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APL747

Homework - 2

Q.1. 1) $f_Z(z) = \alpha \beta z^{\beta-1} e^{-\alpha z^\beta}$; $z \geq 0$

For $z < 0$: $F_Z(z) = P(Z < z) = 0$

For $z \geq 0$: $F_Z(z) = \int_0^z \alpha \beta z^{\beta-1} e^{-\alpha z^\beta} dz$

Let $t = \alpha z^\beta \Rightarrow dt = \alpha \beta z^{\beta-1} dz$; $t' = \alpha z^\beta \Rightarrow z = (t/\alpha)^{1/\beta} = t'$

$\therefore F_Z(z) = \int_0^{t'} e^{-t} dt = -e^{-\alpha z^\beta}\Big|_0^z = 1 - e^{-\alpha z^\beta}$

Now, $y = 1 - e^{-\alpha z^\beta}$

for inverse (F^{-1}): $z = 1 - e^{-\alpha y^\beta} \Rightarrow e^{-\alpha y^\beta} = 1 - z \Rightarrow -\alpha y^\beta = \ln(1-z)$

$\therefore y = \left[\ln \frac{1}{(1-z)^{1/\alpha}}\right]^{1/\beta} \Rightarrow F^{-1}(u^{(i)}) = \left[\ln \frac{1}{(1-u^{(i)})^{1/\alpha}}\right]^{1/\beta} = \left[\frac{-1}{\alpha} \ln(1-u^{(i)})\right]^{1/\beta}$

$\therefore x^{(i)} \sim F^{-1}(u^{(i)})$

2) We have already been ~~pro~~ given the cdf of the distribution.

$F_Z(z) = e^{-e^{-\alpha(z-\mu)}}$; $-\infty < z < \infty$

Now,

$z = e^{-e^{-\alpha(y-\mu)}} \Rightarrow \ln z = -e^{-\alpha(y-\mu)} \Rightarrow \ln(-\ln z) = -\alpha(y-\mu)$

$\Rightarrow y = \mu - \frac{1}{\alpha} \ln(-\ln z)$

$\therefore F^{-1}(u^{(i)}) = \mu - \frac{1}{\alpha} \ln(-\ln u^{(i)})$

$\therefore x^{(i)} \sim F^{-1}(u^{(i)})$

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3) $f_Z(z) = \frac{1}{B(p,q)} z^{p-1}(1-z)^{q-1}$; $0 \le z \le 1$

Now,

$$F(z) = \int_0^z \frac{1}{B(p,q)} z^{p-1}(1-z)^{q-1} dz$$

∴ ~~The~~ exact analytical solution of the above integral is very difficult to compute.

$$\therefore \; u^{(i)} B(p,q) = \int_0^z z^{p-1}(1-z)^{q-1} dz$$

~~We use numeric~~

Thus, we use the inbuilt inverse CDF function of the incomplete Beta distribution.

Q.6. According to the perturbation theory,

~~Method~~ ①: Laplace approximation of the vector $\underline{u}$

$\underline{u} = [c \;\; k \;\; \alpha]^T$

Since the three parameters are independent, their mutual covariance is 0.

For a LogNormal distribution $(\mu, \sigma) = e^{\mu - \sigma^2} = u_0$

Taking Taylor Series expansion of $\log(p(\underline{u}))$ about u_0:

$$\log(p(u)) = \log(p(u_0)) + \cancel{\frac{d}{du}\log(p(u))\Big|_{u_0}}^{\,0} (u-u_0) + \frac{1}{2}\frac{d^2 \log(p(u))}{du^2}\Big|_{u_0} (u-u_0)^2$$

u_0 is mode

$$\therefore \log(p(u)) = \log(p(u_0)) + \frac{1}{2}\frac{d^2}{du^2}\log(p(u))\Big|_{u_0}$$

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$$\therefore\ p(u) \propto \exp\left[\frac{d^2}{du^2}\log p(u)\Big|_{u_0}\frac{(u-u_0)^2}{2}\right]$$

For log Normal,

$$p(u) = \frac{1}{u\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(\ln u-\mu)^2}{2\sigma^2}\right]$$

$$\therefore\ \log(p(u)) = -\log u - \frac{1}{2}\log(2\pi\sigma^2) - \frac{(\ln u-\mu)^2}{2\sigma^2}$$

$$\frac{d}{du}\log(p(u)) = \left[-\frac{1}{u} - \frac{(\ln u-\mu)}{\sigma^2}\left(\frac{1}{u}\right)\right] = -\frac{1}{u}\left[+1+\frac{\ln u-\mu}{\sigma^2}\right]$$

$$\therefore\ \cancel{\frac{(u-u_0)^2}{2}\frac{d^2}{du^2}\log(p(u)) = \frac{-1}{2}}$$

$$\frac{d^2}{du^2}\log(p(u))\Big|_{u_0} = \cancel{\frac{-1}{u^2}\left(1+\frac{1}{\sigma^2}\right)}\ \frac{1}{u^2}\left(1+\frac{\ln u-\mu}{\sigma^2}\right) - \frac{1}{u}\left[\frac{1}{u\sigma^2}\right]$$

$$= \frac{1}{u_0^2}\left[1+\frac{\ln u_0-\mu}{\sigma^2} - \frac{1}{\sigma^2}\right]$$

For $u_0 = e^{\mu-\sigma^2}$:
$$\frac{d^2}{du^2}\log p(u)\Big|_{u_0} = \frac{1}{e^{2(\mu-\sigma^2)}\sigma^2}\left[\cancel{\sigma^2}-1-\cancel{\mu}+\cancel{\mu}-\cancel{\sigma^2}\right]$$

$$= \frac{-1}{e^{2(\mu-\sigma^2)}\sigma^2}$$

$$\therefore\ \mathrm{Var}[u] = -\left(\frac{-1}{e^{2(\mu-\sigma^2)}\sigma^2}\right)^{-1} = \sigma^2 e^{2(\mu-\sigma^2)}$$

$$\therefore\ p(\underline{u}) = \mathcal{N}(\underline{u}_0, \underline{\underline{\Sigma}})$$

where,
$$\underline{u}_0 = \left[e^{\mu_c-\sigma_c^2}\ \ e^{\mu_k-\sigma_k^2}\ \ e^{\mu_\alpha-\sigma_\alpha^2}\right]^T$$

$$\underline{\underline{\Sigma}} = \begin{bmatrix} \sigma_c^2 e^{2(\mu_c-\sigma_c^2)} & 0 & 0 \\ 0 & \sigma_k^2 e^{2(\mu_k-\sigma_k^2)} & 0 \\ 0 & 0 & \sigma_\alpha^2 e^{2(\mu_\alpha-\sigma_\alpha^2)} \end{bmatrix}$$

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For the given system:

Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

$$\therefore \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -cy_2 - ky_1 - \alpha y_1^3 + A_1 \sin(\pi t) A_2 \sin(\pi t) \end{bmatrix}$$

Initial Condⁿs: $y_1(0) = 0.1$; $y_2(0) = 0$

Now,

$$\frac{d}{dt}\left(\frac{\partial y_1}{\partial u_j}\right) = \frac{\partial}{\partial u_j}\left(\frac{dy_1}{dt}\right) = \frac{\partial (y_2)}{\partial u_j} = \frac{\partial y_2}{\partial u_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial y_1}{\partial u_j}\right) = \frac{\partial y_2}{\partial u_j} \quad \text{—— 3 eqⁿs :} \quad j = 1, 2, 3$$

$$\frac{d}{dt}\left(\frac{\partial y_2}{\partial c}\right) = \frac{\partial}{\partial c}\left(\frac{dy_2}{dt}\right) = \frac{\partial}{\partial c}\left(-cy_2 - ky_1 - \alpha y_1^3 + A_1 A_2 \sin(\pi t)\sin(2\pi t)\right)$$

$$= -y_2 - c\frac{\partial y_2}{\partial c} - k\frac{\partial y_1}{\partial c} - 3\alpha y_1^2 \frac{\partial y_1}{\partial c}$$

Ily. $\frac{d}{dt}\left(\frac{\partial y_2}{\partial k}\right) = -c\frac{\partial y_2}{\partial k} - y_1 - k\frac{\partial y_1}{\partial k} - 3\alpha y_1^2 \frac{\partial y_1}{\partial k}$

$$\frac{d}{dt}\left(\frac{\partial y_2}{\partial \alpha}\right) = -c\frac{\partial y_2}{\partial \alpha} - k\frac{\partial y_1}{\partial \alpha} - y_1^3 - 3\alpha y_1^2 \frac{\partial y_1}{\partial \alpha}$$

For initial value: $\frac{d}{dt}\left(\frac{\partial y_1(0)}{\partial u_j}\right) = \frac{d}{dt}\left(\frac{\partial y_2(0)}{\partial u_j}\right) = 0 \quad j = 1, 2, 3$