APL321 : Introduction to Computational Fluid Dynamics

Indian Institute of Technology Delhi
Spring 2024
Prof. Amitabh Bhattacharya

Lab 4

Aditya Agrawal (2021AM10198)

1. The x coordinates corresponding to $N = 20$ are denoted by $X_i; i = 1, .., 21$ and those corresponding to $N = 80$ by $x_j; j = 1, .., 81$. We define $\Delta X_i = X_i - X_{i-1}$. For a constant stretching ration β, it is known that:

$$\Delta x_n = \Delta x_2 \beta_{80}^{(n-1)};\ \Delta X_N = \Delta X_2 \beta_{20}^{(N-1)}$$

From the given condition, the equations deduced are given by:

$$X_i = x_{4i-3}$$

$$\Delta X_i = \Delta x_{4i-6} + \Delta x_{4i-5} + \Delta x_{4i-4} + \Delta x_{4i-3} = \Delta x_{4i-6}(\tfrac{\beta_{80}^4 - 1}{\beta_{80} - 1}) = \Delta x_2(\tfrac{\beta_{80}^4 - 1}{\beta_{80} - 1})\beta_{80}^{4i-8}$$

$$\implies \frac{\Delta X_{i+1}}{\Delta X_i} = \frac{\Delta x_2(\frac{\beta_{80}^4 - 1}{\beta_{80} - 1})\beta^{4i-4}}{\Delta x_2(\frac{\beta_{80}^4 - 1}{\beta_{80} - 1})\beta^{4i-8}} = \beta_{80}^4$$

$$\implies \beta_{20} = \beta_{80}^4 \implies \beta_{80} = \beta_{20}^{(\frac{1}{4})} = 1.024$$

Similarly, it can be shown that $\beta_{320} = \beta_{20}^{\frac{1}{16}} = 1.006$

2. First of all, the Finite Volume Method (FVM) is used to compute the diffusion flux under the Central Differencing Scheme (CDS). The integral approximation for one face of the control volume is given by:

$$F_d^e = \int_{S_e} \Gamma \nabla \phi . \boldsymbol{n} dS \approx \left(\Gamma \frac{\partial \phi}{\partial x}\right)_e \Delta y = \frac{\Gamma \Delta y}{x_E - x_P}(\phi_E - \phi_P)$$

Thus, the entire flux integral on the surface of the control volume is computed as:

$$F_d = F_d^e + F_d^w + F_d^n + F_d^s$$

Upon substituting the values derived above, the integral takes the form:

$$F_d = A_E^d \phi_E + A_W^d \phi_W + A_P^d \phi_P + A_N^d \phi_N + A_S^d \phi_S$$

where, $A_E^d = \frac{\Gamma \Delta y}{x_E - x_P}, A_W^d = \frac{\Gamma \Delta y}{x_P - x_W}, A_N^d = \frac{\Gamma \Delta x}{y_N - y_P}, A_S^d = \frac{\Gamma \Delta x}{y_P - y_S}$ and $A_P^d = -(A_E^d + A_W^d + A_N^d + A_S^d)$.

Now, the convective integral flux for the interior nodes is determined using the CDS and first order UDS schemes given below:

$$F_c^e = \int_{S_e} \rho \phi \boldsymbol{v} . \boldsymbol{n} dS \approx \dot{m}_e \phi_e$$

where, $\dot{m}_e = (\rho u_x)_e \Delta y$

The approximation for the value of ϕ at the mid-point of each face is given by:

For **CDS:**

$$\phi_e = \lambda_e\phi_E + (1-\lambda_e)\phi_P$$

where, $\lambda_e = \frac{x_e - x_P}{x_E - x_P}$

For **UDS:**

$$\dot{m}_e\phi_e = \max(\dot{m}_e,\, 0)\phi_P + \min(\dot{m}_e, 0)\phi_E$$

The flux approximation is thus given by:

$$F_c = \sum_e \dot{m}_e\phi_e$$
$$\implies F_c = A^c_E\phi_E + A^c_W\phi_W + A^c_P\phi_P + A^c_N\phi_N + A^c_S\phi_S$$

where,
For **CDS:** $A^c_E = \dot{m}_e\lambda_e, A^c_W = \dot{m}_w\lambda_w, A^c_N = \dot{m}_n\lambda_n, A^c_S = \dot{m}_s\lambda_s$ and $A^c_P = -(A^c_E + A^c_W + A^c_N + A^c_S)$.

For **UDS:** $A^c_E = \min(\dot{m}_e,\, 0)$, $A^c_W = \min(\dot{m}_w,\, 0)$, $A^c_N = \min(\dot{m}_n,\, 0)$, $A^c_S = \min(\dot{m}_s,\, 0)$ and $A^c_P = -(A^c_E + A^c_W + A^c_N + A^c_S)$.

Note that we have used the mass conservation principle ($\sum_e \dot{m}_e = 0$) to derive the expression of A^c_P.

Now that the convective and diffusive integral terms have been evaluated, they are substituted into the original 2D convection diffusion equation to give the form:

$$A_E\phi_E + A_W\phi_W + A_P\phi_P + A_N\phi_N + A_S\phi_S = 0$$

where, $A_e = A^c_e - A^d_e$

Therefore, the final coefficients of the discretized convection diffusion equation are given by:

For **CDS:**

$$A_E = \dot{m}_e\lambda_e - \frac{\Gamma\Delta y}{x_E - x_P} \;;\; A_W = \dot{m}_w\lambda_w - \frac{\Gamma\Delta y}{x_P - x_W}$$
$$A_N = \dot{m}_n\lambda_n - \frac{\Gamma\Delta x}{y_N - y_P} \;;\; A_S = \dot{m}_s\lambda_s - \frac{\Gamma\Delta x}{y_P - y_S}$$
$$A_P = \frac{\Gamma\Delta y}{x_E - x_P} + \frac{\Gamma\Delta y}{x_P - x_W} + \frac{\Gamma\Delta x}{y_N - y_P} + \frac{\Gamma\Delta x}{y_P - y_S} - (\dot{m}_e\lambda_e + \dot{m}_w\lambda_w + \dot{m}_n\lambda_n + \dot{m}_s\lambda_s)$$

For **UDS:**

$$A_E = \min(\dot{m}_e,\, 0) - \frac{\Gamma\Delta y}{x_E - x_P} \;;\; A_W = \min(\dot{m}_w,\, 0) - \frac{\Gamma\Delta y}{x_P - x_W}$$
$$A_N = \min(\dot{m}_n,\, 0) - \frac{\Gamma\Delta x}{y_N - y_P} \;;\; A_S = \min(\dot{m}_s,\, 0) - \frac{\Gamma\Delta x}{y_P - y_S}$$
$$A_P = \frac{\Gamma\Delta y}{x_E - x_P} + \frac{\Gamma\Delta y}{x_P - x_W} + \frac{\Gamma\Delta x}{y_N - y_P} + \frac{\Gamma\Delta x}{y_P - y_S} - (\min(\dot{m}_e,\, 0) + \min(\dot{m}_w,\, 0) + \min(\dot{m}_n,\, 0) + \min(\dot{m}_s,\, 0))$$

3. The boundary nodes with Dirichlet boundary conditions, i.e. nodes (1,j) and (i,N), except the corner points are solved first.

   WEST BOUNDARY
   For the nodes (1,j), the boundary values is $\phi_w = 1 - y$. Their corresponding diffusive terms will change only on the west surface. The modification is given as follows:

$$F^w_d = \int_{S_w} \Gamma\nabla\phi.\boldsymbol{n}dS \approx -\left(\Gamma\frac{\partial\phi}{\partial x}\right)_w \Delta y = -\frac{\Gamma\Delta y}{x_P - x_w}(\phi_P - \phi_w(y))$$
$$\implies F_d = \frac{\Gamma\Delta y}{x_E - x_P}\phi_E + \frac{\Gamma\Delta x}{y_N - y_P}\phi_N + \frac{\Gamma\Delta x}{y_P - y_S}\phi_S + \frac{\Gamma\Delta y}{x_P - x_w}\phi(0,y) - \left(\frac{\Gamma\Delta y}{x_E - x_P} + \frac{\Gamma\Delta x}{y_N - y_P} + \frac{\Gamma\Delta x}{y_P - y_S} + \frac{\Gamma\Delta y}{x_P - x_w}\right)\phi_P$$

Therefore, changes are $A_W^d = 0$; $A_P^d = -(A_E^d + A_N^d + A_S^d + \frac{\Gamma\Delta y}{x_P - x_w})$; $Q_P^d = \frac{\Gamma\Delta y}{x_P - x_w}\phi(0, y)$

For the convective term, since we already have the value of ϕ_w at the boundary and $\dot{m}_w = 0$ due to the wall, we do not need interpolation and thus,

$$F_c^w = \int_{S_w} \rho\phi \boldsymbol{v}.\boldsymbol{n} dS \approx \dot{m}_w \phi_w(y) = 0$$
$$\implies F_c = \dot{m}_e\phi_e + \dot{m}_n\phi_n + \dot{m}_s\phi_s$$

After substituting ϕ_e, the changes are $A_W^c = 0$; $A_P^c = -(A_E^c + A_N^c + A_S^c)$
Thus, the final modifications needed are: $A_W = 0$, $A_P \longrightarrow A_P + A_W + \frac{\Gamma\Delta y}{x_P - x_w}$, $Q_P \longrightarrow Q_P + \frac{\Gamma\Delta y}{x_P - x_w}\phi(0, y)$

NORTH BOUNDARY

For the nodes (i,N), the boundary value of $\phi_n = 0$ is already known. Their corresponding diffusive terms will change only on the north surface. The modification is given as follows:

$$F_d^n = \int_{S_n} \Gamma\nabla\phi.\boldsymbol{n} dS \approx -\left(\Gamma\frac{\partial\phi}{\partial y}\right)_n \Delta x = -\frac{\Gamma\Delta x}{y_n - y_P}(\phi_P - \phi_n) = -\frac{\Gamma\Delta x}{y_n - y_P}\phi_P$$
$$\implies F_d = \frac{\Gamma\Delta y}{x_E - x_P}\phi_E + \frac{\Gamma\Delta x}{y_P - y_S}\phi_S + \frac{\Gamma\Delta y}{x_P - x_W}\phi_W - \left(\frac{\Gamma\Delta y}{x_E - x_P} + \frac{\Gamma\Delta x}{y_P - y_S} + \frac{\Gamma\Delta y}{x_P - x_W} + \frac{\Gamma\Delta x}{y_n - y_P}\right)\phi_P$$

Therefore, changes are $A_N^d = 0$; $A_P^d = -(A_E^d + A_W^d + A_S^d)$

For the convective term, since we already have the value of $\phi_n = 0$ at the boundary, we do not need interpolation and thus,

$$F_c^n = \int_{S_n} \rho\phi \boldsymbol{v}.\boldsymbol{n} dS \approx \dot{m}_n \phi_n = 0$$
$$\implies F_c = \dot{m}_e\phi_e + \dot{m}_w\phi_w + \dot{m}_s\phi_s$$

After substituting ϕ_e, the changes are $A_N^c = 0$; $A_P^c = -(A_E^c + A_W^c + A_S^c)$
Thus, the final modifications needed are: $A_N = 0$, $A_P \longrightarrow A_P + A_N + \frac{\Gamma\Delta x}{y_n - y_P}$

Now, the boundary nodes with Neumann boundary conditions, i.e., nodes (N,j) and (i,1), except the corner points are solved.

SOUTH BOUNDARY

For the nodes (i,1), the derivative of ϕ w.r.t. y is given to be 0. Their corresponding diffusive terms will change only on the south surface. The modification is given as follows:

$$F_d^s = \int_{S_s} \Gamma\nabla\phi.\boldsymbol{n} dS \approx -\left(\Gamma\frac{\partial\phi}{\partial y}\right)_s \Delta x = -\Gamma\Delta x \frac{\partial\phi}{\partial y}\Big|_{(x,0)} = 0$$
$$\implies F_d = \frac{\Gamma\Delta y}{x_E - x_P}\phi_E + \frac{\Gamma\Delta x}{y_N - y_P}\phi_N + \frac{\Gamma\Delta y}{x_P - x_W}\phi_W - \left(\frac{\Gamma\Delta y}{x_E - x_P} + \frac{\Gamma\Delta x}{y_N - y_P} + \frac{\Gamma\Delta y}{x_P - x_W}\right)\phi_P$$

Therefore, changes are $A_S^d = 0$; $A_P^d = -(A_E^d + A_N^d + A_W^d)$

For the convective term, the gradient at the boundary is zero and $\dot{m}_s = 0$ for $y = 0$ and thus, the boundary value of ϕ is equal to that of the CV centre. Hence,

$$F_c^s = \int_{S_s} \rho\phi \boldsymbol{v}.\boldsymbol{n} dS \approx \dot{m}_s \phi_P = 0$$
$$\implies F_c = \dot{m}_e\phi_e + \dot{m}_w\phi_w + \dot{m}_n\phi_n$$

After substituting ϕ_e, the changes are $A_S^c = 0$; $A_P^c = -(A_E^c + A_N^c + A_W^c)$
Thus, the final modifications needed are: $A_S = 0$, $A_P \longrightarrow A_P + A_S$

EAST BOUNDARY
For the nodes (N,j), the boundary value of $\frac{\partial \phi}{\partial x}\big|_e = 0$ is already known. Their corresponding diffusive terms will change only on the east surface. The modification is given as follows:

$$F_d^e = \int_{S_e} \Gamma \nabla \phi.\boldsymbol{n} dS \approx \left(\Gamma \frac{\partial \phi}{\partial x}\right)_e \Delta y = \Gamma \Delta y \frac{\partial \phi}{\partial x}\Big|_{(1,y)} = 0$$
$$\implies F_d = \frac{\Gamma \Delta x}{y_N - y_P}\phi_N + \frac{\Gamma \Delta x}{y_P - y_S}\phi_S + \frac{\Gamma \Delta y}{x_P - x_W}\phi_W - \left(\frac{\Gamma \Delta x}{y_N - y_P} + \frac{\Gamma \Delta x}{y_P - y_S} + \frac{\Gamma \Delta y}{x_P - x_W}\right)\phi_P$$

Therefore, changes are $A_E^d = 0$; $A_P^d = -(A_S^d + A_N^d + A_W^d)$

For the convective term, the gradient at the boundary is zero and thus, the boundary value of ϕ is equal to that of the CV centre. Hence,

$$F_c^e = \int_{S_e} \rho \phi \boldsymbol{v}.\boldsymbol{n} dS \approx \dot{m}_e \phi_P$$
$$\implies F_c = \dot{m}_e \phi_P + \dot{m}_w \phi_w + \dot{m}_n \phi_n + \dot{m}_s \phi_s$$

After substituting ϕ_e and applying the mass conservation equation, the changes are $A_E^c = 0$; $A_P^c = -(A_S^c + A_N^c + A_W^c)$
Thus, the final modifications needed are: $A_E = 0$, $A_P \longrightarrow A_P + A_E$

The corner nodes will have changes only in the two directions which are adjacent to the boundary and thus the coefficient of the other two directions remain the same. Hence, the modifications required are:

- (1,1) : $A_W = 0; A_S = 0; A_P \longrightarrow A_P + A_W + \frac{\Gamma \Delta y}{x_P - x_w} + A_S; Q_P \longrightarrow Q_P + \frac{\Gamma \Delta y}{x_P - x_w}\phi(0,0)$
- (1,N) : $A_W = 0; A_N = 0; A_P \longrightarrow A_P + A_N + A_W + \frac{\Gamma \Delta y}{x_P - x_w} + \frac{\Gamma \Delta x}{y_n - y_P}; Q_P \longrightarrow Q_P + \frac{\Gamma \Delta y}{x_P - x_w}\phi(0,1)$
- (N,1) : $A_S = 0; A_E = 0; A_P \longrightarrow A_P + A_S + A_E$
- (N,N) : $A_E = 0; A_N = 0; A_P \longrightarrow A_P + A_N + A_E + \frac{\Gamma \Delta x}{y_n - y_P}$

4. Table for optimal ω and corresponding number of iterations is given below:

Method	**N**	ω_{opt}	**Iterations**
Method 2	20	1.51	49
Method 2	80	1.815	222
Method 2	320	1.9299	1281
Method 1	20	0.6	367
Method 1	80	1.657	493
Method 1	320	1.913	1614

Table 1: Optimal ω and number of iterations for different cases

Isocontour plots for $\phi(x, y)$ for $N = 20, 80$ and 320 for both Method 1 (CDS) and Method 2 (UDS) are given below:

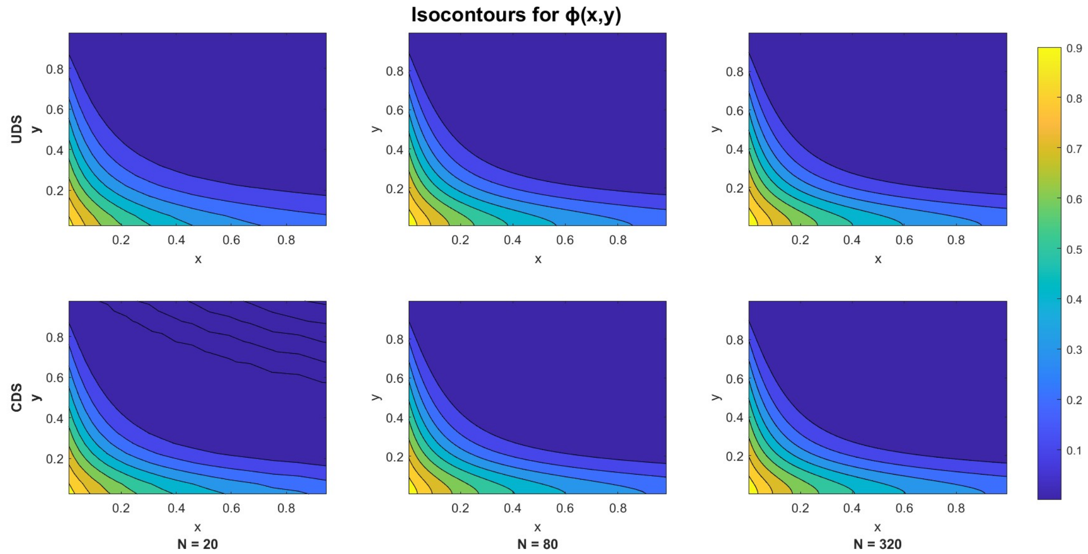

Figure 1: Isocontours of $\phi(x, y)$ for Method 1 and 2

5. ϕ vs x at $y = 0.5$ for methods 1 and 2:

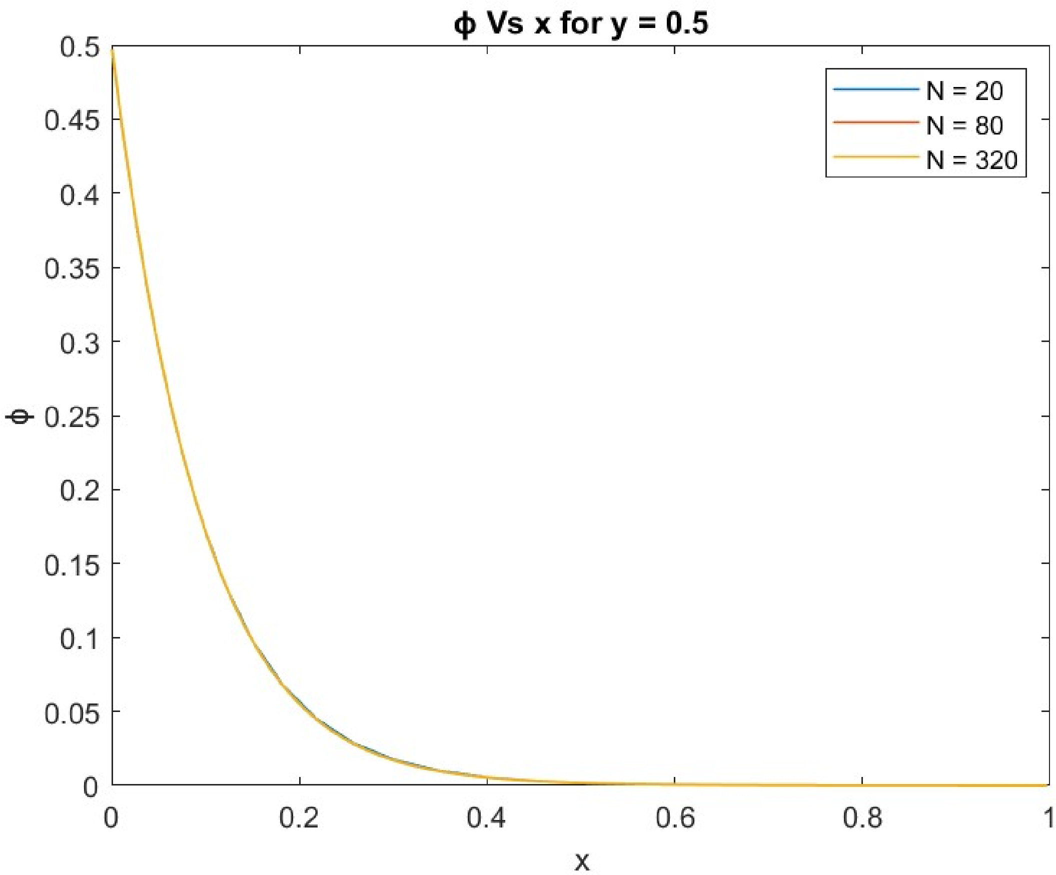

Figure 2: Method 1 (CDS)

→ The plot does not show much variation with grid size since all the three curves overlap, with minute oscillations in the $N = 20$ case. This may be because of coarser grids in case of $N = 20$.

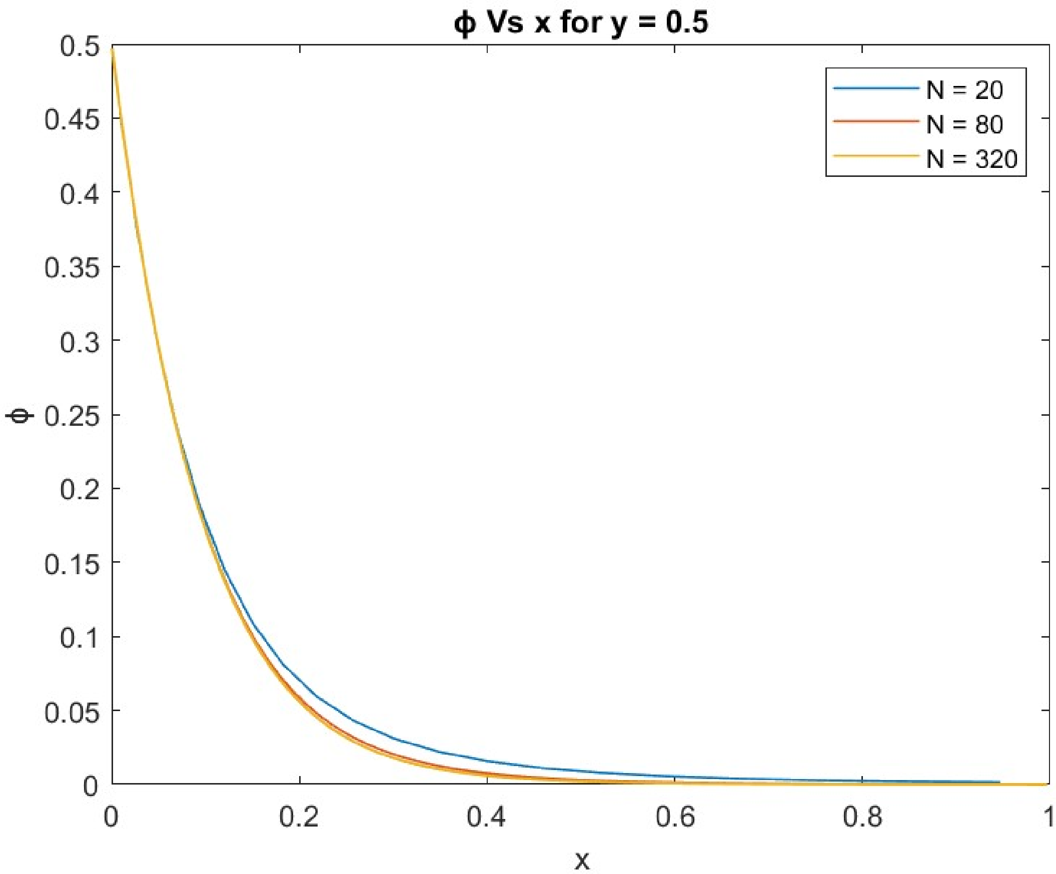

Figure 3: Method 2 (UDS)

→ The plot differs significantly for $N = 20$. This is due to larger errors in the approximation of the values of the mid-point of face of Control Volumes as proposed by the UDS scheme as compared to that of the CDS scheme. These large errors get amplified by the coarser grids in case of $N = 20$ to produce such behaviors.