APL321 : Introduction to Computational Fluid Dynamics
Indian Institute of Technology Delhi
Spring 2024
Prof. Amitabh Bhattacharya

Lab 8

Aditya Agrawal (2021AM10198)

1. The integral form of the Navier-Stokes equation after applying Gauss Divergence theorem is given by the following equations:
   x-dir^n:
   $$\int_S u\boldsymbol{u}.\boldsymbol{n}dS = \frac{1}{Re}\int_S \nabla u.\boldsymbol{n}dS - \int_V \frac{\partial p}{\partial x}dV \tag{1}$$
   y-dir^n:
   $$\int_S v\boldsymbol{u}.\boldsymbol{n}dS = \frac{1}{Re}\int_S \nabla v.\boldsymbol{n}dS - \int_V \frac{\partial p}{\partial x}dV \tag{2}$$
   Solving the LHS of eqn (1) for a CV at (i,j) using integral approximation:
   $$F_c^e = \int_{S_e} u\boldsymbol{u}.\boldsymbol{n}dS \approx u_e u_e \Delta y = \dot{m}_e u_e \tag{3}$$
   Using linear interpolation, we get:
   $$\dot{m}_e \approx \frac{1}{2}(u_{i+1,j} + u_{i,j})\Delta y \tag{4}$$
   Using first order UDS, we get the following approximation for $\dot{m}_e u_e$:
   $$\dot{m}_e u_e = max(0, \dot{m}_e)u_{i,j} + min(0, \dot{m}_e)u_{i+1,j} \tag{5}$$
   Thus, the entire integral is given by:
   $$F_c = \sum_e \dot{m}_e u_e \tag{6}$$
   Now, solving the RHS of eqn (1) for the same CV using integral approximation and CDS:
   $$F_d^e = \frac{1}{Re}\int_{S_e} \nabla u.\boldsymbol{n}dS \approx \frac{1}{Re}\frac{u_{i+1,j} - u_{i,j}}{\Delta x}\Delta y \tag{7}$$
   Thus, the entire integral is given by:
   $$F_d = \sum_e F_d^e \tag{8}$$
   Calculation of the pressure gradient term = Q is given by:
   $$Q = -\int_{y_j}^{y_{j+1}}\int_{X_{i-1}}^{X_i} \frac{\partial p}{\partial x} \approx -\int_{y_j}^{y_{j+1}} \{p(X_i, y) - p(X_{i-1}, y)\}dy \approx \{-p(X_i, Y_j) + p(X_{i-1}, Y_j)\}\Delta y \tag{9}$$
   $$\Rightarrow Q = (p_{i-1,j} - p_{i,j})\Delta y \tag{10}$$

Thus, the final equation takes the form:

$$A_P^u u_{i,j} + A_E^u u_{i+1,j} + A_W^u u_{i-1,j} + A_N^u u_{i,j+1} + A_S^u u_{i,j-1} = Q_P^u \tag{11}$$

where,

$$\begin{gathered}
A_E^u = \frac{1}{Re} - min(0, \dot{m}_e); A_W^u = \frac{1}{Re} - min(0, \dot{m}_w) \\
A_N^u = \frac{1}{Re} - min(0, \dot{m}_n); A_S^u = \frac{1}{Re} - min(0, \dot{m}_s) \\
A_P^u = -\frac{4}{Re} - (max(0, \dot{m}_e) + max(0, \dot{m}_w) + max(0, \dot{m}_n) + max(0, \dot{m}_s)) \\
Q_P^u = (p_{i,j} - p_{i-1,j})\Delta y
\end{gathered}$$

Similarly, eqn (2) takes the following form:

$$A_P^v v_{i,j} + A_E^v v_{i+1,j} + A_W^v v_{i-1,j} + A_N^v v_{i,j+1} + A_S^v v_{i,j-1} = Q_P^v \tag{12}$$

where,

$$\begin{gathered}
A_E^v = \frac{1}{Re} - min(0, \dot{m}_e); A_W^v = \frac{1}{Re} - min(0, \dot{m}_w) \\
A_N^v = \frac{1}{Re} - min(0, \dot{m}_n); A_S^v = \frac{1}{Re} - min(0, \dot{m}_s) \\
A_P^v = -\frac{4}{Re} - (max(0, \dot{m}_e) + max(0, \dot{m}_w) + max(0, \dot{m}_n) + max(0, \dot{m}_s)) \\
Q_P^v = (p_{i,j} - p_{i-1,j})\Delta x
\end{gathered}$$

Substituting u, v and p in equations 11 and 12 by u^*, v^* and p^* gives us the prediction equations.

2. The correction equation can be written as:

$$A_P^u u'_{i,j} + \sum A_{nb} u'_{nb} = (p'_{i,j} - p'_{i-1,j})\Delta y \tag{13}$$

As u'_{nb} converges to 0, we have:

$$u'_{i,j} = \frac{\Delta y}{A_P^u} \tag{14}$$

Similarly,

$$v'_{i,j} = \frac{\Delta x}{A_P^v} \tag{15}$$

3. We have the following equations:

$$u_{i,j} = u^*_{i,j} + u'_{i,j}; v_{i,j} = v^*_{i,j} + v'_{i,j} \tag{16}$$

Using the continuity equation:

$$(u_{i+1,j} - u_{i,j}) + (v_{i,j+1} - v_{i,j}) = 0 \tag{17}$$

Substituting eqn (16) into the above equation gives:

$$\Rightarrow (u'_{i+1,j} - u'_{i,j}) + (v'_{i,j+1} - v'_{i,j}) = -(u^*_{i+1,j} - u^*_{i,j}) - (v^*_{i,j+1} - v^*_{i,j}) \tag{18}$$

$$\Rightarrow d^u_{i+1,j}(p'_{i+1,j} - p'_{i,j}) - d^u_{i,j}(p'_{i,j} - p'_{i-1,j}) + d^v_{i,j+1}(p'_{i,j+1} - p'_{i,j}) - d^v_{i,j}(p'_{i,j} - p'_{i,j-1}) = -(u^*_{i+1,j} - u^*_{i,j}) - (v^*_{i,j+1} - v^*_{i,j})$$

where, $d^u_{i,j} = \frac{\Delta y}{A^u_P}$; $d^v_{i,j} = \frac{\Delta y}{A^v_P}$
The final form of the equation is given by:

$$A^p_P p'_{i,j} + A^p_E p'_{i+1,j} + A^p_N p'_{i,j+1} + A^p_W p'_{i1,j} + A^p_S p'_{i,j1} = Q^p_P \tag{19}$$

where,
$A^p_E = d^u_{i+1,j}$; $A^p_W = d^u_{i,j}$; $A^p_N = d^v_{i,j+1}$; $A^p_S = d^v_{i,j}$
$A^p_P = -(d^u_{i+1,j} + d^u_{i,j} + d^v_{i,j+1} + d^v_{i,j})$; $Q^p_P = -(u^*_{i+1,j} - u^*_{i,j}) - (v^*_{i,j+1} - v^*_{i,j})$

4. (a) Since the residual is not decreasing significantly in magnitude, we have used a total number of 400 iterations for both cases in the outer loop for convergence. The underrelaxation factor being used is 0.001 for both the cases.

   (b) The residual is defined as the sum of the squares of the residual value of the momentum equations in the x and y directions as done in the code of lab 7 provided prof. Amitabh. The plot of the residual against number of iterations for $Re = 50$ is given below:

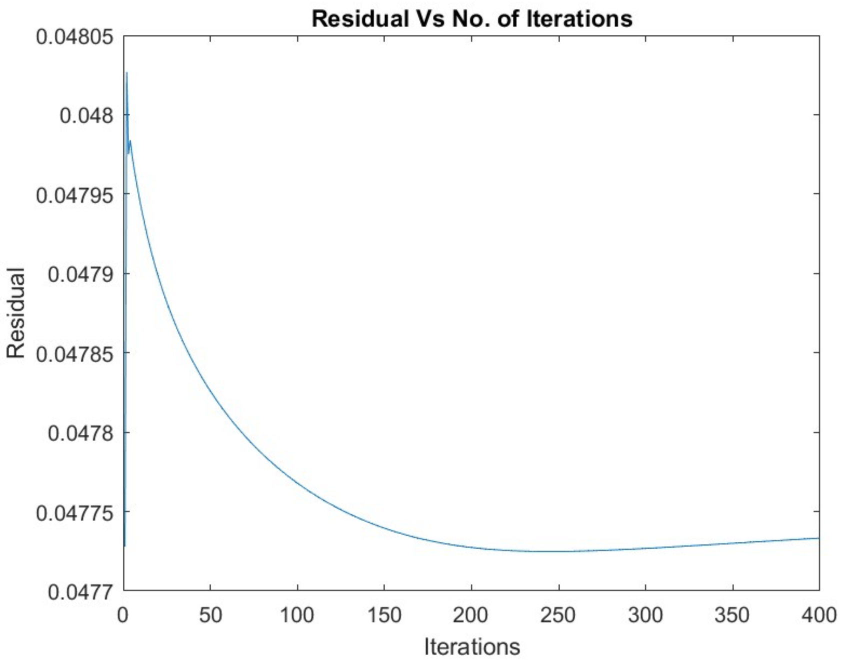

Figure 1: Residual Vs No. of Iterations

   (c) The required isocontours are given below:

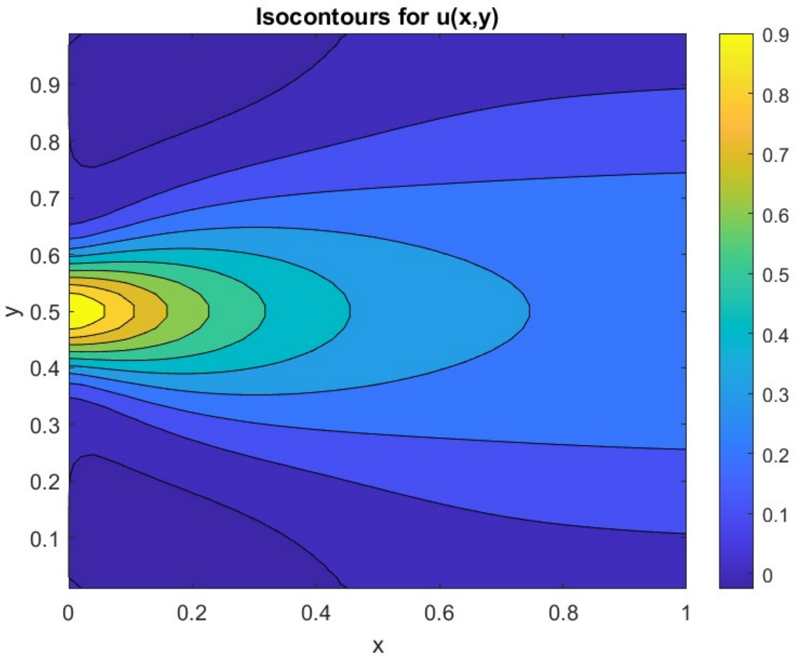

Figure 3: Isocontours for $u(x, y)$ $(Re = 50)$

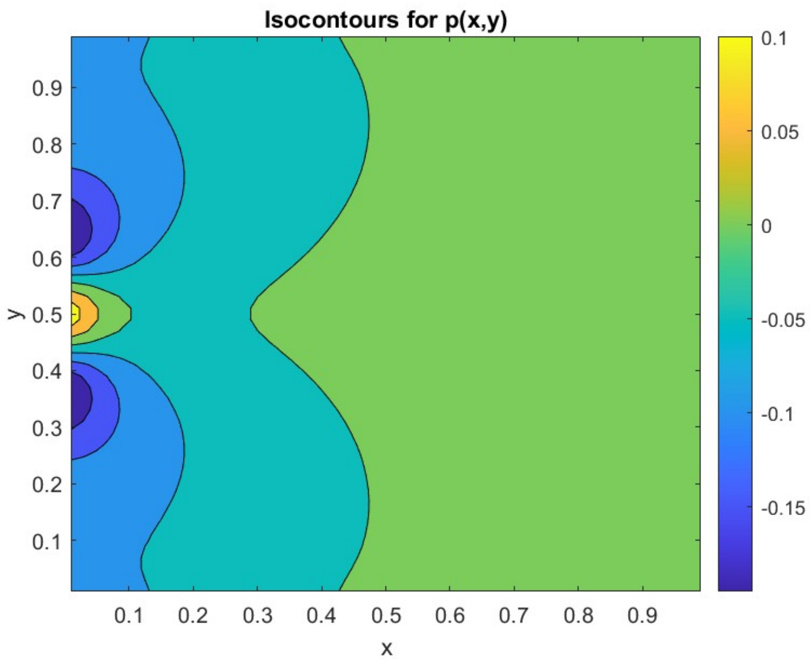

Figure 2: Isocontours for $p(x, y)$ $(Re = 50)$

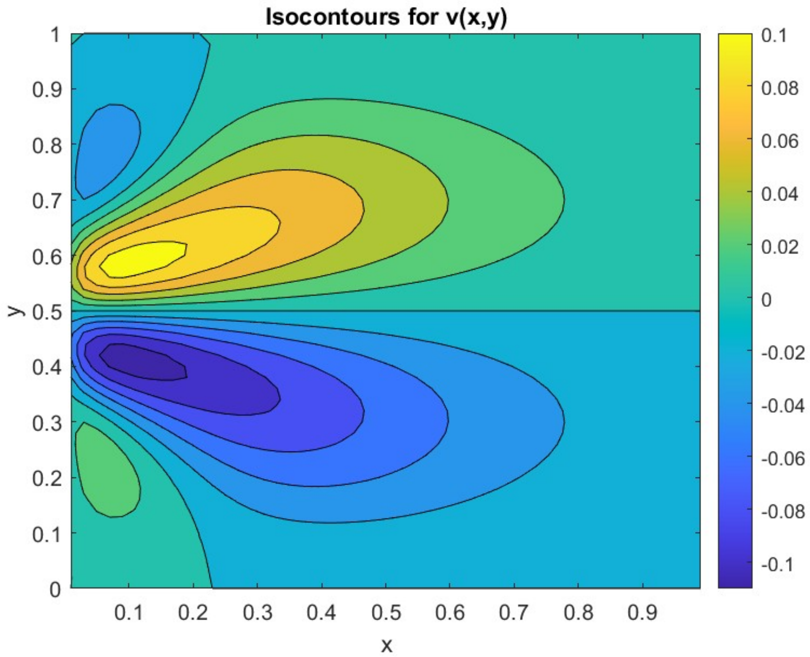

Figure 4: Isocontours for $v(x, y)$ $(Re = 50)$

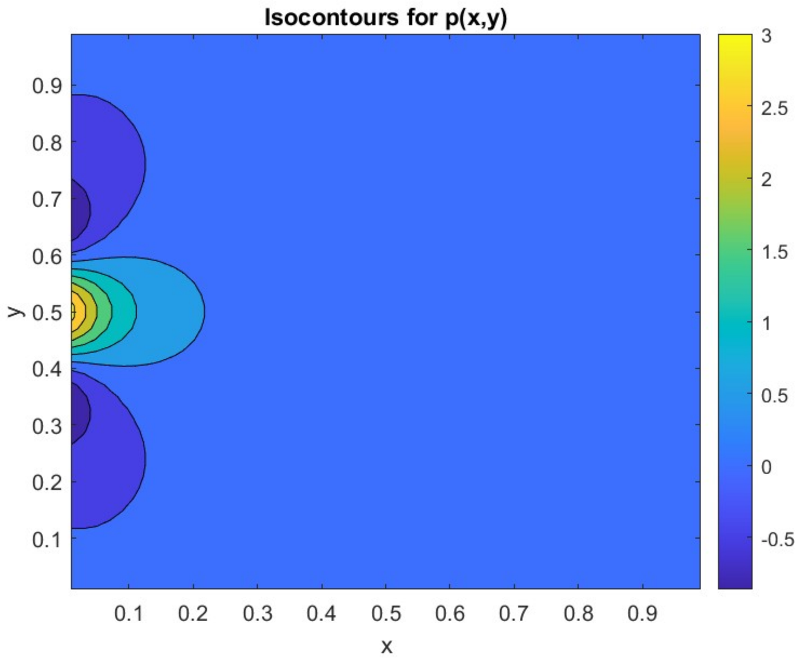

Figure 5: Isocontours for $p(x, y)$ $(Re = 1)$

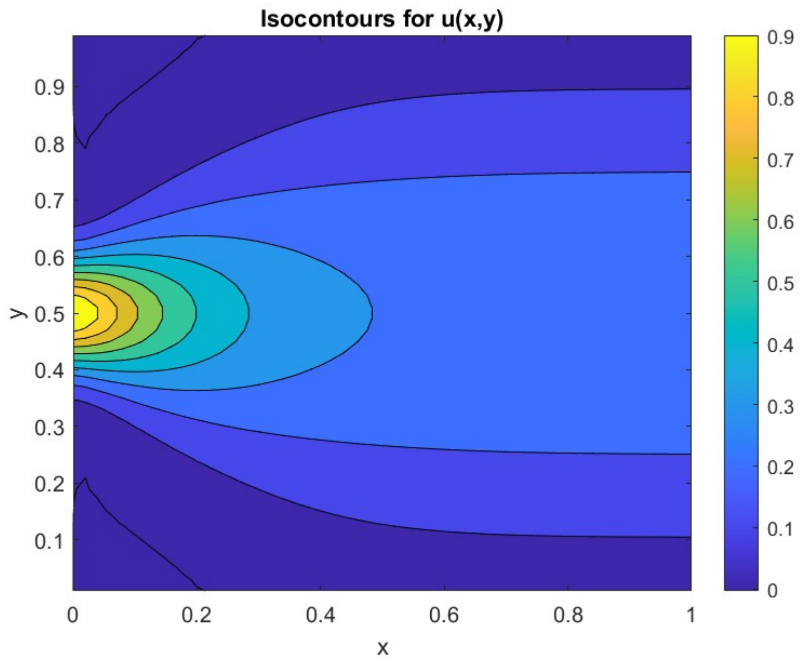

Figure 6: Isocontours for $u(x, y)$ $(Re = 1)$

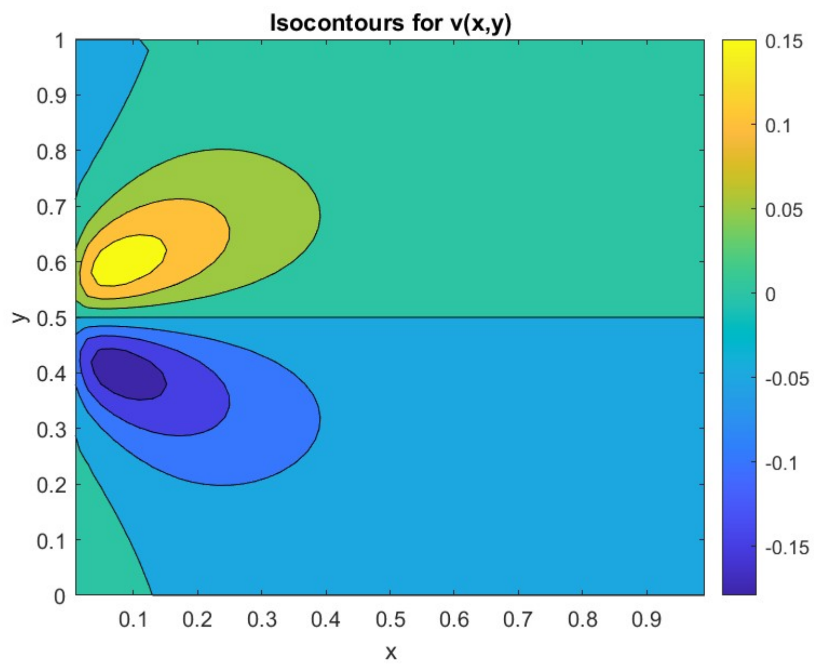

Figure 7: Isocontours for $v(x, y)$ $(Re = 1)$

(d) The required streamline plots are given below:

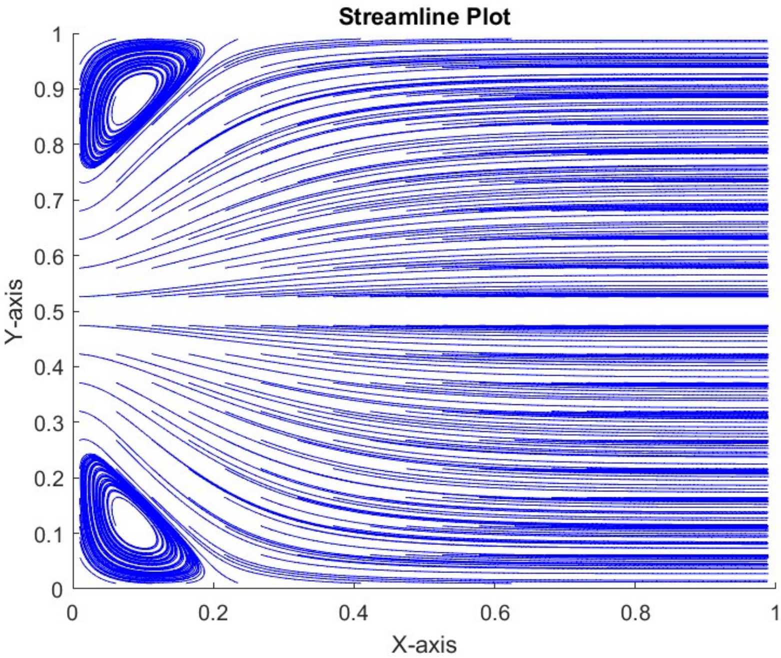

Figure 8: Streamlines for $\vec{u}$ ($Re = 1$)

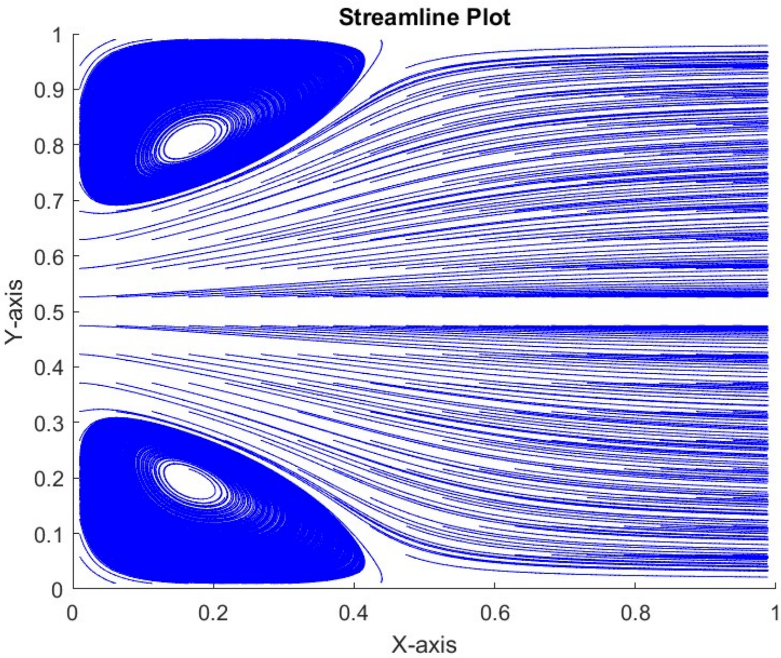

Figure 9: Streamlines for $\vec{u}$ ($Re = 50$)

(e) As observed from the plots, in low Reynolds Number flows, there is very little vortex formation. The flow is more or less smooth and steady. Streamlines are parallel and well-defined. The fluid moves in distinct layers with little mixing between them. The streamlines remain stable and organized.

Whereas, in the other case, the flow becomes chaotic and irregular. Streamlines are no longer parallel and become distorted. There's significant mixing between fluid layers, resulting in eddies, vortices, and turbulence.