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APL410

ASSIGNMENT - 1

(a) The original ~~eq~~ governing equation for the problem is the static linear momentum balance eq^n:

$$\sigma_{ij,j} + b_i = 0 \quad , \partial B_t$$

Multiplying by a test function δv_i,

$$\Rightarrow \delta v_i \sigma_{ij,j} + \delta v_i b_i = 0$$

$$\Rightarrow \int_B \delta v_i \sigma_{ij,j} \, dV + \int_B \delta v_i b_i \, dV = 0$$

$$\Rightarrow \int_B (\delta v_i \sigma_{ij})_{,j} \, dV - \int_B \delta v_{i,j} \sigma_{ij} \, dV + \int_B \delta v_i b_i \, dV = 0$$

$$\Rightarrow \int_{\partial B} \delta v_i \sigma_{ij} n_j \, dS + \int_B \delta v_i b_i \, dV = \int_B \delta v_{i,j} \sigma_{ij} \, dV$$

$$\Rightarrow \int_B \delta v_{i,j} \sigma_{ij} \, dV = \int_B \delta v_i b_i \, dV + \int_{\partial B_t} \delta v_i t_i \, dS \qquad (\delta v_i = 0 \text{ at } \partial B_u)$$

→ This is the weak form of the governing eq^n.

Boundary conditions:

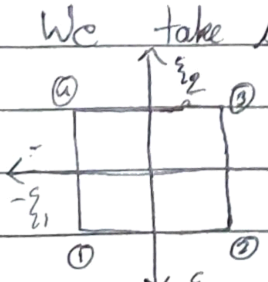
(i) $u_2(x, y=0) = 0$ (ii) $u_1(x \neq 0, y=0)$ is free

(iii) $u_1(x=0, y=0) = 0$ (iv) $\underline{t}(x=(0,L), y) = [0, 0]^T$

(v) $\underline{t}(x, y=L) = [0, ct]^T$

Further simplification of the Weak form:

We take bilinear shape functions for the basis functions of our square element



ξ_1 & ξ_2 are the isoparametric co-ordinates of the master element ~~for~~ of our square elements.

Basis Shape fn^s: (i) $N^1(\xi_1, \xi_2) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$ (ii) $N^2(\xi_1, \xi_2) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$

(iii) $N^3(\xi_1, \xi_2) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$ (iv) $N^4(\xi_1, \xi_2) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$

We know,

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = C_{ijkl}\frac{\partial u_k}{\partial x_l} = C_{ijkl}\,u_{k,l}$$

$$u_i(\underline{x}) = \sum_{a=1}^{N} N^a(\underline{x})\,u_i^a \quad , \quad N^a : \text{Shape fuc. with } N^a(\underline{x}_j) = \delta_{ja},\ j \in \text{nodes}$$
u_i^a : Disp. of node a

$$\delta v_i(\underline{x}) = \sum_{a=1}^{N} N^a(\underline{x})\,\delta v_i^a ,$$

$\therefore$ Weak form of eq^n: $\int_B (N^a \delta v_i^a)_{,j}\, C_{ijkl}\,(N^b u_k^b)_{,l}\, dV = \int_B N^a \delta v_i^a b_i\, dV + \int_{\partial B_t} N^a \delta v_i^a t_i$

$\Rightarrow (K_{aibk} u_k^b - F_i^a)\,\delta v_i^a = 0 \quad \Rightarrow K_{aibk} u_k^b = F_i^a \quad \forall \{a,i\}$ s.t. x_k^a not on ∂

where, $K_{aibk} = \int_B C_{ijkl} N^a_{,j} N^b_{,l}\, dV = \int C_{ijkl} \frac{\partial N^a}{\partial x_j}\frac{\partial N^b}{\partial x_l}\, dV$

$$F_i^a = \int_B b_i N^a\, dV + \int_{\partial B_t} t_i N^a\, dA$$

Note: $\underline{b}$: body force is $\underline{0}$ in our problem.

For numerical integration, we use the 2-point Gaussian Quadrature scheme for a bilinear element:

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi_1,\xi_2)\, d\xi_1\, d\xi_2 \approx \sum_{i=1}^{2}\sum_{j=1}^{2} \omega_i \omega_j f(\xi_i,\xi_j)$$

where, ω_i, ω_j are weights for the i^{th} & j^{th} Gauss pts.
ξ_i, ξ_j are the Gauss points.

For 2x2 integration, $\xi_i, \xi_j = \pm \frac{1}{\sqrt{3}}$, $\omega_i, \omega_j = 1$

(b) The expressions for the elemental stiffness matrix ~~K~~ K_{aibk} & force vector ~~f_i~~ F_i^a were derived in part (a), which are given by:

$$K_{aibk} = \int_B C_{ijkl}\, N^a_{,j}\, N^b_{,l}\, dV = \int C_{ijkl} \frac{\partial N^a}{\partial x_j} \frac{\partial N^b}{\partial x_l}\, dV$$

$$F_i^a = \int_B b_i N^a\, dV + \int_{\partial B_t} t_i N^a\, dS = \int_{\partial B_t} t_i N^a\, dS \quad \{\because b_i = 0\}$$

The Elemental Stiffness Matrix for the Master element with local nodes ① – ④:

$$K^e_{\hat{a} i \hat{b} k} = \int C_{ijkl} \frac{\partial N^{\hat{a}}(x)}{\partial x_j} \frac{\partial N^{\hat{b}}(x)}{\partial x_l}\, dV$$

$$\therefore [K^e] = \begin{bmatrix} K^e_{1111} & K^e_{1112} & K^e_{1121} & \cdots & K^e_{1142} \\ K^e_{1211} & K^e_{1212} & \cdots & \cdots & K^e_{1242} \\ \vdots & \vdots & & \ddots & \\ K^e_{4211} & K^e_{4212} & \cdots & \cdots & K^e_{4242} \end{bmatrix}$$

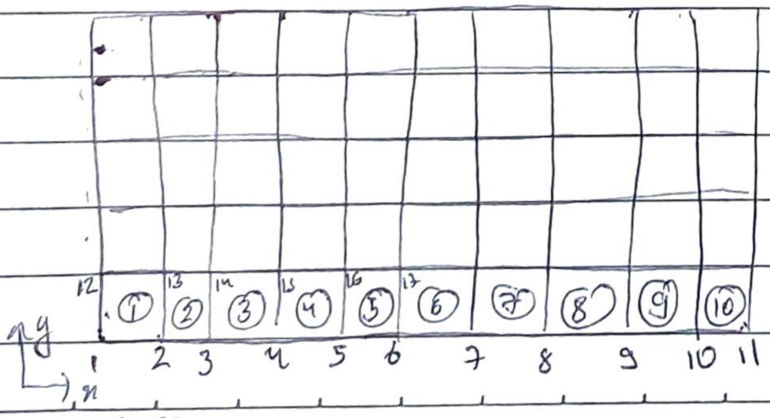
$\therefore [K^e]$ has $= 4 \times 2 = 8$ rows & columns $\Rightarrow$ Size $(K^e) = 8 \times 8$

The Elemental Force Vector is given by:

$$f_i^{\hat{a},e} = \int_{\partial B_t^e} t_i(\underline{x}) N^{\hat{a}}(\underline{x})\, dS$$

$$f^e = \left[f_1^{1,e} \; f_2^{1,e} \; f_1^{2,e} \; f_2^{2,e} \; f_1^{3,e} \; f_2^{3,e} \; f_1^{4,e} \; f_2^{4,e} \right]^T$$

$\therefore$ Size of $\hat{f}^e = (4 \times 2) \times 1 = 8 \times 1$



For element at cell (i,j):
Element No. $= 10(j-1) + i \quad \forall\; i \in (1,10),\; j \in (1,5)$

For node at (i,j):
Node no. $= 10(j-1) + i \quad \forall\; i \in (1,11),\; j \in (1,6)$

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(c) Using the elemental stiffness matrix & force vector, we compute the global stiffness matrix & force ~~...~~ vectors:

$\cancel{K(i,j) = \sum K^e}$ $K_{aibk} = \sum K^e_{\hat{a}i\hat{b}k}$; $f_i^a = \sum f_i^{\hat{a}_i e}$ (Summed over all elements having a node common)

Solving the system $Ku = f$ gives the displacement at all nodes.

(d) We computed the velocity gradient : $\nabla u(i,j) = \sum\sum \frac{\partial N^{\hat{b}}}{\partial \xi_l} \cdot \frac{\partial \xi_l}{\partial x_j} u_i^{\hat{b}}$

Strain : $\varepsilon_{ij} = \frac{1}{2}\left(\nabla u(i,j) + \nabla u(j,i)\right)$

Stress : $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

We computed these values over the quadrature points & found the average over these pts.

Finally the nodal stress, strain values were taken as the average of the previous ~~of~~ values over all elements with that common node.

(e) We just compute the average of displacement, stress & strain over the top edge of the plate.