Indian Institute of Technology Kanpur
Department of Mathematics and Statistics

Single Variable Calculus (MTH 111M)
Semester 2025-2026-I
Exercise Sheet 1

1. Supremum and infimum of subsets of $\mathbb{R}$[†]

1.1. Find the supremum and infimum, whichever exists, of the following subsets of the real line:

(a) $\left\{\frac{1}{3} \pm \frac{n}{3n+1} : n \in \mathbb{N}\right\}$,
(b) $\left\{\frac{m}{|m|+n} : m \in \mathbb{Z}, n \in \mathbb{N}\right\}$,
(c) $\left\{\frac{1}{2^m} + \frac{1}{3^n} + \frac{1}{5^r} : m, n, r \in \mathbb{N}\right\}$,
(d) $\left\{\frac{(n+1)^2}{2^n} : n \in \mathbb{N}\right\}$,
(e) $\{x \in \mathbb{R} : 3x^2 + 3 < 10x\}$,
(f) $\left\{x + \frac{1}{x} : x > 0\right\}$.

1.2. For any $\alpha \in \mathbb{R}$ and $\varnothing \neq A, B \subseteqq \mathbb{R}$, define the following:

$$-A \overset{\text{def}}{=} \{-a : a \in A\},\ \frac{1}{A} \overset{\text{def}}{=} \left\{\frac{1}{a} : a \in A\right\},\ \text{provided } 0 \notin A,$$

$$A + B \overset{\text{def}}{=} \{a + b : a \in A, b \in B\},\ A - B \overset{\text{def}}{=} \{a - b : a \in A, b \in B\},$$

$$\alpha A \overset{\text{def}}{=} \{\alpha a : a \in A\},\ \text{and}\ A \cdot B \overset{\text{def}}{=} \{ab : a \in A, b \in B\}.$$

Show the following:

(a) If A is bounded below then, $-A$ is bounded above and $\sup(-A) = -\inf A$.
(b) If A and B are bounded above then so is $A + B$ and $\sup(A + B) = \sup A + \sup B$.
(c) If A is bounded above and B is bounded below then $A - B$ is bounded above and $\sup(A - B) = \sup A - \inf B$.
(d) If A and B are bounded above then so is $A \cup B$ and $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
(e) If $A, B \subseteqq \mathbb{R}_+$ are bounded then $A \cdot B$ is bounded and $\sup(A \cdot B) = \sup A \cdot \sup B$.
(f) If $A \subseteqq \mathbb{R}_+$ is bounded below and $\inf A > 0$ then $\frac{1}{A}$ is bounded and $\sup\left(\frac{1}{A}\right) = \frac{1}{\inf A}$.
(g) Find the analogues of 1.2.a-1.2.f for infumum.

2. LUB property of $\mathbb{R}$[†]

2.1. (a) Let A be a nonempty and bounded above subset of $\mathbb{R}$. Show that the set of all upper bounds of A is a nonempty and bounded below.
(b) Prove the analogue of (2.1.a) if bounded above is replaced by bounded below.
(c) Prove that LUB and GLB properties of $\mathbb{R}$ are equivalent.

2.2. Let $x \in \mathbb{R}$. Show that there exists a unique integer n satisfying the following:

$$n \leqslant x < n + 1. \tag{2.1}$$

Hint. *Consider the set* $\overset{\text{def}}{=} \{m \in \mathbb{Z} : m \leqslant x\}$. *Do you see that* $S \neq \varnothing$? *(How? Use Archimedian property.) Does* S *have a supremum? If* $\alpha \in \mathbb{R}$ *is the supremum of* S, *there exists* $n \in S$ *such that* $\alpha - 1 < n \leqslant \alpha$. *(Why?) Can you show that this* n *satisfies* (2.1)*? Show further that no other integer can satisfy* (2.1).

Note: The unique integer n satisfying (2.1) is called the *greatest integer* $\leqslant n$ and denoted by $[x]$.

[†]The exercises in these sections will first be discussed during the tutorial sessions. This will enhance your understanding and ensure that you are well-prepared.

2.3. (a) Show that, for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, there exists $m \in \mathbb{Z}$ such that $\left|x - \frac{m}{n}\right| < \frac{1}{n}$.
(b) From (2.3.a), deduce that for any $a, b \in \mathbb{R}$ with $a < b$, there exists at least a rational number r satisfying $a < r < b$, and hence infinitely many.

Note: The property (2.3.b) is often described by saying that the rational numbers are *dense* in the real number system.

2.4. Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{R}$ and $x > 0$. Consider $S \stackrel{\text{def}}{=} \{t \in \mathbb{F} : t > 0 \text{ and } t^2 < x\}$. Show the following:
(a) $\sup S$ exists in $\mathbb{R}$.

Hint. *Choose $n \in \mathbb{N}$ such that $n > \frac{1}{x}$. Then $\frac{1}{n} \in S$ (Why?) Pick any $m > x$. If mis not an upper bound, there would exist $t \in S$ such that $t > m$. Do you now see any problem with that?*

(b) For all $t \in S$ then there exists $n \in \mathbb{N}$ such that $\left(t + \frac{1}{N}\right)^2 < x$.

Hint. *Since $\left(t + \frac{1}{N}\right)^2 = t^2 + \frac{2t}{N} + \frac{1}{N^2} \leqslant t^2 + \frac{2t+1}{N}$, so making $t^2 + \frac{2t+1}{N} < x$ by choosing N suitably will be enough!*

(c) Show that if $t > 0$ and $t^2 > x$ then there exists $N \in \mathbb{N}$ such that $\left(t - \frac{1}{N}\right)^2 > x$.

Hint. *Make similar observation as in* (2.4.b)

(d) Denote $\sup S$ by α. From (2.4.b) and (2.4.c) show that $\alpha^2 = x$.
(e) Show that α is the unique positive number whose square is x.

Note: This unique α w.r.t. the property mentioned above in (2.4.e) α is called the *square root* of 2 and denoted by $\sqrt{x}$ or $x^{\frac{1}{2}}$. Clearly $\sqrt{x} > 0$.

2.5. Show the following:
(a) There is no $\alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$.
(b)∗ $\mathbb{Q}$ does not satisfy the *least upper bound property*, i.e., $\mathbb{Q}$ admits a nonempty bounded above subset which does not have a least upper bound in $\mathbb{Q}$.

Hint. *You may use Exercise 2.4.*

(c) $\mathbb{Q}$ enjoys the Archimedian property, i.e., $\mathbb{N}$ is not bounded above in $\mathbb{Q}$.

Remark. *Recall that in $\mathbb{R}$, the Archimedean property is a consequence of the least upper bound property. Now* (2.5.b) *and* (2.5.c) *show that this is not the situation for $\mathbb{Q}$, which enjoys the Archimedean property in its own right even though the LUB property does not hold!*

2.6. From the density of rationals in $\mathbb{R}$, (2.4.e) and (2.5.a), deduce that for any $x, y \in \mathbb{R}$ with $x < y$, there exists at least an irrational number z satisfying $x < z < y$, and hence infinitely many.

Note: This property is often described by saying that the irrationals are *dense* in the real number system.

3. Additional exercises[‡]

3.1. Prove or disprove the following:

$$\left\{\left(1 + \frac{1}{n}\right)^n : n \in \mathbb{N}\right\} \text{ is bounded above in } \mathbb{R}.$$

[‡]Let's take the opportunity to discuss the additional exercises during the tutorials, as long as time permits.

Hint. *Observe that, for* $0 \leqslant r \leqslant n$, $\binom{n}{r}\frac{1}{n^r} = \frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right) \leqslant \frac{1}{r!}$. *Can you get an upper bound for* $\sum_{r=0}^{n}\frac{1}{r!}$*?*

3.2. Find the supremum and infimum, whichever exists, of the following subsets of $\mathbb{R}$:

(a) $\left\{2^x + 2^{\frac{1}{x}} : x > 0\right\}$,

(b) $\{\sqrt{n} - [\sqrt{n}] : n \in \mathbb{N}\}$,

Hint. *Observe that, for all* $n \in \mathbb{N}$, $[\sqrt{n^2+2n}] = n$. *(Why?)*

(c) $\left\{2(-1)^{n+1} + (-1)^{\frac{n(n+1)}{2}}\left(2+\frac{3}{n}\right) : n \in \mathbb{N}\right\}$,

(d) $\left\{\frac{n-1}{n+1}\cos\frac{2n\pi}{3} : n \in \mathbb{N}\right\}$,

(e) $\left\{\frac{a_1}{a_1+a_2+a_3} + \cdots + \frac{a_{n-2}}{a_{n-2}+a_{n-1}+a_n} + \frac{a_{n-1}}{a_{n-1}+a_n+a_1} + \frac{a_n}{a_n+a_1+a_2} : a_1, \ldots, a_n > 0\right\}$, where $n \geqslant 3$ is an integer.

Hint. *Denote* $a_1 + \cdots + a_n$ *by* s. *It is easy to see that, for any typical element of the above subset, its* i*-th summand is at least* $\frac{a_i}{s}$, *for all* $i = 1, \ldots, n$. *Hence* 1 *is a lower bound. What about the case* $a_k \stackrel{\text{def}}{=} \frac{1}{2^k}$, *for all* $k = 1, \ldots, n$. *On the other hand, observe that* $\frac{a_1}{a_1+a_2+a_3} \leqslant 1 - \frac{a_2}{s} - \frac{a_3}{s}, \ldots, \frac{a_n}{a_n+a_1+a_2} \leqslant 1 - \frac{a_1}{s} - \frac{a_2}{s}$. *This shows that* $n-2$ *is an upper bound. Can you now make thoughtful choices of* a_k*'s?*

3.3. Let $I_n \stackrel{\text{def}}{=} [a_n, b_n]$, $\forall n \in \mathbb{N}$. Assume that, $I_{n+1} \subseteq I_n$, for any $n \in \mathbb{N}$. Show the following:

(a) For any $n, m \in \mathbb{N}$, $a_n \leqslant b_m$.

(b) $\sup\{a_n : n \in \mathbb{N}\}$ and $\inf\{b_n : n \in \mathbb{N}\}$ exist, and $\sup\{a_n : n \in \mathbb{N}\} \leqslant \inf\{b_n : n \in \mathbb{N}\}$.

(c) Find $\bigcap_{n=1}^{\infty} I_n$.

Hint. *Draw picture!*

3.4. Fix a positive integer $D \geqslant 3$. Let $\alpha \in [0,1]$. Show that, for all $n \in \mathbb{N}$, there exists $a_n \in \{0, 1, \ldots, D-1\}$ such that the following holds:

$$\sup\left\{\frac{a_1}{D} + \frac{a_2}{D^2} + \cdots + \frac{a_n}{D^n} : n \in \mathbb{N}\right\} = \alpha.$$

Note: We then say that $\frac{a_1}{D} + \frac{a_2}{D^2} + \cdots + \frac{a_n}{D^n} + \ldots$ is a *representation of* α *to the base* D.

Hint. *Try to define* a_n*'s inductively so that for any* $n \in \mathbb{N}$, *one has*

$$\frac{a_1}{D} + \frac{a_2}{D^2} + \cdots + \frac{a_n}{D^n} \leqslant \alpha < \frac{a_1}{D} + \frac{a_2}{D^2} + \cdots + \frac{a_n}{D^n} + \frac{1}{D^n}.$$

For instance, observe that if (3.4.) *holds for* $n = 1$ *then* a_1 *must be* $[10x]$. *Next, when* $n = 2$ *then* a_2 *has to be* $\left[10^2(x - \frac{a_1}{10})\right]$ *and so on.*

3.5.$*$ Generalize (2.4.) to the following:

For any $x \geqslant 0$ and $n \in \mathbb{N}$, there exists unique $y \geqslant 0$ such that $y^n = x$.

This unique y is called the *n-th root of x*, and denoted by $\sqrt[n]{n}$ or $x^{\frac{1}{n}}$. Note that $\sqrt[n]{x} \geqslant 0$.

Hint. *Proceed like* (2.4.).