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Aptitude Advanced

Geometry I

eBook 01

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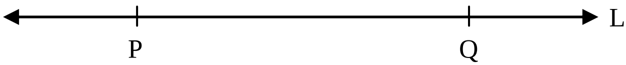
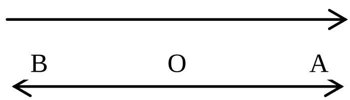
Chapter 1: Lines

Before discussing geometry, there is a need to understand 'Point'. A point is any entity with a location in a plane but without any length, area or volume. A straight line is an infinitely thin, infinitely long, straight geometrical object made up of infinite number of points.



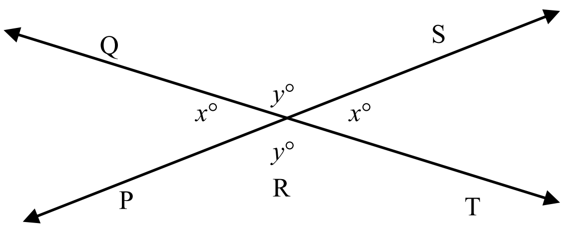
The line above can be referred to as line PQ or line L. The part of the line from P to Q is called a line segment. P and Q are the endpoints of the segment. The notation PQ is used to denote both the segment and the length of the segment. The intention of the notation can be determined from the context.

Ray: A line with one end point is called a ray. The end point is called the origin. Two rays, which lie on the same line and have only the origin as a common point are called opposite rays. Rays OA and OB are opposite rays.



Plane: A plane is a flat surface. It has length and width but no thickness.

Intersecting Lines and Angles: If two lines intersect, the opposite angles are called vertical angles and have the same measure. In the figure given below
Also, $x + y = 180°$ since PRS is a straight line.
$\angle$PRQ and $\angle$ SRT are vertical angles and $\angle$QRS and $\angle$PRT are vertical angles.



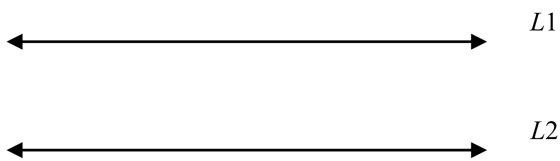
Perpendicular Lines:

An angle that has a measure of 90° is a right angle. If two lines intersect at right angles, the lines are perpendicular. For Ex.:

*L*1, and *L*2 above are perpendicular, denoted by $Ll \perp L2$. A right angle symbol in an angle of intersection indicates that the lines are perpendicular.

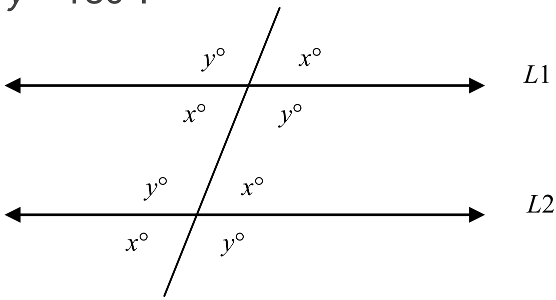
Parallel Lines:

If two lines that are in the same plane do not intersect, the two lines are parallel. In the figure given below,



lines *L*1, and *L*2 are parallel, denoted by $Ll \parallel L2$.

If two parallel lines are intersected by a third line, as shown below, then the angle measures are related as indicated, where
$x + y = 180°$.



1.1 Important points

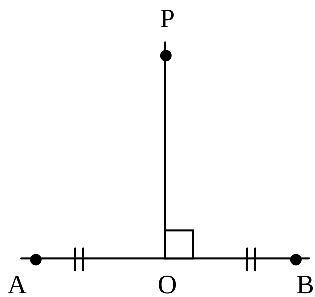
(i) There is one and only one line passing through two distinct points.

(ii) Three or more points are said to be collinear if they lie on a line, otherwise they are said to be non-collinear.

(iii) Two or more lines are said to be coplanar if they lie in the same plane, otherwise they are said to be non-coplanar.

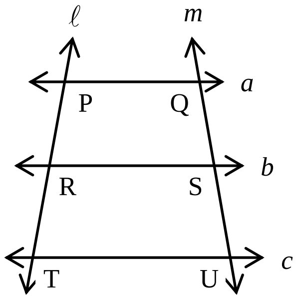
(iv) A line, which is perpendicular to a line segment i.e., intersects at 90° and passes through the midpoint of the segment is called he perpendicular bisector of the segment.

(v) Every point on the perpendicular bisector of a segment is equidistant from the two endpoints of the segment.
Conversely, if any point is equidistant from the two endpoints of the segment, then it must lie on the perpendicular bisector of the segment.
If PO is the perpendicular bisector of segment AB, then, AP = PB.
Also, if AP = PB, then P lies on the perpendicular bisector of segment AB.



(vi) The ratio of intercepts made by three parallel lines on a transversal is equal to the ratio of the corresponding intercepts made on any other transversal by the same parallel lines.

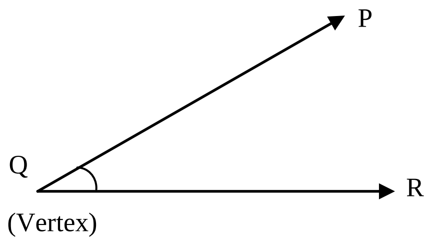
If line $a \parallel$ line $b \parallel$ line c and line ℓ and line m are two transversals, then $\frac{PR}{RT} = \frac{QS}{SU}$.



Chapter 2: Angles

When two rays have the same starting or end points, they form an angle and the common end point is called vertex.

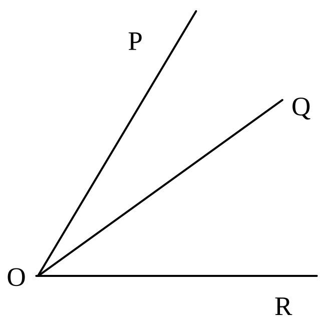
Angle is also defined as the measure of rotation of a ray. For the purpose of Trigonometry, the measure of rotation is termed positive if it is in the anticlockwise direction.



In the figure, ray PQ and QR from angle∠PQR

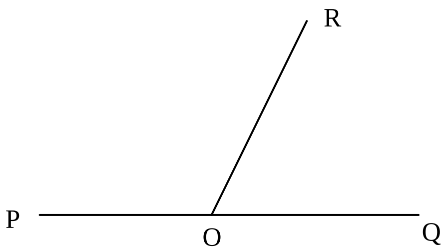
2.1 Types of Angles

(i) Adjacent Angles:



$\angle POQ$ and $\angle QOR$ are called adjacent angles, because they have a common side and their interiors are disjoint.

(ii) Linear Pair:



$\angle POR$ and $\angle ROQ$ are said to form a linear pair because they have a common side and other two sides are opposite rays $\angle POR + \angle ROQ = 180^0$

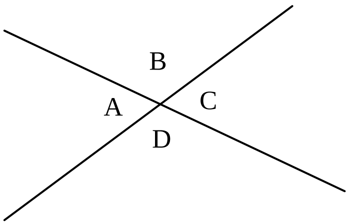
$\Leftrightarrow \angle POR$ and $\angle POQ$ form a linear pair

(iii) An angle greater than 180°, but less than 360° is called a reflex angle.

(iv) Two angles whose sum is 90° are called complementary angles.

(v) Two angles having a sum of 180° are called supplementary angles.

(vi) When two lines intersect, two pairs of vertically opposite angles are equal. The sum of 2 adjacent angles is 180°.



As given in the above diagram $\angle A = \angle C$ & $\angle B = \angle D$.
Secondly $\angle A + \angle B = 180°$ & $\angle C + \angle D = 180°$.

Two lines are parallel to each other if

- They are parallel to a 3rd line.
- They are opposite sides of a rectangle/ square/ rhombus/ parallelogram.
- If they are perpendicular to a 3rd line.
- If one of them is a side of the triangle & other joins the midpoints of the remaining two sides.
- If one of them is a side of a triangle & other divides other 2 sides proportionately.

Two lines are perpendicular to each other if

- They are arms of a right-angle triangle.
- If the adjacent angles formed by them are equal and supplementary.
- They are adjacent sides of a rectangle or a square.
- If they are diagonals of a rhombus.
- If one of them is a tangent & other is radius of the circle through the point of contact.
- If the sum of their squares is equal to the square of line joining their ends.

Two angles are said to be equal if

- They are vertically opposite angles.
- Their arms are parallel to each other.
- They are the corresponding angles of two congruent triangles.
- They are the opposite angles of a parallelogram.
- They are the angles of an equilateral triangle.
- They are the angles of a regular polygon.
- They are in same segment of a circle.
- One of them lies between a tangent & a chord thorough the point of contact & other is in the alternate segment, in a circle.

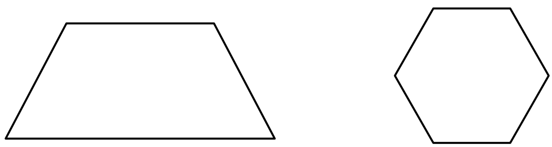
Two sides are equal to each other if

- They are corresponding sides of two congruent triangles.
- They are sides of an equilateral triangle.
- They are opposite sides of a parallelogram.
- They are the sides of a regular polygon.
- They are radii of the same circle.
- They are chords equidistant from centre of circle.
- They are tangents to a circle from an external point

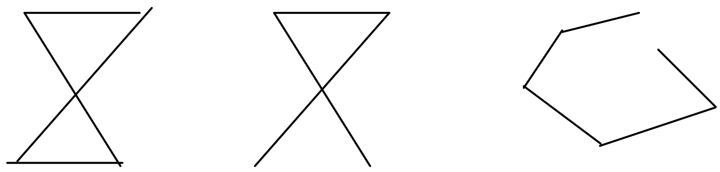
Chapter 3: Polygons

A polygon is a closed plane figure formed by three or more line segments, called the sides of the polygon. Each side intersects exactly two other sides at their endpoints. The points of intersection of the sides are vertices. The term "polygon" will be used to mean a convex polygon, that is, a polygon in which each interior angle has a measure of less than 180°.

The following figures are polygons:



The following figures are not polygons:



Convex polygon: A polygon in which none of the interior angles is more than 180° is called a convex polygon.

Concave polygon: A polygon in which at least one of the interior angles is more than 180° is called a concave polygon.

Regular polygon: A polygon which has all its sides and angles equal is called a regular polygon.

A polygon with three sides is a triangle; with four sides, a quadrilateral; with five sides, a pentagon; and with six sides, a hexagon.

The sum of the interior angles of a triangle is 180°. In general, the sum of the interior angles of a polygon with n sides is equal to $(n - 2)180°$. For Ex., this sum for a pentagon is $(5 - 2)180° = (3)180° = 540°$.

3.1 Properties of polygons

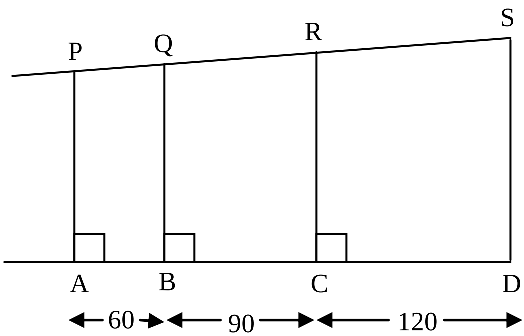
- If the number of sides of the polygon is n, then the sum of all interior angles = $(2n - 4)90°$.
- An interior angle + corresponding exterior angle = 180°.
- The sum of all the exterior angles = 360°.

- The number of diagonals in a polygon of n sides = $\frac{n(n-1)}{2} - n$

Ex. 1. An angle is twice its complement. Find the angle.

Sol. If the complement is $x°$, then angle = $2x°$.
So $2x + x = 90°$ $\therefore$ $x = 30°$
$\therefore$ The angle is $2 \times 30 = 60°$.

Ex. 2. In the figure, if PS = 360, find PQ, QR and RS
.



Sol. PA, QB, RC and SD are perpendicular to AD. Hence, they are parallel. So, the intercepts are proportional.

$$\therefore \frac{AB}{BD} = \frac{PQ}{QS}$$

$\therefore \frac{60}{210} = \frac{x}{360 - x} \quad \therefore \frac{2}{7} = \frac{x}{360 - x}$

$\therefore x = \frac{720}{9} = 80$ $\therefore$ PQ = 80

$\therefore$ QS = 360 – 80 = 280

Again, $\frac{BC}{CD} = \frac{QR}{RS}$

$\therefore \frac{90}{120} = \frac{y}{280 - y} \quad \therefore \frac{3}{4} = \frac{y}{280 - y}$

$\therefore$ $7y = 280 \times 3$ $\therefore$ $y = 120$

$\therefore$ QR = 120 $\therefore$ SR = 280 – 120 = 160.

Ex. 3. Each interior angle of a regular polygon is 120°, how many sides the polygon is having.

Sol. The sum of interior angles of a polygon is equal to (2n-4) rt. angles.

$\Rightarrow$ Each interior angle = $\frac{(2n-4)}{n}$ rt. angles.

$\Rightarrow$ $\frac{(2n-4)}{n}$ x $90^0 = 120^0$.

=> $\left(\frac{2n-4}{n}\right)3 = 4,$

$6n - 12 = 4n \Rightarrow 2n = 12, \Rightarrow n = 6$. Given polygon has six sides.

Ex. 4. A polygon is having 10 sides. What would be the sum of all of its interior angles?

Sol. The no. of sides = 10. Sum of internal angles = $(2 \times 10 - 4)\, 90^0 = 1440^0$.

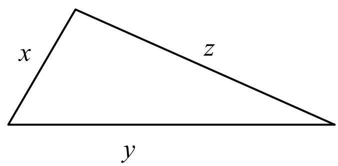
Ex. 5. Which angle is its own compliment?

Sol. The compliment of an angle is obtained by subtracting it for 90^0. Compliment of $45^0 = 90^0 - 45^0 = 45^0$

Chapter 4: Triangles

The plane figure bounded by the union of three lines, which join three non collinear points, is called a triangle.

There are several special types of triangles with important properties. But one property that all triangles share is that the sum of the lengths of any two of the sides is greater than the length of the third side or difference of the lengths of any two of the sides is less than the length of the third side, as illustrated below.



So $x + y > z$, $x + z > y$, and
$y + z > x$.

4.1 Types of Triangles

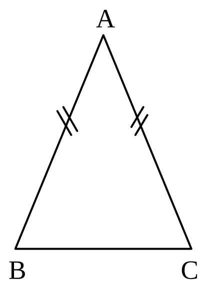
With regard to their sides, triangles are of three types:

(i) **Scalene Triangle:** A triangle in which none of the three sides is equal is called a scalene triangle.

(ii) **Isosceles Triangle:** A triangle in which at least two sides are equal is called an isosceles triangle. In an isosceles triangle, the angles opposite to the congruent sides are congruent.

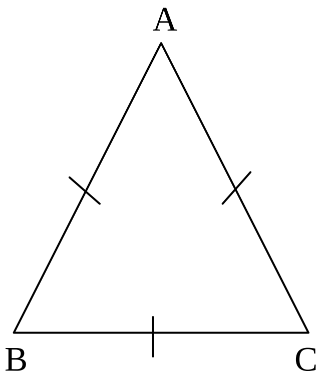
Conversely, if two angles of a triangle are congruent, then the sides opposite to them are congruent.

In Δ ABC, AB = AC,
$\angle$ ABC = $\angle$ ACB



(iii) **Equilateral Triangle:** A triangle in which all the three sides are equal is called an equilateral triangle. In an equilateral triangle, all the angles are congruent and equal to 60°.

In Δ ABC, AB = BC = AC.
$\angle$ ABC = $\angle$ BCA = $\angle$ CAB = 60°

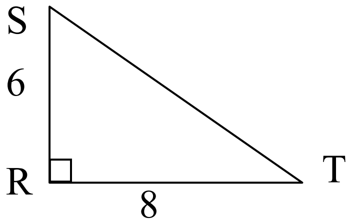


With regard to their angles, triangles are of five types:

(i) **Acute triangle:** If all the three angles of a triangle are acute i.e., less than 90°, then the triangle is an acute-angled triangle.

(ii) **Obtuse triangle:** If any one angle of a triangle is obtuse i.e., greater than 90°, then the triangle is an obtuse-angled triangle. The other two angles of the obtuse triangle will be acute.

(iii) **Right Triangle:** A triangle that has a right angle is a right triangle. In a right triangle, the side opposite the right angle is the hypotenuse, and the other two sides are the legs. An important theorem concerning right triangles is the Pythagorus theorem, which states: In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

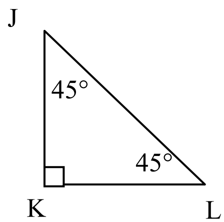


In the figure above, ΔRST is a right triangle, so $(RS)^2 + (RT)^2 = (ST)^2$. Here, RS = 6 and RT = 8, so ST = 10, since $6^2 + 8^2 = 36 + 64 = 100 = (ST)^2$ and ST = $\sqrt{100}$. Any triangle in which the lengths of the legs are in the ratio 3 : 4 is a right triangle. In general, if a, b, and c are the lengths of the sides of a triangle in which $a^2 + b^2 = c^2$, then the triangle is a right triangle. There are some standard Pythagorean triplets, which are repeatedly used in the questions. It is better to remember these triplets by heart.

♠ 3, 4, 5 ♠ 5, 12, 13 ♠ 7, 24, 25 ♠ 8, 15, 17
♠ 9, 40, 41 ♠ 11, 60, 61 ♠ 12, 35, 37 ♠ 16, 63, 65
♠ 20, 21, 29 ♠ 28, 45, 53.

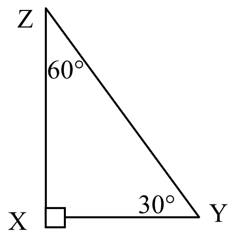
Any multiple of these triplets will also be a triplet i.e. when we say 3, 4, 5 is a triplet, if we multiply all the numbers by 2, it will also be a triplet i.e. 6, 8, 10 will also be a triplet.

(iv) **45°- 45° - 90° Triangle:** If the angles of a triangle are 45°, 45° and 90°, then the perpendicular sides are $1/\sqrt{2}$ times the hypotenuse. In a 45°- 45°- 90° triangle, the lengths of the sides are in the ratio 1 : 1 : √2. For Ex., in ΔJKL, if JL = 2, then JK = √2 and KL = √2.

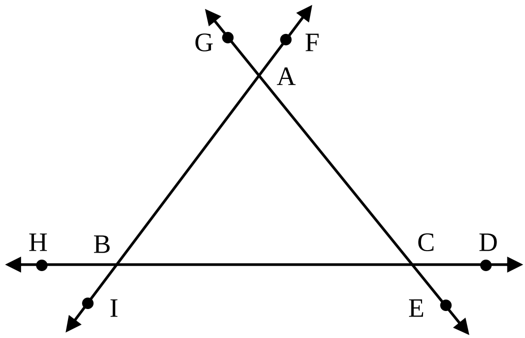


(v) **30°- 60° - 90° Triangle:** In 30°- 60° - 90° triangle, the lengths of the sides are in the ratio 1: √3 : 2. For Ex., in ΔXYZ, if XZ = 3, then XY = 3√3 and YZ = 6. In short, the following formulas can be applied to calculate the two sides of a 30°- 60°-90° triangle, when the third side is given.

Side opposite to 30°
= ½ of hypotenuse.
Side opposite to 60°
= √3/2 of hypotenuse.



4.2 Important properties of triangles



(i) The sum of the three interior angles of a triangle is 180°.
In Δ ABC, ∠ ABC + ∠ BAC + ∠ ACB = 180°

(ii) The sum of an interior angle and the adjacent exterior angle is 180°.
In figure on previous page, ∠ ABC + ∠ ABH = 180°
∠ ABC + ∠ CBI = 180°

(iii) Two exterior angles having the same vertex are congruent.
In figure on previous page, $\angle$ GAB $\cong$ $\angle$ FAC

(iv) The measure of an exterior angle is equal to the sum of the measures of the two interior angles (called remote interior angles) of the triangle, not adjacent to it.

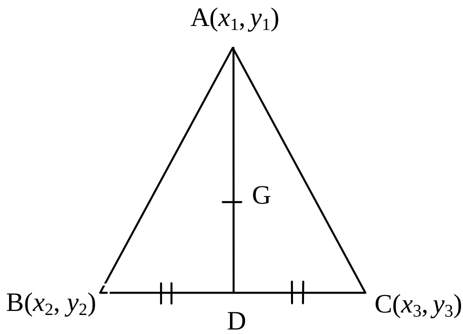
Altitude: The altitude of a triangle is the segment drawn from a vertex perpendicular to the side opposite that vertex. Relative to that vertex and altitude, the opposite side is called the base.

The area of a triangle is equal to: (the length of the altitude) × (the length of the base) / 2.

Centroid:

The segment joining a vertex & midpoint of the opposite side is called median of a triangle. There are three medians & they meet in a single point called, centroid of the triangle. The centroid divides medians in the ratio of 2 : 1.

The following formula is applied to calculate the length of the median. The sum of the squares of two sides = 2[median^2 + (½ 3rd side)^2]



(i) If A(x_1, y_1); B(x_2, y_2); C(x_3, y_3) are the co-ordinates of a triangle , then co-ordinates of centroid G are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

(ii) The median divides the triangle in two equal parts (not necessarily congruent)

(iii) The centroid divides the median in the ratio 2 : 1 with the larger part towards the vertex. So AG : GD = 2 : 1.

➢ Some other important properties

(i) If Δ ABC, is an isosceles triangle with AB $\cong$ AC, then the angle bisector of $\angle$ BAC is the perpendicular bisector of the base BC and is also the median to the base.

Area of an isosceles triangle $= \frac{c}{4}\sqrt{4a^2 - c^2}$, where *c* is the unequal side and *a* is one of the equal sides.

The altitudes on the congruent sides are equal i.e., BE = CF.

(ii) For an equilateral triangle,

height $= \frac{\sqrt{3}}{2} \times \text{side}$;

area $= \frac{\sqrt{3}}{4} \times (\text{side})^2$, inradius $= \frac{1}{3} \times \text{height}$;

circumradius $= \frac{2}{3} \times \text{height}$, perimeter = 3 × sides

(iii) In case of triangles, given perimeter, an equilateral triangle has maximum area.

(iv) For a right-angled triangle,

the median to the hypotenuse $= \frac{1}{2} \times \text{hypotenuse}$.

The median to the hypotenuse is also the circum-radius of the triangle.

$\text{Area} = \frac{1}{2} \times \text{Product of perpendicular sides.}$

(v) Area of a triangle = $\sqrt{s(s-a)(s-b)(s-c)} = r \times s = \frac{abc}{4R}$

where, a, b and c are the sides of the triangle,
s = semi perimeter, r = in-radius, R = circum-radius.

(vi) Triangles on the same base and between the same parallel lines are equal in area.

Congruency of triangles: If the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle, then the two triangles are said to be congruent.

Two triangles are congruent if

- Two sides & the included angle of a triangle are respectively equal to two sides & included angle of other triangle (SAS).
- 2 angles & 1 side of a triangle are respectively equal to two angles & the corresponding side of the other triangle (AAS).
- Three sides of a triangle are respectively congruent to three sides of the other triangle (SSS).
- 1 side & hypotenuse of a right-triangle are respectively congruent to 1 side & hypotenuse of other rt. triangle (RHS).

Similarity of triangles: Two triangles are similar if they alike in shape only. The corresponding angles are congruent, but corresponding sides are only proportional. All congruent triangles are similar but all similar triangles are not necessarily congruent.

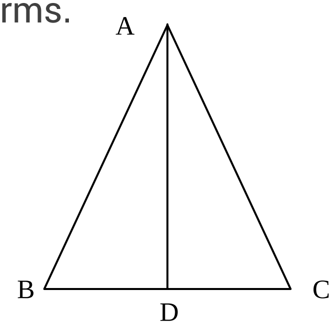
Two triangles are similar if

- Three sides of a triangle are proportional to the three sides of the other triangle (SSS).
- Two angles of a triangle are respectively equal to the two angles of the other triangle (AA).

- Two sides of a triangle are proportional to two sides of the other triangle & the included angles are equal (SAS).

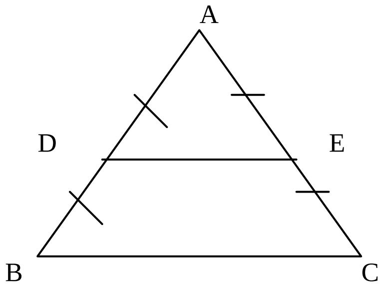
4.3 Some important theorems

(i) **<u>The Angle Bisector Theorem</u>:** The angle bisector divides the opposite side in ratio of the lengths of its adjacent arms.



If AD is the angle bisector, then AB/AC = BD/DC.

(ii) **<u>Midpoint Theorem</u>:** The segment joining the midpoints of any two sides of a triangle is parallel to the third side and is half of the third side.
If AD = DB, AE = EC, then DE is parallel to BC and DE = $\frac{1}{2}BC$.



(iii) **Basic Proportionality Theorem:** If a line is drawn parallel to one side of a triangle and intersects the other sides in two distinct points, then the other sides are divided in the same ratio by it. If DE is parallel to BC, then, $\frac{AD}{DB} = \frac{AE}{EC}$