



APL410

ASSIGNMENT - 1

(a) The original eq governing equation for the problem is the static linear momentum balance eqⁿ:

$$\sigma_{ij,j} + b_i = 0, \quad \partial B_t$$

Multiplying by a test function δv_i ,

$$\Rightarrow \delta v_i \sigma_{ij,j} + \delta v_i b_i = 0$$

$$\Rightarrow \int_B \delta v_i \sigma_{ij,j} dV + \int_B \delta v_i b_i dV = 0$$

$$\Rightarrow \int_B (\delta v_i \sigma_{ij})_{,j} dV - \int_B \delta v_{i,j} \sigma_{ij} dV + \int_B \delta v_i b_i dV = 0$$

$$\Rightarrow \int_{\partial B} \delta v_i \sigma_{ij} n_j dS + \int_B \delta v_i b_i dV = \int_B \delta v_{i,j} \sigma_{ij} dV$$

$$\Rightarrow \int_B \delta v_{i,j} \sigma_{ij} dV = \int_B \delta v_i b_i dV + \int_{\partial B_t} \delta v_i t_i dS \quad (\delta v_i = 0 \text{ at } \partial B_u)$$

→ This is the weak form of the governing eqⁿ.

Boundary conditions:

(i) $u_2(x, y=0) = 0$

(ii) $u_1(x \neq 0, y=0)$ is free

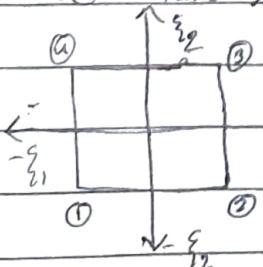
(iii) $u_1(x=0, y=0) = 0$

(iv) $\underline{t}(x=0, y) = [0, 0]^T$

(v) $\underline{t}(x, y=1) = [0, \sigma_0]^T$

Further simplification of the weak form:

We take bilinear shape functions for the basis functions of our square element.



ξ_1 & ξ_2 are the isoparametric co-ordinates of the master element for our square elements.

Basis shape func^s: (i) $N^1(\xi_1, \xi_2) = \frac{1}{4}(1-\xi_1)(1-\xi_2)$ (ii) $N^2(\xi_1, \xi_2) = \frac{1}{4}(1+\xi_1)(1-\xi_2)$

(iii) $N^3(\xi_1, \xi_2) = \frac{1}{4}(1+\xi_1)(1+\xi_2)$ (iv) $N^4(\xi_1, \xi_2) = \frac{1}{4}(1-\xi_1)(1+\xi_2)$

We know,

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} = C_{ijkl} \frac{\partial u_k}{\partial x_l} = C_{ijkl} u_{k,l}$$

$$u_i(\underline{x}) = \sum_{a=1}^N N^a(\underline{x}) u_i^a, \quad N^a: \text{Shape func. with } N^a(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} = \underline{x}_a \\ 0 & \text{else} \end{cases} \quad j \in \text{nodes}$$

u_i^a : Disp. of node a

$$Sv_i(\underline{x}) = \sum_{a=1}^N N^a(\underline{x}) Sv_i^a,$$

$$\therefore \text{Weak form of eqn: } \int_B (N^a Sv_i^a)_{,j} C_{ijkl} (N^b u_k^b)_{,l} dV = \int_B N^a Sv_i^a b_i dV + \int_{\partial B_t} N^a Sv_i^a t_i dA$$

$$\Rightarrow (K_{aibk} u_k^b - F_i^a) Sv_i^a = 0 \Rightarrow K_{aibk} u_k^b = F_i^a \quad \forall \{a, i\} \text{ s.t. } \underline{x}_k^a \text{ not on } \partial$$

$$\text{where, } K_{aibk} = \int_B C_{ijkl} N_{,j}^a N_{,l}^b dV = \int_B C_{ijkl} \frac{\partial N^a}{\partial x_j} \frac{\partial N^b}{\partial x_l} dV$$

$$F_i^a = \int_B b_i N^a dV + \int_{\partial B_t} t_i N^a dA$$

Note: \underline{b} : body force is 0 in our problem.

For numerical integration, we use the 2-point Gaussian Quadrature scheme for a bilinear element:

$$\int_{-1}^1 \int_{-1}^1 f(\xi_1, \xi_2) d\xi_1 d\xi_2 \approx \sum_{i=1}^2 \sum_{j=1}^2 w_i w_j f(\xi_i, \xi_j)$$

where, w_i, w_j are weights for the i^{th} & j^{th} Gauss pts.
 ξ_i, ξ_j are the Gauss points.

For 2x2 integration, $\xi_i, \xi_j = \pm \frac{1}{\sqrt{3}}$, $w_i, w_j = 1$



(b) The expressions for the elemental stiffness matrix K_{aibk} & force vector f_i^a were derived in part (a), which are given by:

$$K_{aibk} = \int_B C_{ijkl} N_j^a N_k^b dV = \int_B C_{ijkl} \frac{\partial N_j^a}{\partial x_l} \frac{\partial N_k^b}{\partial x_l} dV$$

$$f_i^a = \int_B b_i^a N^a dV + \int_{\partial B_f} t_i^a N^a dS = \int_{\partial B_f} t_i^a N^a dS \quad \{b_i^a = 0\}$$

The Elemental Stiffness Matrix for the Master element with ^{local} nodes ①-④:

$$K_{aibk}^e = \int C_{ijkl} \frac{\partial N_j^a(\xi)}{\partial x_l} \frac{\partial N_k^b(\xi)}{\partial x_l} dV$$

$$\therefore [K^e] = \begin{bmatrix} K_{1111}^e & K_{1112}^e & K_{1121}^e & \dots & K_{1142}^e \\ K_{1211}^e & K_{1212}^e & \dots & \dots & K_{1242}^e \\ \vdots & \vdots & \dots & \dots & \vdots \\ K_{4211}^e & K_{4212}^e & \dots & \dots & K_{4242}^e \end{bmatrix}$$

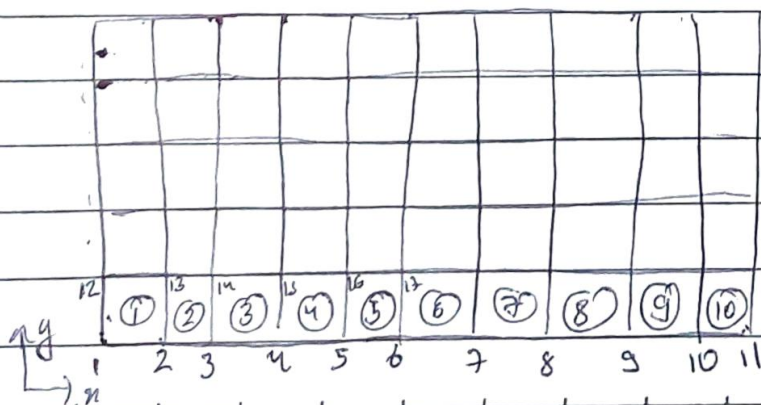
$\therefore [K^e]$ has $= 4 \times 2 = 8$ ~~rows~~ rows & columns \Rightarrow Size $(K^e) = 8 \times 8$

The Elemental Force Vector is given by:

$$f_i^{a,e} = \int_{\partial B_f} b_i^a(\xi) N^a(\xi) dS$$

$$\underline{f}^e = [f_1^{1,e} \quad f_2^{1,e} \quad f_1^{2,e} \quad f_2^{2,e} \quad f_1^{3,e} \quad f_2^{3,e} \quad f_1^{4,e} \quad f_2^{4,e}]^T$$

\therefore Size of $\underline{f}^e = (4 \times 2) \times 1 = 8 \times 1$



For element at cell (ij) :

$$\text{Element No.} = 10(j-1) + i \quad \forall \quad i \in (1,10) \quad j \in (1,5)$$

For node at (ij) :

$$\text{Node no.} = 10(j-1) + i \quad \forall \quad i \in (1,11), j \in (1,6)$$



& force vector

(c) Using the elemental stiffness matrix, we compute the global stiffness matrix & force vectors:

$$K_{ij} = \sum K^e K_{aibk} = \sum K_{aibk}^e \quad ; \quad f_i^a = \sum f_i^{ae} \quad \text{(Summed over all elements having a node common)}$$

Solving the system $Ku = f$ gives the displacement at all nodes.

(d) We computed the velocity gradient: $\nabla u(i,j) = \sum \sum \frac{\partial N_k^b}{\partial x_k} \cdot \frac{\partial u_i^b}{\partial x_j}$

$$\text{Strain: } \epsilon_{ij} = \frac{1}{2} (\nabla u(i,j) + \nabla u(j,i))$$

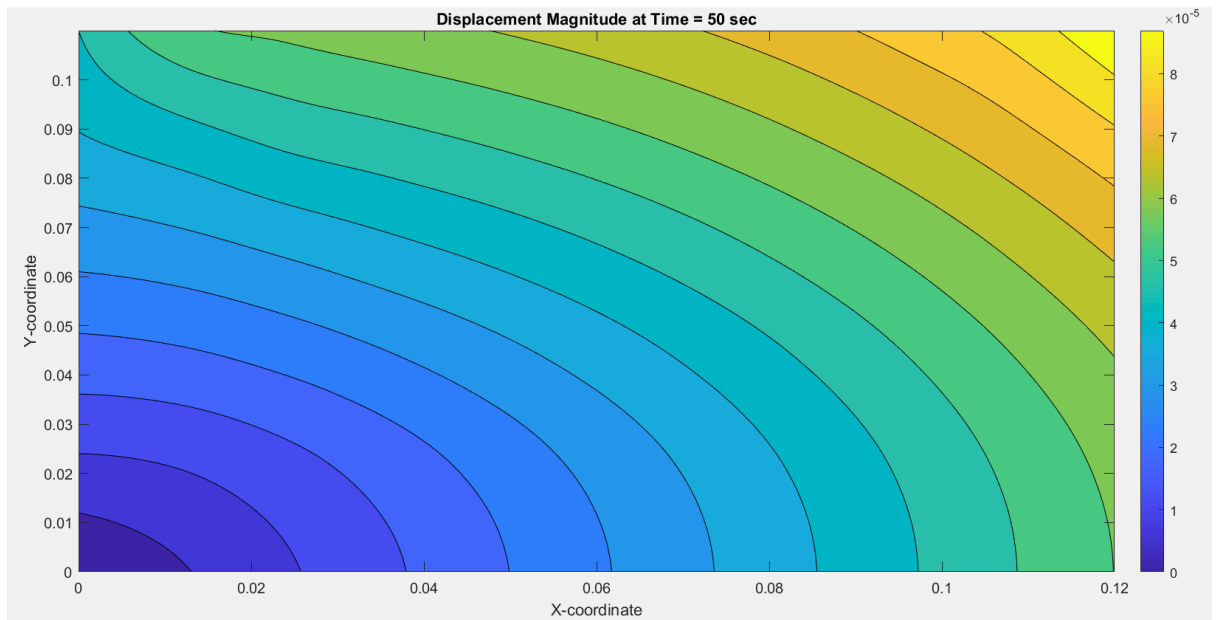
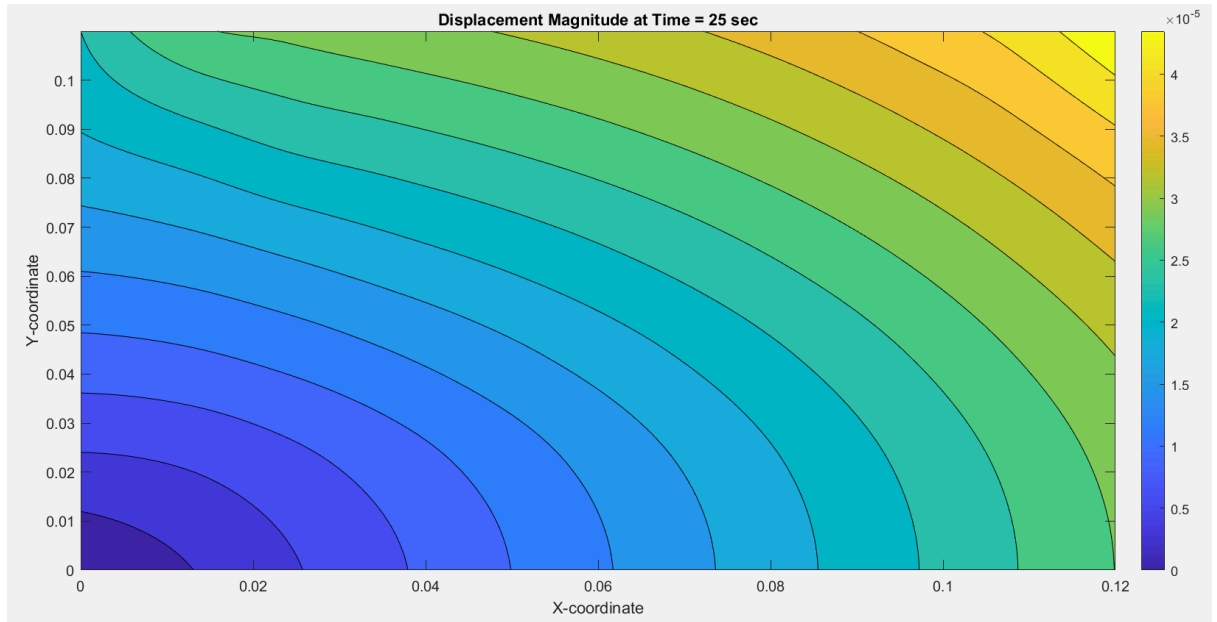
$$\text{Stress: } \sigma_{ij} = C_{ijkl} \epsilon_{kl}$$

We computed these values over the quadrature points & found the average over these pts.

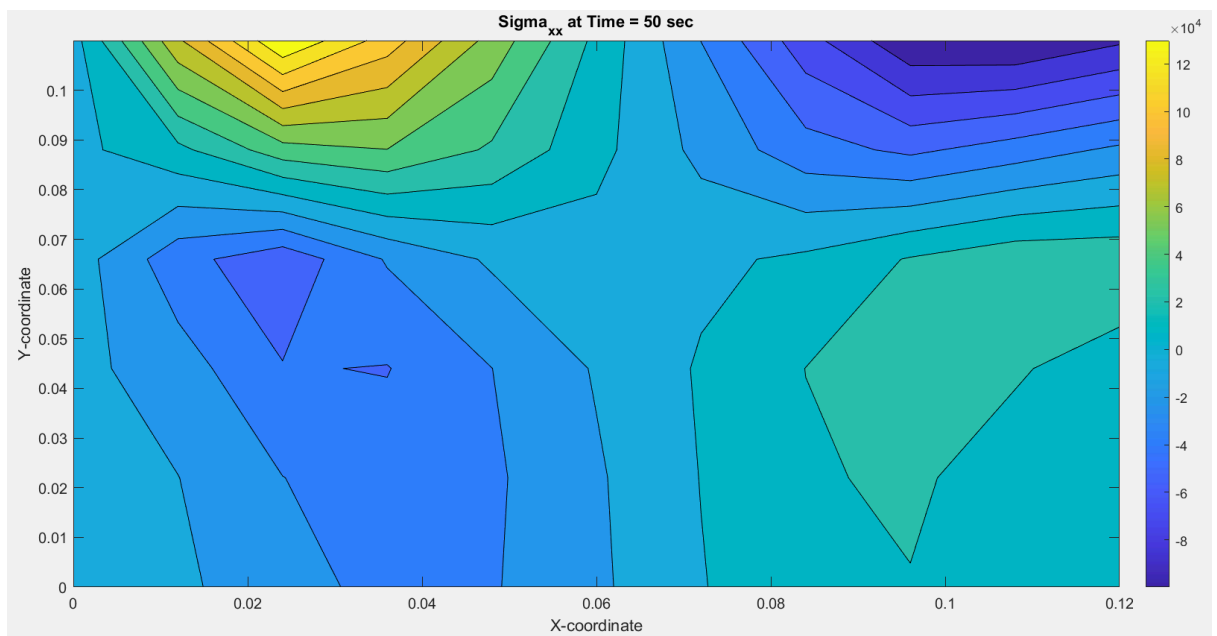
Finally the nodal stress, strain values were taken as the average of the previous values over all elements with that common node.

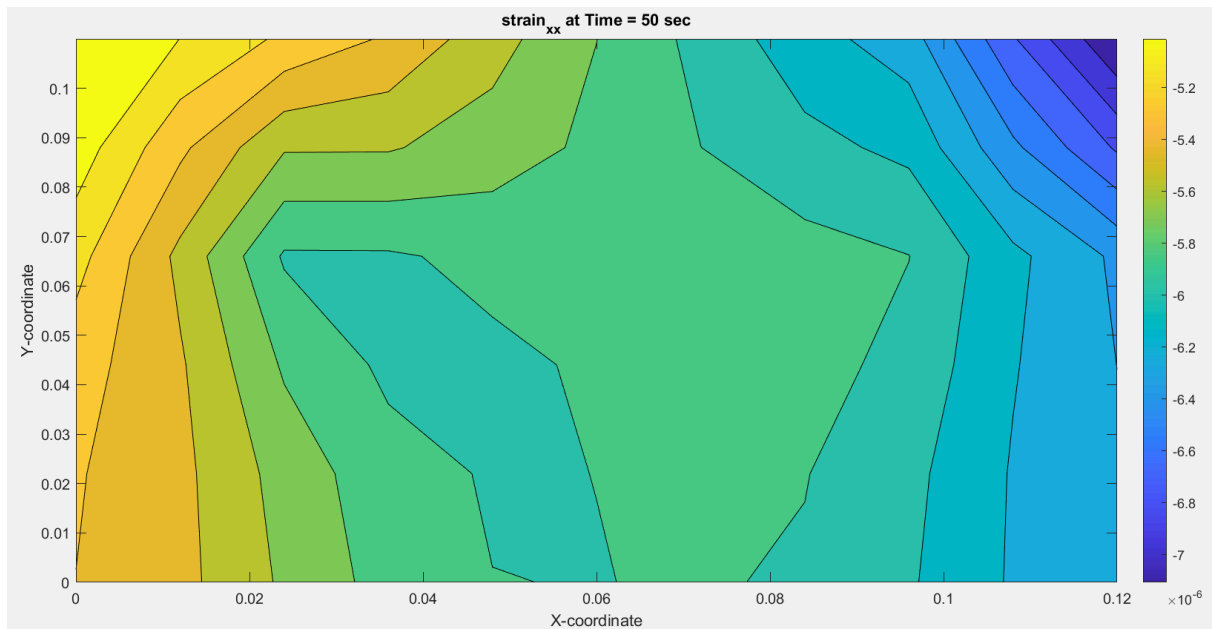
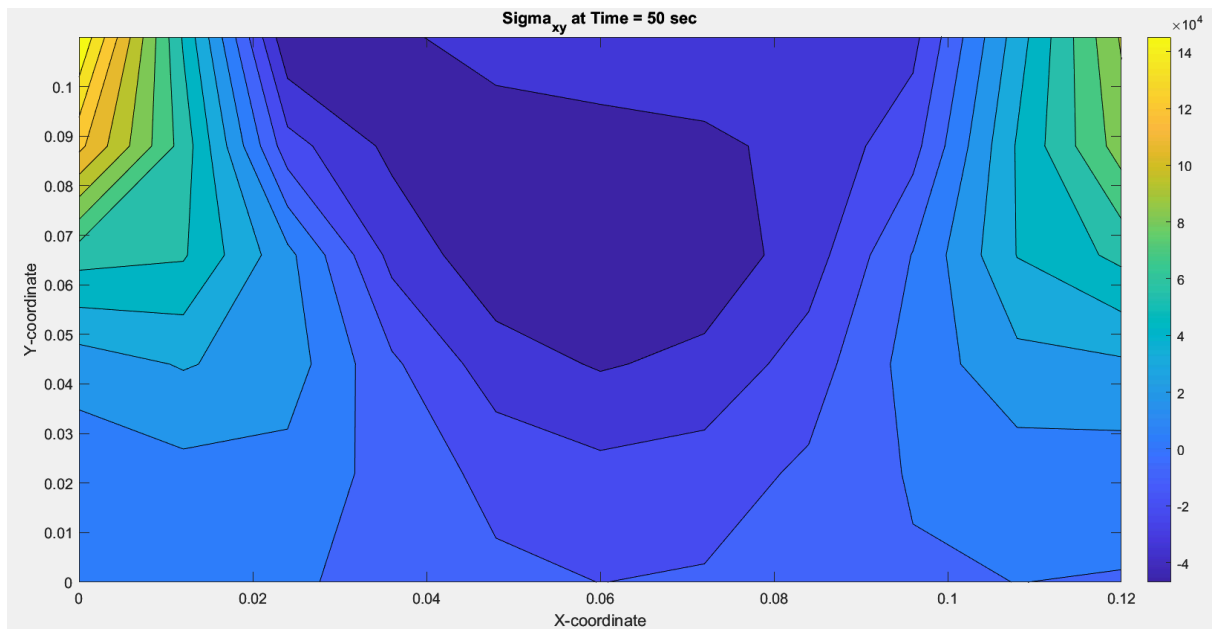
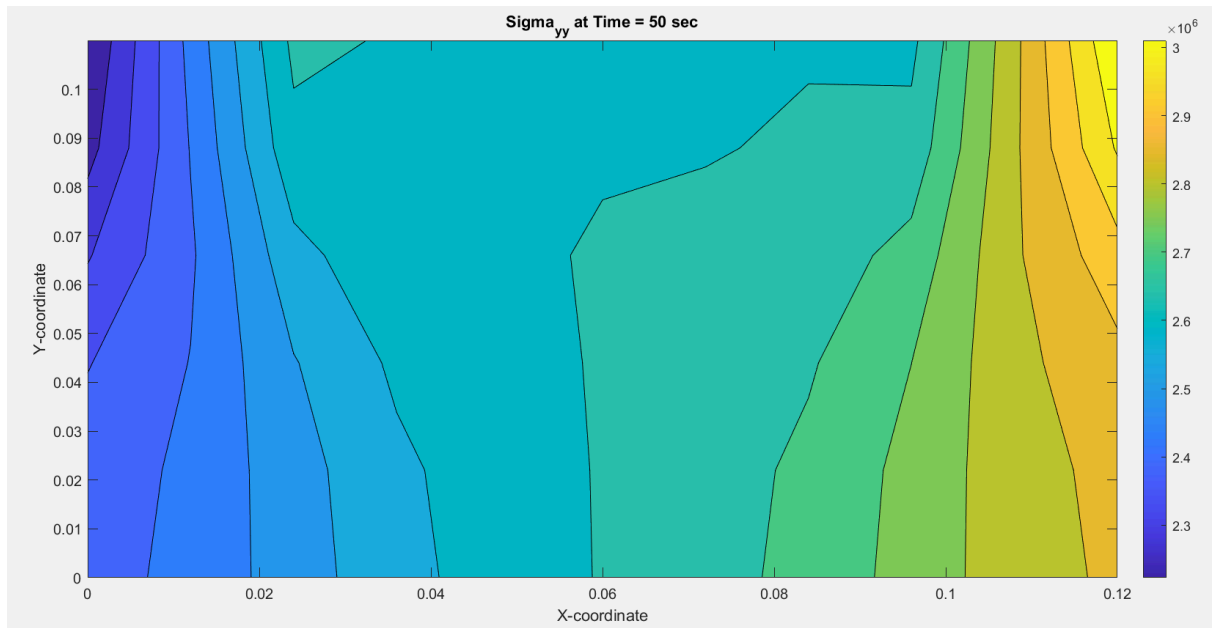
(e) We just compute the average of displacement, stress & strain over the top edge of the plate.

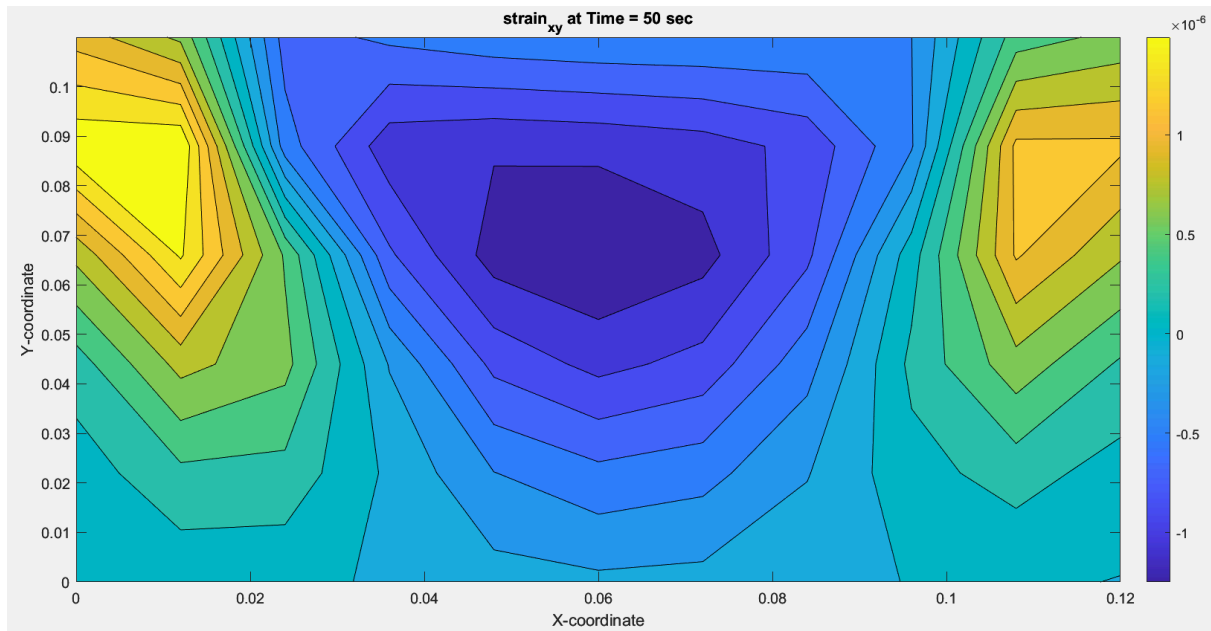
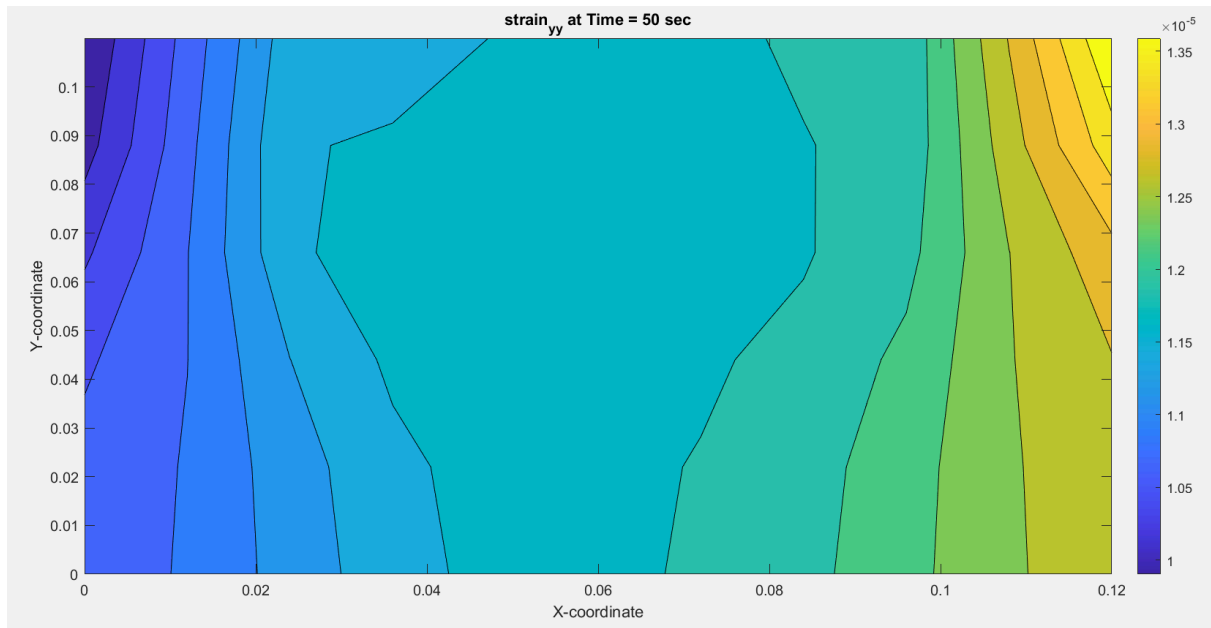
(c)



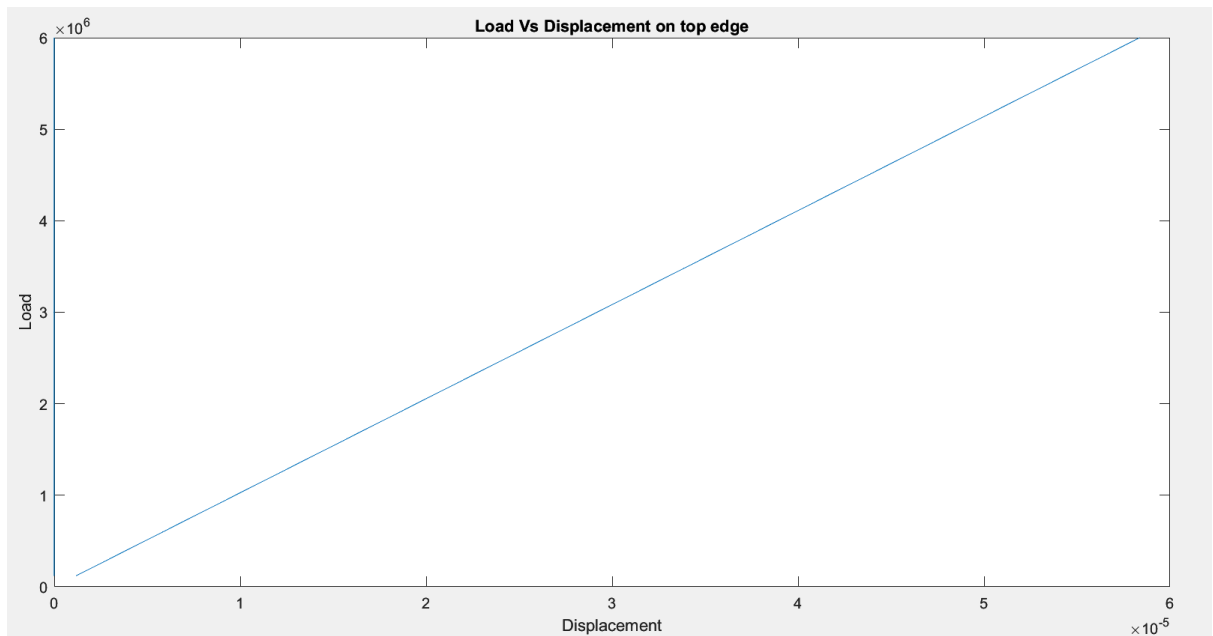
(d)

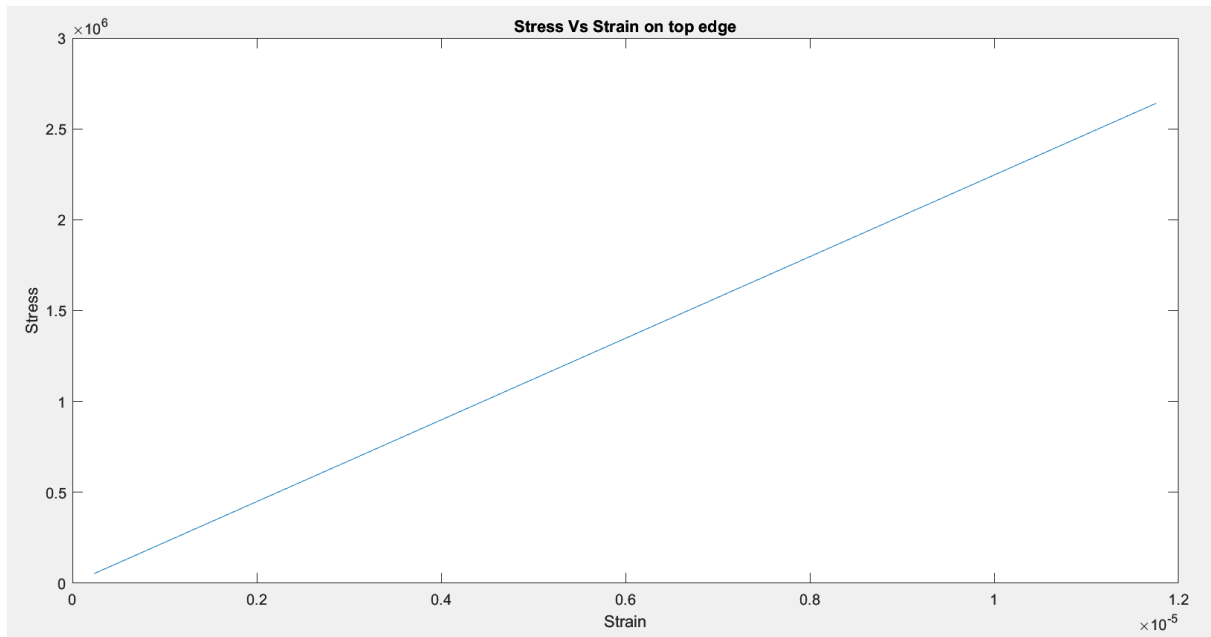






(e)





(f)

