + xy^2 + x^2$ $\qquad$ $= x^2 y$

$\frac{\partial M}{\partial y} = 2xy$ & $\frac{\partial N}{\partial x} = 2xy$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. eq^n is exact

So the Sol^n is $\int M dx + \int N(y) dy = \left(\frac{x^4}{4} + \frac{y^2 x^2}{2} + \frac{x^3}{3}\right) = c$

(b). A switch is Closed in a circuit Containing a battery 'E' a Resistance 'R', an inductance 'L', the Current 'I' build up at a rate given by $L\frac{di}{dt} + Ri = E$ Find i as a function of 'E'. how long it will be, before the Current has reach $\frac{1}{2}$ of initial Values of $E = 6V$ & $R = 100\Omega$ & $L = 0.1H$?

Sol^n Given that,

$$L\frac{di}{dt} + Ri = E \quad (\div L)$$

$\frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$ is in the form of $\frac{di}{dt} + Pi = Q$ (linear eq^n form)

where $P = \frac{R}{L}$ & $Q = \frac{E}{L}$

$$I.F. = e^{\int P.dt} = e^{R/L} = e^{Rt/L}$$

Sol^n is $\quad i \cdot I.F = \int Q\, I.F.\, dt + c$

$$\Rightarrow \quad i \cdot e^{Rt/L} = \int \frac{E}{L} \cdot e^{Rt/L} \cdot dt + c$$

$$\Rightarrow \quad i \cdot e^{Rt/L} = \frac{E}{L} \int e^{Rt/L} dt + c$$

$$i \cdot e^{Rt/L} = \frac{E}{\cancel{L}} \times \frac{e^{Rt/L}}{R/\cancel{L}} + c$$

$$i \cdot e^{Rt/L} = \frac{E}{R} \cdot e^{Rt/L} + c$$

$$i = \frac{E}{R} + c \cdot e^{-Rt/L}$$

Given $E = 6$, $R = 100$, & $L = 0.1$

$i = 0$ when $t = 0$ in eq^n 1

$$0 = \frac{E}{R} + c \quad \Rightarrow \quad c = -\frac{E}{R}$$

Now, put $c = -\frac{E}{R}$ in 1:

$$i = \frac{E}{R} - \frac{E}{R} e^{-Rt/L} \quad \Rightarrow \quad i = \frac{6}{100} - \frac{6}{100} e^{100t/0.1}$$

$$\Rightarrow i = \frac{6}{100}\left(1 - \frac{1}{e^{1000t}}\right) \quad \Rightarrow \quad i = \frac{6}{100} \times \frac{(e^{1000t} - 1)}{e^{1000t}}$$

Current will be Max^n

when $t = \infty$

$$i = 6/100\,[1 - 0]$$

$$i = \frac{6}{100} = 0.06 \text{ Amp}$$

Q(c) Solve the eq^n $(Px-y)(Py+x)=2P$ by reducing in Clauraite form, taking $X=x^2$, $Y=y^2$

Sol^n: Given eq^n

$$(Px-y)(Py+x)=2P \quad -① $$

$$X=x^2 \quad Y=y^2$$

diff^n $dX=2x\,dx$, $dY=2y\,dy \Rightarrow \frac{dX}{dx}=2x$ & $\frac{dY}{dy}=2y$

and $x=\sqrt{X}$, $y=\sqrt{Y}$

Let, $P=\frac{dY}{dX} \Rightarrow \frac{dY}{dy}\cdot\frac{dy}{dx}\cdot\frac{dx}{dX}$

$$\Rightarrow P=2y\cdot p\cdot\frac{1}{2x} \Rightarrow P=\frac{py}{x} \Rightarrow p=\frac{Px}{y} \Rightarrow p=\frac{\sqrt{X}}{\sqrt{Y}}P$$

$\Rightarrow$ Substitute in ①.

$$\left(\frac{\sqrt{X}}{\sqrt{Y}}\cdot P\cdot\sqrt{X}-\sqrt{Y}\right)\left(\frac{\sqrt{X}}{\sqrt{Y}}\cdot P\cdot\sqrt{Y}+\sqrt{X}\right)=\frac{2\sqrt{X}}{\sqrt{Y}}P$$

$$\left(\frac{XP}{\sqrt{Y}}-\sqrt{Y}\right)\left(\sqrt{X}P+\sqrt{X}\right)=\frac{2\sqrt{X}P}{\sqrt{Y}}$$

$$\left(\frac{XP-Y}{\sqrt{Y}}\right)(\sqrt{X})(P+1)=\frac{2\sqrt{X}}{\sqrt{Y}}\cdot P \Rightarrow Y=XP-\frac{2P}{P+1}$$

$$Y=cX-\frac{2c}{c+1} \Rightarrow y^2=cx^2-\frac{2c}{c+1} \Rightarrow \text{diff}^n \text{ w.r.t. } c$$

$$0=x^2\cdot\frac{-[(c+1)\cdot 2-2c]}{(c+1)^2} \Rightarrow \frac{2c+2-2c}{(c+1)^2}=x^2$$

$$\frac{2}{(c+1)^2}=x^2 \Rightarrow \quad c=\frac{\sqrt{2}}{x}-1 \quad \text{from, } y^2=\left(\frac{\sqrt{2}}{x}-1\right)x^2-\frac{2(\sqrt{2}/x-1)}{(\sqrt{2}/x-1+1)}$$

$$\Rightarrow \boxed{c=2\sqrt{2}\cdot x-x^2-2}$$

Q7. (a) Find the least positive values of x such that

(i) $71 \equiv x \pmod 8$ (ii) $78 + x \equiv 3 \pmod 5$ (iii) $89 \equiv (x+3) \pmod 4$

(i) $71 \equiv x \pmod 8 \Rightarrow 8 \mid 71 - x \quad x = 7 \Rightarrow 8 \mid 71 - 7 \Rightarrow \underline{8 \mid 64}$

$\therefore \boxed{x = 7}$

(ii). $78 + x \equiv 3 \pmod 5 \Rightarrow 5 \mid 78 + x - 3 \Rightarrow 5 \mid 75 + x \therefore$ if $x = 0$

$\Rightarrow 5 \mid 75 \quad \therefore \boxed{x = 0}$

(iii). $89 \equiv (x+3) \pmod 4$

$\Rightarrow 4 \mid 89 - x - 3 \Rightarrow 4 \mid 86 - x \quad x = 2 \Rightarrow 4 \mid 86 - 2 \Rightarrow \underline{4 \mid 84}$

$\therefore \boxed{x = 2}$

(b). Find Remainder when $(349 \times 74 \times 36)$ is divided by 3.

$\Rightarrow (349 \times 74 \times 36) \equiv x \pmod 3$

$\Rightarrow 349 \equiv a \pmod 3 \Rightarrow 349 \equiv 1 \pmod 3$

$$\begin{array}{r} 116 \\ 3 \overline{)\,349\,} \\ 348 \\ \hline 1 \end{array} \qquad \begin{array}{r} 25 \\ 3 \overline{)\,74\,} \\ 75 \\ \hline -1 \end{array}$$

$74 \equiv b \pmod 3 \Rightarrow 74 \equiv -1 \pmod 3$

$36 \equiv x \pmod 3 \Rightarrow 36 \equiv 0 \pmod 3$

$\Rightarrow (349 \times 74 \times 36) \equiv 1 \times -1 \times 0 \pmod 3$

$\Rightarrow (349 \times 74 \times 36) \equiv 0 \pmod 3$

$\therefore$ Remainder is $\underline{0}$

(c) Solve: $2x + 6y \equiv 1 \pmod 7$ & $4x + 2y \equiv 2 \pmod 7$

as $ax + by \equiv r \pmod n$ & $cx + dy \equiv s \pmod n$

$\therefore a = 2,\ b = 6\quad c = 4\quad d = 2\quad r = 1\quad s = 2\quad n = 7$

This will have Solⁿ iff $\gcd(ad - bc, n) = 1$

$ad - bc = 2 \times 2 - 6 \times 4 = 4 - 24 = -20.$

$\therefore \underline{\gcd(-20, 7) = 1}$

<u>elimination</u> of y.

$2x + 6y \equiv 1 \pmod 7) \times 2$
$4x + 2y \equiv 2 \pmod 7) \times 6$

$\Rightarrow$

$4x + \cancel{12y} \equiv 2 \pmod 7$
$-\ 24x \pm \cancel{12y} \equiv 12 \pmod 7$

$-20x \equiv -10 \pmod 7) \times -1$

$\Rightarrow +20x \equiv +10 \pmod 7$ $\Rightarrow$ $c \cdot 20 \pmod 7 = 1$ $\therefore c = 1$

$\Rightarrow 20 \pmod{}$

20
40
60
80
100

Q8. (a) (i) Find last digit of 7^{2013} (ii) last digit of 13^{37}

(i) let $a=7$, $n=10$ $\phi(n)=4$

$\therefore$ by Euler's Theorem

$\Rightarrow a^{\phi(n)} \equiv 1 \pmod{n} \Rightarrow 7^4 \equiv 1 \pmod{10}$

$\Rightarrow (7^4)^{503} \equiv (1)^{503} \pmod{10} \Rightarrow 7^{2012} \cdot 7 \equiv 1 \cdot 7 \pmod{10}$

$\Rightarrow 7^{2013} \equiv 7 \pmod{10}$

$\therefore \boxed{\text{Last digit} = 7}$

(ii). let $a=13$, $n=10$, $\phi(n)=4$ (odd Noumbers which are Not divisible by n)

$\Rightarrow a^{\phi(n)} \equiv 1 \pmod{10} \Rightarrow 13^4 \equiv 1 \pmod{10}$

$\Rightarrow (13^4)^9 \equiv (1)^9 \pmod{10} \Rightarrow 13^{36} \equiv 1 \pmod{10}$

$\Rightarrow 13^{36} \cdot 13 \equiv 1 \cdot 13 \pmod{10} \Rightarrow 13^{37} \equiv 13 \pmod{10}$

$\Rightarrow 13^{37} \equiv 3 \pmod{10}$

$\therefore \boxed{\text{Last digit} = 3}$

(b). Find Remainder when 2^{1000} is divided by 13?

By Fermat's Little theorem, $P=13$, $a=2$ Clearly $\underline{P \nmid a}$

$\therefore a^{P-1} \equiv 1 \pmod{P} \Rightarrow 2^{12} \equiv 1 \pmod{13}$

$(2^{12})^{83} \equiv (1)^{83} \pmod{13} \Rightarrow 2^{996} \equiv 1 \pmod{13}$

$2^{996} \cdot 2^4 \equiv 1 \cdot 2^4 \pmod{13} \Rightarrow 2^{1000} \equiv 16 \pmod{13}$

$\Rightarrow 2^{1000} \equiv 3 \pmod{13}$

$\boxed{\therefore \text{ Remainder is } 3}$

C. Find Remainder when 14! is divided by 17.

$\therefore$ By Wilson's Theorem if P is prime then

$(P-1)! \equiv -1 \pmod P$

here, $P = 17$, $\Rightarrow (17-1)! \equiv -1 \pmod P \Rightarrow 16! \equiv -1 \pmod{17}$

$\Rightarrow 16 \times 15 \times 14! \equiv -1 \pmod{17} \Rightarrow -1 \times -2 \equiv -1 \pmod{17}$

$\left\{ \begin{array}{l} 16 \equiv x \pmod{17} \Rightarrow 16 \equiv -1 \pmod{17} \\ 15 \equiv y \pmod{17} \Rightarrow 15 \equiv -2 \pmod{17} \end{array} \quad -1 \equiv z \pmod{17} \Rightarrow -1 \equiv 16 \pmod{17} \right\}$

$\Rightarrow -1 \times -2 \times 14! \equiv 16 \pmod{17}$

$\Rightarrow 14! \equiv \frac{16}{2} \pmod{17} \Rightarrow \underline{14! \equiv 8 \pmod{17}}$

$\therefore \boxed{\text{Remainder is } 8}$

Q9. (a). Find the Rank of the Matrix. $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

$$\Rightarrow \begin{array}{cc} \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \end{array}$$

$R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - 3R_1$, $R_4 \to 2R_4 - 6R_1$

$$\begin{array}{lrrrr} 2R_2 \to & 2 & -2 & -4 & -8 \\ (-)\ R_1 \to & -2 & -3 & \mp 1 & \mp 1 \\ \hline & 0 & -5 & -3 & -7 \end{array}$$

$$\begin{array}{lrrrr} 2R_3 \to & 6 & 2 & 6 & -4 \\ (-)\ 3R_1 \to & -6 & -9 & \mp 3 & \mp 3 \\ \hline & 0 & -7 & 9 & -1 \end{array}$$

$$\begin{array}{lrrrr} 2R_4 \to & 12 & 6 & 0 & -14 \\ (-)\ 6R_1 \to & -12 & -18 & \mp 6 & \mp 6 \\ \hline & 0 & -12 & 6 & -8 \end{array}$$

Q9(a). Cont..

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & -7 & 9 & -1 \\ 0 & -12 & 6 & -8 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$R_3 \to -5R_3 + 7R_2$

$R_4 \to -5R_4 + 12R_2$

$$\begin{array}{lcccc} -5R_3 \to & 0 & \cancel{35} & -45 & 5 \\ (+) \; 7R_2 \to & 0 & \cancel{-35} & -21 & -49 \\ \hline & 0 & 0 & -66 & -44 \end{array}$$

$$\begin{array}{lcccc} -5R_4 \to & 0 & \cancel{60} & -30 & 40 \\ (+) \; 12R_2 \to & 0 & \cancel{-60} & -36 & -84 \\ \hline & 0 & 0 & -66 & -44 \end{array}$$

$$\Rightarrow \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & 0 & -66 & -44 \\ 0 & 0 & -66 & -44 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & -1 & -1 \\ 0 & -5 & -3 & -7 \\ 0 & 0 & -66 & -44 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A$$

$\therefore \boxed{\rho(A) = 3}$ **Rank**

(b). Solve the System of Linear eq^n by Gaus-Jordon Method,

$x + y + z = 10, \; 2x - y + 3z = 19, \; x + 2y + 3z = 22$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 2 & -1 & 3 & 19 \\ 1 & 2 & 3 & 22 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{array}{lcccc} R_2 \to & \cancel{2} & -1 & 3 & 19 \\ (-) \; 2R_1 \to & \cancel{2} & 2 & 2 & 20 \\ \hline & 0 & -3 & 1 & -1 \end{array}$$

$$\begin{array}{lcccc} R_3 \to & \cancel{1} & 2 & 3 & 22 \\ (-) \; R_1 \to & \cancel{1} & 1 & 1 & 10 \\ \hline & 0 & 1 & 2 & 12 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 1 & 2 & 12 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \to -3R_3 - R_2$

$$\begin{array}{lcccc} -3R_3 \to & 0 & \cancel{-3} & -6 & -36 \\ (-) \; R_2 \to & 0 & \cancel{-3} & 1 & -1 \\ & & + & - & + \\ \hline & 0 & 0 & -7 & -35 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 10 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & -7 & -35 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & : & 10 \\ 0 & -3 & 1 & : & -1 \\ 0 & 0 & -1 & : & -5 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$R_2 \to -R_2 - R_3$

$R_1 \to -R_1 - R_3$

$$\begin{array}{rrrrr} -R_2 \to & 0 & 3 & \cancel{-1} & 1 \\ (-)\ R_3 \to & 0 & 0 & \overset{+}{-}1 & \overset{+}{-}5 \\ \hline & 0 & 3 & 0 & 6 \\ & 0 & 1 & 0 & 2 \end{array}$$

$$\begin{array}{rrrrr} -R_1 \to & -1 & -1 & \cancel{-1} & -10 \\ (-)\ R_3 \to & 0 & 0 & \overset{+}{-}1 & \overset{+}{-}5 \\ \hline & -1 & -1 & 0 & -5 \end{array}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 0 & : & -5 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & -1 & : & -5 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$\Rightarrow \quad R_1 \to R_1 + R_2$

$$\Rightarrow \begin{array}{rrrrr} R_1 \to & -1 & \cancel{-1} & 0 & -5 \\ (+)\ R_2 \to & 0 & \cancel{1} & 0 & 2 \\ \hline & -1 & 0 & 0 & -3 \end{array}$$

$$\Rightarrow \quad A = \begin{bmatrix} -1 & 0 & 0 & : & -3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & -1 & : & -5 \end{bmatrix} \quad \therefore \boxed{\begin{matrix} x = 3 \\ y = 2 \\ z = 5 \end{matrix}}$$

Solⁿ by Gaus - Jordan. Method or, Jodi

(C) For What Values of λ & μ the system of linear eqⁿ

$2x + 3y + 5z = 9$, $7x + 3y - 2z = 8$, $2x + 3y + \lambda z = \mu$ has

(i) No Solⁿ, (ii) Unique Solⁿ, (iii) Infinite No. of Solⁿ.

$$\Rightarrow \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 2 & 3 & \lambda & : & \mu \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$R_2 \to 2R_2 - 7R_1$

$R_3 \to 2R_3 - 2R_1$

$$\begin{array}{rrrrr} 2R_2 \to & \cancel{14} & 6 & -4 & 16 \\ (-)\ 7R_1 \to & \cancel{14} & \overset{-}{21} & \overset{-}{35} & \overset{-}{63} \\ \hline & 0 & -15 & -39 & -47 \end{array}$$

$$\begin{array}{rrrrr} 2R_3 \to & \cancel{4} & \cancel{6} & 2\lambda & 2\mu \\ (-)\ 2R_1 \to & \cancel{4} & \overset{-}{6} & \overset{-}{10} & \overset{-}{18} \\ \hline & 0 & 0 & 2\lambda - 10 & 2\mu - 18 \end{array}$$

Q9(c) Contd.

$$\left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 0 & -15 & -39 & -47 \\ 0 & 0 & 2\lambda-10 & 2\mu-18 \end{array}\right]$$

(i) No Sol^n when, $\rho(A) \neq \rho(A:B)$

$2\lambda - 10 = 0 \Rightarrow \boxed{\lambda = 5}$

$2\mu - 18 \neq 0 \Rightarrow \boxed{\mu \neq 9}$

(ii) Unique Sol^n $\rho(A) = \rho(A:B) =$ No of Unknown

$\Rightarrow 2\lambda - 10 \neq 0 \Rightarrow \boxed{\lambda \neq 5}$

$2\mu - 18 \neq 0 \Rightarrow \boxed{\mu \neq 9}$

(iii) Infinite Sol^n, $\rho(A) = \rho(A:B) <$ No of Unknown

$2\lambda - 10 = 0 \Rightarrow \boxed{\lambda = 5}$

$2\mu - 18 = 0 \Rightarrow \boxed{\mu = 9}$

Q10. (a). Solve the following system of linear eq^n by Gauss-Seidal Method

$10x + y + z = 12$, $x + 10y + z = 12$, $x + y + 10z = 12$

$\therefore$ given eq^n's are in diagonally dominating form.

$$x = \frac{12 - y - z}{10}, \quad y = \frac{12 - x - z}{10}, \quad z = \frac{12 - x - y}{10}$$

Let $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$

$$x^{(1)} = \frac{12}{10} = 1.2, \quad y^{(1)} = \frac{12 - 1.2 - 0}{10} = 1.08, \quad z^{(1)} = \frac{12 - 1.2 - 1.08}{10} = 0.9$$

$$x^{(2)} = \frac{12 - 1.08 - 0.9}{10} = 1.002, \quad y^{(2)} = \frac{12 - 1.002 - 0.9}{10} = 1.009$$

$$z^{(2)} = \frac{12 - 1.002 - 1.009}{10} = 0.998$$

$$x^{(3)} = \frac{12 - 1.009 - 0.998}{10} = 0.999, \quad y^{(3)} = \frac{12 - 0.999 - 0.998}{10} = 1.000$$

$$z^{(3)} = \frac{12 - 0.999 - 1.000}{10} = 1.000$$

$$\therefore \boxed{x \approx 1, \ y \approx 1, \ z \approx 1}$$

(b) Solve the following system of equations by Gauss-Elimination method,

$$x+y+z=9,\quad x-2y+3z=8,\quad 2x+y-z=3$$

$$\Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_2 \to R_2 - R_1 \qquad R_3 \to R_3 - 2R_1$

$$\begin{array}{r|cccc} R_2 \to & 1 & -2 & 3 & 8 \\ (-)\ R_1 \to & 1 & 1 & 1 & 9 \\ \hline & 0 & -3 & 2 & -1 \end{array} \qquad \begin{array}{r|cccc} R_3 \to & 2 & 1 & -1 & 3 \\ (-)\ 2R_1 \to & 2 & 2 & 2 & 18 \\ \hline & 0 & -1 & -3 & -15 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

$R_3 \to -3R_3 + R_2$

$$\begin{array}{r|cccc} -3R_3 \to & 0 & 3 & 9 & 45 \\ (+)\ R_2 \to & 0 & -3 & 2 & -1 \\ \hline & 0 & 0 & 11 & 44 \end{array} \Rightarrow \left[\begin{array}{cccc} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 11 & 44 \end{array}\right]$$

$\therefore z=4, \quad -3y + 8 = -1 \Rightarrow -3y = -9 \Rightarrow -3y = -9$

$y = 3$

$x + 4 + 3 = 9 \Rightarrow x = 2$

$\therefore \boxed{x=2,\ y=3,\ z=4}$

(c). Using Modern Mathematical Tool write a Code to find the largest Eigen Value of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ by Power method?

```
import numpy as np
A = np.array([[1,1,3],[1,5,1],[3,1,1]])
x0 = np.array([1,1,1])
max_iter = 1000
tol = 1e-8
for i in range(max_iter):
    x1 = np.dot(A, x0)
    lambda1 = np.max(x1)
    x1 = x1/lambda1
    err = np.linalg.norm(x1-x0)
    x0 = x1
    if err<tol:
        break
print("Largest eigen Value of A:", lambda1)
print("EigenVector Corresponding to lambda1:", x0)
```