Homework 2
Propagating uncertainty using sampling and perturbation methods

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APL 747: Uncertainty Quantification and Propagation
February 18, 2024
Submission due on Marh 04, 2024

Instructions: (I) You are allowed to discuss in a group; however, you must submit your own handwritten homework copy (no computer-typed submission will be accepted). Further, copying homework is forbidden. If found copied, the homework submission will be considered invalid for all the students involved. (II) ***Write all the steps***, including your reasoning and the formulae you referred to, if any. **In the absence of sufficient reasoning, step marks will not be awarded**. (III) For code submission, make sure you ***fix a random seed*** at the beginning of your code and provide comments in between the lines. (IV) Submit all the files in a single zip folder named as: '**FirstName_EntryNumber.zip**'.

Question 1. It has been shown that random variables from any desired distribution can be generated using the uniform random variables through inverse transformation. Therefore, using the concept, derive the expressions for generating random deviates of Z having the following distributions: [Marks = (3+3+3) for derivation, (2+2+2) for code, Total = 15]

1) Weibull distribution: $f_Z(z) = \alpha\beta z^{\beta-1}e^{-\alpha z^{\beta}};\ z \geq 0$,
2) Type I Extremal (largest) distribution: $f_Z(z) = \exp[-\exp(-\alpha(z-\mu))];\ -\infty \leq z \leq \infty$,
3) Beta distribution: $f_Z(z) = \frac{1}{\mathrm{B}(p,q)} z^{p-1}(1-z)^{q-1};\ 0 \leq z \leq 1$,

where $\mathrm{B}(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and Γ is the gamma function. With the derived expressions, write simple computer codes to draw 1000 samples from each distribution. For drawing samples, consider α=1 and β=1 for Weibull distribution, α=0.25 and μ=3 for Type I extremal distribution, and p=q=0.5 for Beta distribution. Compare the results with the standard in-built library functions (you may use random number generators from 'numpy', 'scipy', etc.)

Question 2. Consider the limit-state equation: [Marks = 5 for derivation + 5 for code, Total = 10]

$$F = Z_1 Z_2 - Z_3,$$

where Z_1 and Z_2 correlated random variables. Further, note that all Z_i are normal random variables. The mean $\boldsymbol{\mu_Z} \in \mathbb{R}^3$ and the covariance matrix $\boldsymbol{\Sigma_Z} \in \mathbb{R}^{3\times 3}$ are given as,

$$\boldsymbol{\mu_Z} = \begin{bmatrix} 1.2220 \\ 1.0500 \\ 0.6200 \end{bmatrix},\ \&\ \boldsymbol{\Sigma_Z} = \begin{bmatrix} 0.0222 & 0.0111 & 0 \\ 0.0111 & 0.0110 & 0 \\ 0 & 0 & 0.0308 \end{bmatrix}$$

Determine the distribution of the limit-state function F. With the derived equation, write a computer code (Monte Carlo method) to draw samples and plot the PDF and CDF.
[Hint]You may use eigen-value decompositon to make Z_i independent.

Question 3. Write an efficient generalized code (must be written in function format) for Latin Hypercube sampling (LHS). The code should be able to sample n-dimensional random variable (RV) vector within a given range $[a,b]$, where $n \geq 1$. The code with the highest efficiency will be awarded the highest mark, and others will be awarded based on the decreasing order of efficiency relative to the highest mark. [Marks = 15]

Marking criteria:

- Whether it can sample scalar RV's. [Marks = 5]
- Whether it is generalizable to any vector-valued RV. [Marks = 5]
- The computational time and memory efficiency. [Marks = 5]

Question 4. Consider the damped Duffing oscillator (an ODE system): [Marks = 25]

$$\ddot{X}(t) + c\dot{X}(t) + kX(t) + \alpha X^3(t) = A_1 \sin(2\pi \mathrm{t}) \mathrm{A}_2 \sin(\pi t),\ t \in [0, T],\ \boldsymbol{X}(0) = \boldsymbol{X}_0.$$

It is given that the oscillator parameters c, k, and α are normal random variables with $c \sim \mathcal{N}(\mu_c = 2, \sigma_c^2 = 0.5^2)$, $k \sim \mathcal{N}(\mu_k = 1000, \sigma_k^2 = 100^2)$, and $\alpha \sim \mathcal{N}(\mu_\alpha = 100000, \sigma_\alpha^2 = 2500^2)$. Other parameters are fixed at $A_1 = 100$ and $A_2 = 0.1$. The initial conditions are $\{X_0, \dot{X}_0\} = [0.1, 0]$. The system responses are to be computed up to $T = 5$ sec with $\Delta t = 0.0001$ sec. The problem is to quantify the uncertainty in the $X(t)$ and $\dot{X}(t)$ using,

1) Monte Carlo sampling (MCS), [Marks = 5]
2) Importance sampling (IS), [Marks = 10]
3) Latin hypercube sampling (LHS), [Marks = 10]

The uncertainty can be quantified by estimating the mean and 95% confidence interval for $X(t)$ and $\dot{X}(t)$. Perform the following studies:

- For MCS, consider 1000 samples.
- For IS, obtain results in 100, 500, and 1000 samples. Also, estimate the convergence rate of your proposal distribution. By doing so, justify your choice about the proposal distribution.
- For LHS, obtain results in 100, 500, and 1000 samples using the code developed in Question 3.

Plot both the mean and 95% confidence interval for each case.

[Note] For data generation, see the driver code 'HW2_Duffing.py' provided in the LINK. You are not allowed to modify the data generation file 'HW2_utils_data.py'.

Question 5. Consider the diffusion dynamics of Burgers' equation (a PDE system): [Marks = 25]

$$\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \nu \frac{\partial^2 u(x,t)}{\partial x^2}, && t \in [0,T],\ x \in [-1,1], \\
&u(-1,t) = u(1,t), && t \in [0,T], \\
&u(x,0) = u_0, && x \in [-1,1].
\end{aligned}$$

The dynamic viscosity ν is given as $\nu = \frac{0.01}{\kappa}$ and the initial conditions are sampled as u_0=sin(ηx), where κ and η are normal random variables with $\kappa \sim \mathcal{N}(\mu_\kappa = \pi, \sigma_\kappa^2 = (0.25\pi)^2)$ and $\eta \sim \mathcal{N}(\mu_\eta = \pi, \sigma_\eta^2 = (0.005\pi)^2)$. The system responses are to be computed on a spatial grid of $\mathbb{R}^{256}$ up to $T = 1$ sec with $\Delta t = 0.001$ sec. Quantify the uncertainty in $u(t,x)$ using,

1) Monte Carlo sampling (MCS), [Marks = 5]
2) Importance sampling (IS), [Marks = 10]
3) Latin hypercube sampling (LHS) [Marks = 10]

Similarly to the above, quantify the uncertainty by estimating the mean and 95% confidence interval. Obtains the results for the following cases:

- For MCS, consider 1000 samples.
- For IS, obtain results in 500 samples. Similar to the previous case, estimate the convergence rate of your proposal distribution and justify your choice about the proposal distribution.
- For LHS, obtain results in 500 samples.

Plot both the mean and 95% confidence interval at (I) $u(x = 0.25, t)$, (II) $u(x = 0.5, t)$, and (III) $u(x = 0.75, t)$.
[Note]For data generation, see the driver code 'HW2_1D_Burgers.py' provided in the LINK. You are not allowed to modify the data generation file 'HW2_utils_data.py'.

Question 6. Consider the nonlinear Duffing oscillator in Question 4. Instead of assuming the oscillator parameters c, k, and α to be normal random variables, consider $c \sim \text{LogNormal}(\mu_c = 0.5, \sigma_c^2 = 0.1^2)$, $k \sim \text{LogNormal}(\mu_k = 2, \sigma_k^2 = 0.25^2)$, and $\alpha \sim \text{LogNormal}(\mu_\alpha = 5, \sigma_\alpha^2 = 1)$. Other parameters are kept same at $A_1 = 100$ and $A_2 = 0.1$. The initial conditions are also kept same at $\{X_0, \dot{X}_0\} = [0.1, 0]$. The system responses are to be computed up to $T = 5$ sec with $\Delta t = 0.0001$ sec.

With the given random variables and problem statement, use the perturbation theory and Laplace approximation to quantify the uncertainty in the system responses $X(t)$ and $\dot{X}(t)$. Derive the necessary adjoint equations for computing gradient and solve the coupled system of equations. [Marks = 10]
[Note]For data generation, use the same 'HW2_Duffing.py' provided in the LINK.