

Supplemental Material - Why planar cracks fragment into echelon cracks

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In this supplemental material, we provide the details of material modeling, including the values of the material parameters and details of numerical simulations. Captions for movies of numerical simulations are also included.

Material behavior of hard brittle materials. For linear elastic isotropic materials such as Graphite, the strain energy density function is written as

$$\mathcal{W}(\mathbf{E}(\mathbf{u})) = \frac{E}{2(1+\nu)} \operatorname{tr} \mathbf{E}^2 + \frac{E\nu}{2(1+\nu)(1-2\nu)} (\operatorname{tr} \mathbf{E})^2, \quad (1)$$

where E is the Young's modulus and ν is the Poisson's ratio, and $\mathbf{E}(\mathbf{u})$ is the infinitesimal strain tensor

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}(\mathbf{F} + \mathbf{F}^T - 2\mathbf{I}),$$

with \mathbf{F} being the deformation gradient tensor. The stress tensor at any material point \mathbf{X} and time $t \in [0, T]$ is given by

$$\mathbf{S}(\mathbf{X}, t) = \frac{\partial \mathcal{W}}{\partial \mathbf{E}}(\mathbf{E}).$$

The elastic constants can be measured with uniaxial tension tests. A number of standard tests, such as the Compact Tension test, also exist for measuring fracture toughness. The strength surface measurement can be conducted by carrying out experiments on thin tubes subjected to a combination of axial force and inner pressure [1]. Such experiments were performed by Sato [2] on Graphite. The reported values are listed in the table below.

Table I. Mechanical properties of Graphite.

Property	Symbol	Value
Young's modulus	E	9.8 GPa
Poisson's ratio	ν	0.13
Fracture toughness	G_c	91 J/m ²
Tensile strength	σ_{ts}	27 MPa
Shear strength	σ_{ss}	23 MPa

Material behavior of soft brittle materials. Non-linear elastic materials such as PDMS are nearly incompressible and show a strain stiffening behavior. A non-Gaussian strain energy function, such as the Lopez-Pamies function, can be used to model their elastic be-

havior:

$$\begin{aligned} \mathcal{W}(\mathbf{F}) = & \sum_{r=1}^2 \frac{3^{1-\alpha_r}}{2\alpha_r} \mu_r [(\mathbf{F} \cdot \mathbf{F})^{\alpha_r} - 3^{\alpha_r}] \\ & - \sum_{r=1}^2 \mu_r \ln(\det \mathbf{F}) + \frac{\kappa}{2} (\det \mathbf{F} - 1)^2, \end{aligned} \quad (2)$$

where, μ_1 and μ_2 are shear modulus parameters, such as total shear modulus $\mu = \mu_1 + \mu_2$, κ is the bulk modulus, and α_1 , α_2 are strain stiffening parameters. The first Piola-Kirchhoff stress at any material point \mathbf{X} and time $t \in [0, T]$ is given by

$$\mathbf{S}^{(1)}(\mathbf{X}, t) = \frac{\partial \mathcal{W}}{\partial \mathbf{F}}(\mathbf{F}). \quad (3)$$

We define the strength surface in terms of the Biot stress tensor $\mathbf{S} = (\mathbf{S}^{(1)T} \mathbf{R} + \mathbf{R}^T \mathbf{S}^{(1)})/2$, where \mathbf{R} is the rigid rotation tensor defined through a polar decomposition of the deformation gradient $\mathbf{F} = \mathbf{R}\mathbf{U}$, with \mathbf{U} being the right stretch tensor [3].

Fracture toughness can be measured easily from a pure shear test, and tensile strength from a uniaxial tensile test. The hydrostatic strength needs to be measured from a test such as the poker chip test [4]. We adopt the elastic properties from [5] and approximate fracture properties used in [3], and list them in table below.

Table II. Mechanical properties of PDMS.

Property	Symbol	Value
Modulus parameter	μ_1	0.42 MPa
Modulus parameter	μ_2	0.07 MPa
Stiffening parameter	α_1	0.03
Stiffening parameter	α_2	7.2
Bulk modulus	κ	50 MPa
Fracture toughness	G_c	10 J/m ²
Tensile strength	σ_{ts}	0.1 MPa
Hydrostatic strength	σ_{hs}	0.125 MPa

Numerical simulation details. The governing partial differential equations are solved using the finite element method. It is essential that the length scale ε be fully resolved, so a fine mesh size is needed. We construct an unstructured mesh of size $h = \varepsilon/4$. Note that in some previous work [6] studying the echelon crack formation, a very coarse mesh size of $h = \varepsilon$ was utilized, which would affect the accuracy of results. Also, in the previous work, the regularization length scale was tied to the material

length scale. In contrast, ε is a free parameter in our strength-constrained formulation and can be chosen to be as small as needed.

The governing equations are solved iteratively with the fixed-point iteration method. The governing equation for the phase field needs to be solved along with two constraints. The first constraint enforces that the phase field z lies between 0 and 1. The second constraint enforces irreversibility of phase field once a crack has formed. We make use of a penalty method to enforce both constraints. The details of the numerical implementation can be found in Kumar et al. [7, 8]. We have also made available an open-source FEniCS implementation of the numerical scheme for the three-dimensional notched plate on GitHub [9].

Caption for Movie S1-2: A uniform antiplane shear displacement is applied on a plate made out of Graphite—a linear elastic brittle material—and crack growth is simulated with a strength-constrained phase-field model. Crack contours are shown at a value of phase field $z = 0.1$. S1 shows a 3D view of crack growth, and S2 shows a 2D view looking through the crack plane.

Caption for Movie S3-4: A uniform antiplane shear displacement is applied on a plate made out of PDMS—a nonlinear elastic brittle material—and crack growth is simulated with a strength-constrained phase-field model. Crack contours are shown at a value of phase field $z = 0.1$. S1 shows a 3D view of crack growth, and S2 shows a 2D view looking through the crack plane.

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