DYNAMICS OF MACHINERY
(Code : 402042)

Semester VII- Mechanical and Automobile Engineering (Savitribai Phule Pune University)

Strictly as per the New Credit System Syllabus (2019 Course) of Savitribai Phule Pune University w.e.f. academic year 2022-2023

R. B. Patil

M.E. (Mech) (Design Engg.)
Former Professor,

Former Head of Mechanical Engineering
Department, Former Principal,
Army Institute of Technology,
Pune, Maharashtra.

Dr. F. B. Sayyad

DME, B.E (Mechanical Engg.), M.E (Design Engg.)
Ph.D.(Mechanical Vibration), FISME, SFSIESRP.

Principal

Dr D Y Patil School of Engineering,
Dr Ajeenkya DY Patil Knowledge City,
Charoli Bk, Lohegaon, Pune – 412105
fbsayyad8@gmail.com

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R. B. Patil, Dr. F. B. Sayyad

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First Printed in India : *June 2007 (Pune University)*
First Edition : *August 2022* ***(As per SPPU 2019 Course)***

***ISBN** : 978-93-5563-200-5*

Published By

TECHKNOWLEDGE PUBLICATIONS

Printed @

37/2, Ashtavinayak Industrial Estate,
Near Pari Company,
Narhe, Pune, Maharashtra State, India.
Pune - 411041

Head Office

B/5, First floor, Maniratna Complex, Taware Colony,
Aranyeshwar Corner, Pune - 411 009.
Maharashtra State, India

Ph : 91-20-24221234, 91-20-24225678.

Email : info@techknowledgebooks.com,

Website : www.techknowledgebooks.com

Subject Code : 402042
Book Code : PO229A

We dedicate this Publication soulfully and wholeheartedly,
in loving memory of our beloved founder director,
Late Shri. Pradeepji Lalchandji Lunawat,
who will always be an inspiration, a positive force and strong
support behind us.

"My work is my prayer to God"

- Lt. Shri. Pradeepji L. Lunawat

Soulful Tribute and Gratitude for all Your
Sacrifices, Hardwork and 40 years of Strong Vision...

PREFACE

Dear Students,

*It gives a great pleasure to present this book on '**Dynamics of Machinery**'. This book, which has been written for the final year students of mechanical engineering, comprehensively covers the two important aspects of Dynamics of Machinery, namely, vibrations and balancing. It covers in details the vibration measurement, analysis and control techniques.*

The main objective of this book is to bridge the gap between the reference books written by the renowned international authors and the requirements of undergraduate students.

The book has been presented in a simple language without compromising the quality of text. The concepts have been developed from the fundamentals. The main emphasis has been given on explaining the concepts rather than merely providing the information. Every concept is illustrated with the help of number of solved examples, which are arranged in the order of increasing degree of difficulty.

*We present this book in the loving memory of **Late Shri. Pradeepji Lunawat**, our source of inspiration and a strong foundation of "**TechKnowledge Publications**". He will always be remembered in our heart and motivate us to achieve our new milestone*

*We would like to express our thanks to **Shri. Shital Bhandari, Shri. Arunoday Kumar** and **Shri. Chandroday Kumar** of 'TechKnowledge Publications' for their efforts in publishing this book. We are also thankful to **Seema Lunawat** for technology enhanced reading, E-books support and also to all highly skilled professionals staff of **TechKnowledge Publications** for their constant hard work during preparation of this book. Our thanks are due to all those people who have directly or indirectly contributed to this book.*

Last, but not least, our special thanks go to all our students whose continuous feedback while teaching this subject is the source of inspiration behind this book.

A feedback in the form of suggestions and comments from the readers for further improvement of the book will be highly appreciated.

- R.B. Patil

– F. B. Sayyad

fbsayyad8@gmail.com

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SYLLABUS

Savitribai Phule Pune University

Board of Studies - Mechanical and Automobile Engineering

Final Year Mechanical Engineering (2019 pattern)

Dynamics of Machinery (402042)

Teaching Scheme		Credits		Examination Scheme	
Theory	3 Hrs./Week	Theory	3	In-Semester	30 Marks
Practical	2 Hrs./Week	Practical	1	End-Semester	70 Marks
				Oral	25 Marks

Pre-requisites : Strength of Materials, Engineering Mechanics, Kinematics of Machinery, Engineering Mathematics and Numerical Methods

Course Objectives :

1. To conversant with balancing problems of machines.
2. To understand mechanisms for system control – Gyroscope.
3. To understand fundamentals of free and forced vibrations.
4. To develop competency in understanding of vibration in Industry.
5. To develop analytical competency in solving vibration problems.
6. To understand the various techniques of measurement and control of vibration and noise.

Course Outcomes :

On completion of the course, students will be able to -

CO1 ***APPLY*** *balancing technique for static and dynamic balancing of multi cylinder inline and radial engines.*

CO2 ***ANALYZE*** *the gyroscopic couple or effect for stabilization of Ship, Airplane and Four wheeler vehicles.*

CO3 ***ESTIMATE*** *natural frequency for single DOF un-damped & damped free vibratory systems.*

CO4 ***DETERMINE*** *response to forced vibrations due to harmonic excitation, base excitation and excitation due to unbalance forces.*

CO5 ***ESTIMATE*** *natural frequencies, mode shapes for 2 DOF un-damped free longitudinal and torsional vibratory systems.*

CO6 ***DESCRIBE*** *noise and vibration measuring instruments for industrial / real life applications along with suitable method for noise and vibration control.*

In Semester

Unit 1 : Balancing

Static and dynamic balancing, balancing of rotating masses in single and several planes, primary and secondary balancing of reciprocating masses, balancing in single cylinder engines, balancing in multi-cylinder in-line engines, direct and reverse cranks method -radial and V engines. Introduction to Balancing machines - Types, Classification and Methods **Refer Chapter - 1**

Unit 2 : Gyroscope

Introduction, Precessional angular motion, Gyroscopic couple, Effect of gyroscopic couple on an airplane, Effect of gyroscopic couple on a naval ship during steering, pitching and rolling, Stability of a Four Wheel drive moving in a curved path, Stability of a two wheel vehicle taking a turn, Effect of gyroscopic couple on a disc fixed rigidly at a certain angle to a rotating shaft. **Refer Chapter - 2**

End Semester

Unit 3 : Single Degree of Freedom Systems – Free Vibration

Fundamentals of Vibration : Elements of a vibratory system, vector representation of S.H.M., degrees of freedom, Introduction to Physical and Mathematical modeling of vibratory systems: Bicycle, Motor bike and Quarter Car. types of vibration, equivalent stiffness and damping, formulation of differential equation of motion (Newton, D'Alembert and energy method)

Un-damped free vibrations : Natural frequency for longitudinal, transverse and torsional vibratory systems.

Damped free vibrations : Different types of damping, Viscous damping - over damped, critically damped and under damped systems, initial conditions, logarithmic decrement, Dry friction or coulomb damping - frequency and rate of decay of oscillations. **Refer Chapter - 3**

Unit 4 : Single Degree of Freedom Systems - Forced Vibrations

Forced vibrations of longitudinal and torsional systems, Frequency Response to harmonic excitation, excitation due to rotating and reciprocating unbalance, base excitation, magnification factor, Force and Motion transmissibility, Quality Factor. Half power bandwidth method, Critical speed of shaft having single rotor of un-damped systems. **Refer Chapter - 4**

Unit 5 : Two Degree of Freedom Systems – Un-damped Vibrations

Free vibration of spring coupled systems - longitudinal and torsional, torsionally equivalent shafts, natural frequency and mode shapes, Eigen value and Eigen vector by Matrix method, Combined rectilinear and angular motion, Vibrations of Geared systems. Refer Chapter - 5

Unit 6 : Measurement and Control of Vibrations, Introduction to Noise

A) **Measurement** : Vibration Measuring Instruments, Accelerometers, Impact hammer, Vibration shakers, Vibration Analyzer, Vibration based condition monitoring, Analysis of Vibration Spectrum, Standards related to measurement of vibration.

B) **Control** : Vibration control methods - passive, semi active and active vibration control, control of excitation at the source, control of natural frequency, Vibration isolators, Tuned Dynamic Vibration Absorbers.

C) **Noise** : Fundamentals of noise, Sound concepts, Decibel Level, Logarithmic addition, subtraction and averaging, sound intensity, noise measurement, Noise control at the Source, along the path and at the receiver, Reverberation chamber, Anechoic Chamber, Noise standards. (Unit VI – Only theoretical treatment) Refer Chapter - 6

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UNIT - I

Chapter 1 : Balancing 1-1 to 1-62

Syllabus : Static and Dynamic Balancing, Balancing of Rotating Masses in Single and Several Planes, Primary and Secondary Balancing of Reciprocating Masses, Balancing in Single Cylinder Engines, Balancing in Multicylinder In-Line Engines, Direct and Reverse Cranks Method - Radial and V Engines. Introduction to Balancing machines - Types, Classification and Methods

UNIT II

Chapter 2 : Gyroscope 2-1 to 2-54

Syllabus : Introduction, Precessional angular motion, Gyroscopic couple, Effect of gyroscopic couple on an airplane, Effect of gyroscopic couple on a naval ship during steering, pitching and rolling, Stability of a Four Wheel drive moving in a curved path, Stability of a two wheel vehicle taking a turn, Effect of gyroscopic couple on a disc fixed rigidly at a certain angle to a rotating shaft.

UNIT III

Chapter 3 : Single Degree of Freedom Systems : Free Vibrations 3-1 to 3-85

Syllabus : Fundamentals of Vibration : Elements of a Vibratory System, Vector Representation of S.H.M., Degrees of Freedom, Introduction to Physical and Mathematical Modeling of Vibratory Systems : Bicycle, Motor Bike and Quarter Car. Types of Vibration, Equivalent Stiffness and Damping, Formulation of Differential Equation of Motion (Newton, D'Alembert and Energy Method) ;

Undamped Free Vibrations : Natural Frequency for Longitudinal, Transverse and Torsional Vibratory Systems.

Damped Free Vibrations : Different Types of Damping, Viscous Damping - Over Damped, Critically Damped and Under Damped Systems, Initial Conditions, Logarithmic Decrement, Dry Friction or Coulomb Damping - Frequency and Rate of Decay of Oscillations.

UNIT IV

Chapter 4 : Single Degree of Freedom Systems : Forced Vibrations 4-1 to 4-41

Syllabus : Forced vibrations of Longitudinal and Torsional Systems, Frequency Response to Harmonic Excitation, Excitation due to Rotating and Reciprocating Unbalance, Base Excitation, Magnification Factor, Force and Motion Transmissibility, Quality Factor, Half Power Bandwidth Method, Critical Speed of Shaft Having Single Rotor of Undamped Systems.

UNIT V

Chapter 5 : Two Degree of Freedom Systems : Undamped Free Vibrations 5-1 to 5-51

Syllabus : Free Vibration of Spring Coupled Systems - Longitudinal and Torsional, Torsionally Equivalent Shafts, Natural Frequency and Mode Shapes, Eigen Value and Eigen Vector by Matrix Method, Combined Rectilinear and Angular Motion, Vibrations of Geared Systems.

UNIT VI

Chapter 6 : Measurement and Control of Vibrations and Introduction to Noise 6-1 to 6-59

Syllabus : Fundamentals of Noise, Sound Concepts, Decibel Level, White Noise, Weighted Sound Pressure Level, Logarithmic Addition, Subtraction and Averaging, Sound Intensity, Noise Measurement, Sound Fields, Octave Band, Sound Reflection, Absorption and Transmission, Acoustic Material & its Characteristics, Noise Control at Source, Along the Path and at the Receiver, Pass-by-Noise, Reverberation Chamber, Anechoic Chamber, Human Exposure to Noise and Noise Standards.

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3

UNIT III

SINGLE DEGREE OF FREEDOM SYSTEMS : FREE VIBRATIONS

Syllabus

Fundamentals of Vibration : Elements of a Vibratory System, Vector Representation of S.H.M., Degrees of Freedom, Introduction to Physical and Mathematical Modeling of Vibratory Systems : Bicycle, Motor Bike and Quarter Car. Types of Vibration, Equivalent Stiffness and Damping, Formulation of Differential Equation of Motion (Newton, D'Alembert and Energy Method) ;

Undamped Free Vibrations : Natural Frequency for Longitudinal, Transverse and Torsional Vibratory Systems.

Damped Free Vibrations : Different Types of Damping, Viscous Damping - Over Damped, Critically Damped and Under Damped Systems, Initial Conditions, Logarithmic Decrement, Dry Friction or Coulomb Damping - Frequency and Rate of Decay of Oscillations.

TOPICS

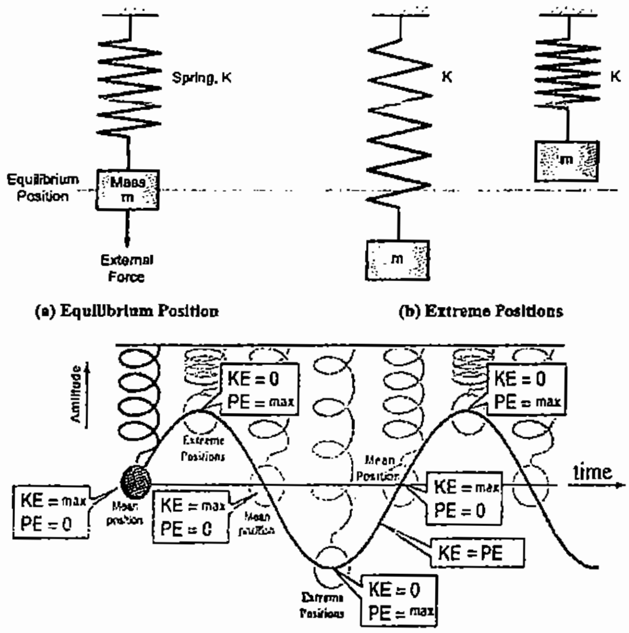
PART I : FUNDAMENTALS OF VIBRATIONS

3.1 INTRODUCTION TO VIBRATIONS

- **Vibrations** : When any elastic body such as spring, shaft or beam, is displaced from the equilibrium position by the application of external forces and then released, it commences cyclic motion. *Such cyclic motion of a body or a system, due to elastic deformation under the action of external forces, is known as* **vibration**. All bodies possessing mass and elasticity are capable of vibration.

3.1.1 Phenomenon of Vibration :

- When the elastic body is displaced from its equilibrium position, work is done by the external force in producing the displacement or deformation against the internal elastic forces, which resist deformation. This work done is stored in the body as an **elastic strain energy** (also called as internal **potential energy**).
- When external force is removed, the resisting internal elastic forces cause the body to restore to its equilibrium position. As the body moves towards the equilibrium position, elastic strain energy is gradually converted to **kinetic energy.**
- At the instant body reaches its equilibrium position, complete elastic strain energy is converted to kinetic energy.
- At equilibrium position as body possesses maximum kinetic energy, the motion of body continues until the complete kinetic energy is absorbed in doing the work against the internal elastic forces and kinetic energy is again converted to elastic strain energy. Thus as the body reaches extreme position, it possesses the maximum elastic strain energy. The body again begins to return to its equilibrium position and oscillations or vibrations repeated indefinitely. The motion of spring - mass system, shown in Fig. 3.1.1, is an example of vibration.



(c) Motion of Spring-mass system

Fig. 3.1.1 : Spring-mass System

3.1.2 Causes of Vibrations :

(i) **Unbalance forces** and **couples** in the machine parts.

(ii) **External excitation forces** applied on the system.

(iii) **Dry friction** between two mating surfaces.

(iv) **Winds** may cause vibrations in certain systems such as telephone lines, electricity lines, etc.

(v) **Earthquakes** also cause vibrations in civil structures like buildings, dams, etc.

3.1.3 Disadvantages of Vibrations :

(i) It creates **excessive stresses** in machine parts.

(ii) It leads to **loosening** of assembled parts.

(iii) It may lead to partial or complete **failure of machine parts**.

(iv) It creates **undesirable noise.**

3.1.4 Advantages of Vibrations :

(i) All **musical instruments** work on phenomenon of vibration.

(ii) **Vibrating screens, shakers and conveyors** work on phenomenon of vibrations.

(iii) In **stress relieving equipment**, vibrations are useful.

3.1.5 Methods of Reducing Effects of Undesirable Vibrations :

Some of the methods used for reducing the effects of undesirable are as follows :

(i) By **removing unbalanced forces and couples** in machine parts, which cause vibrations.

(ii) By placing the machinery on proper type of vibration **isolators**.

(iii) By putting the **sound proof screens** or **glass**, if noise is created due to vibrating parts.

(iv) By using **shock absorbers**.

3.2 TERMINOLOGY AND BASIC CONCEPTS

Some of the basic concepts used in vibration study are discussed below :

1. Simple Harmonic Motion (S.H.M.) :

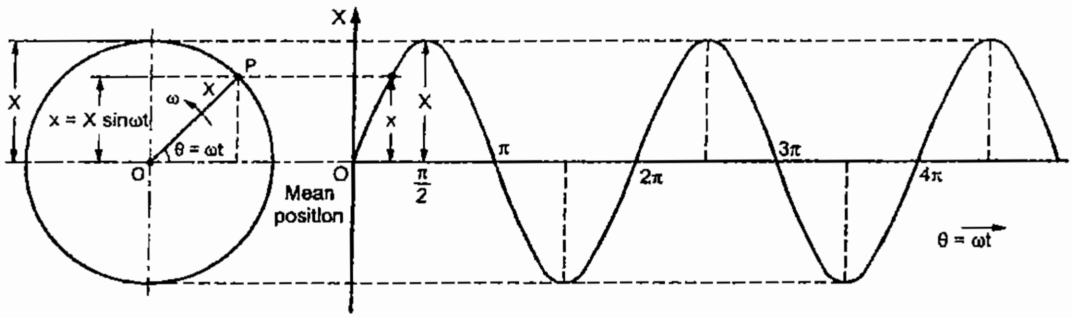


Fig. 3.2.1 : Simple Harmonic Motion

Let, x = displacement of point from mean position after time 't'.

X = maximum displacement of point from mean position.

From Fig. 3.2.1;

- **Displacement of point :**

$$x = X \sin \theta$$

or $$x = X \sin \omega t \qquad ...(3.2.1)$$

- **Velocity of point :**

$$\dot{x} = \frac{dx}{dt}$$

or $$\dot{x} = \omega X \cos \omega t$$

- **Acceleration of point :**

$$\ddot{x} = \frac{d^2 x}{dt^2} = -\omega^2 X \sin \omega t$$

or $$\ddot{x} = -\omega^2 x \quad ... [\because x = X \sin \omega t] \qquad ...(3.2.2)$$

- Acceleration $\propto$ Displacement From Mean Position.
- **Simple harmonic motion** : A motion, whose acceleration is proportional to displacement from mean position and is directed towards the mean position, is known as **simple harmonic motion**.
- **Fundamental equation of SHM :**

The Equation (3.2.2) can be written as,

$$\ddot{x} + \omega^2 x = 0 \qquad ...(3.2.3)$$

- The Equation (3.2.3) is known as **fundamental equation of simple harmonic motion.**

2. Time Period (t_p) :

Time period is the time required to complete one cycle (2π).

Mathematically,

$$t_p = \frac{2\pi}{\omega}, \quad \text{sec} \qquad ...(3.2.4)$$

3. Frequency (f) :

The number of cycles per unit time is known as frequency. It is a reciprocal of time period.

$$f = \frac{1}{t_p} = \frac{1}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi}, \quad \text{Hz} \qquad ...(3.2.5)$$

4. Amplitude (X) :

It is the maximum displacement of a vibrating body from its mean position.

5. Stiffness of spring (K) :

It is the force required to produce unit displacement in the direction of applied force.

$$K = \frac{F}{\delta}, \quad \text{N/m} \qquad ...(3.2.6)$$

where, K = stiffness of spring, N/m

F = force applied on spring, N

δ = deflection of spring, m.

6. Degrees of Freedom (D.O.F) :

- *The minimum number of independent co-ordinates required to specify the motion or configuration of a system at any instant is known as* ***degrees of freedom*** *of that system.*
- In general, degrees of freedom (D.O.F.) is equal to the number of independent displacements that are possible.
- **Examples of D.O.F :** The Fig. 3.2.2 shows examples of one, two and three degrees of freedom (D.O.F) systems.

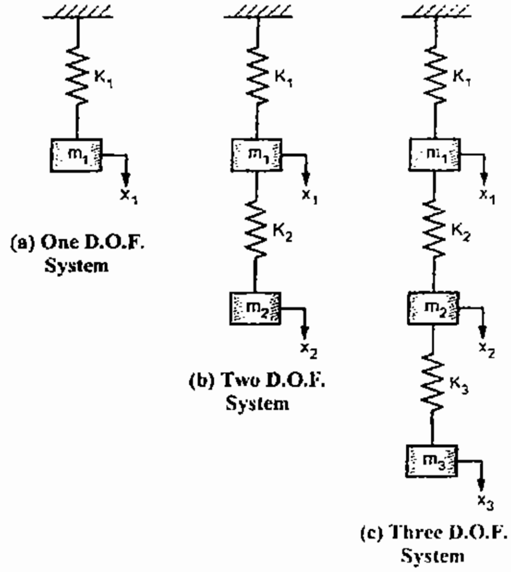


Fig. 3.2.2 : Degree of Freedom (D.O.F)

7. Damping :

Damping *is the resistance to the motion of the vibrating body, which causes a vibrating body to come to rest or equilibrium position.*

8. Damping Coefficient (c) :

University Question

Q. Define the damping coefficient related to vibrations.

SPPU : Dec. 12, Dec. 13, May 14, Dec. 16, Dec. 17

Damping coefficient *is the damping force or resisting force developed per unit velocity.* Mathematically,

$$c = \frac{F}{v}, \frac{N}{m/sec} \text{ or N-sec/m} \qquad ...(3.2.7)$$

where, F = Force applied on damper or damping force in N.

v = Velocity of viscous fluid in m/sec.

9. Resonance :

When the frequency of external excitation force acting on a body is equal to the natural frequency of a vibrating body, the body starts vibrating with excessively large amplitude. Such state is known as ***resonance****.*

3.3 ELEMENTS OF VIBRATORY SYSTEM

- Any vibratory system contains :
  (i) **Spring :** For storing internal potential energy;
  (ii) **Mass :** For storing kinetic energy; and
  (iii) **Damper :** For gradually dissipating energy.

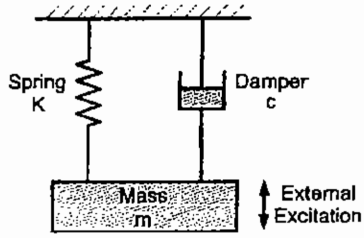
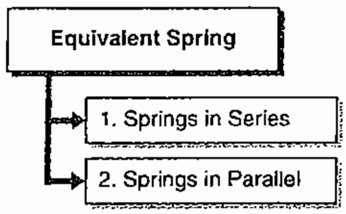


Fig. 3.3.1 : Elements of a Vibratory System

- Fig 3.3.1 shows a basic vibratory system. The energy enters the system with the application of external force known as **external excitation.** The excitation disturbs the mass from its mean position and mass starts vibrating (oscillating) between two extreme positions. During the vibration (oscillation), the kinetic energy is converted into potential energy and potential energy is converted into kinetic energy. This sequence goes on repeating and the system continues to vibrate.
- If some damping is provided to oppose the motion of mass, then some amount of energy is dissipated in each cycle of vibration due to damping effect. Due to this, the vibrations decay gradually and system will come to its static equilibrium or mean position.

3.4 EQUIVALENT SPRINGS

In many practical applications, more than one springs may be used. To convert such system into equivalent mathematical model, it is necessary to replace springs in system by one equivalent spring.



1. Springs in Series :

- Fig. 3.4.1(a) shows a system with two springs in series having stiffness K_1 and K_2. For springs in series :

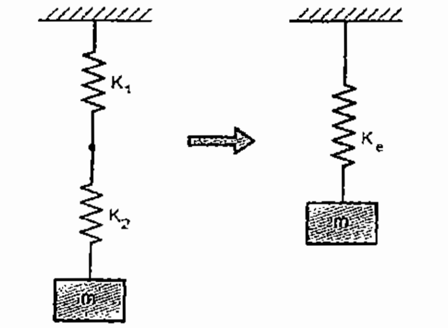
(i) $\begin{bmatrix} \text{Deflection of} \\ \text{Equivalent} \\ \text{Spring} \end{bmatrix} = \begin{bmatrix} \text{Deflection} \\ \text{of} \\ \text{Spring 1} \end{bmatrix} + \begin{bmatrix} \text{Deflection} \\ \text{of} \\ \text{Spring 2} \end{bmatrix}$

$$\delta = \delta_1 + \delta_2.$$

(ii) $\begin{bmatrix}\text{Force on}\\ \text{Equivalent}\\ \text{Spring}\end{bmatrix} = \begin{bmatrix}\text{Force}\\ \text{on}\\ \text{Spring 1}\end{bmatrix} = \begin{bmatrix}\text{Force}\\ \text{on}\\ \text{Spring 2}\end{bmatrix}$

$$mg = m_1 g = m_2 g$$

- The system of two springs in series is to be replaced by an equivalent spring having stiffness K_e [Fig. 3.4.1(b)].



(a) Actual System (b) Equivalent System

Fig. 3.4.1 : Spring in Series

$$\begin{matrix}\text{Deflection of}\\ \text{equivalent}\\ \text{spring}\end{matrix} = \begin{matrix}\text{Deflection of}\\ \text{equivalent}\\ \text{spring 1}\end{matrix} + \begin{matrix}\text{Deflection of}\\ \text{equivalent}\\ \text{spring 2}\end{matrix}$$

$$\delta = \delta_1 + \delta_2 \quad \text{...(a)}$$

$$\therefore \frac{mg}{K_e} = \frac{m_1 g}{K_1} + \frac{m_2 g}{K_2} \quad \text{...(b)}$$

But, $mg = m_1 g = m_2 g$ ...(c)

- Substituting Equation (c) in Equation (b), we get,

$$\frac{mg}{K_e} = \frac{mg}{K_1} + \frac{mg}{K_2}$$

$$\frac{1}{K_e} = \frac{1}{K_1} + \frac{1}{K_2} \quad \text{...(3.4.1)}$$

2. Springs in Parallel :

- Fig. 3.4.2(a) shows a system with two springs in parallel having stiffness K_1 and K_2. For springs in parallel :

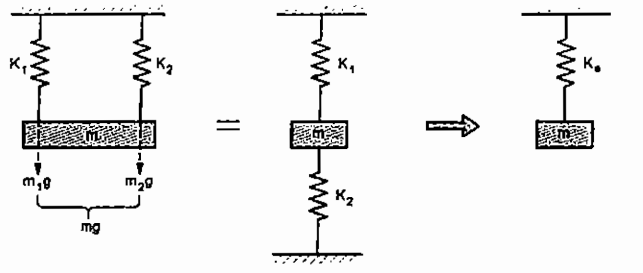
(i) $\begin{bmatrix}\text{Deflection of}\\ \text{Equivalent}\\ \text{Spring}\end{bmatrix} = \begin{bmatrix}\text{Deflection}\\ \text{of}\\ \text{Spring 1}\end{bmatrix} = \begin{bmatrix}\text{Deflection}\\ \text{of}\\ \text{Spring 2}\end{bmatrix}$

$$\delta = \delta_1 = \delta_2$$

(ii) $\begin{bmatrix}\text{Force on}\\ \text{Equivalent}\\ \text{Spring}\end{bmatrix} = \begin{bmatrix}\text{Force}\\ \text{on}\\ \text{Spring 1}\end{bmatrix} + \begin{bmatrix}\text{Force}\\ \text{on}\\ \text{spring 2}\end{bmatrix}$

$$mg = m_1 g + m_2 g$$

- The system of two springs in parallel is replaced by an equivalent spring having stiffness K_e, as shown in Fig. 3.4.2(b).



(a) Actual System (b) Another Form of Actual System (c) Equivalent System

Fig. 3.4.2 : Springs in Parallel

$$mg = m_1 g + m_2 g \quad \text{...(d)}$$

$$\therefore K_e\delta = K_1\delta_1 + K_2\delta_2 \quad \text{...(e)}$$

But $\delta = \delta_1 = \delta_2$ ... (f)

- Substituting Equation (f) in Equation (e), we get,

$$K_e\delta = K_1\delta + K_2\delta$$

$$\therefore \mathbf{K_e = K_1 + K_2} \quad \text{...(3.4.2)}$$

3.5 EQUIVALENT DAMPERS

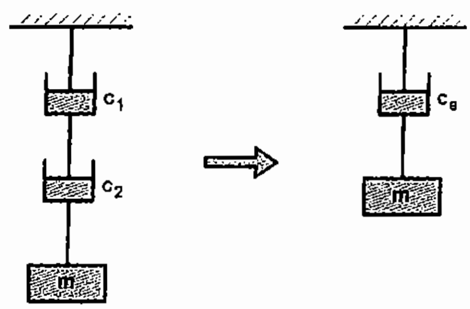
In many applications, similar to springs, number of dampers are also used in combinations. To convert such system into equivalent mathematical model, it is necessary to replace dampers in system by one equivalent damper.

Equivalent Damper
- 1. Dampers in Series
- 2. Dampers in Parallel

1. Dampers in Series :

- Fig. 3.5.1(a) shows a system having two dampers, with damping coefficients c_1 and c_2 in series. For dampers in series,

$$\frac{1}{c_e} = \frac{1}{c_1} + \frac{1}{c_2} \quad \text{...(3.5.1)}$$



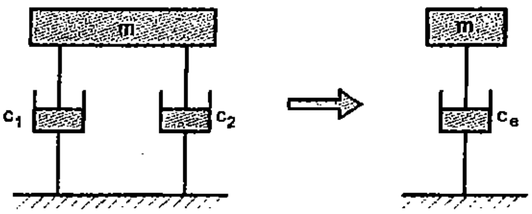
(a) Actual System **(b) Equivalent System**

Fig. 3.5.1 : Dampers in Series

2. Dampers in Parallel :

- Fig. 3.5.2(a) shows a system, having two dampers with damping coefficients c_1 and c_2, in parallel. For Dampers in parallel,

$$c_e = c_1 + c_2 \quad ...(3.5.2)$$



(a) Actual System **(b) Equivalent System**

Fig. 3.5.2 : Dampers in Parallel

3.6 INTRODUCTION TO MODELING

Modelling :

Modeling *is the art of abstracting or representing the object, system or phenomenon.*

Types of Modeling :

1. Physical Modeling
2. Geometric Modeling
3. Mathematical Modeling
4. Combination of Geometric and Mathematical Modeling

1. Physical Modeling :

- **Physical modeling** *is the process of making the actual prototype of an object (or a system) by using the material like : plastic, wood or other easy-to-work materials.*
- The physical model mimic the actual object (or a system). The physical models are mainly useful for understanding the object (or a system).

2. Geometric Modeling :

- **Geometric modeling** *is the process of complete representation of an object (or a system) with the graphical and non-graphical information.*
- It generates the mathematical description of the geometry of an object (or a system) in the computer database and the image of an object (or a system) on the graphics screen.

3. Mathematical Modeling :

- **Mathematical modeling** *is the process of representing a system (or phenomenon or process) in terms of a set of equations.*
- The basis for creating the mathematical model of a system (or phenomenon or process) is the physical laws (such as the Newton's laws and the law of conservation of energy, etc.) that the system elements as a group obey. The mathematical model is solved for variety of input variables, using the computer, to obtain a set of out puts.
- If the mathematical model is complex, the degree of accuracy of solution is high. However, the process is time consuming and the computer system (hardware and software) should have high capability. Many-a-times, the mathematical model is simplified by ignoring some of the relatively unimportant parameters and making some assumptions. This makes the analysis simple.
- **Simulation** *is the process of exploration of a model behavior by varying the input variables.*

4. Combination of Geometric and Mathematical Modeling :

- With the use of computer system and various software, it is possible to combine the geometric and mathematical models.
- This is highly effective in simulation and analysis of the systems.

3.7 PRACTICAL EXAMPLES OF MATHEMATICAL MODELING

Some of the practical examples of mathematical modeling are discussed below :

3.7.1 Mathematical Modeling of Motor Bike :

University Question

Q. Explain with neat diagram mathematical model of a motorbike.

SPPU : May 16, May 17, Oct. 18(In sem), May 19

- Fig. 3.7.1(a) shows a physical system consisting of motor bike with rider.
- **Mathematical model of motor bike and rider** : In order to develop a mathematical model of a physical system consisting of a motor bike and a rider, following parameters are considered :
  1. Mass of rider, m_r
  2. Stiffness of rider, K_r
  3. Damping coefficient of rider, C_r
  4. Mass of vehicle body (except wheels), m_v
  5. Stiffness of rear suspension, K_{s1}

6. Damping coefficient of rear suspension, C_{s1}
7. Stiffness of front suspension, K_{s2}
8. Damping coefficient of front suspension, C_{s2}
9. Mass of each wheel, m_w
10. Stiffness of each tyre K_t

- Fig. 3.7.1(b) shows a mathematical model of a physical system consisting of a motor bike and a rider.

- **Simplified mathematical model of motor bike :**

(i) If the stiffness and damping coefficient of rider are neglected, the mathematical model can be simplified as shown in Fig. 3.7.1(c).

(ii) Assuming that front and rear suspensions are identical, the mathematical model can be further simplified, as shown in Fig. 3.7.1(d).

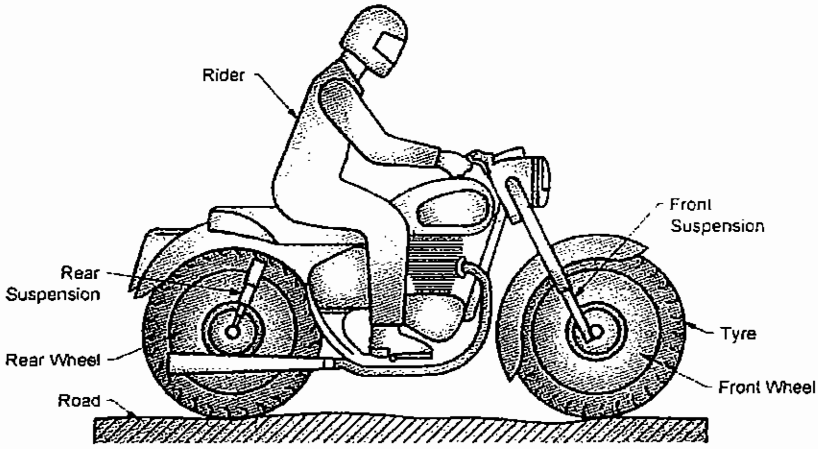
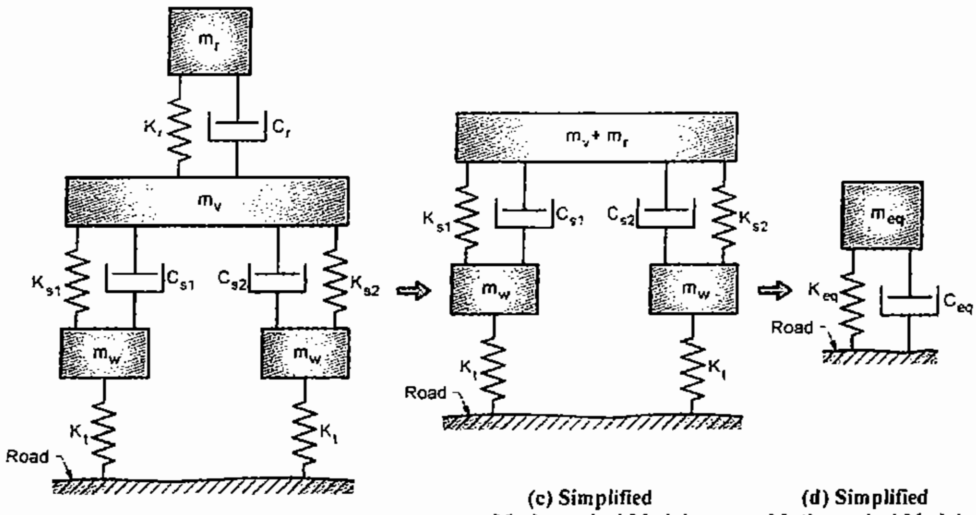


Fig. 3.7.1(a) : Physical System-Motor Bike and Rider



(b) Real Mathematical Model

(c) Simplified Mathematical Model (Neglecting Stiffness and Damping Coefficient of Rider)

(d) Simplified Mathematical Model (Assuming $K_{s1} = K_{s2} = K_s$ and $C_{s1} = C_{s2} = C_s$)

Fig. 3.7.1 : Mathematical Model of Bike

3.7.2 Mathematical Modeling of Bicycle :

University Question

Q. Explain mathematical model of a bicycle with a rider. **SPPU : Dec. 19**

- The mathematical modeling of a bicycle is similar to that of the motor bike, discussed in section 3.7.1. The only difference is, in bicycle the damping coefficient of suspension system is neglected i.e. $C_{s1} = C_{s2} = 0$. Fig. 3.7.2(a) shows a physical system consisting of motor bike with rider.

- **Mathematical model of bicycle** : In order to develop a mathematical model of a physical system consisting of a bicycle and a rider, following parameters are considered :
  1. Mass of rider, m_r
  2. Stiffness of rider, K_r
  3. Damping coefficient of rider, C_r
  4. Mass of bicycle (except wheels), m_v
  5. Stiffness of rear suspension, K_{s1}
  6. Stiffness of front suspension, K_{s2}
  7. Mass of each wheel, m_w
  8. Stiffness of each tyre K_t

  Fig. 3.7.2(a) shows the physical model of bicycle with rider while Fig. 3.7.2(b) shows the mathematical model of bicycle.

- **Simplified mathematical model of bicycle :**

  Fig. 3.7.2(c) and Fig. 3.7.2(d) show the simplified mathematical model of bicyle.

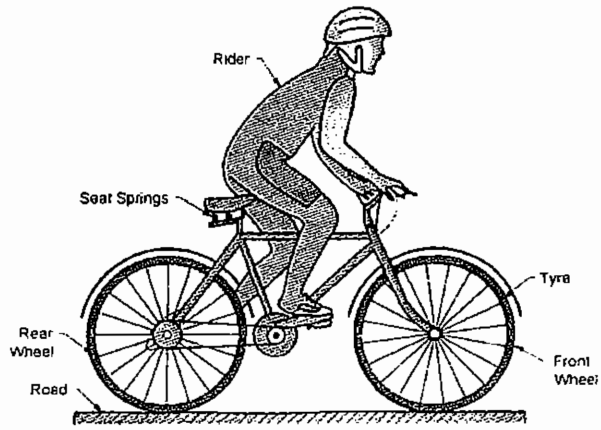
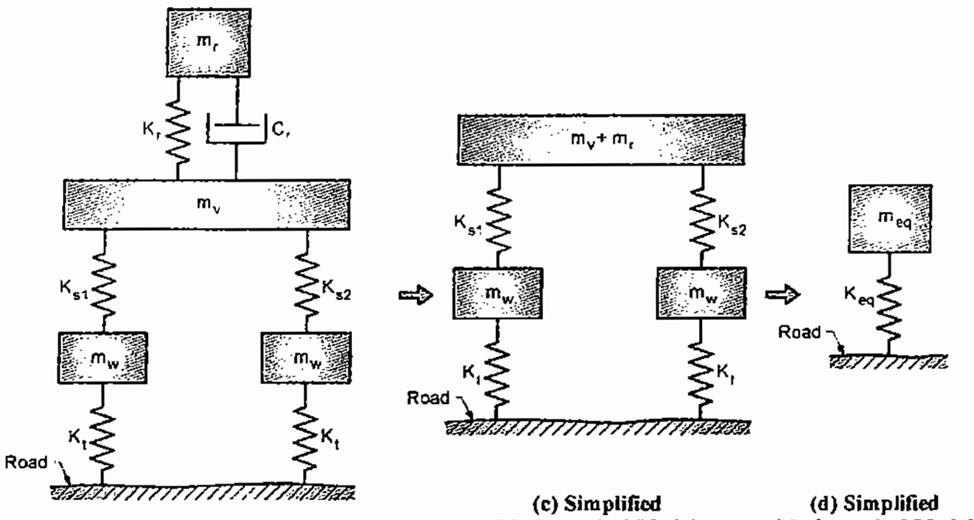


Fig. 3.7.2(a) : Physical Model of Bicycle



(b) Real Mathematical Model

(c) Simplified Mathematical Model (Neglecting Stiffness and Damping Coefficient of Rider)

(d) Simplified Mathematical Model (Assuming $K_{s1} = K_{s2} = K_s$

Fig. 3.7.2 : Mathematical Model of Bicycle

3.7.3 Mathematical Model of Car :

University Question

Q. Explain with neat diagram mathematical model of a quarter car. **SPPU : Oct. 19(In Sem.)**

Fig. 3.7.3(a) shows a physical system of a car.

- **Mathematical model of car** : In order to develop a mathematical model of a physical system of car, following parameters are considered :
  1. Mass of driver and driver seat, m_d
  2. Stiffness of driver seat spring, K_d
  3. Damping coefficient of driver seat, C_d
  4. Stiffness of back rest, K_b

5. Damping coefficient of back rest, C_b
6. Mass of vehicle body, m_v
7. Stiffness of rear suspension, K_{S1}
8. Damping coefficient of rear suspension, C_{s1}
9. Stiffness of front suspension, K_{S2}
10. Damping coefficient of front suspension, C_{s2}
11. Mass of rear wheel and axle, m_{W1}
12. Mass of front wheel and axle, m_{w2}
13. Stiffness of rear tyre, K_{t1}
14. Stiffness of front tyre, K_{t2}

Fig. 3.7.3(b) shows a mathematical model of a car.

Simplified mathematical model of car :

(i) If the stiffness and damping coefficient of driver seat is neglected, the mathematical model can be simplified as shown in Fig. 3.7.3(c).

(ii) Assuming that the front and rear suspensions are identical, the mathematical model can be simplified as shown in Fig. 3.7.3(d).

- The chassis is a largest mass supported on the front and rear axles through the suspension systems. The suspension systems between the chassis and the front and rear axles are represented by springs and dampers.
- The wheels and axles are equivalent to the masses supported by the tyres. The tyres have elasticity, and hence, are represented by the springs between the wheels and the road.
- The driver seat is cushioned from the chassis motion. Therefore, a driver seat is equivalent to a mass supported by a spring and a damper on the chassis.
- There is a friction between the driver seat back and the support.
- This modeling is done with the assumption that the vehicle is traveling at a constant speed and there is no horizontal inertia effect.

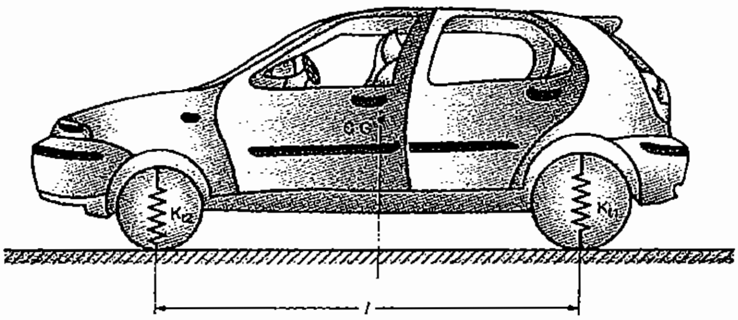
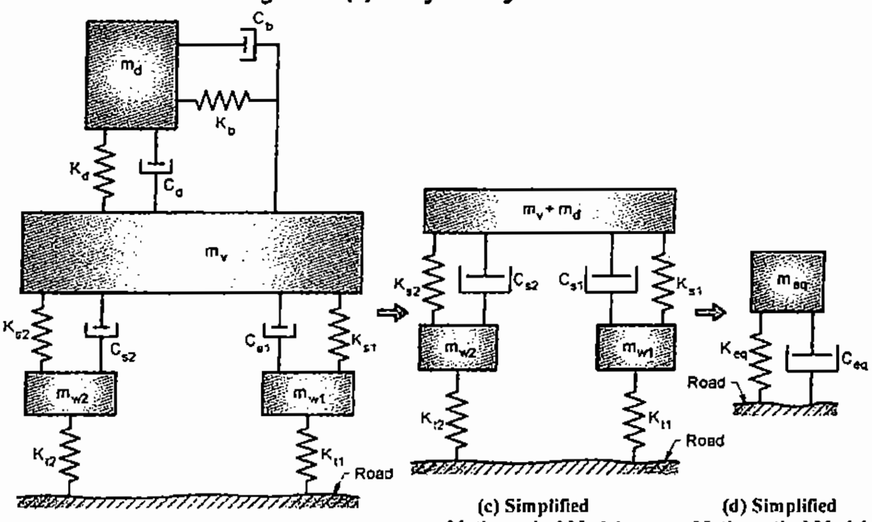


Fig. 3.7.3(a) : Physical system of car



(b) Real Mathematical Model

(c) Simplified Mathematical Model (Neglecting Stiffness and Damping Coefficient of Driver Seat)

(d) Simplified Mathematical Model (Assuming $K_{s1} = K_{s2} = K_s$ and $C_{s1} = C_{s2} = C_s$)

Fig. 3.7.3 : Mathematical Model of Car

3.8 TYPES OF VIBRATIONS

- The vibrations can be classified in several ways [Fig. 3.8.1]. Some of the important classifications are discussed below :

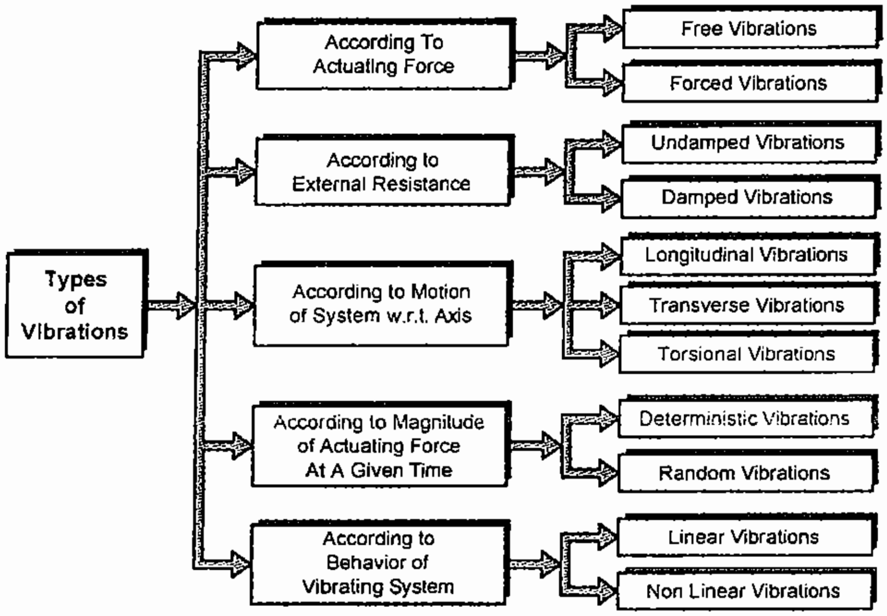


Fig. 3.8.1 : Classification of Vibrations

1. According to Actuating Force :

(i) Free Vibrations :

University Questions

Q. Explain : Free vibrations. **SPPU : Dec. 12**

Q. What is natural frequency? **SPPU : May 19**

- *If the external force is removed after giving an initial displacement to the system, then the system vibrates on its own due to internal elastic forces. Such type of vibrations are knows as* ***free vibrations****. The frequency of free vibrations is* known as **free** or **natural frequency (f_n)**.

  Example : Oscillation of simple pendulum.

(ii) Forced Vibrations :

University Question

Q. Explain : Forced vibrations. **SPPU : Dec. 12**

- *If a system or a body is subjected to a periodic external excitation force, then the resulting vibrations are known as* ***forced vibrations***.
- **Example :** Vibrations of I.C. engine, electric motor, centrifugal pump, etc.

2. According to External Resistance :

(i) Undamped Vibrations :

- *If there is no external resistance to the vibration of a system, then such vibrations are known as* ***undamped vibrations****[Fig.3.8.2].* The negligible resistance (damping) is also considered as an undamped condition.
- **Example :** Simple pendulum with negligible air resistance is an example of undamped free vibrations.
- Theoretically, free undamped vibrations continue indefinitely because there is no external resistance (damping).

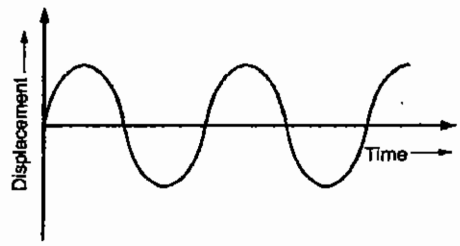


Fig. 3.8.2 : Undamped Vibrations

(ii) Damped Vibration :

University Question

Q. Explain : Damped Vibrations. **SPPU : Dec. 12**

- *If an external resistance is provided to the vibrating system, then such vibrations are known as* ***damped vibrations*** *[Fig. 3.8.3].* The external resistance is called damping.
- Due to damping there is reduction in amplitude of vibration every cycle and eventually vibrations die out.

Example : Vehicle moving over a rough road with shock absorber

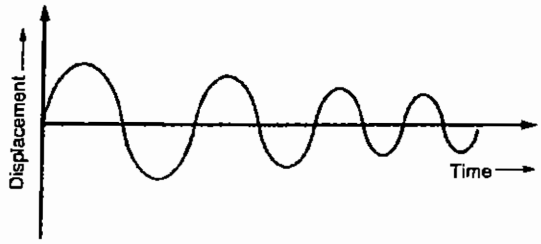


Fig. 3.8.3 : Damped Vibrations

3. According to Motion of System With Respect to Axis :

(i) Longitudinal Vibrations :

> **University Question**
> **Q.** With neat diagram, explain longitudinal vibrations.
> **SPPU : May 13**

- *If the vibration is along the axis of shaft, then the vibrations are known as* **longitudinal vibrations.**
- In longitudinal vibrations, [Fig. 3.8.4] the shaft is subjected to alternate direct tensile and compressive stresses.

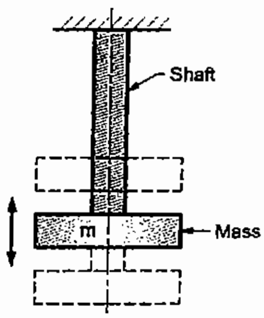


Fig. 3.8.4 : Longitudinal Vibrations

(ii) Transverse Vibrations :

> **University Question**
> **Q.** With neat diagram, explain transverse vibrations.
> **SPPU : May 13**

- *If the vibration is perpendicular to the axis of shaft, then the vibrations are known as* **transverse vibrations**.
- In transverse vibrations [Fig. 3.8.5], the shaft is subjected to alternate bending stresses.

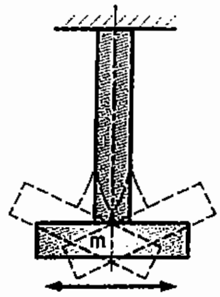


Fig. 3.8.5 : Transverse Vibrations

(iii) Torsional Vibrations :

- *If the vibration is about the axis of shaft such that the shaft gets twisted and untwisted alternately, then the vibrations are known as* ***torsional vibration.***
- In torsional vibrations [Fig. 3.8.6], the shaft is subjected to torsional shear stresses.

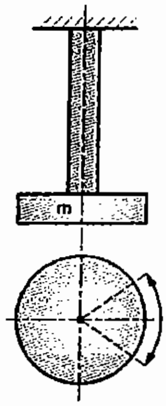


Fig. 3.8.6 : Torsional Vibrations

4. According to Magnitude of Actuating Force at a Given Time :

(i) Deterministic Vibration :

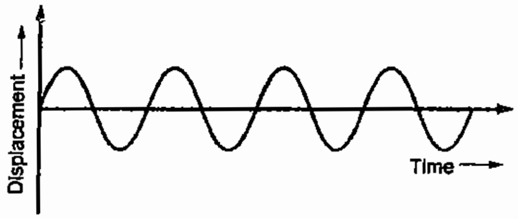
- *If the magnitude of the external excitation force acting on a vibrating system is known at any given time, then the excitation is known as* ***deterministic*** *and the resulting vibrations are known as* ***deterministic vibrations.*** The characteristic curve for such vibrations is shown in Fig. 3.8.7(a)

Example : Vibrations caused by rotating unbalanced mass.

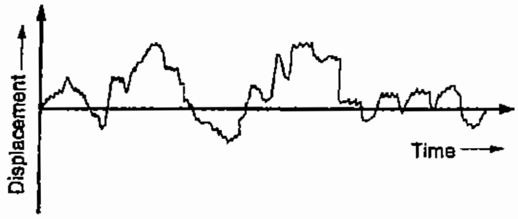
(ii) Random Vibrations :

- *If the magnitude of the external excitation force acting on a vibrating system cannot be predicted at any given time, then the excitation is known is non-deterministic or random and the resulting vibrations are known as* ***random vibrations.*** The characteristic curve for such vibrations is shown in Fig. 3.8.7(b).

Example : Due to earthquake a building is subjected to random vibrations.



(a) Deterministic (Periodic) Vibrations



(b) Random Vibrations

Fig. 3.8.7 : Deterministic and Random Vibrations

5. According to Behaviour of Vibrating System

(i) Linear Vibrations

- In a vibrating system, if basic components i.e. spring, mass and damper, behave linearly then the resulting vibrations are known as **linear vibrations**.
- In linear vibrations, the differential equation governing the motion of the vibrating system is linear.

  E.g. : Vibrations of spring - mass system along the axis of system.

(ii) Non-linear Vibrations

- If any of the three basic components of the vibrating system behave non-linearly, then the resulting vibrations are known as **non-linear vibrations**.
- In non-linear vibrations, the differential equation governing the motion of the vibrating system is non-linear.

  E.g. : Motion of spring-mass system in transverse direction (along the direction perpendicular to the axis of system)

PART II : UNDAMPED FREE VIBRATIONS

3.9 INTRODUCTION TO UNDAMDED FREE VIBRATIONS

- **Undamped free vibrations :** If the external force is removed after giving an initial displacement to the system, the system vibrates on its own due to internal elastic forces. Such vibrations are known as **free vibrations**; and if there is no external artificial resistance (damping) to the vibrations then such vibrations are known as **undamped free vibrations**. In most of the free vibrations there is always certain amount of damping (like air resistance) associated with the system. However, if the damping is very small, for all practical purpose it can be neglected and the vibrations are considered as **undamped free vibrations.**
- **Resonance :** When the frequency of external excitation force acting on a body is equal to the natural frequency of a vibrating body, the amplitude of vibrations becomes excessively large. Such state is known as **resonance.** The resonance is dangerous and it may lead to the failure of the part. Therefore to avoid the resonance condition, designer should know the natural frequency of machine or its parts which are subjected to external excitation force.

3.10 METHODS TO DETERMINE NATURAL FREQUENCY OF UNDAMPED FREE LONGITUDINAL VIBRATIONS

University Question

Q. What are the methods to determine the equation of motion for the vibratory system ? Explain any one method with example. **SPPU : May 13**

- The natural frequency of any body or a system depends upon :

  (i) geometrical parameters of body, and

  (ii) mass properties of body.
- Following three methods are used to determine the natural frequency of the given body or system :

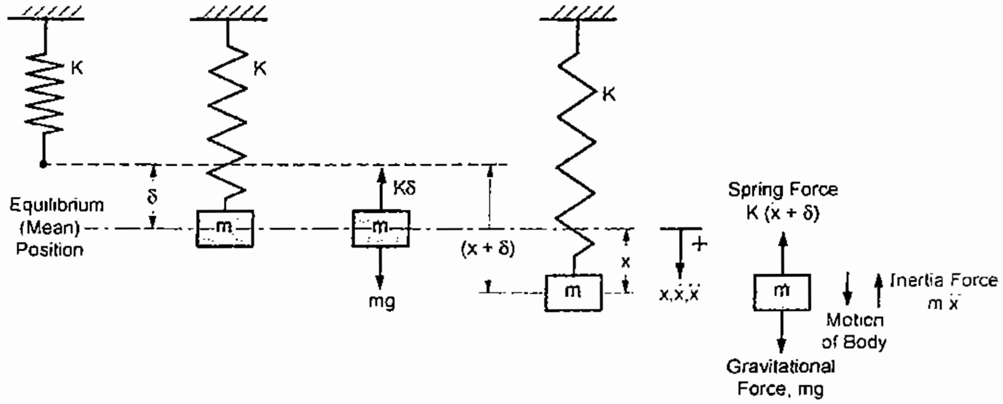
1.	Equilibrium Method (D'Alembert's Principle)
2.	Energy Method
3.	Rayleigh's Method

3.10.1 Equilibrium Method (D'Alembert's Principle) :

University Question

Q. Derive a relation for determining the solution in case of free under damped vibrations. Show amplitude versus time plot.

SPPU : Dec. 11



(a) Spring in Unstrained Position (b) Spring in Strained Position (c) F.B.D. of Mass at Mean Position (d) Displaced Position (e) F.B.D. of Mass at Displaced Position

Fig. 3.10.1 : Equilibrium Method

D'Alembert's Principle :

- According to **D'Alembert's principle**, a body or a system which is not in static equilibrium due to acceleration it possesses, can be brought to static equilibrium by introducing the inertia force on it.
- **D'Alembert's Principle** is used for developing the equation of motion for vibrating system which is further used to find the natural frequency of the vibrating system.
- **Inertia force** : Inertia Force = Mass × Acceleration
- Direction of inertia force is opposite to that of acceleration.

Spring-Mass System :

- Consider a spring-mass system, constrained to move along the axis of the spring, as shown in Fig. 3.10.1(a).

Let, m = mass suspended from the spring end, kg.

K = stiffness of the spring, N/m.

δ = deflection of the spring due to weight mg, m.

x = displacement given to the mass, by application of initial external force, from mean position, m.

- A spring of negligible mass, in an unstrained position [Fig. 3.10.1(a)].
- When mass '**m**' is attached to the free end of spring, it will deflect by 'δ' due to gravitational force '**mg**' [Fig. 3.10.1(b)]. The F.B.D. of mass at equilibrium (mean) position is shown in Fig. 3.10.1(c).

$$mg = K\delta \quad ...(a)$$

- Let, the spring and mass system is subjected to one time initial external force, due to which it will be displaced by 'x' from the mean position, as shown in Fig. 3.10.1(d). The F.B.D. of mass at the displaced position is shown in Fig. 3.10.1(e).

Equation of Motion :

- **Forces acting on mass :**

(i)	Inertia force, $m\ddot{x}$ (upwards)
(ii)	Spring force or restoring force, $K(x+\delta)$ (upwards)
(iii)	Gravitational force, mg (downwards)

- **D'Alembert's principle :**

$$\sum [\text{Inertia Force + External Forces}] = 0$$

From Fig. 3.10.1(e),

$$\therefore \quad m\ddot{x} + K(x+\delta) - mg = 0$$

...[Taking upward forces as + ve and downward forces as – ve]

$$m\ddot{x} + Kx + K\delta - mg = 0$$

$$m\ddot{x} + Kx = 0 \; [mg = k\delta \text{ from Equation (a)}]$$

$$\therefore \quad \ddot{x} + \frac{K}{m}x = 0 \quad ...(3.10.1)$$

- **Fundamental equation of simple harmonic motion :**

$$\ddot{x} + \omega_n^2 x = 0 \quad ...(b)$$

- Comparing Equation (3.10.1) and Equation (b),

$$\omega_n^2 = \frac{K}{m}$$

$$\text{or} \quad \omega_n = \sqrt{\frac{K}{m}}\ \text{, rad/s} \qquad ...(3.10.2)$$

where, ω_n = circular natural frequency, rad/s.

- **Natural frequency of vibrations (f_n) :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}\ \text{, Hz} \qquad ...(3.10.3)$$

- Also, from Equation (a),

$$mg = K\delta$$

$$\therefore \quad \frac{K}{m} = \frac{g}{\delta} \qquad ...(c)$$

- Substituting Equation (c) in Equation (3.10.1),

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}\ \text{, Hz.} \qquad ...(3.10.4)$$

- **Time period (t_p) :**

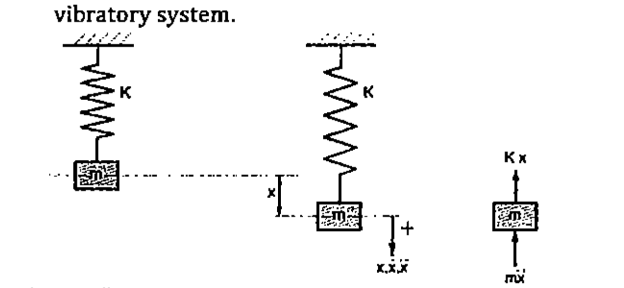
$$t_p = \frac{1}{f_n} = \frac{1}{\frac{1}{2\pi}\sqrt{\frac{K}{m}}}$$

$$\text{or} \quad t_p = 2\pi\sqrt{\frac{m}{K}}\ \text{, s.} \qquad ...(3.10.5)$$

Note : It is important to note that, the displacement (x), velocity ($\dot{x}$) and acceleration ($\ddot{x}$) are considered positive in downward direction and negative in upward direction, from the equilibrium (or mean) position. Hence, if at any instant $\ddot{x}$ is negative, it means the acceleration is in upward direction.

Equation of Motion-Simplified Method :

- While writing the equation of motion for any vibratory system with spring mass system, the forces acting on the system are considered beyond equilibrium position.
- Since mg = Kδ at mean position, therefore these forces are neglected while writing equation of motion of any vibratory system.



(a) Mean or Equilibrium Position (b) Displaced Position (c) F.B.D in Displaced Position

Fig. 3.10.2 : Equilibrium Method

- **D'Alembert's principle :**

$$\sum [\text{ Inertia Force + External Forces }] = 0$$

- From Fig. 3.10.2(c),

$$m\ddot{x} + Kx = 0$$

$$\therefore \quad \ddot{x} + \frac{K}{m}x = 0 \qquad ...(d)$$

- Equation (d) is same as Equation (3.10.1)

3.10.2 Energy Method :

Principle of Conservation of Energy :

- In energy method, law of conservation of energy is used to write the equation of motion and determine the natural frequency.
- Law of conservation of energy frequency :

$$\text{Total Energy} = \text{Constant}$$

$$\textbf{KE + PE = constant} \qquad ...(3.10.6)$$

Equation of Motion :

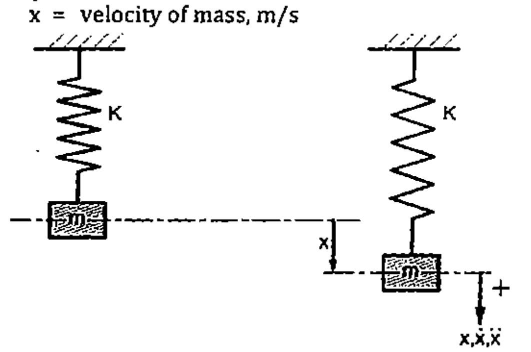
- Differentiating Equation (3.10.6) with respect to 't',

$$\frac{d}{dt}(KE + PE) = 0 \qquad ...(3.10.7)$$

- **K.E. of system :**

$$KE = \frac{1}{2}m\dot{x}^2 \qquad ...(3.10.8)$$

where, $\dot{x}$ = velocity of mass, m/s



(a) Mean or Equilibrium Position (b) Displaced Position

Fig. 3.10.3 : Spring-mass System

- **P.E. of system :**

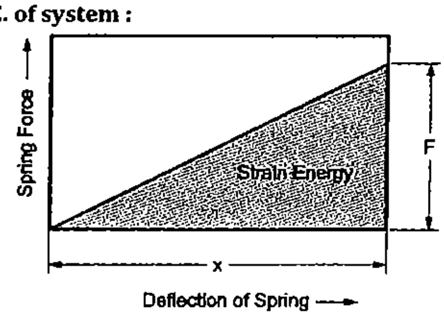


Fig. 3.10.4 : Forces – Deflection Diagram for Spring

$\therefore$ PE = Strain Energy

= Area Under Force - Deflection Diagram

$$= \frac{1}{2} Fx = \frac{1}{2} (Kx)\, x \quad ...[F = Kx]$$

or
$$PE = \frac{1}{2} Kx^2 \quad ...(3.10.9)$$

- Substituting Equations (3.10.8) and (3.10.9) in Equation (3.10.7) we get,

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} Kx^2\right) = 0$$

$$\therefore \frac{1}{2} m\, (2\, \dot{x}\, \ddot{x}) + \frac{1}{2} K\, (2x\, \dot{x}) = 0$$

$$(m\, \ddot{x})\, \dot{x} + (Kx)\, \dot{x} = 0$$

$$m\, \ddot{x} + Kx = 0$$

$$\therefore \quad \ddot{x} + \frac{K}{m} x = 0 \quad ...(3.10.10)$$

- **Fundamental equation of simple harmonic motion :**

$$\dot{x} + \omega_n^2 x = 0 \quad ...(e)$$

- Comparing Equation (3.10.10) with Equation (e), we get,

$$\omega_n^2 = \frac{K}{m}$$

or
$$\omega_n = \sqrt{\frac{K}{m}}\ , \text{rad/s} \quad ...(f)$$

- **Natural frequency of vibration (f_n) :**

$$f_n = \frac{\omega_n}{2\pi}$$

or
$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}},\ \text{Hz} \quad ...(g)$$

- **Time period (t_p) :**

$$t_p = \frac{1}{f_n} = \frac{1}{\frac{1}{2\pi}\sqrt{\frac{K}{m}}}$$

or
$$t_p = 2\pi\sqrt{\frac{m}{K}}\ , s \quad ...(h)$$

3.10.3 Rayleigh's Method :

Principle of Conservation of Energy :

- This is an extension of energy method, which is developed by Lord Rayleigh.
- **Principle of conservation of energy :**

$$\text{Total Energy} = \text{Constant}$$

$$(\text{Total Energy})_{\text{mean position}} = (\text{Total Energy})_{\text{extreme position}}$$

$$(KE + PE)_m = (KE + PE)_e$$

or
$$(KE)_m + (PE)_m = (KE)_e + (PE)_e \quad ...(i)$$

where, m = Mean position

e = Extreme position

- **At mean position :** At mean position, kinetic energy is maximum while potential energy is zero.

$$\therefore (KE)_m = (KE)_{max} \text{ and } (PE)_m = 0 \quad ...(j)$$

- **At extreme position :** At extreme position, kinetic energy is zero while potential energy is maximum

$$\therefore (KE)_e = 0 \text{ and } (PE)_e = (PE)_{max} \quad ...(k)$$

- Substituting equations (j) and (k) in equation (i),

$$(KE)_m + 0 = 0 + (PE)_e$$

$$\therefore \quad (KE)_{max} = (PE)_{max} \quad ...(3.10.11)$$

Equation of Motion :

- Therefore, according to Lord Rayleigh's, the *maximum kinetic energy which is at the mean position is equal to maximum potential energy which is at the extreme position.*
- Consider spring-mass system as shown in Fig. 3.10.2.
- **Displacement of body :**

$$x = X \sin \omega_n t \quad ...(l)$$

where, x = displacement of the body from the mean position after time 't' sec.

X = maximum displacement of body from the mean position or amplitude of vibration, m

ω_n = circular natural frequency, rad/s

- **Velocity of body :**

$$\dot{x} = \frac{dx}{dt} = \omega_n X \cos \omega_n t \quad ...(m)$$

- **Maximum velocity which is at mean position (i.e., at t = 0) :**

$$\dot{x}_{max} = \omega_n X \quad ...(n)$$

- **Maximum kinetic energy :**
- The maximum kinematic energy, which is at mean position (at t = 0) is given by,

$$(KE)_{max} = \frac{1}{2} m\, (\dot{x}_{max})^2 = \frac{1}{2} m\, \omega_n^2 X^2 \quad ...(o)$$

- **Potential energy :**

$$PE = \frac{1}{2} Kx^2$$

- **Maximum potential energy :**

The maximum potential energy which is at extreme position (at x = X), is given by,

$$(PE)_{max} = \frac{1}{2} KX^2 \quad ...(p)$$

- Substituting Equations (o) and (p) in Equation (3.10.11) we get,

$$\frac{1}{2} m\, \omega_n^2 X^2 = \frac{1}{2} KX^2$$

$$\therefore \omega_n^2 = \frac{K}{m}$$

or $$\omega_n = \sqrt{\frac{K}{m}},\ \text{rad/s} \qquad ...(q)$$

- **Natural frequency of vibrations (f_n) :**

$$f_n = \frac{\omega_n}{2\pi}$$

or $$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}\ ,\ \text{Hz} \qquad ...(r)$$

- **Time period (t_p) :**

$$t_p = \frac{1}{f_n} = \frac{1}{\frac{1}{2\pi}\sqrt{\frac{K}{m}}}$$

or $$t_p = 2\pi\sqrt{\frac{m}{K}}\ ,\ \text{s} \qquad ...(s)$$

3.11 NATURAL FREQUENCY OF UNDAMPED FREE TRANSVERSE VIBRATIONS

University Question

Q. A light cantilever of length l, has a mass M, fixed at its free end. Find the frequency of its lateral vibrations in the vertical plane. **SPPU : Dec. 19**

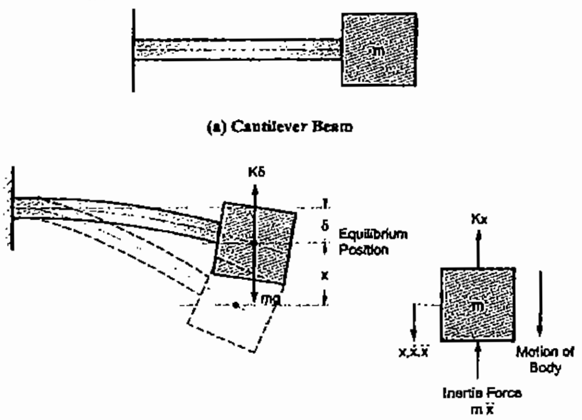
- Consider a cantilever beam of negligible mass carrying a concentrated mass 'm' at free end, as shown in Fig. 3.11.1(a).

Let, K = stiffness of mass, N/m

δ = static deflection of the beam due to mass attached at the end, m

x = displacement of the mass from mean position after applying initial external force, m

- **Transverse vibrations :** If a beam is given a deflection perpendicular to its axis, it oscillates or vibrates in a direction perpendicular to the axis of beam. Such vibrations are known as **transverse vibrations**.



(a) Cantilever Beam

(b) Cantilever Beam Before and After Giving Intial Displacement

(c) F.B.D. of Mass

Fig. 3.11.1 : Undamped Free Transverse Vibrations

- **At equilibrium (mean) position :** Due to gravitational force **'mg'**, the cantilever beam is deflected by 'δ' as shown in Fig. 3.11.1(b).

$$mg = K\delta \qquad ...(a)$$

- **Forces acting on beam :** Let, the system is subjected to one time initial external force due to which it will displaced by 'x' from equilibrium (mean) position as shown in Fig. 3.11.1(b) by dotted line.

Forces acting on the mass beyond equilibrium (mean) position, on the mass are :

(i) Inertia force, $m\ddot{x}$ (upwards)
(ii) Resisting or restoring force, Kx (upwards)

- **D'Alembert's principle :**

$\sum$ [Inertia Force + External Forces] = 0

From Fig. 3.11.1,

$$\therefore \quad m\ddot{x} + Kx = 0$$

[Taking upward force as + ve downward force as – ve]

$$\therefore \quad \ddot{x} + \frac{K}{m}x = 0 \qquad ...(b)$$

- Comparing Equation (b) with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K}{m}$$

or $$\omega_n = \sqrt{\frac{K}{m}},\ \text{rad/s} \qquad ...(c)$$

- **Natural frequency of vibrations (f_n) :**

$$f_n = \frac{\omega_n}{2\pi}$$

or $$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}\ ,\ \text{Hz} \qquad ...(d)$$

- From Equation (a),

$$\frac{K}{m} = \frac{g}{\delta},\ \text{Hz} \qquad ...(e)$$

- Substituting the Equation (e) in Equation (d), we get,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}\ ,\ \text{Hz} \qquad ...(f)$$

- **Static defection for different types of beams :**

Let W = load at the free end, N.

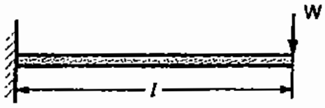
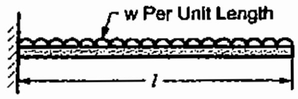
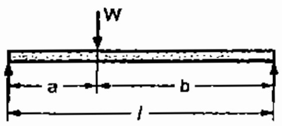
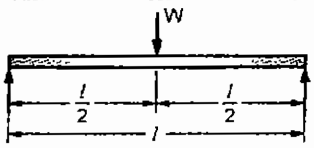
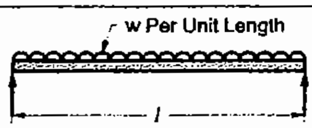
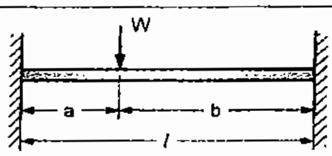
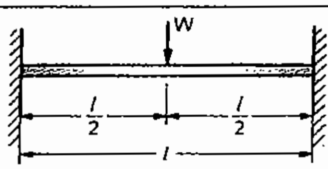
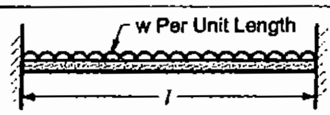
l = length of shaft beam, m.

E = Young's modulus for the material of beam, N/m^2.

I = moment of inertia of beam about horizontal axis, m^4.

Table 3.11.1 gives static deflection for various types of beams :

Table 3.11.1 : Static Deflection for Various Types of Beams

Sr. No.	Type of Beam	Configuration	Deflection, δ
1.	**Cantilever beam with point load**		$\delta = \frac{Wl^3}{3EI}$ (at free end)
2.	**Cantilever beam with U.D.L.**		$\delta = \frac{wl^4}{8EI}$ (at free end)
3.	**Simply supported beam with an eccentric point load**		$\delta = \frac{Wa^2 b^2}{3EIl}$ (at point load)
4.	**Simply supported beam with a central point load**		$\delta = \frac{wl^3}{48EI}$ (at center)
5.	**Simply supported beam with U.D.L.**		$\delta = \frac{5}{384} \times \frac{wl^4}{EI}$ (at center)
6.	**Fixed beam with eccentric point load**		$\delta = \frac{Wa^3 b^3}{3EIl^3}$ (at load)
7.	**Fixed beam with a central point load**		$\delta = \frac{Wl^3}{192EI}$ (at center)
8.	**Fixed beam with U.D.L.**		$\delta = \frac{wl^4}{384EI}$ (at center)

3.12 NATURAL FREQUENCY OF UNDAMPED FREE TORSIONAL VIBRATIONS

- Consider a disc having mass moment of inertia 'I' suspended on a shaft with negligible mass, as shown in Fig. 3.12.1.

Let, θ = angular displacement of disc from mean position, rad.

m = mass of the disc, kg

k = Radius of gyration of the disc, m

I = mass moment of inertia of the disc = mk^2, kg-m^2

K_t = torsional stiffness of the shaft, N-m/rad

- **Torsional vibrations :** If the disc is given an angular deflection about the axis of shaft, it oscillates or vibrates about the axis of shaft. Such vibrations are known as **torsional vibrations.**
- **Torques acting on disc :**

(i)	Inertia torque, $I\ddot{\theta}$ (anticlockwise)
(ii)	Restoring torque due to torsional stiffness of shaft, $K_t\,\theta$ (anticlockwise)

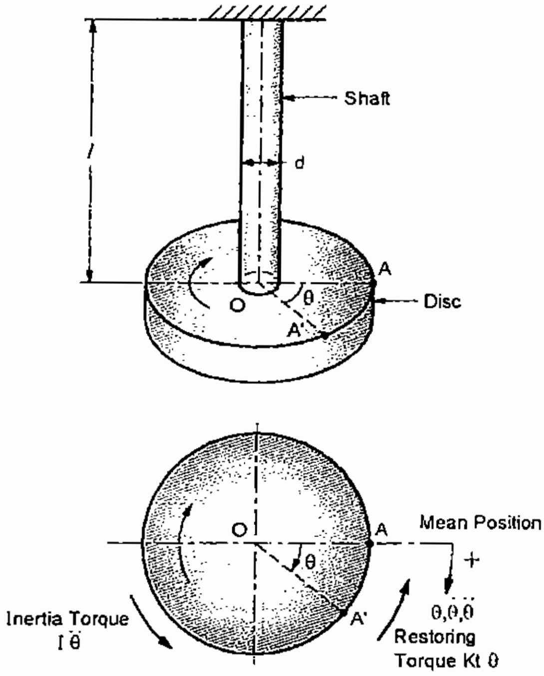


Fig. 3.12.1 : Undamped Free Torsional Vibration

- **D'Alembert's principle :**

$$\sum [\text{Inertia Torque} + \text{External Torques}] = 0$$

From Fig. 3.12.1,

$$\therefore\ I\ddot{\theta} + K_t\theta = 0$$

$$\therefore\ \ddot{\theta} + \frac{K_t}{I}\theta = 0 \qquad ...(3.12.1)$$

- **Fundamental equation of simple harmonic motion :**

$$\ddot{\theta} + \omega_n^2\theta = 0 \qquad ...(a)$$

- Comparing Equation (3.12.1) with Equation (a), we get,

$$\omega_n^2 = \frac{K_t}{I}$$

or $$\omega_n = \sqrt{\frac{K_t}{I}}\ \text{, rad/s} \qquad ...(3.12.2)$$

- **Natural frequency of torsional vibrations is (f_n) :**

$$f_n = \frac{\omega_n}{2\pi}$$

or $$f_n = \frac{1}{2\pi}\sqrt{\frac{K_t}{I}}\ \text{, Hz} \qquad ...(3.12.3)$$

- **Time period (t_p) :**

$$t_p = \frac{1}{f_n} = \frac{1}{\frac{1}{2\pi}\sqrt{\frac{K_t}{I}}}$$

or $$t_p = 2\pi\sqrt{\frac{I}{K_t}}\ \text{, s} \qquad ...(3.12.4)$$

3.12.1 Torsional Stiffness (K_t) :

- Torsional stiffness of shaft (K_t) is defined as the torque required to produce unit angular deflection in the direction of applied torque. Mathematically,

$$K_t = \frac{T}{\theta}\ \text{N-m/rad}$$

or $$K_t = \frac{GJ}{l} = \frac{G\pi d^4}{32\,l}\text{, N-m / rad} \qquad ...(3.12.5)$$

where, G = modulus of rigidity for the shaft material, N/m^2

J = polar moment of inertia of shaft = $\frac{\pi}{32}d^4$, m^4

l = length of the shaft, m

d = diameter of the shaft, m

3.12.2 Parameters for Linear and Torsional Vibrations :

Table 3.12.1 : Parameters for Linear and Torsional vibrations

Parameters	Linear vibration		Torsional vibration	
	Symbol	Unit	Symbol	Unit
Displacement	x	m	θ	rad
Velocity	$\dot{x}$	m/s	$\dot{\theta}$	rad/s
Acceleration	$\ddot{x}$	m/s^2	$\ddot{\theta}$	rad/s^2
Inertia force or torque	$m\ddot{x}$	N	$I\ddot{\theta}$	N-m
Stiffness	K	N/m	K_t	N-m/rad
Kinetic Energy	$KE = \frac{1}{2}m\dot{x}^2$	N-m	$KE = \frac{1}{2}I\dot{\theta}^2$	N-m
Potential Energy	$PE = \frac{1}{2}Kx^2$	N-m	$PE = \frac{1}{2}K_t\theta^2$	N-m
Natural Frequency	$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$	Hz	$f_n = \frac{1}{2\pi}\sqrt{\frac{K_t}{I}}$	Hz

Ex. 3.12.1 : A spring mass system has spring stiffness 'K' N/m and a mass of 'm' kg. It has natural frequency of vibration of 12 Hz. If an extra 2 kg mass is coupled to m, the natural frequency of system reduces by 2 Hz. Determine K and m. **SPPU - Dec. 11, Dec. 12, May 16, 6 Marks**

Soln. :

- **Initial natural frequency of system :**

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}\ , \text{Hz}$$

$$12 = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$\therefore\ \frac{K}{m} = 5684.89$$

$$\text{or}\quad K = 5684.89\ m \qquad ...(a)$$

- **Natural frequency of system with extra mass :**

When extra mass of 2 kg coupled, then natural frequency is reduced by 2 Hz i.e. natural frequency becomes 10 Hz. Therefore,

$$10 = \frac{1}{2\pi}\sqrt{\frac{K}{m+2}}$$

$$\therefore\ \frac{K}{(m+2)} = 3947.84$$

$$\text{or}\quad K = 3947.84\,(m+2)$$

$$K = 3947.84\ m + 7895.68 \qquad ...(b)$$

- **Stiffness (K) and mass (m) :**

From Equations (a) and (b),

$$5684.89\ m = 3947.84\ m + 7895.68$$

$$\therefore\ \mathbf{m = 4.54\ kg} \qquad ...\text{Ans.}$$

$$\text{Now,}\quad K = 5684.89\ m = 5684.89 \times 4.54$$

$$\therefore\ \mathbf{K = 25840.40\ N/m} \qquad ...\text{Ans.}$$

Ex. 3.12.2 : A truck weighing 150 kN and traveling at 2 m/sec impacts with a buffer spring which compresses 1.25 cm per 10 kN. The maximum compression of the spring will be ? **SPPU - Dec. 11,4 Marks**

Soln. :

Given : $mg = 150 \times 10^3\ N$; $\dot{x}_{max} = 2\ m/s$;

$\delta = 1.25\ cm$; $F = 10\ kN$

$$K = \frac{F}{\delta} = \frac{10 \times 10^3}{1.25 \times 10^{-2}}\ N/m = 8 \times 10^5\ N/m$$

$$\text{Now,}\quad m = \frac{150 \times 10^3}{9.81} = 15290.2\ kg$$

According to Rayleigh's Method,

$$(KE)_{max} = (PE)_{max}$$

$$\frac{1}{2}\, m\, \dot{x}^2_{max} = \frac{1}{2} K X^2$$

$$\frac{1}{2} \times 15290.52 \times (2)^2 = \frac{1}{2} \times 8 \times 10^5 \times X$$

$$X = 0.0764\ m$$

$$\text{or}\quad \mathbf{X = 7.64\ cm} \qquad ...\text{Ans.}$$

Ex. 3.12.3 : Find the natural frequency of vibration of the system shown in Fig. P. 3.12.3. Use following data :

$K_1 = 1000$ N/m, $K_2 = 1000$ N/m, $K_3 = 2000$ N/m, m = 10 kg.

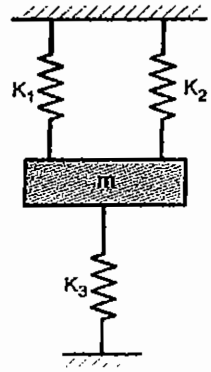
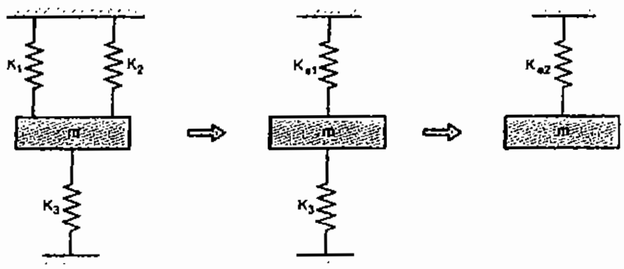


Fig. P. 3.12.3

Soln. :



(a) (b) (c)

Fig. P. 3.12.3

- **From Fig. P. 3.12.3(a) :**

The springs $K_1$and K_2 are in parallel. Therefore, their equivalent spring stiffness $\mathbf{K_{e1}}$ is given by,

$$K_{el} = K_1 + K_2 = 1000 + 1000$$

$$= 2000\ N/m$$

- **From Fig. P. 3.12.3(b) :**

The springs K_{el}, and K_3 are in parallel. Therefore, their equivalent spring stiffness K_{e2} is given by,

$$K_{e2} = K_{el} + K_3 = 2000 + 2000$$

$$= 4000\ N/m$$

- **From Fig. P. 3.12.3(c) :**

The natural frequency for spring-mass system is given by,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_{e2}}{m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{4000}{10}}$$

$$\text{or}\quad \mathbf{f_n = 3.18\ Hz} \qquad ...\text{Ans.}$$

Ex. 3.12.4 : For the system shown in Fig. P. 3.12.4(a), if $K_1 = 2400$ N/m, $K_2 = 1600$ N/m, $K_3 = 3600$ N/m and $K_4 = K_5 = 500$ N/m, find the mass m such that the system will have a natural frequency of 10 Hz.

SPPU - Oct. 19 (In Sem), 6 Marks

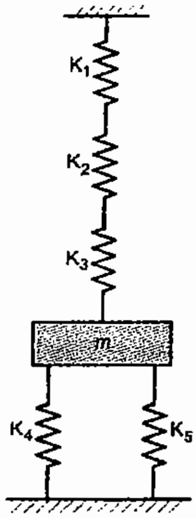
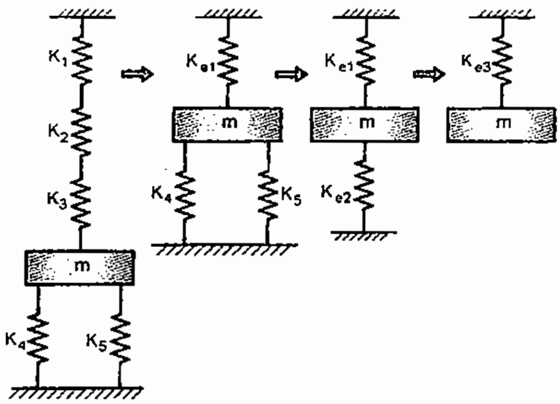


Fig. P. 3.12.4(a)

Soln. :

Given : $f_n = 10$ Hz.



(b) (c) (d) (e)

Fig. P. 3.12.4

- **From Fig. P. 3.12.4(b) :**

  The springs K_1, K_2, K_3 are in series. Therefore, their equivalent stiffness K_{e1} is given by,

$$\frac{1}{K_{el}} = \frac{1}{K_1} + \frac{1}{K_2} + \frac{1}{K_3}$$

$$\therefore \quad \frac{1}{K_{el}} = \frac{1}{2000} + \frac{1}{1500} + \frac{1}{3000}$$

$$K_{el} = 666.66 \text{ N/m}$$

- **From Fig. P. 3.12.4(c) :**

  The springs K_4 and K_5 are in parallel. Therefore, their equivalent stiffness K_{e2} is given by,

$$K_{e2} = K_4 + K_5$$

$$= 500 + 500 = 1000 \text{ N/m}$$

- **From Fig. P. 3.12.4(d) :**

  The spring K_{el} and K_{e2} are in parallel. Therefore, their equivalent stiffness K_{e3} is given by,

$$K_{e3} = K_{el} + K_{e2}$$

$$= 666.66 + 1000$$

$$= 1666.66 \text{ N/m}$$

- **From Fig. P. 3.12.4(e) :**

  The natural frequency for spring-mass system is given by,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_{e3}}{m}}$$

$$\therefore \quad 10 = \frac{1}{2\pi}\sqrt{\frac{1666.66}{m}}$$

$$\therefore \quad \mathbf{m = 0.4221 \ kg} \qquad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.5 : For the system shown in following Fig. P. 3.12.5(A) : $k_1 = 2000$ N/m, $k_2 = 1500$ N/m, $k_3 = 3000$ N/m and $k_4 = k_5 = 500$ N/m. Find mass (M), such that the system has a natural frequency of 10 Hz.

SPPU : Oct. 19(In Sem.), 6 Marks

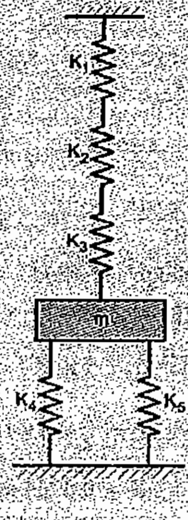


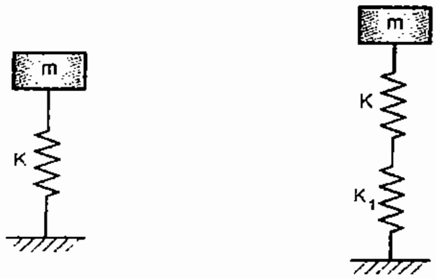
Fig. P. 3.12.5(A)

Ex. 3.12.5 : A spring mass system with mass 'm' kg and stiffness 'K' N/m has a natural frequency of 'f' Hz. Determine the value of stiffness 'K_1' of another spring which when arranged in conjunction with spring of Stiffness 'K' in series will lower the natural frequency by 20%.

SPPU - May 14, 6 Marks

Soln. :

Consider Fig. P. 3.12.5



(a) First Case **(b) Second Case**

Fig. P. 3.12.5

- First case ($f_{n1} = f$) :

$$f_{n1} = \sqrt{\frac{K}{m}}$$

$$\therefore \quad f = \sqrt{\frac{K}{m}}$$

$$\therefore\ K = f^2\, m \qquad \text{...(a)}$$

- Second case ($f_{n2} = 0.8\,f$) :

$$f_{n2} = \sqrt{\frac{K_e}{m}}$$

$$\therefore \quad 0.8\,f = \sqrt{\frac{K_e}{m}}$$

$$0.64\,f^2 = \frac{K_e}{m}$$

$$K_e = 0.64\,f^2 m \qquad \text{...(b)}$$

- From equations (a) and (b),

$$\frac{K}{K_e} = \frac{1}{0.64}$$

$$\therefore \quad \frac{1}{K_e} = \frac{1}{0.64\,K}$$

- **Springs in series :**

$$\frac{1}{K_e} = \frac{1}{K} + \frac{1}{K_1}$$

$$\frac{1}{0.64\,K} = \frac{1}{K} + \frac{1}{K_1}$$

$$\therefore \quad \frac{1}{K_1} = \frac{1}{0.64\,K} - \frac{1}{K} = \frac{0.56}{K}$$

$$K_1 = \frac{K}{0.56}$$

$$= \mathbf{1.7857\,K} \qquad \text{...Ans.}$$

Ex. 3.12.6 : For the mathematical model shown in Fig. P. 3.12.6(a), determine the equivalent stiffness.

SPPU - Dec. 14, 8 Marks, Dec. 18(In sem), 5 Marks

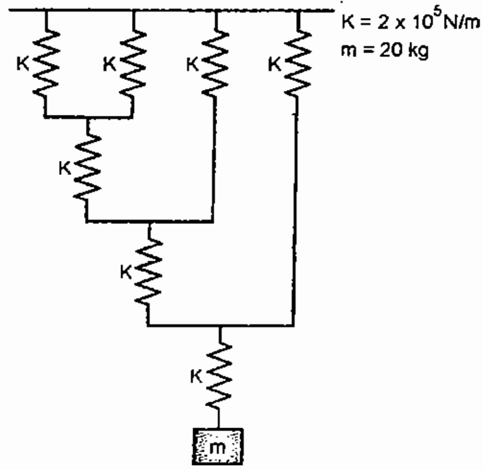
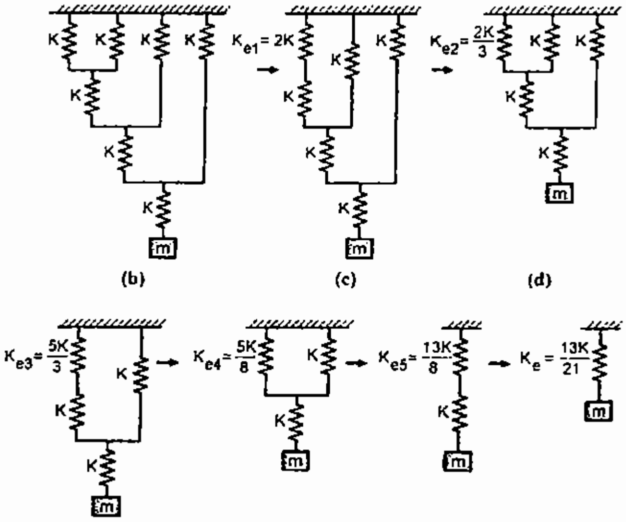


Fig. P. 3.12.6(a)

Soln. :



(b) (c) (d)

(e) (f) (g) (h)

Fig. P. 3.12.6

- **From Fig. P. 3.12.6(b) :**

  For springs K and K in parallel, $K_{e1} = K + K = 2K$

- **From Fig. P. 3.12.6(c) :**

  For springs K_{e1} and K in series,

$$\frac{1}{K_{e2}} = \frac{1}{K_{e1}} + \frac{1}{K}$$

$$\frac{1}{K_{e2}} = \frac{1}{2K} + \frac{1}{K} = \frac{3}{2K}$$

$$\therefore \quad K_{e2} = \frac{2K}{3}$$

- **From Fig. P. 3.12.6(d) :**

  For springs K_{e2} and K in parallel,

$$K_{e3} = K_{e2} + K = \frac{2K}{3} + K = \frac{5K}{3}$$

- **From Fig. P. 3.12.6(e) :**

  For springs K_{e3} and K in series,

$$\frac{1}{K_{e4}} = \frac{1}{K_{e3}} + \frac{1}{K} = \frac{1}{5K/3} + \frac{1}{K} = \frac{3}{5K} + \frac{1}{K} = \frac{8}{5K}$$

$$\therefore \; K_{e4} = \frac{5K}{8}$$

- **From Fig. P. 3.12.6(f) :**

  For springs K_{e4} and K in parallel

$$K_{e5} = K_{e4} + K = \frac{5K}{8} + K = \frac{13\,K}{8}$$

- **From Fig. P. 3.12.6(g) :**

  For springs K_{e5} and K is series,

$$\frac{1}{K_e} = \frac{1}{K_{e5}} + \frac{1}{K} = \frac{1}{13\,K/8} + \frac{1}{K}$$

$$= \frac{8}{13K} + \frac{1}{K} = \frac{21}{13\,K}$$

$$\therefore \; K_e = \frac{13\,K}{21}$$

- **From Fig. P. 3.12.6(h) :**

  Natural frequency of the system is,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_e}{m}} = \frac{1}{2\pi}\sqrt{\frac{\frac{13K}{21}}{m}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{13 \times 2 \times 10^5}{21 \times 20}}$$

$$f_n = \mathbf{12.52\ Hz} \qquad \text{...Ans.}$$

Ex. 3.12.7 : Determine the natural frequency of the system shown in Fig. P. 3.12.7(a).

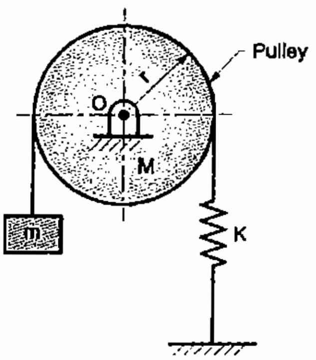


Fig. P. 3.12.7(a)

Soln. :

- If the mass 'm' is displaced through a small linear distance 'x', the pulley will rotate through an angle 'θ' in an anticlockwise direction as shown in Fig. P. 3.12.7(c).

- **From Fig. P. 3.12.7(c) :**

Angular displacement of pulley = θ
Angular velocity of pulley = $\dot{\theta}$
Angular acceleration of pulley = $\ddot{\theta}$
Linear displacement of mass = $x = r\theta$
Linear velocity of mass = $\dot{x} = r\dot{\theta}$
Linear acceleration of mass = $\ddot{x} = r\ddot{\theta}$

[I] Equilibrium Method

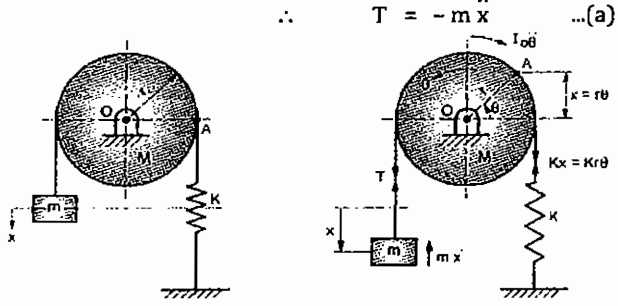
From Fig. P. 3.12.7(c) ;

- **Linear motion of mass 'm' :**

$$\sum[\text{Inertia force + External forces}] = 0$$

$$\therefore \; m\ddot{x} + T = 0$$

$$\therefore \; T = -m\ddot{x} \qquad \text{...(a)}$$



(b) Equilibrium Position **(c) Displaced Position**

Fig. P. 3.12.7

Let, I_o = moment of inertia of pulley, kg-m²

M = mass of pulley, kg

r = radius of pulley, m

- **Rotary motion of pulley :**

$$\sum[\text{Inertia torque + External torques}] = 0$$

$$I_o\ddot{\theta} + Kr\theta \cdot r - T \cdot r = 0 \qquad \text{...(b)}$$

Substituting Equation (a) in Equation (b),

$$I_o\ddot{\theta} + Kr^2\theta + m\ddot{x}r = 0$$

$$I_o\ddot{\theta} + Kr^2\theta + mr^2\ddot{\theta} = 0 \qquad \text{...}[\because \ddot{x} = r\ddot{\theta}]$$

$$\left(I_o + mr^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\left(\frac{1}{2}Mr^2 + mr^2\right)\ddot{\theta} + Kr^2\theta = 0 \;\text{...}\left[\because I_o = \frac{1}{2}Mr^2 \text{ for pulley}\right]$$

$$\ddot{\theta} + \left(\frac{Kr^2}{\frac{1}{2}Mr^2 + mr^2}\right)\theta = 0$$

$$\ddot{\theta} + \left(\frac{\mathbf{K}}{\frac{\mathbf{M}}{2} + \mathbf{m}}\right)\theta = 0 \qquad \text{...(b)}$$

- **Natural circular frequency :** This Equation (b) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K}{\left(\frac{M}{2}+m\right)}$$

$$\therefore \quad \omega_n = \sqrt{\frac{K}{\left(\frac{M}{2}+m\right)}}, \text{ rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K}{\frac{M}{2}+m}}, \text{ Hz} \quad \text{...Ans.}$$

[II] Energy Method

- From Fig. P. 3.12.7(c);
- K.E. of system :
  - Linear K.E. of mass $= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{\theta}^2$
  - Rotary K.E. of pulley $= \frac{1}{2}I_o\omega^2 = \frac{1}{2}I_o\dot{\theta}^2$
  - Total kinetic energy is,

$$KE = \frac{1}{2}m\,r^2\dot{\theta}^2 + \frac{1}{2}I_o\dot{\theta}^2$$

- P.E. of system :
  - Potential energy of spring $= \frac{1}{2}Kx^2 = \frac{1}{2}Kr^2\theta^2$

$$\therefore \quad PE = \frac{1}{2}K\,r^2\theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}I_o\dot{\theta}^2 + \frac{1}{2}Kr^2\theta^2\right) = 0$$

$$\frac{1}{2}mr^2\,2\,\dot{\theta}\,\ddot{\theta} + \frac{1}{2}I_o\,2\dot{\theta}\,\ddot{\theta} + \frac{1}{2}Kr^2\,2\,\theta\,\dot{\theta} = 0$$

$$mr^2\ddot{\theta} + \frac{1}{2}Mr^2\ddot{\theta} + Kr^2\theta = 0$$

$$\text{...}\left[\because I_o = \frac{1}{2}Mr^2 \text{ for pulley}\right]$$

$$m\ddot{\theta} + \frac{M}{2}\ddot{\theta} + K\theta = 0$$

$$\left(\frac{M}{2}+m\right)\ddot{\theta} + K\theta = 0$$

$$\ddot{\theta} + \left(\frac{K}{\frac{M}{2}+m}\right)\theta = 0 \quad \text{...(c)}$$

- **Natural circular frequency :** This Equation (c) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K}{\left(\frac{M}{2}+m\right)}$$

$$\therefore \quad \omega_n = \sqrt{\frac{K}{\left(\frac{M}{2}+m\right)}}, \text{ rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K}{\frac{M}{2}+m}}, \text{ Hz} \quad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.8 : Find the natural frequency of the system shown in Fig. P. 3.12.8(a).

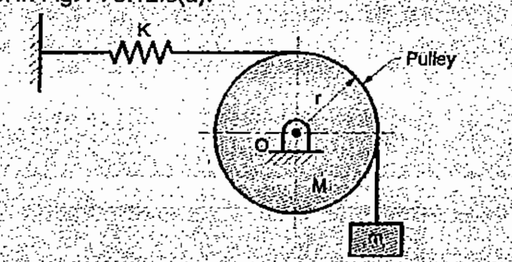


Fig. P. 3.12.8(a)

Ex. 3.12.9 : A mass of 1 kg is suspended by a spring passing over the pulley, as shown in Fig. P. 3.12.9(a). The system is supported horizontally by a spring of stiffness 1 kN/m. Determine the natural frequency of vibration of a system, using following data :

- Mass of pulley, M = 10 kg
- Radius of pulley, R = 50 mm
- Distance of spring from centre of pulley, r = 35 mm

SPPU - Dec. 06, Dec. 13, 10 Marks

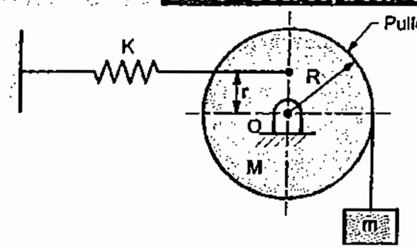


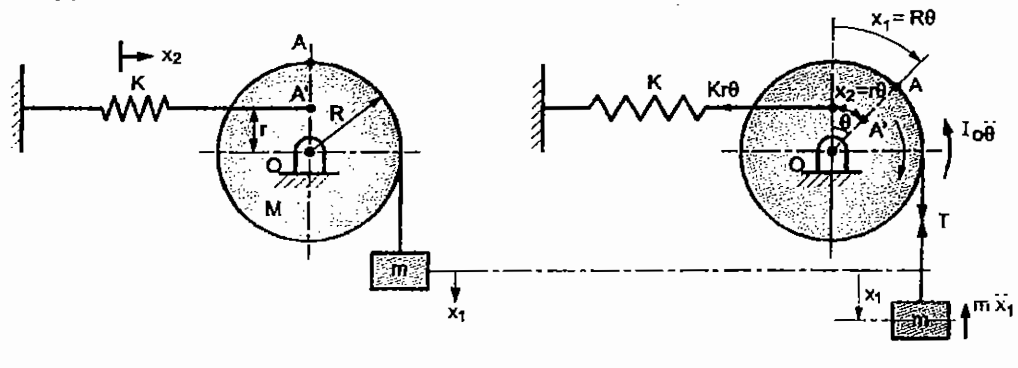
Fig. P. 3.12.9(a)

Soln. :

Given : $m = 1$ kg ; $K = 1000$ N/m ;

$M = 10$ kg ; $R = 0.05$ m ; $r = 0.035$ m.

- The mass 'm' and spring 'K' are not attached on one cord or string. Therefore, consider x_1 be the displacement of mass in downward direction and x_2 be the deflection of spring. The pulley will rotate through an angle 'θ' as shown in Fig. P. 3.12.9(c).



(b) Equilibrium Position **(c) Displaced Position**

Fig. P. 3.12.9

- **From Fig. P. 3.12.9(c) :**

Angular displacement of pulley $= \theta$

Linear displacement of mass $= x_1 = R\theta$

Linear velocity of mass $= \dot{x}_1 = R\dot{\theta}$

Linear acceleration of mass $= \ddot{x}_1 = R\ddot{\theta}$

Deflection of spring $= x_2 = r\theta$

[I] Equilibrium Method

From Fig. P. 3.12.9(c) ;

- **Linear motion of mass 'm' :**

$$\sum [\text{Inertia force + External forces}] = 0$$

$$\therefore\ m\ddot{x}_1 + T = 0$$

$$\therefore\ T = -m\ddot{x}_1 \quad \text{...(a)}$$

- **Rotary motion of pulley :**

$$\sum [\text{Inertia torque + External torques}] = 0$$

$$I_o\ddot{\theta} + Kr\theta \cdot r - T \cdot R = 0 \quad \text{...(b)}$$

Substituting Equation (a) in Equation (b),

$$I_o\ddot{\theta} + Kr^2\theta + m\ddot{x}_1 R = 0$$

$$I_o\ddot{\theta} + Kr^2\theta + mR^2\ddot{\theta} = 0 \quad \text{...}[\because \ddot{x}_1 = R\ddot{\theta}]$$

$$(I_o + mR^2)\ddot{\theta} + Kr^2\theta = 0$$

$$\left(\frac{1}{2}MR^2 + mR^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\text{...}\left[\because I_o = \frac{1}{2}MR^2 \text{ for pulley}\right]$$

$$\ddot{\theta} + \left(\frac{Kr^2}{\frac{MR^2}{2} + mR^2}\right)\theta = 0$$

$$\ddot{\theta} + \left[\frac{Kr^2}{\left(\frac{M}{2} + m\right)R^2}\right]\theta = 0 \quad \text{...(c)}$$

- **Natural circular frequency :** This Equation (c) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{Kr^2}{\left(\frac{M}{2} + m\right)R^2}$$

$$\therefore\ \omega_n = \sqrt{\frac{Kr^2}{\left(\frac{M}{2} + m\right)R^2}}$$

$$\text{or}\quad \omega_n = \frac{r}{R}\sqrt{\frac{K}{\frac{M}{2} + m}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{r}{2\pi R}\sqrt{\frac{K}{\frac{M}{2} + m}}\ \text{, Hz}$$

$$\therefore\ f_n = \frac{0.035}{2\pi \times 0.05}\sqrt{\frac{1000}{\frac{10}{2} + 1}}$$

or $f_n =$ **1.4382 Hz** ...Ans.

[II] Energy Method

From Fig. P. 3.12.9(c);

- **K.E. of system :**
  - Linear K.E. of mass $= \frac{1}{2} m \dot{x}_1^2 = \frac{1}{2} mR^2 \dot{\theta}^2$
  - Rotary K.E. of pulley $= \frac{1}{2} I_o \omega^2 = \frac{1}{2} I_o \dot{\theta}^2 = \frac{1}{4} MR^2 \dot{\theta}^2$
  - Total kinetic energy is,

$$KE = \frac{1}{2} mR^2 \dot{\theta}^2 + \frac{1}{2} MR^2 \dot{\theta}^2$$

- **P.E. of system :**
  - Potential energy of spring $= \frac{1}{2} Kx_2^2 = \frac{1}{2} Kr^2 \theta^2$

$$\therefore PE = \frac{1}{2} K r^2 \theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2} mR^2 \dot{\theta}^2 + \frac{1}{4} MR^2 2\dot{\theta} + \frac{1}{2} Kr^2 \theta^2\right) = 0$$

$$\frac{1}{2} mR^2\, 2\, \dot{\theta}\, \ddot{\theta} + \frac{1}{4} MR^2\, 2\dot{\theta}\, \ddot{\theta} + \frac{1}{2} Kr^2\, 2\, \theta\, \dot{\theta} = 0$$

$$mR^2 \ddot{\theta} + \frac{1}{2} MR^2 \ddot{\theta} + Kr^2 \theta = 0$$

$$\left(mR^2 + \frac{MR^2}{2}\right)\ddot{\theta} + Kr^2 \theta = 0$$

$$\therefore \quad \ddot{\theta} + \left(\frac{Kr^2}{\frac{MR^2}{2} + mR^2}\right)\theta = 0 \quad \text{...(d)}$$

- **Natural circular frequency :** This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{Kr^2}{\left(\frac{M}{2} + m\right) R^2}$$

$$\therefore \omega_n = \sqrt{\frac{Kr^2}{\left(\frac{M}{2} + m\right) R^2}}$$

$$\text{or} \quad \omega_n = \frac{r}{R}\sqrt{\frac{K}{\frac{M}{2} + m}} \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{r}{2\pi R}\sqrt{\frac{K}{\frac{M}{2} + m}} \text{, Hz}$$

$$= \frac{0.035}{2\pi \times 0.05}\sqrt{\frac{1000}{\frac{10}{2} + 1}}$$

$$\text{or } f_n = 1.4382 \text{ Hz} \quad \text{...Ans.}$$

Ex. 3.12.10 : Find the natural frequency of the system shown in Fig. P. 3.12.10(a).

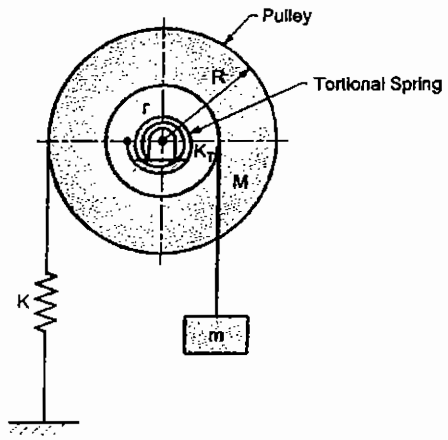
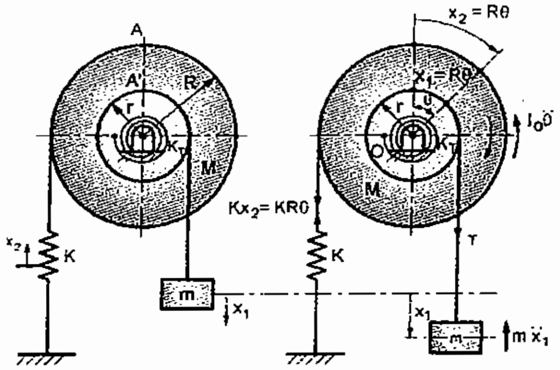


Fig. P. 3.12.10(a)

Soln. :

- If the mass displaced through a small linear distance 'x_1', the pulley will rotate through an angle 'θ' and 'x_2' will be the deflection of spring, as shown in Fig. P. 3.12.10(b)(c).



(b) Equilibrium Position (c) Displaced Position
Fig. P. 3.12.10

- **From Fig. P. 3.12.10(c) :**

Angular displacement of pulley	$= \theta$
Linear displacement of mass	$= x_1 = r\theta$
Velocity of mass	$= \dot{x}_1 = r\dot{\theta}$
Acceleration of mass	$= \ddot{x}_1 = r\ddot{\theta}$
Deflection of spring	$= x_2 = R\theta$

[I] Equilibrium Method

From Fig. P. 3.12.10(c) :

- **Linear motion of mass 'm' :**

$$\sum \text{[Inertia force + External forces]} = 0$$

$$\therefore\ m\ddot{x}_1 + T = 0$$

$$\therefore\ T = -m\ddot{x}_1 \qquad \text{...(a)}$$

- **Rotary motion of pulley :**

$$\sum \text{[Inertia torque + External torques]} = 0$$

$$\therefore\ I_o\theta + KR\theta \cdot R + K_T\theta - T \cdot r = 0 \qquad \text{...(b)}$$

Substituting equation (a) in Equation (b),

$$I_o\ddot{\theta} + KR^2\theta + K_T\theta + m\ddot{x}_1 r = 0$$

$$I_o\ddot{\theta} + KR^2\theta + K_T\theta + mr^2\ddot{\theta} = 0 \qquad \text{...}[\because \ddot{x}_1 = r\ddot{\theta}]$$

$$\ddot{\theta} + (KR^2 + K_T)\theta = 0$$

$$\left(\frac{1}{2}MR^2 + mr^2\right)\ddot{\theta} + (KR^2 + K_T)\theta = 0$$

$$\text{...}\left[\because I_o = \frac{1}{2}MR^2 \text{ for pulley}\right]$$

$$\therefore\ \ddot{\theta} + \left(\frac{KR^2 + K_T}{\frac{MR^2}{2} + mr^2}\right)\theta = 0 \qquad \text{...(c)}$$

- **Natural circular frequency :** This Equation (C) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{KR^2 + K_T}{\left(\frac{MR^2}{2} + mr^2\right)}$$

$$\therefore\ \omega_n = \sqrt{\frac{KR^2 + K_T}{\left(\frac{MR^2}{2} + mr^2\right)}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or}\quad f_n = \frac{1}{2\pi}\sqrt{\frac{KR^2 + K_T}{\left(\frac{MR^2}{2} + mr^2\right)}}\ \text{, Hz} \qquad \text{...Ans.}$$

[II] Energy Method

From Fig. P. 3.12.10(c) ;

- **K. E. of system :**
  - Linear K. E. of mass $= \frac{1}{2}m\dot{x}_1^2 = \frac{1}{2}mr^2\dot{\theta}^2$
  - Rotary K. E. of pulley $= I_o\omega^2 = \frac{1}{2}I_o\dot{\theta}^2 = \frac{1}{4}MR^2\dot{\theta}^2$
  - Total kinetic energy is,

$$KE = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{4}MR^2\dot{\theta}^2$$

- **P. E. of system :**
  - P. E. linear spring $= \frac{1}{2}Kx_2^2 = \frac{1}{2}KR^2\theta^2$
  - P. E. of torsional spring $= \frac{1}{2}K_T\theta^2$
  - Total potential energy is,

$$PE = \frac{1}{2}KR^2\theta^2 + \frac{1}{2}K_T\theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore\ \frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}KR^2\theta^2 + \frac{1}{2}K_T\theta^2\right) = 0$$

$$\frac{1}{2}mr^2\, 2\dot{\theta}\ddot{\theta} + \frac{1}{4}MR^2\, 2\dot{\theta}\ddot{\theta} + \frac{1}{2}KR^2\, 2\theta\dot{\theta} + \frac{1}{2}K_T\, 2\theta\dot{\theta} = 0$$

$$mr^2\ddot{\theta} + \frac{MR^2}{2}\ddot{\theta} + KR^2\theta + K_T\theta = 0$$

$$\therefore\ \left(mr^2 + \frac{MR^2}{2}\right)\ddot{\theta} + (KR^2 + K_T)\theta = 0$$

$$\therefore\ \ddot{\theta} + \left(\frac{KR^2 + K_T}{\frac{MR^2}{2} + mr^2}\right)\theta = 0 \quad \text{..(d)}$$

- **Natural Circular frequency :** This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we can write,

$$\omega_n^2 = \frac{KR^2 + K_T}{\left(\frac{MR^2}{2} + mr^2\right)}$$

$$\therefore\ \omega_n = \sqrt{\frac{KR^2 + K_T}{\left(\frac{MR^2}{2} + mr^2\right)}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or}\quad f_n = \frac{1}{2\pi}\sqrt{\frac{KR^2 + K_T}{\left(\frac{MR^2}{2} + mr^2\right)}}\ \text{, Hz} \qquad \text{...Ans.}$$

Ex. 3.12.11 : Find the natural frequency of oscillation for the roller rolling on horizontal surface without slipping, as shown in Fig. P. 3.12.11(a). The mass of roller is 5 kg, radius of roller is 50 mm and stiffness of spring is 2000 N/m. What would be the new frequency of oscillation, if radius of roller is made 100 mm without changing the mass ?

SPPU : May 03, Oct. (In sem), 6 Marks

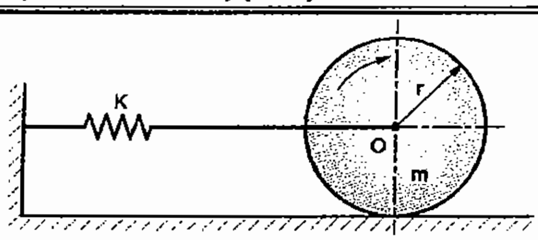


Fig. P. 3.12.11(a)

Soln. :

- If the roller is displaced through a small linear distance 'x', it will also rotate through an angle 'θ', as shown in Fig. P. 3.12.11(b).
- Let 'F_r' be the frictional force acting at the point of contact between roller and surface.

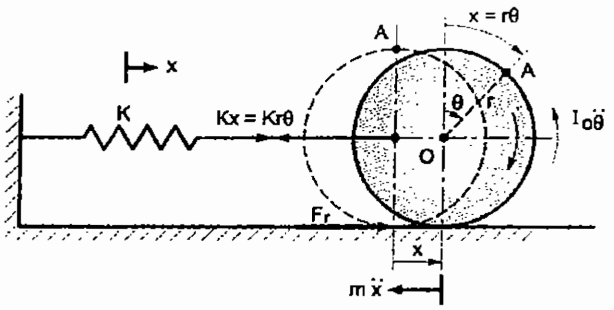


Fig. P. 3.12.11(b) : Displaced Position

- **From Fig. P. 3.12.11(b) :**

$$\text{Angular displacement of roller} = \theta$$
$$\text{Linear displacement of roller} = x = r\theta$$
$$\text{Linear velocity of roller} = \dot{x} = r\dot{\theta}$$
$$\text{Linear acceleration of roller} = \ddot{x} = r\ddot{\theta}$$

[I] Equilibrium Method

From Fig. P. 3.12.11(b) ;

- **Linear motion of roller :**

$$\sum \text{[Inertia force + External forces]} = 0$$

$$m\ddot{x} + Kr\theta - Fr = 0$$

$$\therefore \quad Fr = (m\ddot{x} + Kr\theta) \quad \text{...(a)}$$

- **Rotary motion of roller :**

$$\sum \text{[Inertia torque + External torques]} = 0$$

$$\therefore \quad I_0\ddot{\theta} + F_r \cdot r = 0 \quad \text{...(b)}$$

Substitute Equation (a) in Equation (b),

$$\frac{1}{2}mr^2\ddot{\theta} + (m\ddot{x} + Kr\theta)r = 0$$

$$\frac{1}{2}mr^2\ddot{\theta} + mr^2\ddot{\theta} + Kr^2\theta = 0$$

$$\left(\frac{1}{2}mr^2 + mr^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\left(\frac{3}{2}mr^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\ddot{\theta} + \left(\frac{Kr^2}{\frac{3}{2}mr^2}\right)\theta = 0$$

$$\therefore \quad \ddot{\theta} + \left(\frac{2K}{3m}\right)\theta = 0 \quad \text{...(c)}$$

- **Natural circular frequency :** This Equation (c) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{2K}{3m}$$

$$\omega_n = \sqrt{\frac{2K}{3m}}\text{, rad/sec} \quad \text{...(d)}$$

- **Natural frequency :**

$$\text{Now,} \quad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2K}{3m}}\text{, Hz}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\times 2000}{3\times 5}}$$

$$\text{or} \quad f_n = 2.59 \text{ Hz} \quad \text{...Ans.}$$

From Equation (d), it is seen that, the natural frequency of oscillation of roller is independent of its radius. Hence , even, if the radius is increased to 100 mm (by keeping the mass same), the natural frequency remains unchanged.

[II] Energy Method

From Fig. P. 3.12.11(a);

- **K. E. of system :**
  - o Linear of K.E. roller $= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mr^2\dot{\theta}^2$
  - o Rotary K. E. of roller $= \frac{1}{2}I_o\omega^2 = \frac{1}{2}I_o\dot{\theta}^2 = \frac{1}{4}mr^2\dot{\theta}^2$

Total kinetic energy is,

$$KE = \frac{1}{2}m\,r^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\theta}^2$$

- **P. E. of system :**

Potential energy of the spring $= \frac{1}{2}Kx^2 = \frac{1}{2}Kr^2\theta^2$

$$PE = \frac{1}{2}K\,r^2\theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{4}mr^2\dot{\theta}^2 + \frac{1}{2}Kr^2\theta^2\right) = 0$$

$$\frac{1}{2}mr^2\,2\,\dot{\theta}\,\ddot{\theta} + \frac{1}{4}mr^2\,2\,\dot{\theta}\,\ddot{\theta} + \frac{1}{2}Kr^2\,2\,\theta\,\dot{\theta} = 0$$

$$mr^2\ddot{\theta} + \frac{1}{2}mr^2\ddot{\theta} + Kr^2\theta = 0$$

$$\left(mr^2 + \frac{1}{2}mr^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\left(\frac{3\,mr^2}{2}\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\ddot{\theta} + \left[\frac{Kr^2}{\left(\frac{3}{2}mr^2\right)}\right]\theta = 0$$

$$\ddot{\theta} + \left(\frac{2K}{3m}\right)\theta = 0 \quad \text{...(e)}$$

- **Natural circular frequency :**

This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{2K}{3m}$$

$$\therefore \omega_n = \sqrt{\frac{2K}{3m}}\text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{2K}{3m}}\text{, Hz} \quad \text{...(f)}$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{2\times 2000}{3\times 5}}$$

$$\mathbf{f_n = 2.59\ Hz} \quad \text{...Ans.}$$

- From Equation (f) it is seen that, the natural frequency of oscillation of roller is independent of its radius. Hence , even, if the radius is increased to 100 mm (by keeping the mass same), the natural frequency remains unchanged.

Ex. 3.12.12 : A homogeneous solid cylinder of mass 'm' is linked by a spring of constant 'k' N/m as shown in Fig. P. 3.12.12 If it rolls without slipping, show that frequency of oscillations is $w_n = \sqrt{\frac{2K}{3m}}$

SPPU - May 14, 6 Marks, May 15, 5 Marks, Oct. 16(In Sem), 4 Marks

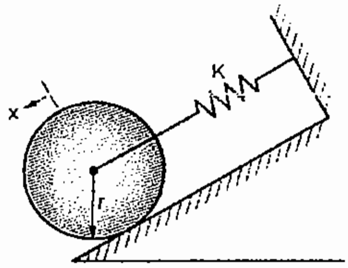


Fig. P. 3.12.12

Soln. : Same as Example 3.12.11

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.13 : A roller, shown in Fig. P. 3.12.13(a), rolls over the surface without slipping. Find the natural frequency of the system. **SPPU - May 02, May 04**

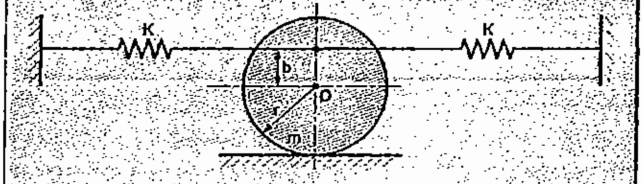


Fig. P. 3.12.13(a)

Ex. 3.12.14 : Determines the natural frequency of the system, shown in Fig. P. 3.12.14(a). The cord may be assumed inextensible and no slip between pulley and cord. **SPPU - Dec. 03**

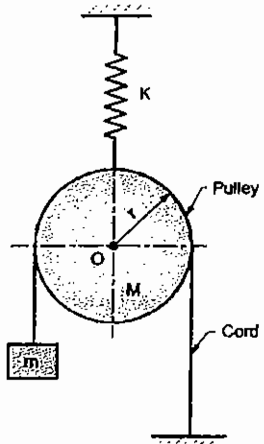
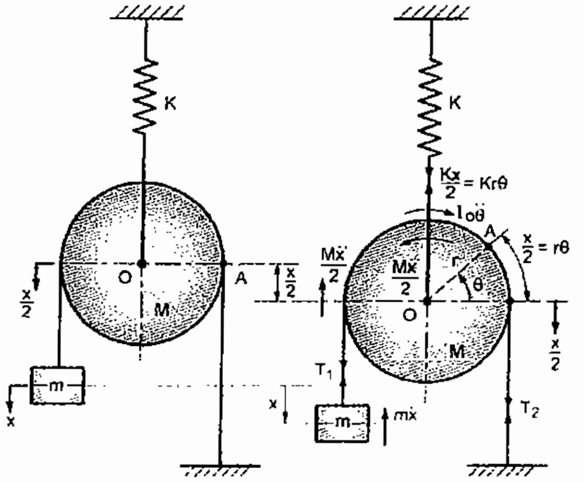


Fig. P. 3.12.14(a)

Soln. :

- When mass 'm' moves by a distance x in downward direction, the center of pulley moves by a distance x/2. Hence, the displacement of mass 'm' left for the rotation of pulley is $(x - \frac{x}{2}) = x/2$.

- Therefore, the rotation of the pulley in anticlockwise direction is, $\theta = \frac{x}{2r}$.



(b) Equilibrium position (c) Displaced position

Fig. P. 3.12.14

- **From Fig. P. 3.12.14(c) :**

Linear displacement of mass	$= x = 2r\theta$
Angular displacement of pulley	$= \theta = \frac{x}{2r}$
Angular velocity of pulley	$= \dot{\theta} = \frac{\dot{x}}{2r}$
Angular acceleration of pulley	$= \ddot{\theta} = \frac{\ddot{x}}{2r}$
Linear velocity of mass	$\dot{x} = 2r\dot{\theta}$
Linear acceleration of mass	$\ddot{x} = 2r\ddot{\theta}$

[I] Equilibrium Method

From Fig. P. 3.12.14(c);

- **Linear motion of mass 'm' :**

$$\sum [\text{Inertia force + External forces}] = 0$$

$$\therefore m\ddot{x} + T_1 = 0$$

$$\therefore T_1 = -m\ddot{x} \qquad ...(a)$$

- **Linear motion of pulley :**

$$\sum [\text{Inertia force + External forces}] = 0$$

$$\therefore M\frac{\ddot{x}}{2} - T_1 + Kr\theta - T_2 = 0$$

$$\therefore M\frac{\ddot{x}}{2} + m\ddot{x} + Kr\theta - T_2 = 0 \quad [\because T_1 = -m\ddot{x}]$$

$$\frac{M\,2r\ddot{\theta}}{2} + m\,2r\ddot{\theta} + Kr\theta - T_2 = 0$$

$$(Mr + 2mr)\ddot{\theta} + Kr\theta - T_2 = 0$$

$$\therefore T_2 = (Mr + 2mr)\ddot{\theta} + Kr\theta \qquad ...(b)$$

- **Rotary motion of pulley :**

$$\sum [\text{Inertia torque + External torques}] = 0$$

$$\therefore I_o\ddot{\theta} + T_2 r - T_1 r = 0 \qquad ...(c)$$

Substituting equations (a) and (b) in Equation (c),

$$I_o\ddot{\theta} + [(Mr + 2mr)\ddot{\theta} + Kr\theta]r + m\ddot{x}r = 0$$

$$I_o\ddot{\theta} + (Mr^2 + 2mr^2)\ddot{\theta} + Kr^2\theta + m\,2r\ddot{\theta}r = 0$$

$$(I_o + Mr^2 + 2mr^2 + 2mr^2)\ddot{\theta} + Kr^2\theta = 0$$

$$(I_o + Mr^2 + 4mr^2)\ddot{\theta} + Kr^2\theta = 0$$

$$\left(\frac{1}{2}Mr^2 + Mr^2 + 4mr^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$[\because I_o = \frac{1}{2}Mr^2 \text{ for pulley}]$$

$$\left(\frac{3}{2}Mr^2 + 4mr^2\right)\ddot{\theta} + Kr^2\theta = 0$$

$$\ddot{\theta} + \left(\frac{Kr^2}{\frac{3}{2}Mr^2 + 4mr^2}\right)\theta = 0$$

$$\therefore \ddot{\theta} + \left(\frac{2K}{3M + 8m}\right)\theta = 0 \qquad ...(d)$$

- **Natural circular frequency** : This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion we get,

$$\omega_n^2 = \frac{2K}{3M + 8m}$$

$$\therefore \omega_n = \sqrt{\frac{2K}{3M + 8m}}, \text{ rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or } f_n = \frac{1}{2\pi}\sqrt{\frac{2K}{3M + 8m}}, \text{ Hz} \qquad ...\text{Ans.}$$

[II] Energy Method

From Fig. P. 3.12.14(c);

- **K.E. of system :**
  - o Linear K.E. of mass $= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m(2r\dot{\theta})^2 = 2mr^2\dot{\theta}^2$
  - o Rotary K.E. of pulley $= \frac{1}{2}I_o\omega^2 = \frac{1}{2}I_o\dot{\theta}^2 = \frac{1}{4}Mr^2\dot{\theta}^2$

- Linear K.E. of pulley $= \frac{1}{2} M \left(\frac{\dot{x}}{2}\right)^2 = \frac{1}{8} M \dot{x}^2$

  $$= \frac{1}{8} M (2 r \dot{\theta})^2 = \frac{1}{2} M r^2 \dot{\theta}^2$$

- Total kinetic energy is,

  $$KE = 2 mr^2 \dot{\theta}^2 + \frac{1}{4} Mr^2 \dot{\theta}^2 + \frac{1}{2} Mr^2 \dot{\theta}^2$$

- **P.E. of system :**
  - Potential energy of spring $= \frac{1}{2} K \left(\frac{x}{2}\right)^2 = \frac{1}{8} K x^2$

  $$= \frac{1}{8} K (2 r \theta)^2 = \frac{1}{2} K r^2 \theta^2$$

  $$PE = \frac{1}{2} K r^2 \theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{1}{4} M r^2 \dot{\theta}^2 + \frac{1}{2} M r^2 \dot{\theta}^2 + 2 m r^2 \dot{\theta}^2 + \frac{1}{2} K r^2 \theta^2\right) = 0$$

$$\frac{1}{4} M r^2 2 \dot{\theta} \ddot{\theta} + \frac{1}{2} M r^2 2 \dot{\theta} \ddot{\theta} + 2 m r^2 2 \dot{\theta} \ddot{\theta} + \frac{1}{2} K r^2 2\theta \dot{\theta} = 0$$

$$\frac{1}{2} M r^2 \ddot{\theta} + M r^2 \ddot{\theta} + 4 m r^2 \ddot{\theta} + K r^2 \theta = 0$$

$$\frac{1}{2} M r^2 \ddot{\theta} + M r^2 \ddot{\theta} + 4 m r^2 \ddot{\theta} + K r^2 \theta = 0$$

$$\left(\frac{3}{2} M r^2 + 4 m r^2\right) \ddot{\theta} + K r^2 \theta = 0$$

$$\ddot{\theta} + \left(\frac{K r^2}{\frac{3}{2} M r^2 + 4 m r^2}\right) \theta = 0$$

$$\therefore \ddot{\theta} + \left(\frac{2 K}{3 M + 8 m}\right) \theta = 0 \qquad \text{...(e)}$$

- **Natural circular frequency :**

  This is Equation (e) the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion we get,

$$\omega_n^2 = \frac{2 K}{3 M + 8 m}$$

$$\therefore \omega_n = \sqrt{\frac{2 K}{3 M + 8 m}}, \text{ rad/s}$$

- **Natural frequency**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or } f_n = \frac{1}{2\pi} \sqrt{\frac{2 K}{3 M + 8 m}}, \text{ Hz ...Ans.}$$

Ex. 3.12.15 : A cylinder of mass 10 kg and radius 50 mm is suspended from an inextensible, cord, as shown in Fig. P. 3.12.15(a). One end of the cord is attached directly to a rigid support, while the other end is attached to a spring having stiffness 500 N/m. Determine the natural frequency of the system. What would be the new spring stiffness required, if mass of cylinder is reduced to 5 kg for same natural frequency of oscillation ? **SPPU - Dec. 03**

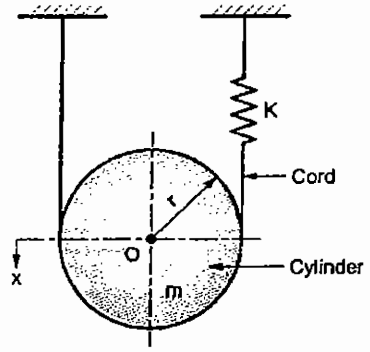
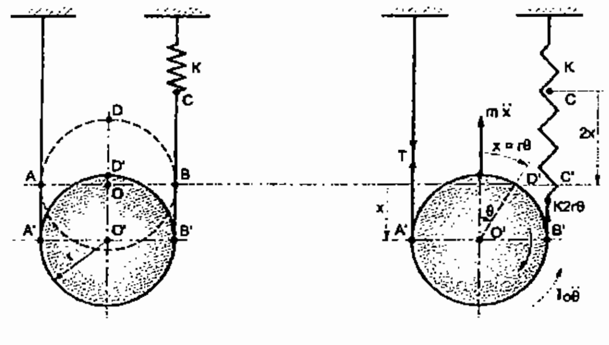


Fig. P. 3.12.15(a)

Soln. :

- If the cord is fully extensible and elastic, the cylinder will move by x distance i.e. points A and B will move to A′ and B′ as shown in Fig. P. 3.12.15(b) and spring may deflect at all.
- If the cord is inextensible, then spring will deflect by ‘2x’, as shown in Fig. P. 3.12.15(c).



(b) When cord in fully extensible **(c) when cord is fully inextensible**

Fig. P. 3.12.15

- **From Fig. P. 3.12.15(c) :**

Angular displacement of cylinder	$= \theta$
Linear displacement of cylinder	$= x = r\theta$
Linear velocity of cylinder	$= \dot{x} = r\dot{\theta}$
Linear acceleration of cylinder	$= \ddot{x} = r\ddot{\theta}$
Deflection of spring	$= 2x = 2 r\theta$

[I] Equilibrium Method

From Fig. P. 3.12.15(c);

- **Linear motion of cylinder :**

$$\sum \text{[Inertia force + External forces]} = 0$$

$$\therefore\ m\ddot{x} + K\,2\,r\,\theta + T = 0$$

$$\therefore\ T = -(m\ddot{x} + 2\,K\,r\,\theta) \qquad \text{...(a)}$$

- **Rotary motion of cylinder :**

$$\sum \text{[Inertia torque + External torques]} = 0$$

$$\therefore\ I_o\ddot{\theta} + 2\,K\,r\,\theta\cdot r - T\cdot r = 0 \qquad \text{...(b)}$$

Substituting Equation (a) in Equation (b),

$$I_o\ddot{\theta} + 2\,K\,r^2\,\theta + (m\ddot{x} + 2\,K\,r\,\theta)\,r = 0$$

$$\therefore\ I_o\ddot{\theta} + 2K\,r^2\theta + m\ddot{x}\,r + 2K\,r^2\theta = 0$$

$$I_o\ddot{\theta} + m\,r^2\,\ddot{\theta} + 4\,K\,r^2\theta = 0 \qquad [\because\ m\ddot{x} = m\,r\,\ddot{\theta}]$$

$$\frac{1}{2}mr^2\,\ddot{\theta} + m\,r^2\,\ddot{\theta} + 4\,K\,r^2\theta = 0$$

$$[\because\ I_o = \frac{1}{2}m\,r^2 \text{ for pulley}]$$

$$\left(\frac{1}{2}m + m\right)\ddot{\theta} + 4\,K\,\theta = 0$$

$$\left(\frac{3}{2}m\right)\ddot{\theta} + 4\,K\,\theta = 0$$

$$\ddot{\theta} + \left(\frac{4\,K}{\frac{3}{2}m}\right)\theta = 0$$

$$\therefore\ \ddot{\theta} + \left(\frac{8\,K}{3\,m}\right)\theta = 0 \qquad \text{...(c)}$$

- **Natural circular frequency :**

This Equation (c) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{8\,K}{3\,m}$$

$$\therefore\ \omega_n = \sqrt{\frac{8\,K}{3\,m}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\,\pi} = \frac{1}{2\,\pi}\sqrt{\frac{8\,K}{3\,m}}\,\text{Hz}$$

$$= \frac{1}{2\,\pi}\sqrt{\frac{8\times 500}{3\times 100}}$$

$$\therefore\ f_n = \mathbf{1.8377\ Hz} \qquad \text{...Ans.}$$

- If mass of cylinder is made 5 kg, then the required new spring stiffness for same natural frequency is calculated as follows :

$$1.8377 = \frac{1}{2\,\pi}\sqrt{\frac{8\times K}{3\times 5}}$$

$$K = 249.99$$

$$\text{or}\quad K \simeq \mathbf{250\ N/m} \qquad \text{...Ans.}$$

[II] Energy Method

From Fig. P. 3.12.15(c) ;

- **K. E. of system :**
  - Translatory K.E. of cylinder $= \frac{1}{2}m\,\dot{x}^2 = \frac{1}{2}m\,r^2\,\dot{\theta}^2$
  - Rotary K.E. of cylinder $= \frac{1}{2}I_o\,\omega^2 = \frac{1}{2}I_o\,\dot{\theta}^2 = \frac{1}{4}m\,r^2\,\dot{\theta}^2$
  - Total kinetic energy is,

$$KE = \frac{1}{2}m\,r^2\,\dot{\theta}^2 + \frac{1}{4}m\,r^2\,\dot{\theta}^2$$

- **P.E. of system :**
  - Potential energy of spring

$$= \frac{1}{2}K\,(2x)^2 = \frac{1}{2}K\,(2r\theta)^2 = 2\,Kr^2\,\theta^2$$

$$PE = 2\,Kr^2\,\theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2}m\,r^2\,\dot{\theta}^2 + \frac{1}{4}m\,r^2\,\dot{\theta}^2 + 2\,K\,r^2\,\theta^2\right) = 0$$

$$\frac{1}{2}m\,r^2\,2\,\dot{\theta}\,\ddot{\theta} + \frac{1}{4}m\,r^2\,2\,\dot{\theta}\,\ddot{\theta} + 2\,K\,r^2\,2\,\theta\cdot\dot{\theta} = 0$$

$$m\,\ddot{\theta} + \frac{1}{2}m\,\ddot{\theta} + 4\,K\,\theta = 0$$

$$\left(\frac{3}{2}m\right)\ddot{\theta} + 4\,K\,\theta = 0$$

$$\ddot{\theta} + \left(\frac{4\,K}{\frac{3}{2}m}\right)\theta = 0$$

$$\therefore\ \ddot{\theta} + \left(\frac{8\,K}{3\,m}\right)\theta = 0 \qquad \text{...(d)}$$

- **Natural circular frequency :**

  This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion we get,

$$\omega_n^2 = \frac{8\,K}{3\,m}$$

$$\therefore \quad \omega_n = \sqrt{\frac{8\,K}{3\,m}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{8\,K}{3\,m}}\ \text{Hz} = \frac{1}{2\pi}\sqrt{\frac{8\times 500}{3\times 100}}$$

or $f_n = 1.8377$ Hz ...Ans.

- If mass of cylinder is made 5 kg, then required new spring stiffness for same natural frequency is calculated as follows :

$$1.8377 = \frac{1}{2\pi}\sqrt{\frac{8\times K}{3\times 5}}$$

$$\therefore \quad K = 249.99$$

or $K \simeq 250$ N/m ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.16 : Assuming that the cord is inextensible and neglecting mass of pulley, determine the natural frequency of the system, shown in Fig. P. 3.12.16(a).

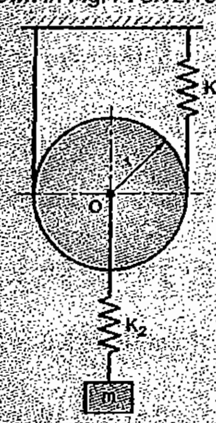


Fig. P. 3.12.16(a)

Ex. 3.12.17 : Find the natural frequency of the system shown in Fig. P. 3.12.17(a). Neglect the mass of pulleys and assume the cord is inextensible. **SPPU- May 13, 8 Marks**

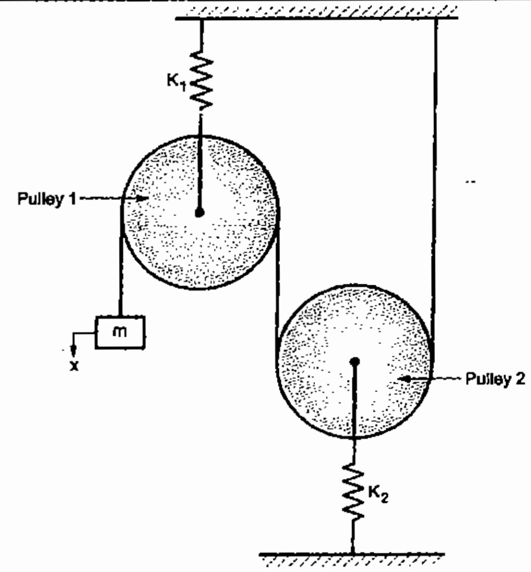


Fig. P. 3.12.17(a)

Soln. :

[I] Equilibrium Method

- **From Fig. P. 3.12.17(b) :**

Tension in cord = T

Displacement of mass = x

Displacement of pulley 1 and spring 1 = x_1

Displacement of pulley 2 and spring 2 = x_2

- **Displacement of mass 'm' (x) :** If pulley 2 is fixed, then displacement of mass m = 2 x_1

  If pulley 1 is fixed, then displacement of mass m = 2 x_2

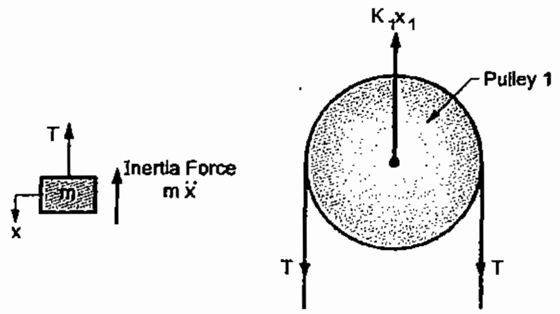
  But both pulleys are movable. Hence, total displacement of mass 'm' is

  x = 2 [Deflection of spring 1 + Deflection of spring 2]

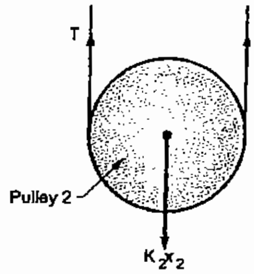
$$x = 2\,(x_1 + x_2) \qquad \text{...(a)}$$

- **Linear motion of pulley 1 :**

  From Fig. P. 3.12.17(b)(ii);



(i) F.B.D of mass **(ii) F.B.D. of pulley 1**



(iii) F.B.D of pulley 2

Fig. P. 3.12.17(b)

$$\sum [\text{inertia force + External forces}] = 0$$

$$K_1 x_1 - 2T = 0$$

$$\therefore \quad x_1 = \frac{2T}{K_1} \qquad ...(b)$$

- **Linear motion of pulley 2 :**

  From Fig. P.3.12.17(b)(iii);

$$\sum [\text{inertia fore + External force}] = 0$$

$$2T - K_2 x_2 = 0$$

$$\therefore \quad x_2 = \frac{2T}{K_2} \qquad ...(c)$$

- **Linear motion of mass 'm' :**
  - Substituting Equations (b) and (c) in Equation (a), we get,

$$x = 2\left(\frac{2T}{K_1} + \frac{2T}{K_2}\right) = 4T\left(\frac{1}{K_1} + \frac{1}{K_2}\right)$$

$$= 4T\left(\frac{K_1 + K_2}{K_1 K_2}\right)$$

$$\therefore \quad T = \left[\frac{K_1 K_2}{4(K_1 + K_2)}\right]x \qquad ...(d)$$

From Fig. P. 3.12.17(b)(i);

$$\sum [\text{Inertia force + External forces}] = 0$$

$$m\ddot{x} + T = 0 \qquad ...(e)$$

substituting equation (d) in Equation (e),

$$\therefore \quad m\ddot{x} + \left[\frac{K_1 K_2}{4(K_1 + K_2)}\right]x = 0$$

$$\therefore \quad \ddot{x} + \left[\frac{K_1 K_2}{4m(K_1 + K_2)}\right]x = 0 \qquad ...(f)$$

- **Natural circular frequency :** This Equation (f) is the differential equation of motion for a given system. Comparing this equation with the fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K_1 K_2}{4m(K_1 + K_2)}, \text{rad/s}$$

$$\therefore \quad \omega_n = \sqrt{\frac{K_1 K_2}{4m(K_1 + K_2)}}, \text{rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_1 K_2}{4m(K_1 + K_2)}} \text{Hz} \qquad ...\text{Ans.}$$

[II] Alternate Solution

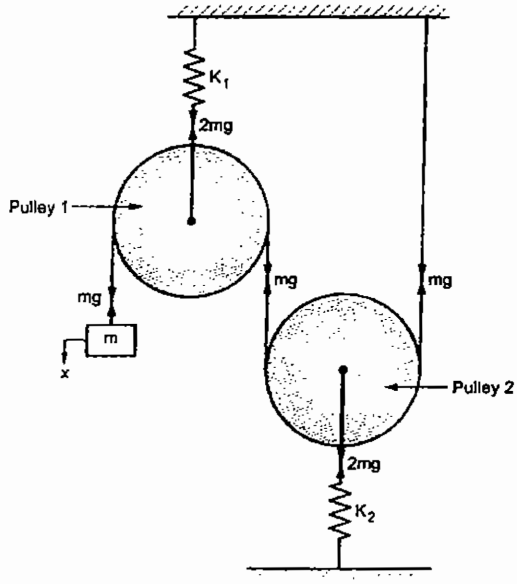


Fig. P. 3.12.17(c)

- **From Fig. P. 3.12.17(c) :**

$$\text{Force in spring } 1 = 2\,mg$$

$$\text{Static deflection of spring 1, } \delta_1 = \frac{2\,mg}{K_1}$$

$$\text{Force in spring } 2 = 2mg,$$

$$\text{Static deflection of spring 2, } \delta_2 = \frac{2\,mg}{K_2}$$

- **Static deflection of mass m :**

$$\delta = 2(\delta_1 + \delta_2) = 2\left(\frac{2\,mg}{K_1} + \frac{2\,mg}{K_2}\right)$$

$$\therefore \quad \delta = 4mg\left(\frac{1}{K_1} + \frac{1}{K_2}\right)$$

$$\therefore \quad \delta = 4mg\left(\frac{K_1 + K_2}{K_1 K_2}\right)$$

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{g}{4\,mg\left(\frac{K_1 + K_2}{K_1 K_2}\right)}}$$

or

$$\omega_n = \sqrt{\frac{K_1 K_2}{4m(K_1 + K_2)}}, \text{rad/s}$$

- **Natural frequency :**

$$\therefore \quad f_n = \frac{\omega_n}{2\pi}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_1 K_2}{4\,m\,(K_1 + K_2)}}\,\text{Hz} \qquad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.18 : Determine the natural frequency of vibration of the spring mass system, taking into account the mass of spring. **SPPU - May 02**

Ex. 3.12.19 : Find the natural frequency of vibration of the system, shown in Fig. P. 3.12.19(a). **SPPU - May 07, Dec. 07**

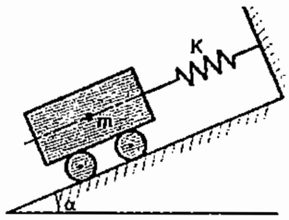
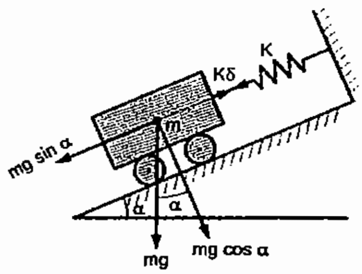
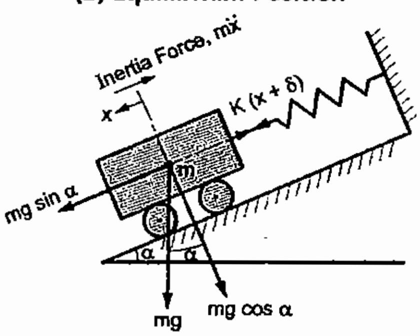


Fig. P. 3.12.19(a)

Soln. :



(b) Equilibrium Position



(c) Displaced Position

Fig. P. 3.12.19

Fig. P. 3.12.19(a) ;

$$mg \sin\alpha = K\delta. \qquad \text{...(a)}$$

[I] Equilibrium Method

From Fig. P. 3.12.19(c);

- **Motion of System :**

$$\sum [\text{Inertia force + External force}] = 0$$

$$\therefore \quad m\ddot{x} + K(x + \delta) - mg \sin\alpha = 0$$

$$m\ddot{x} + Kx + K\delta - mg\sin\alpha = 0 \qquad \text{...(b)}$$

substituting Equation (a) in Equation (b)

$$m\ddot{x} + Kx = 0$$

$$\therefore \quad \ddot{x} + \left(\frac{K}{m}\right)x = 0 \qquad \text{...(c)}$$

- **Natural circular frequency :** This Equation (c) is the differential equation of motion for given system. Comparing this equation of with fundamental equation simple harmonic motion, we get,

$$\omega_n^2 = \frac{K}{m}$$

$$\therefore \quad \omega_n = \sqrt{\frac{K}{m}}\,,\ \text{rad/s} \qquad \text{...(d)}$$

But,

$$K\delta = mg\sin\alpha$$

$$\therefore \quad \frac{K}{m} = \frac{g\sin\alpha}{\delta} \qquad \text{...(e)}$$

Substituting Equation (e) in Equation (d), we get,

$$\omega_n = \sqrt{\frac{g\sin\alpha}{\delta}},\ \text{rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}} \quad \text{Hz} \qquad \text{...Ans.}$$

$$f_n = \frac{\omega_n}{2\pi}$$

$$\therefore \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g\sin\alpha}{\delta}},\ \text{Hz} \qquad \text{...Ans.}$$

[II] Energy Method

- **K.E. of system :**

The kinetic energy of mass is,

$$KE = \frac{1}{2}m\dot{x}^2.$$

- **P.E. of system :**

The potential energy of spring is,

$$PE = \frac{1}{2}Kx^2.$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}Kx^2\right) = 0$$

$$\frac{1}{2}m\,2\dot{x}\ddot{x} + \frac{1}{2}K2x\dot{x} = 0$$

$$m\ddot{x} + Kx = 0$$

$$\therefore \ddot{x} + \left(\frac{K}{m}\right)x = 0 \qquad \text{...(f)}$$

- **Natural circular frequency:** This Equation (f) is the differential equation of motion for given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K}{m}$$

$$\therefore \omega_n = \sqrt{\frac{K}{m}}, \text{rad/s} \qquad \text{...(g)}$$

But, $K\delta = mg\sin\alpha$

$$\therefore \frac{K}{m} = \frac{g\sin\alpha}{\delta} \qquad \text{...(h)}$$

Substituting Equation (h) in Equation (g), we get,

$$\therefore \omega_n = \sqrt{\frac{g\sin\alpha}{\delta}}, \text{rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}, \text{Hz}$$

$$f_n = \frac{\omega_n}{2\pi}$$

$$\therefore f_n = \frac{1}{2\pi}\sqrt{\frac{g\sin\alpha}{\delta}}, \text{Hz} \qquad \text{...Ans.}$$

Examples for Practice

Refer our website for complete solution of following examples

Ex. 3.12.20 : The circular disc of mass 'm' and radius 'r' is pivoted, as shown in Fig. P. 3.12.20(a). At equilibrium condition, both springs are under tension. Find the natural frequency of vibration of system.

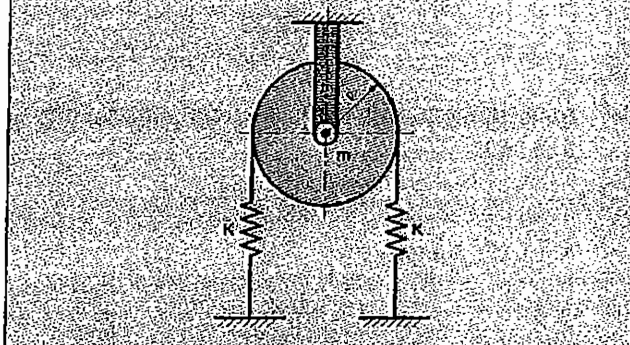


Fig. P. 3.12.20

Ex. 3.12.21 : An electric motor supported by 4 springs, each having stiffness K, as shown in Fig. P. 3.12.21(a). If the mass moment of inertia of motor about the axis of rotation is I_o, find its natural frequency of vibrations.

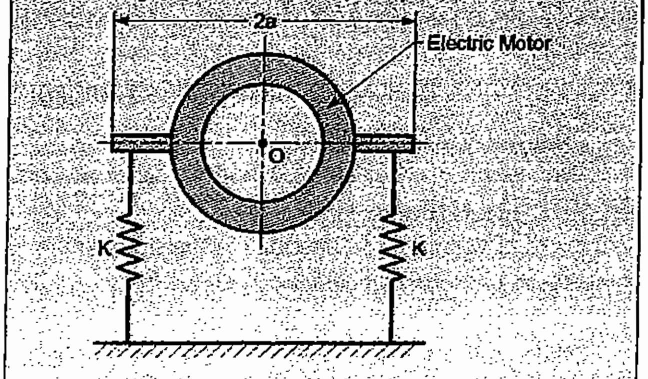
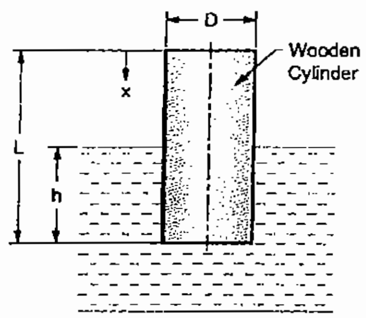


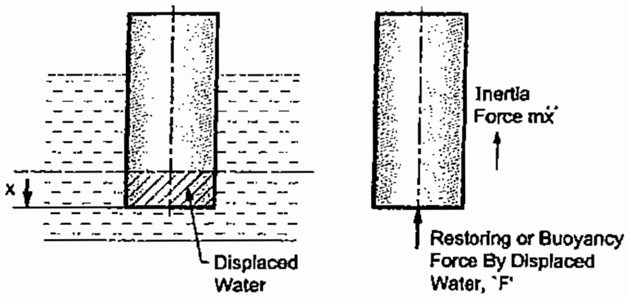
Fig. P. 3.12.21

Ex. 3.12.22 : A wooden cylinder of length 'L' , diameter 'D' and specific gravity 'S_{cyl}' floats vertically in water. Find the frequency of oscillation when it depressed slightly and released. Also find the frequency of oscillation when water is replaced with salt water with specific gravity of 1.2.

Soln. :



(a) Before Depressing Cylinder



(b) After Depressing Cylinder **(c) F.B.D.**

Fig. P. 3.12.22

I. Equilibrium Method

- **Mass of cylinder :**

$$m = \left[\begin{matrix}\text{Volume} \\ \text{of cylinder}\end{matrix} \times \begin{matrix}\text{Specific gravity} \\ \text{of cylinder}\end{matrix} \times \begin{matrix}\text{Density} \\ \text{of water} \\ (\rho)\end{matrix}\right]$$

$$\therefore \; m = \frac{\pi}{4} D^2 \times L \times S_{cyl} \times 1000 \quad ...[\because \rho = 1000 \text{ kg/m}^3]$$

- **Buoyancy force on cylinder :**
  (i) When cylinder is depressed through distance 'x', mass of displaced water is,

$$m_w = \left[\begin{matrix}\text{Volume of} \\ \text{displaced} \\ \text{water}\end{matrix} \times \begin{matrix}\text{Specific} \\ \text{gravity of} \\ \text{water}\end{matrix} \times \begin{matrix}\text{Density} \\ \text{of water} \\ (\rho)\end{matrix}\right]$$

$$m_w = \frac{\pi}{4} D^2 \times x \times S_{liq} \times 1000 \quad ...[\because \rho = 1000 \text{ kg/m}^3]$$

  (ii) Restoring force or buoyancy force acting on cylinder due to displaced water is,

$$F = m_w \times g = \frac{\pi}{4} D^2 \times x \times S_{liq} \times 1000 \times g$$

- **Motion of cylinder :**
  Fig. P. 3.12.22(c);

$$\sum [\text{Inertia force + External force}] = 0$$

$$\therefore \; m\ddot{x} + F = 0$$

$$\therefore \; \frac{\pi}{4} D^2 \times L \times S_{cyl} \times 1000 \times \ddot{x} + \frac{\pi}{4} D^2 \times x \times S_{liq} \times 1000 \times g = 0$$

$$L \times S_{cyl} \times \ddot{x} + S_{liq} \times g \times X = 0$$

$$\therefore \; \ddot{x} + \left(\frac{S_{liq} \times g}{S_{cyl} \times L}\right) x = 0 \quad ...(a)$$

- **Natural circular frequency :**
  This Equation (a) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{S_{liq} \times g}{S_{cyl} \times L}$$

$$\therefore \; \omega_n = \sqrt{\frac{S_{liq} \times g}{S_{cyl} \times L}}, \text{ rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{S_{liq} \times g}{S_{cyl} \times L}}, \text{ Hz} \quad ...(b)$$

- **Natural frequency in water :** Specific gravity of water is one, (i.e. $S_{liq} = 1$) therefore, from Equation (b), natural frequency in water is,

$$\therefore \; f_{n\omega} = \frac{1}{2\pi}\sqrt{\frac{g}{S_{cyl} \times L}}, \text{ Hz} \quad ...\textbf{Ans.}$$

- **Natural frequency in salt water :** Specific gravity of salt water is 1.2 (i.e. $S_{liq} = 1.2$), therefore, from Equation (b), natural frequency in salt water is,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{1.2g}{S_{cyl} \times L}}, \text{ Hz} \quad ...\textbf{Ans.}$$

Examples for Practice

Refer our website for complete solution of following examples

Ex. 3.12.23 : A U-tube manometer having column length '*l*' contains liquid with its both ends open to atmosphere. Find the natural frequency of oscillations of liquid column, when tube is slightly displaced.

Ex. 3.12.24 : A V-tube manometer having liquid column of length '*l*' has both ends open to atmosphere, as shown in Fig. P. 3.12.24(a). Find the natural frequency of oscillation of liquid column when tube is slightly displaced.

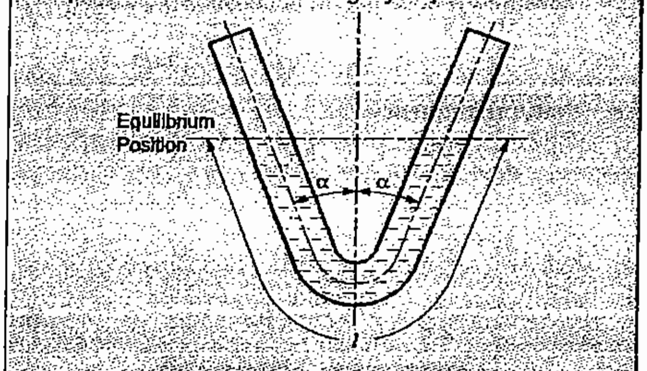


Fig. P. 3.12.24(a)

Ex. 3.12.25 : Determine the natural frequency of cantilever beam with mass 'm' placed at free end. Neglect the mass of cantilever beam.

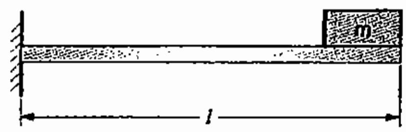


Fig. P. 3.12.25

Soln. :

- **Deflection of cantilever beam with point load W at free end :**

$$\delta = \frac{W l^3}{3 EI} = \frac{mgl^3}{3 EI}$$

- **Natural frequency :**

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{g}{\frac{mgl^3}{3 EI}}}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{3\,EI}{ml^3}}\,,\ \text{Hz} \qquad \text{...Ans.}$$

Ex. 3.12.26 : A shaft of length 0.75 m and diameter 50mm, supported freely at the ends, is carrying a weight 90 N at 0.25 m from one end. Find the natural frequency of transverse vibration. Assume $E = 200\ \text{GN/m}^2$.

SPPU - Dec. 06

Soln. :

$l = 0.75$ m ; $d = 0.05$ m ;

$W = 90$ N ; $a = 0.25$ m ;

$E = 200\ \text{GN/m}^2$.

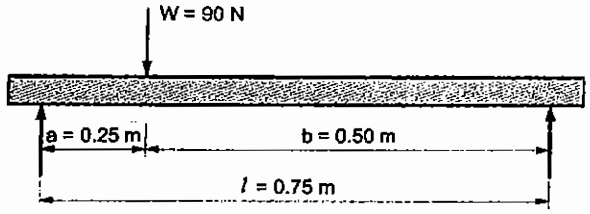


Fig. P. 3.12.26

- **Moment of inertia of beam :**

$$I = \frac{\pi}{64}\times d^4 = \frac{\pi\,(0.05)^4}{64} = 0.307\times10^{-6}\ \text{m}^4$$

- **Static deflection at load point for simply supported beam :**

$$\delta = \frac{Wa^2 b^2}{3\,E\,I\,l}$$

$$= \frac{90\times0.25^2\times0.50^2}{3\times200\times10^9\times0.307\times10^{-6}\times0.75}$$

$$\delta = 1.079\times10^{-5}\ \text{m}$$

- **Natural frequency of transverse vibration :**

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{1}{2\pi}\sqrt{\frac{9.81}{1.079\times10^{-5}}}$$

$$\text{or} \quad f_n = \mathbf{156.24\ Hz} \qquad \text{...Ans.}$$

Ex. 3.12.27 : Find the natural frequency of oscillation of a simple pendulum, shown in Fig. P. 3.12.27(a).

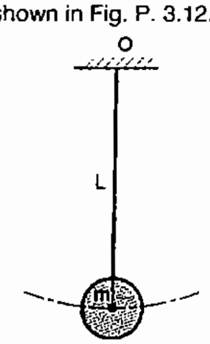


Fig. P. 3.12.27(a) : Simple Pendulum

Soln. :

Let, I_o = mass M.I. of simple pendulum about O

or $I_o = m\,L^2$, kg-m^2 ...(a)

[I] Equilibrium Method

From Fig. 3.12.27(b) ;

- **Angular motion about 'O' :**

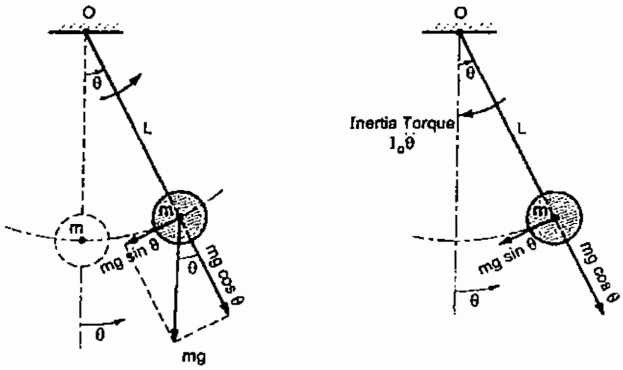
$$\sum [\text{Inertia torque} + \text{External torques}] = 0$$

$$\therefore\ I_o\ddot{\theta} + mg\sin\theta\times L = 0$$

$$I_o\ddot{\theta} + mg\,L\,\theta = 0$$

$$[\because\ \theta \text{ is small, } \sin\theta \simeq \theta\,]$$

$$\therefore\ \ddot{\theta} + \frac{mgL}{I_o}\theta = 0 \qquad \text{...(b)}$$



(b) Displaced Position (c) F.B.D of System

Fig. P. 3.12.27

- **Natural circular frequency :** This Equation (b) is the differential equation of motion of a simple pendulum. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{mgL}{I_o}$$

$$\therefore\ \omega_n = \sqrt{\frac{mgL}{I_o}} \qquad \text{...(c)}$$

- Substituting Equation (a) in Equation (c),

$$\omega_n = \sqrt{\frac{mgL}{mL^2}}$$

$$\text{or} \quad \omega_n = \sqrt{\frac{g}{L}}\,,\ \text{rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\therefore\ f_n = \frac{1}{2\pi}\sqrt{\frac{g}{L}}\,,\ \text{Hz} \qquad \text{...Ans.}$$

[II] Energy Method

From Fig. P. 3.12.27(d);

- **K. E. of mass 'm' :**

$$KE = \frac{1}{2} I_o \dot{\theta}^2$$

- **P. E. of mass 'm' :**

$$PE = mg(L - L\cos\theta) = mgL(1 - \cos\theta)$$

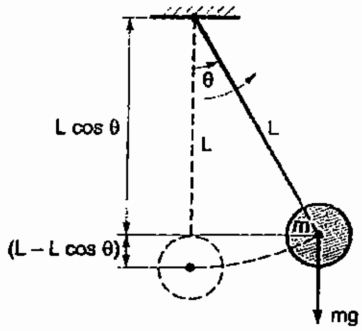


Fig. P. 3.12.27(d)

- **Energy Method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left[\frac{1}{2} I_o \dot{\theta}^2 + mgL(1 - \cos\theta)\right] = 0$$

$$\frac{d}{dt}\left[\frac{1}{2} I_o \dot{\theta}^2 + mgL - mgL\cos\theta\right] = 0$$

$$\frac{1}{2} I_o 2\dot{\theta}\ddot{\theta} + mgL\sin\theta\,\dot{\theta} = 0$$

$$I_o\ddot{\theta} + mgL\,\theta = 0 \quad ...[\because \theta \text{ is small, } \sin\theta \simeq \theta]$$

$$\therefore \ddot{\theta} + \left(\frac{mgL}{I_o}\right)\theta = 0 \qquad ...(d)$$

- **Natural circular frequency :** This Equation (d) is the differential equation of motion of a simple pendulum. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{mgL}{I_o}$$

$$\therefore \omega_n = \sqrt{\frac{mgL}{I_o}} \qquad ...(e)$$

Substituting Equation (a) in Equation (e),

$$\omega_n = \sqrt{\frac{mgL}{mL^2}}$$

$$\text{or} \quad \omega_n = \sqrt{\frac{g}{L}}\text{, rad/s}$$

- **Natural frequency :**

Now, $f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$, Hz ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.28 : Determine the natural frequency of oscillation of the simple pendulum, shown in Fig. P. 3.12.28(a), considering the mass of rod. **SPPU - Dec. 13, 8 Marks**

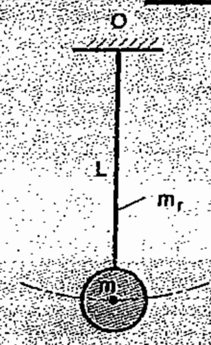


Fig. P. 3.12.28(a)

Ex. 3.12.29 : Find the differential equation of motion and the natural frequency of vibration for a system, shown in Fig. P. 3.12.29(a). **SPPU- Dec. 11, May 15, 8 Marks**

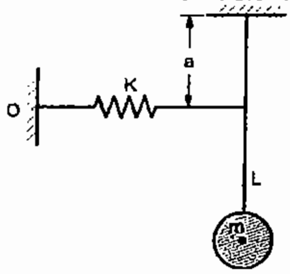
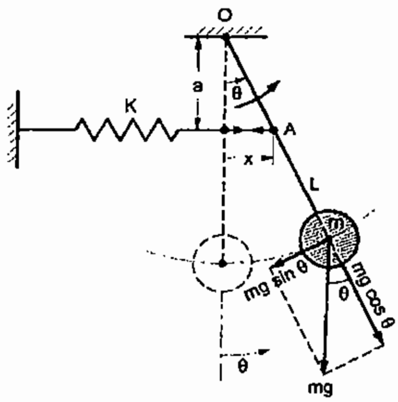


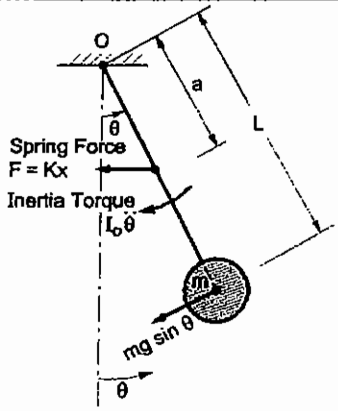
Fig. P. 3.12.29(a)

Soln. :

- Fig. P. 3.12.29(b) shows the system when mass 'm' is displaced through an angle, 'θ' due to which the spring is stretched by distance 'x'.



(b) Displaced Position



(c) F.B.D. of System

Fig. P. 3.12.29

- From Fig. P. 3.12.29 (b) :

Deflection of spring $= x \approx a\,\theta$

Spring force or Restoring force by spring, $F = Kx = Ka\theta$

Mass moment of inertia about O, $I_o = mL_2$

[II] Equilibrium Method

- **Angular motion about point 'O' :**

From Fig. P. 3.12.29(c) ;

$$\sum \text{[Inertia torque + External torques]} = 0$$

$$\therefore\ I_o\ddot{\theta} + mg\sin\theta \times L + Kx \times a = 0$$

$$I_o\ddot{\theta} + mgL\theta + Ka^2\theta = 0$$

$$\text{...[}\because \sin\theta \simeq \theta \text{ and } x \simeq a\,\theta\text{]}$$

$$I_o\ddot{\theta} + (mgL + Ka^2)\,\theta = 0$$

$$\therefore\ \ddot{\theta} + \left(\frac{mgL + Ka^2}{I_o}\right)\theta = 0 \quad \text{...(a)}$$

- **Natural circular frequency :** This Equation (a) is the differential equation of motion of a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{mgL + Ka^2}{I_o}$$

$$\therefore\ \omega_n = \sqrt{\frac{mgL + Ka^2}{I_o}} = \sqrt{\frac{mgL + Ka^2}{mL^2}}\text{, rad/sec}$$

$$\text{...[For a point mass at distance L, } I_o = mL^2\text{]}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{mgL + Ka^2}{mL^2}}\text{, Hz} \quad \text{...Ans.}$$

[II] Energy Method

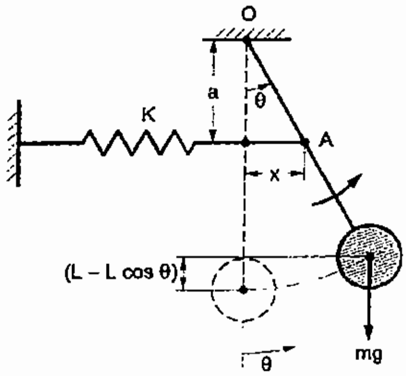


Fig. P. 3.12.29(d)

From Fig. P. 3.12.29(d) :

- **K. E. of system :**

Kinetic energy of mass $= \frac{1}{2} I_o \dot{\theta}^2$

$$\therefore\ KE = \frac{1}{2} I_o \dot{\theta}^2$$

- **P. E. of system :**

Potential energy of mass = mg (L – L cos θ)

Potential energy of spring $= \frac{1}{2}Kx^2 = \frac{1}{2}Ka^2\theta^2$

Total potential energy is,

$$PE = mg(L - L\cos\theta) + \frac{1}{2}Ka^2\theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore\ \frac{d}{dt}\left(\frac{1}{2}I_o\dot{\theta}^2 + mg(L - L\cos\theta) + \frac{1}{2}Ka^2\theta^2\right) = 0$$

$$\frac{1}{2}I_o 2\dot{\theta}\ddot{\theta} + mgL\sin\theta\,\dot{\theta} + \frac{1}{2}Ka^2 2\theta\,\dot{\theta} = 0$$

$$I_o\ddot{\theta} + mgL\theta + Ka^2\theta = 0$$

$$I_o\ddot{\theta} + (mgL + Ka^2)\,\theta = 0$$

$$\therefore\ \ddot{\theta} + \left(\frac{mgL + Ka^2}{I_o}\right)\theta = 0 \quad \text{...(b)}$$

- **Natural circular frequency :** This Equation (b) is the differential equation of motion of a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{mgL + Ka^2}{I_o}$$

$$\therefore \quad \omega_n = \sqrt{\frac{mgL + Ka^2}{I_o}}$$

$$\text{or} \quad \omega_n = \sqrt{\frac{mgL + Ka^2}{mL^2}}, \text{rad/sec}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{mgL + Ka^2}{mL^2}}, \text{Hz} \qquad \text{...Ans.}$$

Ex. 3.12.30 : Find the natural frequency of vibration for the system shown in Fig. P. 3.12.30(a). Establish the condition for system to be non vibratory, in terms of 'b'.

SPPU- Dec. 16, Dec. 17, 4 Marks

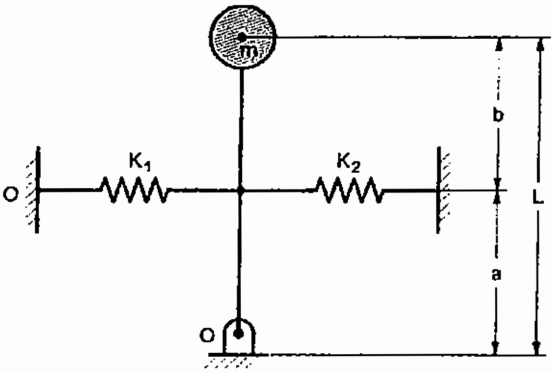
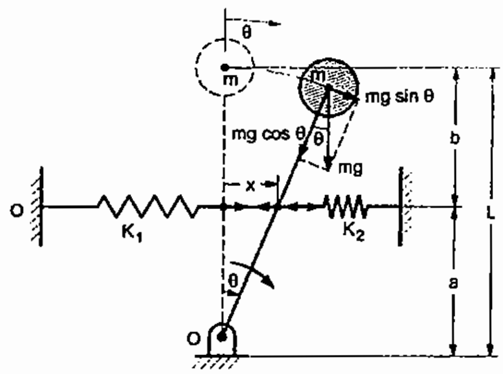


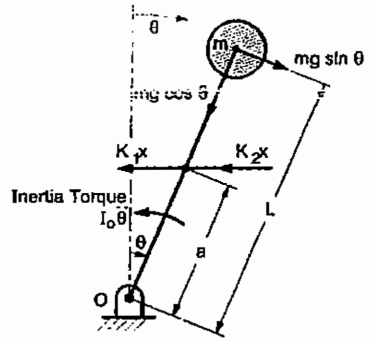
Fig. P. 3.12.30(a)

Soln. :

- Fig. P. 3.12.30 shows the system when mass 'm' is displaced through an angle 'θ' due to which the spring K_1 is stretched by distance 'x' and the spring 'K_2' is compressed by distance 'x'.



(b) Displaced Position



(c) F.B.D. of System

Fig. P. 3.12.30

- **From Fig. P. 3.12.30(b) :**

Deflection in both springs $= x \simeq a\,\theta$

Spring force or restoring force by spring, $K_1 = K_1 x = K_1 a\,\theta$

Spring force or restoring force by spring $K_2 = K_2 x = K_2 a\,\theta$

Mass moment of moment of inertia about O, $I_o = m\,L^2$

[I] Equilibrium Method

From Fig. P. 3.12.30(b) ;

- **Angular motion of body about point 'O' :**

$$\sum [\text{Inertia torque + External torques}] = 0$$

$$\therefore \quad I_o\,\ddot{\theta} + (K_1 \times a + K_2 \times a) - mg\,L\sin\theta = 0$$

$$\therefore \quad I_o\,\ddot{\theta} + (K_1\,a\,\theta\,a + K_2\,a\,\theta\,a) - mg\,L\sin\theta = 0$$

$$I_o\,\ddot{\theta} + (K_1 + K_2)\,a^2\,\theta - mgL\theta = 0$$

$$\text{...}[\because \sin\theta \simeq \theta]$$

$$\therefore \quad \ddot{\theta} + \left[\frac{(K_1 + K_2)\,a^2 - mgL}{I_o}\right]\theta = 0 \qquad \text{...(a)}$$

- **Natural circular frequency :** This Equation (a) is the differential equation of motion for a given system. Comparing this equation with the fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{(K_1 + K_2)\,a^2 - mgL}{I_o}$$

$$\therefore \quad \omega_n = \sqrt{\frac{(K_1 + K_2)\,a^2 - mgL}{I_o}}$$

$$\text{or} \quad \omega_n = \sqrt{\frac{(K_1 + K_2)\,a^2 - mgL}{mL^2}}, \text{rad/sec}$$

- Natural frequency :

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{(K_1 + K_2)\,a^2 - mgL}{mL^2}}, \text{Hz} \qquad \text{...Ans.}$$

- **Condition for system to be non-vibratory :**

  for non vibration motion, $f_n \le 0$.

$$\frac{1}{2\pi}\sqrt{\frac{(K_1+K_2)a^2 - mgL}{mL^2}} \le 0$$

$$\therefore \quad \frac{(K_1+K_2)a^2}{mL^2} - \frac{mgL}{mL^2} \le 0$$

$$\therefore \quad \frac{(K_1+K_2)a^2}{mL^2} \le \frac{g}{L}$$

$$\therefore \quad a^2 \le \frac{gmL}{(K_1+K_2)}$$

$$\therefore \quad a \le \sqrt{\frac{mgL}{K_1+K_2}} \qquad \text{...(b)}$$

but $\quad a = L - b$

Hence, Equation (b) can be written as,

$$L - b \le \sqrt{\frac{mgL}{K_1+K_2}}$$

$$\therefore \quad L - \sqrt{\frac{mgL}{K_1+K_2}} \le b$$

$$\therefore \quad \mathbf{b \ge L - \sqrt{\frac{mgL}{K_1+K_2}}} \qquad \text{...(c)} \quad \text{...Ans.}$$

[II] Energy Method

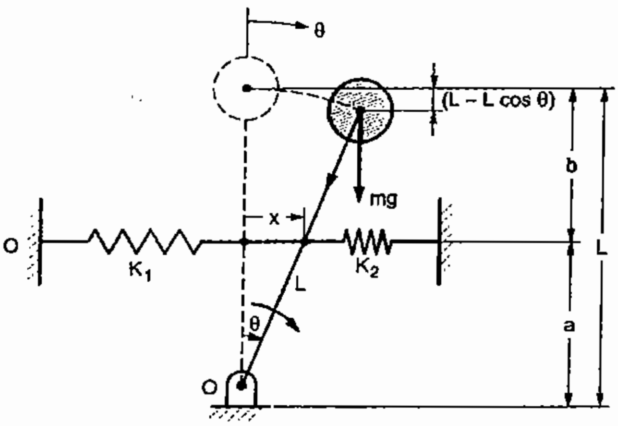


Fig. P. 3.12.30(d)

From Fig. P. 3.12.30(d);

- **K. E. of system :**

  Kinetic energy of mass, $KE = \frac{1}{2} I_o \dot{\theta}^2$

- **P. E. of system :**

$$\text{Potential energy of mass} = -mg(L - L\cos\theta)$$

$$\text{Potential energy of spring 1} = \frac{1}{2}K_1 x^2 = \frac{1}{2}K_1 a^2\theta^2$$

$$\text{Potential energy of spring 2} = \frac{1}{2}K_2 x^2 = \frac{1}{2}K_2 a^2\theta^2$$

Total potential energy is,

$$PE = -mg(L - L\cos\theta) + \frac{1}{2}K_1 a^2\theta^2 + \frac{1}{2}K_2 a^2\theta^2$$

- **Energy Method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \quad \frac{d}{dt}\left(\frac{1}{2}I_o\dot{\theta}^2 - mg(L - L\cos\theta) + \frac{1}{2}K_1 a^2\theta^2 + \frac{1}{2}K_2 a^2\theta^2\right) = 0$$

$$\frac{1}{2}I_o 2\dot{\theta}\ddot{\theta} - mgL\sin\theta\,\dot{\theta} + \frac{1}{2}K_1 a^2 2\theta\dot{\theta} + \frac{1}{2}K_2 a^2 2\theta\dot{\theta} = 0$$

$$I_o\ddot{\theta} - mgL\theta + (K_1+K_2)a^2\theta = 0$$

$$\text{...}[\because \theta \text{ is small}, \sin\theta \simeq \theta]$$

$$I_o\ddot{\theta} + [(K_1+K_2)a^2 - mgL]\theta = 0$$

$$\therefore \quad \mathbf{\ddot{\theta} + \left[\frac{(K_1+K_2)^2 - mgL}{I_o}\right] = 0} \qquad \text{...(d)}$$

- **Natural circular frequency :** This Equation (d) is the differential equation of motion for a given system, comparing this equation with the fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{(K_1+K_2)a^2 - mgL}{I_o}$$

$$\therefore \quad \omega_n = \sqrt{\frac{(K_1+K_2)a^2 - mgL}{I_o}}$$

$$\text{or} \quad \mathbf{\omega_n = \sqrt{\frac{(K_1+K_2)a^2 - mgL}{mL^2}}\text{, rad/sec}}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad \mathbf{f_n = \frac{1}{2\pi}\sqrt{\frac{(K_1+K_2)a^2 - mgL}{mL^2}}\text{, Hz}} \qquad \text{...Ans.}$$

Ex. 3.12.31 : Determine the natural frequency of vibration of a system shown in Fig. P. 3.12.31(a).

SPPU - Aug. 17(In Sem), 6 Marks

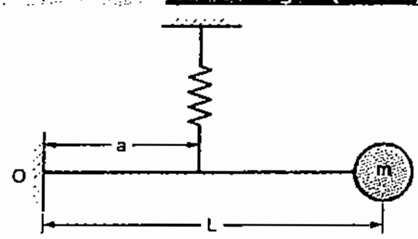


Fig. P. 3.12.31(a)

Soln. :

- Mass moment of inertia of system, $I_o = mL^2$
- **Static deflection of system :** Due to weight 'mg' the system is initially deflected by small angle 'ϕ' and spring 'K' is deflected by distance 'δ', as shown in Fig. P. 3.12.31(b).

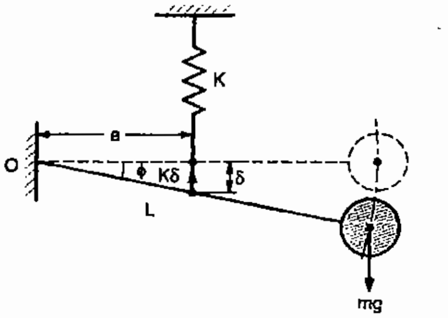


Fig. P. 3.12.31(b) : Initial Deflection of System

For equilibrium,

$$mg \times L \simeq K\delta \times a \qquad ...(a)$$

When system is deflected through an angle 'θ', the spring is stretched by distance 'x' and total deflection of spring is $(x + \delta)$, as shown in Fig. P. 3.12.31(b).

[I] Equilibrium Method

From Fig. P. 3.12.31(d) ;

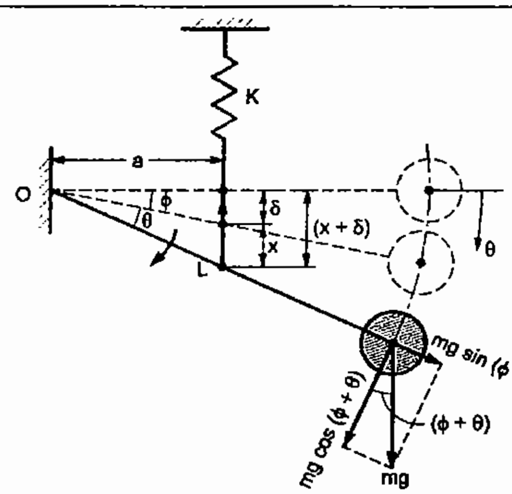
- **Angular motion about print 'O' :**

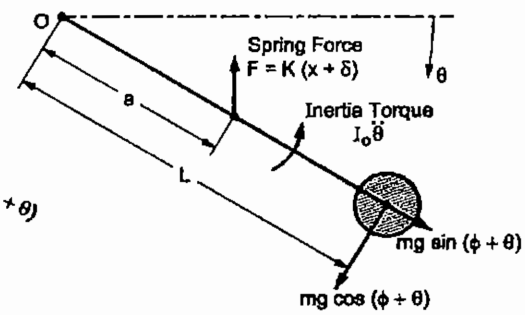
$$\sum \text{[Inertia torque + External torques]} = 0$$

$$\therefore \; I_o \ddot{\theta} + K(x+\delta)\,a - mg\cos(\phi+\theta)\,\theta \times L = 0$$

$$I_o \ddot{\theta} + Kxa + K\delta a - mgL = 0$$

$$...[\because \cos(\phi+\theta) \approx 1 \simeq 1]$$





(c) Displaced Position **(d) F.B.D. of System**

Fig. P. 3.12.31

$$I_o \ddot{\theta} + Kxa + mgL - mgL = 0 \qquad ...[\because \text{from Equation (a) } K\delta a \simeq mgL]$$

$$I_o \ddot{\theta} + Kxa = 0$$

$$I_o \ddot{\theta} + Ka^2\theta = 0 \qquad ...[\because x \simeq a\theta]$$

$$\therefore \; \ddot{\theta} + \left(\frac{Ka^2}{I_o}\right)\theta = 0 \qquad ...(b)$$

- **Natural circular frequency :** This Equation (b) is the differential equation of motion for a given system. Comparing this equation with the fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{Ka^2}{I_o}$$

$$\therefore \; \omega_n = \sqrt{\frac{Ka^2}{I_o}}$$

$$\text{or } \omega_n = \sqrt{\frac{Ka^2}{mL^2}} \text{ , rad/sec}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{Ka^2}{mL^2}}, \text{Hz} \quad \text{...Ans.}$$

[II] Energy Method

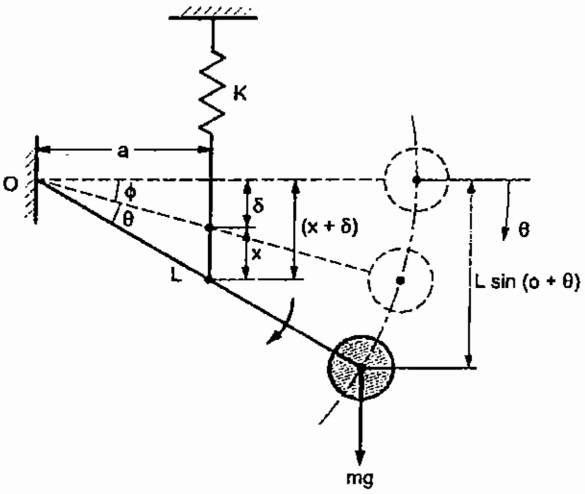


Fig. P. 3.12.31(e)

From Fig. P. 3.12.31(e) ;

- K. E. of system :

Kinetic energy of mass ,

$$KE = \frac{1}{2} I_o \dot{\theta}^2$$

- **P. E. of system :**

$$\text{Potential energy of mass} = -mgL\sin(\phi + \theta)$$
$$= -mgL\sin\theta$$

[as ϕ is small, it is neglected]

$$\text{Potential energy of spring} = \frac{1}{2}K(x+\delta)^2$$

Total potential energy is,

$$PE = -mgL\sin\theta + \frac{1}{2}K(x+\delta)^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \frac{d}{dt}\left[\frac{1}{2}I_o\dot{\theta}^2 - mgL\sin\theta + \frac{1}{2}K(x+\delta)^2\right] = 0$$

$$\frac{1}{2}I_o 2\dot{\theta}\ddot{\theta} - mgL\cos\theta\,\dot{\theta} + \frac{1}{2}K\,2(x+\delta)\dot{x} = 0$$

$$I_o\dot{\theta}\ddot{\theta} - mgL\dot{\theta} + K(x+\delta)\dot{x} = 0 \text{...}[\because \cos\theta \simeq 1]$$

$$I_o\dot{\theta}\ddot{\theta} - mgL\dot{\theta} + K(a\theta+\delta)a\dot{\theta} = 0$$

$$\text{...}[\because x = a\theta \text{ and } \dot{x} = a\dot{\theta}]$$

$$I_o\dot{\theta}\ddot{\theta} - mgL\dot{\theta} + Ka^2\theta\dot{\theta} + K\delta a\dot{\theta} = 0$$

$$\therefore I_o\ddot{\theta} - mgL + Ka^2\theta + K\delta a = 0$$

$$I_o\ddot{\theta} + Ka^2\theta = 0$$

$$\text{...}[\because \text{from Equation (i) } K\delta a \simeq mgL]$$

$$\therefore \ddot{\theta} + \left(\frac{Ka^2}{I_o}\right)\theta = 0 \quad \text{...(c)}$$

- **Natural circular frequency :** This Equation (c) is the differential equation of motion for a given system. Comparing this equation with the fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{Ka^2}{I_o}$$

$$\therefore \omega_n = \sqrt{\frac{Ka^2}{I_o}}$$

$$\therefore \omega_n = \sqrt{\frac{Ka^2}{mL^2}} \text{ , rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{Ka^2}{mL^2}}, \text{Hz} \quad \text{...Ans.}$$

Ex. 3.12.32 : Determine the natural frequency of vibration for a system, shown in Fig. P. 3.12.32(a). Assume the mass of the beam as 'm'. **SPPU - Dec. 03**

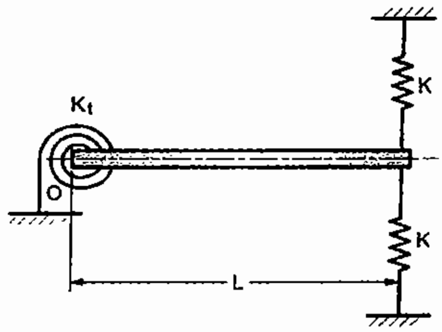
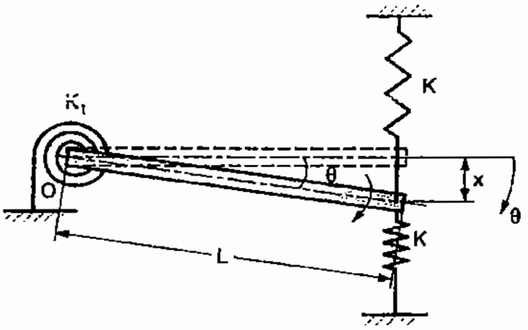


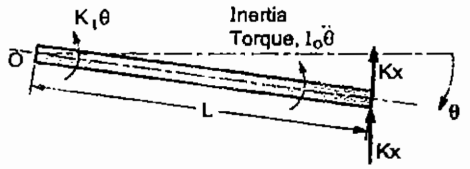
Fig. P. 3.12.32(a)

Soln. :

- Fig. P. 3.12.32(a) shows the system when beam is deflected through an angle 'θ' due to which the upper spring is stretched by distance 'x' and lower spring is compressed by same distance 'x'.



(b) Displaced Position



(c) F.B.D. of Beam

Fig. P. 3.12.32

- **Mass M. I. of beam about O :**

$$I_o = I_G + m\left(\frac{L}{2}\right)^2$$

$$= \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{16mL^2}{48}$$

$$\text{or} \quad I_o = \frac{mL^2}{3} \qquad \text{...(a)}$$

[I] Equilibrium Method

From Fig. P. 3.12.32(c);

- **Motion about point 'O' :**

$$\sum [\text{Inertia torque + External torques}] = 0$$

$$I_o\ddot{\theta} + 2Kx \times L + K_t\theta = 0$$

$$I_o\ddot{\theta} + 2KL^2\theta + K_t\theta = 0 \quad \text{...}[\because x \simeq L\,\theta]$$

$$I_o\ddot{\theta} + (2KL^2 + K_t)\theta = 0$$

$$\therefore \quad \ddot{\theta} + \left(\frac{2\,KL^2 + K_t}{I_o}\right) = 0 \qquad \text{...(b)}$$

- **Natural circular frequency :** This Equation (b) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{2\,KL^2 + K_t}{I_o}$$

$$\therefore \omega_n = \sqrt{\frac{2\,KL^2 + K_t}{I_o}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{2\,KL^2 + K_t}{I_o}}\text{, Hz} \qquad \text{...(c)}$$

Substituting Equation (a) in Equation (c),

$$f_n = \frac{1}{2\pi}\sqrt{\frac{2\,KL^2 + K_t}{mL^2/3}}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{3(2\,KL^2 + K_t)}{mL^2}}\text{, Hz} \qquad \text{...Ans.}$$

[II] Energy Method

From Fig. P. 3.12.32(b) ;

- **K.E. of system :**

Kinetic energy of beam is,

$$KE = \frac{1}{2} I_o \dot{\theta}^2$$

- **P. E. of system :**

$$\text{Potential energy of two linear springs} = \frac{1}{2}[2\,Kx^2]$$

$$= Kx^2 = KL^2\theta^2$$

$$\text{Potential energy of torsional spring} = \frac{1}{2}K_t\theta^2$$

Total potential energy is,

$$PE = KL^2\theta^2 + \frac{1}{2}K_t\theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \quad \frac{d}{dt}\left(\frac{1}{2}I_o\dot{\theta}^2\ KL^2\theta^2 + \frac{1}{2}K_t\theta^2\right) = 0$$

$$\frac{1}{2}I_o 2\dot{\theta}\ddot{\theta} + KL^2\,2\theta\dot{\theta} + \frac{1}{2}K_t\,2\theta\dot{\theta} = 0$$

$$I_o\ddot{\theta} + 2KL^2\theta + K_t\theta = 0$$

$$I_o\ddot{\theta} + (2KL^2 + K_t)\theta = 0$$

$$\therefore \quad \ddot{\theta} + \left(\frac{2KL^2 + K_t}{I_o}\right)\theta = 0 \qquad \text{...(d)}$$

- **Natural circular frequency :** This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{2\,KL^2 + K_t}{I_o}$$

$$\therefore \omega_n = \sqrt{\frac{2\,KL^2 + K_t}{I_o}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or } f_n = \frac{1}{2\pi}\sqrt{\frac{2\,KL^2 + K_t}{I_o}}, \text{Hz} \quad ...(e)$$

Substituting Equation (a) in Equation (e),

$$\therefore \quad f_n = \frac{1}{2\pi}\sqrt{\frac{2\,KL^2 + K_t}{mL^2/3}}$$

$$\text{or } f_n = \frac{1}{2\pi}\sqrt{\frac{3(2\,KL^2 + K_t)}{mL^2}}, \text{Hz} \quad ...\text{Ans.}$$

Ex. 3.12.33 : Determine the natural frequency of vibration for a system, shown in Fig. P. 3.12.33(a). Take mass of the beam as 5 kg.

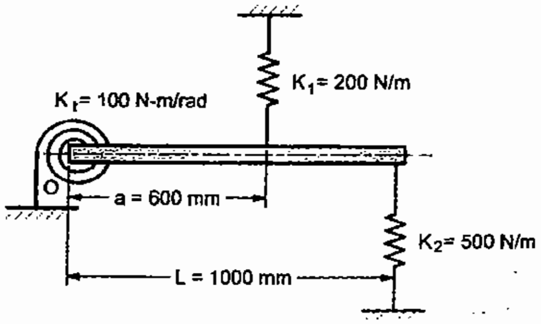


Fig. P. 3.12.33(a)

Soln. :

- Fig. P. 3.12.33(a) shows the system when beam is deflected through an angle 'θ' due to which upper spring is stretched by distance 'x_1' and lower spring is compressed by distance 'x_2'.

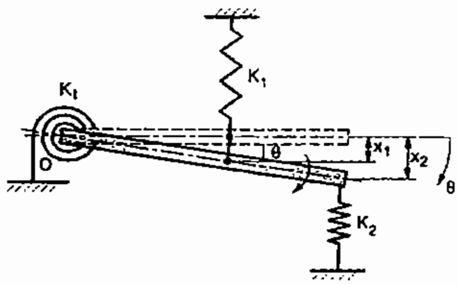
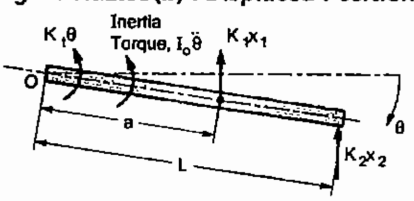


Fig. P. 3.12.33(b) : Displaced Position



(c) F.B.D. of Beam
Fig. P. 3.12.33

- **From Fig. P. 3.12.33(c) :**

Spring force in upper spring	= $K_1 x_1 = K_1 a\theta$
Spring force in lower spring	= $K_2 x_2 = K_2 L\theta$

- **Mass M. I of beam about O :**

$$I_o = I_G + m\left(\frac{L}{2}\right)^2$$

$$= \frac{mL^2}{12} + \frac{mL^2}{4} = \frac{16mL^2}{48}$$

$$\therefore \quad I_o = \frac{mL^2}{3} \quad ...(a)$$

[I] Equilibrium Method

From Fig. P. 3.12.33(c);

- **Motion about 'O' :**

$$\sum [\text{Inertia torque + External torques}] = 0$$

$$I_o\ddot{\theta} + K_1 x_1 a + K_2 x_2 L + K_t\theta = 0$$

$$I_o\ddot{\theta} + K_1 a^2\theta + K_2 L^2\theta + K_t\theta = 0$$

$$I_o\ddot{\theta} + \left(K_1 a^2 + K_2 L^2 + K_t\right)\theta = 0$$

$$\therefore \quad \ddot{\theta} + \left(\frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}\right)\theta = 0 \quad ...(b)$$

- **Natural circular frequency :**

This Equation (b) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}$$

$$\therefore \quad \omega_n = \sqrt{\frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}}, \text{rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or } f_n = \frac{1}{2\pi}\sqrt{\frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}} \quad ...(c)$$

- **Substituting Equation (a) in Equation (c),**

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_1 a^2 + K_2 L^2 + K_t}{\frac{mL^2}{3}}}$$

$$\text{or } f_n = \frac{1}{2\pi}\sqrt{\frac{3\left(K_1 a^2 + K_2 L^2 + K_t\right)}{mL^2}}, \text{Hz} \quad ...\text{Ans.}$$

Now, $K_1 = 200$; $K_2 = 500$ N/mm ;

$K_t = 100$ N-m/rad ; $a = 0.6$ m ;

$L = 1.0$ m ; $m = 5$ kg

$$\therefore \ f_n = \frac{1}{2\pi}\sqrt{\frac{3\,(200 \times 0.6^2 + 500 \times 1.0^2 + 100)}{5 \times 1^2}}$$

$$\text{or} \quad f_n = \mathbf{3.196\ Hz} \qquad \text{...Ans.}$$

[II] Energy Method

From Fig. P. 3.12.33(c);

- **K. E. of system :**

Kinetic energy of beam is, $KE = \frac{1}{2} I_o \dot{\theta}^2$

- **P. E. of system :**

Potential energy of upper spring $= \frac{1}{2} K_1 x_1^2 = \frac{1}{2} K_1 a^2 \theta^2$

Potential energy of lower spring $= \frac{1}{2} K_2 x_2^2 = \frac{1}{2} K_2 L^2 \theta^2$

Potential energy of torsional spring $= \frac{1}{2} K_t \theta^2$

Total potential energy is,

$$PE = \frac{1}{2} K_1 a^2 \theta^2 + \frac{1}{2} K_2 L^2 \theta^2 + \frac{1}{2} K_t \theta^2$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\therefore \ \frac{d}{dt}\left(\frac{1}{2} I_o \dot{\theta}^2 + \frac{1}{2} K_1 a^2 \theta^2 + \frac{1}{2} K_2 L^2 \theta^2 + \frac{1}{2} K_t \theta^2\right) = 0$$

$$\frac{1}{2} I_o 2\dot{\theta}\ddot{\theta} + \frac{1}{2} K_1 a^2 2\theta\dot{\theta} + \frac{1}{2} K_2 L^2 2\theta\dot{\theta} + \frac{1}{2} K_t 2\theta\dot{\theta} = 0$$

$$I_o \ddot{\theta} + K_1 a^2 \theta + K_2 L^2 \theta + K_t \theta = 0$$

$$I_o \ddot{\theta} + \left(K_1 a^2 + K_2 L^2 + K_t\right)\theta = 0$$

$$\therefore \ \ddot{\theta} + \left(\frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}\right)\theta = 0 \qquad \text{...(d)}$$

- **Natural circular frequency :** This Equation (d) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}$$

$$\therefore \ \omega_n = \sqrt{\frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}} \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{K_1 a^2 + K_2 L^2 + K_t}{I_o}} \text{, Hz} \qquad \text{...(e)}$$

Substituting Equation (a) in Equation (e),

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K_1 a^2 + K_2 L^2 + K_t}{\frac{mL^2}{3}}}$$

$$\text{or} \quad f_n = \frac{1}{2\pi}\sqrt{\frac{3\left(K_1 a^2 + K_2 L^2 + K_t\right)}{mL^2}} \text{, Hz} \qquad \text{...Ans.}$$

Now, $K_1 = 200$; $K_2 = 500$ N/mm ;

$K_t = 100$ N-m/rad ; $a = 0.6$ m ;

$L = 1.0$ m ; $m = 5$ kg

$$\therefore \ f_n = \frac{1}{2\pi}\sqrt{\frac{3\,(200 \times 0.6^2 + 500 \times 1.0^2 + 100)}{5 \times 1^2}}$$

$$\text{or} \quad f_n = \mathbf{3.196\ Hz} \qquad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.12.34 : Find the natural frequency of vibration for the system, shown in Fig. P. 3.12.34(a). Neglect the mass of cantilever beam. Also, find the natural frequency of vibration when :

(i) K = ∞; and

(ii) I = ∞.

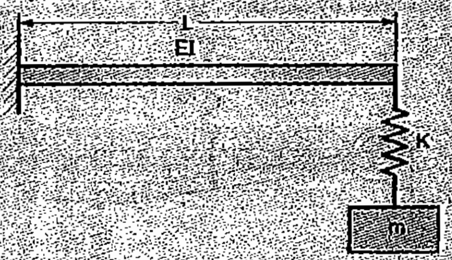


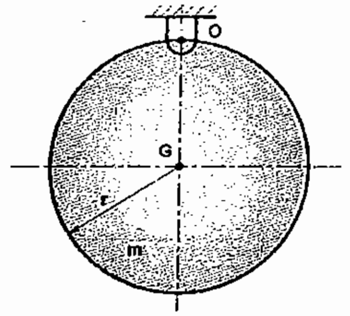
Fig. P. 3.12.34(a)

Ex. 3.12.35 : Determine the natural frequency of oscillation of a cylindrical disc suspended from a point on its circumference. **SPPU - May 12, 6 Marks**

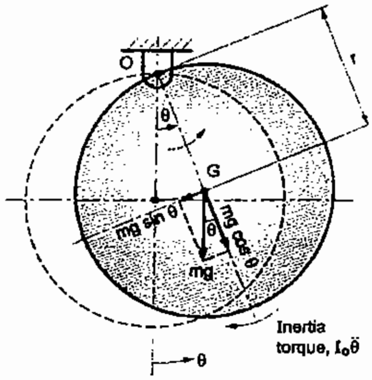
Soln. :

- **Mass M. I. of disc about 'O' :**

$$I_O = I_G + mr^2 = \frac{mr^2}{2} + mr^2 = \frac{3mr^2}{2} \qquad \text{...(a)}$$



(a) Equilibrium Position



(b) Displaced Position

Fig. P. 3.12.35

[I] Equilibrium Method

From Fig. P. 3.12.35(b) ;

- **Angular motion about 'O' :**

$$\Sigma\,[\text{Inertia torque + External torques}] = 0$$

$$\therefore\ Io\,\theta + mg\sin\theta \times r = 0$$

$$Io\,\theta + mg\theta r = 0 \ldots[\because \sin\theta \simeq \theta]$$

$$\therefore\ \theta + \left(\frac{mgr}{Io}\right)\theta = 0 \qquad \ldots(b)$$

- **Natural circular frequency :** This Equation (b) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{mgr}{I_o}$$

$$\therefore\ \omega_n = \sqrt{\frac{mgr}{I_o}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgr}{I_o}}\ \text{, Hz} = \frac{1}{2\pi}\sqrt{\frac{mgr}{\frac{3mr^2}{2}}}$$

or $$f_n = \frac{1}{2\pi}\sqrt{\frac{2g}{3r}}\ \text{, Hz}$$

[II] Energy Method

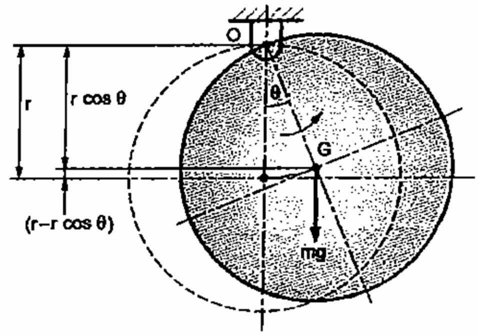


Fig. P. 3.12.35(c)

From Fig. P. 3.12.35(c);

- **K. E. of system :**

The kinetic energy of the cylindrical disc is,

$$KE = \frac{1}{2} I_o \dot{\theta}^2$$

- **P. E. of system :**

The potential energy of the cylindrical disc is,

$$PE = mg\,(r - r\cos\theta)$$

- **Energy method :**

$$\frac{d}{dt}(KE + PE) = 0$$

$$\frac{d}{dt}\left[\frac{1}{2} I_o \dot{\theta}^2\ mg\,(r - r\cos\theta)\right] = 0$$

$$\frac{1}{2} I_o\, 2\,\dot{\theta}\,\ddot{\theta} + mgr\sin\theta\,\dot{\theta} = 0$$

$$I_o\,\ddot{\theta} + mgr\sin\theta = 0$$

$$I_o\,\ddot{\theta} + mgr\theta = 0 \ldots[\because \sin\theta \simeq \theta]$$

$$\ddot{\theta} + \left(\frac{mgr}{I_o}\right)\theta = 0 \qquad \ldots(c)$$

- **Natural circular frequency :** This Equation (c) is the differential equation of motion for a given system. Comparing this equation with fundamental equation of simple harmonic motion, we get,

$$\omega_n^2 = \frac{mgr}{I_o}$$

$$\therefore\ \omega_n = \sqrt{\frac{mgr}{I_o}}\ \text{, rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{mgr}{I_o}} = \frac{1}{2\pi}\sqrt{\frac{mgr}{\frac{3mr^2}{2}}}$$

or $$f_n = \frac{1}{2\pi}\sqrt{\frac{2g}{3r}}\ \text{, Hz} \qquad \ldots\textbf{Ans.}$$

Examples for Practice

Refer our website for complete solution of following examples

Ex. 3.12.36 : Determine the natural frequency of oscillation of a semicircular disc suspended its center.

Ex. 3.12.37 : Determine the natural frequency of oscillation of a half solid cylinder of mass 'm' and radius 'r', when it is slightly displaced from mean position and released.

Ex. 3.12.38 : A flywheel, having a mass of 20 kg, is allowed to swing as pendulum about a knife-edge at the inner side of the rim. If the time period for one oscillation is 1.2 seconds, determine the mass moment of inertia of the flywheel about its geometric axis. Take inner radius of flywheel as 0.15 m.

Ex. 3.12.39 : A uniform rod of mass 'm' is supported, as shown in Fig. P. 3.12.39(a). Determine the frequency of the resulting motion. **SPPU - Dec. 06**

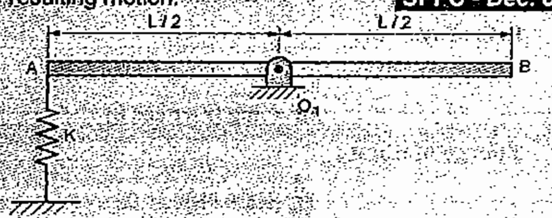


Fig. P. 3.12.39(a)

Ex. 3.12.40 : Determine the natural frequency of oscillation of the system shown in Fig. P. 3.12.40(a).

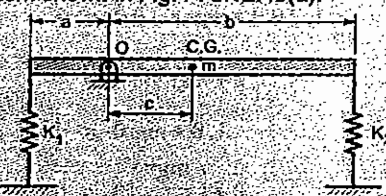


Fig. P. 3.12.40(a)

Ex. 3.12.41 : A slender bar of length 'L' is supported symmetrically on two rollers rotating in opposite direction with the same speed, as shown in Fig. P. 3.12.41(a). The coefficient of friction between the bar and the roller material is 'μ'. Write the equation of motion for the system and find the natural frequency of oscillation of the slender bar. What will be the effect on the system, if the direction of rotation of rollers is reversed ? **SPPU - May 05**

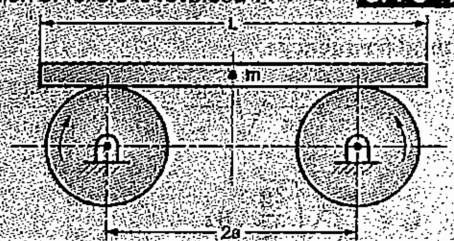


Fig. P. 3.12.41(a)

Ex. 3.12.42 : L-shaped rod is suspended, as shown in Fig. P. 3.12.42(a). The length of each side is '*l*' while mass of each side is m. Determine the natural frequency of oscillations of the system.

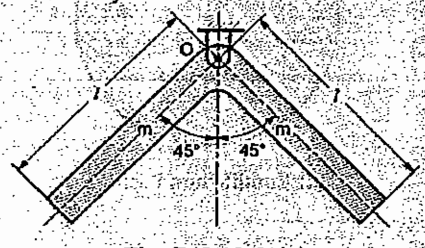


Fig. P. 3.12.42(a)

Ex. 3.12.43 : The slender rod of length 'L' having mass 'm' is oscillating without slipping on a curved cylindrical surface of radius 'r', as shown in Fig. P. 3.12.43(a). Find the natural frequency of oscillation of the rod.

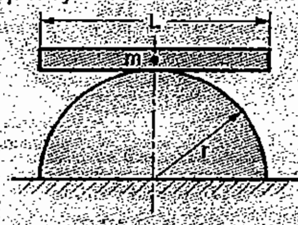


Fig. P. 3.12.43(a)

Ex. 3.12.44 : Find the natural frequency of vibration of the system shown in Fig. P. 3.12.44(a).

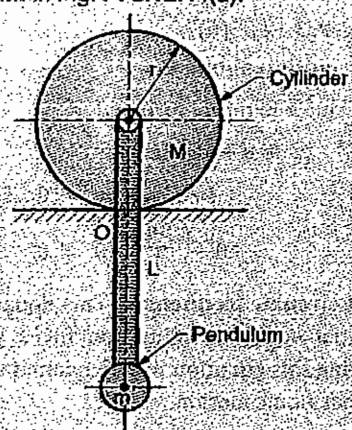


Fig. P. 3.12.44(a)

Ex. 3.12.45 : A cylinder of mass 'm' and radius 'r' rolls without slipping on a concave cylindrical surface of radius 'R'. Find the natural frequency of oscillations. **SPPU - Dec. 15, 4 Marks**

Part III : Damped Free Vibrations

3.13 DAMPING

- **Damped vibrations** : In a vibratory system, if an external resistance is provided so as to reduce the amplitude of vibrations, the vibrations are known as **damped vibrations**.
- **Damping** : The external resistance which is provided to reduce the amplitude of vibrations is known as **damping**.
- **Damper** : The damper is a unit which absorbs the energy of the vibratory system, thereby reducing the amplitude of vibration.
- **Important parameters in damped free vibrations :**

(i) Frequency of damped vibrations
(ii) Rate of decay of amplitude

3.14 TYPES OF DAMPINGS

Based on the method of providing the resistance to the vibrations, the dampings are classified in to three types [Fig. 3.14.1] :

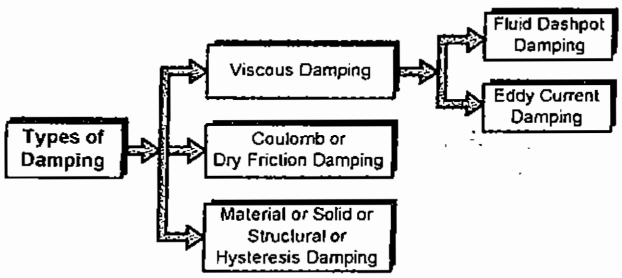


Fig 3.14.1 : Types of Dampings

3.14.1 Viscous Damping :

Principle of Viscous Damping :

- This is the most commonly used damping method to reduce the amplitude of vibrations. When system vibrates in a fluid medium like: air, gas, water or oil, the resistance is offered by the fluid to the vibrating body.
- In viscous, damping the damping resistance depends upon :
  (a) Relative velocity of vibrating body and
  (b) Parameters of the damper (like: viscosity of the fluid used in damper and dimensions of damper).

Viscous Damping :

In viscous damping, the two surfaces having relative motion are separated by a viscous fluid film. This fluid-film offers resistance to motion of one surface with respect to another surface. This resistance is known as **damping resistance** or **damping force.**

Damping Resistance :

- According to Newton's law of viscosity, the damping resistance is given by ,

$$F = \frac{\mu A V}{y} = \left(\frac{\mu A}{y}\right) \dot{x}$$

$$\text{or} \quad F = c\,\dot{x} \qquad \text{...(3.14.1)}$$

$$\text{i.e.} \quad F \propto \dot{x} \qquad \text{...(3.14.2)}$$

Where, F = damping resistance or damping force, N
v = $\dot{x}$ = relative velocity between two surfaces, m/s
c = damping coefficient or damping force per unit velocity, N-s/m

- From Equation (3.14.1) it is seen that, in **viscous damping**, the damping resistance or damping force is directly proportional to the relative velocity. Therefore, the viscous damping is a linear type damping.

Damping Coefficient :

- The constant of proportional **'c'** in Equation (3.14.1) is known as **damping coefficient** and is defined as the damping force per unit velocity.
- The value of damping coefficient 'c' depends upon :
  (a) The viscosity of fluid
  (b) The dimensions of damper.

Examples of Viscous Damping :

(i) Fluid dashpot damping
(ii) Eddy current damping

(i) Fluid dashpot damping :

- The fluid dashpot damper consist of a piston moving inside a cylinder filled with viscous fluid, as shown in Fig. 3.14.2.

- The expression for a damping coefficient 'c' of a fluid dashpot damper can be obtained by using a theory of fluid flow through a rectangular slot [hydrostatic lubrication]
- **Flow rate of fluid through a finite slot :**

$$Q = \frac{\Delta P\, b\, h^3}{12\,\mu\, l} \qquad ...(a)$$

Where, ΔP = pressure difference, N/ m²

b = width of the slot across the flow, m

l = length of the slot in the direction of flow, m

h = thickness of the slot or fluid-film, m

μ = absolute viscosity of the fluid, N-s/m²

- **Pressure difference across finite slot :**

$$\Delta P = \frac{12\mu\, l\, Q}{b\, h^3} \qquad ...(b)$$

- **Parameters of fluid dashpot damper (Fig. 3.14.2) :**

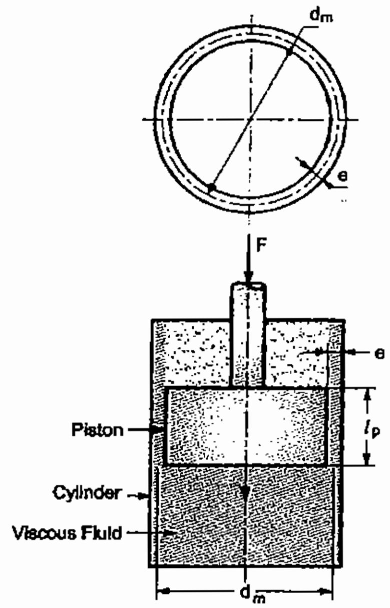


Fig. 3.14.2 : Fluid Dashpot Damper

Let, A_p = cross-sectional area of the piston, m²

l_p = length of piston m

e = radial clearance between the piston and cylinder, m

d_m = mean diameter of an annular area between the piston and cylinder, m

μ = absolute viscosity of the fluid, N-s/ m²

ΔP = pressure difference across the two ends of the piston, N/ m²

F = damping resistance or damping force or force acting on the piston, N

Q = flow rate of fluid through, an annular area between the piston and cylinder, m³/s

v = $\dot{x}$ = relative velocity or velocity of piston, m/s

- Equation (b) can be applied for the flow of fluid through an annular area between the piston and cylinder in fluid dashpot damper by substituting following parameters :

$$\left.\begin{aligned} b &= \pi\, d_m \\ l &= l_p \\ h &= e \\ \text{and} \quad Q &= V \cdot A_p = \dot{x} \cdot A_p \end{aligned}\right\} \qquad ...(c)$$

- **Pressure difference in fluid dashpot - damper :** Substituting Equation (c) in Equation (b), the pressure difference across an annular area between the piston and cylinder in fluid dashpot damper is given by,

$$\Delta P = \frac{12\,\mu l_p\, \dot{x}\, A_p}{\pi\, d_m\, e^3} \qquad ...(d)$$

- **Damping resistance or force acting on piston :**

$$F = \Delta P \,.\, A_p \qquad ...(e)$$

$$\therefore \quad F = \left[\frac{12\,\mu\, l_p\, A_p^2}{\pi\, d_m\, e^3}\right]\dot{x} \qquad ...(f)$$

Again, $$F = c\,\dot{x} \qquad ...(g)$$

- **Damping coefficient :** Comparing Equations (f) and (g), the damping coefficient 'c' is given by,

$$c = \left[\frac{12\,\mu\, l_p\, A_p^2}{\pi\, d_m\, e^3}\right] \text{N-s/m} \qquad ...(3.14.3)$$

- **Factors affecting damping coefficient 'c' :**
  (a) Viscosity of fluid (μ);
  (b) Clearance between piston and cylinder (e);
  (c) Length of piston (l_p) ; and
  (d) Diameter of piston (d_p) .

(ii) Eddy current damping :

- Eddy current damping is based on the principle of magnetic flux. It consist of a magnet and non - ferrous metal, as shown in Fig. 3.14.3.

- When the plate moves between north and south poles of a magnet, in a direction perpendicular to magnetic flux, a current is induced in the plate and it is proportional to the velocity of the plate. This current is in the form of eddy current that sets up a magnetic field in a direction opposing the original magnetic field that causes them. Thus, there is resistance to the motion of the plate in a magnetic field which results in damping.

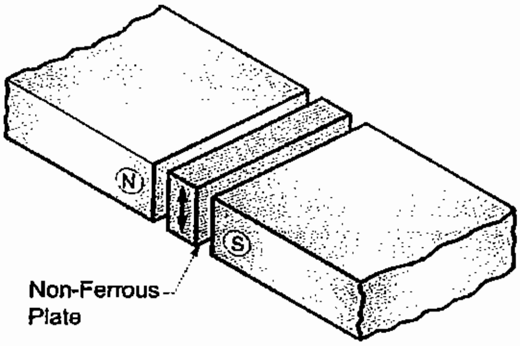


Fig. 3.14.3 : Eddy Current Damper

- This type of damping is used in vibrometers and in vibration control devices.

3.14.2 Coulomb or Dry Friction Damping :

University Question

Q. Define the term : Coulomb damping.

SPPU : Dec. 12, Dec. 13, May 14

- This type of damping occurs due to friction between two rubbing surfaces which are dry or unlubricated.
- The force of friction acting on each of the two mating surfaces is given by,

$$F_r = \mu R_N$$

where, μ = coefficient of friction

R_N = normal reaction between two mating surfaces

- The variation of 'μ' with respect to sliding velocity 'v' for different surface conditions is shown in Fig. 3.14.4. From Fig. 3.14.4 it is seen that, for ideally smooth surfaces, coefficient of friction 'μ' is independent of velocity. For rough surfaces, coefficient of friction 'μ' decreases some what initially with the increase in velocity, and then is practically constant. For all practical purpose 'μ' is taken as constant throughout the velocity range.

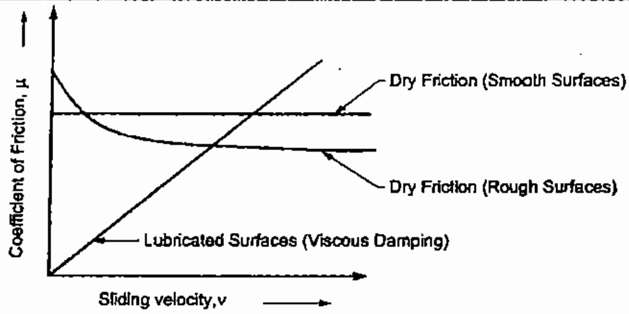


Fig. 3.14.4 : Coefficient of Friction Versus Sliding Velocity

3.14.3 Material or Solid or Structural or Hysteresis Damping :

- This type of damping occurs in all vibrating systems due to elasticity of material. The amount of such damping is very small. When materials are deformed, energy is absorbed and dissipated to surrounding in the form of heat. This effect is due to the internal friction of the molecules of material.
- When a body with material damping is subjected to vibrations, the stress- strain diagram for a vibrating body is not a straight line but forms a hysteresis loop, as shown in Fig. 3.14.5.
- The area under hysteresis loop represents the energy dissipated due to molecular friction per cycle per unit volume of the body. The size of hysteresis loop depends upon the type of a material of the vibrating body.

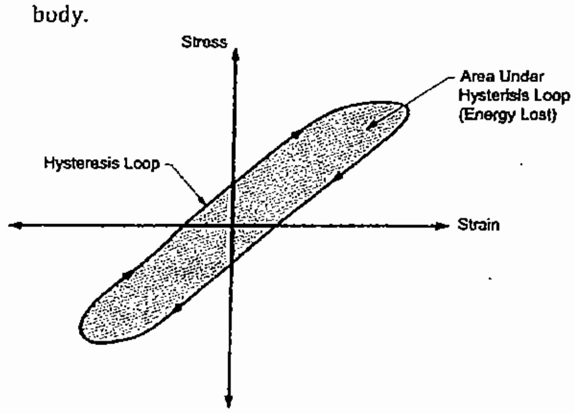


Fig. 3.14.5 : Hysteresis Loop for Elastic Material Subjected to Vibrations

Ex. 3.14.1 : A Dashpot is used in a vibrating system to damp out the vibrations. The diameter of cylinder is 85 mm and is filled completely with an oil having viscosity of 40 cP. The length and diameter of piston are 60 mm and 80 mm respectively . Calculate the damping coefficient of a dashpot. If the damping resistance is 250 N, determine the velocity of piston.

Soln. :

Given : $D = 0.085$ m ; $\mu = 40$ cP ;

$l_P = 0.06$ m ; $d_P = 0.08$ m ;

$F = 250$ N .

- **Viscosity of oil :**

$$\mu = 40 \text{ cP} = 40 \times 10^{-9} \text{ N-/mm}^2$$

$$= \frac{40 \times 10^{-9}}{10^6} \text{ N-/m}^2$$

$$\text{or} \quad \mu = 40 \times 10^{-3} \text{ N-s/m}^2$$

- **Radial clearance between the piston and cylinder :**

$$e = \frac{D - d_P}{2} = \frac{0.085 - 0.08}{2} = 0.0025 \text{ m}$$

- **Mean diameter of annular area between piston and cylinder :**

$$d_m = \frac{D + d_P}{2} = \frac{0.085 + 0.08}{2} = 0.0825 \text{ m}$$

- **Cross-sectional area of piston :**

$$A_P = \frac{\pi}{4} d_P^2 = \frac{\pi}{4} \times (0.08)^2 = 5.026 \times 10^{-3} \text{ m}^2$$

- **Damping coefficient of dashpot :**

$$C = \frac{12 \mu l_P A_P^2}{\pi d_m e^3} = \frac{12 \times 40 \times 10^{-3} \times 0.06 \times (5.026 \times 10^{-3})^2}{\pi \times 0.0825 \times (0.0025)^3}$$

or $c = 179.68$ **N-s/m** ...**Ans.**

- **Velocity of piston :**

$$F = c\dot{x}$$

$$\therefore \quad 250 = 179.68 \times \dot{x}$$

$$\therefore \quad \dot{x} = 3.39 \text{ m/s}$$

Ex. 3.14.2 : The following data refers to a fluid dashpot damper :

Diameter of piston = 60 mm

Length of piston = 50 mm

Radial clearance between piston and cylinder = 2 mm

If the vibrating system exerts a force of 400 N on dashpot at the relative velocity of 2.6 m/s, determine the absolute viscosity of the fluid to be used in dashpot.

Soln. :

Given : $d_P = 0.06$ m ; $l_P = 0.05$ m ;

$e = 0.002$ m ; $F = 400$ N ;

$v = 2.6$ m/s.

- **Diameter of cylinder :**

$$D = d_P + 2e = 0.06 + 2 \times 0.002 = 0.064 \text{ m}$$

- **Mean diameter of annular area between piston and cylinder :**

$$d_m = \frac{D + d_P}{2} = \frac{0.064 + 0.06}{2} = 0.062 \text{ m}$$

- **Cross -sectional area of piston :**

$$A_P = \frac{\pi}{4} d_P^2 = \frac{\pi}{4} \times (0.06)^2$$

$$= 2.8274 \times 10^{-3} \text{ m}^2$$

- **Damping coefficient :**

$$F = cv$$

$$\therefore \quad 400 = c \times 2.6$$

$$\therefore \quad c = 153.85 \text{ N-s/m}$$

- **Absolute viscosity :**

$$c = \frac{12 \mu l_P A_P^2}{\pi d_m e^3}$$

$$\therefore \quad 153.85 = \frac{12 \mu \times 0.05 \times (2.8274 \times 10^{-3})}{\pi \times 0.062^2 \times (0.002)^3}$$

$$\therefore \quad \mu = 8.2054 \times 10^{-6} \text{ N-s/m}^2$$

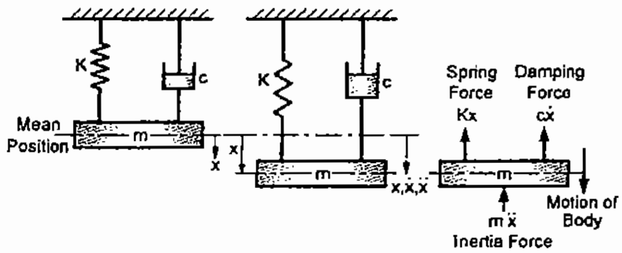
$$\therefore \quad \mu = 8.2054 \times 10^{-12} \text{ N-s/mm}^2$$

$$\therefore \quad \mu = 8.2054 \times 10^{-3} \text{ cp}$$

3.15 DAMPED FREE VIBRATIONS WITH VISCOUS DAMPING

Spring - Mass - Dashpot System :

Fig. 3.15.1(a) shows schematic mathematical model of spring-mass-dashpot system, called as viscous damped system.



(a) Equilibrium or Mean Position **(b) Displaced Position** **(c) F.B.D. of Mass in Displaced position**

Fig. 3.15.1 : Damped Free Vibration with Viscous Damping

- The system is displaced in downward direction through a distance 'x' from the equilibrium (or mean) position, as shown in Fig. 3.15.1(b).

Let, m = mass of the body, kg

K = stiffness of the spring, N/m

c = damping coefficient, N-s/m

x = displacement of the body from equilibrium (mean) position, m

$\dot{x}$ = velocity of the body, m/s

$\ddot{x}$ = acceleration of the body, m/s^2

- **Forces acting on mass in displaced position [Fig. 3.15.1.(c)] :**

(i) Inertia force, $m\ddot{x}$ (upward)
(ii) Damping force, $c\dot{x}$ (upward)
(iii) Spring force, Kx (upward).

Note : The weight of body 'mg' is not considered because, it is nullified by the spring force 'kδ' due to static deflection (i.e. $mg = k\delta$).

Differential Equation of Motion for Damped Free Vibration :

- According to D'Alembert's principle,

$$\therefore \Sigma \text{ Inertia force + External forces } = 0$$

$$\therefore m\ddot{x} + c\dot{x} + Kx = 0 \quad ...(3.15.1)$$

This Equation (3.15.1) is the fundamental differential equation of motion for a single degree of freedom system having damped free vibrations.

Solution of Differential Equation :

- The Equation (3.15.1) is a linear differential equation of the second order and its solution can be written in the form,

$$x = e^{St} \quad ...(a)$$

where, e = base of natural logarithms = 2.71828

S = constant to be determined

t = time, s

- Differentiating Equation (a) with respect to 't', we get,

$$\dot{x} = S.e^{St} \quad ...(b)$$

- Again differentiating Equation (b) with respect to 't', we get,

$$\ddot{x} = S^2.e^{St} \quad ...(c)$$

- Substituting Equations (a), (b) and (c) in Equation (3.15.1),

$$\therefore m S^2 e^{St} + c S e^{St} + K e^{St} = 0$$

$$\therefore mS^2 + cS + K = 0 \quad ...(3.15.2)$$

- **Two roots of equation of damped free vibrations :** The Equation (3.15.2) is called as **characteristic equation** of the damped free vibration system. It is in the form of a quadratic equation for which two roots S_1 and S_2 are :

$$S_{1,2} = \frac{-c \pm \sqrt{c^2 - 4\,mK}}{2m}$$

or

$$S_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} \quad ...(3.15.3)$$

i.e.

$$S_1 = \frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} \quad ...(3.15.4)$$

and,

$$S_2 = \frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} \quad ...(3.15.5)$$

- Two solutions to differential Equation (3.15.1) :

$$\left.\begin{aligned} x &= e^{S_1 t} \\ \text{and} \quad x &= e^{S_2 t} \end{aligned}\right\} \quad ...(d)$$

- General solution to the differential Equation (3.15.1),

$$x = A\,e^{S_1 t} + B\,e^{S_2 t} \quad ...(3.15.6)$$

where, A and B are arbitrary constants which are determined from the initial conditions.

Parameters in Damped Free Vibrations :

For further analysis of general equation of motion for damped free vibrations, given by Equation (3.15.6), it is desirable to define following two terms :

1. Critical Damping Coefficient (c_c)
2. Damping Factor or Damping Ratio (ξ)

1. Critical Damping Coefficient (c_c) :

University Question

Q. Define the term : Critical damping coefficient, related to vibrations **SPPU : Dec. 12, May 16, Dec. 16, Dec. 17**

- **Critical damping coefficient :** Critical damping coefficient 'c_c' is that value of damping coefficient 'c' at which the frequency of free damped vibrations is zero and the motion is aperiodic. The critical damping coefficient 'c_c' is also defined as that value of the damping coefficient 'c' that makes the expression within the radial sign of Equation (3.15.3) to zero, thereby giving two equal roots of 'S' (i.e. S_1 and S_2).
- **At $c = c_c$:**

$$\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} = 0$$

$$\therefore \sqrt{\left(\frac{c_c}{2m}\right)^2 - \frac{K}{m}} = 0$$

$$\left(\frac{c_c}{2m}\right)^2 = \frac{K}{m}$$

$$\frac{c_c}{2m} = \sqrt{\frac{K}{m}}$$

$$\frac{c_c}{2m} = \omega_n$$

$$\text{or} \quad c_c = 2m \cdot \omega_n \qquad ...(3.15.7)$$

where, ω_n = natural frequency of undamped free vibrations, rad/s

$$= \sqrt{\frac{K}{m}}$$

- Again from Equation (3.15.7),

$$c_c = 2m\sqrt{\frac{K}{m}}$$

$$\text{or} \quad c_c = 2\sqrt{Km} \qquad ...(3.15.8)$$

- From Equations (3.15.7) and (3.15.8) it is seen that :
  (i) Critical damping coefficient 'cc' is only dependent on the mass of body and the stiffness of the spring.
  (ii) Critical damping coefficient is independent of the damping coefficient 'c' and hence actual damping condition.

2. Damping Factor or Damping Ratio (ξ) :

University Question

Q. Define the term Damping factor.

SPPU : Dec. 12, Dec. 13, May 16, Dec. 16, Dec. 17, Oct. 18(In sem), May 19, Oct. 19(In Sem.)

Damping factor or **damping ratio** 'ξ' is defined as the ratio of the damping coefficient to the critical damping coefficient. Mathematically,

$$\xi = \frac{c}{c_c} \qquad ...(3.15.9)$$

3.15.1 General Solution to Differential Equation and Types of Damped Systems :

University Questions

Q. With neat sketches, explain underdamped, over damped and critically damped systems.

SPPU : May 12, May 15

Q. Discuss time-displacement plots for over-damped, critically-damped and under-damped system, with zero initial displacement.

SPPU : May 13

Q. Explain with displacement-time plot, the overdamped, critically-damped and under-damped vibratory systems. Give suitable examples.

SPPU : May 16, Aug. 17(In Sem)

Q. Draw and explain displacement-time curves for over damped system, critically damped and under damped vibratory system.

SPPU : May 19

Two Roots of General Solution :

- From Equation (3.15.9)

$$\xi = \frac{c}{c_c}$$

$$\text{or} \quad \xi = \frac{c}{2m\,\omega_n} \qquad [\text{ as } c_c = 2m\,\omega_n]$$

$$\therefore \quad \frac{c}{2m} = \xi\,\omega_n \qquad ...(e)$$

$$\text{and} \quad \omega_n^2 = \frac{K}{m} \qquad ...(f)$$

- Substituting Equations (e) and (f) in Equation (3.15.3), we get,

$$S_{1,2} = -\xi\,\omega_n \pm \sqrt{\xi^2\,\omega_n^2 - \omega_n^2}$$

$$\therefore S_{1,2} = \left[-\xi \pm \sqrt{\xi^2 - 1}\right]\omega_n$$

$$\therefore S_1 = \left[-\xi + \sqrt{\xi^2 - 1}\right]\omega_n \qquad ...(g)$$

$$\text{and } S_2 = \left[-\xi - \sqrt{\xi^2 - 1}\right]\omega_n \qquad ...(h)$$

Types or Cases of Damped Systems :

- Depending upon the value of damping factor or damping ratio 'ξ', there are three types or cases of damped systems :

Case 1 : Over-Damped System ($\xi > 1$)
Case 2 : Critically Damped System ($\xi = 1$)
Case 3 : Under-Damped System ($\xi < 1$)

1. Over-Damped System ($\xi > 1$) :

- **Over-damped system** : If the damping factor 'ξ' is greater than one or the damping coefficient 'c' is greater than critical damping coefficient then the system is said to be **over -damped**. For over-damped system ;

$$\xi > 1 \text{ or } c > c_c \qquad ...(i)$$

- **Two roots for over-damped system**

$$\left.\begin{aligned} S_1 &= \left[-\xi+\sqrt{\xi^2-1}\right]\omega_n \\ S_2 &= \left[-\xi-\sqrt{\xi^2-1}\right]\omega_n \end{aligned}\right\} \qquad ...(j)$$

In this case two roots S_1 and S_2 real and negative.

- **Solution to differential Equation (3.15.1) in over-damped system :**

$$\boxed{x = A\,e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n t} + B\,e^{\left[-\xi-\sqrt{\xi^2-1}\right]\omega_n t}} \qquad ...(3.15.10)$$

- **Values of constants A and B :** The values of constants **A** and **B**, in Equation (3.15.10), are determined from initial conditions as follows :

$$\text{at } t = 0; \qquad x = X_0 \qquad ...(k)$$

$$\text{at } t = 0; \qquad \dot{x} = 0 \qquad ...(l)$$

From Equation (3.15.10),

$$x = A\,e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega n\, t} + B\,e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega^n t} \qquad ...(m)$$

Differentiating Equation (m) with respect to 't', we get,

$$\dot{x} = A\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n\,e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n t} + B\left[-\xi-\sqrt{\xi^2-1}\right]\omega_n\,e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n t} \qquad ...(n)$$

Substituting Equation (k) in Equation (m),

$$X_0 = A + B \qquad ...(o)$$

Substituting Equation (*l*) in Equation (n),

$$0 = A\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n + B\left[-\xi-\sqrt{\xi^2-1}\right]\omega_n \qquad ...(p)$$

From Equations (o) and (p), we get,

$$\left.\begin{aligned} A &= \frac{\left[\xi+\sqrt{\xi^2-1}\right]X_0}{2\sqrt{\xi^2-1}} \\ \text{and} \quad B &= \frac{\left[-\xi+\sqrt{\xi^2-1}\right]X_0}{2\sqrt{\xi^2-1}} \end{aligned}\right\} \qquad ...(q)$$

- **General solution to differential equation in over damped system :**

Substituting Equation (q) in Equation (3.15.10), we get,

$$x = \frac{X_0}{2\sqrt{\xi^2-1}}\left\{\left[\xi+\sqrt{\xi^2-1}\right]e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega^n t} + \left[-\xi+\sqrt{\xi^2-1}\right]e^{\left[-\xi+\sqrt{\xi^2-1}\right]\omega_n t}\right\} \qquad ...(3.15.11)$$

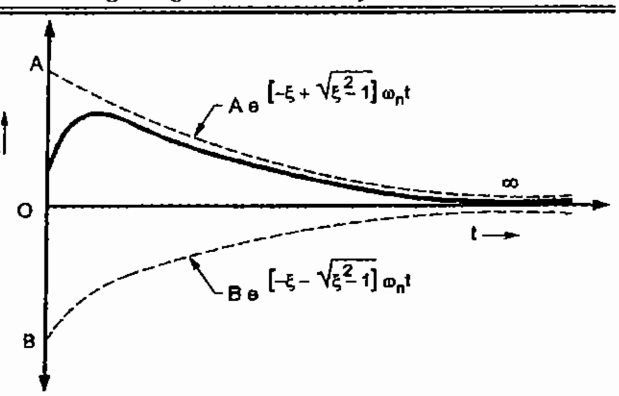


Fig. 3.15.2 : Displacement Versus Time Curve for Over-Damped System

The motion represented by Equation (3.15.11), in over-damped system, is a **periodic motion**. Therefore, the system is non-vibratory. Once the system is displaced from the equilibrium (mean) position, it will take infinite time to come back to the equilibrium position (Fig. 3.15.2). As most of the systems do not have so much of damping, this type of motion is rarely encountered.

2. Critically Damped System ($\xi = 1$) :

University Question

Q. Explain significance of critical damping. Give any two applications of critical damping. **SPPU : Dec. 15**

- **Critically damped system :** If the damping factor 'ξ' is equal to one or the damping coefficient 'c' is equal to the critical damping coefficient 'c_c', the system is said to be **critically damped.**

- **Two roots for critically damped system :**

$$S_1 = -\omega_n \quad ...[\text{Putting } \xi = 1 \text{ in Equation (g)}] \qquad ...(r)$$

$$S_2 = -\omega_n \quad ...[\text{Putting } \xi = 1 \text{ in Equation (h)}] \qquad ...(s)$$

- The two roots are real and equal. As the roots are equal, in critically damped system, solution to the differential Equation (3.15.1)

$$x = A\,e^{S_1 t} + B\,t\,e^{S_2 t}$$

$$\therefore \quad x = A\,e^{-\omega_n t} + B\,t\,e^{-\omega_n t}$$

$$\text{or} \quad \boxed{x = (A + Bt)\,e^{-\omega_n t}} \qquad ...(3.15.12)$$

- **Values of constants A and B :** The values of constants A and B, in Equation (3.15.12), are determined from initial conditions as follows :

$$\text{at } t = 0 \quad ; \quad x = X_0 \qquad ...(r)$$

$$\text{at } t = 0 \quad ; \quad \dot{x} = 0 \qquad ...(s)$$

From Equation (3.15.12),

$$x = (A + Bt)\, e^{-\omega_n t} \qquad ...(t)$$

Differentiating Equation (t) with respect to 't', we get,

$$\dot{x} = B e^{-\omega_n t} - (A + Bt)\,\omega_n e^{-\omega_n t} \qquad ...(u)$$

Substituting Equation (r) in Equation (t),

$$X_0 = A$$

$$\therefore\; A = X_0$$

Substituting Equation (s) in Equation (u),

$$0 = B - A\,\omega_n$$

$$\therefore\; B = A\,\omega_n$$

$$\text{or}\quad B = X_0\,\omega_n \qquad ...(v)$$

- **General solution to differential equation in critically damped system :**

  Substituting Equations (u) and (v) in Equation (3.15.12), we get,

$$x = \left[X_0 + X_0\,\omega_n\, t\right] e^{-\omega_n t}$$

or

$$x = X_0\left[1 + \omega_n t\right] e^{-\omega_n t} \qquad ...(3.15.13)$$

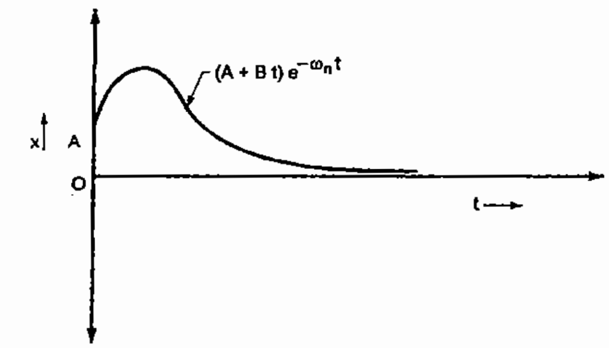


Fig. 3.15.3 : Displacement Versus Time Curve for Critically Damped System

- From Equation (3.15.13) it is seen that, as time 't' increases, displacement 'x' decreases.(As $t \to \infty$, $x \to 0$). Fig. 3.15.3 shows the displacement versus time curve for critically damped system.
- The displacement versus time curve for critically damped system lies below any of the curves for over damped system. The motion represented by Equation (3.15.13), in critically damped system, is also a periodic. Therefore, the system is non-vibratory.
- In critically damped system, once the system is disturbed it will move back rapidly close to its equilibrium position in shortest possible time, after that it will take infinite time to come exactly to equilibrium position.
- **Application of critically damped system :** Application of critical damping is in hydraulic door closer, in which it is necessary that the door to return to its original position, after it has been opened, within shortest possible time.

3. **Under-Damped System ($\xi < 1$) :**

- **Under-damped system :** If the damping factor 'ξ' is less than one or the damping coefficient 'c' is less than the critical damping coefficient 'c_c', the system is said to be **under damped**.
- **Two roots for under - damped system :** In such case, the two roots S_1 and S_2 are complex conjugate (imaginary) and are given by,

$$S_1 = \left[-\xi + i\sqrt{1-\xi^2}\right]\omega_n \qquad ...(w)$$

and

$$S_2 = \left[-\xi - i\sqrt{1-\xi^2}\right]\omega_n \qquad ...(x)$$

where, $i = \sqrt{-1}$ is the imaginary unit of the complex root.

- **Solution to differential equation in under-damped system**

$$x = A e^{\left[-\xi + i\sqrt{1-\xi^2}\right]\omega_n t} + B e^{\left[-\xi - i\sqrt{1-\xi^2}\right]\omega_n t}$$

$$\therefore\; x = e^{-\xi\omega_n t}\left\{A e^{\left[+i\sqrt{1-\xi^2}\right]\omega_n t} + B e^{\left[-i\sqrt{1-\xi^2}\right]\omega_n t}\right\} \qquad ...(3.15.14)$$

putting $\boxed{\sqrt{1-\xi^2}\cdot\omega_n = \omega_d}$ in Equation (3.15.14), we get,

$$\therefore\; x = e^{-\xi\omega_n t}\left[A e^{i\omega_d t} + B e^{-i\omega_d t}\right] \qquad ...(3.15.15)$$

According to Euler's theorem, the above Equation (3.15.15) can also be written as,

$$\boxed{x = X e^{-\xi\omega_n t}\sin(\omega_d t + \phi)} \qquad ...(3.15.16)$$

where, X and ϕ are constants to be determined from initial conditions.

- The Equation (3.15.16) describe the simple harmonic motion of angular frequency 'ω_d' and amplitude $X e^{-\xi\omega_n t}$, which decreases exponentially with increase in time, as shown in Fig. 3.15.4. Thus the resultant motion is oscillatory having frequency 'ω_d' and decreasing amplitude $X e^{-\xi\omega_n t}$, which ultimately dies out after some considerable time.
- **Natural frequency of damped vibrations :**

$$\omega_d = \left[\sqrt{1-\xi^2}\right]\omega_n \qquad ...(3.15.17)$$

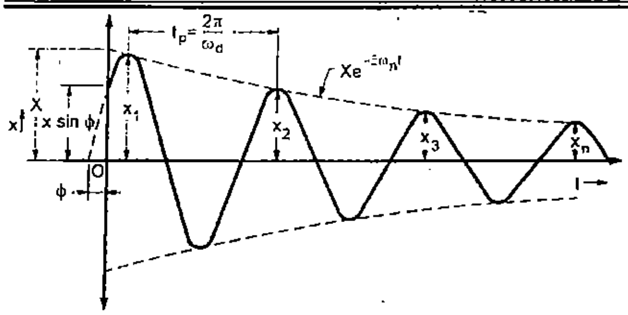


Fig. 3.15.4 : Displacement Versus Time Curve for Under Damped System

- In Equation (3.15.17), as $\xi < 1$, the natural frequency of damped vibrations 'ω_d' is always less than the natural frequency of undamped vibrations 'ω_n'
- **Time period for damped vibrations :**

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}, \text{ s.} \qquad ...(3.15.18)$$

Comparison of Responses of Various Types of Damping Conditions :

University Question

Q. Draw Displacement versus Time curve for different damping conditions. **SPPU : Dec. 18**

- Fig. 3.15.5 shows the comparison of responses of various types of damping conditions.

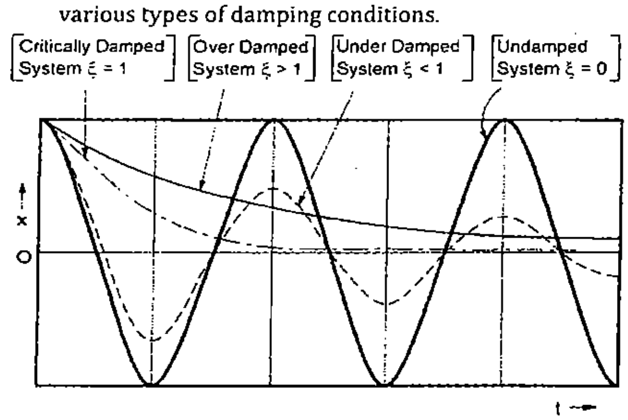


Fig. 3.15.5 : Displacement Versus Time Plot for Various Types of Damping Conditions

3.16 LOGARITHMIC DECREMENT (δ)

University Questions

Q. Define the term : Logarithmic decrement, related to vibration. **SPPU : Dec. 13, May 14, May 15, May 16, Dec. 16, Dec. 17**

Q. What is logarithmic decrement? Derive an expression for the same. **SPPU : Dec. 12, Dec. 14, May 15**

- **Two important parameters indicative of damped free vibrations :**

(i) Natural frequency of damped vibrations; and

(ii) Rate of decay of amplitude.

- **Measurement of rate of decay :** The rate of decay of amplitude is measured by parameter known as **logarithmic decrement.** The rate of decay of amplitude is proportional to the amount of damping present in a system. The larger the damping, the grater will be the rate of decay.
- **Logarithmic decrement (δ) : Logarithmic decrement** *is defined as the natural logarithm of, the ratio of any two successive amplitudes on the same side of the mean position.*

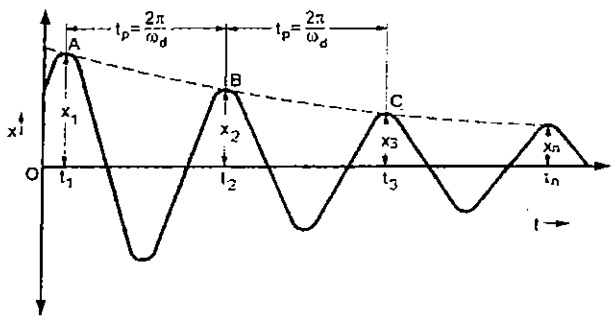


Fig. 3.16.1 : Displacement Versus Time Curve for Under Damped System

- Fig. 3.16.1 shows the free vibrations of an under-damped system. Let A and B are the corresponding points on the two successive cycles where displacement is maximum.
- **Periodic time (t_P) :**

$$t_P = t_2 - t_1 \qquad ...(a)$$

$$= \frac{2\pi}{\omega_d}$$

or

$$t_P = \frac{2\pi}{\left(\sqrt{1-\xi^2}\right)\omega_n} \qquad ...(3.16.1)$$

- **Amplitudes at time 't_1' and 't_2' :**

$$x_1 = Xe^{-\xi\omega_n t_1} \sin[\omega_d t_1 + \phi] \qquad ...(b)$$

$$x_2 = Xe^{-\xi\omega_n t_2} \sin[\omega_d t_2 + \phi]$$

$$= Xe^{-\xi\omega_n (t_1+t_p)} \sin[\omega_d(t_1 + t_p) + \phi]$$

$$...\left[\because t_2 = t_1 + t_p\right]$$

$$= Xe^{-\xi\omega_n (t_1+t_p)} \sin[\omega_d t_1 + \omega_d t_p + \phi]$$

$$= Xe^{-\xi\omega_n (t_1+t_p)} \sin\left[\omega_d t_1 + \omega_d \frac{2\pi}{\omega_d} + \phi\right]$$

$$= Xe^{-\xi\omega_n (t_1+t_p)} \sin[\omega_d t_1 + 2\pi + \phi]$$

or $x_2 = Xe^{-\xi\omega_n(t_1+t_p)}\sin[\omega_d t_1 + \phi]$ ...(c)

- **Logarithmic decrement (δ) :**

From Equations (b) and (c),

$$\frac{x_1}{x_2} = \frac{Xe^{-\xi\omega_n t_1}\sin[\omega_d t_1 + \phi]}{Xe^{-\xi\omega_n(t_1+t_p)}\sin[\omega_d t_1 + \phi]}$$

$$= e^{-\xi\omega_n(t_1 - t_1 - t_p)}$$

or $$\frac{x_1}{x_2} = e^{\xi\omega_n t_p} \quad ...(d)$$

Hence, the logarithmic decrement is given by,

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) \quad ...(e)$$

Substituting Equation (d) in Equation (e), we get

$$\delta = \log_e(e^{\xi\omega_n t_p}) = \xi\omega_n t_p$$

$$= \xi\omega_n \frac{2\pi}{\omega_d} \quad ...\left(\because t_p = \frac{2\pi}{\omega_d}\right)$$

$$\therefore \delta = \xi\omega_n \frac{2\pi}{(\sqrt{1-\xi^2})\,\omega_n}$$

$$...\left[\because \omega_d = (\sqrt{1-\xi^2})\,\omega_n\right]$$

$$\therefore \delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad ...(3.16.2)$$

$$\text{or } \xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \quad ...(3.16.2(a))$$

- **Logarithmic decrement alternative expression (δ) :** The logarithmic decrement can also be determined as follows :

$$\delta = \log_e\left[\frac{X_0}{X_1}\right] = \log_e\left[\frac{X_1}{X_2}\right]$$

$$= \log_e\left[\frac{X_2}{X_3}\right] = ...= \log_e\left[\frac{X_{n-1}}{X_n}\right]$$

$$\therefore \quad n\delta = \log_e\left[\frac{X_0}{X_1}\right] + \log_e\left[\frac{X_1}{X_2}\right] + \log_e\left[\frac{X_2}{X_3}\right] + \log_e\left[\frac{X_{n-1}}{X_n}\right]$$

$$= \log_e\left[\frac{X_0}{X_1}\cdot\frac{X_1}{X_2}\cdot\frac{X_2}{X_3}\cdots\frac{X_{n-1}}{X_n}\right]$$

or $$n\delta = \log_e\left[\frac{X_0}{X_n}\right]$$

$$\therefore \quad \delta = \frac{1}{n}\log_e\left[\frac{X_0}{X_n}\right] \quad ...(3.16.3)$$

Where, X_0 = amplitude at the starting position,

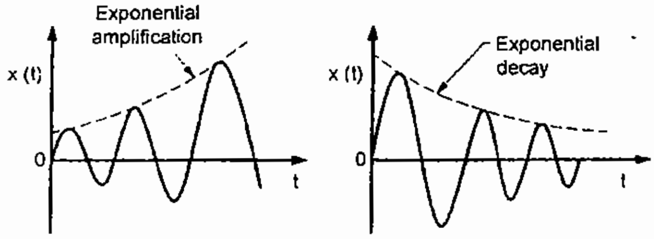
X_n = amplitude after 'n' cycles

3.17 Negative Damping

University Question

Q. Explain the significance of negative damping. **SPPU : May 17**

- In case of positive damping, vibrating body gradually loose their energy and thus loose their amplitude of vibration and gradually die away.
- But in case of negative damping, vibrating body gradually gains their energy and amplitude of vibration will goes on increasing and become infinitely large in time.
- A system with positive damping is called to be dynamically stable shown in Fig. 3.17.1(a), whereas one with negative damping is known as dynamically unstable shown in Fig. 3.17.1(b).



(a) Dynamically unstable (b) Dynamically stable

Fig. 3.17.1

- In case of positive damping, the damping force does negative work, the mechanical energy is converted into heat usually in the dashpot fluid.
- The energy is taken from the vibrating system. Each successive vibration has less amplitude and less kinetic energy.
- In case of negative damping the damping force (which is now a driving force) does positive work on the system.
- The work done by that force during a cycle is converted into the additional kinetic energy of the increased vibration.
- A negative damping phenomenon occurs due to reduction in dry frictional forces with increase in rubbing velocity.
- The friction coefficient dictates the frictional forces, which in turn influences the damping conditions in the system.
- It is seen in Fig. 3.17.1(c) that the coefficient of friction reduces with increase in velocity at low speeds and again rises at higher speeds.

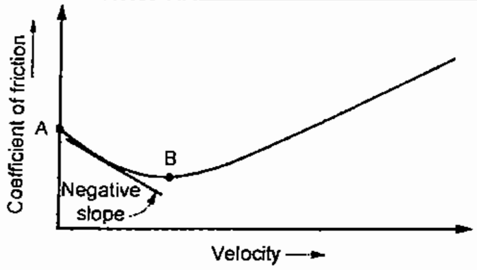


Fig. 3.17.1(c) : Coefficient of Friction Versus Velocity

- It can be seen that the plot has region AB where the slope of the curve is negative and hence in this region the damping coefficient would be negative. In this case, the governing equation of motion would be modified to :

$$m\ddot{x} - C\dot{x} + Kx = 0$$

- In the region AB, the negative damping coefficient creates a propulsive force instead of a damping force. This force is also proportional to velocity and leads to unstable motion.

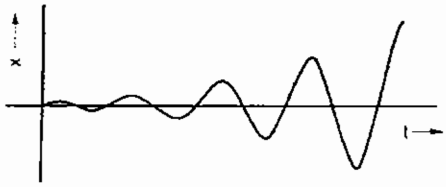


Fig. 3.17.1(d) : Unstable motion

- An unstable motion is the case where the disturbance causes the amplitude to be built up with time. Fig. 3.17.1(d) shows an unstable motion.
- As the damping coefficient is negative, the damping factor would be negative and the study of equation of motions for different values of ξ indicate that instead of decay in amplitude, the motion builds up and hence the system would be inherently unstable.
- Oscillations of this type can often be observed in belt and pulley absorption brakes. The action of a bow on the strings of a violin.

3.18 DAMPED FREE TORSIONAL VIBRATIONS

Shaft - Disc - Damper System :

- Consider a shaft-disc-damper system shown in Fig. 3.18.1.

Let, I = mass moment of inertia of disc about an axis of rotation, kg-m^2

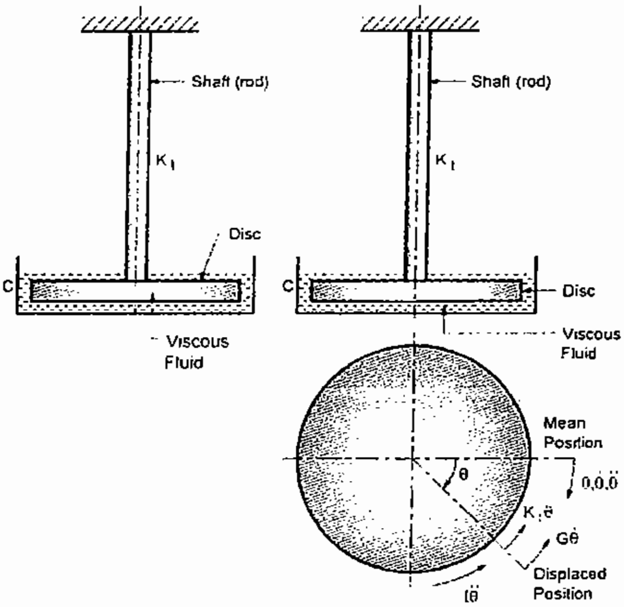
K_t = torsional stiffness of the shaft (rad), N- m/rad

c_t = torsional damping coefficient, N-m-s/rad

θ = angular displacement of the disc from the mean position, rad

$\dot{\theta}$ = angular velocity of the disc, rad/s

$\ddot{\theta}$ = angular acceleration of the disc, rad/s^2



(a) Actual System **(b) Equivalent System**

Fig. 3.18.1 : Damped Free Torsional Vibrations

- **Torques acting on disc in displaced position [Fig. 3.18.1(b)] :**

(i) Inertia torque (Anti-clockwise)
(ii) Damping torque (Anti-clockwise)
(iii) Spring torque (Anti-clockwise)

Differential Equation of Motion :

- According to D'Alembert's principle,

$$\Sigma \text{[Inertia torque + External torque]} = 0$$

$$I\ddot{\theta} + c_t\dot{\theta} + K_t\theta = 0 \qquad ...(3.18.1)$$

- This Equation (3.18.1) is the fundamental differential equation of motion for a single degree of freedom system having damped free torsional vibrations. This Equation (3.18.1) for the torsional system is similar to the Equation (3.16.1) for the longitudinal system.

3.18.1 Comparison of Damped Free Longitudinal Vibrations and Torsional Vibrations :

The parameters used in damped free longitudinal vibrations and torsional vibrations are given in Table 3.18.1.

Table 3.18.1 : Damped Free Longitudinal and Torsional Vibrations

Sr. No.	Parameter	Longitudinal vibrations	Torsional vibrations
1.	Displacement	x, m	θ, rad
2.	Velocity	$\dot{x}$, m/s	$\dot{\theta}$, rad/s
3.	Acceleration	$\ddot{x}$, m/s^2	$\ddot{\theta}$, rad/s^2
4.	Mass/Mass Moment of Inertia	m, kg	I, kg-m^2
5.	Stiffness	K, N/m	K_t, N-m/rad
6.	Damping Coefficient	c, N-s/m	c_t, N-m-s/rad
7.	Differential Equation of Motion	$m\ddot{x} + c\dot{x} + Kx = 0$	$I\ddot{\theta} + c_t\dot{\theta} + K_t\theta = 0$
8.	Critical Damping Coefficient	$c_c = 2\sqrt{Km} = 2m\,\omega_n$, N-s/m	$c_{ct} = 2\sqrt{K_t I} = 2I\,\omega_n$, N-m-s/rad
9.	Damping Factor or Damping Ratio	$\xi = \frac{c}{c_c}$	$\xi = \frac{c_t}{c_{ct}}$
10.	Two Roots of Characteristic Equation	$S_{1,2} = \left[-\xi \pm \sqrt{\xi^2 - 1}\right]\omega_n$	$S_{1,2} = \left[-\xi \pm \sqrt{\xi^2 - 1}\right]\omega_n$
11.	Equation of Motion for Underdamped System	$x = X e^{-\xi\omega_n t}\sin[\omega_d t + \phi]$, m where, $\omega_d = \sqrt{1-\xi^2}\,\omega_n$, rad/s	$\theta = \beta e^{-\xi\omega_n t}\sin[\omega_d t + \phi]$, rad where, $\omega_d = \left(\sqrt{1-\xi^2}\right)\omega_n$, rad/s
12.	Logarithmic Decrement	$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$ $\delta = \frac{1}{n}\log_e\left[\frac{x_0}{x_n}\right]$	$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$ $\delta = \frac{1}{n}\log_e\left[\frac{\beta_0}{\beta_n}\right]$

Ex. 3.18.1 : A 500 kg vehicle is mounted on springs such that its static deflection is 1.5 mm. What is the damping coefficient of viscous damper to be added to the system in parallel with the springs, such that the system is critically damped ? **SPPU - May 14, 6 Marks**

Soln. :

Given : $m = 500$ kg ; $\delta = 1.5$ mm $= 1.5 \times 10^{-3}$ m

- **Natural circular frequency of system :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{g}{\delta}}$$

$$= \sqrt{\frac{9.81}{1.5 \times 10^{-3}}}$$

$$\therefore \omega_n = 80.87 \text{ rad/s}$$

- **Damping coefficient :**

$$c = c_c = 2\,m\,\omega_n$$

$$= 2 \times 500 \times 80.87$$

or $$c = \mathbf{80.87 \times 10^3\ N\text{-}s/m}$$

Ex. 3.18.2 : For a spring-mass-damper system, m = 1.5 kg, K = 5000 N/m. Determine the magnitude of coefficient of damping 'c' for the system to be critically damped. **SPPU - May 12, 2 Marks**

Soln. :

- For critically damped system,

$$c = c_C = 2m\,\omega_n = 2m\sqrt{\frac{K}{m}}$$

$$\therefore \frac{c}{2\,m} = \sqrt{\frac{K}{m}}$$

$$\therefore \quad \frac{c}{2 \times 1.5} = \sqrt{\frac{5000}{1.5}}$$

$$\therefore \quad c = c_c = 173.20 \text{ N-sec/m} \quad \text{...Ans.}$$

Ex. 3.18.3 : A wheel is mounted on a steel shaft ($G = 83 \times 10^9$ N/m²) of length 1.5 m and radius 0.80 cm the wheel is rotated 5° and released. The period of oscillation is observed as 2.3 seconds. Determine the mass moment of inertia of the wheel. **SPPU - May 17, 4 Marks**

Soln. :

Given : $G = 83 \times 10^9 \text{ N/m}^2, \quad l = 1.5 \text{ m},$

$r = 0.80 \text{ cm} = 0.008\text{m}$

$\therefore d = 0.016$

According to D'Alembert's principle,

$$\therefore \quad I\ddot{\theta} + K_t\theta = 0$$

$$\therefore \quad \ddot{\theta} + \frac{K_t}{I}\theta = 0$$

The fundamental equation of simple harmonic motion is,

$$\ddot{\theta} + \omega_n^2\theta = 0 \quad ; \quad \omega_n^2 = \frac{K_t}{I}$$

$$\omega_n = \sqrt{\frac{K_t}{I}}, \text{ rad/s}$$

But $$\omega_n = \frac{2\pi}{t_p} = \frac{2\pi}{2.3} = 2.73 \text{ rad/sec}$$

$$I = \frac{K_t}{\omega_n} = \frac{GJ}{l\,\omega_n^2} = \frac{\pi \times (0.016)^4 \times 83 \times 10^9}{32 \times 1.5 \times 2.73^2}$$

$$= \mathbf{47.77 \text{ kg.m}^2} \quad \text{...Ans.}$$

Ex. 3.18.4 : The restroom door shown in Fig. P. 3.18.4 is equipped with a torsional spring with 25 Nm/rad as stiffness and a torsional viscous damper. The door has a mass of 60 kg and a centroidal moment of inertia about an axis parallel to the axis of the door's rotation is 7.2 kg-m². Assuming that the system is critically damped, determine the damping coefficient. **SPPU - May 17, 6 Marks**

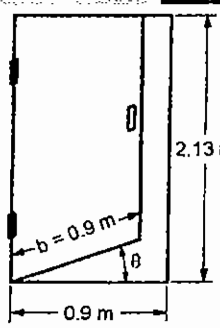


Fig. P. 3.18.4

Soln. :

Given : $K_t = 25 \text{ Nm/rad}, \quad m = 60 \text{ kg},$

$I_c = 7.2 \text{ kg.m}^2, \quad b = 0.9 \text{ m}$

$$\therefore \quad I = I_c + \frac{m\,b^2}{4} = 7.2 + \frac{60 \times 0.9^2}{4} = \mathbf{19.35 \text{ kg.m}^2}$$

System is critically damped, $\xi = 1$

$$\text{But } \xi = \frac{c_t}{c_{ct}} \qquad \therefore c_t = c_{ct}$$

$$\therefore c_{ct} = 2\sqrt{K_t I} = 2 \times \sqrt{25 \times 19.35}$$

$$= \mathbf{43.99 \text{ Nm.s/rad}} \quad \text{...Ans.}$$

Ex. 3.18.5 : The amplitude decreases to 0.33 of the initial value after four consecutive cycles. The damping factor will be ? **SPPU - Dec. 11, 2 Marks**

Soln. :

Given : $n = 4 \quad ; \quad x_4 = 0.33\,x_0$

1. **Logarithmic Decrement :**

$$\therefore \quad \frac{x_0}{x_4} = \frac{1}{0.33} = 3.03$$

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = \frac{1}{4}\log_e\left(\frac{x_0}{x_4}\right) \quad \text{...[When } n = 4 \text{]}$$

$$= \frac{1}{4}\log_e(3.03)$$

or $\delta = 0.2771$

2. **Damping factor :**

The damping factor is,

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.2771}{\sqrt{4\pi^2 + (0.2771)^2}}$$

or $\xi = \mathbf{0.044}$ ...Ans.

Ex. 3.18.6 : A spring-mass-damper system has a mass of 4 kg, a spring of stiffness 300 N/m and damping coefficient of 35 N sec/m. Determine :

(i) the natural frequency of damped vibration ; and

(ii) the natural frequency of the system, if instead of viscous damping dry friction damping is present.

Soln. :

Given : $m = 4 \text{ kg} \quad ; \quad K = 300 \text{ N/m};$

$c = 35 \text{ N-s/m}.$

1. **Natural Frequency of Damped Vibrations :**

- **Natural circular frequency of system :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{300}{4}} = 8.66 \text{ rad/s}$$

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2m\,\omega_n} = \frac{35}{2 \times 4 \times 8.66} = 0.50$$

- **Natural circular frequency of damped vibrations :**

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 8.66\sqrt{1-(0.5)^2}$$

$$= 7.49 \text{ rad/s}$$

- **Natural frequency of damped vibrations :**

$$f_d = \frac{\omega_d}{2\pi} = \frac{7.49}{2\pi}$$

or $f_d = \mathbf{1.19\ Hz}$ ...Ans.

2. Natural Frequency With Dry Friction Damping :

The natural frequency of the system under dry friction damping (i.e. when there is no damper) is,

or $$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{300}{4}}$$

or $\omega_n = \mathbf{8.66\ rad/sec.}$ ...Ans.

Ex. 3.18.7 : An under-damped shock absorber is to be designed for an automobile. It is required that initial amplitude is to be reduced to 1/16[th] in one cycle. The mass of the automobile is 200 kg and damped period of vibration is 1 sec. Find the necessary stiffness and damping constants of shock absorbers. **SPPU - May 15, 5 Marks**

Solution :

Given : $m = 200$ kg ; $t_p = 1$ s ;

$\frac{x_0}{x_1} = 16$; $n = 1$.

1. Logarithmic Decrement :

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = 1\log_e\left(\frac{x_0}{x_1}\right) = 1\log_e(16)$$

or $\delta = 2.7725$

2. Damping Factor :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$2.7725 = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\sqrt{1-\xi^2} = 2.2665\,\xi$$

$$1-\xi^2 = 5.136\,\xi^2$$

$$1 = 6.136\,\xi^2$$

$$\therefore \xi = 0.4037$$

3. Stiffness of Spring :

$$\omega_d = \frac{2\pi}{t_p} = \frac{2\pi}{1} = 6.28 \text{ rad/sec}$$

Again, $$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\therefore 6.28 = \omega_n\sqrt{1-(0.4037)^2}$$

$$\therefore \omega_n = 6.86 \text{ rad/sec}$$

Now, $$\omega_n = \sqrt{\frac{K}{m}}$$

$$\therefore 6.86 = \sqrt{\frac{K}{200}}$$

$\therefore$ $K = \mathbf{9423.45\ N/m}$ ...Ans.

4. Damping Coefficient :

$$\xi = \frac{c}{c_c}$$

or $$\xi = \frac{c}{2\,m\,\omega_n}$$

$$\therefore 0.4037 = \frac{c}{2 \times 200 \times 6.86}$$

$\therefore$ $c = \mathbf{1107.75\ N\text{-}sec/m}$ ...Ans.

Ex. 3.18.8 : An under-damped shock absorber is to be designed for a motorcycle of mass 200 kg, such that during a road bump, the damped period of vibration is limited to 2 seconds and the amplitude of vibration should reduce to one sixteenth in one cycle. Find the spring stiffness and damping coefficient of the shock absorber. **SPPU - Dec. 19, 5 Marks**

Soln. :

Given : $m = 200$ kg ; $t_p = 2$ s ;

$\frac{x_0}{x_1} = 16$; $n = 1$.

1. Logarithmic Decrement :

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = 1\log_e\left(\frac{x_0}{x_1}\right) = 1\log_e(16)$$

or $\delta = 2.7725$

2. Damping Factor :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$2.7725 = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\sqrt{1-\xi^2} = 2.2665\,\xi$$

$$1-\xi^2 = 5.136\,\xi^2$$

$$1 = 6.136\,\xi^2$$

$$\therefore \xi = 0.4037$$

3. Stiffness of Spring :

$$\omega_d = \frac{2\pi}{t_p} = \frac{2\pi}{2} = 3.14 \text{ rad/sec}$$

Again, $\omega_d = \omega_n \sqrt{1-\xi^2}$

$\therefore \quad 3.14 = \omega_n \sqrt{1-(0.4037)^2}$

$\therefore \quad \omega_n = 3.43 \text{ rad/sec}$

Now, $\omega_n = \sqrt{\dfrac{K}{m}}$

$\therefore \quad 3.43 = \sqrt{\dfrac{K}{200}}$

$\therefore \quad K = \mathbf{2352.98\ N/m}$ ...Ans.

4. Damping Coefficient :

$$\xi = \frac{c}{c_c}$$

or
$$\xi = \frac{c}{2\,m\,\omega_n}$$

$$\therefore \quad 0.4037 = \frac{c}{2 \times 200 \times 3.43}$$

$\therefore \quad c = \mathbf{553.87\ N\text{-}sec/m}$ ...Ans.

Ex. 3.18.9 : A shock absorber is to be designed so that its overshoot is 10% of the initial displacement when released. Determine the damping factor. Also find the overshoot if the damping factor is reduced to 50%.

SPPU - Dec. 16, Dec. 17, 6 Marks

Soln. :

Given : $\dfrac{x_2}{x_1} = 0.1$

(i) Damping factor :

$$\frac{x_1}{x_2} = 10$$

$$\delta = \log_e\left(\frac{x_1}{x_2}\right) = \log_e(10)$$

$$\delta = 2.30258$$

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}\ ;$$

$$2.30258 = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\sqrt{1-\xi^2} = 2.729\,\xi$$

$$\therefore \quad 1-\xi^2 = 7.447\,\xi^2$$

$$1 = 8.447\,\xi^2$$

$\xi = \mathbf{0.344}$ ...Ans.

(ii) Damping factor reduced to 50% :

$$\therefore \quad \xi' = \frac{\xi}{2} = 0.172$$

$$\delta' = \frac{2\pi\xi'}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.172}{\sqrt{1-0.172^2}} = \frac{1.808}{0.985}$$

$$\delta' = 1.095$$

Now, $\delta' = \log_e\left(\dfrac{x_1}{x_2}\right)$

$$1.095 = \log_e\left(\frac{x_1}{x_2}\right)$$

$$\frac{x_1}{x_2} = 2.989$$

$$\therefore \quad \frac{x_2}{x_1} = 0.3345$$

overshoot = **33.45 %** ...Ans.

Ex. 3.18.10 : An under damped shock absorber is to be designed for a motorcycle of mass 200 kg such that during a road bump, the damped period of vibration is limited to 2 sec and amplitude of vibration should reduce to one-sixth in one cycle. Find spring stiffness and the damping coefficient of shock absorber.

SPPU - Dec. 13, May 14, Oct. 16 (In Sem), 6 Marks

Soln. :

Given : $m = 200$ kg ; $t_p = 2s$;

$\dfrac{x_0}{x_1} = 6$; $n = 1$.

1. Logarithmic Decrement :

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = 1\log_e\left(\frac{x_0}{x_1}\right) = 1\log_e(6)$$

or
$$\delta = 1.79$$

2. Damping Factor :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

or
$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{1.79}{\sqrt{4\pi^2+1.79^2}} = 0.2739$$

3. Stiffness of Spring :

$$\omega_d = \frac{2\pi}{t_p} = \frac{2\pi}{2} = 3.1415 \text{ rad/sec}$$

Again, $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\therefore \quad 3.1415 = \omega_n\sqrt{1-(0.2739)^2}$$

$$\therefore \quad \omega_n = 3.26 \text{ rad/sec}$$

Now, $\omega_n = \sqrt{\dfrac{K}{m}}$

$$\therefore \quad 3.26 = \sqrt{\frac{K}{200}}$$

$\therefore \quad K = \mathbf{2133.89\ N/m}$ ...Ans.

4. Damping Coefficient :

$$\xi = \frac{c}{c_c}$$

or $$\xi = \frac{c}{2\,m\,\omega_n}$$

$$\therefore \quad 0.2739 = \frac{c}{2 \times 200 \times 3.26}$$

$$\therefore \quad c = 357.16 \text{ N-sec/m} \qquad \text{...Ans.}$$

Ex. 3.18.11 : In a single degree freedom viscously damped vibrating system, the suspended mass of 20 kg makes 50 oscillations in 20 seconds. The amplitude of natural vibrations decreases to one fourth of the initial value after 4 oscillations. Determine :

(i) The logarithmic decrement.

(ii) Damping factor

(iii) Damping coefficient. **SPPU - May 18, 6 Marks**

Soln. :

Given : Mass, m = 20 kg, Number of oscillations, n = 4

1. Logarithmic decrement :

Let, x_0 = Initial amplitude,

x_4 = Final amplitude after 4 oscillations

$$\therefore \quad x_4 = \frac{x_0}{4}$$

or $$\frac{x_0}{x_4} = 4$$

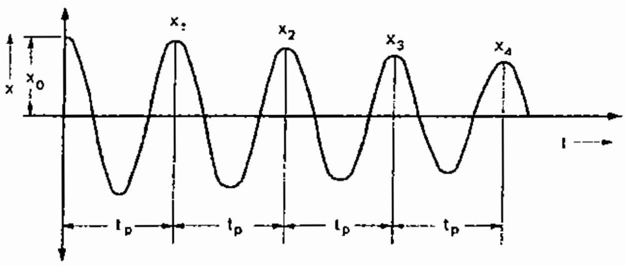


Fig. P. 3.18.11

The logarithmic decrement is,

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = \frac{1}{4}\log_e\left(\frac{x_0}{x_4}\right) \qquad \text{...(when n = 4)}$$

$$= \frac{1}{4}\log_e(4)$$

or $$\delta = 0.3465 \qquad \text{...Ans.}$$

2. Damping factor :

The damping factor is,

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.3465}{\sqrt{4\pi^2 + (0.3465)^2}}$$

or $$\xi = 0.055 \qquad \text{...Ans.}$$

3. Spring stiffness :

The natural frequency of damped vibrations is,

$$f_d = \frac{N}{t_p} = \frac{50}{20}$$

or $$f_d = 2.5 \text{ Hz}$$

The natural circular frequency of damped vibrations is,

$$\omega_d = f_d \times 2\pi = 2.5 \times 2\pi = 15.70 \text{ rad/s}$$

but, $$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\therefore \quad 15.70 = \omega_n\sqrt{1-(0.055)^2}$$

$$\therefore \quad \omega_n = 15.72 \text{ rad/s}$$

Now, $$\omega_n = \sqrt{\frac{K}{m}}$$

$$\therefore \quad 15.72 = \sqrt{\frac{K}{20}}$$

$$\therefore \quad K = 4942.36 \text{ N/m} \qquad \text{...Ans.}$$

4. Damping coefficient :

$$\xi = \frac{c}{c_c}$$

or $$\xi = \frac{c}{2m\,\omega_n}$$

or $$0.055 = \frac{c}{2 \times 20 \times 15.72}$$

or $$c = 34.58 \text{ N-s/m} \qquad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.18.12 : In a damped vibrating system, the mass having 20 kg makes 40 oscillations in 25 sec. The amplitude of natural vibrations decreases to one eighth of the initial value after 8 oscillations. Determine :

(i) the logarithmic decrement;

(ii) the damping factor and damping coefficient; and

(iii) the spring stiffness.

Ex. 3.18.13 : In a damped free vibrations, mass is 2 kg, and spring stiffness is 100 N/m. It is observed that an initial amplitude of 100 mm is reduced to 1 mm in 10 oscillations. Find :

(i) the damping constant c; and

(ii) the natural frequency of vibrations. **SPPU - Dec. 02**

Soln. :

Given : m = 2 kg; K = 100 N/m; x_0 = 100 mm; x_{10} = 1 mm; n = 10.

1. **Logarithmic Decrement :**

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = \frac{1}{10}\log_e\left(\frac{x_0}{x_{10}}\right)$$

$$\therefore \ \delta = \frac{1}{10}\log_e\left(\frac{100}{1}\right) = 0.4605$$

2. **Natural Frequency :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{100}{2}} = 7.071 \text{ rad/s}$$

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{7.071}{2\pi} = \mathbf{1.125 \ Hz} \quad \text{...Ans.}$$

- **Natural circular frequency of damped vibrations :**

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 7.071\sqrt{1-(0.0731)^2}$$
$$= 7.052 \text{ rad/s}$$

- **Natural frequency of damped vibration :**

$$f_d = \frac{\omega_d}{2\pi} = \frac{7.052}{2\pi}$$

$$\text{or} \quad f_d = \mathbf{1.1223 \ Hz} \quad \text{...Ans.}$$

3. **Damping Constant (c) :**

The damping factor is,

$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{0.4605}{\sqrt{4\pi^2+(0.4605)^2}} = 0.0731$$

$$\text{Now,} \quad \xi = \frac{c}{2m\,\omega_n}$$

$$\therefore \ 0.0731 = \frac{c}{2\times 2\times 7.071}$$

$$\therefore \ c = \mathbf{2.067 \ n\text{-}s/m} \quad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.18.14 : A mass, suspended from a helical spring, vibrates in a viscous fluid medium whose resistance varies directly with the speed. It is observed that, the frequency of damped vibration is 120 per minute and the amplitude decreases to 20% of its initial value in one complete cycle. Find the frequency of the free undamped vibrations of the system.

Ex. 3.18.15 : A mass of 1 kg is supported on a spring of 9800 N/m and has a dashpot having damping coefficient of 6 N-sec/m. Find the damped natural frequency. Also find the logarithmic decrement and amplitude after 4 cycles, if the initial displacement is 5 mm.

Soln. :

Given : m = 1 kg ; K = 9800 N/m ; c = 6 N-sec/m ; x_0 = 5 mm ; n = 4.

1. **Damped Natural Frequency :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{9800}{1}} = 98.99 \text{ rad/s}$$

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2m\,\omega_n} = \frac{6}{2\times 1\times 98.99} = 0.030$$

- **Circular frequency of damped vibrations :**

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\text{or} \quad \omega_d = 98.99\sqrt{1-(0.030)^2} = 98.95 \text{ rad/s}$$

- **Frequency of damped vibrations :**

$$f_d = \frac{\omega_d}{2\pi} = \frac{98.99}{2\pi} = \mathbf{15.74 \ Hz} \quad \text{...Ans.}$$

2. **Logarithmic Decrement (δ) :**

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi\times 0.030}{\sqrt{1-(0.030)^2}} = \mathbf{0.1885} \quad \text{...Ans.}$$

3. **Amplitude After Four Cycles :**

$$\text{but,} \quad \delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = \frac{1}{4}\log_e\left(\frac{x_0}{x_4}\right) \quad \text{...(when n = 4)}$$

$$\therefore \ 0.1885 = \frac{1}{4}\log_e\left(\frac{x_0}{x_4}\right)$$

$$\therefore \ 0.1885\times 4 = \log_e\left(\frac{x_0}{x_4}\right)$$

$$0.7543 = \log_e\left(\frac{x_0}{x_4}\right)$$

$$\therefore \ 2.126 = \frac{x_0}{x_4}$$

$$\therefore \ x_4 = \frac{5}{2.126}$$

$$\text{or} \quad x_4 = \mathbf{2.35 \ mm} \quad \text{...Ans.}$$

Ex. 3.18.16 : A machine of 75 kg mass is mounted on three springs, each of stiffness 10 N/mm and is fitted with a dashpot to damp out vibrations. During vibrations, it is found that the amplitude of vibration diminishes from 40 mm to 6 mm in two complete cycles. Determine :

(i) the resistance of dashpot at unit velocity;

(ii) the frequency ratio of damped vibrations to undamped vibrations; and

(ii) the time period of damped vibrations. **SPPU- May 07**

Soln. :

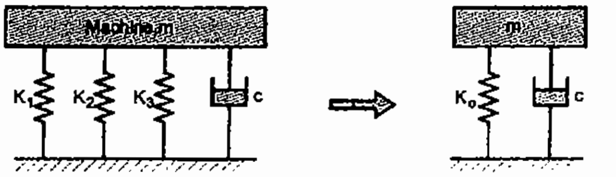
Given : $m = 75\text{ kg}$; $K = 10\text{ N/mm}$;

$x_0 = 40\text{ mm}$; $x_2 = 6\text{ mm}$;

$n = 2.$

1. Equivalent Stiffness :

- The machine is mounted on 3 springs. It is assumed that these 3 springs are in parallel as shown in Fig. P. 3.18.16.



(a) Actual System (b) Equivalent System

Fig. P. 3.18.16

- Equivalent stiffness of spring is,

$$K_e = K_1 + K_2 + K_3 = 3 \times K = 3 \times 10 = 30\text{ N/mm}$$

$$\therefore\ K_e = 30 \times 10^3\text{ N/m}$$

2. Natural Circular Frequency :

$$\omega_n = \sqrt{\frac{K_e}{m}} = \sqrt{\frac{30 \times 10^3}{75}} = 20\text{ rad/s}$$

3. Logarithmic Decrement :

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right) = \frac{1}{2}\log_e\left(\frac{x_0}{x_2}\right) \quad \text{...(when n = 2)}$$

$$= \frac{1}{2}\log_e\left(\frac{40}{6}\right) = 0.9485$$

4. Resistance of Dashpot at Unit Velocity :

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.9485}{\sqrt{4\pi^2 + (0.9485)^2}} = 0.1492$$

Again, $$\xi = \frac{c}{c_c} = \frac{c}{2m\,\omega_n}$$

$$\therefore\ 0.1492 = \frac{c}{2 \times 75 \times 20}$$

$$\therefore\ \mathbf{c = 447.80\text{ N-s/m}} \quad \text{...Ans.}$$

- This is the required resistance of dashpot at unit velocity.

5. Frequency Ratio of Damped Vibrations to Undamped Vibrations :

- **Circular frequency of damped vibration :**

$$\omega_d = \omega_n\sqrt{1 - \xi^2} = 20\sqrt{1 - (0.1492)^2} = \mathbf{19.77\text{ rad/s}}$$

- **Frequency ratio of damped vibrations to undamped vibrations :**

$$\frac{\omega_d}{\omega_n} = \frac{19.77}{20}$$

or $$\frac{\omega_d}{\omega_n} = 0.9888 \quad \text{...Ans.}$$

6. Time Period of Damped Vibration :

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{19.77}$$

or $$t_p = 0.3178\text{ s} \quad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.18.17 : A flywheel of mass 10 kg and radius of gyration 0.3 m makes torsional rotations under a torsion spring of stiffness 5Nm/rad. A viscous damper is fitted and it is found that the amplitude is reduced by a factor 100 over any two successive cycles. Determine :

(i) the damping factor ;

(ii) the damping coefficient ;

(iii) the damped frequency ; and

(iv) the periodic time of oscillations.

SPPU- Dec. 15, 4 Marks

Ex. 3.18.18 : The successive amplitudes of vibrations of a vibratory system, as obtained under free vibrations are : 0.69, 0.32, 0.19 and 0.099 units respectively. Determine the damping ratio of the system. If the damping ratio is doubled, what would be the amplitude ratio then?

Soln. :

Given :

$$\frac{x_0}{x_1} = \frac{0.69}{0.32}; \qquad \frac{x_1}{x_2} = \frac{0.32}{0.19};$$

$$\frac{x_2}{x_3} = \frac{0.19}{0.099}.$$

1. **Damping Ratio :**

- **Logarithmic decrement :**

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right)$$

$$\delta = \frac{1}{3}\log_e\left(\frac{x_0}{x_3}\right) \qquad ...[\because n = 3]$$

$$\delta = \frac{1}{3}\log_e\left(\frac{x_0}{x_1}\times\frac{x_1}{x_2}\times\frac{x_2}{x_3}\right)$$

$$\delta = \frac{1}{3}\log_e\left(\frac{0.69}{0.32}\times\frac{0.32}{0.19}\times\frac{0.19}{0.099}\right)$$

$$\delta = 0.647$$

- **Damping ratio :**

$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{0.647}{\sqrt{4\pi^2+(0.647)^2}} = 0.10$$

2. **Amplitude Ratio, when Dumping Ratio is Doubled :**

- **Logarithmic decrement :**

If damping ratio is doubled,

$$\xi_1 = 0.10 \times 2 = 0.20$$

$$\xi_1 = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

$$0.2 = \frac{\delta}{\sqrt{4\pi^2+\delta^2}}$$

$$\therefore \quad \delta = 1.31$$

- **Amplitude ratio :**

$$\text{Now,} \quad \delta = \frac{1}{n}\log_e\left(\frac{x_o}{x_1}\right)$$

$$1.31 = \log_e\left(\frac{x_o}{x_1}\right) \qquad ...[\because n = 1]$$

$$\left(\frac{x_o}{x_1}\right) = \mathbf{3.706} \qquad \textbf{...Ans.}$$

Ex. 3.18.19 : The disc of a torsional pendulum has a mass moment of inertia of 0.06 kg-m^2. The brass shaft attached to it is of 100 mm diameter and 400 mm long. When the pendulum is vibrating, the observed amplitudes on the same side of the rest position for successive cycles are 9°, 6° and 4°. Find :

(i) the logarithmic decrement;

(ii) the damping torque at unit velocity; and

(iii) the periodic time of vibration. Assume modulus of rigidity as 4.4×10^{10} N / m^2.

What would the frequency be, if the disc is removed from viscous fluid ? **SPPU - Dec. 06**

Soln. :

Given : $I = 0.06$ kg-m^2 ; $d = 0.1$ m ; $L = 0.4$ m ; $\frac{\beta_0}{\beta_1} = \frac{9}{6} = \frac{3}{2}$; $\frac{\beta_1}{\beta_2} = \frac{6}{4} = \frac{3}{2}$; $G = 4.4 \times 10^{10}$ N/m^2

1. **Logarithmic Decrement :**

$$\frac{\beta_0}{\beta_2} = \frac{\beta_0}{\beta_1}\times\frac{\beta_1}{\beta_2} = \frac{3}{2}\times\frac{3}{2} = \frac{9}{4}$$

$$\delta = \frac{1}{n}\log_e\left(\frac{\beta_0}{\beta_n}\right) = \frac{1}{2}\log_e\left(\frac{\beta_0}{\beta_2}\right) \qquad ...(\text{when } n = 2)$$

$$= \frac{1}{2}\log_e\left(\frac{9}{4}\right) = \mathbf{0.405} \qquad \textbf{...Ans.}$$

2. **Damping Torque at Unit Velocity :**

- **Damping factor :**

$$\xi = \frac{\delta}{\sqrt{4\pi^2+\delta^2}} = \frac{0.405}{\sqrt{4\pi^2+(0.405)^2}} = \mathbf{0.0642}$$

- **Torsional stiffness of shaft :**

$$K_t = \frac{GJ}{L} = \frac{4.4\times10^{10}\frac{\pi}{32}(0.1)^4}{0.4}$$

$$K_t = 1.08 \times 10^6 \text{ NM / rad}$$

- **Natural circular frequency of system :**

$$\omega_n = \sqrt{\frac{K_t}{I}} = \sqrt{\frac{1.08\times10^6}{0.06}} = 4242.64 \text{ rad/s}$$

- **Damping torque at unit velocity :**

$$\text{Now,} \quad \xi = \frac{c_t}{2I\omega_n}$$

$$\therefore \quad 0.0643 = \frac{c_t}{2\times0.06\times4242.64}$$

$$c_t = \mathbf{32.73\ Nm\text{-}sec/rad} \qquad \textbf{...Ans.}$$

3. **Time Period of Damped Vibrations :**

- **Circular frequency of damped vibrations :**

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 4242.64\sqrt{1-0.0643^2}$$

$$= 4233.86 \text{ rad/s}$$

- **Time period of damped vibrations :**

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{4233.86} = \mathbf{0.00148\ sec} \qquad \textbf{...Ans.}$$

4. **Frequency, if Disc Removed From Viscous Fluid :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{4242.64}{2\pi}$$

$$= \mathbf{675.24\ Hz} \qquad \textbf{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.18.20 : A disc of torsion pendulum has a moment of inertia of 0.05 kg-m^2 and is immersed in a viscous fluid. During vibration of pendulum, the observed amplitudes on the same side of the neutral axis for successive cycles are found to decay 50 % of the initial value. Determine :

(i) the logarithmic decrement ;

(ii) the damping torque per unit velocity ; and

(iii) the periodic time of vibration.

Assume $G = 4.5 \times 10^{10}$ N/m^2 for the material of shaft for shaft d = 0.10 m and l = 0.50 m.

Ex. 3.18.21 : A vibrating system consists of a mass of 50 kg, a spring of stiffness 30 kN/m and a damper. The damping provided is only 20% of the critical value. Determine :

i) the damping factor;

ii) the critical damping coefficient;

iii) the natural frequency of damped vibration;

iv) the logarithmic decrement; and

v) the ratio of two consecutive amplitudes.

SPPU - May 09

Soln. :

Given : $m = 50$ kg ;

$K = 30 \text{ kN/m} = 30 \times 10^3 \text{ N/m}$;

$c = 0.2\, c_c$.

(i) Natural circular frequency of system :

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{30 \times 10^3}{50}} = \mathbf{24.5 \text{ rad/sec}}$$

(ii) Damping factor :

$$\xi = \frac{c}{c_c} = \frac{0.2\, c_c}{c_c} = \mathbf{0.2} \qquad \text{...Ans.}$$

(iii) Critical damping coefficient :

$$c_c = 2\, m\, \omega_n = 2 \times 50 \times 24.5$$

$$= \mathbf{2450 \text{ N-s/m}} \qquad \text{...Ans.}$$

(iv) Natural circular frequency of damped vibration :

$$\omega_d = \omega_n \sqrt{1 - \xi^2} = 24.5 \sqrt{1 - (0.2)^2}$$

$$= \mathbf{24 \text{ rad/sec}} \qquad \text{...Ans.}$$

(v) Natural frequency of damped vibration :

$$f_d = \frac{\omega_d}{2\pi} = \frac{24}{2\pi} = \mathbf{3.82 \text{ Hz}} \qquad \text{...Ans.}$$

(vi) Logarithmic decrement :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.2}{\sqrt{1-(0.2)^2}}$$

$$= \mathbf{1.2825} \qquad \text{...Ans.}$$

(vii) Ratio of two successive amplitudes :

$$\delta = \frac{1}{n} \log_e \left(\frac{x_0}{x_n}\right) = 1 \log_e \left(\frac{x_0}{x_1}\right)$$

$$\frac{x_0}{x_1} = e^{\delta} = e^{1.2825} = \mathbf{3.6} \qquad \text{...Ans.}$$

Ex. 3.18.22 : In a single degree damped vibrating system, a suspended mass of 8 kg makes 30 oscillations in 18 sec. The amplitude decreases to 0.25 of the initial value after 5 oscillations. Determine :

(i) the stiffness of the spring ;

(ii) the logarithmic decrement ;

(iii) the damping factor ; and

(iv) the damping coefficient.

SPPU - May 08, Dec. 14, 8 Marks

Soln. :

$m = 8$ kg ; $N = 30$;

$t_p = 18$ s ; $x_5 = 0.25$;

$n = 5$.

1. Logarithmic Decrement :

$$\frac{x_0}{x_5} = \frac{1}{0.25}$$

$$\delta = \frac{1}{n} \log_e \left(\frac{x_0}{x_n}\right) = \frac{1}{5} \log_e \left(\frac{x_0}{x_5}\right) \quad \text{...(when n = 5)}$$

$$\delta = \frac{1}{5} \log_e \left(\frac{1}{0.25}\right)$$

$$\delta = \mathbf{0.278} \qquad \text{...Ans.}$$

2. Damping Factor :

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = \frac{0.278}{\sqrt{4\pi^2 + 0.278}}$$

$$= \mathbf{0.0442} \qquad \text{...Ans.}$$

3. Stiffness of Spring :

$$f_d = \frac{N}{t_p} = \frac{30}{18} = 1.66 \text{ Hz}$$

$$\therefore \quad \omega_d = 2\pi f_n = 2\pi \times 1.66$$

$$= 10.47 \text{ rad / sec}$$

but

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

$$10.47 = \omega_n \sqrt{1 - 0.0442}$$

$$\omega_n = 10.48$$

Now, $\omega_n = \sqrt{\frac{K}{m}}$

$\therefore \quad 10.48 = \sqrt{\frac{K}{8}}$

$\therefore \quad K = 878 \text{ N/m}$

or $\quad \mathbf{K = 0.878 \text{ N/mm}}$ ...Ans.

4. Damping Coefficient :

$$\xi = \frac{c}{c_c} = \frac{c}{2\, m\omega_n}$$

$$\therefore \quad 0.0442 = \frac{C}{2 \times 8 \times 10.47}$$

$c = 7.40$ **N-s/m** ...Ans.

Ex. 3.18.23 : A body of mass 5 kg is supported on a spring of stiffness 1960 N/m and has dashpot connected to it, which produces a resistance of 1.96 N at a velocity of 1 m/sec. In what ratio will be amplitude of vibration be reduced after 5 cycles ? **SPPU - Dec. 08, Dec. 11, Dec. 18, 6 Marks**

Soln. :

Given : $m = 5$ kg ; $K = 1960$ N/m ; $F = 1.96$ N ; $V = 1$ m/s ; $n = 5$

Ratio of amplitude of vibration after 5 cycles $= \frac{x_0}{x_5}$

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1960}{5}} = 19.8 \text{ rad/sec}$$

- **Damping coefficient**

$$c = \frac{F}{V} = \frac{1.96}{1} = 1.96 \text{ N-s/m}$$

- **Critical damping coefficient :**

$$c_c = 2\, m\, \omega_n = 2 \times 5 \times 1.98 = 198$$

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{1.96}{198} = 9.89 \times 10^{-3}$$

- **Logarithmic Decrement :**

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 9.89 \times 10^{-3}}{\sqrt{1-(9.89 \times 10^{-3})^2}}$$

$$\delta = 0.0621$$

- **Ratio of amplitudes :**

$$\delta = \frac{1}{n} \log_e \frac{x_0}{x_n}$$

$$0.0621 = \frac{1}{5} \log_e \left[\frac{x_0}{x_5}\right]$$

$$\therefore \quad \log_e \left[\frac{x_0}{x_5}\right] = 0.3105$$

$$\frac{x_0}{x_5} = e^{0.3105}$$

$\therefore \quad \frac{x_0}{x_5} = \mathbf{1.364}$ ...Ans.

Ex. 3.18.24 : A mass of 2 kg is supported on a spring of 3 kN/m and has a dashpot having damping coefficient of 5 N-sec/m. If the initial displacement of 8 mm is given, find :

(i) the damped natural frequency ;

(ii) the logarithmic decrement ; and

(iii) the amplitude after 3 cycles. **SPPU - May 12, 6 Marks**

Soln. :

Given : $m = 2$ kg ; $K = 3000$ N/m ; $c = 5$ N-sec/m ; $x_0 = 8$ mm.

1. Damped Natural Frequency :

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{3000}{2}}$$

or $\quad \omega_n = 38.72$ rad/s

- **Damping factor :**

$$\xi = \frac{c}{c_c}$$

or $$\xi = \frac{c}{2m\,\omega_n} = \frac{5}{2 \times 2 \times 38.72}$$

$$\xi = 0.032$$

- **Circular frequency of damped vibrations :**

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

$$= 38.72 \sqrt{1-(0.032)^2}$$

or $\quad \omega_d = 38.70$ rad/s

- **Frequency of damped vibrations :**

$$f_d = \frac{\omega_d}{2\pi} = \frac{38.70}{2\pi}$$

or $\quad \mathbf{f_d = 6.15 \text{ Hz}}$ ...Ans.

2. Logarithmic Decrement (δ) :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.032}{\sqrt{1-(0.032)^2}}$$

or $\quad \mathbf{\delta = 0.2011}$ ...Ans.

3. Amplitude After Three Cycles :

but, $\quad \delta = \frac{1}{n} \log_e \left(\frac{x_0}{x_n}\right)$

$\delta = \frac{1}{3} \log_e \left(\frac{x_0}{x_3}\right)$ ...(when n = 3)

$$\therefore \quad 0.2011 = \frac{1}{3} \log_e \left(\frac{x_0}{x_3}\right)$$

$$0.2011 \times 3 = \log_e\left(\frac{x_0}{x_3}\right)$$

$$\therefore \quad 0.6033 = \log_e\left(\frac{x_0}{x_3}\right)$$

$$\therefore \quad 1.8281 = \frac{x_0}{x_3}$$

$$\therefore \quad x_4 = \frac{8}{1.8281}$$

or $\quad x_4 = \mathbf{4.3760\ mm}$ ...**Ans.**

Ex. 3.18.25 : A vibrating system is defined by the following parameters : m = 3 kg, k = 100 N/m and c = 3 N-s/m.

Determine :

(i) Critical damping Coefficient;

(ii) the damping factor ;

(iii) the natural frequency of damped vibration ;

(iv) the logarithmic decrement ;

(v) the ratio of two consecutive amplitudes ; and

(vi) the number of cycles after which the original amplitude is reduced to 20%.

SPPU - May 15, Oct. 16 (In Sem.), 5 Marks, Oct. 18 (In Sem.), Oct. 19(In Sem.), 6 Marks

Soln. :

Given : $m = 3$ kg ; $K = 100$ N/m ; $C = 3$ N-s/m ; $x_n = 0.2\, x_0$.

1. Natural Circular Frequency of System :

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{100^3}{3}}$$

$$= \mathbf{5.77\ rad/sec} \qquad \text{... Ans.}$$

2. Critical damping Coefficient

$$c_c = 2m\omega_n$$

$$= 2 \times 3 \times 5.77 = \mathbf{34.62} \qquad \text{... Ans.}$$

2. Damping Factor :

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{3}{2 \times 3 \times 5.77}$$

$$= \mathbf{0.058} \qquad \text{...Ans.}$$

3. Natural Circular Frequency of Damped Vibration :

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$= 5.77\sqrt{1-(0.086)^2}$$

$$= 5.74 \text{ rad/sec}$$

4. Natural Frequency of Damped Vibration :

$$f_d = \frac{\omega_d}{2\pi} = \frac{5.74}{2\pi} = \mathbf{0.914\ Hz} \qquad \text{... Ans.}$$

5. Logarithmic Decrement :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} = \frac{2\pi \times 0.086}{\sqrt{1-(0.086)^2}}$$

$$= \mathbf{0.5423} \qquad \text{...Ans.}$$

6. Ratio of Two Consecutive Amplitudes :

$$\delta = \frac{1}{n}\log_e\left(\frac{X_0}{X_n}\right) = 1 \times \log_e\left(\frac{x_0}{x_1}\right)$$

$$\therefore \quad \frac{x_0}{x_1} = e^{\delta} = e^{0.5423} = \mathbf{1.71} \qquad \text{...Ans.}$$

7. Number of cycles after which the original amplitude is reduced to 20% :

$$\delta = \frac{1}{n}\log_e\left(\frac{X_0}{X_n}\right) = \frac{1}{n}\log_e\left(\frac{X_0}{0.2\,X_0}\right)$$

$$= \frac{1}{n}\log_e\left(\frac{1}{0.2}\right)$$

$$\delta = \frac{1}{n} \times 1.6094$$

$$\therefore \quad 0.5423 = \frac{1}{n} \times 1.6094$$

$$n = 2.96$$

$$n \simeq 3 \text{ cycles} \qquad \text{...Ans.}$$

Ex. 3.18.26 : Two slabs of isolators, natural rubber and felt, are kept between a solid mass of 10 kg and the floor as shown in Fig. P. 3.18.26(a). The natural rubber slabs has a stiffness of 3000 N/m and an equivalent viscous damping coefficient of 100 N-sec/m. The felt slab has a stiffness of 12000 N/m and an equivalent viscous damping coefficient of 330 N-sec/m. Determine the undamped and the damped natural frequencies of the system in vertical direction. Neglect the mass of the isolators. **SPPU - May 13, 8 Marks**

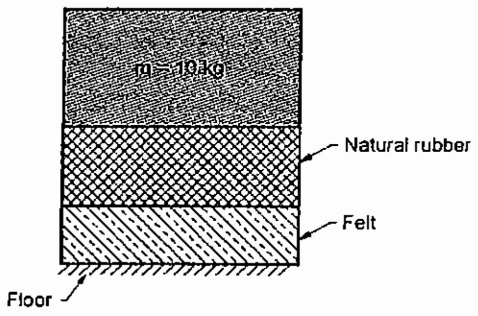


Fig. P. 3.18.26(a)

Soln. :

Given : $K_1 = 3000$ N/m ; $c_1 = 100$ N-sec/m ;
$K_2 = 12000$ N/m ; $c_2 = 330$ N-sec/m ;
$m = 10$ kg.

- Fig. P. 3.18.26(a) shows the given system. The isolators being in series, the system can be schematically represented by Fig. P. 3.18.26(b) and Fig. P. 3.18.26(c).

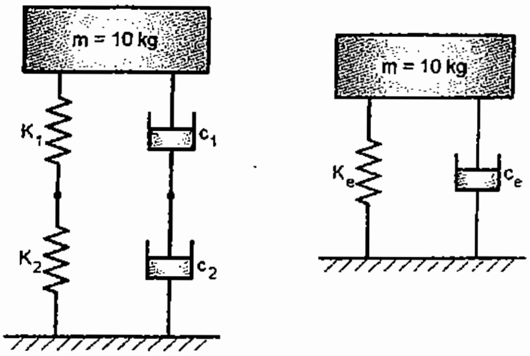


Fig. P. 3.18.26

1. **Undamped Natural Frequency :**

- **Equivalent spring stiffness (K_e) :**

$$\frac{1}{K_e} = \frac{1}{K_1} + \frac{1}{K_2} = \frac{1}{3000} + \frac{1}{12000}$$

$$\text{or} \quad K_e = 2400 \text{ N/m}$$

- **Equivalent damping coefficient (c_e) :**

$$\frac{1}{c_e} = \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{100} + \frac{1}{330}$$

$$\text{or} \quad c_e = 76.74 \text{ N-sec/m}$$

- **Natural circular frequency of undamped vibrations :**

$$\omega_n = \sqrt{\frac{K_e}{m}} = \sqrt{\frac{2400}{10}} = 15.49 \text{ rad/s}$$

- **Undamped natural frequency of system (f_n) :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{15.49}{2\pi} = \mathbf{2.46 \text{ Hz}} \qquad \text{...Ans.}$$

2. **Damped Nature Frequency of System (f_d) :**

- **Damping ratio :**

$$\xi = \frac{c_c}{2\sqrt{K_e \cdot m}} = \frac{76.744}{2\sqrt{2400 \times 10}} = 0.247$$

- **Damped circular frequency :**

$$\omega_d = \sqrt{1 - \xi^2} \times \omega_n$$

$$\therefore \omega_d = \sqrt{1 - (0.247)^2} \times 15.49$$

$$= 15 \text{ rad/s}$$

- **Natural frequency of damped vibrations ,**

$$f_d = \frac{\omega_d}{2\pi} = \frac{15}{2\pi} = \mathbf{2.38 \text{ Hz}} \qquad \text{...Ans.}$$

Ex. 3.18.27 : A machine weighs 18 kg and is supported on spring and dashpots. The total stiffness of springs is 12 N/mm and damping coefficient is 0.2 N-s/mm. The system is initially at rest and a velocity of 120 mm/s is imparted to the mass. Determine :

(i) the displacement and velocity of mass as a function of time; and

(ii) the displacement and velocity after 0.5 sec.

Soln. :

Given :
$$m = 18 \text{ kg} \; ;$$
$$K = 12 \text{ N/mm} = 12 \times 10^3 \text{ N/m} \; ;$$
$$c = 0.2 \text{ N-s/mm} = 0.2 \times 10^3 \text{ N-s/m} \; ;$$
$$\text{At } t = 0 \; ; \quad x = 0,$$
$$\text{At } t = 0 \; ; \quad \dot{x} = 120 \text{ mm/s} = 120 \times 10^{-3} \text{ m/s.}$$

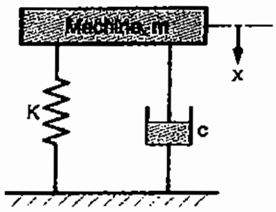


Fig. P. 3.18.27

1. **Natural Circular Frequency and Damping Factor :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{12 \times 10^3}{18}}$$

$$= \mathbf{25.82 \text{ rad/s.}}$$

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2m\,\omega_n} = \frac{0.2 \times 10^3}{2 \times 18 \times 25.82}$$

$$= \mathbf{0.2151}$$

- **Natural circular frequency of damped vibration :**

$$\omega_d = \left[\sqrt{1 - \xi^2}\right] \omega_n \text{, rad/s}$$

$$= \left[\sqrt{1 - (0.2151)^2}\right] \times 25.82$$

$$\text{or} \quad \omega_d = 25.216 \text{ rad/s}$$

2. **Equations for Displacement and Velocity :**

- **Equation of motion for under-damped system :**

$$x = X e^{-\xi \omega_n t} \sin(\omega_d t + \phi)$$

$$= X e^{-\xi \omega_n t} \sin(\omega_d t + \phi)$$

$$= X e^{(-0.2151 \times 25.82) t} \sin(25.216\, t + \phi)$$

$$\text{or} \quad x = X e^{-5.55 t} \sin(25.216\, t + \phi) \qquad \text{...(a)}$$

- **Equation of velocity for under-damped system :**

Differentiating above equation with respect to 't', we get,

$$\dot{x} = X \sin(25.216\,t + \phi)(-5.55)\,e^{-5.55t} + X e^{-5.55t} \cos(25.216\,t + \phi)(25.216) \quad ...(b)$$

- **Constants ϕ and X :**

(i) Substituting t = 0 and x = 0 in Equation (a),

$$0 = X \sin \phi$$

$X \neq 0$; therefore,

$$\phi = 0 \quad ...(c)$$

(ii) On substituting t = 0 and $\dot{x} = 120 \times 10^{-3}$ m/s in Equation (b), we get,

$$120 \times 10^{-3} = -5.55\,X \sin\phi + X \cos\phi\,(25.216)$$

$$\therefore\ 120 \times 10^{-3} = -5.55\,X \sin(0) + X \cos(0)(25.216)$$

$$\therefore\ 120 \times 10^{-3} = X(25.216)$$

$$\therefore\ X = 0.00475 \quad ...(d)$$

- **Equation of motion (displacement) :**

Substituting Equation (c), and (d) in Equation (a),

$$x = \mathbf{0.00475\,e^{-5.55t} \sin(25.216\,t)} \quad ...Ans.$$

Equation of velocity :

The Equation (b), for velocity, can be written as,

$$\dot{x} = -5.55 \times 0.00475 \times e^{-5.55t} \sin(25.216\,t) + 0.00475\,e^{-5.55t} \sin(25.216) \cos(25.216\,t)$$

$$\dot{x} = \mathbf{0.00475\,e^{-5.55t}[-5.55 \sin(25.216\,t) + (25.216)\cos(25.216\,t)]} \quad ...Ans.$$

> **Note :** It is important to note that, the angle ($\omega_d t + \phi$) is in radians. Hence while calculating sin or cos of angle, either radian mode should be used in calculator or angle ($\omega_d t + \phi$) should be converted to degrees.

3. **Displacement and Velocity after 0.5 s :**

- **Displacement after 0.5 sec :**

$$x = 0.00475\,e^{-5.55 \times 0.5} \sin(25.216 \times 0.5)$$

$$= 1.2325 \times 10^{-5}\ \text{m}$$

or $\quad x = \mathbf{0.012325\ mm} \quad$ ...Ans.

- **Velocity after 0.5 sec :**

$$\dot{x} = 0.00475 \times e^{-5.55 \times 0.5}[-5.55 \sin(25.216 \times 0.5) + (25.216)\cos(25.216 \times 0.5)]$$

$$\dot{x} = 7.393 \times 10^{-3}\ \text{m/s}$$

$$\dot{x} = \mathbf{7.393\ mm/s} \quad ...Ans.$$

Ex. 3.18.28 : Consider the spring-mass-damper system in which mass is given a velocity of 0.1 m/s. What will be the subsequent displacement and velocity of the mass, if damping coefficient is 100 Ns/m, spring stiffness is 3000 N/m and mass of 20 kg; $F_0 \sin \omega t = 0$. **SPPU - Dec. 15, 6 Marks**

Soln. :

Given : c = 100 N-s/m ; K = 3000 N/mm ;

m = 20 kg;

At t = 0 ; x = 0;

At t = 0 ; $\dot{x}$ = 120 mm/s = 0.1 m/s.

1. **Natural Circular Frequency and Damping Factor :**

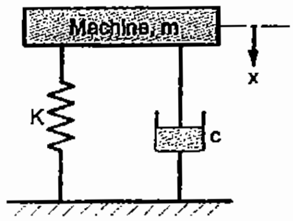


Fig. P. 3.18.28

- **Natural circular frequency of the system :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\left(\frac{3000}{20}\right)} = 12.25\ \text{rad/sec.}$$

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2m\,\omega_n} = \frac{100}{2 \times 20 \times 12.25} = 0.2$$

- **Natural circular frequency of damped vibration :**

$$\omega_d = \left[\sqrt{1-\xi^2}\right]\omega_n,\ \text{rad/s} = \left[\sqrt{1-(0.2)^2}\right] \times 12.25$$

or $\quad \omega_d = 12$ rad/s

2. **Equations for Displacement and Velocity :**

- **Equation of motion for under-damped system :**

$$x = X e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$$

$$= X e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$$

$$= X e^{(-0.2 \times 12.25)t} \sin(12t + \phi)$$

or $\quad x = X e^{-2.45t} \sin(12\,t + \phi) \quad$ ...(a)

- **Equation of velocity :**

  Differentiating above equation with respect to 't', we get,

$$\dot{x} = X \sin(12t + \phi)(-2.45)\, e^{-2.45t} + Xe^{-2.45t} \cos(12t + \phi)(12) \qquad \text{...(b)}$$

- **Constants ϕ and X :**

  (1) Substituting t = 0 and x = 0 in Equation (a), we get,

$$0 = X \sin \phi$$

  $X \neq 0$; therefore,

$$\phi = 0 \qquad \text{...(c)}$$

  (ii) On substituting t = 0 and $\dot{x} = 0.1 \times 10^{-3}$ m/s in Equation (b), we get,

$$0.1 \times 10^{-3} = -2.45\, X \sin\phi + X \cos\phi\, (12)$$

$$\therefore \quad 0.1 \times 10^{-3} = -2.45\, X \sin(0) + X \cos(0)(12)$$

$$\therefore \quad 0.1 \times 10^{-3} = X(12)$$

$$\therefore \quad X = 8.16 \times 10^{-3}$$

- **Equation of motion :**

  Substituting Equation (c) and (d) in Equation (a), for displacement is,

$$x = 0.00816\, e^{-2.45t} \sin(12t) \qquad \text{...Ans.}$$

- **Equation of velocity :**

  The Equation (b), for velocity, can be written as,

$$\dot{x} = -2.45 \times 0.00816 \times e^{-2.45t} \sin(12t) + 0.00816\, e^{-2.45t} \sin(12) \cos(25.12t)$$

Ex. 3.18.29 : The spring-mass-damper system has spring stiffness 10 kN/m, viscous damping coefficient 1500 N-s/m and a mass of 7 kg. If the mass is displaced by 0.01 m and released with a velocity of 10 m/sec in the direction of return motion, determine :

(i) an expression for the displacement x of mass in terms of time t ; and

(ii) the displacement of mass after 0.02 second.

SPPU - May 12, 8 Marks

Soln. :

Given : $K = 10\ \text{kN/m} = 10 \times 10^3\ \text{N/m}$;

$c = 1500$ N-s/m ; $m = 7$ kg ;

At t = 0 ; x = 0.01 m ;

At t = 0 ; $\dot{x} = -10$ m/s

...(Negative sign indicates direction of motion is return)

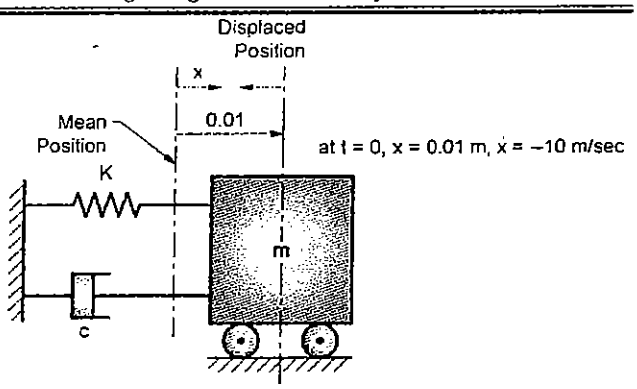


Fig. P. 3.18.29

1. **Natural Circular Frequency and Damping Factor :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{10 \times 10^3}{7}} = 37.79 \text{ rad/s}$$

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2m\,\omega_n} = \frac{1500}{2 \times 7 \times 37.79} = 2.83$$

2. **Expression for Displacement :**

   Since $\xi > 1$, the system is over damped. Therefore the equation of motion of over damped system is,

$$x = A\, e^{\left[-\xi + \sqrt{\xi^2 - 1}\right]\omega_n t} + B\, e^{\left[-\xi - \sqrt{\xi^2 - 1}\right]\omega_n t}$$

$$= +B\, e^{\left[-2.83 - \sqrt{2.83^2 - 1}\right] 37.79t}\, A\, e^{\left[-2.83 + \sqrt{2.83^2 - 1}\right] 37.79t}$$

or $$x = A\, e^{(-6.89)t} + B\, e^{(-206.99)t} \qquad \text{...(a)}$$

- **Differentiating Equation (a) w.r.t. time, we get,**

$$\dot{x} = -6.89\, A\, e^{-6.89t} - 206.99\, B\, e^{-206.99t} \qquad \text{...(b)}$$

- **Substituting t = 0 and x = 0.01 in Equation (a), we get,**

$$0.01 = A\,e^0 + B\,e^0$$

$$\therefore \quad 0.01 = A + B \qquad \text{...(c)}$$

  Substituting t = 0 and $\dot{x} = -10$ in Equation (b), we get,

$$-10 = -6.89\, A\, e^0 - 206.99\, B\, e^0$$

$$-10 = -6.89\, A - 206.99\, B \qquad \text{...(d)}$$

- **Solving Equations (c) and (d), we get,**

$$A = -0.04$$

$$B = 0.05$$

- **Substituting these values of A and B in Equation (a), we get,**

$$x = -0.04\, e^{-(6.89)t} + 0.05\, e^{-(206.99)t} \qquad \text{...Ans.}$$

  at t = 0.02 the displacement is,

$$x = -0.04\, e^{-6.89 \times 0.02} + 0.05\, e^{-206.99 \times 0.02}$$

or $$x = -0.034 \text{ m} \qquad \text{...Ans.}$$

Negative sign indicates the displacement is on opposite side of mean position.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 3.18.30 : An over damped system, shown in Fig. P. 3.18.30, has a spring of stiffness 14 kN/m, a viscous damper having damping coefficient of 1400 N-s/m and a mass of 8.6 kg. It is at rest in its static equilibrium position when it receives an impulse force acting to the right that imparts an initial instantaneous velocity of 25 m/sec to the mass.

(i) Determine an expression for the displacement 'x' of the mass in terms of time 't'.

(ii) What will be the maximum displacement of the mass from the initial position ?

(iii) What length of time will be required for the mass to attain the position of maximum displacement ?

SPPU - May 05

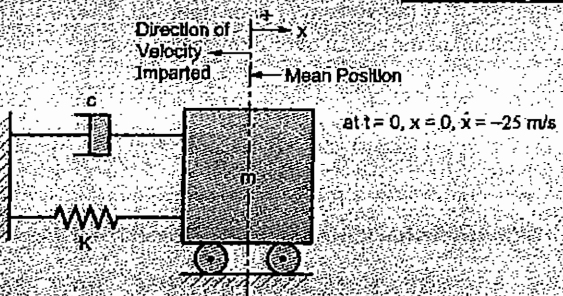


Fig. P. 3.18.30

Ex. 3.18.31 : In a spring mass-damper system, m = 10 kg, K = 16 kN/m and c = 1600 N-s/m. If the mass is displaced by 0.1 m and released with a velocity of 2 m/sec in the direction of return motion, determine :

(i) the circular frequency ;

(ii) the damping factor ; and

(iii) the displacement after 1/100 sec.

SPPU - Dec. 12, 12 Marks

Ex. 3.18.32 : A spring-mass-dashpot system consists of spring of stiffness 400 N/m and the mass of 4 kg. The mass is displaced 20 mm beyond the equilibrium position and released. Find the equation of motion of the mass, if the damping coefficient of the dashpot is :

(i) 160 N-s/m ; (ii) 80 N-s/m **SPPU - Dec. 04**

Ex. 3.18.33 : A gun of cannon is so designed that, on firing the barrel recoils against a spring. A dashpot at the end of the recoil allows the barrel to come back to its initial position within the minimum time without any oscillations. The gun barrel has a mass of 500 kg and a recoil spring of stiffness of 300 N/mm. If the barrel recoils 1 m on firing, Determine :

(i) the initial recoil velocity of the gun barrel ; and

(ii) the damping coefficient of the dashpot engaged at the end of the recoil stroke.

What type of damping it is ? **SPPU : Dec. 05**

Soln. :

Given : $m = 500$ kg ;

$K = 300$ N/mm $= 300 \times 10^3$ N /m.

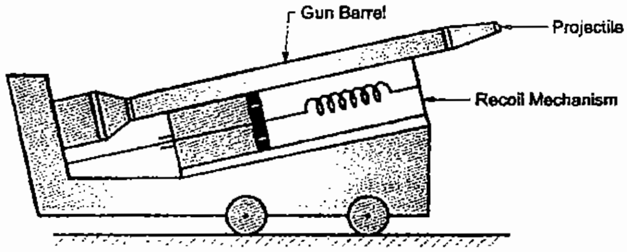


Fig. P. 3.18.33

1. **Initial Recoil Velocity of Gun Barrel :**

- **Natural circular frequency of system :**

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$= \sqrt{\frac{300 \times 10^3}{500}} = 24.49 \text{ rad/s}$$

- **Maximum Potential energy in spring :**

$$PE = \frac{1}{2} K x^2 = \frac{1}{2} \times 300 \times 10^3 \times 1^2$$

or $$P.E. = 150{,}000 \text{ N-m}$$

- **Kinetic energy of barrel :**

$$K.E. = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} \times 500 \times \dot{x}^2$$

or $$K.E. = 250\,\dot{x}^2, \text{ N-m}$$

$$(K.E.)_{max} = (P.E.)_{max}$$

$$\therefore \quad 250\,\dot{x}^2 = 150000$$

$$\therefore \quad \dot{x}^2 = 600$$

or $$\dot{x} = \mathbf{24.49 \text{ m/sec}}$$ ...Ans.

2. **Damping Coefficient :**

- **The critical damping coefficient is,**

$$c_c = 2m\,\omega_n = 2 \times 500 \times 24.49$$

$$c_c = \mathbf{24.49 \times 10^3 \text{ N-s/m}}$$ ...Ans.

- The dashpot allows barrel to come back to its initial position within the minimum time without oscillations. Hence, the system must be critically damped system.

$$\therefore \quad \xi = 1$$

$$\therefore \quad \frac{c}{c_c} = 1$$

$$\therefore \quad c = c_c$$

$$\text{or} \quad c = 24.49 \times 10^3 \text{ N-s/m} \quad \text{...Ans.}$$

Ex. 3.18.34 : A door along with door-closing system, shown in Fig. P. 3.18.34(a), has a moment of inertia of 25 kg-m^2 about the hinge axis. If the stiffness of torsional spring is 20 N-m/rad, determine the most suitable value of the damping coefficient. **SPPU - May 07**

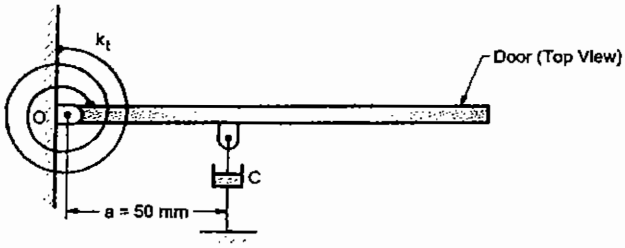
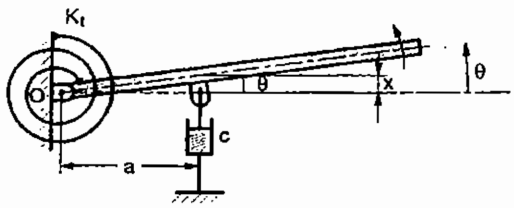


Fig. P. 3.18.34(a)

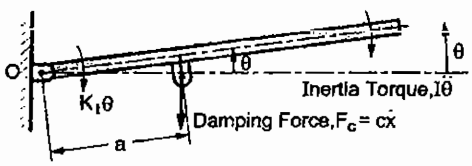
Soln. :

1. **Differential Equation of Motion :**

- From Fig. P. 3.18.34(b) shows the system when door is deflected through an angle 'θ'.
- From Fig. P. 3.18.34(c) ;



(b) Displaced Position



(c) F.B.D
Fig. P. 3.18.34

$$\Sigma \,[\text{ Inertia torque + External torques }] = 0$$

$$I_o \ddot{\theta} + c\,\dot{x}\,a + K_t \theta = 0$$

$$I_o \ddot{\theta} + c\,a\,\dot{\theta}\,a + K_t \theta = 0 \quad \text{...}[\because x = a\theta \text{ and } \dot{x} = a\,\dot{\theta}\,]$$

$$I_o \ddot{\theta} + c\,a^2 \dot{\theta} + K_t \theta = 0 \quad \text{...(a)}$$

- The above Equation (a) can be written as,

$$I_o \ddot{\theta} + c_{te} \dot{\theta} + K_{te} \theta = 0$$

$$\text{where,} \quad c_{te} = c\,a^2 \,; \quad \text{and} \quad K_{te} = K_t$$

2. **Critical Damping Coefficient :**

For critically damped torsional vibrations,

$$c_{te} = 2 \times \sqrt{K_{te} I_o}$$

$$\therefore \quad c_c a^2 = 2 \times \sqrt{K_t I_o}$$

$$c_c = \frac{2}{a^2}\sqrt{K_t I_o} = \frac{2}{0.05^2}\sqrt{20 \times 25}$$

$$\therefore \quad c_c = 17888.54 \text{ N sec/m} \quad \text{...Ans.}$$

Ex. 3.18.35 : Derive an equation of motion for the system shown in Fig. P. 3.18.35(a). If m = 1.5 kg, K = 4900 N/m, a = 6 cm and b = 14 cm, determine the value of c for which the system is critically damped.

SPPU - Aug 15 (In Sem), 4 Marks, Oct 16 (In Sem), 5 Marks

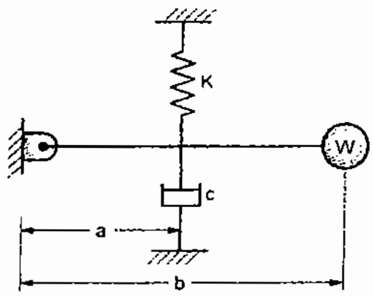
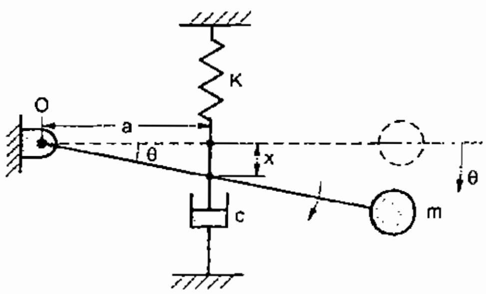


Fig. P. 3.18.35(a)

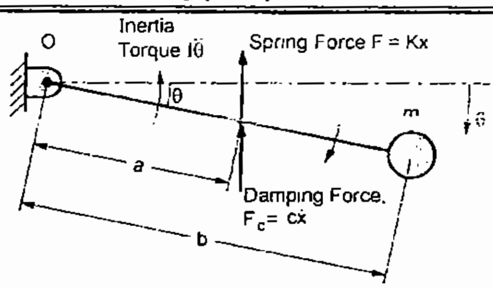
Soln. :

1. **Differential Equation of Motion :**

Fig. P. 3.18.35 shows the system when mass '**m**' is deflected through a angle 'θ' , due to which the spring '**K**' will be stretched by a distance x = a θ.



(b) Displaced Position



(c) F.B.D
Fig. P. 3.18.35

From Fig. P. 3.18.35(c) ;

$$\Sigma\,[\text{ Inertia torque + External torques }] = 0$$

$$\therefore\; I_0\ddot{\theta} + c\dot{x}a + Kxa = 0$$

$$\therefore\; I_0\ddot{\theta} + ca^2\dot{\theta} + Ka^2\theta = 0 \qquad \text{...(a)}$$

$$\text{... [} \because\; x = a\theta, \text{ and } \dot{x} = a\dot{\theta} \text{]}$$

$$\text{or } \ddot{\theta} + \left(\frac{ca^2}{I_0}\right)\dot{\theta} + \left(\frac{Ka^2}{I_0}\right)\theta = 0 \qquad \text{...(b)}$$

- This is the fundamental differential equation of motion for single degree of freedom of a system having viscous damping. This is a linear differential equation of the second order and its solution can be written as substituting :

$$\theta = e^{St}$$

$$\dot{\theta} = S e^{St}$$

$$\text{and } \quad \ddot{\theta} = S^2 e^{St}$$

Therefore Equation (b) can be written as,

$$S^2 e^{St} + \frac{ca^2}{I_0} S e^{St} + \frac{Ka^2}{I_0} e^{St} = 0$$

$$\therefore\; S^2 + \left(\frac{ca^2}{I_0}\right)S + \left(\frac{Ka^2}{I_0}\right) = 0 \qquad \text{...(c)}$$

- **Two roots of equation :**

The Equation (c) is the quadric equation for which the two roots are :

$$S_{1,2} = \frac{\left(\frac{ca^2}{I_0}\right) \pm \sqrt{\left(\frac{ca^2}{I_0}\right)^2 - 4\left(\frac{Ka^2}{I_0}\right)}}{2}$$

$$S_{1,2} = \frac{ca^2}{2I_0} \pm \sqrt{\left(\frac{ca^2}{2I_0}\right)^2 - \left(\frac{Ka^2}{I_0}\right)}$$

2. Critical Damping Coefficient :

The system is critically damped when,

$$\left(\frac{ca^2}{2I_0}\right)^2 = \frac{Ka^2}{I_0}$$

$$\therefore\; \left(\frac{c_c a^2}{2I_0}\right)^2 = \frac{Ka^2}{I_0} \qquad \text{...[} \because\; c = c_c \text{]}$$

$$\therefore\; \frac{c_c^2 a^4}{4I_0^2} = \frac{Ka^2}{I_0}$$

$$\therefore\; c_c^2 = \frac{Ka^2}{I_0} \times \frac{4I^2}{a^4} = \frac{4KI}{a^2}$$

$$\therefore\; c_c = \frac{2\sqrt{KI}}{a} = \frac{2\sqrt{Kml^2}}{a} \qquad \text{...[} \because\; I = mb^2 \text{]}$$

$$\text{or } \; \mathbf{c_c = \frac{2b}{a}\sqrt{Km}} \qquad \text{...Ans.}$$

$$\text{But, } \; \xi = \frac{c}{c_c} = \frac{c}{\frac{2b}{a}\sqrt{Km}} = \frac{ca}{2b\sqrt{Km}}$$

3. Frequency of Damped Vibrations :

- **Circular damped frequency :**

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \sqrt{\omega_n^2 - \omega_n^2\xi^2}$$

$$= \sqrt{\frac{Ka^2}{mb^2} - \frac{Ka^2}{mb^2} \times \left(\frac{ca}{2b\sqrt{Km}}\right)^2}$$

$$\text{...}\left[\because\; \omega_n = \sqrt{\frac{Ka^2}{mb^2}}\right]$$

$$= \sqrt{\frac{Ka^2}{mb^2} - \frac{Ka^2}{mb^2} \times \frac{c^2a^2}{4b^2Km}}$$

$$= \sqrt{\frac{Ka^2}{mb^2} - \frac{c^2a^4}{4m^2b^4}}$$

$$\text{or } \quad \omega_d = \frac{a}{b}\sqrt{\frac{K}{m} - \frac{c^2a^2}{4m^2b^2}}$$

- **Frequency of damped vibration :**

$$f_d = \frac{\omega_d}{2\pi}$$

$$\text{or } \quad \mathbf{f_d = \frac{1}{2\pi}\frac{a}{b}\sqrt{\frac{K}{m} - \frac{c^2a^2}{4m^2b^2}}, \text{ Hz}} \qquad \text{...Ans.}$$

4. Damping Coefficient :

When m = 1.5 kg, K = 4900 N/m, b = 0.14 m and a = 0.06 m.

The critical damping coefficient is,

$$c_c = \frac{2b}{a}\sqrt{Km} = \frac{2 \times 0.14}{0.06}\sqrt{4900 \times 1.5}$$

$$\text{or } \quad c_c = 400 \text{ N-s/m} \qquad \text{...Ans.}$$

5. Alternate Solution of Equation (a) :

From Equation (a),

$$I_o\ddot{\theta} + c\,a^2\dot{\theta} + K\,a^2\theta = 0 \qquad ...(d)$$

Equation (d) can be written as,

$$I_o\ddot{\theta} + c_{te}\dot{\theta} + K_{te}\theta = 0$$

where $I_o = m\,b^2$; $c_{te} = c\,a^2$

and $K_{te} = K\,a^2$

For critically damped torsional vibrations,

$$c_{te} = 2\sqrt{K_{te} I_o}$$

$$\therefore \quad c_c\,a^2 = 2\sqrt{K\,a^2\,m\,b^2}$$

$$\therefore \quad c_c = \frac{2}{a^2}\sqrt{K\,a^2\,m\,b^2} = \frac{2\,a\,b}{a^2}\sqrt{K\,m}$$

$$\text{or} \quad c_c = \frac{2\,b}{a}\sqrt{K\,m} \qquad ...\text{Ans.}$$

3.19 DAMPED FREE VIBRATIONS WITH COULOMB OR DRY FRICTION DAMPING

University Question

Q. What is Coulomb Damping ? **SPPU : May 19**

Coulomb Damping :

Coulomb damping is the damping that occurs when two dry or unlubricated surfaces slide against each other. The frictional force, which is constant in magnitude, acts as damping force. Therefore, coulomb or dry friction damping is also known as **constant damping**.

Spring-Mass System with Coulomb Damping :

Consider a spring-mass system, show in Fig. 3.19.1, with mass **'m'** sliding on a dry surface. Let μ be the coefficient of dry friction between the two surfaces. In equilibrium position, the spring us unstretched and no fractional force acts on the mass.

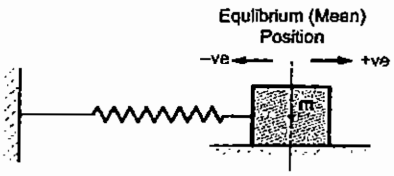


Fig. 3.19.1 : Spring-mass System with Coulomb or Dry Friction Damping

- **Forces Acting on mass in displaced position :**

(i) Spring force, Kx (opposite to displacement x)
(ii) Friction force, Fr (opposite to velocity $\dot{x}$)
(iii) Inertia force, $m\ddot{x}$ (opposite to acceleration $\ddot{x}$)

- **Sign convention :** The displacement x, velocity $\dot{x}$ and acceleration $\ddot{x}$ are taken as positive towards the right side of the equilibrium (or mean) position and taken as negative towards the left side of the equilibrium (or mean) position.

Equation of Motion During Four Quarters :

In one cycle, the motion of mass can be divided into four quarters.

First Quarter :	**Mass is on right side of equilibrium position and moving towards right**
Second Quarter :	**Mass is on right side of equilibrium position and moving towards left.**
Third Quarter :	**Mass is on left side of equilibrium position and moving towards left**
Fourth Quarter :	**Mass is on left side of equilibrium position and moving towards right**

- **First Quarter :** Mass is on right side of equilibrium position and moving towards right

From Fig. 3.19.2(a) ;

$$\Sigma\,[\text{ Inertia force + External forces }] = 0$$

$$-m\ddot{x} - Kx - F_r = 0$$

$$\text{or} \quad m\ddot{x} + Kx + F_r = 0 \qquad ...(a)$$

- **Second Quarter :** Mass is on right side of equilibrium position and moving towards left.

From Fig. 3.19.2(b) ;

$$\Sigma\,[\text{ Inertia force + External forces }] = 0$$

$$-m\ddot{x} - Kx + F_r = 0$$

$$\text{or} \quad m\ddot{x} + Kx - F_r = 0 \qquad ...(b)$$

- **Third Quarter :** Mass is on left side of equilibrium position and moving towards left

From Fig. 3.19.2(c) ;

$$\Sigma\,[\text{ Inertia force + External forces }] = 0$$

$$\text{or} \quad m(-\ddot{x}) + K(-x) - F_r = 0$$

$$\text{or} \quad m\ddot{x} + Kx - F_r = 0 \qquad ...(c)$$

- **Fourth Quarter :** Mass is on left side of equilibrium position and moving towards right

From Fig. 3.19.2(d) ;

$$\Sigma\,[\text{ Inertia force + External forces }] = 0$$

$$m(-\ddot{x}) + K(-x) - F_r = 0$$

$$\text{or} \quad m\ddot{x} + Kx + F_r = 0 \qquad ...(d)$$

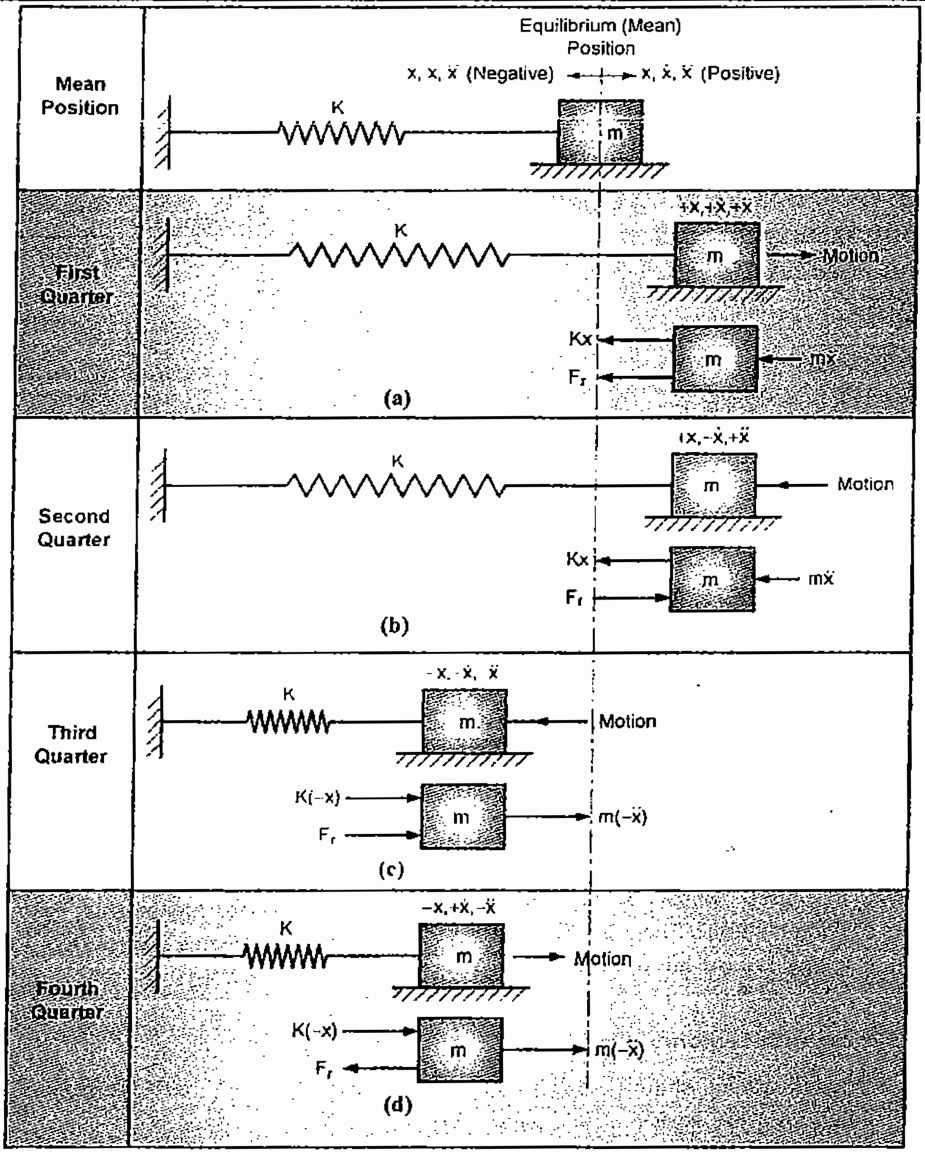


Fig. 3.19.2

Natural Frequency of Motion :

- **First and fourth quarter of cycle :**

  The differential Equations of motion for first and fourth quarter of cycle Equations (a) and (d) are similar and can be written as,

$$m\ddot{x} + Kx + F_r = 0$$

$$\therefore \quad \ddot{x} + \frac{K}{m}x + \frac{F_r}{m} = 0$$

$$\ddot{x} + \frac{K}{m}\left[x + \frac{F_r}{K}\right] = 0 \qquad ...(e)$$

$$\left.\begin{aligned} \text{Let,} \quad y &= x + \frac{F_r}{m} \\ \therefore \quad \ddot{y} &= \ddot{x} \end{aligned}\right\} \qquad ...(f)$$

Substituting Equation (f) in Equation (e), we get,

$$\ddot{y} + \frac{K}{m}y = 0 \qquad ...(g)$$

The Equation (g) represents the simple harmonic motion about,

$$y = 0$$

$$\text{i.e.} \quad x + \frac{F_r}{K} = 0$$

$$\text{i.e.} \quad x = -\frac{F_r}{K} \qquad \text{....[Refer Fig. 3.19.3]}$$

The natural frequency of damped vibrations for this part of cycle is,

$$\omega_n = \sqrt{\frac{K}{m}} \qquad ...(h)$$

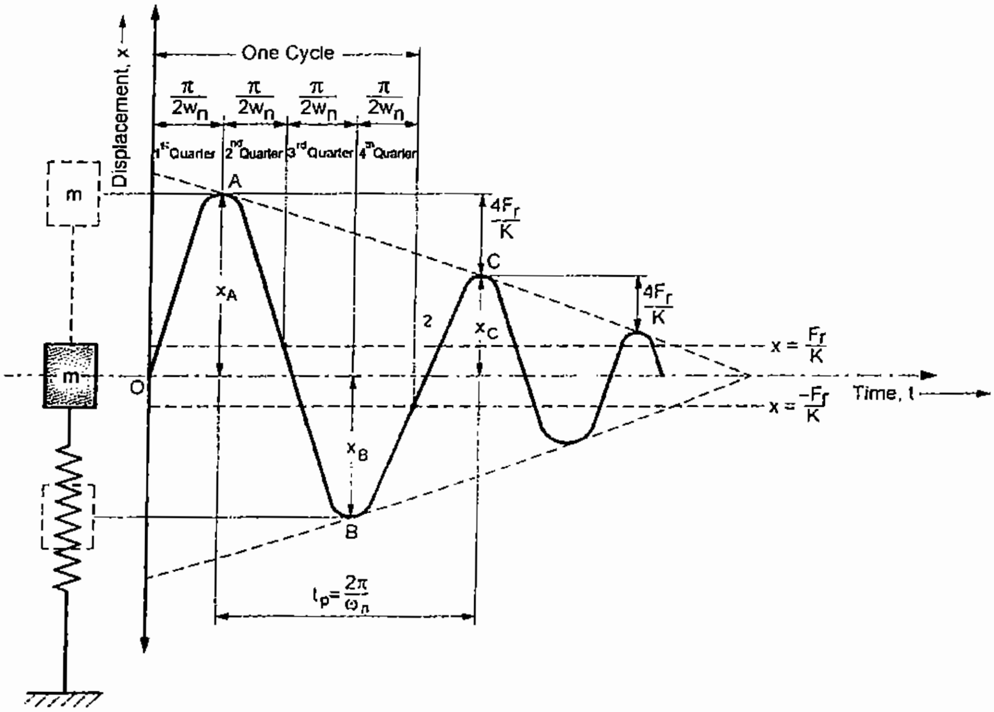


Fig. 3.19.3 : Displacement vs Time Plot for Coulomb or Dry Friction Damping

- **Second and third quarter of cycle :**

  The differential Equations of motion for second and third quarter of cycle [Equations (b) and (c)] are similar and can be written as,

$$m\ddot{x} + Kx - F_r = 0$$

$$\therefore \quad \ddot{x} + \frac{K}{m}x - \frac{F_r}{m} = 0$$

$$\ddot{x} + \frac{K}{m}\left[x - \frac{F_r}{K}\right] = 0 \qquad ...(i)$$

$$\left.\begin{aligned} \text{Let, } y &= x - \frac{F_r}{m} \\ \therefore \ddot{y} &= \ddot{x} \end{aligned}\right\} \qquad ...(j)$$

Substituting Equations (j) in Equation (i) we get,

$$\ddot{y} + \frac{K}{m}y = 0 \qquad ...(k)$$

The Equation (k) represents the simple harmonic motion about,

$$y = 0$$

$$\text{i.e.} \quad x - \frac{F_r}{K} = 0$$

$$\text{i.e. } x = \frac{F_r}{K} \qquad \text{..... [Refer Fig. 3.19.3]}$$

The natural frequency, of damped vibrations for this part of cycle is,

$$\omega_n = \sqrt{\frac{K}{m}} \qquad ...(l)$$

- From Equations (h) and (*l*) it is seen that, the natural frequency of damped vibrations for the system with coulomb damping is same as that of natural frequency of undamped vibrations

  The time period is given by,

$$t_p = \frac{2\pi}{\omega_n}, s$$

3.19.1 Rate of Decay of Vibrations :

University Questions

Q. Derive a relation to determine the loss of amplitude per cycle in case of Coulomb damping. **MU : May 14**

Q. Derive that the loss of amplitude per cycle for coloumb damping is given by 4 F/K. **MU : Dec. 14, May 19**

- Fig. 3.19.3 shows a rate of decay of vibrations i.e. loss of amplitude per cycle with coulomb damping. Consider cycle starting from point A (i.e. extreme left position).
- **Total energy of system at point A (strain energy) :**

$$U_A = \frac{1}{2} K X_A^2 \qquad ... (m)$$

At point A, the velocity of mass is zero, hence, the kinetic energy is zero.

- **Total energy of system at point B :**

  As the system moves from point A to point B (i.e. from extreme left position to extreme right position), because of coulomb damping, there is decay of vibrations. At point B, the amplitude reduces to 'X_B'. At point B, the velocity of mass 'm' in again zero, hence the total energy of the system, which is strain energy is given by,

$$U_B = \frac{1}{2} K X_B^2 \quad \text{... (n)}$$

- **Loss of energy in half cycle from point A to point B :**

$$\Delta U = U_A - U_B = \frac{1}{2} K X_A^2 - \frac{1}{2} K X_B^2$$

or
$$\Delta U = \frac{1}{2} K (X_A^2 - X_B^2) \quad \text{... (3.19.1)}$$

- **Work done against frictional force in half cycle from point A to point B :**

  The loss of energy is due to work done against frictional force. This work done against frictional force is given by,

$$W = \text{Force} \times \text{Distance traveled}$$

or
$$\mathbf{W = F_r (X_A + X_B)} \quad \text{... (3.19.2)}$$

- **Loss of amplitude in half cycle form point A to point B :**

  From Equations (3.19.1) and (3.19.2),

$$\text{Loss of energy} = \text{Work done against friction}$$

$$\Delta U = W$$

$$\frac{1}{2} K (X_A^2 - X_B^2) = F_r (X_A + X_B)$$

$$\frac{1}{2} K (X_A - X_B) (X_A + X_B) = F_r (X_A + X_B)$$

$$\therefore \quad X_A - X_B = \frac{2 F_r}{K}$$

The loss of amplitude in half cycle from point A to point B is,

$$X_A - X_B = \frac{2 F_r}{K} \quad \text{...(3.19.3)}$$

- **Loss of amplitude in one cycle :**

  Similarly the loss of amplitude in half cycle from point B to point C is,

$$X_B - X_C = \frac{2 F_r}{K} \quad \text{... (3.19.4)}$$

  Hence total loss of amplitude in one cycle is,

$$\Delta = X_A - X_C = (X_A - X_B) + (X_B - X_C)$$

$$= \frac{2 F_r}{K} + \frac{2 F_r}{K}$$

or
$$\Delta = \frac{4 F_r}{K} \quad \text{...(3.19.5)}$$

Thus, in coulomb damping, the difference between any two successive amplitudes is constant and is given by,

$$\Delta = \frac{4 F_r}{K}, \text{ m} \quad \text{... (3.19.6)}$$

3.19.2 Damped Free Torsional Vibrations with Coulomb or Dry Friction Damping :

- **Differential equation of motion with coulomb or dry friction damping :**

$$I \ddot{\theta} + K_t \theta \pm T_r = 0 \quad \text{... (a)}$$

Note : + sign for first and further quarters of cycle, and – sign for second and third quarters of cycle.

where, I = mass moment of inertia of disc under torsional vibrations about an axis, kg-m²

K_t = torsional stuffiness of rod or shaft supporting the disc, N-m/rad

T_r = Frictional torque due to coulomb damping, N-m

θ = angular displacement of disc from mean position, rad

$\dot{\theta}$ = angular velocity of disc, rad/s

$\ddot{\theta}$ = angular acceleration of disc, rad/s²

- **Natural frequency of damped torsional vibrations :**

$$\omega_n = \sqrt{\frac{K_t}{I}}, \text{ rad/s} \quad \text{...(3.19.7)}$$

- **Loss of amplitude in one cycle :**

$$\Delta = \frac{4 T_r}{K_t}, \text{ rad} \quad \text{...(3.19.8)}$$

Ex. 3.19.1 : A vibrating system having 50 kg mass and spring stiffness of 500 N/m is damped by coulomb damping with limiting frictional force equal to 6 N. If the mass is given initial displacement of 300 mm, determine :

(i) the loss of amplitude per cycle;

(ii) the number of cycles before stopping;

(iii) the time elapsed before stopping; and

(iv) the distance at which mass stops from the mean position **SPPU - Dec. 02**

Soln. :

Given : m = 50 kg ; K = 500 N/m ;

F_r = 6 N ; x_0 = 300 mm.

1. Loss of Amplitude Per Cycle :

The loss of amplitude per cycle is,

$$\Delta = \frac{4F_r}{K} = \frac{4 \times 6}{500} = 48 \times 10^{-3} \text{ m}$$

or Δ = **48 mm** ...Ans.

2. Distance at which Mass Stops from Mean Position :

- In determining the number of cycles before stopping, it is assumed that the system comes at the equilibrium position. Therefore,

$$x_n = 0$$

Now, $x_n = x_0 - n\Delta$

$$\therefore \; 0 = x_0 - n\Delta = 300 - n \times 48$$

$\therefore$ n = **6.25 half cycles** ...Ans.

- The mass will complete at least six full cycles. At the end of six full cycles, the amplitude is given by,

$$x_6 = x_0 - 6\Delta = 300 - 6 \times 48$$

$$x_6 = 12 \text{ mm}$$

- The spring force acting on the mass when the mass is at a distance of 12 mm form the mean position is,

$$F_S = K \times x_6 = 500 \times 12 \times 10^{-3} = 6 \text{ N} = F_r$$

- At a distance of 12 mm from the mean position, the spring force is equal to the frictional force and hence, the mass stops there.

$\therefore$ x = **12 mm** ...Ans.

3. Number of Cycles Before Stopping :

The mass will come to the rest at a distance of 12 mm from the mean position, after completing the six cycles. Therefore,

n = **6** ...Ans.

4. Time Elapsed Before Stopping :

The time elapsed before the mass stops is,

$$t = n \times \frac{2\pi}{\omega_n} = 6 \times \frac{2\pi}{\sqrt{\frac{K}{m}}} = \frac{6 \times 2\pi}{\sqrt{\frac{500}{50}}}$$

or t = **11.922 s** ...Ans.

Ex. 3.19.2 : A horizontal spring mass system with coulomb damping has a mass of 5 kg attached to a spring of stiffness 980 N/m. If the coefficient of friction is 0.25, calculate :
(i) the frequency of free oscillations;
(ii) the number of cycles corresponding to 50% reduction in amplitude if the initial amplitude is 5 cm; and
(iii) the time taken to achieve this 50% reduction

SPPU - May 10, Dec. 16, Dec. 17, 6 Marks

Soln. :

m = 5 kg ; K = 980 N/m ;
μ = 0.25 ; x_o = 5 cm = 0.05 m.

1. Frequency of Free Vibrations :

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{980}{5}}$$

or ω_n = **14 rad/s** ...Ans.

- **Natural frequency :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{14}{2\pi}$$

$$= 2.23 \text{ Hz}$$ ...Ans.

2. Number of Cycles for 50 % Reduction in Amplitude :

- **Loss of amplitude per cycle :**

$$\Delta = \frac{4F_r}{K} = \frac{4\mu R_N}{K} = \frac{4\mu m g}{K}$$

$$\Delta = \frac{4 \times 0.25 \times 5 \times 9.81}{980} = 0.05 \text{ m}$$

- **Amplitude at 50% of the initial amplitude :**

$$x_n = 0.5 \times x_o = 0.5 \times 5$$

$$= 2.5 \text{ cm} = 0.025 \text{ m}$$

- **Number of cycles corresponding to 50% reduction in amplitude :**

We know that,

$$x_n = x_o - n\Delta$$

$$\therefore \; 0.025 = 0.05 - n \times 0.05$$

n = **0.5 cycles** ...Ans.

3. Time Taken to Achieve 50% Reduction :

$$t = n\frac{2\pi}{\omega_n} = 0.5 \times \frac{2\pi}{14}$$

or t = **0.22 s** ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex. 3.19.3 : A horizontal spring mass system with Coulomb damping has a mass of 0.5 kg attached to a spring of stiffness 980 N/m. If the coefficient of friction is 0.025, calculate
(i) The number of cycles corresponding to 50 % reduction in amplitude, if the initial amplitude is 50 mm.
(ii) The time taken to achieve this 50 % reduction.

SPPU - May 16, 6 Marks

Ex. 3.19.4 : A mass placed on rough surface is attached to a spring and is given an initial displacement of 120 mm from its equilibrium position. After completing 8 cycles of oscillations in 2 sec, the final position of mass is found to be 10 mm from its equilibrium position. Find the coefficient of friction between the surface and mass. **SPPU - May 12, 6 Marks**

Soln. :

$$x_0 = 120 \text{ mm} = 0.12 \text{ m};$$
$$x_n = 10 \text{ mm} = 0.01 \text{ m};$$
$$n = 8.$$

- **Time period for oscillation :**

$$t_p = \frac{2}{8} = 0.25 \text{ s}$$

- **Value of K/m :**

$$\omega_n = \frac{2\pi}{t_p}$$

$$\text{But } \omega_n = \sqrt{\frac{K}{m}}$$

$$\therefore \sqrt{\frac{K}{m}} = \frac{2\pi}{t_p}$$

$$\sqrt{\frac{K}{m}} = \frac{2\pi}{0.25}$$

$$\sqrt{\frac{K}{m}} = 25.13 \text{ rad/s}$$

$$\text{or } \frac{K}{m} = 631.51$$

- **Coefficient of friction :**

Mass will stops at 10 mm from, its equilibrium position (i.e. $x_n = 10$ mm), and it will take 8 cycles. Therefore,

$$x_n = x_0 - n\frac{4F_r}{K}$$

$$0.01 = 0.12 - (8)\frac{4F_r}{K} = 0.12 - \frac{32 \times \mu R_n}{K}$$

$$\therefore 0.01 = 0.12 - \frac{32 \times \mu mg}{K} \quad ...[\because R_n = W = mg]$$

$$0.01 = 0.12 - \frac{32 \times \mu g}{K/m}$$

$$= 0.12 - \frac{32 \times \mu \times 9.81}{631.51}$$

$$\therefore \mu = \mathbf{0.2213} \quad \text{...Ans.}$$

Ex. 3.19.5 : A body of mass 100 kg is suspended on a leaf spring. The system is then made to vibrate and its natural frequency measured is 7 rad/s. It is observed that if the initial amplitude is 48 mm, the subsequent amplitudes are 32 mm and 20 mm. Determine spring stiffness and coulomb damping force. **SPPU - Dec. 15, 6 Marks**

Soln. :

Given : $m = 100$ kg ; $\omega_n = 7$ rad/s;

$x_0 = 48$ mm ; $x_{1/2} = 32$ mm;

$x_1 = 20$ mm.

1. Stiffness of Spring :

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\therefore 7 = \sqrt{\frac{K}{100}}$$

$$\therefore K = \mathbf{4900 \text{ N/m}}$$

$$= \mathbf{4.9 \text{ N/mm}} \quad \text{...Ans.}$$

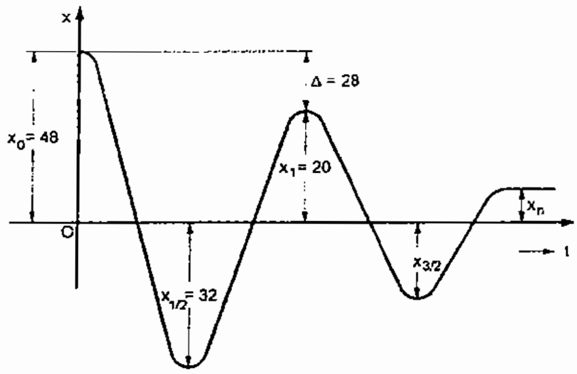


Fig. P. 3.19.5

2. Damping Force :

The loss of amplitude per half cycle is,

$$\frac{\Delta}{2} = x_0 - x_{1/2} = 48 - 32 = 16 \text{ mm}$$

The loss of amplitude per cycle is,

$$\Delta = x_0 - x_1 = 48 - 20 = 28 \text{ mm}$$

$$\text{Therefore} \quad \Delta = \frac{4F_r}{K}$$

$$\therefore 28 = \frac{4 \times F_r}{4.9}$$

$$F_r = \mathbf{34.3 \text{ N}} \quad \text{...Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 3.19.6 : A mass of 500 kg is suspended with a spring. The system vibrates with a natural frequency of 3 rad/s. If the initial amplitude is 24 mm and subsequent half amplitudes are 20 and 16 mm, determine the stiffness of spring and coulomb damping force. Also find the number of cycles corresponding to 50% reduction of its initial amplitude.

Exercise

1. What are the various methods of vibration analysis ?
2. Explain Equilibrium method to find the natural frequency of vibratory system?
3. Explain energy method for finding the natural frequency of vibratory system?
4. Explain Rayleigh's method for finding the natural frequency of vibratory system.
5. Find the natural frequency of the system shown in Fig Q. 5.

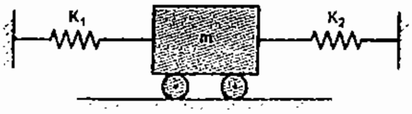


Fig. Q. 5

$$\left(\text{Ans. : } \frac{1}{2\pi}\sqrt{\frac{K_1 + K_2}{m}}\right)$$

6. Find the natural frequency of the system shown in Fig. Q. 6.

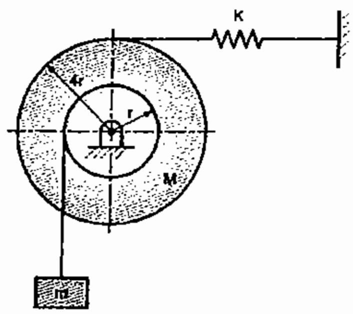


Fig. Q. 6

$$\left(\text{Ans. : } f_n = \frac{1}{2\pi}\sqrt{\frac{16\,K}{m + 8m}}\right)$$

7. Find the natural frequency of the system shown in Fig. Q. 7.

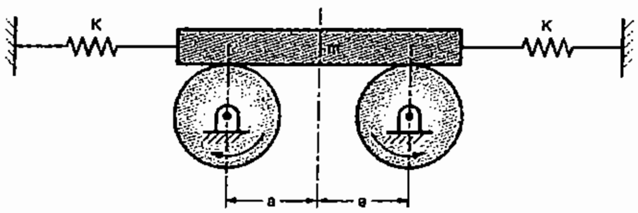


Fig. Q. 7

$$\left(\text{Ans. : } f_n = \frac{1}{2\pi}\sqrt{\frac{2\,\mu\,g\,K}{a}}\right)$$

8. Find the equation of motion of the uniform rigid bar of length l and having mass 'm' as shown in Fig. Q. 8. Also find the natural frequency of the system

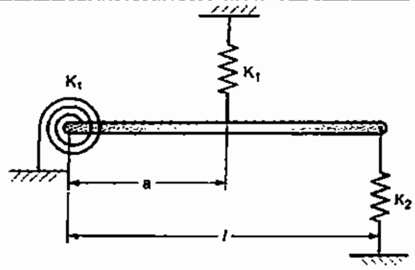


Fig. Q. 8

$$\left(\text{Ans. : } f_n = \frac{1}{2\pi}\sqrt{\frac{K_T + K_1\,a^2 + K_2 l^2}{\frac{1}{2} m\, l^2}}\right)$$

9. The inclined manometer shown in Fig. Q. 9 is used to measure pressure. If the total length of mercury in the tube is l, find an expression for the natural frequency of oscillation of the mercury.

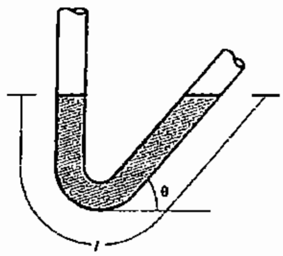


Fig. Q. 9

$$\left(\text{Ans. : } f_n = \frac{1}{2\pi}\sqrt{\frac{2\,g}{l}}\right)$$

10. What is vibration ? Explain the phenomenon of vibration.
11. What are the various causes of vibrations? How the effects of undesirable vibrations can be reduced ?
12. What are the advantages and disadvantages of vibrations ?
13. Define the following terms :

    (i) Time period (ii) Simple harmonic motion

    (iii) Frequency (iv) Amplitude
14. What do you understand by degrees of freedom ?
15. What are the basic elements of a vibratory system ?
16. Explain the concept of modeling of a system, with example.
17. Explain, with neat sketches, the following terms :

    (i) Equivalent spring (ii) Equivalent damper

18. What are the various types of vibrations ?

19. Distinguish between longitudinal, transverse and torsional vibration.

20. What do you mean by damping ? What are the various type of damping ?

21. Explain dry friction or coulomb damping.

22. Derive an expression for the displacement of spring-mass-damper system in case of

(i) over-damped system

(ii) under-damped system

(iii) critically damped system

23. What is "over-damping" and "under-damping" of a system ?

24. Define logarithmic decrement and derive an expression for it ?

25. Define the following terms :

(i) Damping factor

(ii) Damping coefficient

(iii) Critical damping coefficient

26. Find the natural frequency of damped vibrations of the system shown in Fig. Q. 26. Neglect the mass of rod.

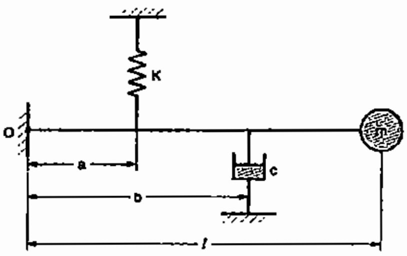


Fig. Q. 26

$$\left(\textbf{Ans. : } f_d = \frac{a}{2\pi l}\sqrt{\frac{k}{m} - \frac{b^4 c^2}{4 l^2 m^2 a^2}}\right)$$

27. A weightless rigid rod of length '4a' is provided at one end, and supported on a spring of 980 N/m at other end as, shown in Fig. Q. 27. At the mid of rod, the mass of 5 kg is attached. If the damper having damping coefficient of 98 N.-s/m is placed between the mid of mass and spring, find :

(i) the damped natural frequency of the system.

(ii) the critical damping coefficient of damper.

(iii) the logarithmic decrement.

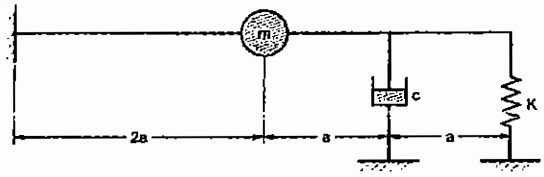


Fig. Q. 27

(**Ans. :** f_d = 2.75 Hz, c_c = 0.127, δ = 8.05)

28. A typical spring mass-damper system is having a mass of 10 kg, spring of stiffness 1000 N/m and damping coefficient of 150 N-s/m. Determine :

(i) the damping factor ; and

(ii) the circular damped frequency.

(**Ans. :** ξ = 0.75, ω_d = 6.62 rad / s)

29. A mass of 20 kg is displaced by 50 mm from its mean position then it subjected a coulombs friction of 50 N. A spring of stiffness 1000 N/m is attached to the mass. Determine ;

(i) the number of half cycles elapsed before the mass comes to rest ;

(ii) the time elapsed before the mass comes to rest ; and

(iii) the final extension of the spring.

(**Ans. :** n = 5, t_p = 0.7025, x = 19.60 mm)

30. A shock absorber is to be designed so that its overshoot is 10% of the initial displacement when released. Determine the logarithmic decrement and damping factor. If the damping factor is reduced to one half of this value, what will be the overshoot ?

(**Ans. :** δ = 2.29, ξ = 0.3435, 33.45 %)

31. The disc of a torsional pendulum has a mass moment of inertia of 0.068 kg.m^2 and is immersed in a viscous fluid. It is supported by a shaft having diameter of 1cm and length of 38 cm. The modulus of rigidity of shaft material is 40 GN/m^2. When pendulum is under oscillation, the amplitudes on the same side of the rest position for successive cycles are 5°, 3° and 1.8°. Determine :

(i) the logarithmic decrement

(ii) the damping coefficient at unit velocity ;

(iii) periodic time of the vibration ;

(iv) what would be the frequency of the vibrations, if the disc is removed from the viscous fluid ?

(**Ans. :** δ =0.511, c_t = 0.431 N–m / s / rad, t_p = 0.1619 s, f_n = 6.20 Hz)

32. A gun barrel of mass 545 kg has a recoil spring of stiffness 297000 N/m. The barrel recoils 1.2 m on firing. Determine.

(i) the initial recoil velocity of the gun barrel ;

(ii) the critical damping coefficient of dashpot which is engaged at the end of there coil stroke.

(**Ans. :** $v = 28.01 \text{m/s}$, $c_c = 25445.2 \text{ N s/m}$)

□□□

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UNIT 4

4

SINGLE DEGREE OF FREEDOM SYSTEMS : FORCED VIBRATIONS

Syllabus

Forced vibrations of Longitudinal and Torsional Systems, Frequency Response to Harmonic Excitation, Excitation due to Rotating and Reciprocating Unbalance, Base Excitation, Magnification Factor, Force and Motion Transmissibility, Quality Factor, Half Power Bandwidth Method, Critical Speed of Shaft Having Single Rotor of Undamped Systems.

TOPICS

4.1 INTRODUCTION TO FORCED VIBRATIONS

- **Forced vibrations :** *If the system vibrates under the influence of external periodic force, the vibrations are known as* **forced vibrations.**
- **Examples of forced Vibrations :** Vibrations of air compressors, I.C. engines, turbines, pumps, etc.
- **Source of external periodic force :** All the rotating machines like : compressors, pumps, turbines etc. are, though, carefully designed and manufactured, still there are some unbalanced forces acting on such machines which act as harmonic excitation forces.
- Resonance : When frequency of external excitation force coincides with the natural frequency of a system, it results in resonance and the system starts vibrating dangerously with the large amplitude.
- **Remedies to avoid resonance :** In order to avoid the resonance either the operational speed is changed, or dimensions are changed to change its natural frequency. In addition, sufficient amount of damping may be provided to keep the vibrating amplitude as low as possible and components are designed to withstand the dynamic loads.

4.2 FORCED DAMPED VIBRATIONS WITH CONSTANT HARMONIC EXCITATION

University Question

Q. Derive an expression for magnification factor for steady state amplitude of vibration subjected to external excitation F sin ωt. **SPPU : May 14, May 16**

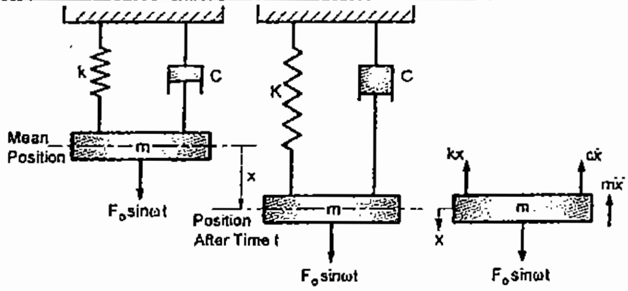
Spring-Mass-Damper System Excited by Harmonic Force :

- Consider the system having spring-mass-damper system excited by a harmonic force $\mathbf{F_0 \sin \omega t}$, as shown in Fig. 4.2.1.

where, F_o = magnitude of external exciting harmonic force, N

ω = circular frequency of external exciting force, rad/s

- **Forces acting on mass in displaced position [Fig. 4.2.1] :**
  (i) External harmonic force, $F_o \sin \omega t$, (downwards)
  (ii) Inertia force, $m\ddot{x}$ (upwards)
  (iii) Damping force, $\dot{x}$ (upwards)
  (iv) Spring force, Kx (upwards)



(a) Equilibrium Position (b) Displaced Position (c) F.B.D. of Mass

Fig. 4.2.1

Differential Equation of Motion for Forced Damped Vibrations :

Consider F.B.D. of mass, shown in Fig. 4.2.1(c). According to D' Alembert's principle,

Σ [Inertia force + External force] = 0

$$m\ddot{x} + c\dot{x} + Kx - F_o \sin \omega t = 0$$

$$m\ddot{x} + c\dot{x} + Kx = F_o \sin \omega t \quad ...(4.2.1)$$

Solution of Differential Equation :

- The complete solution to the linear, second order differential Equation (4.2.1) is

$$x = x_c + x_p \quad ...(4.2.2)$$

- The solution of this equation consist of two parts :

1. Complementary Function (x_c)
2. Particular Integral (x_p)

4.2.1 Complementary Function (X_c) :

- The first part is complementary function and is obtained by considering no force condition i.e. considering Equation (4.2.1) as,

$$m\ddot{x} + c\dot{x} + kx = 0 \quad ...(a)$$

- This Equation (a) is same as the equation obtained for damped free vibration system. The solution of Equation (a) is given by,

$$x_c = X_1 e^{-\xi \omega_n t} \cdot \sin [\omega_d t + \phi_1]$$

or
$$x_c = X_1 e^{-\xi \omega_n t} \cdot \sin [\sqrt{1-\xi^2}\, \omega_n t + \phi_1] \quad ...(4.2.3)$$

where, x_c = complementary function of x, m

ξ = damping ratio or Damping factor

ω_d = natural circular frequency of damped free vibrations, rad/s

ω_n = natural circular frequency of undamped free vibrations, rad/s t = time, s

X_1, ϕ_1 = constants, to be determined from initial conditions

4.2.2 Particular Integral (x_p) :

The second part is the particular integral, which is of form,

$$x_p = X \sin (\omega t - \phi)$$

The particular integral can be obtained by following two methods :

1. Analytical Method (Differential Equation Method)
2. Graphical Method

1. Analytical Method (Differential Equation Method) :

- **Particular integral (x_p) :**

$$x_p = X \sin (\omega t - \phi) \quad ...(4.2.4)$$

where, x_p = particular integral, m

X = amplitude of steady state vibrations of the system, m

ϕ = phase angle or phase difference. It is the angle by which the displacement vector $\vec{X}$ lags the force vector $\vec{F_0}$ [Fig. 4.2.2].

ω = circular frequency of external exciting harmonic force, rad/s

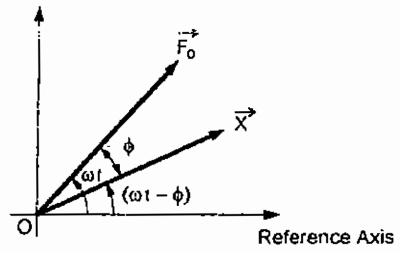


Fig. 4.2.2 : Phase Angle or Phase Difference

- **Velocity ($\dot{x}_p$) :**

$$\frac{dx_p}{dt} = \frac{d}{dt}[X \sin (\omega t - \phi)]$$

$$\therefore \dot{x}_p = \omega X \cos (\omega t - \phi) \quad ...(b)$$

or $$\dot{x}_p = \omega X \sin \left(\omega t - \phi + \frac{\pi}{2}\right)$$

- **Acceleration ($\ddot{x}_p$) :**

$$\frac{d^2x_p}{dt^2} = \frac{d}{dt}[\omega X \cos (\omega t - \phi)]$$

$$\ddot{x}_p = -\omega^2 X \sin (\omega t - \phi)$$

or $$\ddot{x}_p = \omega^2 X \sin (\omega t - \phi + \pi) \quad ...(c)$$

- **Substitution of x_p, $\dot{x}_p$ and $\ddot{x}_p$ in Equation (4.2.1) :**

Substituting Equations (4.2.4), (b) and (c) in Equation (4.2.1), we get,

$$m \omega^2 X \sin (\omega t - \phi + \pi) + c \omega X \sin \left(\omega t - \phi + \frac{\pi}{2}\right) + K X \sin (\omega t - \phi) = F_0 \sin \omega t \quad ...(d)$$

$$\therefore -m \omega^2 X \sin (\omega t - \phi) + c\omega X \cos (\omega t - \phi) + KX \sin (\omega t - \phi) = F_0 \sin \omega t$$

$$(K - m\omega^2) X \sin (\omega t - \phi) + c \omega X \cos (\omega t - \phi) = F_0 \sin \omega t$$

$$(K - m\omega^2) X [\sin \omega t \cdot \cos \phi - \cos \omega t \times \sin \phi] + c \omega X [\cos \omega t \cdot \cos \phi + \sin \omega t \cdot \sin \phi] = F_0 \sin \omega t$$

or $$[(K - m\omega^2) X \cos \phi + c \omega X \sin \phi] \sin \omega t + [c\omega X \cos \phi - (K - m\omega^2) X \sin \phi] \cos \omega t = F_0 \sin \omega t$$

Comparing the coefficients of **sinωt** and **cosωt** on left hand and right hand side separately, we get,

$$(K - m\omega^2) X \cos \phi + c \omega X \sin \phi = F_0 \quad ...(e)$$

and $$\mathbf{c \omega X \cos \phi - (K - m\omega^2) X \sin \phi = 0} \quad ...(f)$$

- **Phase angle (ϕ) :**

From Equation (f),

$$c \omega X \cos \phi = (K - m\omega^2) X \sin \phi$$

$$\therefore \frac{\sin \phi}{\cos \phi} = \frac{c\omega X}{(K - m\omega^2) X}$$

or $$\tan \phi = \frac{c \omega}{K - m\omega^2} \quad ...(g)$$

$$\therefore \phi = \tan^{-1}\left[\frac{c \omega}{K - m\omega^2}\right] \quad ...(h)$$

or $$\phi = \tan^{-1}\left[\frac{c\omega / K}{1 - \frac{m\omega^2}{K}}\right] \quad ...(i)$$

Now, $$\frac{m\omega^2}{K} = \frac{\omega^2}{K/m} = \frac{\omega^2}{\omega_n^2} = \left(\frac{\omega}{\omega_n}\right)^2$$

or $$\frac{m\omega^2}{K} = \left(\frac{\omega}{\omega_n}\right)^2 \quad ...(j)$$

and $$\frac{c\omega}{K} = \frac{c}{c_c} \cdot \frac{c_c\,\omega}{K} = \frac{2\xi \cdot \sqrt{Km} \cdot \omega}{K}$$

$$= 2\xi \cdot \sqrt{m/K} \cdot \omega = \frac{2\xi\omega}{\sqrt{K/m}}$$

or $$\frac{c\omega}{K} = 2\xi\frac{\omega}{\omega_n} \quad ...(k)$$

Substituting Equations (j) and (k) in Equation (i), we get,

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \quad ...(4.2.5)$$

- **Amplitude of vibration (X) :**

From Equation (e),

$$[(K - m\omega^2)\cos\phi + c\omega\sin\phi]\,X = F_0$$

$$\therefore \quad X = \frac{F_0}{(K - m\omega^2)\cos\phi + c\omega\sin\phi} \quad ...(4.2.6)$$

The Equation (g) can be represented in the vector form, as shown in Fig. 4.2.3.

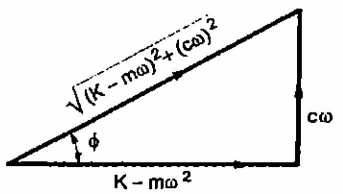


Fig. 4.2.3

From Fig. 4.2.3,

$$\sin\phi = \frac{c\omega}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \quad ...(l)$$

and $$\cos\phi = \frac{K - m\omega^2}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \quad ...(m)$$

Substituting values of $\sin\phi$ and $\cos\phi$ from Equations (*l*) and (m) in Equation (4.2.6),

$$X = \frac{F_0}{\frac{(K - m\omega^2)(K - m\omega^2)}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} + \frac{c\omega \cdot c\omega}{\sqrt{(K - m\omega^2)^2 + (K - m\omega^2)^2}}}$$

$$= \frac{F_0}{\frac{(K - m\omega^2)^2 + (c\omega)^2}{\sqrt{(K - m\omega^2) + (c\omega)^2}}} = \frac{F_0\sqrt{(K - m\omega^2)^2 + (c\omega)^2}}{(K + m\omega^2)^2 + (c\omega)^2}$$

or $$X = \frac{F_0}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \quad ...(n)$$

Again, $$X = \frac{F_0 / K}{\sqrt{\left(1 - \frac{m\omega^2}{K}\right)^2 + \left(\frac{c\omega}{K}\right)^2}} \quad ...(o)$$

Substituting Equations (j) and (k) in Equation (o), amplitude of vibrations is,

$$X = \frac{F_0 / K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad ...(4.2.7)$$

$$X = \frac{X_{st}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad ...(4.2.8)$$

where, $X_{st} = \frac{F_0}{K}$ = deflection due to force F_0 or zero frequency deflection or static deflection

- **Phase Angle (ϕ) :**

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

- **Particular integral (x_p) :**

$$x_p = X\sin(\omega t - \phi)$$

or $$x_p = \frac{F_0\sin(\omega - \phi)}{K\left[\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2\left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}\right]} \quad ...(4.2.9)$$

where, $$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

2. Graphical method :

- **Particular integral (x_p) :**

$$x_p = X\sin(\omega t - \phi) \quad ...(4.2.10)$$

- **Velocity ($\dot{x}_p$) :**

$$\frac{dx_p}{dt} = \frac{d}{dt}[X\sin(\omega t - \phi)]$$

$$\therefore \quad \dot{x}_p = \omega X\cos(\omega t - \phi)$$

or $$\dot{x}_p = \omega X\sin\left(\omega t - \phi + \frac{\pi}{2}\right) \quad ...(p)$$

- **Acceleration ($\ddot{x}_p$) :**

$$\frac{d^2x_p}{dt^2} = \frac{d}{dt}[\omega X\cos(\omega t - \phi)]$$

$$\therefore \quad \ddot{x}_p = -\omega^2 X\sin(\omega t - \phi)$$

or $$\ddot{x}_p = \omega^2 X\sin(\omega t - \phi + \pi) \quad ...(q)$$

- **Substitution of x_p, $\dot{x}_p$ and $\ddot{x}_p$ in Equation (4.2.1) :**

  Substituting Equations (4.2.10), (p) and (q) in Equation (4.2.1), we get,

$$m\,\omega^2 X \sin(\omega t - \phi + \pi) + c\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + K X \sin(\omega t - \phi) = F_0 \sin \omega t$$

$$-m\omega^2 X \sin(\omega t - \phi) + c\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right) + K X \sin(\omega t - \phi) = F_0 \sin \omega t$$

$$\therefore F_0 \sin \omega t + m\omega^2 X \sin(\omega t - \phi) - c\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right) - K X \sin(\omega t - \phi) = 0 \quad ...(4.2.11)$$

- **Forces acting on body :**

(i) External harmonic force [$+ F_0 \sin \omega t$]

(ii) Inertia force [$+ m\omega^2 X \sin(\omega t - \phi)$]

(iii) Damping force [$- c\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right)$]

(iv) Spring force [$- K X \sin(\omega t - \phi)$]

The vector sum of these forces is zero. Fig. 4.2.4(a) shows the representation of these forces and Fig. 4.2.4(b) shows the vector addition of these four forces.

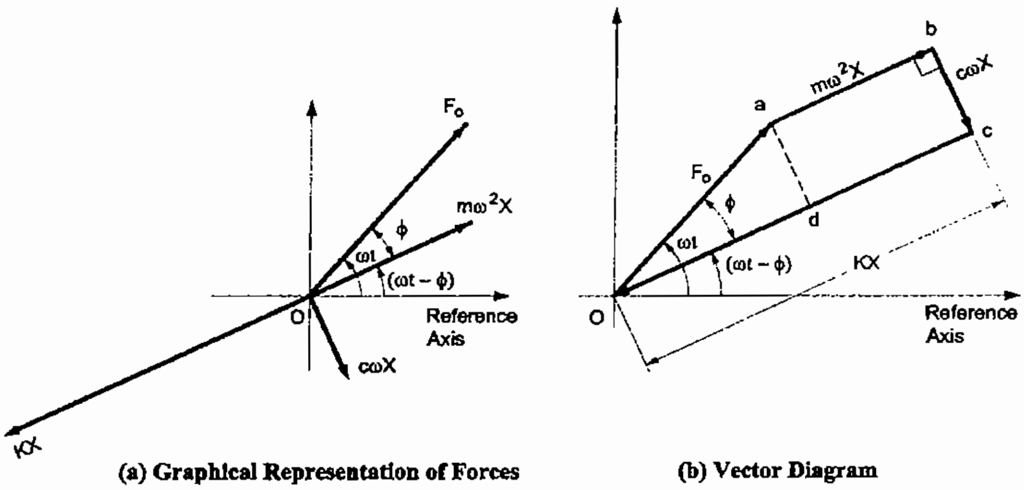


Fig. 4.2.4 : Graphical Method

- **Phase Angle (ϕ) :**

  From Fig. 4.2.4(b) ;

$$\tan \phi = \frac{ad}{od} = \frac{bc}{oc - dc} = \frac{bc}{oc - ab} = \frac{c\,\omega X}{KX - m\omega^2 X} = \frac{c\,\omega / K}{1 - \frac{m\omega^2}{K}}$$

or

$$\tan \phi = \left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore \quad \phi = \tan^{-1}\left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \quad ...(4.2.12)$$

- **Amplitude of vibration (X) :**

  Again from Fig. 4.2.4(b),

$$oa = \sqrt{(od)^2 + (ad)^2}$$

$$F_0 = \sqrt{[KX - m\omega^2 X]^2 + [c\omega X]^2}$$

or $$F_0 = X\sqrt{[K - m\omega^2]^2 + [c\omega]^2}$$

$$\therefore \quad X = \frac{F_0}{K\sqrt{\left[1 - \frac{m\omega^2}{K}\right]^2 + \left[\frac{c\omega}{K}\right]^2}}$$

or $$X = \frac{F_0/K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad ...(4.2.13)$$

- **Particular integral (x_p) :**

$$x_p = X \sin(\omega t - \phi)$$

or $$x_p = \frac{F_0 \sin(\omega t - \phi)}{K\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

where, $$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] \quad ...(4.2.14)$$

4.2.3 Complete Solution for Differential Equation :

University Question

Q. Explain the terms : transient vibration and steady state vibration, related to forced vibration. **SPPU : May 13**

Two Parts of Complete Solution :

(i) Transient Vibrations : The first part of the complete solution i.e. complementary function, is seen to decay with time and vanishes ultimately. This part is called as **transient vibrations**. The transient vibrations take place at the damped frequency of the system (ω_d), as shown in Fig. 4.2.5(a). The complementary function is given by,

$$x_c = X_1 e^{-\xi\omega_n t} \cdot \sin\left[\sqrt{1-\xi^2}\,\omega_n t + \phi_1\right] \quad ...(4.2.15)$$

(ii) Steady State Vibrations : The second part of complete solution i.e. particular integral is seen to be a sinusoidal vibration with a constant amplitude and is called as steady state vibrations. The **steady state vibrations** takes place at the frequency of excitation (i.e. at ω), as shown in Fig. 4.2.5(b). The particular integral is given by,

$$x_p = \frac{F_0 \sin(\omega t - \phi)}{K\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad ...(4.2.16)$$

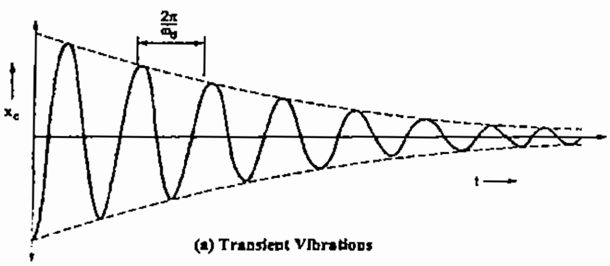
Complete Solution for Differential Equation :

- The complete solution to an underdamped system subjected to sinusoidal excitation is given by,

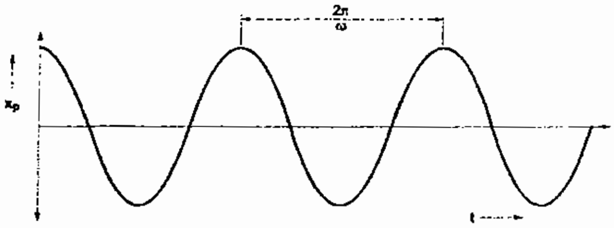
$$x = x_c + x_p$$

$$\therefore \quad x = X_1 e^{-\xi\omega_n t} \cdot \sin\left[\sqrt{1-\xi^2}\,\omega_n t + \phi_1\right] + \frac{F_0 \sin(\omega t - \phi)}{K\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad ...(4.2.17)$$

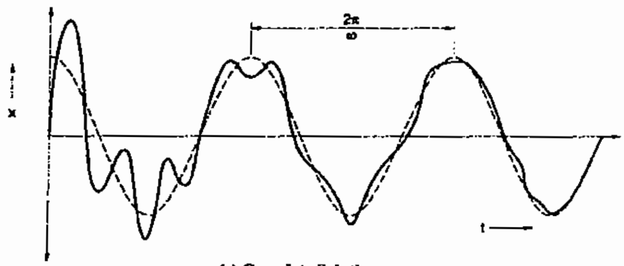
- The complete solution is obtained by superposition of transient and steady state vibrations, as shown in Fig. 4.2.5(c).



(a) Transient Vibrations



(b) Steady State Vibrations



(c) Complete Solution

Fig. 4.2.5 : Forced Damped Vibrations with Constant Harmonic Excitation

- After the transient vibrations die out, the complete solution consist of steady state vibrations only.
- The two constants X_1 and ϕ_1 can be determined by applying the initial conditions to the complete solution given by Equation (4.2.17).

4.2.4 Magnification Factor (MF) :

University Questions

Q. Explain the term magnification factor and obtain expression for it. **SPPU : Dec. 12**

Q. Write short note on : Magnification factor. **SPPU : Dec. 13**

Magnification factor, amplification factor, amplitude ratio or dynamic magnifier *is defined as the ratio of the amplitude of steady state vibrations 'X' to the zero frequency deflection 'X_{st}'.* (deflection due to force F_0). It is denoted by **M.F.**

$$\text{M.F.} = \frac{\text{Amplitude of steady state vibration}}{\text{Zero frequency deflection}}$$

$$\therefore \quad \text{M.F.} = \frac{X}{X_{st}}$$

$$= \frac{F_0/K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \times \frac{1}{F_0/K}$$

or $$\textbf{M. F.} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...(4.2.18)}$$

$$\therefore \quad X = X_{st} \cdot \text{M. F.} \quad \text{...(4.2.19)}$$

4.2.5 Frequency Response Curves :

University Questions

Q. Explain frequency response curve and phase frequency curve. **SPPU : Dec. 12, May 16**

Q. Plot the frequency response curves and draw any four conclusions from the same. **SPPU : May 15**

Q. Plot magnification factor versus frequency ratio curve for different damping conditions and write concluding remarks. **SPPU : Dec. 15**

Q. Explain with neat sketch, effect of damping on magnification factor for different forcing frequencies and hence justify that dynamic systems are to be operated at high speed as is possible. **SPPU : May 18**

Q. Explain frequency response curve with neat labelled diagram. **SPPU : Dec. 18, Oct. 19(In Sem.), Dec. 19**

Q. Discuss the response curve with frequency ratio. **SPPU : May 19**

- **Frequency response curve :** The plot of magnification factor (**M.F.**) versus frequency ratio (ω / ω_n) is known as **frequency response curve** (Fig. 4.2.6).

- Frequency response curves is plotted using Equation (4.2.18).

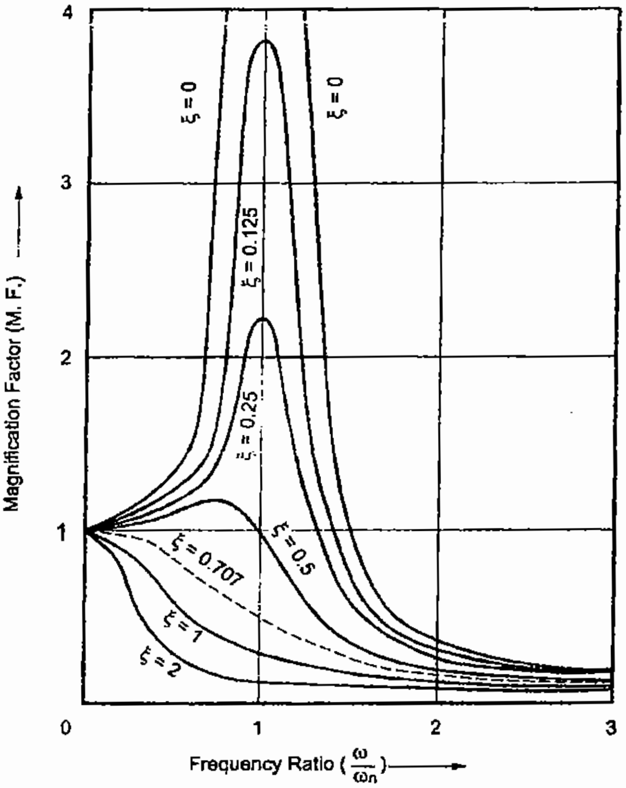


Fig. 4.2.6 : Frequency Response Curve For Different Damping Conditions

- **Observations made from frequency response curve :**

(i) The magnification factor (M.F.) is maximum when $(\omega/\omega_n) = 1$. This condition is known as **resonance**

(ii) As the damping factor (ξ) decreases, the maximum value of magnification factor (M.F.) increases.

(iii) When there is no damping ($\xi = 0$), the magnification factor (M.F.) reaches infinity at $(\omega/\omega_n) = 1$. However, the system may go into destruction much before that.

(iv) At zero frequency of excitation (i.e. at $\omega = 0$) the magnification factor (M.F.) is unity for all values of damping factors. In other words, the damping does not have any effect on magnification factor at zero frequency of excitation.

(v) At very high frequency of excitation, the magnification factor (M.F.) tends to zero.

(vi) For damping factor (ξ) more than 0.707, the magnification factor is below unity.

4.2.6 Plot of Phase Angle (ϕ) Versus Frequency Ratio (ω / ω_n) :

University Question

Q. Discuss the response curve and variation of phase angle with frequency ratio. **SPPU : May 19**

- The plot of phase angle (ϕ) versus frequency ratio (ω/ω_n) for different damping conditions is shown in Fig. 4.2.7. These curves are plotted using Equation (4.2.14).

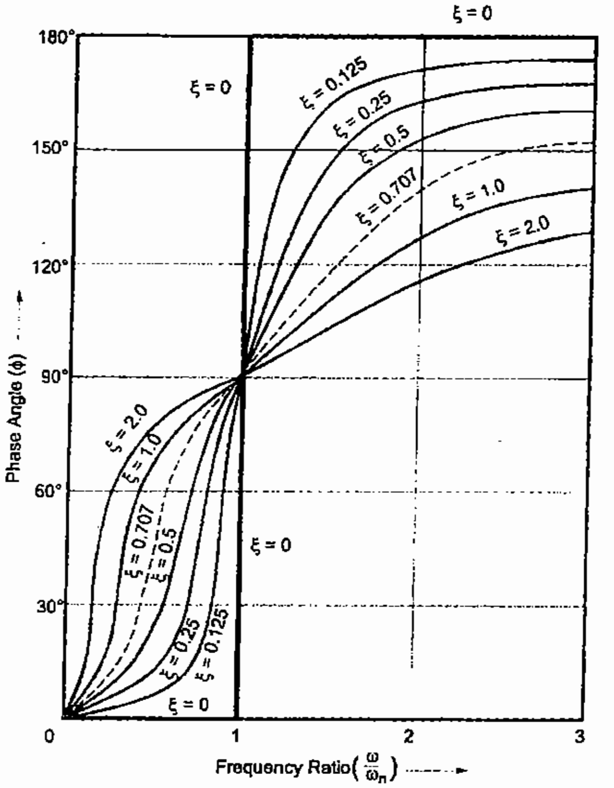


Fig. 4.2.7 : Phase Angle Versus Frequency Ratio for Different Damping Conditions

- **Observations made from phase angle vs frequency ratio curves :**
  (i) The phase angle varies from 0° at low frequency ratio to 180° at very high frequency ratio.
  (ii) At resonance frequency (i.e. at $\omega = \omega_n$) the phase angle is 90° and damping does not have any effect on phase angle.
  (iii) At frequency ratio (ω / ω_n) less than unity, higher the damping factor-higher is the phase angle; whereas at frequency (ω / ω_n) greater than unity, higher the damping factor-lower is the phase angle. In other words, below resonance frequency, the phase angle increases with increase in damping factor; whereas above resonance frequency, the phase angle decreases with increase in damping factor.
  (iv) The variation in phase angle is because of damping. If there is no damping, ($\xi = 0$) the phase angle is either 0° or 180° and at resonance the phase angle suddenly changes from 0° to 180°.

Ex. 4.2.1 : A spring-mass damper system has a mass of 80 kg suspended from a spring having stiffness of 1000 N/m. and a viscous damper with a damping coefficient of 80 N-s/m. If the mass is subjected to a periodic disturbing force of 50 N at undamped natural frequency, determine :

(1) the undamped natural frequency ;

(2) the damped natural frequency ;

(3) the amplitude of forced vibrations of mass ; and

(4) the phase difference between force and displacement.

Soln. :

Given : $m = 80$ kg ; $K = 1000$ N/m ;
$c = 80$ N-sec/m ; $F_0 = 50$ N ;
$\omega = \omega_n$.

1. Undamped Natural Circular Frequency (ω_n) :

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1000}{80}}$$

or $\omega_n = 3.53$ rad/s ...Ans.

2. Damped Natural Frequency (ω_d) :

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n} = \frac{80}{2 \times 80 \times 3.53}$$

or $\xi = 0.1416$

- **Damped natural circular frequency :**

$$\omega_d = \omega_n \sqrt{1 - \xi^2} = 3.53\sqrt{1 - (0.1416)^2}$$

or $\omega_d = 3.49$ rad/s ...Ans.

3. Amplitude of Forced Vibrations (X) :

$$X = \frac{F_0/K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} = \frac{50 / 1000}{\sqrt{[1 - (1)^2]^2 + [2 \times 0.1416 \times 1]^2}}$$

or $X = 0.1765$ m ...Ans.

4. Phase Difference (ϕ) :

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left[\frac{2\times 0.1416\times 1}{1-(1)^2}\right]$$

$$= \tan^{-1}\left[\frac{0.2832}{0}\right] = \tan^{-1}(\infty)$$

or $\phi = 90°$ **...Ans.**

Ex. 4.2.2 : A spring-mass-damper systems is subjected to a harmonic force. The amplitude is found to be 0.02 m at resonance and 0.01 m at a frequency 0.75 times the resonant frequency. Find the damping ratio of the system.

SPPU - Aug. 15 (In Sem), Oct. 16 (In Sem), 4 Marks

Soln. :

Given : $X = 0.02$ m at $\omega = \omega_n$

and $X = 0.01$ m at $\omega = 0.75\,\omega_n$

$$X = \frac{F_0/K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\therefore X = \frac{X_{st}}{\sqrt{[1-r^2]^2+[2\xi r]^2}} \qquad ...(a)$$

where, $r = \frac{\omega}{\omega_n}$

1. At $\omega = \omega_n$:

$$r = \frac{\omega}{\omega_n} = 1$$

$\therefore$ Equation (a) becomes,

$$X = \frac{X_{st}}{2\xi}$$

$$\therefore 0.02 = \frac{X_{st}}{2\xi} \qquad ...(b)$$

2. At $\omega = 0.75\,\omega_n$:

$$\therefore r = \frac{\omega}{\omega_n} = 0.75$$

$\therefore$ Equation (a) becomes,

$$X = \frac{X_{st}}{\sqrt{[1-r^2]^2+[2\xi r]^2}}$$

$$0.01 = \frac{X_{st}}{\sqrt{[1-(0.75)^2]^2+[2\times\xi\times 0.75]^2}}$$

$$0.01 = \frac{X_{st}}{\sqrt{0.191+2.25\xi^2}} \qquad ...(c)$$

3. Damping ratio (ξ) :

Dividing equation (b) by equation (c) ;

$$\frac{0.02}{0.01} = \frac{X_{st}}{2\xi}\times\frac{\sqrt{0.191+2.25\xi^2}}{X_{st}}$$

$$\therefore 4\xi = \sqrt{0.191+2.25\xi^2}$$

$$16\xi^2 = 0.191 + 2.25\,\xi^2$$

$$13.75\,\xi^2 = 0.1914$$

$$\xi = \mathbf{0.118} \qquad \textbf{...Ans.}$$

Ex. 4.2.3 : In a vibrating system, a mass of 3 kg is suspended by a spring of stiffness 1200 N/m and it is subjected to a harmonic excitation of 20 N. If the viscous damper is provided with the damping coefficient of 75 N-s/m, determine :

(1) the resonance frequency ;
(2) the phase angle at resonance ;
(3) the amplitude at resonance ;
(4) the frequency corresponding to peak amplitude ; and
(5) the damped frequency. **SPPU - Dec. 07**

Soln. :

Given : $m = 3$ kg ; $K = 1200$ N/m ;
$F_0 = 20$ N ; $c = 75$ N-sec/m .

1. Resonance Frequency (ω_n) :

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1200}{3}} = 20 \text{ rad/s}$$

- **Resonant frequency :**

$$\omega = \omega_n = 20 \text{ rad / s} \qquad \textbf{...Ans.}$$

2. Phase Angle of Resonance (ϕ) :

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{2\xi\times 1}{1-(1)^2}\right] = \tan^{-1}[\infty]$$

or $\phi = 90°$ **...Ans.**

3. Amplitude of resonance (X) :

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

$$= \frac{75}{2\times 3\times 20} = 0.625$$

- **Amplitude at resonance :**

$$X = \frac{F_0 / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{20/1200}{\sqrt{[1-(1)^2]^2 + [2\times 0.625\times 1]^2}}$$

or $\mathbf{X = 0.0133\ m}$ ...Ans.

4. **Frequency Corresponding to Peak Amplitude :**

- The frequency at which the amplitude becomes maximum is called as **frequency of peak amplitude (ω_p)**.

- Now, $$X = \frac{F_0 / K}{\sqrt{\left[1-\left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega_p}{\omega_n}\right]^2}}$$

where, ω_p = frequency at peak amplitude, rad/s

$$X = \frac{X_{st}}{\sqrt{\left[1-\left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega_p}{\omega_n}\right]^2}}$$

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega_p}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega_p}{\omega_n}\right]^2}} \quad ...(a)$$

- Putting $\frac{\omega_p}{\omega_n}$ =a, in Equation (a), we get,

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{[1-a^2]^2 + [2\xi a]^2}} \quad ...(b)$$

For maximum amplitude,

$$\frac{d}{da}\left(\frac{X}{X_{st}}\right) = 0$$

$$\frac{d}{da}\left(\frac{X}{X_{st}}\right) = \frac{2(1-a^2)(-2a) + 2(2\xi a)(2\xi)}{2[(1-a^2)^2 + (2\xi a)^2]^{3/2}} = 0$$

$$\therefore \quad 0 = \frac{-4a + 4a^3 + 8a\xi^2}{2[(1-a^2)^2 + (2\xi a)^2]^{3/2}}$$

$$= -4a + 4a^3 + 8a\xi^2$$

$$0 = -1 + a^2 + 2\xi^2$$

$$\therefore \quad a^2 = 1 - 2\xi^2$$

$$\therefore \quad a = \sqrt{1-2\xi^2}$$

$$\therefore \quad \frac{\omega_p}{\omega_n} = \sqrt{1-2\xi^2}$$

$$\therefore \quad \omega_p = \omega_n\sqrt{1-2\xi^2} = 20\sqrt{1-2(0.625)^2}$$

or $\boldsymbol{\omega_p = 9.35\ rad/s}$ ...Ans.

5. **Damped Natural Circular Frequency :**

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 20\sqrt{1-(0.625)^2}$$

$\therefore$ $\boldsymbol{\omega_d = 15.61\ rad/sec}$ ...Ans.

Ex. 4.2.4 : A vibrating system having mass 1 kg is suspended by a spring of stiffness 1000 N/m and it is put to harmonic excitation of 10 N. Assuming viscous damping, determine :

(i) the resonant frequency

(ii) the phase angle at resonance

(iii) the amplitude at resonance

(iv) the corresponding to the peak amplitude and

(v) damped frequency

Take damping coefficient, C = 40 N-S/m.

SPPU - Oct. 19(In Sem.), 6 Marks

Soln. :

Given : m = 1 kg ; K = 1000 N/m ; F_0 = 10 N ; c = 40 N-sec/m .

1. **Resonance Frequency (ω_n) :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1000}{1}} = 31.62 \text{ rad/s}$$

- **Resonant frequency :**

$\boldsymbol{\omega = \omega_n = 31.62\ rad/s}$ ...Ans.

2. **Phase Angle of Resonance (ϕ) :**

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left[\frac{2\xi\times 1}{1-(1)^2}\right] = \tan^{-1}[\infty]$$

or $\boldsymbol{\phi = 90^\circ}$ ...Ans.

3. **Amplitude of resonance (X) :**

- **Damping factor :**

$$\xi = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{40}{2\times 1\times 31.62} = 0.6325$$

- **Amplitude at resonance :**

$$X = \frac{F_0 / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{10/1000}{\sqrt{[1-(1)^2]^2 + [2\times 0.6325\times 1]^2}}$$

or $\mathbf{X = 7.90\times 10^{-3}\ m}$ ...Ans.

4. **Frequency Corresponding to Peak Amplitude :**

- The frequency at which the amplitude becomes maximum is called as **frequency of peak amplitude (ω_p).**
- Now, $X = \dfrac{F_o/K}{\sqrt{\left[1-\left(\frac{\omega_p}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega_p}{\omega_n}\right]^2}}$

where, ω_p = frequency at peak amplitude, rad/s

$$X = \frac{X_{st}}{\sqrt{\left[1-\left(\frac{\omega_p}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega_p}{\omega_n}\right]^2}}$$

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega_p}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega_p}{\omega_n}\right]^2}} \quad ...(a)$$

- Putting $\frac{\omega_p}{\omega_n}$ = a, in Equation (a), we get,

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{[1-a^2]^2+[2\xi a]^2}} \quad ...(b)$$

For maximum amplitude,

$$\frac{d}{da}\left(\frac{X}{X_{st}}\right) = 0$$

$$\frac{d}{da}\left(\frac{X}{X_{st}}\right) = \frac{2(1-a^2)(-2a)+2(2\xi a)(2\xi)}{2[(1-a^2)^2+(2\xi a)^2]^{3/2}} = 0$$

$$\therefore \quad 0 = \frac{-4a+4a^3+8a\xi^2}{2[(1-a^2)^2+(2\xi a)^2]^{3/2}} = -4a+4a^3+8a\xi^2$$

$$0 = -1+a^2+2\xi^2$$

$$\therefore \quad a^2 = 1-2\xi^2$$

$$\therefore \quad a = \sqrt{1-2\xi^2}$$

$$\therefore \quad \frac{\omega_p}{\omega_n} = \sqrt{1-2\xi^2}$$

$$\therefore \quad \omega_p = \omega_n\sqrt{1-2\xi^2} = 31.62\sqrt{1-2(0.6325)^2}$$

or ω_p = **14.13 rad/s** ...Ans.

5. **Damped Natural Circular Frequency :**

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 31.62\sqrt{1-(0.6325)^2}$$

$\therefore$ ω_d = **24.49 rad / sec** ...Ans.

Ex. 4.2.5 : In a forced vibratory system, a body having 2 kg mass vibrates in a viscous fluid. The harmonic exciting force of 20 N acting on the mass results in a resonance amplitude of 15 mm with a period of 0.15 sec. Determine the damping coefficient of viscous fluid. If the system is excited by the same harmonic force but at a frequency of 5 cycles/s, what will be the amplitudes of forced vibrations with and without damper ? **SPPU - May 08**

Soln. :

Given : m = 2 kg ; F_o = 20 N ;

X = 15 mm = 0.015 m ; t_p = 0.15 s ;

f_1 = 5 cycle / s.

1. **Damping Coefficient (c) :**

- **Circular frequency of forced vibrations :**

$$\omega = \frac{2\pi}{t} = \frac{2\pi}{0.15} = 41.88 \text{ rad/s}$$

- **At resonance :**

$$\therefore \quad \omega_n = \omega = 41.88 \text{ rad/s}$$

- Now, $\omega_n = \sqrt{\frac{K}{m}}$

$$\therefore \quad 41.88 = \sqrt{\frac{K}{2}}$$

$$\therefore \quad K = 3507.86 \text{ N/m}$$

- **Amplitude of vibrations :**

$$X = \frac{F_0/K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{F_0/K}{\sqrt{[1-(1)^2]^2+[2\xi\times 1]^2}} \quad [\because \omega = \omega_n]$$

$$= \frac{F_0/K}{2\xi} = \frac{F_0/K}{2\times\frac{c}{2m\omega_n}} \quad ...\left[\because \xi = \frac{c}{2m\omega_n}\right]$$

or $$X = \frac{F_0\cdot 2m\omega_n}{2Kc}$$

$$\therefore \quad 0.015 = \frac{20\times2\times2\times41.88}{2\times3507.86\times c}$$

$\therefore$ c = **31.83 N-s/m** ...Ans.

2. **Amplitude of Forced Vibrations Under New Condition :**

- **Damping factor :**

$$\xi = \frac{c}{2m\omega_n} = \frac{31.83}{2\times2\times41.88} = 0.190$$

- **Frequency of vibrations :**

$$f_1 = \frac{\omega_1}{2\pi}$$

$$\therefore \quad 5 = \frac{\omega}{2\pi}$$

$$\therefore \quad \omega_1 = 31.41 \text{ rad/s}$$

- **Amplitude of forced vibrations at ω = 31.41 rad/s and with damper :**

$$X_1 = \frac{F_0/K}{\sqrt{\left[1-\left(\frac{\omega_1}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega_1}{\omega_n}\right]^2}}$$

$$= \frac{20/3507.86}{\sqrt{\left[1-\left(\frac{31.41}{41.88}\right)^2\right]^2+\left[2\times 0.190\times\frac{31.41}{41.88}\right]^2}}$$

or $X_1 = 0.0109\text{ m}$ ...Ans.

- **Amplitude of forced vibration at ω = 31.41 rad/s and without damper :**

$$X_2 = \frac{F_0/K}{1-\left(\frac{\omega_1}{\omega_n}\right)^2}$$

$$= \frac{20/3507.86}{1-\left(\frac{31.41}{41.88}\right)^2}$$

or $X_2 = 0.0130\text{ m.}$ ...Ans.

- **Percentage increase in amplitude of vibrations when damper is removed :**

$$= \frac{X_2 - X_1}{X_1}\times 100$$

$$= \frac{0.0130-0.0109}{0.0109}\times 100$$

$= 19.56\%$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 4.2.6 : A machine part of mass 4 kg vibrates in a viscous fluid. Find the damping coefficient when a harmonic exciting force of 50 N results in resonant amplitude of 250 mm with a period of 0.4 sec. If the excitation frequency is 2 Hz, find the percentage increase in the amplitude of forced vibration when the damper is removed.

SPPU - Aug. 15(In Sem), Oct. 16(In Sem), 6 Marks

Ex. 4.2.7 : The damped natural frequency of a system, as obtained from a free vibration test is 9.8 Hz. During the forced vibration test with constant exciting force on the same system, the maximum amplitude of vibration is found to be at frequency of 9.6 Hz. Find the damping factor for the system and its natural frequency.

SPPU - May 13, 6 Marks

Ex. 4.2.8 : A 45 kg machine is mounted on four parallel springs each of stiffness 2×10^5 N/m. When the machine operates at 32 Hz, the machine's steady- state amplitude is measured as 1.5 mm. What is the magnitude of the excitation force provided to the machine at this speed ?

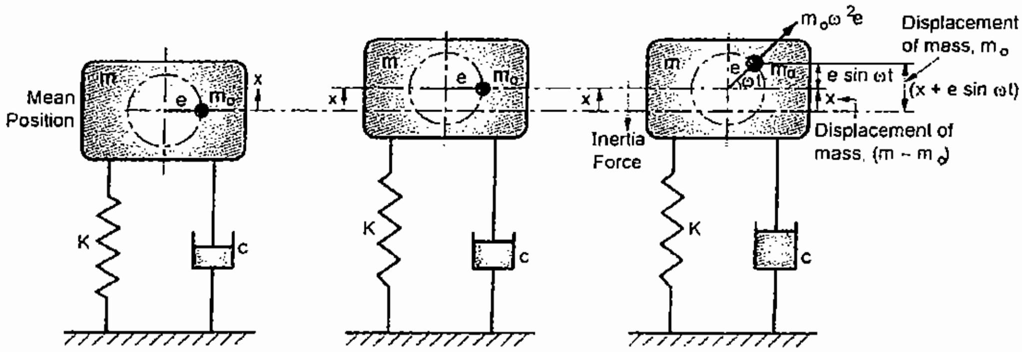
4.3 FORCED VIBRATIONS DUE TO ROTATING UNBALANCE

University Questions

Q. Derive an expression for determining the amplitude of vibration for a system having a rotating or reciprocating unbalance. **SPPU : Dec. 11**

Q. Explain, forced vibration with rotating unbalance. **SPPU : May 13, Dec. 14**

☞ **Spring-Mass-Damper Systems with rotating Unbalance :**



(a) Equilibrium Position (b) Displaced Position with Angle of Rotation of Unbalanced Rotating Mass Equal to Zero (c) Displaced Positon with Angle of Rotation of Rotation of Unbalanced Rotating Mass Equal to 'ωt'

Fig. 4.3.1 : Rotating Unbalance

- All rotating machines such as : Centrifugal pumps, turbines, electric motor, etc, have some amount of unbalance left in them even after balancing. This unbalance is a common source of forced vibrations.

Consider a machine with a rotating unbalance supported on a spring and a damper shown in Fig. 4.3.1

Let, m = total mass of machine including unbalanced mass, kg

m_o = unbalanced mass, kg

e = eccentricity of the unbalanced mass, m

ω = angular velocity of rotation of unbalanced mass (i.e. circular frequency of external excitation force), rad/s

K = spring stiffness of the support, N/m

c = damping coefficient, N-s/m

Forces Acting on System :

Let,

- Total vertical displacement of mass of machine excluding unbalanced rotating mass $(m - m_o) = x$.
- Total vertical displacement of unbalanced rotating mass 'm_o' $= (x + e \sin \omega t)$

(i) Inertia force :

- Inertia force due to mass of machine excluding unbalanced rotating mass $(m - m_o) = (m - m_o)\ddot{x}$
- Inertia force due to unbalanced rotating mass 'm_o'

$$= m_o \frac{d^2}{dt^2}(x + e \sin \omega t)$$

$$= m_o(\ddot{x} - \omega^2 e \sin \omega t)$$

$$= (m_o \ddot{x} - m_o \omega^2 e \sin \omega t)$$

(ii) Spring force :

Spring force = Kx

(iii) Damping force :

Damping force = $c\dot{x}$

Differential Equation of Motion :

- Consider F. B. D. of machine shown in Fig. 4.3.2.

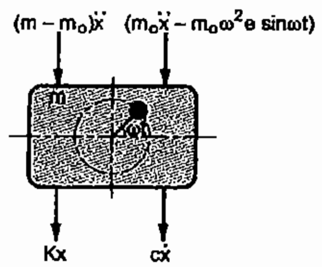


Fig. 4.3.2 : F.B.D. of Machine

$$\sum [\text{Inertia force + External forces}] = 0$$

$$\therefore (m - m_o)\ddot{x} + (m_o\ddot{x} - m_o\omega^2 e \sin \omega t) + c\dot{x} + Kx = 0$$

$$\therefore m\ddot{x} - m_o\ddot{x} + m_o\ddot{x} - m_o\omega^2 e \sin \omega t + c\dot{x} + Kx = 0$$

or
$$\mathbf{m\ddot{x} + c\dot{x} + Kx = m_o\omega^2 e \sin \omega t} \quad \text{...(4.3.1)}$$

This Equation (4.3.1) is the linear, second order differential equation, of motion for a forced damped vibrations due to rotating unbalance. This Equation (4.3.1) is similar to Equation (4.2.1),

i.e.
$$m\ddot{x} + c\dot{x} + Kx = F_o \sin \omega t$$

Hence,
$$F_0 = m_o\omega^2 e \quad \text{... (4.3.2)}$$

Complete Solution for Differential Equation :

$$x = x_c + x_p$$

$$\therefore \quad \mathbf{x = X_1 e^{-\xi\omega_n t} \sin[\sqrt{1-\xi^2}\,\omega_n t + \phi_1]} + \frac{\mathbf{m_o e \omega^2 \sin(\omega t - \phi)}}{K\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...(4.3.3)}$$

(i) Amplitude of steady state vibrations (X) :

$$X = \frac{m_o e \omega^2 / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{m_o e \omega^2 / m\omega_n^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...}[\because K = m\omega_n^2]$$

$$\therefore \quad X = \frac{\frac{m_o e}{m}\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$\therefore \frac{X}{\left(\frac{m_o e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...(4.3.4)}$$

This Equation (4.3.4) gives the **dimensionless steady-state amplitude.**

(ii) Phase angle (ϕ) :

$$\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \quad \text{...(4.3.5)}$$

4.3.1 Plot of Dimensionless Steady-State Amplitude Versus Frequency Ratio :

- The plot of $\left(\frac{X}{m_0 e / m}\right)$ versus **frequency ratio** $\left(\frac{\omega}{\omega_n}\right)$ for different damping condition is shown in Fig. 4.3.3. These curves are plotted with the help of Equation (4.3.4).
- **Observations made from Plot :**
  (i) When the speed is zero, the dimensionless steady-state amplitude $\left(\frac{X}{m_0 e / m}\right)$ is zero. Hence, all curves start from origin.
  (ii) At resonance $\left(\text{i.e. when } \frac{\omega}{\omega_n} = 1\right)$, $\left(\frac{X}{m_0 e / m}\right) = \frac{1}{2\xi}$. Hence, the dimensionless amplitude is limited only by the damping present in the system.
  (iii) When the frequency ratio (ω / ω_n) is very large, the dimensionless amplitude $\left(\frac{X}{m_0 e / m}\right)$ trends to unity i.e. one.

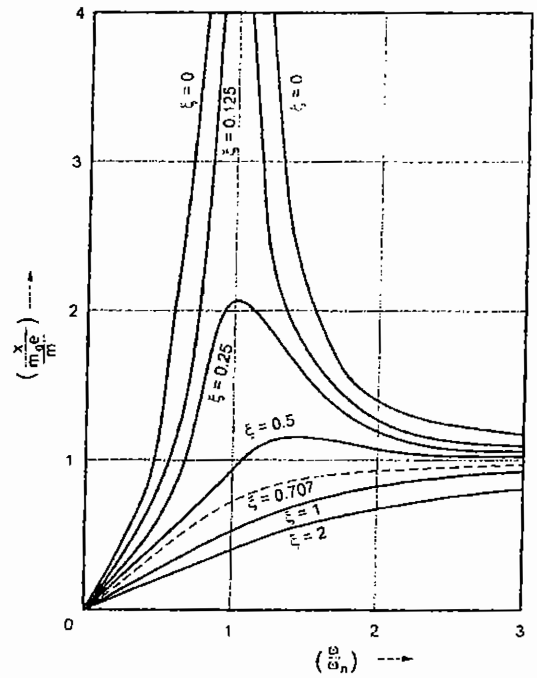


Fig. 4.3.3 : A Plot of Dimensionless Amplitude $\left(\frac{X}{m_0 e / m}\right)$ Vs Frequency Ratio $\left(\frac{\omega}{\omega_n}\right)$

4.4 FORCED VIBRATIONS DUE TO RECIPROCATING UNBALANCE

University Question

Q. Write a short note on : Forced vibrations due to reciprocating unbalance. **SPPU : Dec. 16**

- **Examples of reciprocating unbalance :** The unbalance in reciprocating machines such as : I.C engines, reciprocating compressors, reciprocating pumps, etc is an another source of forced vibrations.
- **Reciprocating machine (Fig. 4.4.1) :**

Let,
m = total mass of the reciprocating machine, kg
m_o = mass of reciprocating parts, kg
l = length of connecting rod, m
r = length of crank, m
n = obliquity ratio = l / r
ω = angular velocity of crank, rad/s
ωt = angle made by crank with horizontal reference axis.

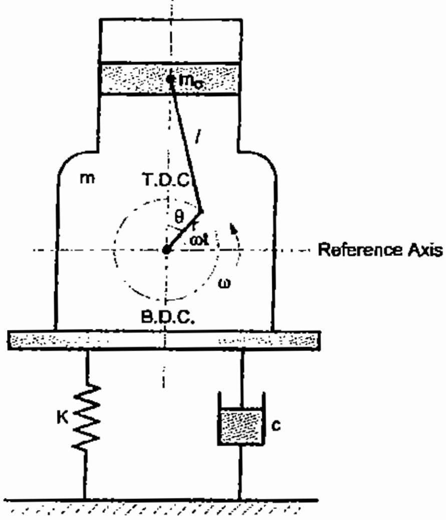


Fig. 4.4.1 : Reciprocating Unbalance

- **Inertia force due to mass of reciprocating parts :**

The inertia force due to the mass of reciprocating parts acts as the excitation force in reciprocating machines. The inertia force due to mass of reciprocating parts is given by,

$$F_I = m_o r \omega^2 \left[\sin \omega t + \frac{\sin 2\omega t}{n}\right] \quad ...(4.4.1)$$

As 'n' is large $\left[\frac{\sin 2\omega t}{n}\right]$ is very small and hence, can neglected.

$$\therefore \mathbf{F_I = m_o r \omega^2 \sin \omega t} \quad ...(4.4.2)$$

From Equation (4.4.2) it is seen that, the exciting force due to reciprocating unbalance is same as due to rotating unbalance, discussed in earlier section.

Conclusion : Therefore, analysis of the rotating unbalance is also applicable to reciprocating unbalance.

Ex. 4.4.1 : The rotating machine, having total mass of 20 kg, is having an eccentric mass of 1.5 kg with eccentricity of 25 mm. The machine rotates at 720 r.p.m. If the amplitude of vibrations, which is 20 mm, lags the eccentric mass by 90°, determine :

(i) the natural circular frequency of the system;

(ii) the damping factor; and

(iii) the amplitude and phase angle when eccentric mass rotates at 1440 r.p.m. **SPPU - May 07**

Soln. :

Given : $m = 20$ kg ; $m_o = 1.5$ kg ; $e = 25$ mm $= 0.025$ m ; $N = 720$ r.p.m. ; $\phi = 90°$; $X = 20$ mm $= 0.02$ m

1. **Natural Circular Frequency (ω_n) :**

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 720}{60} = 75.39 \text{ rad/sec}$$

$$\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore \quad 90° = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore \tan(90°) = \left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore \infty = \left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore \quad 1-\left(\frac{\omega}{\omega_n}\right)^2 = 0$$

$$\therefore \quad \left(\frac{\omega}{\omega_n}\right)^2 = 1$$

$$\therefore \quad \frac{\omega}{\omega_n} = 1$$

$$\therefore \quad \omega_n = \omega$$

or $\quad \omega_n = 75.39$ rad/s. ...Ans.

2. **Damping Factor (ξ) :**

$$\frac{X}{\left(\frac{m_o e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

At resonance (i.e. when $\omega = \omega_n$),

$$\frac{X}{\left(\frac{m_o e}{m}\right)} = \frac{1}{2\xi}$$

$$\xi = \frac{m_o e}{2\,m\,X} = \frac{1.5 \times 0.025}{2 \times 20 \times 0.02}$$

or $\quad \xi = 0.046$ ...Ans.

3. **Amplitude at 1440 r.p.m. (X_1) :**

$$\omega_1 = \frac{2\pi \times 1440}{60} = 150.79 \text{ rad/s.}$$

$$X_1 = \frac{\left(\frac{m_o e}{m}\right)\left(\frac{\omega_1}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega_1}{\omega_n}\right)^2\right]^2+\left[2\xi\left(\frac{\omega_1}{\omega_n}\right)\right]^2}}$$

$$= \frac{\left(\frac{1.5 \times 0.025}{20}\right)\left(\frac{150.79}{75.39}\right)^2}{\sqrt{\left[1-\left(\frac{150.79}{75.39}\right)^2\right]^2+\left[2 \times 0.046 \times \left(\frac{150.79}{75.39}\right)\right]^2}}$$

$$X_1 = 0.002495 \text{ m}$$

or $\quad X_1 = 2.49$ mm ...Ans.

4. **Phase Angle at 1440 r.p.m. (ϕ_1) :**

The phase angle at $\omega = 150.79$ rad/s is,

$$\phi_1 = \tan^{-1}\left[\frac{2\xi\frac{\omega_1}{\omega_n}}{1-\left(\frac{\omega_1}{\omega_n}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{2 \times 0.046 \times \left(\frac{150.79}{75.39}\right)}{1-\left(\frac{150.79}{75.39}\right)^2}\right]$$

$$= \tan^{-1}[-0.06133]$$

$$\therefore \quad \phi_1 = 176.49° \text{ or } 356.49° \quad ...\text{Ans.}$$

Ex. 4.4.2 : A system having rotating unbalance has total mass of 25 kg. The unbalanced mass of 1 kg rotates with a radius 0.04 m. It has been observed that, at a speed of 1000 rpm, the system and eccentric mass have a phase difference of 90° and the corresponding amplitude is 0.015 m. Determine :

(i) the natural frequency of the system ;

(ii) the damping factor

(iii) the amplitude at 1500 r.p.m. ; and

(iv) the phase angle at 1500 r.p.m.

SPPU - May 15, 8 Marks, Oct 16 (In Sem), 4 Marks

Soln. :

Given : $m = 25$ kg ; $m_o = 1$ kg ;

$e = 40$ mm $= 0.04$ m ; $N = 1000$ r.p.m.;

$\phi = 90°$; $X = 15$ mm $= 0.015$ m.

1. **Natural Circular Frequency (ω_n) :**

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1000}{60} = 104.71 \text{ rad/sec}$$

$$\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore\ 90° = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore \tan(90°) = \left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore\ \infty = \left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

$$\therefore 1-\left(\frac{\omega}{\omega_n}\right)^2 = 0$$

$$\therefore \left(\frac{\omega}{\omega_n}\right)^2 = 1$$

$$\therefore \frac{\omega}{\omega_n} = 1$$

$$\therefore \omega_n = \omega$$

or $\omega_n =$ **104.71 rad/s.** ...Ans.

2. **Damping Factor (ξ) :**

$$\frac{X}{\left(\frac{m_o e}{m}\right)} = \frac{\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

At resonance $\omega = \omega_n$, hence

$$\therefore \frac{X}{\left(\frac{m_o e}{m}\right)} = \frac{1}{2\xi}$$

$$\xi = \frac{m_o e}{2\,m\,X} = \frac{1\times 0.04}{2\times 25\times 0.015} = \mathbf{0.0533}$$ ...Ans.

3. **Amplitude at 1500 r.p.m. (X_1) :**

$$\omega_1 = \frac{2\pi N}{60} = \frac{2\pi \times 1500}{60} = 157.07 \text{ rad/s.}$$

$$X_1 = \frac{\left(\frac{m_o e}{m}\right)\left(\frac{\omega_1}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega_1}{\omega_n}\right)^2\right]^2+\left[2\xi\left(\frac{\omega_1}{\omega_n}\right)\right]^2}}$$

$$= \frac{\left(\frac{1\times 0.04}{25}\right)\left(\frac{157.07}{104.71}\right)^2}{\sqrt{\left[1-\left(\frac{157.07}{104.71}\right)^2\right]^2+\left[2\times 0.0533\times\left(\frac{157.07}{104.71}\right)\right]^2}}$$

$$X_1 = 0.001791 \text{ m}$$

or $X_1 =$ **1.79 mm** ...Ans.

4. **Phase Angle at 1500 r.p.m. (ϕ_1) :**

$$\phi_1 = \tan^{-1}\left[\frac{2\xi\frac{\omega_1}{\omega_n}}{1-\left(\frac{\omega_1}{\omega_n}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{2\times 0.533\times\left(\frac{157.01}{104.71}\right)}{1-\left(\frac{157.01}{104.71}\right)^2}\right]$$

$$= \tan^{-1}[-0.12792]$$

$\therefore\ \phi_1 =$ **172.71° or 352.71°** ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex. 4.4.3 : The vibrating system is displayed for vibration characteristics. The total mass of the system is 30 kg. At the speed of 900 rpm, system and eccentric mass have a phase difference of 90° and corresponding amplitude is 18 mm. If the eccentric unbalance mass of 1.2 kg has radius of rotation 45 mm, determine :

(i) the natural frequency ;

(ii) the damping factor, ξ ;

(iii) the amplitude at 1550 rpm ; and

(iv) the phase angle at 1550 rpm.

SPPU - Dec. 14, 10 Marks

Ex. 4.4.4 : When a single cylinder engine of total mass 300 kg is placed on four springs, each spring is compressed by 2 mm. A dashpot, offering 400 N of damping force at relative velocity of 200 mm/sec, is attached to the engine to damp out the vibrations. The reciprocating mass of the engine is 20 kg and stroke of the piston is 130 mm. If the engine is running at 1500 r.p.m, find the amplitude of steady-state vibrations, neglecting secondary unbalance.

SPPU - Dec. 02, May 10

Soln. :

Given : $m = 300\text{ kg}$; $\delta = 2\text{ mm} = 0.002\text{ m}$;

$F = 400\text{ N}$; $v = 0.2\text{ m/s}$;

$m_o = 20\text{ kg}$; $S = 2r = 130\text{ mm}$;

$N = 1500\text{ r.p.m.}$

$$\therefore \quad r = \frac{S}{2} = \frac{130}{2} = 65\text{ mm} = 0.065\text{ m}$$

Damping coefficient,

$$c = \frac{F}{v} = \frac{400}{0.2} = 2000\text{ N-sec/m.}$$

$$\omega = \frac{2\pi \times 1500}{60} = 157.058\text{ rad/s.}$$

1. **Damping Factor (ξ) :**

- **Load taken by each spring** $= \dfrac{mg}{4}$.
- **Stiffness of each spring :**

$$K = \frac{mg/4}{\delta} = \frac{300 \times 9.81}{4 \times 0.002}$$

$$= 0.367875 \times 10^6\text{ N/m}$$

- **Equivalent stiffness of four springs :**

$$K_e = 4K = 4 \times 0.367875 \times 10^6$$

or

$$K_e = \frac{300 \times 9.81}{0.002}$$

$$= 1.4715 \times 10^6\text{ N/m.}$$

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K_e}{m}}$$

$$= \sqrt{\frac{1.4715 \times 10^6}{300}}$$

$$= 70.03\text{ rad/s}$$

- **Damping factor :**

$$\xi = \frac{c}{2m\omega_n}$$

$$= \frac{2000}{2 \times 300 \times 70.03}$$

$$\xi = 0.0476$$

2. **Amplitude of Forced Vibrations (X) :**

$$X = \frac{\left(\frac{m_o r}{m}\right)\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$= \frac{\left(\frac{20 \times 0.065}{300}\right)\left(\frac{157.08}{70.03}\right)^2}{\sqrt{\left[1-\left(\frac{157.08}{70.03}\right)^2\right]^2 + \left[2 \times 0.0476 \times \frac{157.08}{70.03}\right]^2}}$$

$$X = 0.00540\text{ m}$$

or $X = 5.40\text{ mm}$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 4.4.5 : A single cylinder vertical petrol engine of total mass 320 kg is mounted on a steel chassis and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine have a mass of 24 kg and move through a vertical stroke of 150 mm with SHM. A dashpot attached to the system offers a resistance of 490 N at a velocity of 0.3 m/s. Determine :

(i) the speed of driving shaft at resonance; and

(ii) the amplitude of steady state vibrations when the driving shaft of the engine rotates at 480r.p.m.

SPPU - Dec. 16, Aug 17 (In Sem), 6 Marks

Ex. 4.4.6 : A 70 kg machine has a rotating unbalance of 0.15 kg-m. The machine operates at 125 Hz and is mounted on a foundation of equivalent stiffness 2000 kN/m and damping ratio 0.12. What is the machine's steady - state amplitude ? **SPPU - May 12, 4 Marks**

Ex. 4.4.7 : A vertical single stage compressor has a mass of 500 kg, mounted on springs having stiffness of 1.96×10^5 N/m and a dashpot having damping factor of 0.2. The rotating parts are completely balanced and the equivalent reciprocating parts weigh 2 kg. The stroke of the compressor is 0.2 m. Find the dynamic amplitude of vertical motion and the phase difference between the motion and the excitation force, if the operating speed is 200 r.p.m ? **SPPU - May 09, May 14, 8 Marks**

4.5 VIBRATION TRANSMISSIBILITY

Transmission of Vibrations :

The vibrations (force or motion) are transmitted :

(i) From the machine to the supporting structure on which the machine is mounted; or

(ii) From the supporting structure to the machine.

Terms Used to Specify Transmission of Vibrations :

The transmission of the vibrations can be specified / measured by one of the following the terms :

Terms in Vibration Transmissibility
- 1. Force Transmissibility
- 2. Motion Transmissibility

4.5.1 Force Transmissibility (T_r) :

University Questions

Q. Define isolation factor or transmissibility ratio. Write down the equation for the same in relation with magnification factor. **SPPU : Dec. 11**

Q. Explain the term : Force transmissibility. **SPPU : Dec. 12, Dec. 13, May 15, May 16**

- **Force Transmissibility** : Force transmissibility is defined as *the ratio of the force transmitted to the supporting structure or foundation,* F_T *to that force impressed upon the system,* F_0. Force transmissibility measures the effectiveness of the vibration isolating material.

$$\text{Force Transmissibility} = \frac{\text{Force transmitted to the foundation}}{\text{Force impressed upon the system}}$$

$$\therefore\ T_r = \frac{F_T}{F_0} \qquad \text{...(4.5.1)}$$

- **Spring Mass-Damper system :**

Consider the mass '**m**' is supported on the foundation by means of an isolator and excited by the external force $F_0 \sin \omega t$, as shown in Fig. 4.5.1(a).

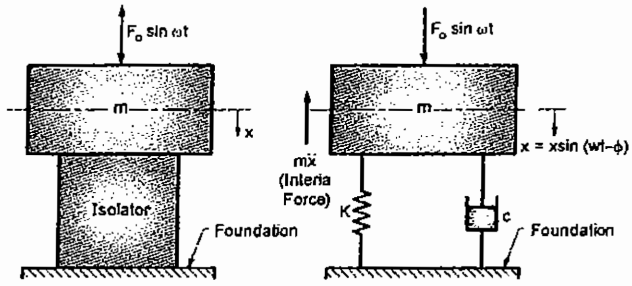
The displacement of mass is,

$$x = X \sin(\omega t - \phi)$$

Hence,

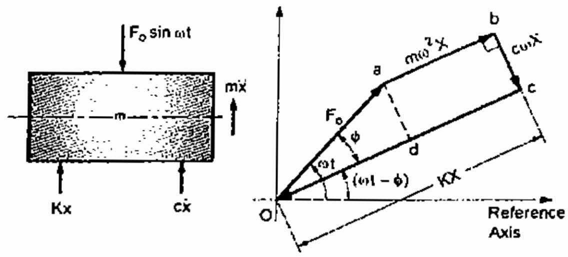
$$\dot{x} = X\omega \cos(\omega t - \phi) = -X\omega \sin\left(\omega t - \phi + \frac{\pi}{2}\right)$$

$$\ddot{x} = -X\omega^2 \sin(\omega t - \phi)$$



(a) Actual System (b) Equivalent System

Fig. 4.5.1



(c) F.B.D. of Mass (d) Vector Diagram of forces Acting on Mass

Fig. 4.5.1 : Forces Acting on Mass

Let,

K = equivalent stiffness of isolator, N/m

c = damping coefficient of isolator, N-s/m

- **Force Acting on Mass :**

(i) External impressed force, $F_0 \sin \omega t$ (downwards)

(ii) Inertia force, $m\ddot{x} = m\omega^2 X \sin(\omega t - \phi)$ (upward)

(iii) Damping force,

$$c\dot{x} = -\left(\omega X \sin\left(\omega t - \phi + \frac{\pi}{2}\right)\right) \text{ (upwards)}$$

(iv) Spring force, Kx (upwards) = $KX \sin(\omega t - \phi)$(upwards)

Fig. 4.5.1(c) shows the F.B.D. of mass and Fig. 4.5.1(d) shows the vector diagram of forces acting on the mass.

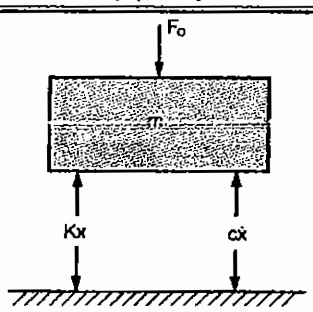
- **Force Transmitted to Foundation :**

Out of four forces acting on mass, the following two forces are transmitted to the foundation [Fig. 4.5.2(a)]

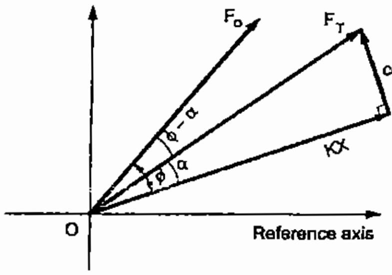
(i) Spring force Kx (downwards).

(ii) Damping force, $c\dot{x}$ (downwards).

The total force transmitted to the foundation, F_T is the vector sum of these two forces acting on the foundation. These two forces i.e. spring force and damping force are 90° out of phase with each other, as shown in Fig. 4.5.2(b). Fig. 4.5.2(b) shows the force transmitted to the foundation 'F_T' and force impressed upon the mass, 'F_0'.



(a) F.B.D. of Foundation



(b) Vector Diagram of Transmitted and Impressed Forces

Fig. 4.5.2 : Force Transmitted to Foundation

- From Fig. 4.5.2(b), force transmitted to foundation is given by,

$$F_T = \sqrt{(KX)^2 + (c\,\omega\, X)^2}$$

or
$$F_T = X\sqrt{K^2 + (c\,\omega)^2} \qquad ...(a)$$

Substituting the value of X form Equation (4.2.7) in Equation (a), we get,

$$F_T = \left[\frac{F_o/K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}\right]\sqrt{K^2+(c\omega)^2}$$

$$= \frac{F_o\sqrt{1+\left(\frac{c\omega}{K}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\text{or } F_T = \frac{F_o\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \qquad ...(b)$$

$$\left[\because \frac{c\omega}{K} = 2\xi\frac{\omega}{\omega_n}\right]$$

- **Force Transmissibility (T_r) :**

  The force transmissibility is given by,

$$T_r = \frac{F_T}{F_o}$$

  From Equation (b),

$$T_r = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \qquad ...(4.5.2)$$

- **Angle of Lag ($\phi - \alpha$) :**

  The angle through which the transmitted force F_T lags the impressed force F_o is $(\phi - \alpha)$

$$\text{where, } \alpha = \tan^{-1}\left[\frac{c\,\omega\, X}{KX}\right] = \tan^{-1}\left[\frac{c\,\omega}{K}\right]$$

$$\text{or } \alpha = \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right] \qquad ...(4.5.3)$$

  Therefore, the angle of lag is given by,

$$(\phi - \alpha) = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right] \qquad ...(4.5.4)$$

4.5.2 Motion Transmissibility :

University Question

Q. Explain the term : Motion transmissibility.

SPPU : Dec. 13, May 15

- **Motion Transmissibility** : In case of the forced vibrations due to the excitation of the support, motion transmissibility is defined as *the ratio of absolute amplitude of the mass (body) to the amplitude of the base excitation*. Therefore,

$$T_r = \frac{X}{Y} \qquad ...(4.5.5)$$

where, X = absolute amplitude of the mass (body)

Y = amplitude of the base excitation

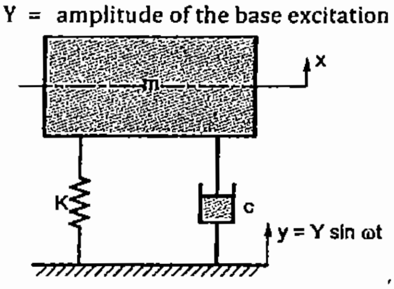


Fig. 4.5.3 : Motion Transmissibility

- Substituting Equation (4.5.2) in Equation (4.5.5), the motion transmissibility is given by,

$$T_r = \frac{X}{Y} = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \qquad ...(4.5.6)$$

- Thus, the transmissibility is same whether it is force transmission or motion transmission.
- **Phase Angle or Angle of Lag ($\phi - \alpha$) :**

The phase angle or angle of lag is given by,

$$(\phi - \alpha) = \tan^{-1}\left[\frac{2\xi \frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi \frac{\omega}{\omega_n}\right] \quad ...(4.5.7)$$

4.5.3 Transmissibility Versus Frequency Ratio :

University Questions

Q. Explain transmissibility versus frequency ratio curve for different amount of damping. **SPPU : May 13**

Q. Explain significance of Force transmissibility versus Frequency ratio curve. **SPPU : Dec. 13, Dec. 14, Dec. 15**

Q. Draw transmissibility curves for different damping conditions . Give the significance of these curves. **SPPU : Oct. 19(In Sem.)**

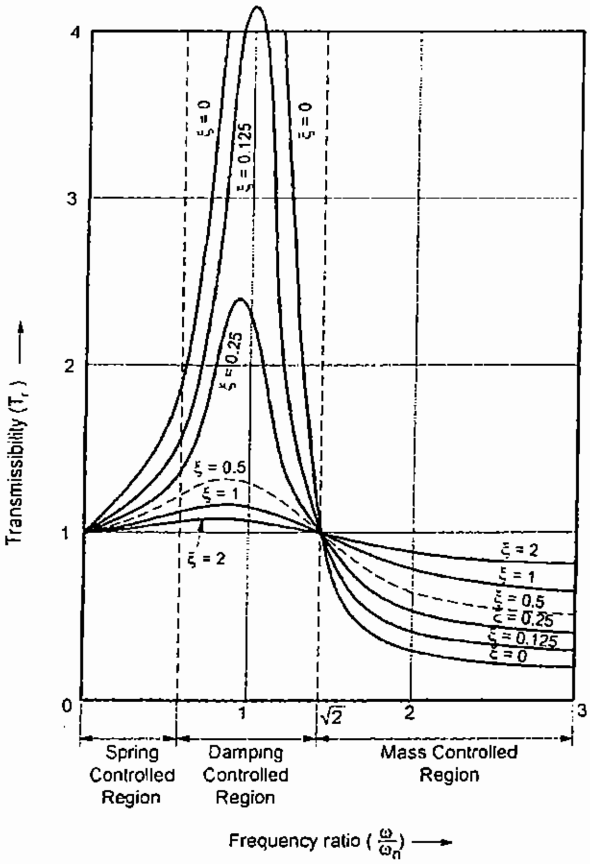


Fig. 4.5.4 : Transmissibility Versus Frequency Ratio

The plot of transmissibility (T_r) versus frequency ratio (ω/ω_n) for different damping conditions is shown in Fig. 4.5.4. The curves are plotted using Equation (4.5.2).

- **Observation From Transmissibility Versus Frequency Ratio Curves :**

1. All the curves start from the unit value of transmissibility. At frequency ratio $\left(\frac{\omega}{\omega_n}\right) = 0$, transmissibility, $T_r = 1$.
2. When $(\omega/\omega_n) = 1$, the transmissibility is maximum. In this conditions, damping controls the transmissibility.
3. When $(\omega/\omega_n) < \sqrt{2}$, the transmissibility is greater than one. This means, the transmitted force is always greater than the impressed exciting force. In this range, the greater amount of damping gives lower transmissibility.
4. When $(\omega/\omega_n) = \sqrt{2}$, the transmissibility is equal to one. This means, the transmitted force is equal to the impressed exciting force, irrespective of damping. At this value of frequency ratio (ω/ω_n), the transmissibility is independent of the damping.
5. When $(\omega/\omega_n) > \sqrt{2}$, the transmissibility is less than one. This means, the transmitted force is always less than the impressed exciting force. The better vibration isolation is possible in this range. In this range, greater amount of damping gives greater transmissibility, and hence, damping is unfavorable.
6. The transmissibility, T_r tends to zero as the frequency ratio $\left(\frac{\omega}{\omega_n}\right)$ tends to infinity.
7. In order to have low value of transmissibility, the operation of vibrating system generally kept in the range $(\omega/\omega_n) > \sqrt{2}$. In this range, zero damping will be ideally suitable as this would give extremely low value of transmissibility. But since the system has to pass through the resonance (i.e. $\omega = \omega_n$) in reaching the operating point and zero damping will give very high transmissibility (though for a moment only), some amount of damping is generally incorporated in the system.

- **Regions of Transmissibility Vs Frequency Ratio Curve :**

The transmissibility curve can be divided into three distinct frequency regions, as shown in Fig. 4.5.4.

(i) Spring controlled region
(ii) Damping controlled region
(iii) Mass controlled region

(i) Spring controlled region :

The region where (ω/ω_n) is small, is called **spring controlled region.** In this region, the larger value of spring stiffness gives high value of natural frequency (ω_n) and consequently lower frequency ratio (ω/ω_n).

(ii) Damping controlled region :

The middle region is called **damping controlled region** which should be generally avoided. When damping is zero and $\omega = \omega_n$, the transmissibility tends to infinity. Therefore, some amount of damping is generally incorporated in the system.

(iii) Mass controlled region :

The region where (ω/ω_n) is large called **mass controlled region.** The larger value of mass gives lower value of natural frequency (ω_n) and consequently higher frequency ratio (ω/ω_n).

Ex. 4.5.1 : A machine having a mass of 100 kg is running at a speed of 1500 r.p.m. It is mounted on a spring of stiffness 10^6 N/m and damper having damping factor 0.3. A 3 kg piston within the machine has a reciprocating motion with a stroke of 100 mm. Determine :
(i) the amplitude of vibrations of the machine; and
(ii) the force transmitted to the foundation.

SPPU - Dec. 04, Dec. 07

Soln. :

Given : $m = 100$ kg ; $N = 1500$ r.p.m; $K = 10^6$ N/m ; $\xi = 0.3$; $m_o = 3$ kg ; $S = 100$ mm.

Now, $r = \dfrac{S}{2} = 50 \text{ mm} = 0.05 \text{ m}$;

and $\omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 1500}{60} = 157.08$ rad/sec.

1. **Amplitude of Forced Vibrations due to Reciprocating Unbalance (X) :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{10^6}{100}} = 100 \text{ rad/s.}$$

- **Frequency ratio :**

$$\frac{\omega}{\omega_n} = \frac{157.08}{100} = 1.5708$$

- **Amplitude of forced vibrations due to reciprocating unbalance :**

$$X = \frac{\left(\frac{m_o r}{m}\right)\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{\left(\frac{3 \times 0.05}{100}\right)(1.5708)^2}{\sqrt{[1-(1.5708)^2]^2 + [2 \times 0.3\,(1.5708)]^2}}$$

$\therefore \quad X = 2.12 \times 10^{-3}$ m

or $\quad X = 2.12$ mm ...Ans.

2. **Impressed Force (F_o) :**

$$F_o = m_o\omega^2 r = 3 \times (157.08)^2 \times 0.05$$

or $\quad F_o = 3700.64$ N

3. **Force Transmissibility (T_r) :**

$$T_r = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{\sqrt{1 + (2 \times 0.3 \times 1.5708)^2}}{\sqrt{[1-(1.5708)^2]^2 + [2 \times 0.3\,(1.5708)]^2}}$$

or $\quad T_r = 0.7879$

4. **Transmitted Force (F_T) :**

$$T_r = \frac{F_T}{F_o}$$

$$0.7879 = \frac{F_T}{3700.64}$$

$\therefore \quad F_T = 2915.99$ N ...Ans.

Ex. 4.5.2 : A centrifugal fan weighs 400 N and has a rotating unbalance of 250 N-cm. The damper used has a damping factor of 0.2. Specify the stiffness of the spring used for mounting such that only 10% of the unbalanced force is transmitted to the floor. The fan is running at a constant speed of 800 r.p.m. **SPPU - Dec. 03**

Soln. :

Given : $m = \dfrac{400}{9.81} = 40.77$ kg;

$m_o = \dfrac{250}{9.81} \times 10^{-2} = 0.25$ kg-m;

$\xi = 0.2$; $T_r = 10\% = 0.1$;

$N = 800$ r.p.m.

$\therefore \quad \omega = \dfrac{2\pi N}{60} = \dfrac{2\pi \times 800}{600} = 83.77$ rad/s.

1. Frequency Ratio (ω / ω) :

$$T_r = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}} \quad ...(a)$$

Putting $\frac{\omega}{\omega_n} = a$ in Equation (a), we get,

$$T_r = \frac{\sqrt{1 + (2\xi a)^2}}{\sqrt{[1 - a^2]^2 + [2\xi a]^2}}$$

$$\therefore \quad 0.1 = \frac{\sqrt{1 + (2 \times 0.2\, a)^2}}{\sqrt{(1 - a^2)^2 + (2 \times 0.2 \times a)^2}}$$

$$\therefore \quad 0.1 = \frac{\sqrt{1 + 0.16\, a^2}}{\sqrt{1 + a^4 - 2a^2 + 0.16\, a^2}}$$

$$\therefore \quad (0.1)^2 = \frac{1 + 0.16\, a^2}{1 + a^4 - 1.8a^2}$$

$$\therefore \quad 1 + a^4 - 1.84\, a^2 = \frac{1 + 0.16\, a^2}{(0.1)^2}$$

$$\therefore \quad 1 + a^4 - 1.84\, a^2 = 100 + 16\, a^2$$

$$a^4 - 17.84\, a^2 - 99 = 0$$

$$\therefore \quad a^2 = \frac{17.84 \pm \sqrt{(17.84)^2 + 4(1)(99)}}{2}$$

$$= \frac{17.84 \pm 26.73}{2}$$

$$\text{or} \quad a^2 = 22.28 \text{ or } - 4.44$$

Taking positive value,

$$\therefore \quad a^2 = 22.28$$

$$\therefore \quad a = 4.72$$

$$\text{or} \quad \frac{\omega}{\omega_n} = 4.72$$

2. Natural Circular Frequency (ω_n) :

$$\frac{\omega}{\omega_n} = 4.72$$

$$\therefore \quad \frac{83.77}{\omega_n} = 4.72$$

$$\therefore \quad \omega_n = 17.74 \text{ rad/s.}$$

3. Stiffness of spring (K) :

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\therefore \quad 17.74 = \sqrt{\frac{K}{40.77}}$$

$$\therefore \quad (17.74)^2 = \frac{K}{40.77}$$

$$\therefore \quad K = 12830.63 \text{ N/m} \quad ...\text{Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 4.5.3 : A refrigerator unit of mass 30 kg is to be supported by three springs. The unit operates at 500 r.p.m. If 10% of the shaking force of refrigerator unit is to be transmitted to the supporting structure, determine the spring constant. **SPPU - May 08**

Ex. 4.5.4 : A mass 25 kg is placed on an elastic foundation. A sinusoidal force of magnitude 25 N is applied to the machine. A frequency sweep reveals that the maximum steady state amplitude of 1.3 mm occurs when the period of response is 0.22 seconds. Determine the equivalent stiffness and damping ratio of the foundation. **SPPU - May 17, 6 Marks**

Soln. :

Given : $F_0 = 25$ N ; $m = 25$ kg ;

$X_{max} = 1.3$ mm ; $t_p = 0.22$ sec

For linear system, the frequency of response is same as frequency of excitation.

$\therefore$ Excitation frequency $= \omega = 2\pi f = 2\pi / t_p = 28.6$ rad/sec

Thus X_{max} = occurs when $\omega = 28.6$ rad/s

- Condition for maximum amplitude to occur

$$r = \sqrt{1 - 2\xi^2} = \frac{\omega}{\omega_n}$$

$$\therefore \quad \omega_n = \frac{\omega}{\sqrt{1 - \xi^2}} = \frac{28.6}{\sqrt{1 - \xi^2}} \quad ...(1)$$

- Also we have,

$$\frac{X}{X_{st}} = \frac{1}{\sqrt{[1 - r^2]^2 + [2\xi r]^2}}$$

For $x_{max} = r = \sqrt{1 - 2\xi^2}$

$$\frac{X_{max}}{X_{st}} = \frac{1}{\sqrt{[1 - (1 - 2\xi^2)]^2 + [4\xi^2 (1 - 2\xi^2)}}$$

$$= \frac{1}{2\xi \sqrt{1 - \xi^2}}$$

$$\frac{X_{max} K}{F_0} = \frac{1}{2\xi \sqrt{1 - \xi^2}} \quad ;$$

$$\frac{X_{max} m \omega_n^2}{F_0} = \frac{1}{2\xi \sqrt{1 - \xi^2}}$$

$$25 \times 0.013 \times \frac{\omega_n^2}{25} = \frac{1}{2\xi \sqrt{1 - \xi^2}}$$

Now substitute for ω_n^2 from Equation (1),

$$0.013 \times \frac{28.6}{\sqrt{1-2\xi^2}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$\frac{1.0633}{\sqrt{1-2\xi^2}} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

Squaring and rearranging,

$$\xi^4 - \xi^2 + 0.117 = 0$$

$$Z^2 - Z + 0.117 = 0$$

Where $\xi^2 = Z$

Solving the quadratic equation

$$\xi = 0.368, 0.93$$

The larger value of ξ is to be discarded because the amplitude would be maximum only for $\xi < 0.707$

$$\therefore \quad \xi = \mathbf{0.368}$$

$\therefore$ Natural frequency $\omega_n = \dfrac{\omega}{\sqrt{(1-2(0.368)^2)}} = 33.5$ rad/sec

Stiffness of the foundation,

$K = m\,\omega_n^2 = 25\,(33.5)^2 = \mathbf{28.05 \times 10^3\ N/m}$ ...**Ans.**

Ex. 4.5.5 : A machine of mass 60 kg is placed on four springs. The mass of reciprocating parts of a machine is 3 kg which moves through a stroke of 100 mm. The speed of crank is 800 r.p.m. If the damping is introduced into the system to reduce the amplitudes of successive vibrations by 20%, determine :

(i) the stiffness of each spring, if the damper is removed and the force transmitted to the foundation is $\left(\frac{1}{10}\right)^{th}$ of the impressed force;

(ii) the force transmitted to the foundation at 800 r.p.m; and

(iii) the force transmitted to the foundation at resonance.

SPPU - May 07

Soln. :

Given : $m = 60$ kg ; $m_0 = 3$ kg ;

$S = 2r = 0.1$ m ; $N = 800$ r.p.m. ;

$r = \dfrac{S}{2} = 0.05$ m.

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 800}{60} = 83.77 \text{ rad/s.}$$

Let, First amplitude = x_0.

Second amplitude, $x_1 = 0.8\,x_0$

1. Equivalent Stiffness of Spring (K_e) :

- **Natural Circular Frequency :**

$$T_r = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

In absence of damping (i.e. $\xi = 0$),

$$T_r = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2}} = \frac{1}{\sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2 - 1\right]^2}}$$

or
$$T_r = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1}$$

$$\therefore \quad \frac{1}{10} = \frac{1}{\left(\frac{83.77}{\omega_n}\right)^2 - 1} \qquad \therefore \quad \frac{701.74}{\omega_n^2} - \frac{1}{10} = 1$$

$$\therefore \quad \omega_n^2 = 637.94$$

$$\therefore \quad \omega_n = 25.25 \text{ rad/s}$$

- **Equivalent Spring Stiffness :**

$$\omega_n = \sqrt{\frac{K_e}{m}}$$

or
$$\omega_n = \sqrt{\frac{K_e}{m}} \qquad \therefore \quad 25.25 = \sqrt{\frac{K_e}{60}}$$

$\therefore$ $K_e = \mathbf{38384.78\ N/m}$ ...**Ans.**

2. Stiffness of Each Spring (K) :

Four springs, each having stiffness k, are used. Assuming springs in parallel,

$$K_e = 4K$$

$$38384.78 = 4K$$

$\therefore K = \mathbf{9596.19\ N/m}$ ...**Ans.**

3. Logarithmic Decrement (δ) :

$$\delta = \frac{1}{n}\log_e\left(\frac{x_0}{x_n}\right)$$

$$\therefore \quad \delta = \log_e\left(\frac{x_0}{x_1}\right) \quad \text{..... (when n = 1)}$$

$$= \log_e\left(\frac{x_0}{0.8\,x_0}\right) = \log_e\left(\frac{1}{0.8}\right)$$

or $\delta = 0.2231$

4. Damping Ratio (ξ) :

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$0.2231 = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\therefore \quad \xi = 0.035$$

5. Transmitted Force at 800 r.p.m. :

- **Impressed force at 800 r.p.m. :**

$$F_o = m_o \omega^2 r = 3 \times (83.77)^2 \times 0.05$$

or $$F_o = 1052.61 \text{ N}$$

- **Frequency ratio :**

$$\frac{\omega}{\omega_n} = \frac{83.77}{25.25} = 3.32$$

- **Transmitted force at 800 r.p.m. :**

$$T_r = \frac{F_T}{F_o} = \frac{\sqrt{1 + \left(2\xi \frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi \frac{\omega}{\omega_n}\right]^2}}$$

$$\therefore \frac{F_T}{1052.61} = \frac{\sqrt{1 + (2 \times 0.035 \times 3.32)^2}}{\sqrt{[1 - (3.32)^2]^2 + [2 \times 0.035 \times 3.32]^2}}$$

$\therefore$ $F_T = $ **23.85 N** ...**Ans.**

6. Transmitted Force at Resonance :

- **Impressed force at resonance (i.e. at $\omega = \omega_n$) :**

$$F_o = m_o \omega_n^2 r = 3 \times (25.25)^2 \times 0.05$$

or $$F_o = 95.63 \text{ N}$$

- **Transmitted force at resonance (i.e. at $\omega = \omega_n$) :**

$$T_r = \frac{F_T}{F_o} = \frac{\sqrt{1 + (2\xi)^2}}{2\xi}$$

$$\therefore \frac{F_T}{95.63} = \frac{\sqrt{1 + (2 \times 0.035)^2}}{2 \times 0.035}$$

$\therefore$ $F_T = $ **1369.48 N** ...**Ans.**

Examples for Practice

Refer our website for complete solution of following examples

Ex. 4.5.6 : A machine weighing 100 kg is supported on 4 springs. It has 80 mm stroke and it runs at 1000 r.p.m. If the springs are symmetrically placed with respect to C.G. of the machine, neglecting damping, determine the combined stiffness of the springs such that the force transmitted to the foundation is 1/25 times the impressed force. It is found that, damping however small, reduces the amplitude of successive vibrations by 25% Determine :

(i) the force transmitted to foundation at 1000 r.p.m;

(ii) the force transmitted to foundation at resonance; and

(iii) the amplitude of vibration at resonance, if weight of the reciprocating parts is 2 kg.

SPPU - Dec. 06, Dec. 12

Ex. 4.5.7 : An electric motor weighs 25 kg and is mounted on a rubber pad which deflects by 1 mm to motor weight. The rotor weighs 5 kg, has an eccentricity of 0.1 mm and rotates at 1500 r.p.m. Find the amplitude of vibration of the motor and the force transmitted to the foundation under the following conditions :

(i) there is no damping.

(ii) damping factor = 0.1

SPPU - Dec. 06, Oct. 18(In sem),6 Marks

Ex. 4.5.8 : The weight of an electric motor is 125 N and it runs at 1500 rpm. The armature weighs 35 N and its CG lies 0.05 cm from the axis of rotation. The motor is mounted on 5 springs of negligible damping so that the force transmitted is 1/11th of the impressed force. Assuming the weight of the motor is equally distributed among the 5 springs, determine :

(i) the stiffness of each spring ;

(ii) the dynamic force transmitted to the base at operating speed ; and

(iii) the natural frequency of the system. **SPPU - May 14**

Ex. 4.5.9 : A machine of mass 68 kg is mounted on springs of stiffness 11,000 N/cm with a damping factor of 0.2. A piston, within the machine, of mass 2 kg had a reciprocating motion with a stroke of 7.5 cm and a speed of 3000 r.p.m. Assuming the motion of the piston to be simple harmonic, determine :

(i) The amplitude of vibration of machine;

(ii) The phase angle with respect to exciting force;

(iii) The transmissibility and the force transmitted to the foundation; and

(iv) The phase angle of the transmitted force with respect to the exciting force. **SPPU - Dec. 08**

Soln. :

Data : $m = 68 \text{ kg}$;

$K = 11000 \text{ N/cm} = 1100 \times 10^2 \text{ N/m}$;

$\xi = 0.2$; $m_0 = 2 \text{ kg}$;

$S = 75 \text{ mm}$; $N = 3000 \text{ r.p.m.}$

$$r = \frac{S}{2} = \frac{75}{2} = 37.5 \text{ mm}$$

$$\omega = \frac{2\pi N}{\omega} = \frac{2\pi \times 3000}{60}$$

$$= 314.16 \text{ rad/sec}$$

1. **Amplitude of Vibration :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{11000}{68}}$$

$$= 127.18 \text{ rad/sec}$$

- **Frequency ratio :**

$$\frac{\omega}{\omega_n} = \frac{314.16}{127.18} = 2.47$$

- **Amplitude of vibration :**

$$X = \frac{\left(\frac{m_0 r}{m}\right)\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{\left(\frac{2 \times 0.0375}{68}\right)(2.47)^2}{\sqrt{\left[1-(2.47)^2\right]^2 + \left[2 \times 0.2 \times 2.47\right]^2}}$$

$$= 1.3 \times 10^{-3} \text{ m}$$

or $X = 1.3$ mm

2. **Phase Angle w.r.t. Exciting Force :**

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] = \tan^{-1}\left[\frac{2 \times 0.2 \times 2.47}{1-(2.47)^2}\right]$$

or $\phi = 169.04°$ or $349.04°$

3. **Transmitted Force :**

- **Impressed force :**

$$F_0 = m_0 \omega^2 r$$

$$= 2 \times (314.16)^2 \times 0.0375 = 7402.24 \text{ N}$$

- **Transmissibility :**

$$T_r = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{\sqrt{1+(2 \times 0.2 \times 2.47)^2}}{\sqrt{\left[1-(2.47)^2\right]^2 + \left[2 \times 0.2 \times 2.47\right]^2}}$$

$$T_r = 0.2705$$

- **For transmitted force :**

$$T_r = \frac{F_T}{F_0}$$

$$0.2705 = \frac{F_T}{F_0}$$

$$F_T = 0.2705\, F_0 = 0.2705 \times 7402.24$$

$$F_T = 2002.3 \text{ N}$$ ...Ans.

4. **Phase Angle of Transmitted Force w.r.t. Exciting Force :**

$$(\phi - \alpha) = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right]$$

$$(\phi - \alpha) = \tan^{-1}\left[\frac{2 \times 0.2 \times 2.47}{1-(2.47)^2}\right] - \tan^{-1}\left[2 \times 0.2 \times 2.47\right]$$

$$(\phi - \alpha) = 124.38° \text{ or } 304.38°$$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 4.5.10 : A machine of 100 kg is supported on spring of total stiffness 700 kN/m and has an unbalanced rotating element, which results in a disturbing force of 300 N at a speed of 2500 rpm. Assuming a damping factor of 0.25, determine :

(i) the amplitude of vibration ;

(ii) the transmissibility ; and

(iii) the transmitted force. **SPPU - Dec. 13, 8 Marks**

Ex. 4.5.11 : A 75 kg machine is mounted on springs of stiffness 11.76×10^5 N/m with an assumed damping factor of 0.20. A 2 kg piston within the machine has a reciprocating motion with a stroke of 0.08 m and a speed of 3000 cycles per minutes. Assuming the motion of the piston to be harmonic, determine the amplitude of vibration of the machine and the vibratory force transmitted to the foundation. **SPPU - Oct. 19(In Sem.), 6 Marks**

Ex. 4.5.12 : An instrument panel of an aircraft is mounted on isolators. The isolator has a negligible damping and it deflects 6 mm under the weight of 30 kg. Find the percentage of motion transmitted to the instrument board, if the vibration of the aircraft is at 2400 r.p.m.

Ex. 4.5.13 : A body of mass 80 kg is suspended from a spring which deflects 20 mm due to the mass. If the body is subjected to a periodic disturbing force of 700 N and of frequency equal to 0.63 times the natural frequency, find :

(i) The amplitude of forced vibration ;

(ii) Transmissibility ;

(iii) The dynamic Magnification Factor ; and

(iv) The force transmitted to support.

Ex. 4.5.14 : A machine having a mass of 1000 kg is mounted on the rubber pad having stiffness of 2000 kN/m and equivalent viscous damping coefficient of 1050 N-sec/m. If the machine is subjected to external disturbing harmonic force of 0.6 kN at the frequency of 6π rad/sec, determine :

(i) The amplitude of vibration of machine

(ii) The maximum force transmitted to the foundation because of unbalance force

(iii) The transmissibility ; and

(iv) the magnification factor.

Ex. 4.5.15 : A vibratory body of mass 150 kg is supported on springs of total stiffness 1050 kN/m. It has a rotating unbalanced force of 525 N at a speed of 6000 r.p.m. If the damping factor is 0.3,

Determine :

(i) the amplitude caused by the unbalance ;

(ii) its phase angle ; and

(ii) the transmissibility .

Ex. 4.5.16 : A reciprocating air compressor has a mass of 1000 kg running at 1500 r.p.m. The equivalent reciprocating parts of compressor are of 1 kg and stroke length is 0.2m. To reduce the effect of vibration, isolators of rubber, having static deflection of 2 mm under compressor weight and an estimated damping factor of 0.2 are used. determine :

(i) the amplitude of vibration of compressor ;

(ii) the force transmitted to foundation ;

(iii) the phase lag ;

(iv) the phase angle between transmitted force ; and exciting force.

(v) the speed at which maximum amplitude of vibration would occur. **SPPU - May 12, 8 Marks**

Ex. 4.5.17 : A machine having a mass of 100 kg is mounted on the springs and damper. The total stiffness of the springs is 7.84×10^5 N/m while the damping ratio of the damper is 0.2. A vertical harmonic force F = 392 sin (314.15t) N acts on the machine. For the steady state vibrations of the system, determine :

(i) the amplitude of vibration of the machine ;

(ii) the transmissibility ;

(iii) the transmitted force ; and

(iv) the phase difference between motion and the exciting force. **SPPU - Dec. 11, 10 Marks**

Ex. 4.5.18 : An exhaust fan, rotating at 1000 rpm, is to be supported by four springs, each having a stiffness of K. If only 10 percent of the unbalanced force of the fan is to be transmitted to the base, what should be the value of K? Assume the mass of the exhaust fan to be 40 kg. **SPPU - May 18, 8 Marks**

Ex. 4.5.19 : A machine of one tone is acted upon by an external force of 2450 N at a frequency of 1500 r.p.m. To reduce the effects of vibration, isolator of rubber having a static deflection of 2mm under the machine load and an estimated damping ratio $\xi = 0.2$ is used. Determine :

(i) the force transmitted to foundation.

(ii) the amplitude of vibration of machine, and

(iii) the phase lag. **SPPU - May 16**

Ex. 4.5.20 : A single cylinder vertical petrol engine of total mass 400 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2.5 mm. The reciprocating parts of the engine have a mass of 5 kg and move through a vertical stroke of 120 mm with SHM. A dashpot provided, the damping resistance of which is directly proportional to the velocity and amounts to 20 N at 1m/s, if a steady state vibrations has been reached.

determine :

a) The amplitude of forced vibration when the driving shaft of engine rotates at 540 rpm

b) The maximum dynamic force transmitted to the ground through chassis frame (which behaves as a spring), through the dashpot and through the chassis frame and dashpot together.

c) The driving shaft speed at which resonance will occur. **SPPU - Oct. 18(In sem), 10 Marks**

Soln. :

$$m = 400 \text{ kg} \quad ; \quad \delta = 2.5 \text{ mm} = 0.0025 \text{ m} ;$$

$$m_o = 5 \text{ kg} \quad ; \quad S = 120 \text{ mm} = 0.12 \text{ m};$$

$$F = 20 \text{ kN} = 20 \times 10^3 \text{ N};$$

$$v = 1 \text{ m/s} \quad ; \quad N = 540 \text{ r.p.m.};$$

$$r = \frac{S}{2} = \frac{0.12}{2} = 0.06 \text{ m}$$

- **Damping resistance :**

$$c = \frac{F}{v} = \frac{20 \times 10^3}{1} = 20 \times 10^3 \text{ N-sec. m}$$

- **Load taken by each spring** $= \dfrac{mg}{4}$

- **Stiffness of each spring :**

$$K = \frac{mg/4}{\delta}$$

$$= \frac{400 \times 9.81}{4 \times 0.0025} = 392.4 \text{ N/m}$$

- **Equivalent stiffness of four springs :**

$$K_e = 4K = 4 \times 392.4$$

or $$K_e = 15.696 \times 10^5 \text{ N/m}$$

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K_e}{m}} = \sqrt{\frac{15.696 \times 10^5}{400}}$$

$$= 62.64 \text{ rad/sec}$$

- **Damping factor (ξ) :**

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 540}{60}$$

$$= 56.54 \text{ rad/sec}$$

and $$\frac{\omega}{\omega_n} = \frac{56.54}{62.64} = 0.90$$

$$\therefore \quad \xi = \frac{C}{2m\,\omega_n} = \frac{20 \times 10^3}{2 \times 400 \times 62.64}$$

$$\therefore \quad \xi = 0.399$$

1. **Speed of driving shaft at resonance :**

$$N_n = \frac{\omega_n \times 60}{2\pi} = \frac{62.64 \times 60}{2\pi}$$

$$\mathbf{N_n = 598.16 \text{ rpm.}} \quad \text{...Ans.}$$

2. **Amplitude of steady vibration (X) :**

$$X = \frac{\left(\frac{m_o r}{m}\right)\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$X = \frac{\left(\frac{5 \times 0.06}{400}\right)(0.90)^2}{\sqrt{[1-0.90^2]^2 + [2 \times 0.399 \times 0.90]^2}}$$

$$= \frac{6.075 \times 10^{-4}}{\sqrt{0.0361 + 0.5158}}$$

$$\mathbf{X = 8.17 \times 10^{-4} \text{ m}}$$

OR $$\mathbf{X = 0.8177 \text{ mm}}$$

3) **The maximum force transmitted to the ground is ;**

$$F_t = KX = 15.696 \times 10^5 \times 8.17 \times 10^{-4} \text{ m}$$

$$= \mathbf{1282.36 \text{ N}} \quad \text{...Ans.}$$

The dynamic force transmitted to the foundation through dashpot is,

$$F_d = c\dot{x} = c\omega x = 20 \times 10^3 \times 56.54 \times 8.17 \times 10^{-4}$$

$$= 923.86 \text{ N}$$

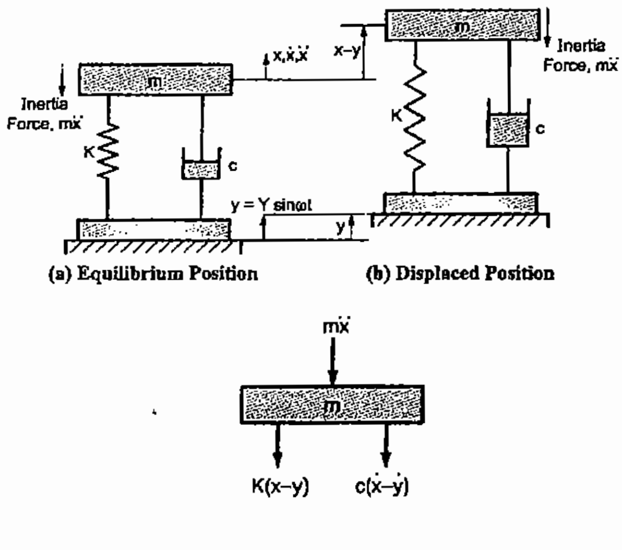
The spring force F_t & the force in dashpot F_d are out of phase by 90° and therefore maximum force transmitted to the ground through chassis & dashpot both is,

$$F = \sqrt{F_t^2 + F_d^2} = \sqrt{(1282.36)^2 + (923.86)^2}$$

$$\mathbf{F = 1580.49 \text{ N}} \quad \text{... Ans.}$$

4.6 FORCED VIBRATIONS DUE TO EXCITATION OF SUPPORT INSTEAD OF MASS

- In many cases, the excitation of the system is through the support or base instead of being applied to the mass. For example, the control panel of a machine. The vibrations of control panel is due to the excitation of the base frame.
- In such case, the support or base is considered to be excited by a sinusoidal motion $y = Y \sin \omega t$ as shown in Fig. 4.6.1.
- The analysis of forced vibrations due to excitation of support can be done by considering **absolute amplitude** and **relative amplitude** of mass with respect to the support or base.



(c) F.B.D. of Mass

Fig. 4.6.1 : Forced Vibration Due to Excitation of Support

4.6.1 Absolute Amplitude :

University Question

Q. Derive an expression for determining the absolute amplitude of vibration in case of the excitation of the base. **SPPU : Dec. 11**

Basic Parameters :

Let, y = absolute harmonic displacement of support $= Y \sin \omega t$

x = absolute displacement of the mass '**m**'

$$y = Y \sin \omega t$$

$$\therefore \quad \dot{y} = Y \omega \cos \omega t$$

Deflection of spring $= (x - y)$

Velocity between two ends of damper $= (\dot{x} - \dot{y})$

Differential Equation of Motion :

- The F.B.D. of mass in displaced position is shown in Fig. 4.6.1(c). The differential equation of motion can be written as,

$$m\ddot{x} + c(\dot{x} - \dot{y}) + K(x - y) = 0 \quad \text{...(a)}$$

$$\therefore \quad m\ddot{x} + c\dot{x} - c\dot{y} + Kx - Ky = 0$$

$$m\ddot{x} + c\dot{x} + Kx = Ky + c\dot{y}$$

$$m\ddot{x} + c\dot{x} + Kx = K(Y \sin \omega t) + c(Y \omega \cos \omega t)$$

$$m\ddot{x} + c\dot{x} + Kx = Y(K \sin \omega t + c\omega \cos \omega t)$$

$$\therefore m\ddot{x} + c\dot{x} + Kx = Y\sqrt{K^2 + c^2\omega^2}\left(\frac{K}{\sqrt{K^2 + c^2\omega^2}} \sin \omega t + \frac{c\omega}{\sqrt{K^2 + c^2\omega^2}} \cos \omega t\right)$$

$$\text{or } m\ddot{x} + c\dot{x} + Kx = Y\sqrt{K^2 + c^2\omega^2}\,[\cos\alpha \sin \omega t + \sin \alpha \cos \omega t]$$

where, $$\cos \alpha = \frac{K}{\sqrt{K^2 + c^2\omega^2}} \quad \text{...(b)}$$

and $$\sin \alpha = \frac{c\omega}{\sqrt{K^2 + c^2\omega^2}} \quad \text{...(c)}$$

$$\therefore \quad m\ddot{x} + c\dot{x} + Kx = Y\sqrt{K^2 + c^2\omega^2}\,\sin(\omega t + \alpha) \quad \text{...(4.6.1)}$$

- Equation (4.6.1) is known as **absolute differential equation of motion** which is similar to Equation (4.2.1) i.e. $m\ddot{x} + c\dot{x} + Kx = F_o \sin \omega t$.

where, $$F_o = Y\sqrt{K^2 + c^2\omega^2} \quad \text{...(4.6.2)}$$

Complete Solution to Differential Equation :

- The complete solution of the Equation (4.6.1) is given by,

$$x = x_c + x_p \quad \text{...(d)}$$

where, $x_c = X_1 e^{-\xi\omega_n t} \sin(\omega_d t + \phi)$

$$x_p = X \sin(\omega t + \alpha - \phi)$$

or $$x_p = X \sin[\omega t - (\phi - \alpha)] \quad \text{...(e)}$$

X = steady state absolute amplitude

$\phi - \alpha$ = phase angle

1. Steady-state absolute amplitude (X) :

- The steady-state absolute amplitude '**X**' is given by,

$$X = \frac{Y\sqrt{K^2 + c^2\omega^2}\,/\,K}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

[From Equation (4.2.9)]

$$X = \frac{Y\sqrt{1 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad \text{...(4.6.3)}$$

or $$\frac{X}{Y} = \frac{\sqrt{1 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...(4.6.4)}$$

- **Observations made from Equation (4.6.4) :**

(i) **When $\omega << \omega_n$, then $\frac{X}{Y} \simeq 1$** : It means that the complete system can move as a rigid body at low frequencies.

(ii) **When $\omega >> \omega_n$, then $\frac{X}{Y} \simeq 0$** : It means that the mass is stationary at high frequencies.

2. Phase Angle $(\phi - \alpha)$:

- From Equation (b) and (c) we can write

$$\tan \alpha = \frac{c\omega}{K} = 2\xi\frac{\omega}{\omega_n}$$

$$\therefore \quad \alpha = \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right]$$

- From Equation (4.2.12),

$$\phi = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right]$$

- **Phase angle :** The displacement of the mass (i.e. x) lags that of the support (i.e. y) by an angle $(\phi - \alpha)$. Hence the phase angle is,

$$(\phi - \alpha) = \tan^{-1}\left[\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] - \tan^{-1}\left[2\xi\frac{\omega}{\omega_n}\right] \quad ...(4.6.5)$$

4.6.2 Relative Amplitude :

Basic Parameters :

Let, z = relative displacement of mass with respect to the support

$= (x - y)$

$$\left.\begin{aligned} \therefore \dot{z} &= (\dot{x} - \dot{y}) \\ \therefore \ddot{z} &= (\ddot{x} - \ddot{y}) \\ \text{and} \quad \ddot{x} &= \ddot{z} + \ddot{y} \end{aligned}\right\} \quad ...(f)$$

Differential Equation of Motion :

- The differential Equation of motion for the system is,

$$m\ddot{x} + c(\dot{x} - \dot{y}) + K(x - y) = 0 \quad ...(g)$$

- Substituting Equations (f) in Equation (g), we get,

$$m(\ddot{z} + \ddot{y}) + c\dot{z} + Kz = 0$$

$$m\ddot{z} + c\dot{z} + Kz = -m\ddot{y}$$

$$\therefore m\ddot{z} + c\dot{z} + Kz = m\omega^2 Y \sin\omega t ...[\because \ddot{y} = -Y\omega^2 \sin\omega t] \quad ...(4.6.6)$$

- Equation (4.6.6) is known as **relative equation of motion** which is similar to Equation (4.2.1)

i.e. $m\ddot{x} + c\dot{x} + Kx = F_o \sin\omega$

Hence, $F_o = m\omega^2 Y$.

Complete Solution of Differential Equation :

$$z = z_c + z_p \quad ...(h)$$

$$\left.\begin{aligned} \text{where,} \quad z_c &= Z_1 e^{-\xi\omega_n t} \sin[\omega_d t + \phi_1] \\ z_p &= Z\sin(\omega t - \phi) \end{aligned}\right\} \quad ...(i)$$

z = steady state relative amplitude

ϕ = phase angle

(i) Steady-state relative amplitude (Z) :

The steady-state amplitude Z is given by the Equation similar to Equation (4.2.4). The steady-state relative amplitude is given by,

$$Z = \frac{m\omega^2 Y / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$Z = \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad ...(4.6.7)$$

(ii) Phase angle :

The angle by which the relative displacement 'z' lags the support displacement 'y' is given by,

$$\phi = \tan^{-1}\left[\frac{2\xi\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right] \quad ...(4.6.8)$$

Ex. 4.6.1 : The suspension system of a vehicle has a spring constant of 500 kN/m and a damping ratio of 0.5. The vehicle has a speed of 80 km/hr. The road surface varies sinusoidal with an amplitude of 10 cm and a wavelength of 5 m. If the mass of a vehicle is 1000 kg, determine its amplitude of oscillations. **SPPU - May 04**

Soln. :

Given : $K = 500 \times 10^3$ N/m ; $\xi = 0.5$;

$v = 80$ km/hr ; $Y = 10$ cm $= 0.1$m ;

$\lambda = 5$ m ; $m = 1000$ kg.

Speed of vehicle, $v = 80$ km/hr

$$= \frac{80 \times 10^3}{3600} = 22.22 \text{ m/sec}$$

1. Circular Frequency of Forced Vibrations (ω):

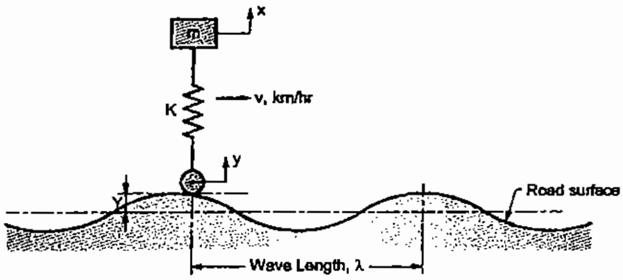


Fig. P. 4.6.1

- **Time period :**

$$t_p = \frac{\text{Wavelength}}{\text{Velocity}} = \frac{\lambda}{v}$$

$$= \frac{5}{22.22} = 0.225 \text{ s}$$

- **Circular frequency of forced vibrations :**

$$\omega = \frac{2\pi}{t_p} = \frac{2\pi}{0.225}$$

$$= 27.92 \text{ rad/s.}$$

2. **Vertical Amplitude of Vibration of Vehicle (X) :**

- **Natural circular frequency :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{500 \times 10^3}{1000}}$$

$$= 22.36 \text{ rad/s}$$

- **Steady-state amplitude due to excitation of support :**

$$X = \frac{Y\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{0.1\sqrt{1+\left(2\times 0.5\times\frac{27.92}{22.36}\right)^2}}{\sqrt{\left[1-\left(\frac{27.92}{22.36}\right)^2\right]^2+\left[2\times 0.5\times\left(\frac{27.92}{22.36}\right)\right]^2}}$$

or $X = 0.11$ m ...Ans.

Ex. 4.6.2 : A vehicle has a mass of 500 kg and the total spring constant of its suspension system is 19600 N/m. The profile of the road may be approximated as a sine wave of amplitude 10 mm and a wavelength of 1.5 m. Determine :

1. Critical speed of the vehicle ; and
2. Amplitude of steady state motion of mass :
   (a) when driven at critical speed and without damping.
   (b) When driven at critical speed and having damping factor of 0.5.
   (c) when driven at 50 km/hr and having damping factor 0.4. **SPPU - May 03**

Soln. :

Given : $m = 500$ kg ; $K = 19600$ N/m ; $Y = 10$ mm $= 0.01$ m ; $\lambda = 1.5$ m.

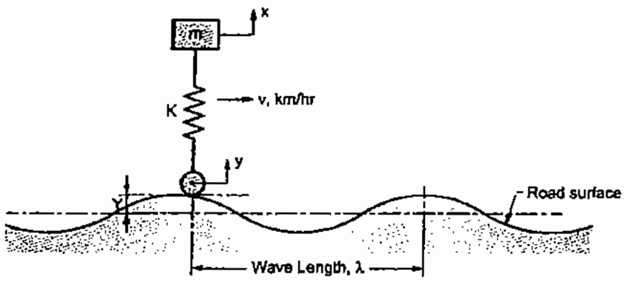


Fig. P. 4.6.2

1. **Critical Speed of Vehicle (V_c) :**

Natural circular frequency :

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{19600}{500}} = 6.261 \text{ rad/s.}$$

$$\text{Time period} = \frac{\text{Wavelength}}{\text{Velocity}}$$

$$\therefore \quad t_p = \frac{\lambda}{v}$$

$$\therefore \quad \frac{2\pi}{\omega} = \frac{\lambda}{v}$$

- **The critical speed of vehicle will be at resonance. (i.e. when $\omega = \omega_n$)**

$$\therefore \quad \frac{2\pi}{\omega_n} = \frac{\lambda}{v_c}$$

$$\therefore \quad \frac{2\pi}{6.261} = \frac{1.5}{v_c}$$

$$\therefore \quad v_c = 1.4947 \text{ m/s}$$

$$\therefore \quad v_c = \frac{1.4947 \times 3600}{1000}$$

$$\therefore \quad v_c = 5.3809 \text{ km/hr} \quad \text{...Ans.}$$

2. **Amplitude of Steady-State Motion (X) :**

The steady-state amplitude due to excitation to support is,

$$X = \frac{Y\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...(a)}$$

(i) When $\xi = 0$ and $\omega = \omega_n$:

$$X = \frac{0.01 \times \sqrt{1+0}}{\sqrt{[1-1]^2+0}} = \infty$$

$$\therefore \quad X = \infty \quad \text{...Ans.}$$

(ii) When $\xi = 0.5$ and $\omega = \omega_n$:

$$X = \frac{Y\sqrt{1+(2\xi)^2}}{2\xi}$$

$$= \frac{0.01\sqrt{1+(2\times 0.5)^2}}{2\times 0.5}$$

$$= 14.142 \times 10^{-3} \text{ m}$$

or $X = 14.142$ mm ...Ans.

(iii) When $\xi = 0.4$ and $v = 50$ km/hr :

$$v = \frac{50 \times 1000}{360} = 13.88 \text{ m/s}$$

We know that,

$$t_p = \frac{\lambda}{v}$$

$$\therefore \quad \frac{2\pi}{\omega} = \frac{\lambda}{v}$$

$$\frac{2\pi}{\omega} = \frac{1.5}{13.88}$$

$\therefore \quad \omega = 58.17$ rad/s

The steady-state amplitude is,

$$X = \frac{Y\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{0.01\sqrt{1+\left(2\times0.4\times\frac{58.17}{6.261}\right)^2}}{\sqrt{\left[1-\left(\frac{58.17}{6.261}\right)^2\right]^2+\left[2\times0.4\left(\frac{58.17}{6.261}\right)\right]^2}}$$

$$= 8.68\times10^{-4}\text{ m}$$

or $X = \mathbf{0.868\ mm}$ ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex. 4.6.3 : A vehicle has a mass of 490 kg and the total spring constant of suspension system is 58800 N/m. If the profile of a road may be approximated to a sine wave of amplitude 40 mm and wave length 4 m, determine :

(i) the a critical speed of vehicle ;

(ii) the amplitude of steady state vibration of the mass when the vehicle is driven at critical speed and damping factor = 0.5 ; and

(iii) the amplitude of steady state motion of mass when the vehicle is driven at 57 km/hr and damping factor = 0.5.

SPPU - May 03

Ex. 4.6.4 : The static deflection of an automobile spring under its weight is 10 cm. Find the critical speed when the trailer is traveling over a road with a profile approximated by a sine wave of amplitude 8 cm and wavelength of 16 m. If damping factor is ξ = 0.05, what will be the amplitude of vibration at 75 km/hr ?

SPPU - Dec.14, Dec.15, Dec. 18(In sem), 10 Marks

Soln. :

Given : δ = 10 cm = 0.1 m ; Y = 8 cm = 0.08 m ;

λ = 16 m ; ξ = 0.05.

$$v = 70\text{ km/hr} = \frac{70\times1000}{3600} = 19.44\text{ m/s}$$

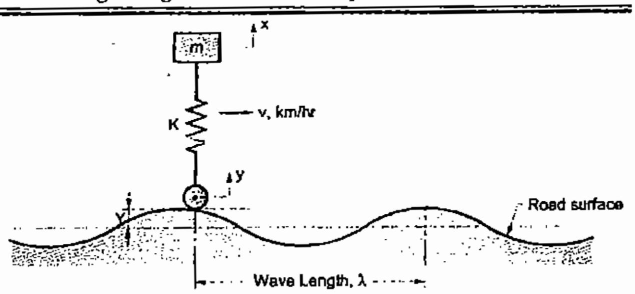


Fig. P. 4.6.4

1. **Frequency Ratio$\left(\frac{\omega}{\omega_n}\right)$:**

$$\therefore \omega_n = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.1}} = 9.90\text{ rad/s}$$

$$\text{Time period} = \frac{\text{Wavelength}}{\text{Velocity}}$$

$$\therefore t_p = \frac{\lambda}{v}$$

$$\therefore \frac{2\pi}{\omega} = \frac{\lambda}{v}$$

$$\therefore \frac{2\pi}{\omega} = \frac{16}{19.44}$$

$$\therefore \omega = 8.82\text{ rad/s.}$$

$$\therefore \frac{\omega}{\omega_n} = \frac{8.82}{990} = 0.89$$

2. **Steady - state Amplitude (X) :**

For forced vibrations due to excitation of support, the steady-state amplitude is given by,

$$X = \frac{Y\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad ...(a)$$

$$= \frac{0.08\times\sqrt{1+(2\times0.05\times0.89)^2}}{\sqrt{[1-(0.89)^2]^2+[2\times0.05\times0.89]^2}}$$

$$= 0.3551$$

$X = \mathbf{355.1\ mm}$ ...Ans.

3. **Critical Speed of Vehicle (v_c) :**

At the critical speed of vehicle, $\omega = \omega_n$, therefore,

$$\frac{2\pi}{\omega_n} = \frac{\lambda}{v_c}$$

$$\therefore \frac{2\pi}{9.90} = \frac{16}{v_c}$$

$$\therefore v_c = 25.21\text{ m/sec} = \frac{25.21\times3600}{1000}$$

or $v_c = \mathbf{90.75\ km/hr}$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 4.6.5 : The springs of an automobile trailer are compressed 0.1 m under its own weight. Find the critical speed when the trailer is travelling over a road with a profile approximated by a sine wave of amplitude 0.08 m and wavelength of 14 m. What will be the amplitude of vibration at 60 km/hour ?

SPPU - May 13, 6 Marks, May 14, 8 Marks

Ex. 4.6.6 : A motor car moving with a speed of 100 km/hr has a gross mass of 1500 kg. It passes over a rough road which has a sinusoidal surface with an amplitude of 75 mm and a wavelength of 5 m. The suspension system has a spring constant of 500 N/mm and damping ratio of 0.5. Determine the displacement amplitude of the car and time lag.

SPPU - May 02, Dec. 13, 8 Marks

Ex. 4.6.7 : A machine of 200 N is supported by a spring and dashpot. The spring is stretched by 100 mm due to weight of machine and the dashpot has a coefficient of damping 800 N-s/m. If the support is vibrating with an amplitude of 20 mm and frequency of 8 rad/sec, determine:

(i) the amplitude of machine ;

(ii) the relative amplitude between machine and support ; and

(iii) the amplitude of machine when dashpot is removed and frequency of support is (a) 8 rad/sec (b) equal to natural frequency of machine.

Ex. 4.6.8 : An automobile, weighing 9.8 kN when fully loaded and 2.45 kN when empty, vibrates in a vertical direction while travelling at 96 km/hr on a rough road having a sinusoidal wave form with an amplitude Y and wavelength 4.88m. Assuming that the automobile can be modeled as a single degree-of-freedom system with stiffness 350 kN/m and the damping factor 0.5. Determine the amplitude ratio of the vehicle when fully loaded and when empty.

SPPU - Oct 16 (In Sem), 6 Marks

Ex. 4.6.9 : An instrument of 50 kg mass is located in an airplane cabin which vibrates at 2000 r.p.m. with an amplitude of 0.1 mm. Determine the stiffness of the four springs to be used as supports for the instrument to reduce its amplitude to 0.005 mm. Also calculate the maximum total load for which each spring must be designed.

SPPU - May 16

Soln. :

Given : $m = 50$ kg ; $N = 2000$ r.p.m. ; $Y = 0.1$ mm ; $X = 0.005$ mm

1. **Transmissibility :**

and $$T_r = \frac{X}{Y} = \frac{0.005}{0.1} = 0.05$$

2. **Natural Circular Frequency (ω_n) :**

$$\therefore \omega = \frac{2\pi \times 2000}{60} = 209.43 \text{ rad/sec.}$$

$$T_r = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad \text{...(a)}$$

- Since damping is not present (i.e. $\xi = 0$), Equation (a) becomes,

$$T_r = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2}} = \frac{1}{\sqrt{\left[\left(\frac{\omega}{\omega_n}\right)^2-1\right]^2}}$$

or $$T_r = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2-1}$$

$$\therefore 0.05 = \frac{1}{\left(\frac{209.43}{\omega_n}\right)^2-1}$$

$$\therefore \frac{2193.04}{\omega_n^2} - 0.05 = 1 ;$$

$$\therefore \omega_n^2 = 2088.60$$

$$\therefore \omega_n = 45.70 \text{ rad/s.}$$

3. **Spring Constant (K) :**

- Four springs having stiffness K are used. Assuming the springs are in parallel, the equivalent spring stiffness is,

$$K_e = 4K$$

$$\omega_n = \sqrt{\frac{K_e}{m}}$$

or $$\omega_n = \sqrt{\frac{4K}{m}}$$

$$\therefore 45.70 = \sqrt{\frac{4K}{30}}$$

$$\therefore K = 15663.67 \text{ N/m} \quad \text{...Ans.}$$

4. **Relative Amplitude (Z) :**

$$Z = \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\left(\frac{\omega}{\omega_n}\right)^2-1} \qquad [\because \xi = 0]$$

$$= \frac{0.1\times10^{-3}\times4.58^2}{(4.58)^2-1} \quad \left[\because \frac{\omega}{\omega_n}=\frac{209.43}{45.70}=4.58\right]$$

$$\therefore \quad Z = 1.05\times10^{-4}\ \text{m}$$

5. **Dynamic Load on Each Isolator (F_s) :**

The spring force is,

$$F_s = KZ = 15663.67\times1.05\times10^{-4}$$

or $\quad F_s = $ **1.64 N** ...Ans.

Ex. 4.6.10 : A radio set of 20 kg mass must be isolated from a machine vibrating with an amplitude of 0.05 mm at 500 cpm. The set is mounted on four isolators, each having a spring scale of 31400 N/m and damping factor of 392 N-s/m.

(i) What is the amplitude of vibration of the radio ?

(ii) What is the dynamic load on each isolator due to vibration ? **SPPU - Dec. 10**

Soln. :

Given : $m = 20$ kg ; $Y = 0.05$ mm ;

$N = 500$ r.p.m. ; $K = 31400$ N/m ;

$K_e = 4\times31400 = 125600$; $c = 392$ N-s/m.

$\therefore \ c_e = 4\times392 = 1568$ N-sec/m

$$\omega = \frac{2\pi N}{60} = \frac{2\times\pi\times500}{60} = 52.35\ \text{rad/s}$$

1. **Amplitude of Vibration :**

- **Natural frequency of system :**

$$\omega_n = \sqrt{\frac{K_e}{m}} = \sqrt{\frac{125600}{20}} = 79.24\ \text{rad/s}$$

- **Damping factor :**

$$\xi = \frac{c_e}{2m\omega_n} = \frac{392\times4}{2\times20\times(79.24)} = 0.4946$$

- **Amplitude of vibration (X) :**

$$\therefore \quad X = \frac{Y\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{0.05\sqrt{1+\left(2\times0.496\times\frac{52.35}{79.24}\right)^2}}{\sqrt{\left[1-\left(\frac{52.35}{79.24}\right)^2\right]^2+\left[2\times0.4946\times\frac{52.35}{79.24}\right]^2}}$$

$X = $ **0.069 mm** ...Ans.

2. **Dynamic Load :**

- **Relative amplitude (Z) :**

$$Z = \frac{Y\left(\frac{\omega}{\omega_n}\right)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$= \frac{0.05\left(\frac{52.35}{79.24}\right)^2}{\sqrt{\left[1-\left(\frac{52.35}{79.24}\right)^2\right]^2+\left[2\times0.4946\times\frac{52.35}{79.24}\right]^2}}$$

$\therefore \quad Z = 0.025$ mm

- **Spring force :**

$$F_s = K_e\times Z$$
$$= 125600\times0.025 = 3140$$

- **Damping force :**

$$F_d = c_e\,\omega Z$$
$$= 1508\times52.35\times(0.025)$$
$$= 2052.12$$

- **Dynamic Load :**

$$\therefore \quad F = \sqrt{F_s^2+F_d^2} = 3751.10$$

$F = $ **3.75 N** ...Ans.

4.7 QUALITY FACTOR (Q) AND BANDWIDTH (Δω)

University Questions

Q. Define quality factor and state its significance in frequency response curve. **SPPU : Dec. 14**

☞ **Magnification Factor :**

For a system subjected to forced vibrations, the value of magnification factor or amplitude ratio is given by,

$$M.F = \frac{X}{X_{st}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \qquad ...(4.7.1)$$

...(from Equation 4.2.18)

Quality Factor (Q) :

- The magnification factor or amplitude ratio of a system at resonance (i.e. $\omega = \omega_n$) is called as **quality factor** or **Q factor** of the system.

Substituting $\omega = \omega_n$ in Equation (4.7.1) by,

$$Q = \frac{X}{X_{st}} = \frac{1}{2\xi} \quad ...(4.7.2)$$

Bandwidth :

- Fig. 4.7.1 shows a response curve of amplitude ratio (X/X_{st}) versus frequency ratio (ω/ω_n) for values of ξ less than 0.1. Let, r_1 and r_2 are the frequency ratio where, the amplitude or amplification ratio falls to $Q/\sqrt{2}$ and are called as **half power points.**
- *The difference between the frequencies associated with the half power points r_1, and r_2 is called as **bandwidth** of the system.*

$$\Delta\omega = (\omega_2 - \omega_1)$$

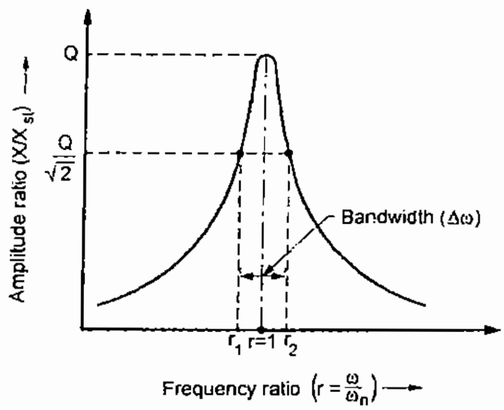


Fig 4.7.1 : Response curve showing bandwidth at half power points

Half Power Points :

Frequency ratio r_1 and r_2 : To find the values of frequency ratio r_1 and r_2, substitute $X/X_{st} = Q/\sqrt{2}$ in Equation (4.7.1),

$$\therefore \frac{Q}{\sqrt{2}} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\text{or} \quad \frac{Q}{\sqrt{2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \quad ...\left[\text{Putting } \frac{\omega}{\omega_n} = r\right]$$

$$\text{or} \quad \frac{1}{2\xi\sqrt{2}} = \frac{1}{\sqrt{(1-r^2)^2+(2\xi r)^2}} \quad ...(4.7.3)$$

Squaring both sides of Equation (4.7.3) we get,

$$r^4 - (2-4\xi^2)r^2 + (1-8\xi^2) = 0 \quad ...(4.7.4)$$

Solving quadratic Equation (4.7.4) in r^2 we get,

$$r^2 = (1-2\xi^2) \pm 2\xi\sqrt{1-\xi^2} \quad ...(4.7.5)$$

For small values of ξ, $(1-2\xi^2) \approx 1$ and $\sqrt{1-\xi^2} \approx 1$,
Therefore, Equation (4.7.5) becomes,

$$r^2 = 1 \pm 2\xi \quad ...(4.7.6)$$

Using binomial expansion,

$$r = \sqrt{(1 \pm 2\xi)} = 1 \pm \xi \pm \text{higher power terms of } \xi$$

Neglecting the higher power terms of ξ, the values of r_1 and r_2 are approximated as,

$$r_1 = \frac{\omega_1}{\omega_n} = (1-\xi) \quad ...(4.7.7)$$

$$\text{and} \quad r_2 = \frac{\omega_2}{\omega_n} = (1+\xi) \quad ...(4.7.8)$$

Expression for Bandwidth ($\Delta\omega$) :

$$\therefore \quad \omega_1 = (1-\xi)\omega_n \quad ...(4.7.9)$$

$$\text{and} \quad \omega_2 = (1+\xi)\omega_n \quad ...(4.7.10)$$

Therefore, the bandwidth ($\Delta\omega$) is given by,

$$\Delta\omega = \omega_2 - \omega_1 = [(1-\xi)\omega_n - (1+\xi)\omega_n]$$

$$\Delta\omega = 2\xi\omega_n \quad ...(4.7.11)$$

Form Equation (4.7.2) and (4.7.11), we can write,

$$\frac{\omega_n}{\Delta\omega} = \frac{\omega_n}{(\omega_2-\omega_1)} = \frac{\omega_n}{2\xi\omega_n} = \frac{1}{2\xi} = Q \quad ...(4.7.12)$$

4.8 INTRODUCTION TO CRITICAL SPEED OF SHAFTS

University Questions

Q. What do you mean by whirling of shaft ?

SPPU : Dec. 11, May 15, May 18

Q. Explain critical speed of shaft carrying single rotor.

SPPU : Oct. 18(In sem)

- **Whirling of shaft :** When a rotor is mounted on a shaft, its center of gravity usually does not coincide with the axis of rotation of the shaft. This center of gravity is normally displaced from the axis of rotation, although the amount of displacement may be very small. As a result of this initial eccentricity of the center of gravity from the axis of rotation, shaft is subjected to a centrifugal force when it begins to rotate.
- This centrifugal force acts radially outwards, which makes the shaft to bend in the direction of eccentricity

of the C. G. This further increases the eccentricity, and hence the magnitude of centrifugal force.

- In this way the effect is cumulative and ultimately the shaft may fail. Because of this unbalanced centrifugal force, a shaft starts vibrating violently in the direction perpendicular to the axis of the shaft. This phenomenon is known as **whirling of shaft.**
- **Critical speed (whirling speed) :** The speed at which the shaft starts to vibrate violently in the direction perpendicular to the axis of the shaft is known as **critical speed** or **whirling speed.**
- **Causes of displacement of centre of gravity of Whirling of shafts :**
  1. Eccentric mounting of the rotor on the shaft
  2. Lack of straightness of the shaft,
  3. Bending of shaft under the action of gravity in case of horizontal shaft,
  4. Non-homogeneous rotor material, and
  5. Unbalanced magnetic pull in case of electrical machinery.

4.9 CRITICAL SPEED OF SHAFT CARRYING SINGLE ROTOR WITHOUT DAMPING

University Questions

Q. Derive an expression for deflection of vertical shaft with a single rotor without damping. **SPPU : Dec. 11, May 15**

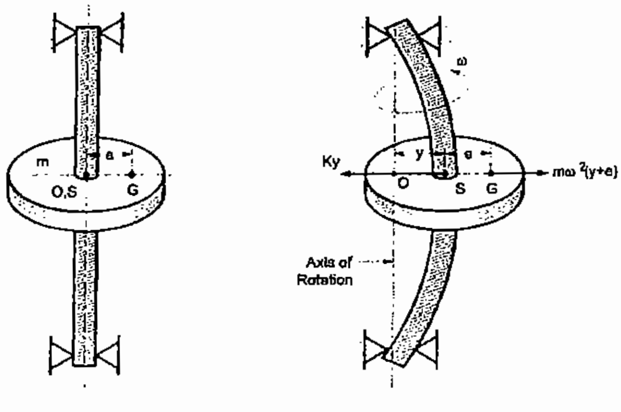
Q. Explain the whirling of the shaft, carrying a single rotor, without damping. Show that the deflection of the shaft is given by the expressions : $y = \dfrac{\left(\dfrac{\omega}{\omega_n}\right)^2 e}{\left[1-\left(\dfrac{\omega}{\omega_n}\right)^2\right]}$ **SPPU : May 12**

Q. What is the critical speed of the rotor without damping? Explain with the neat sketch. **SPPU : May 14**

Q. Obtain expression for excessive transverse vibrations of a simply supported shaft rotating at N rpm with a rotor of mass 'm' having eccentricity 'e'. **SPPU : May 18**

Vertical Rotating Shaft :

- Consider a vertical shaft having negligible inertia and carrying a single rotor, as shown in Fig. 4.9.1.
- Fig. 4.9.1(a) shows the shaft in stationary condition, while Fig. 4.9.1(b) shows the shaft in rotating condition.



(a) Shaft in Stationary Condition (b) Shaft in Rotating Condition

Fig. 4.9.1

Let, O = point of intersection of bearing centre line with the rotor. It is the point on the axis of rotation,

S = geometric centre of the rotor

G = centre of gravity of the rotor

m = mass of the rotor, kg

e = eccentricity of the rotor i.e. distance between the C.G. of rotor and geometric center 'S', m

y = deflection of geometric center 'S' from point 'O' due to centrifugal force, m

K = transverse stiffness of the shaft, N/m

ω = angular speed of the shaft, rad/s

ω_n = natural circular frequency of lateral or transverse vibrations of the shaft, rad/s

- **Forces Acting on Shaft in Rotating condition :**

There are two forces acting on the rotating shaft :

1. **Centrifugal Force = m ω^2 (y + e)** (radially outward through point G)
2. **Restoring Force = Ky** (radially inward through point G)

Equation of Motion :

- In equilibrium condition,

Centrifugal force = Restoring force

$$\therefore \quad m\omega^2 (y + e) = Ky$$

$$\therefore \quad m\omega^2 y + m\omega^2 e = Ky$$

$$Ky - m\omega^2 y = m\omega^2 e$$

$$y(K - m\omega^2) = m\omega^2 e$$

$$\therefore \quad y = \frac{m\omega^2 e}{K - m\omega^2} = \frac{\frac{m\omega^2 e}{K}}{1 - \frac{m\omega^2}{K}} = \frac{\frac{\omega^2 e}{K/m}}{1 - \left(\frac{\omega^2}{K/m}\right)}$$

$$\text{or} \quad y = \frac{\left(\frac{\omega}{\omega_n}\right)^2 e}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \qquad \left[\because \omega_n^2 = \frac{K}{m}\right] \qquad ...(4.9.1)$$

- **Observations from Equation (4.9.1) :**
  (i) As angular speed of the shaft 'ω' increases, the deflection of shaft 'y' increases.
  (ii) When $\omega = \omega_n$, the deflection of shaft y becomes infinity.

Critical Speed or Whirling Speed (ω_c) :

- *The speed of shaft at which the deflection of the shaft tends to be infinity is known as* **critical speed** or **whirling speed.**
- Critical speed or whirling speed of shaft is given by,

$$\omega_c = \omega_n$$

$$\text{or} \quad \omega_c = \sqrt{\frac{K}{m}}\text{ , rad/s} \qquad ...(4.9.2)$$

$$\text{or} \quad \omega_c = \sqrt{\frac{g}{\delta}}\text{ , rad/s}$$

$$\text{or} \quad N_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

$$\text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}\text{ r.p.s.} \qquad ...(4.9.3)$$

Where, N_c = Critical speed, in r.p.s.

δ = Static deflection of the shaft, m

- **Deflection of Shaft :** Equation (4.9.1) can be written as,

$$y = \frac{(\omega/\omega_c)^2 e}{1 - (\omega/\omega_c)^2} \qquad ...(4.9.4)$$

4.9.1 Ranges of Shaft Speed :

University Question

Q. Discuss the effect of shaft speed $\omega < \omega_c$, $\omega = \omega_c$ and $\omega > \omega_c$. ω_c is the critical speed of the shaft. **SPPU : May 13**

- From Equation (4.9.4) it is seen that, there are three ranges of shaft speed 'ω' :

1. Shaft speed (ω) < Critical speed (ω_c)
2. Shaft speed (ω) = Critical speed (ω_c)
3. Shaft speed (ω) > Critical speed (ω_c)

1. Shaft speed (ω) < Critical speed (ω_c)

- When the speed of shaft is less than the critical speed (i.e. $\omega < \omega_c$), the deflection of shaft 'y' is positive.
- In this speed range, the deflection of shaft 'y' and eccentricity 'e' are on opposite side of the geometric centre of the rotor 'S'.
- This means, the rotor rotates with heavy side outwards, as shown in Fig. 4.9.2(a).
- In this speed range, the deflection of shaft 'y' increases with shaft speed 'ω'

2. Shaft speed (ω) = Critical speed (ω_c) :

When the speed of shaft is equal to the critical speed (i.e. $\omega = \omega_c$), the deflection of shaft 'y' tends to be infinity and the shaft vibrates with large amplitude. This may lead to the failure of the shaft.

3. Shaft speed (ω) > Critical speed (ω_c) :

- When the speed of shaft is greater than the critical speed (i.e. $\omega > \omega_c$), the deflection of shaft 'y' is negative.
- In this speed range, the deflection of shaft 'y' and eccentricity 'e' are on the same side of the geometric centre of the rotor 'S'. This means, the rotor rotates with light side outwards, as shown in Fig. 4.9.2(b).
- In this speed range, $|y| > |e|$. As the shaft speed 'ω' increases, the deflection of shaft 'y' approaches $-e$.
- When $\omega >> \omega_c$, $y \approx -e$; which means that the centre of gravity of rotor 'G' approaches the axis of rotation 'O' and the rotor rotates about its C.G, as shown in Fig. 4.9.2(c). This principle is used in running high speed turbines by speeding up the rotor rapidly beyond the critical speed. When 'y' approaches the value of '$-e$', the rotor runs steadily.

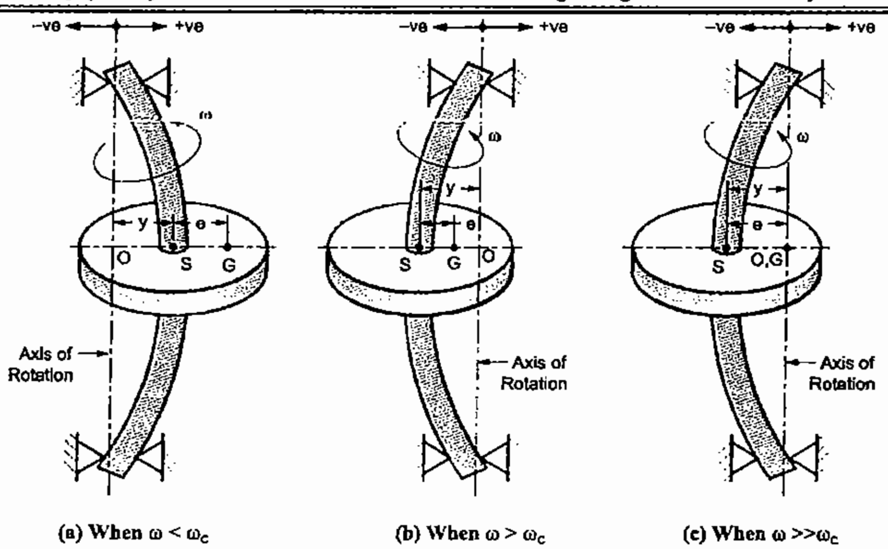


Fig. 4.9.2 : Ranges of Shaft Speed

Ex. 4.9.1 : A vertical shaft of 20 mm in diameter and 600 mm long is held in short bearings at the ends. A 5 kg disc is mounted on the shaft midway between the bearings. CG of the disc is 0.5 mm away from the axis of the shaft. If the allowable tensile stress for the shaft is 70 N/mm², determine :

(i) the critical speed of the shaft ; and

(ii) the range of the speed which is not safe.

Take E = 200 GPa **SPPU - Dec. 07**

Soln. :

Given : $d = 0.02$ m ; $l = 0.6$ m ;

$m = 5$ kg ; $e = 0.5\text{ mm} = 0.5 \times 10^{3}$ m ;

$\sigma_b = 70 \times 10^6\ \text{N/m}^2$; $E = 200 \times 10^9\ \text{N/m}^2$.

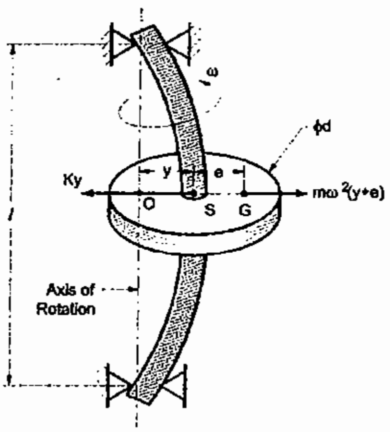


Fig. P. 4.9.1(a)

1. **Static Deflection of Shaft (δ) :**

Since the shaft is supported in short bearings, it may be assumed that shaft is simply supported.

$$\therefore \quad \delta = \frac{W l^3}{48\,EI} = \frac{mg\,l^3}{48\,E \times \frac{\pi d^4}{64}}$$

$$= \frac{5 \times 9.81 \times (0.6)^3}{48 \times 200 \times 10^9 \times \frac{\pi}{64}(0.02)^4}$$

or $\quad \delta = 1.4051 \times 10^{-4}$ m

2. **Critical Speed of Shaft (N_c) :**

$$\omega_c = \sqrt{\frac{g}{\delta}}$$

$$= \sqrt{\frac{9.81}{1.4051 \times 10^{-4}}} = 264.23 \text{ rad/s}$$

$$\therefore \quad N_c = \frac{\omega_c}{2\pi} = \frac{264.23}{2\pi} = 42.05 \text{ r.p.s.}$$

$\therefore \quad N_c = 42.05 \times 60$ r.p.m

or $\quad N_c =$ **2523.13 r.p.m.** ...**Ans.**

3. **Unsafe Dynamic Load for Shaft :**

$$\sigma_b = \frac{M}{Z} = \frac{M}{\pi d^3 / 32}$$

$$= \frac{32\,M}{\pi d^3} = \frac{32}{\pi d^3} \times \frac{W_d\, l}{4}$$

$$\left[\because \text{For simply supported beam}, M = \frac{W_d\, l}{4}\right]$$

or $$\sigma_b = \frac{8\, W_d\, l}{\pi d^3}$$ M : Bending moment

$$70 \times 10^6 = \frac{8 \times W_d \times 0.6}{\pi \times (0.02)^3}$$

$\therefore \quad W_d =$ **366.52 N**

4. Unsafe Speed Range :

- **Deflection of shaft due to dynamic load 'W_d' :**

$$y = \frac{W_d l^3}{48 EI} = \frac{366.51 \times (0.6)^3}{48 \times 2 \times 10^{11} \times \frac{\pi}{64}(0.02)^4}$$

or $y = 1.0499 \times 10^{-3}$ m

- The value of 'y' will be positive when $\omega < \omega_c$, and negative when $\omega > \omega_c$. Hence, taking positive as well as negative value of y,

$$\pm y = \frac{(\omega / \omega_n)^2 e}{1 - (\omega / \omega_n)^2} = \frac{(\omega / \omega_c)^2 e}{1 - (\omega / \omega_c)^2}$$

or $$\pm y = \frac{(N / N_c)^2 e}{1 - (N / N_c)^2}$$

$$\pm 1.0499 \times 10^{-3} = \frac{(N / N_c)^2 \times 0.5 \times 10^{-3}}{1 - (N / N_c)^2}$$

$$\therefore \quad \pm 2.09 = \frac{(N / N_c)^2}{1 - (N / N_c)^2}$$

- **For positive value :**

$$+2.09 = \frac{\left(\frac{N}{N_c}\right)^2}{1 - \left(\frac{N}{N_c}\right)^2}$$

$$2.09 - 2.09 (N_1 / N_c)^2 = (N_1 / N_c)^2$$

$$2.09 = 3.09 (N_1 / N_c)^2$$

$$\therefore \quad (N_1 / N_c)^2 = 0.6795$$

$$\therefore \quad (N_1 / N_c) = 0.8243$$

$$\therefore \quad N_1 = 0.8243 N_c = 0.8243 \times 2523.13$$

or $N_1 =$ **2079.93 r.p.m** ...Ans.

- **For negative value :**

$$-2.09 = \frac{\left(\frac{N}{N_c}\right)^2}{1 - \left(\frac{N}{N_c}\right)^2}$$

$$-2.09 + 2.09 (N_2 / N_c)^2 = (N_2 / N_c)^2$$

$$-2.09 = -1.09 (N_2 / N_c)^2$$

$$(N_2 / N_c)^2 = 1.9174$$

$$(N_2 / N_c) = 1.3847$$

$$N_2 = 1.3847 N_c = 1.3847 \times 2523.13$$

or $N_2 =$ **3493.81 r.p.m.** ...Ans.

- **Unsafe Speed :** From 2079.93 r.p.m. to 3493.81 r.p.m.

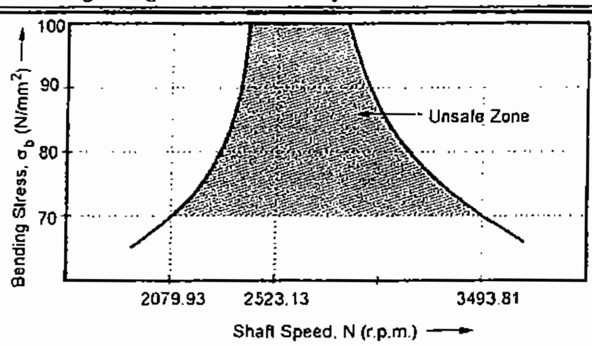


Fig. P. 4.9.1(b)

Fig. P. 4.9.1(b) shows unsafe speed range.

Example for Practice

Refer our website for complete solution of following example

Ex. 4.9.2 : A vertical shaft 12.5 mm in diameter rotates in sleeve bearing and a disc of mass 15 kg is attached to the shaft at mid-span. The span of the shaft between bearings is 0.5 m. The mass center of the disc is 0.5 mm from the axis of the shaft. Determine critical speed of rotation of shaft. What is range of speed in which bending stress in shaft will exceed 125 N/mm². Take $E = 2 \times 10^5$ N-mm². **SPPU - May 09**

Ex. 4.9.3 : A rotor having a mass of 6 kg is mounted midway on a simply supported shaft of diameter 10 mm and length 400 mm. The center a gravity of rotor is 0.02 mm away from the geometric center of the rotor. If the rotor rotates at 2500 rpm, find the amplitude of steady state vibrations and the dynamic force transmitted to the bearings. Assume for shaft material, E = 200 GPa. **SPPU - May 05**

Soln. :

Given : $m = 6$ kg ; $d = 0.01$ m ;

$l = 0.4$ m ; $e = 0.02 \times 10^{-3}$ m ;

$N = 2500$ r.p.m. ; $E = 200 \times 10^9$ N/m²

$$\therefore \omega = \frac{2 \pi N}{60} = \frac{2 \pi \times 2500}{60} = 261.79 \text{ rad/s}$$

1. Static Deflection of shaft (δ) :

The static deflection of simply supported shaft is,

$$\delta = \frac{W l^3}{48 EI} = \frac{mg l^3}{48 E \frac{\pi d^4}{64}} = \frac{6 \times 9.81 \times (0.4)^3}{48 \times 2 \times 10^{11} \times \frac{\pi}{64} (0.01)^4}$$

or $\delta = 0.7993 \times 10^{-3}$ m

2. **Critical Speed of Shaft (ω_c) :**

$$\omega_c = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.7993 \times 10^{-3}}}$$

or $\quad \omega_c = \mathbf{110.77\ rad/s}$

3. **Amplitude of Steady-State Vibrations (y) :**

$$\frac{\omega}{\omega_c} = \frac{261.79}{110.77} = 2.3631$$

$$y = \frac{(\omega / \omega_c)^2 e}{1 - (\omega / \omega_c)^2} = \frac{(2.3631)^2 \times 0.02 \times 10^{-3}}{1 - (2.3631)^2}$$

$$y = -0.02436 \times 10^{-3}\ m$$

$\therefore \quad y = \mathbf{-0.02436\ mm}$ ...Ans.

Negative sign indicates that the deflection is out of phase with centrifugal force. In other words, the deflection of shaft and eccentricity are on the same side of the geometric centre of the rotor 'S'.

4. **Dynamic Load :**

- **Dynamic load due to deflection of shaft :**

$$W_d = Ky = 73631.07 \times 0.02436 \times 10^{-3}$$

or $\quad W_d = \mathbf{1.793\ N}$ ...Ans.

- **Load due to self weight for horizontal shaft :**

$$W = mg = 6 \times 9.81 = 58.86$$

5. **Net Dynamic Load on Bearings :**

- **If shaft is vertical :**

$$R_b = \frac{W_d}{2} = \frac{1.793}{2}$$

or $\quad R_b = \mathbf{0.896\ N}$ ...Ans.

- **If shaft is horizontal :**

$$R_b = \frac{W + W_d}{2} = \frac{58.86 + 1.793}{2}$$

or $\quad R_b = \mathbf{30.32\ N}$ ...Ans.

Ex. 4.9.4 : A disc of mass 4 kg is mounted midway between bearings which may be assumed to be simply supported. The bearing span is 0.48 m. The steel shaft which is horizontal, is 9mm in diameter. The CG of the disc is displaced 3 mm form the geometric center. If the shaft rotates at 760 r.p.m., find the maximum stress in the shaft. Take $E = 1.96 \times 10^{11}\ N/m^2$. **SPPU - Dec. 10**

Soln. :

Given : $m = 4\ kg$; $l = 0.48\ m$;
$d = 9 \times 10^{-3}\ m$; $e = 3 \times 10^{-3}\ m$;
$N = 760\ r.p.m.$; $E = 1.96 \times 10^{11}\ N/m^2$.

$$\omega = \frac{2\pi N}{60} = \frac{2 \times \pi \times 760}{60} = 79.58\ rad/sec$$

$$\text{and } I = \frac{\pi d^4}{64} = \frac{\pi \times (9 \times 10^{-3})^4}{64} = 3.22 \times 10^{-10}$$

1. **Static Deflection of Shaft (δ) :**

$$\therefore \delta = \frac{W l^3}{48\ EI} = \frac{79.58 \times (0.48)^3}{48 \times 1.96 \times 10^{11} \times (3.22 \times 10^{-10})}$$

$$\delta = 0.00143\ m$$

2. **Critical Speed of Shaft (ω_c) :**

$$\omega_c = \sqrt{\frac{g}{\delta}} = \sqrt{\frac{9.81}{0.00143}}$$

$$= 82.82\ rad/sec$$

3. **Amplitude of Steady-State Vibrations (y) :**

$$\therefore \quad y = \frac{(\omega / \omega_c)^2 e}{1 - (\omega / \omega_c)^2} = \frac{(79.58/82.82)^2 \times 3 \times 10^{-3}}{1 - (79.58/82.82)^2}$$

or $\quad y = 0.0352\ m$

4. **Dynamic Load on Shaft :**

- **Dynamic load on shaft due to deflection (W_d) :**

$$y = \frac{W_d\, l^3}{48\ EI}$$

$$0.0352 = \frac{W_d \times (0.48)^3}{48 \times 1.96 \times 10^{11} \times 3.22 \times 10^{-10}}$$

$$\therefore \quad W_d = 965.3\ N$$

- **Maximum dynamic load on shaft :**

$$\therefore \quad W_{dmax} = W + W_d = mg + W_d$$

or $\quad W_{dmax} = 4 \times 9.81 + 965.3 = 1004.54\ N$

5. **Maximum Stress in Shaft :**

$$\sigma_b = \frac{M}{Z} = \frac{8\ W_{dmax}\ l}{\pi\ d^3} = \frac{8 \times 965.3 \times 0.48}{\pi \times (9 \times 10^{-3})^3}$$

or $\quad \sigma_{bmax} = \mathbf{16.84 \times 10^8\ N/m^2}$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 4.9.5 : The rotor of a turbo super charger weighing 9 kg is keyed to the center of a 25 mm diameter steel shaft 40 cm between bearings. If the shaft material has a density of 8 gm/cm^3, determine :

(i) the critical speed of shaft;

(ii) the amplitude of vibration of rotor at a speed of 3200 r.p.m, if the eccentricity is 0.015 mm; and

(iii) the vibratory force transmitted to the bearings at this speed. Assume the shaft to be simply supported.

SPPU - Dec. 08

Ex. 4.9.6 : A rotor has a mass of 10 kg and is mounted midway on a 20 mm diameter horizontal shaft supported at its ends by two ball bearings. The bearings are one meter apart. The shaft rotates at 2000 rpm. If the center of mass of rotor is 0.02 mm away from the geometric axis of the rotor due to certain manufacturing defect, find the amplitude of the steady state vibration and the dynamic force transmitted to the bearing. Also calculate the maximum bending stress in the shaft (Assume E = 200 GPa for shaft material). **SPPU - May 11**

Exercise

1. Derive the expressions for :
   (i) amplitude of steady-state vibrations, and
   (ii) phase angle; for a spring-mass-damper system subjected to an external periodic force $f_0 \sin \omega t$.

2. What do you understand by transient vibrations and steady-state vibrations ?

3. Explain the term magnification factor and obtain expression for it.

4. Explain salient features of frequency response curve and phase frequency curve.

5. A mass 'm' is supported by a spring of stiffness K and a damper having damping constant 'c'. If the support is moving with a motion $y = Y \sin \omega t$, derive the expressions for :
   (i) absolute amplitude of vibrations of the mass; and
   (ii) relative amplitude of vibrations of the mass.

6. A machine of 50 kg is supported on an elastic structure of total stiffness 20 kN/m. The damping factor is 0.2. A simple harmonic disturbing force acts on the machine, the force is 60 sin 10 t newtons. Find the **amplitude** of the vibrations and the phase angle.
   [**Ans. :** $X = 3.86$ mm, $\phi = 14.93°$]

1. Write **short** notes on :
   (i) Forced vibrations due to rotating unbalance.
   (ii) Forced vibrations due to reciprocating unbalance.

2. A vehicle moves over the road surface making approximately sinusoidal profile with a wavelength 14 m and amplitude of 80 mm. The spring of vehicle is compressed by 0.1 m under its own weight. Determine :
   (i) the critical speed of vehicle; and
   (ii) the amplitude of vibrations, when vehicle moves at a speed of 60 km/hr.
   [**Ans.** : $V_c = 79.5$ km/hr, $X = 0.186$ m]

1. What is vibration isolation ? What are the various types of isolating materials used for isolation?

2. Explain the following terms :
   (i) Vibration isolation
   (ii) Force transmissibility

3. Derive an expression for force transmissibility.

4. Explain salient features of transmissibility Vs frequency curve ?

5. A machine of 100 kg is supported on isolator of stiffness 800 kN/m and has a rotating unbalance mass which results in a disturbing force of 400 N at a **speed** of 3000 r.p.m. If the damping factor is 0.25, determine :
   (i) the amplitude of vibration.
   (ii) the force transmitted.
   [**Ans. :** $X = 0.04$ mm, $F_T = 35.2$ N]

6. A **machine** of mass 75 kg is mounted on an isolator having a stiffness 1200×10^3 N/m and a damping factor 0.2. A reciprocating part of 2 kg has 80 mm stroke. If the crank speed is 3000 r.p.m., determine :
   (i) the amplitude of machine
   (ii) the phase angle; and
   (iii) the force transmitted to the foundation
   [**Ans. :** $X = 1.25$ mm, $\phi = 169°$, $F_T = 2132$ N]

7. A centrifugal compressor weighing 981 N has a rotating unbalanced of 0.1 kg-m. The isolator has a damping factor of 0.2. The compressor runs at 1500 r.p.m. If the isolator transmit only 10% of the unbalanced force to the foundation, determine its spring stiffness.
   [**Ans. :** $K = 11 \times 10^4$ N/m]

8. What do you mean by critical speed of shaft ? State its significance.

9. Explain the method to determine the critical speed of shaft carrying single rotor, neglecting damping.

10. Why shafts are run above critical speed in case of high speed applications ?

11. Explain the method to determine the critical speed of shaft carrying single rotor, considering damping.

12. Explain various phase relationships for shaft carrying single rotor with damping ?

13. Explain the term half frequency whirl. Derive the expression for it .

14. What are the various methods to determine the critical speed of a shaft carrying multiple rotors ?

15. Explain Dunkerley's method to determine the natural frequency of shaft carrying number of points load.

16. A shaft of 12 mm diameter rotates in long bearing having rotor of mass 15 kg at mid span. The length of shaft is 500 mm. While the CG of rotor is 0.6 mm from axis of the shaft. Determine the critical speed of shaft. Also determine the speed range in which the bending stress in shaft will exceed 110 N/mm^2. Take E = 200 GN / m^2.

[Ans. : N_c = 1420.08 rpm, N_1 = 1098.76 rpm N_2 = 2473.45 rpm]

17. A rotor of mass 12 kg mounted on 24 mm diameter horizontal shaft supported in two short bearings which are 1 m apart. The shaft rotates at 2500 r.p.m. If the CG of the rotor is 0.8 mm away from the axis of shaft, find the dynamic force transmitted to the bearings. Take E = 200 GN / m^2. **[Ans. : 136.13 N]**

❑❑❑

5

UNIT V

TWO DEGREE OF FREEDOM SYSTEMS : UNDAMPED FREE VIBRATIONS

Syllabus

Free Vibration of Spring Coupled Systems - Longitudinal and Torsional, Torsionally Equivalent Shafts, Natural Frequency and Mode Shapes, Eigen Value and Eigen Vector by Matrix Method, Combined Rectilinear and Angular Motion, Vibrations of Geared Systems

TOPICS

5.1 INTRODUCTION TO TWO DEGREES OF FREEDOM SYSTEMS

University Questions

Q. Explain two degree of freedom system with any two practical examples. **SPPU : May 13**

Q. What is 2 DOF systems? **SPPU : May 19**

- **Two degrees of freedom system :** *The system which requires independent two co-ordinates to specify its motion or configuration at any instant is known as* ***two degrees of freedom system.***
- **Examples of two degrees of freedom systems :**

(i) Two springs and two masses system, which requires two independent co-ordinates x_1 and x_2 to specify the motion from equilibrium position [Fig. 5.1.1(a)].

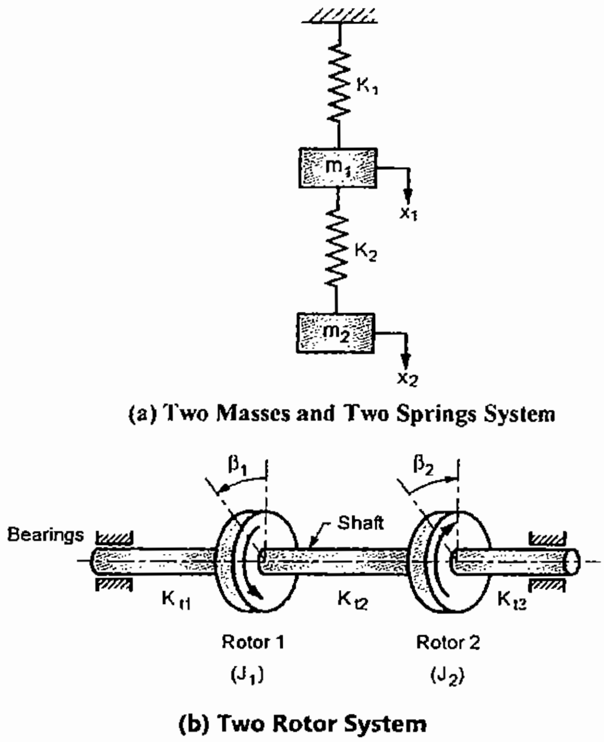


Fig. 5.1.1 : Two Degrees of Freedom Systems

(ii) Two rotors mounted an a shaft, which requires two angular displacements θ_1 and θ_2 to specify the motion from equilibrium position. [Fig. 5.1.1(b)].

- **Coupled differential equations :** A systems having two degrees of freedom have two equations of motion, one for each mass. Such equations are called **coupled differential equations.** Each equation involves both the co-ordinates.
- **Natural frequencies :** If a harmonic motion is assumed for each co-ordinate, the two equations of motion give two natural frequencies for the system. Thus, a two degrees of freedom system has two natural frequencies. Therefore, two degrees of freedom system has two modes of vibrations corresponding to the two natural frequencies.

5.2 UNDAMPED FREE LONGITUDINAL VIBRATIONS OF TWO DEGREES OF FREEDOM SYSTEM

University Question

Q. Explain principle modes of vibration with respect to 2 DOF translational system. **SPPU : Dec. 13**

Two Springs - Two Masses System :

- Consider a spring mass system having two degrees of freedom, as shown in Fig. 5.2.1.

Let, x_1 = displacement of mass m_1 after time 't', m

x_2 = displacement of mass m_2 after time 't', m

- **Assumptions made in analysis of two degrees of freedom system :**

(i) Both the masses are constrained to move in the direction of spring axis.

(ii) The springs are weightless.

(iii) $x_2 > x_1$

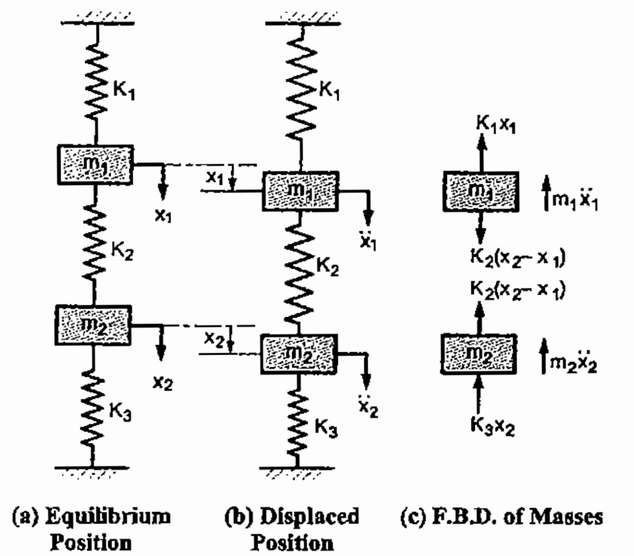


Fig. 5.2.1 : Spring Mass System having Two Degrees of Freedom

Differential Equations of Motion for Two Springs - Two Masses System :

- **Two differential equations of motion :**
  From Fig. 5.2.1(c);

$$m_1 \ddot{x}_1 + K_1 x_1 - K_2 (x_2 - x_1) = 0 \quad ...(5.2.1)$$

$$m_2 \ddot{x}_2 + K_2 (x_2 - x_1) + K_3 x_2 = 0 \quad ...(5.2.2)$$

- Rearranging the terms of Equations (5.2.1) and (5.2.2) we get,

$$m_1 \ddot{x}_1 + (K_1 + K_2) x_1 - K_2 x_2 = 0 \quad ...(5.2.3)$$

$$m_2 \ddot{x}_2 + (K_2 + K_3) x_2 - K_2 x_1 = 0 \quad ...(5.2.4)$$

Solution of Differential Equation of Motion :

- **Solutions for x_1 and x_2 :**

$$x_1 = X_1 \sin \omega t \quad ...(5.2.5)$$

$$x_2 = X_2 \sin \omega t \quad ...(5.2.6)$$

where, ω = frequency of vibration, rad/s

X_1 = amplitude of vibration of mass m_1

X_2 = amplitude of vibration of mass m_2

Therefore,

$$\ddot{x}_1 = -X_1 \omega^2 \sin \omega t \quad ...(5.2.7)$$

$$\ddot{x}_2 = -X_2 \omega^2 \sin \omega t \quad ...(5.2.8)$$

- Substituting Equations (5.2.5), (5.2.6) and (5.2.7) in Equation (5.2.3), we get,

$$-m_1 X_1 \omega^2 \sin \omega t + (K_1 + K_2) X_1 \sin \omega t - K_2 X_2 \sin \omega t = 0$$

$$\therefore \quad -m_1 X_1 \omega^2 + (K_1 + K_2) X_1 - K_2 X_2 = 0$$

$$\therefore \quad [-m_1 \omega^2 + (K_1 + K_2)] X_1 - K_2 X_2 = 0$$

$$\therefore \quad [(K_1 + K_2) - m_1 \omega^2] X_1 = K_2 X_2$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{K_2}{[(K_1 + K_2) - m_1 \omega^2]} \quad ...(5.2.9)$$

- Substituting Equations (5.2.5), (5.2.6) and (5.2.8) in Equation (5.2.4), we get,

$$-m_2 \omega^2 X_2 \sin \omega t + (K_2 + K_3) X_2 \sin \omega t - K_2 X_1 \sin \omega t = 0$$

$$[-m_2\omega^2 + (K_2 + K_3)] X_2 - K_2 X_1 = 0$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{[(K_2 + K_3) - m_2\omega^2]}{K_2} \quad ...(5.2.10)$$

Equations (5.2.9) and (5.2.10) give mode shapes for the system.

Frequency equation :

From Equations (5.2.9) and (5.2.10),

$$\frac{K_2}{[(K_1 + K_2) - m_1 \omega^2]} = \frac{[(K_2 + K_3) - m_2\omega^2]}{K_2}$$

$$\therefore \quad [(K_1 + K_2) - m_1\omega^2][(K_2 + K_3) - m_2\omega^2] = K_2^2$$

$$\therefore \quad (K_1 + K_2)(K_2 + K_3) - m_1\omega^2 (K_2 + K_3) - m_2\omega^2(K_1 + K_2) + m_1 m_2 \omega^4 = K_2^2$$

$$\therefore \quad K_1K_2 + K_1K_3 + K_2^2 + K_2K_3 - [m_1(K_2 + K_3) + m_2 (K_1 + K_2)] \omega^2 + m_1 m_2 \omega^4 = K_2^2$$

$$\therefore \quad \mathbf{m_1 m_2 \omega^4 - [m_1(K_2 + K_3) + m_2(K_1 + K_2)]\omega^2 + [K_1K_2 + K_2K_3 + K_3K_1] = 0} \quad ...(5.2.11)$$

The above Equation (5.2.11) is quadratic in ω^2 and gives two values of ω^2. The two positive values of ω give natural frequencies ω_{n1} and ω_{n2} of the system. Therefore, Equation (5.2.11) is called as **frequency equation**.

Special Cases of Two Springs - Two Masses System :

Case I	:	$m_1 = m_2 = m$; $K_1 = K_2 = K$ and $K_3 = 0$
Case II	:	$m_1 = m_2 = m$ and $K_1 = K_3 = K$
Case III	:	$m_1 = m$; $m_2 = 2m$ and $K_1 = K_3 = K$ and $K_2 = 2K$

Case I - $m_1 = m_2 = m$; $K_1 = K_2 = K$ and $K_3 = 0$:

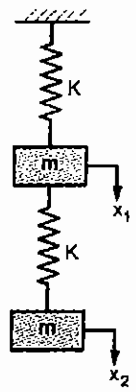


Fig. 5.2.2 : Case I

Refer Fig. 5.2.2 ;

1. Two Natural Frequencies (ω_{n1} and ω_{n2}) :

- Substituting $m_1 = m_2 = m$, $K_1 = K_2 = K$ and $K_3 = 0$, in Equation (5.2.11), we get,

$$m^2\omega^4 - [Km + 2Km] \omega^2 + K^2 = 0$$

$$m^2\omega^4 - 3Km \omega^2 + K^2 = 0$$

$$\therefore \quad \omega^2 = \frac{3Km \pm \sqrt{9K^2 m^2 - 4 K^2 m^2}}{2m^2}$$

$$\therefore \quad \omega^2 = \frac{3 Km}{2m^2} \pm \frac{\sqrt{5K^2 m^2}}{2m^2}$$

$$\therefore \quad \omega^2 = \frac{3K}{2m} \pm \frac{\sqrt{5}}{2} \frac{K}{m}$$

$$\therefore \quad \omega^2 = (1.5 \pm 1.11)\frac{K}{m}$$

$$\therefore \quad \omega_{n1}^2 = 0.39\frac{K}{m} \text{ and } \omega_{n2}^2 = 2.61\frac{K}{m} \quad \text{...(a)}$$

$$\therefore \omega_{n1} = 0.62\sqrt{\frac{K}{m}}, \text{rad/s and } \omega_{n2} = 1.61\sqrt{\frac{K}{m}}, \text{rad/s} \quad \text{...(5.2.12)}$$

- Therefore, ω_{n1} and ω_{n2} are the two natural frequencies of the system, shown in Fig. 5.2.2.

2. Mode Shapes :

- **Principle mode of vibration :** *The motion of a system where every point in a system executes harmonic motion with one natural frequency is called as* ***principal mode of vibration or natural mode of vibration.***
- A system with two degrees of freedom can vibrate in two principal modes of vibrations corresponding to its two natural frequencies
- **Ratio of amplitudes :**

From Equation (5.2.9)

$$\frac{X_1}{X_2} = \frac{K_2}{[(K_1 + K_2) - m_1\omega^2]} \quad \text{...(b)}$$

Substituting $m_1 = m$, $K_1 = K_2 = K$, in Equation (b),

$$\frac{X_1}{X_2} = \frac{K}{2K - m\,\omega^2} \quad \text{...(c)}$$

- **First mode shape (first ratio of amplitudes) :**

Substituting $\omega^2 = \omega_{n1}^2 = 0.39\frac{K}{m}$ [Equation (a)] in Equation (c), we get,

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{K}{2K - m\left(0.39\frac{K}{m}\right)}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{K}{2K - 0.39\,K}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{1}{1.61}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = 0.62 \quad \text{...(d)}$$

The first ratio of the amplitudes of the two masses is + 0.62, which means that the two motions are in phase i.e. the two masses move up or down together such that $\left(\frac{X_1}{X_2}\right)_1 = 0.62$, and natural frequency is ω_{n1} [Fig. 5.2.3(a)].

- **Second mode shape (second ratio of amplitudes) :**

Substituting $\omega^2 = \omega_{n2}^2 = 2.61\frac{K}{m}$ [Equation (a)] in Equation (c), we get,

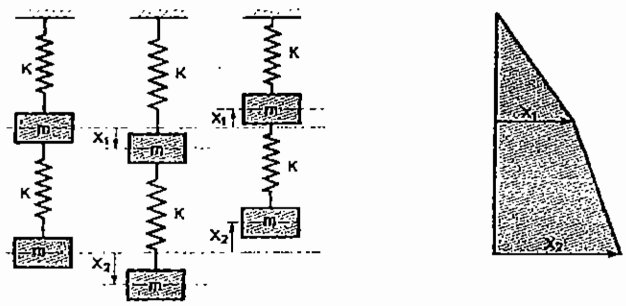
$$\left(\frac{X_1}{X_2}\right)_2 = \frac{K}{2K - m\left(2.61\frac{K}{m}\right)}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = \frac{K}{2K - 2.61\,K}$$

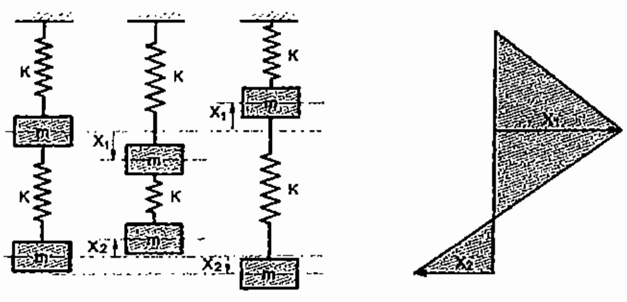
$$\therefore \left(\frac{X_1}{X_2}\right)_2 = \frac{1}{-0.61}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = -1.63 \quad \text{...(e)}$$

The second ratio of the amplitudes of the two masses is - 1.63, which means that the two motions are out of phase, i.e. when one mass moves up the other moves down or vice-versa such that $\left(\frac{X_1}{X_2}\right)_2 = -1.63$, and natural frequency is ω_{n2} [Fig. 5.2.3(b)].



(a) First Mode Shape at ω_{n1} for $\left(\frac{X_1}{X_2}\right)_1 = 0.62$



(b) Second Mode Shape at ω_{n2} for $\left(\frac{X_1}{X_2}\right)_2 = -1.63$

Fig. 5.2.3 : Principal Mode Shapes for Case I

Case II - $m_1 = m_2 = m$ and $K_1 = K_3 = K$:

Refer Fig. 5.2.4;

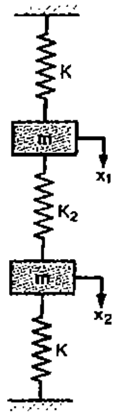


Fig. 5.2.4 : Case II

1. Natural Frequencies (ω_{n1} and ω_{n2}) :

- Substituting $m_1 = m_2 = m$ and $K_1 = K_3 = K$, in Equation (5.2.11), we get,

$$mm\,\omega^4 - [m(K_2 + K) + m(K + K_2)]\omega^2 + [KK_2 + K_2K + KK] = 0$$

$$m^2\omega^4 - 2m(K + K_2)\omega^2 + (K^2 + 2KK_2) = 0$$

$$\therefore \omega^2 = \frac{+2m(K+K_2) \pm \sqrt{4m^2(K+K_2)^2 - 4m^2(K^2+2KK_2)}}{2m^2}$$

$$\therefore \omega^2 = \frac{2m(K+K_2)}{2m^2} \pm \frac{\sqrt{4m^2(K^2+2KK_2+K_2^2) - 4m^2(K^2+2KK_2)}}{2m^2}$$

$$\therefore \omega^2 = \frac{K+K_2}{m} \pm \frac{4m^2K_2^2}{2m^2}$$

$$\therefore \omega^2 = \frac{K+K_2}{m} \pm \frac{2mK_2}{2m^2}$$

$$\therefore \omega^2 = \frac{K+K_2}{m} \pm \frac{K_2}{m}$$

$$\therefore \omega^2 = \left(\frac{K+K_2 \pm K_2}{m}\right)$$

$$\therefore \omega_{n1}^2 = \frac{K}{m} \text{ and } \omega_{n2}^2 = \frac{K+2K_2}{m} \quad \ldots(f)$$

$$\therefore \omega_{n1} = \sqrt{\frac{K}{m}}, \textbf{ rad/s and } \omega_{n2} = \sqrt{\frac{K+2K_2}{m}}, \textbf{ rad/s} \quad \ldots(5.2.13)$$

- Therefore ω_{n1} and ω_{n2} are the two natural frequencies of the system shown in Fig. 5.2.4.

2. Mode Shapes :

From Equation (5.2.9),

$$\frac{X_1}{X_2} = \frac{K_2}{[(K_1+K_2) - m_1\omega^2]} \quad \ldots(g)$$

Substituting $m_1 = m_2 = m$ and $K_1 = K_3 = K$, in Equation (g),

$$\frac{X_1}{X_2} = \frac{K_2}{K+K_2-m\,\omega^2} \quad \ldots(h)$$

- **First mode shape (first ratio of amplitudes) :**

Substituting $\omega^2 = \omega_{n1}^2 = \frac{K}{m}$ [Equation (f)] in Equation (h), we get,

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{K_2}{K+K_2-m\left(\frac{K}{m}\right)}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{K_2}{K_2}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = 1 \quad \ldots(i)$$

The first ratio of the amplitude of the two masses is + 1, which means that the two amplitudes are equal and two motions are in phase i.e. the two masses move up or down together, with frequency ω_{n1} [Fig. 5.2.5(a)].

- **Second mode shape (second ratio of amplitudes) :**

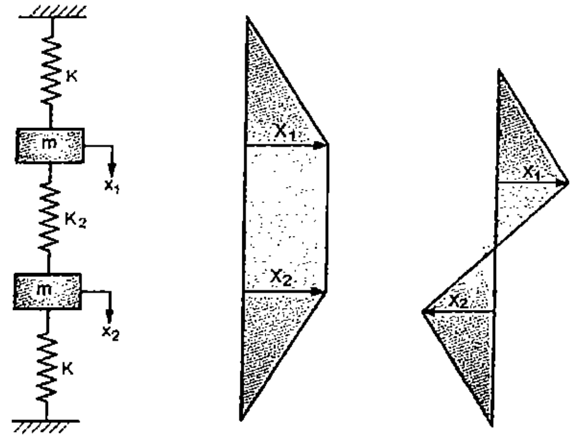
Substitute $\omega^2 = \omega_{n2}^2 = \frac{K+2K_2}{m}$ [Equation (f)] in Equation (h), we get,

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{K_2}{K+K_2-m\left(\frac{K+2K_2}{m}\right)}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = \frac{K_2}{-K_2}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = -1 \quad \ldots(j)$$

The second ratio of the amplitudes of the two masses is - 1, which means that the two amplitudes are equal but two motions are out of phase i.e. when one mass moves up, the other mass moves down or vice-versa, with frequency ω_{n2} [Fig. 5.2.5(b)].



(a) Equilibrium Position (b) First Mode Shape at ω_{n_1} for $\left(\frac{X_1}{X_2}\right)_1 = 1$ (c) Second Mode Shape ω_{n_2} for $\left(\frac{X_1}{X_2}\right)_2 = -1$

Fig. 5.2.5 : Principal Mode Shapes for Case II

Case III - $m_1 = m$; $m_2 = 2m$; $K_1 = K_3 = K$ and $K_2 = 2K$:

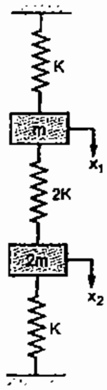


Fig. 5.2.6 : Case III

Refer Fig. 5.2.6

1. Natural Frequencies (ω_{n1} and ω_{n2}) :

- Substituting, $m_1 = m$, $m_2 = 2$ m, $K_1 = K_3 = K$ and $K_2 = 2K$, in Equation (5.2.11), we get,

$$m \cdot 2m\,\omega^4 - [m(2K+K) + 2m(K+2K)]\,\omega^2 + [K\,2K + 2K\,K + KK] = 0$$

$$\therefore \quad 2m^2\omega^4 - [3\,Km + 6\,Km]\omega^2 + [2K^2 + 2K^2 + K^2] = 0$$

$$\therefore \quad 2m^2\omega^4 - 9\,Km\,\omega^2 + 5K^2 = 0$$

$$\therefore \omega^2 = \frac{9\,Km \pm \sqrt{81\,K^2 m^2 - 4\,(2m^2\,5K^2)}}{4m^2}$$

$$\therefore \quad \omega^2 = \frac{9Km}{4m^2} \pm \frac{\sqrt{81\,K^2m^2 - 40K^2m^2}}{4m^2}$$

$$\therefore \quad \omega^2 = \frac{9K}{4m} \pm \frac{\sqrt{41}\,Km}{4m^2} = \frac{9K}{4m} \pm \frac{\sqrt{41}}{4}\frac{K}{m}$$

$$\therefore \quad \omega^2 = \left(\frac{9}{4} \pm \frac{\sqrt{41}}{4}\right)\frac{K}{m} = (2.25 \pm 1.60)\frac{K}{m}$$

$$\therefore \quad \omega_{n1}^2 = (0.65)\frac{K}{m} \text{ and } \omega_{n2}^2 = (3.85)\frac{K}{m} \qquad \text{...(k)}$$

$$\therefore \omega_{n1} = \mathbf{0.80}\sqrt{\frac{K}{m}}\text{, rad/s} \quad \text{and} \quad \omega_{n2} = \mathbf{1.962}\sqrt{\frac{K}{m}}\text{, rad/s} \qquad \text{...(5.2.14)}$$

- Therefore ω_{n1} and ω_{n2} are the two natural frequencies of the system shown in Fig. 5.2.6.

2. Mode Shapes :

From Equation (5.2.9),

$$\frac{X_1}{X_2} = \frac{K_2}{[(K_1 + K_2) - m_1\omega^2]} \qquad \text{...}(l)$$

Substituting, $m_1 = m$, $K_1 = K$ and $K_2 = 2K$, in Equation (l),

$$\frac{X_1}{X_2} = \frac{2K}{[(K + 2K) - m\,\omega^2]}$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{2K}{3K - m\omega^2} \qquad \text{....(m)}$$

- **First mode shape (first ratio of amplitudes) :**

Substituting $\omega^2 = \omega_{n1}^2 = (0.65)\frac{K}{m}$ [Equation (k)] in Equation (m), we get,

$$\left(\frac{X_1}{X_2}\right)_1 = \frac{2K}{3K - m\left(0.65\frac{K}{m}\right)} = \frac{2K}{3K - 0.65\,K}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{2}{2.35}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \mathbf{0.851} \qquad \text{...(n)}$$

The first ratio of the amplitudes of the two masses is + 0.851, which means that the two motions are in phase, i.e. the two masses move up or down together such that $\left(\frac{X_1}{X_2}\right)_1 = 0.851$, and frequency is ω_{n1} [Fig. 5.2.7(b)].

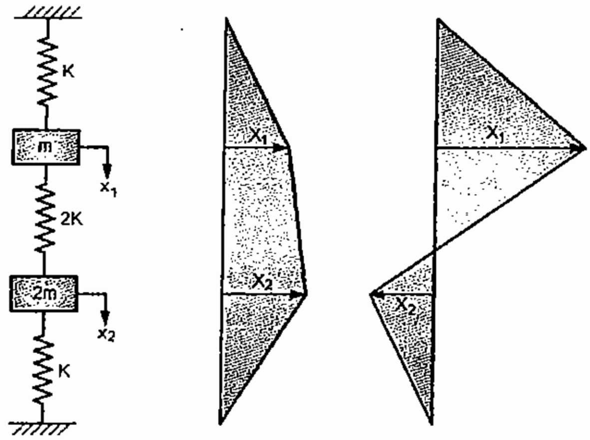
- **Second mode shape (second ratio of amplitudes) :**
- Again substituting $\omega^2 = \omega_{n2}^2 = (3.85)\frac{K}{m}$ [Equation (k)] in Equation (m), we get,

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{2K}{3K - m\left(3.85\frac{K}{m}\right)} = \frac{2K}{3K - 3.85\,K}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = \frac{2K}{-0.85\,K} = \frac{2}{-0.85}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = \mathbf{-2.3529} \qquad \text{...(o)}$$

The second ratio of the amplitudes of the two masses is – 2.3529, which means that the two motions are out of phase i.e. when one mass moves up, the other moves down or vice versa such that $\left(\frac{X_1}{X_2}\right)_2 = \mathbf{-2.3529}$ and frequency is ω_{n2} [Fig. 5.2.7(c)].



(a) Equilibrium Position (b) First Mode Shape at ω_{n1} for $\left(\frac{X_1}{X_2}\right)_1 = 0.851$ (c) Second Mode Shape at ω_{n2} for $\left(\frac{X_1}{X_2}\right)_2 = -2.3529$

Fig. 5.2.7 : Principal Mode Shapes for Case III

5.3 EXAMPLES OF TWO DEGREES OF FREEDOM SYSTEM

The following two systems of two degrees of freedom are discussed in subsequent section :

1. Double Pendulum
2. Two Masses Fixed on Stretched String
3. Torsional System
4. Semi-definite system

5.3.1 Double Pendulum :

Double Pendulum System :

University Questions

Q. Set up the differential equations of motion for the double pendulum shown in Fig.1, using the coordinates x_1 and x_2 and assuming small amplitudes. Find the natural frequencies and ratios of amplitude, if $m_1 = m_2 = m$ and $l_1 = l_2$. **SPPU : Dec. 13**

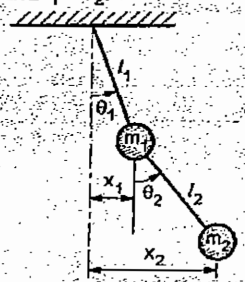


Fig.1

Q. Find the natural frequency of oscillations of the double pendulum as shown in Fig.2. where $m_1 = m_2 = m$ and $l_1 = l_2 = 1$. Draw mode shapes and locate the nodes for each mode of vibration. **SPPU : Dec. 18**

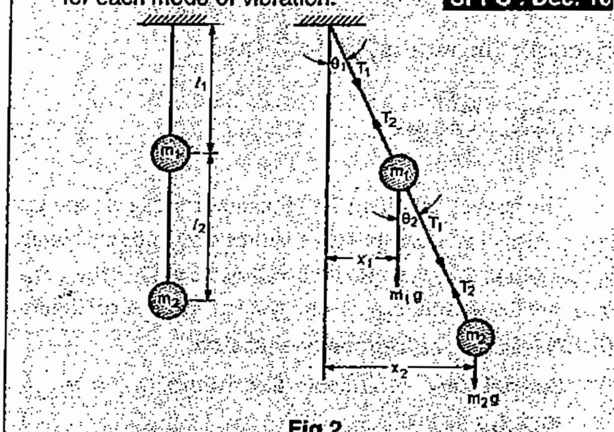


Fig.2

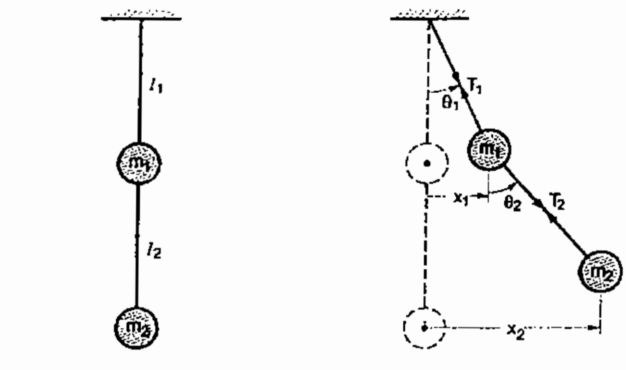
- Consider double pendulum having two point masses m_1 and m_2 suspended by inextensible strings of lengths l_1 and l_2 respectively as shown in Fig. 5.3.1. The masses m_1 and m_2 are considered to have only horizontal motion.

Let, θ_1, θ_2 = angles of upper and lower strings with the vertical,

x_1 and x_2 = horizontal displacements of masses m_1 and m_2 from equilibrium position.

T_1, T_2 = tensions in upper and lower strings

X_1 and X_2 = amplitudes of masses m_1 and m_2



(a) In Equilibrium Position **(b) In Displaced Position**

Fig. 5.3.1 : Double Pendulum

Differential Equations of Motion :

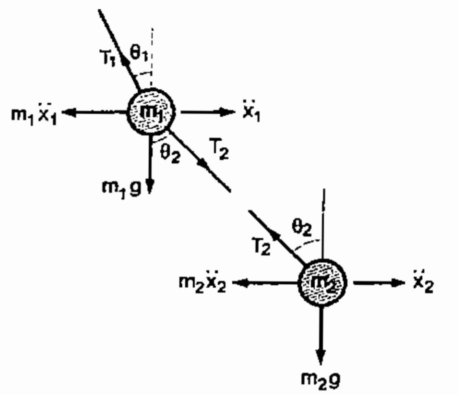


Fig. 5.3.2 : F.B.D. of Two Masses

$\Sigma F_v = 0$:

There is no motion in vertical direction. The vertical components of forces acting on each mass must be balanced.

$$T_1 \cos\theta_1 = m_1 g + T_2 \cos\theta_2 \quad ...(a)$$

$$\text{and } T_2 \cos\theta_2 = m_2 g \quad ...(b)$$

$$\cos\theta_1 = \cos\theta_2 \approx 1 \quad [\text{As } \theta_1 \text{ and } \theta_2 \text{ are small}] \quad ...(c)$$

Substituting Equation (c) in Equations (a) and (b), we get,

$$\therefore \quad T_1 = m_1 g + T_2 \quad \text{...(d)}$$

$$\text{and} \quad T_2 = m_2 g \quad \text{...(e)}$$

Substituting Equation (e) in Equation (d), we get,

$$\therefore \quad T_1 = m_1 g + m_2 g$$

$$\text{or} \quad T_1 = (m_1 + m_2) g \quad \text{...(f)}$$

$\Sigma F_H = 0$

$$m_1 \ddot{x}_1 + T_1 \sin\theta_1 - T_2 \sin\theta_2 = 0 \quad \text{...(5.3.1)}$$

$$\text{and} \quad m_2 \ddot{x}_2 + T_2 \sin\theta_2 = 0 \quad \text{...(5.3.2)}$$

From Fig. 5.3.1(b),

$$\sin\theta_1 = \frac{x_1}{l_1} \quad \text{...(g)}$$

$$\text{and} \quad \sin\theta_2 = \frac{(x_2 - x_1)}{l_2} \quad \text{...(h)}$$

Substituting the values of sin θ_1 and sin θ_2 in Equation (5.3.1) and (5.3.2), we get,

$$m_1 \ddot{x}_1 + T_1 \frac{x_1}{l_1} - T_2 \frac{(x_2 - x_1)}{l_2} = 0$$

$$\text{and} \quad m_2 \ddot{x}_2 + T_2 \frac{(x_2 - x_1)}{l_2} = 0$$

Rearranging the above equations,

$$m_1 \ddot{x}_1 + \left(\frac{T_1}{l_1} + \frac{T_2}{l_2}\right) x_1 - \frac{T_2 x_2}{l_2} = 0 \quad \text{...(i)}$$

$$m_2 \ddot{x}_2 + \frac{T_2 x_2}{l_2} - \frac{T_2 x_1}{l_2} = 0 \quad \text{...(j)}$$

Substituting values of T_1 and T_2 from Equations (f) and (e) in Equations (i) and (j), we get,

$$m_1 \ddot{x}_1 + \left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g x_1 - \frac{m_2}{l_2} g x_2 = 0 \quad \text{...(5.3.3)}$$

$$m_2 \ddot{x}_2 + \frac{m_2}{l_2} g x_2 - \frac{m_2}{l_2} g x_1 = 0 \quad \text{...(5.3.4)}$$

☞ Solution of Differential Equation :

- **Solutions for x_1 and x_2 :**

$$x_1 = X_1 \sin\omega t \quad \text{...(k)}$$

$$x_2 = X_2 \sin\omega t \quad \text{...(l)}$$

where,

X_1, X_2 = amplitudes of vibration of the two masses m_1 and m_2

ω = frequency of vibration rad/s

Therefore,

$$\ddot{x}_1 = -X_1 \omega^2 \sin\omega t \quad \text{...(m)}$$

$$\ddot{x}_2 = -X_2 \omega^2 \sin\omega t \quad \text{...(n)}$$

Substituting Equations (k), (l) and (m) in Equation (5.3.3) we get,

$$-m_1 X_1 \omega^2 \sin\omega t + \left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g X_1 \sin\omega t - \frac{m_2}{l_2} g X_2 \sin\omega t = 0$$

$$\therefore \quad -m_1 X_1 \omega^2 + \left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g X_1 - \frac{m_2}{l_2} g X_2 = 0$$

$$\therefore \quad \left\{\left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g - m_1 \omega^2\right\} X_1 = \frac{m_2}{l_2} g X_2$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{m_2 g / l_2}{\left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g - m_1 \omega^2} \quad \text{...(5.3.5)}$$

- On substituting Equations (k), (l) and (n) in Equation (5.3.4), we get,

$$-m_2 X_2 \omega^2 \sin\omega t + \frac{m_2}{l_2} g X_2 \sin\omega t - \frac{m_2}{l_2} g X_1 \sin\omega t = 0$$

$$\therefore \quad -m_2 X_2 \omega^2 + \frac{m_2}{l_2} g X_2 - \frac{m_2}{l_2} g X_1 = 0$$

$$\therefore \quad \left(\frac{m_2}{l_2} g - m_2 \omega^2\right) X_2 = \frac{m_2}{l_2} g X_1$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{m_2 g / l_2 - m_2 \omega^2}{m_2 g / l_2} \quad \text{...(5.3.6)}$$

- Equations (5.3.5) or (5.3.6), give the mode shapes of the system.
- **Frequency equations :**

From Equations (5.3.5) and (5.3.6),

$$\frac{m_2 g / l_2}{\left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g - m_1 \omega^2} = \frac{\left[\frac{m_2 g}{l_2} - m_2 \omega^2\right]}{m_2 g / l_2}$$

$$\therefore \quad \left\{\left[\frac{m_1 + m_2}{l_1} + \frac{m_2}{l_2}\right] g - m_1 \omega^2\right\}\left\{\frac{m_2 g}{l_2} - m_2 \omega^2\right\} = \left[\frac{m_2 g}{l_2} \cdot \frac{m_2 g}{l_2}\right]$$

$$\left(\frac{m_1 m_2 + m_2^2}{l_1 l_2}\right) g^2 - \frac{(m_1 m_2 + m_2^2)}{l_1} g \omega^2 + \frac{m_2^2 g^2}{l_2^2} - \frac{m_2^2 g \omega^2}{l_2}$$

$$- \frac{m_1 m_2 g \omega^2}{l_2} + m_1 m_2 \omega^4 = \frac{m_2^2 g^2}{l_2^2}$$

$$\mathbf{m_1 m_2 \omega^4 - [m_1 m_2 + m_2^2]\left[\frac{1}{l_1} + \frac{1}{l_2}\right] g \omega^2}$$

$$\mathbf{+ \left[\frac{m_1 m_2}{l_1 l_2} + \frac{m_2^2}{l_2^2}\right] g^2 = 0} \quad \text{...(5.3.7)}$$

The above Equation (5.3.7) is quadratic in ω^2 and gives two values of ω^2 (Two positive values of ω and two negative values of ω). The two positive values of ω give the natural frequencies ω_{n1} and ω_{n2} of the system. Therefore, Equation (5.3.7) is known as **frequency equation**.

☞ Special Case of Double Pendulum ($m_1 = m_2 = m$ and $l_1 = l_2 = l$) :

$$\frac{X_1}{X_2} = \frac{mg / l}{\left(\frac{2m}{l} + \frac{m}{l}\right) g - m \omega^2}$$

$$\text{and} \quad \frac{X_1}{X_2} = \frac{mg / l - m \omega^2}{mg / l}$$

$$\text{or} \quad \frac{X_1}{X_2} = \frac{g/l}{\frac{3g}{l} - \omega^2} \qquad \text{...(o)}$$

$$\text{and} \quad \frac{X_1}{X_2} = \frac{g/l - \omega^2}{g/l} \qquad \text{...(p)}$$

1. Natural Frequencies :

Substituting $m_1 = m_2 = m$ and $l_1 = l_2 = l$ in Equation (5.3.7), we get,

$$m^2\omega^2 - [m^2 + m^2]\left[\frac{2}{l}\right] g\,\omega^2 + \left[\frac{m^2}{l^2} + \frac{m^2}{l^2}\right] g^2 = 0$$

$$\therefore \quad m^2\omega^4 - \frac{4\,m^2}{l} g\omega^2 + \frac{2m^2 g^2}{l^2} = 0$$

$$\therefore \quad \omega^4 - \frac{4g}{l}\omega^2 + \frac{2g^2}{l^2} = 0$$

$$\therefore \quad \omega^2 = \frac{+\frac{4g}{l^2} \pm \sqrt{\frac{16g^2}{l^2} - \frac{4\times 2\,g^2}{l^2}}}{2}$$

$$\therefore \quad \omega^2 = \frac{4g}{2l} \pm \frac{\sqrt{\frac{16g^2}{l^2} - \frac{8\,g^2}{l^2}}}{2} = \frac{2g}{l} \pm \frac{\sqrt{\frac{8g^2}{l^2}}}{2}$$

$$\therefore \quad \omega^2 = \frac{2g}{l} \pm \frac{2\sqrt{2}}{2}\frac{g}{l} = \frac{2g}{l} \pm \sqrt{2}\frac{g}{l}$$

$$\therefore \quad \omega^2 = \left(2 \pm \sqrt{2}\right)\frac{g}{l}$$

$$\therefore \quad \omega_{n1}^2 = \left(2 - \sqrt{2}\right)\frac{g}{l} \text{ and } \omega_{n2}^2 = \left(2 + \sqrt{2}\right)\frac{g}{l}$$

$$\therefore \quad \omega_{n1}^2 = 0.58\frac{g}{l} \text{ and } \omega_{n2}^2 = 3.4142\frac{g}{l}$$

$$\left.\begin{aligned} \therefore \quad \omega_{n1} &= 0.7615\sqrt{\frac{g}{l}}, \text{ rad/s and} \\ \omega_{n2} &= 1.8477\sqrt{\frac{g}{l}}, \text{ rad/s} \end{aligned}\right\} \qquad \text{...(q)}$$

- Therefore, ω_{n1} and ω_{n2} are two natural frequencies of the double pendulum shown in Fig. 5.3.1.

2. Mode Shapes :

From Equation (5.3.6),

$$\frac{X_1}{X_2} = \frac{\frac{m_2 g}{l_2} - m_2^2\omega^2}{\frac{m_2 g}{l_2}} \qquad \text{...(r)}$$

Substituting $m_1 = m_2 = m$ and $l_1 = l_2 = l$ in Equation (r),

$$\frac{X_1}{X_2} = \frac{\frac{mg}{l} - m\omega^2}{\frac{mg}{l}}$$

$$\text{or} \quad \frac{X_1}{X_2} = \frac{\frac{g}{l} - \omega^2}{\frac{g}{l}} \qquad \text{...(s)}$$

- **First mode shape (first ratio of amplitudes) :**

Substituting $\omega^2 = \omega_{n1}^2 = 0.58\frac{g}{l}$ in Equation (s), we get,

$$\left(\frac{X_1}{X_2}\right)_1 = \frac{\frac{g}{l} - 0.58\frac{g}{l}}{\frac{g}{l}}$$

$$\therefore \quad \left(\frac{X_1}{X_2}\right)_1 = 0.42 \qquad \text{...(t)}$$

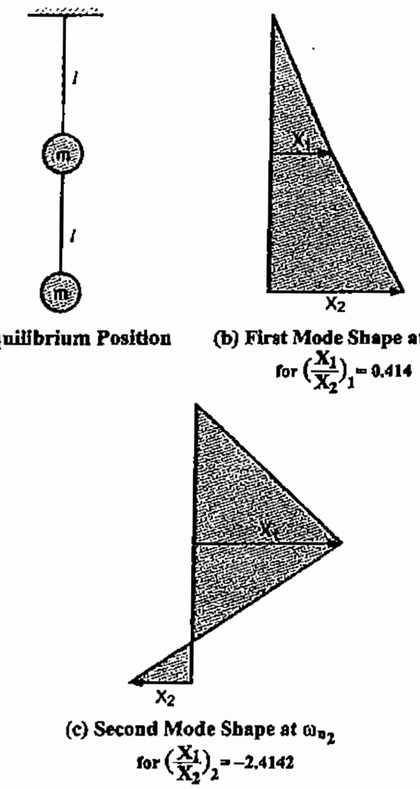
The first ratio of amplitudes of the two masses is + 0.414, which means that the two motions are in phase, i.e. the two masses move left or right together such that $\left(\frac{X_1}{X_2}\right)_1 = 0.414$, and frequency is ω_{n1} [Fig. 5.3.3(b)].

- **Second mode shape (Second ratio of amplitudes) :**

Again substituting $\omega^2 = \omega_{n2}^2 = 3.4142\frac{g}{l}$ in Equation (t), we get,

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{\frac{g}{l} - 3.4142\frac{g}{l}}{\frac{g}{l}}$$

$$\therefore \quad \left(\frac{X_1}{X_2}\right)_2 = -2.4142 \qquad \text{...(u)}$$



(a) Equilibrium Position (b) First Mode Shape at ω_{n_1} for $\left(\frac{X_1}{X_2}\right)_1 = 0.414$

(c) Second Mode Shape at ω_{n_2} for $\left(\frac{X_1}{X_2}\right)_2 = -2.4142$

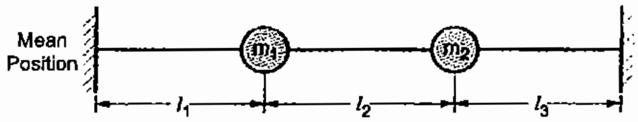
Fig. 5.3.3 : Principal Mode Shapes for Double Pendulum

The second ratio of amplitudes of the two masses is -2.4142 which means that the two motions are out of phase, i.e. when mass moves left the other mass moves right or vice- versa such that $\left(\frac{X_1}{X_2}\right)_2 = -2.4142$, and frequency is ω_{n2} [Fig. 5.3.3(c)].

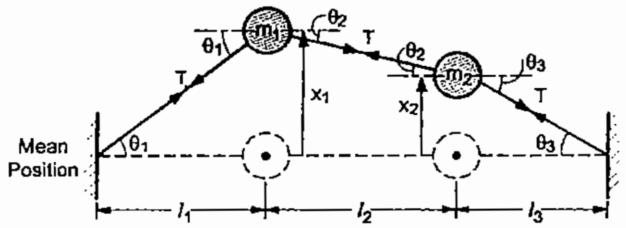
5.3.2 Two Masses Fixed on Stretched String :

Two Mass and String System :

- Consider two masses m_1 and m_2 fixed on a tight string stretched between two fixed supports, as shown in Fig. 5.3.4(a). At any instant the string is stretched and the two masses are displaced be the distances x_1 and x_2 from mean position, as shown in Fig. 5.3.4(b).



(a) Equilibrium Position



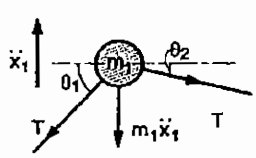
(b) Displaced Position

Fig. 5.3.4 : Two Masses fixed on a Tightly Stretched String

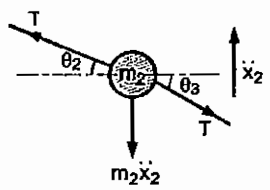
- The amplitudes of vibration of two masses are so small and tension 'T' is so large that it remains appreciably constant during the vibrations of the two masses.

Differential Equation of Motion for Two Masses Fixed on Stretched String :

$\Sigma F_H = 0$ **(Fig. 5.3.5) :**



(a) F.B.D. of Mass m_1



(b) F.B.D. of Mass m_2

Fig. 5.3.5 : F.B.D of Two Masses

The horizontal components of tension **T (i.e. T cos θ_1, T cos θ_2 and T cos θ_3)** are approximately equal to T (since the amplitude of vibration is small; θ_1, θ_2, and θ_3 are also small). Therefore, no resultant force acts along the horizontal direction.

$\Sigma F_v = 0$ **(Fig. 5.3.5) :**

$$m_1 \ddot{x}_1 + T\sin\theta_1 + T\sin\theta_2 = 0 \quad ...(5.3.8)$$

$$m_2 \ddot{x}_2 - T\sin\theta_2 + T\sin\theta_3 = 0 \quad ...(5.3.9)$$

From Fig. 5.3.4(b),

$$\sin\theta_1 \simeq \tan\theta_1 = \frac{x_1}{l_1} \quad ...(a)$$

$$\sin\theta_2 \simeq \tan\theta_2 = \frac{(x_1 - x_2)}{l_2} \quad ...(b)$$

$$\text{and} \quad \sin\theta_3 \simeq \tan\theta_3 = \frac{x_2}{l_3} \quad ...(c)$$

Substituting values of $\sin\theta_1$, $\sin\theta_2$ and $\sin\theta_3$ in Equations (5.3.8) and (5.3.9), we get,

$$m_1 \ddot{x}_1 + T\frac{x_1}{l_1} + T\frac{(x_1 - x_2)}{l_2} = 0$$

$$m_2 \ddot{x}_2 - T\frac{(x_1 - x_2)}{l_2} + T\frac{x_2}{l_3} = 0$$

Rearranging the above equations,

$$m_1 \ddot{x}_1 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)x_1 - \left(\frac{T}{l_2}\right)x_2 = 0 \quad ...(5.3.10)$$

$$m_2 \ddot{x}_2 + \left(\frac{T}{l_2} + \frac{T}{l_3}\right)x_2 - \left(\frac{T}{l_2}\right)x_1 = 0 \quad ...(5.3.11)$$

Solution of Differential Equations of Motion :

- **Solutions for x_1 and x_2:**

$$x_1 = X_1 \sin\omega t \quad ...(d)$$

$$x_2 = X_2 \sin\omega t \quad ...(e)$$

where, X_1 = amplitude of vibration of mass m_1

X_2 = amplitude of vibration of mass m_2

ω = frequency of vibration, rad/s

Therefore,

$$\ddot{x}_1 = -X_1\omega^2 \sin\omega t \quad ...(f)$$

$$\ddot{x}_2 = -X_2\omega^2 \sin\omega t \quad ...(g)$$

- Substituting Equations (d), (e) and (f) in Equation (5.3.10) we get,

$$-m_1X_1\omega^2\sin\omega t + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)X_1\sin\omega t - \left(\frac{T}{l_2}\right)X_2\sin\omega t = 0$$

$$\therefore m_1X_1\omega^2 + \left(\frac{T}{l_1} + \frac{T}{l_2}\right)X_1 - \left(\frac{T}{l_2}\right)X_2 = 0$$

$$\therefore \left[\left(\frac{T}{l_1} + \frac{T}{l_2}\right) - m_1\omega^2\right]X_1 = \left(\frac{T}{l_2}\right)X_2$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{T/l_2}{\left[\left(\frac{T}{l_1}+\frac{T}{l_2}\right)-m_1\omega^2\right]} \qquad ...(5.3.12)$$

- Substituting Equations (d), (e) and (g) in Equation (5.3.11), we get,

$$-m_2X_2\,\omega^2 \sin\omega t + \left(\frac{T}{l_2}+\frac{T}{l_3}\right)X_2\sin\omega t - \left(\frac{T}{l_2}\right)X_1\sin\omega t = 0$$

$$\therefore \quad -m_2X_2\,\omega^2 + \left(\frac{T}{l_2}+\frac{T}{l_3}\right)X_2 - \left(\frac{T}{l_2}\right)X_1 = 0$$

$$\therefore \quad \left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\omega^2\right]X_2 = \left(\frac{T}{l_2}\right)X_1$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\omega^2\right]}{T/l_2} \qquad ...(5.3.13)$$

- Equations (5.3.12) and (5.3.13) gives the mode shapes for the system
- **Frequency equation :**

From Equations (5.3.12) and (5.3.13),

$$\frac{T/l_2}{\left[\left(\frac{T}{l_1}+\frac{T}{l_2}\right)-m_1\omega^2\right]} = \frac{\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\omega^2\right]}{T/l_2}$$

$$\therefore \left[\left(\frac{T}{l_1}+\frac{T}{l_2}\right)-m_1\omega^2\right]\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\omega^2\right] = \left(\frac{T}{l_2}\right)^2$$

$$\frac{T^2}{l_1l_2}+\frac{T^2}{l_1l_3}-\frac{m_2\omega^2T}{l_1}+\frac{T^2}{l_2^2}+\frac{T^2}{l_2l_3}-\frac{m_2\omega^2T}{l_2}-\frac{m_1\omega^2T}{l_2}-\frac{m_1\omega^2T}{l_3}$$

$$+m_1m_2\omega^4 = \frac{T^2}{l_2^2}$$

$$m_1m_2\omega^4 - \left[\left(\frac{1}{l_1}+\frac{1}{l_2}\right)m_2T + \left(\frac{1}{l_2}+\frac{1}{l_3}\right)m_1T\right]\omega^2$$

$$+\left[\frac{1}{l_1l_2}+\frac{1}{l_2l_3}+\frac{1}{l_1l_3}\right]T^2 = 0 \qquad ...(5.3.14)$$

The above Equation (5.3.14) is quadratic in ω^2 and gives two values of ω^2 (two positive values ω and two negative values of ω). The two positive values of ω give the natural frequencies ω_{n1} and ω_{n2} of the system. Therefore Equation (5.3.14) is known as **frequency equation**.

Special Case of Two Masses and Stretched String System($m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = l$) :

1. Natural Frequencies (ω_{n1} and ω_{n2}) :

Substituting $m_1 = m_2 = m$ and $l_1 = l_2 = l_3 = l$ in Equation (5.3.14), we get,

$$m^2\omega^4 - \left[\frac{2mT}{l}+\frac{2mT}{l}\right]\omega^2 + \frac{3T^2}{l^2} = 0$$

$$\therefore \quad m^2\omega^4 - \frac{4mT}{l}\omega^2 + \frac{3T^2}{l^2} = 0$$

$$\therefore \quad \omega^2 = \frac{+\frac{4mT}{l} \pm \sqrt{16\frac{T^2}{l^2}m^2 - 4m^2\frac{3T^2}{l^2}}}{2m^2}$$

$$\therefore \quad \omega^2 = \frac{4mT}{2lm^2} \pm \frac{\sqrt{4\frac{T^2}{l^2}m^2}}{2m^2} = \frac{2T}{lm} \pm \frac{2\,Tm}{2l\,m^2}$$

$$\therefore \quad \omega^2 = \frac{2T}{lm} \pm \frac{T}{lm}$$

$$\therefore \quad \omega_{n1}^2 = \frac{T}{lm} \quad \text{and} \quad \omega_{n2}^2 = \frac{3T}{lm}$$

$$\therefore \quad \omega_{n1} = \sqrt{\frac{T}{lm}}\,\text{, rad/s and } \omega_{n2} = \sqrt{\frac{3T}{lm}}\,\text{, rad/s} \qquad ...(h)$$

- Therefore ω_{n1} and ω_{n2} are the two natural frequencies of the system shown in Fig. 5.3.4.

2. Mode Shapes :

From Equation (5.3.13),

$$\frac{X_1}{X_2} = \frac{\left[\left(\frac{T}{l_2}+\frac{T}{l_3}\right)-m_2\,\omega^2\right]}{\frac{T}{l_2}} \qquad ...(i)$$

Substituting $m_2 = m$ and $l_2 = l_3 = l$ in Equation (i).

- **Ratio of amplitudes :**

$$\frac{X_1}{X_2} = \frac{\left[\frac{2T}{l}-m\omega^2\right]}{(T/l)} \qquad ...(j)$$

- **First mode shape (first ratio of amplitudes) :**

Substituting $\omega^2 = \omega_{n1}^2 = \frac{T}{lm}$ in Equation (j), we get,

$$\left(\frac{X_1}{X_2}\right)_1 = \frac{\left[\frac{2T}{l}-\frac{mT}{lm}\right]}{(T/l)} = \frac{T/l}{T/l}$$

$$\therefore \quad \left(\frac{X_1}{X_2}\right)_1 = 1 \qquad ...(k)$$

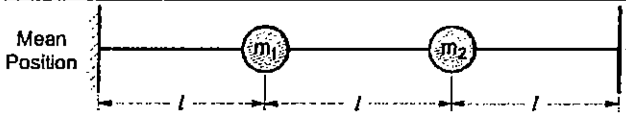
The first ratio of the amplitudes of the two masses is + 1, which means that the two amplitudes are equal and two motions are in phase i.e. the two masses move up or down together with frequency ω_{n1} [Fig. 5.3.6(b)]

- **Second mode shape (second ratio of amplitudes) :**

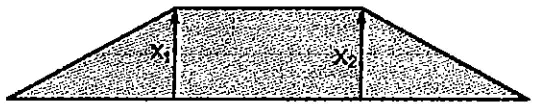
Again substituting $\omega^2 = \omega_{n2}^2 = \frac{3T}{lm}$ in Equation (h), we get,

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{\left[\frac{2T}{l}-\frac{3mT}{lm}\right]}{T/l} = \frac{-T/l}{T/l}$$

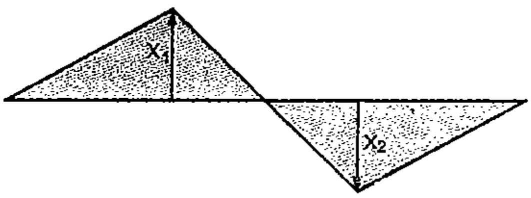
$$\therefore \quad \left(\frac{X_1}{X_2}\right)_2 = -1 \qquad ...(l)$$



(a) In Equilibrium Position



(b) Frist Mode Shape at ω_{n1} for $\left(\frac{X_1}{X_2}\right)_1 = 1$

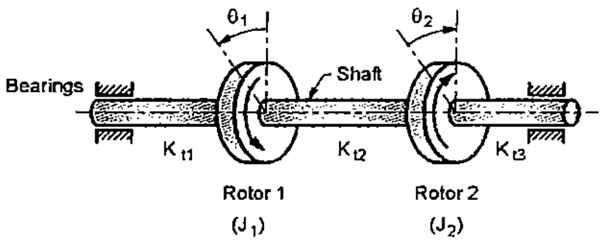


(c) Second Mode Shape at ω_{n2} for $\left(\frac{X_1}{X_2}\right)_2 = -1$

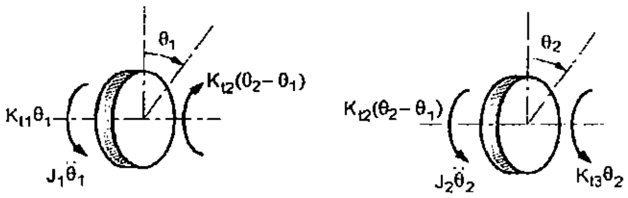
Fig. 5.3.6 : Principal Mode Shapes for Two Masses Fixed on Tightly Stretched String

The second ratio of the amplitude of the two masses is – 1, which means that the two amplitudes are equal and two motions are out of phase i.e. when one mass moves up the other mass moves down or vice-versa, with frequency ω_{n2}. [Fig. 5.3.6(c)]

5.3.3 Torsional System :



(a) Equilibrium Position



(b) F. B. D

Fig. 5.3.7 : Torsional System

- **Differential Equations of Motion for Two Degrees of Freedom Torsional System :**

From Fig. 5.3.7(b);

$$J_1 \ddot{\theta}_1 + K_{t1}\theta_1 - K_{t2}(\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + K_{t2}(\theta_2 - \theta_1) + K_{t3}\theta_2 = 0$$

- **Solution of Differential Equation of Motion :**

- The differential equations as well as solution to differential equations of motion for two degrees of freedom torsional system is identical to longitudinal vibrations of two degrees of freedom system discussed under section 5.2.
- The differential equations as well as solution to differential equations of motion for two degrees of freedom torsional system is obtained by replacing equivalent terms given in Table 5.3.1.

Table 5.3.1 : Longitudinal System and Torsional System

Sr. No.	Parameter in Longitudinal System	Equivalent Parameter in Torsional System
1.	m_1	J_1
2.	m_2	J_2
3.	K_1	K_{t1}
4.	K_2	K_{t2}
5.	K_3	K_{t3}
6.	x_1	θ_1
7.	x_2	θ_2
8.	X_1	ϕ_1
9.	X_2	ϕ_2

Note : Refer Example 5.3.19 for illustration.

5.3.4 Semi-definite System :

University Question

Q. Explain degenerate system with any two examples.

SPPU : Dec. 15, May 18

Q. What is semi definite system ?

SPPU : Oct. 19(In Sem.)

- If one of the natural frequency of the system is zero, then system will not vibrate. In other words there is no relative motion between two masses and the system can be moved as a rigid body. Such systems are known as semi-definite system or degenerate systems.

- The example of such type of system os shown in Fig. 5.3.8

  The two differential equations of motion are,

$$m_1 \ddot{x}_1 - k(x_2 - x_1) = 0 \text{ and } m_2 \ddot{x}_2 + k(x_2 - x_1) = 0$$

The natural frequencies are, $\omega_1 = 0$ and

$$\omega_2 = \sqrt{\frac{(m_1 + m_2)\,k}{m_1 m_2}} \text{ rad/s}$$

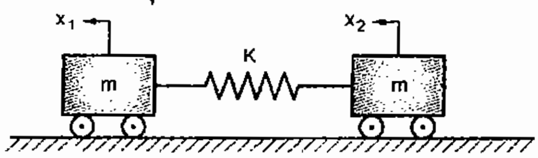


Fig. 5.3.8

- Another example of degenerate system is shown in Fig. 5.3.9.

  The natural frequencies for the system are,

$$\omega_{n1} = 0 \quad \text{and}$$

$$\omega_{n2} = \sqrt{\frac{(m_1 + m_2)\,g}{m_1 l}} \text{ rad/s}$$

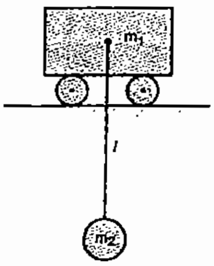


Fig. 5.3.9

Ex. 5.3.1 : Fig. P. 5.3.1(a) shows a vibrating system having two masses $m_1 = 2$ kg, $m_2 = 0.5$ kg and two springs of stiffnesses $K_1 = 30$ N / m, $K_2 = 10$ N / m. Determine :

(i) the two natural frequencies ;

(ii) the ratio of amplitudes for the two modes of vibration ; and

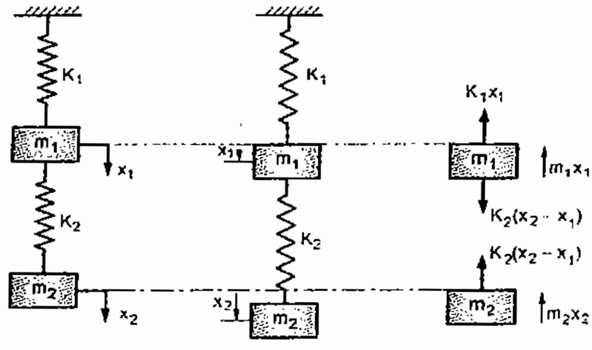
(iii) the principal mode shapes.

Fig. P. 5.3.1(a)

Soln. :

1. Frequency Equation :

- **Assumption :** $x_2 > x_1$.



(b) Equilibrium Position (c) Displaced Position (d) F.B.D. of Masses

Fig. P. 5.3.1

- **Two differential equations of motion :**

  From Fig. P. 5.3.1(d),

$$m_1 \ddot{x}_1 + K_1 x_1 - K_2(x_2 - x_1) = 0 \quad \text{...(a)}$$

and
$$m_2 \ddot{x}_2 + K_2(x_2 - x_1) = 0 \quad \text{...(b)}$$

Rearranging the terms in Equation (a) and (b)

or
$$m_1 \ddot{x}_1 + (K_1 + K_2)x_1 - K_2 x_2 = 0 \quad \text{...(c)}$$

and
$$m_2 \ddot{x}_2 - K_2 x_1 + K_2 x_2 = 0 \quad \text{...(d)}$$

- **Solutions for x_1 and x_2 under steady state conditions :**

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right\} \quad \text{...(e)}$$

Therefore,

$$\left.\begin{aligned} \ddot{x}_1 &= -X_1 \omega^2 \sin \omega t \\ \ddot{x}_2 &= -X_2 \omega^2 \sin \omega t \end{aligned}\right\} \quad \text{...(f)}$$

- Substituting Equations (e) and (f) in Equation (c),

$$-m_1 X_1 \omega^2 \sin \omega t + (K_1 + K_2) X_1 \sin \omega t - K_2 X_2 \sin \omega t = 0$$

$$\therefore \quad -m_1 X_1 \omega^2 + (K_1 + K_2) X_1 - K_2 X_2 = 0$$

$$\therefore \quad [(K_1 + K_2) - m_1 \omega^2] X_1 = K_2 X_2$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{K_2}{[(K_1 + K_2) - m_1 \omega^2]} \quad \text{...(g)}$$

- Substituting Equations (e) and (f) in Equation (d),

$$-m_2 X_2 \omega^2 \sin \omega t - K_2 X_1 \sin \omega t + K_2 X_2 \sin \omega t = 0$$

$$\therefore \quad -m_2 X_2 \omega^2 - K_2 X_1 + K_2 X_2 = 0$$

$$\therefore \quad (K_2 - m_2 \omega^2) X_2 = K_2 X_1$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{(K_2 - m_2\omega^2)}{K_2} \quad ...(h)$$

- Frequency equation :
- From Equations (g) and (h),

$$\frac{K_2}{[(K_1 + K_2) - m_1 \omega^2]} = \frac{(K_2 - m_2\omega^2)}{K_2}$$

$$\therefore \quad [(K_1 + K_2) - m_1\omega^2](K_2 - m_2\omega^2) = K_2^2$$

$$(K_1 + K_2) K_2 - (K_1 + K_2) m_2\omega^2 - m_1\omega^2 K_2 + m_1 m_2\omega^4 = K_2^2$$

$$\therefore m_1 m_2\omega^4 - [(K_1 + K_2) m_2 + m_1 K_2] \omega^2 + (K_1 + K_2) K_2 - K_2^2 = 0$$

$$m_1 m_2\omega^4 - [(K_1 + K_2) m_2 + m_1 K_2] \omega^2 + K_1 K_2 + K_2^2 - K_2^2 = 0$$

$$\omega^4 - \left[\frac{(K_1 + K_2)}{m_1} + \frac{K_2}{m_2}\right] \omega^2 + \frac{K_1 K_2}{m_1 m_2} = 0 \quad ...(i)$$

2. Two Natural Frequencies :

- Putting $m_1 = 2$ kg, $m_2 = 0.5$ kg, $K_1 = 30$ N/m and $K_2 = 10$ N/m in Equation (g),

$$\therefore \omega^4 - \left[\frac{30 + 10}{2} + \frac{10}{0.5}\right] \omega^2 + \frac{30 \times 10}{2 \times 0.5} = 0$$

$$\therefore \quad \omega^4 - 40 \omega^2 + 300 = 0$$

$$\therefore \quad \omega^2 = \frac{+ 40 \pm \sqrt{(40)^2 - 4 \times 300}}{2}$$

$$\therefore \quad \omega^2 = 20 \pm 10$$

$$\therefore \quad \omega_{n1}^2 = 10 \text{ and } \omega_{n2}^2 = 30$$

$$\therefore \quad \omega_{n1} = \mathbf{3.16\ rad/s} \text{ and } \omega_{n2} = \mathbf{5.47\ rad/s} \quad ...\text{Ans.}$$

3. Ratio of Amplitudes :

- **For first mode shape :**

$$\omega^2 = \omega_{n1}^2 = 10$$

$$\frac{X_1}{X_2} = \frac{K_2}{[(K_1 + K_2) - m_1 \omega^2]}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{K_2}{(K_1 + K_2) - m_1 \omega_{n1}^2} = \frac{10}{(30 + 10) - 2 \times 10}$$

$$\text{or} \left(\frac{X_1}{X_2}\right)_1 = \mathbf{0.5} \quad ...\text{Ans.}$$

- **For second mode shape :**

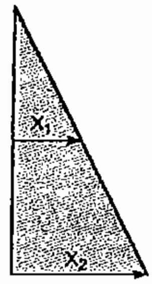
$$\omega^2 = \omega_{n2}^2 = 30$$

$$\therefore \left(\frac{X_1}{X_2}\right)_2 = \frac{K_2}{[(K_1 + K_2) - m_1 \omega_{n2}^2]} = \frac{10}{(30 + 10) - 2 \times 30}$$

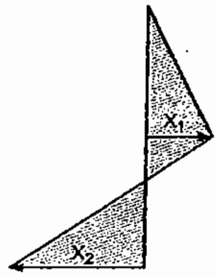
$$\text{or} \left(\frac{X_1}{X_2}\right)_2 = \mathbf{- 0.5} \quad ...\text{Ans}$$

4. Principal mode shapes :

- The two principal mode shapes for given system are shown in Fig. P. 5.3.1(e) and Fig. P. 5.3.1(f).



(e) First Mode Shape at ω_{n1} = 3.16 rad/s for $\left(\frac{X_1}{X_2}\right)_1$ = 0.5



(f) Second Mode Shape at ω_{n2} = 5.47 rad/s for $\left(\frac{X_1}{X_2}\right)_2$ = − 0.5

Fig. P. 5.3.1 : Principal Mode Shapes

Ex. 5.3.2 : Determine the natural frequencies of the system shown in Fig. P. 5.3.2(a). Use following data :
Given : $K_1 = K_2 = 40$ N/m, $K = 60$ N/m, $m_1 = m_2 = 10$ kg

SPPU - Dec. 12,10 Marks

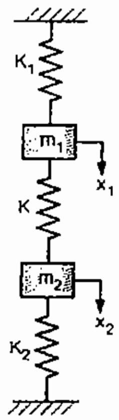


Fig. P. 5.3.2(a)

Soln. :

1. Frequency Equation :

- **Assumptions :**
  (i) The masses are constrained to move in the direction of spring axis.
  (ii) The springs are weightless.
  (iii) $x_2 > x_1$.
- **Two differential equations of motion :**
  From Fig. P. 5.3.2(d),

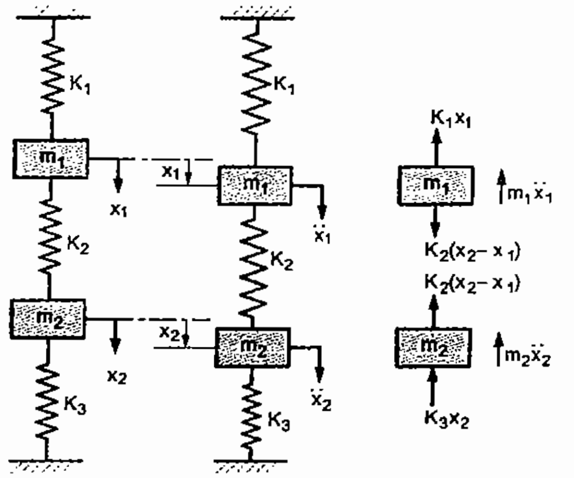
$$m_1 \ddot{x}_1 + K_1 x_1 - K_2 (x_2 - x_1) = 0 \quad ...(a)$$

$$m_2 \ddot{x}_2 + K_2 (x_2 - x_1) + K_3 x_2 = 0 \quad ...(b)$$

- By rearranging the terms of Equations (a) and (b), we get,

$$m_1 \ddot{x}_1 + (K_1 + K_2) x_1 - K_2 x_2 = 0 \quad ...(c)$$

$$m_2 \ddot{x}_2 + (K_2 + K_3) x_2 - K_2 x_1 = 0 \quad ...(d)$$



(b) Equilibrium position **(c) displaced position** **(d) F.B.D. of masses**

Fig. P. 5.3.2 : Spring Mass System having Two Degrees of Freedom

- **Solutions for x_1 and x_2 under steady state condition :**

$$x_1 = X_1 \sin \omega t \quad ...(e)$$

$$x_2 = X_2 \sin \omega t \quad ...(f)$$

- Therefore,

$$\ddot{x}_1 = -X_1 \omega^2 \sin \omega t \quad ...(g)$$

$$\ddot{x}_2 = -X_2 \omega^2 \sin \omega t \quad ...(h)$$

- Substituting Equations (e), (f) and (g) in Equation (c), we get,

$$-m_1 X_1 \omega^2 \sin \omega t + (K_1 + K_2) X_1 \sin \omega t - K_2 X_2 \sin \omega t = 0$$

$$\therefore -m_1 X_1 \omega^2 + (K_1 + K_2) X_1 - K_2 X_2 = 0$$

$$\therefore [-m_1 \omega^2 + (K_1 + K_2)] X_1 - K_2 X_2 = 0$$

$$\therefore [(K_1 + K_2) - m_1 \omega^2] X_1 = K_2 X_2$$

$$\therefore \frac{\mathbf{X_1}}{\mathbf{X_2}} = \frac{\mathbf{K_2}}{\mathbf{[(K_1 + K_2) - m_1 \omega^2]}} \quad ...(i)$$

- Substituting Equations (e), (f) and (h) in Equation (d), we get,

$$-m_2 \omega^2 X_2 \sin \omega t + (K_2 + K_3) X_2 \sin \omega t - K_2 X_1 \sin \omega t = 0$$

$$[-m_2 \omega^2 + (K_2 + K_3)] X_2 - K_2 X_1 = 0$$

$$\therefore \frac{\mathbf{X_1}}{\mathbf{X_2}} = \frac{\mathbf{[(K_2 + K_3) - m_2 \omega^2]}}{\mathbf{K_2}} \quad ...(j)$$

- **Frequency equation :**
  From Equations (i) and (j), we get,

$$\frac{K_2}{[(K_1 + K_2) - m_1 \omega^2]} = \frac{[(K_2 + K_3) - m_2 \omega^2]}{K_2}$$

$$\therefore [(K_1 + K_2) - m_1 \omega^2][(K_2 + K_3) - m_2 \omega^2] = K_2^2$$

$$\therefore (K_1 + K_2)(K_2 + K_3) - m_1 \omega^2 (K_2 + K_3) - m_2 \omega^2 (K_1 + K_2) + m_1 m_2 \omega^4 = K_2^2$$

$$\therefore K_1 K_2 + K_1 K_3 + K_2^2 + K_2 K_3 - [m_1 (K_2 + K_3) + m_2 (K_1 + K_2)] \omega^2 + m_1 m_2 \omega^4 = K_2^2$$

$$\therefore \mathbf{m_1 m_2 \omega^4 - [m_1 (K_2 + K_3) + m_2 (K_1 + K_2)] \omega^2 + [K_1 K_2 + K_2 K_3 + K_3 K_1] = 0} \quad ...(k)$$

- The above Equation (k) is quadratic in ω^2 and gives two values of ω^2, (two positive values of ω and two negative values of ω). The two positive values of ω give natural frequencies ω_{n1} and ω_{n2} of the system. Therefore, Equation (k) is called as **frequency equation.**

2. Two Natural Frequencies :

- Substituting $m_1 = m_2 = m = 10$ kg, $K_1 = K_3 = K = 40$ N/m and $K_2 = 60$ N/m, in Equation (k) becomes,

$$m^2 \omega^4 - [m (K_2 + K) + m (K + K_2)] \omega^2 + [KK_2 + K_2 K + KK] = 0$$

$$m^2 \omega^4 - 2m (K + K_2) \omega^2 + (K^2 + 2KK_2) = 0$$

$$(10)^2 \omega^4 - 2 \times 10 (40 + 60) \omega^2 + (40^2 + 2 \times 40 \times 60) = 0$$

$$100 \omega^4 - 2000 \omega^2 + 6400 = 0$$

$$\omega^4 - 20 \omega^2 + 64 = 0$$

$$\omega^2 = \frac{20 \pm \sqrt{(20)^2 - 4 \times 64}}{2}$$

$$\therefore \omega^2 = \frac{20 \pm 12}{2}$$

$$\therefore \quad \omega_{n1}^2 = 4 \text{ and } \omega_{n2}^2 = 16$$

$$\therefore \quad \omega_{n1} = 2 \text{ rad/s and } \omega_{n2} = 4 \text{ rad/s} \quad ...(l)$$

...**Ans.**

Ex. 5.3.3 : Derive the differential equations of motion for the system shown in Fig. P. 5.3.3(a). It is given that $m_1 = 20$ kg, $m_2 = 35$ kg and $k = 3000$ N/m. Determine :

(i) the natural frequencies;

(ii) the amplitude ratio for two modes; and

(iii) the principal mode shape. **SPPU - May 15, 12 Marks**

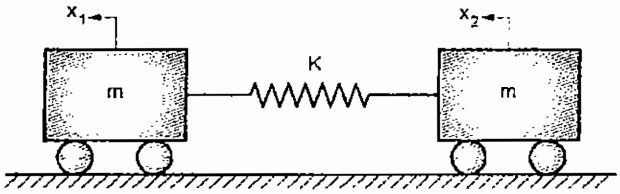
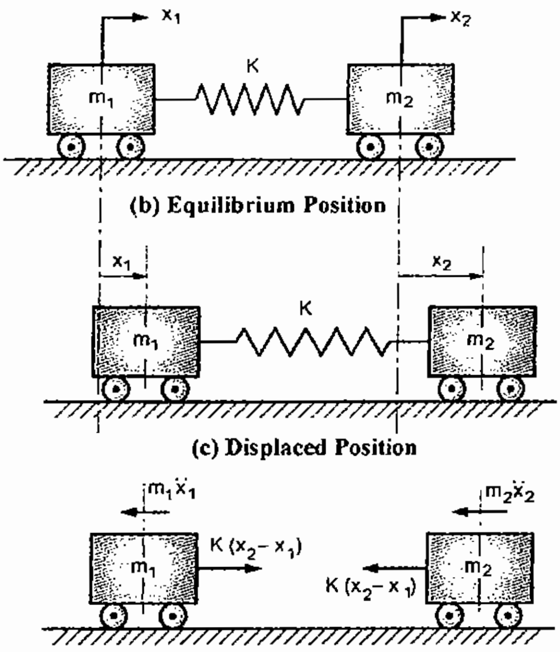


Fig. P. 5.3.3(a)

Soln. :

1. Frequency Equation :

Assumption : : $x_2 > x_1$



(b) Equilibrium Position

(c) Displaced Position

(d) F.B.D. of Masses

Fig. P. 5.3.3

- **Two differential equations of motion :**

  From Fig. P. 5.3.3 (d),

$$m_1 \ddot{x}_1 - K(x_2 - x_1) = 0$$

$$\text{and} \quad m_2 \ddot{x}_2 + K(x_2 - x_1) = 0$$

$$\text{or} \quad m_1 \ddot{x}_1 + K x_1 - K x_2 = 0 \quad ...(a)$$

$$\text{and} \quad m_2 \ddot{x}_2 - K x_1 + K x_2 = 0 \quad ...(b)$$

- **Solutions for x_1 and x_2 under steady state conditions :**

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right\} \quad ...(c)$$

Therefore,

$$\left.\begin{aligned} \ddot{x}_1 &= -X_1 \omega^2 \sin \omega t \\ \ddot{x}_2 &= -X_2 \omega^2 \sin \omega t \end{aligned}\right\} \quad ...(d)$$

- Substituting Equations (c) and (d) in Equation (a),

$$-m_1 X_1 \omega^2 \sin \omega t + K X_1 \sin \omega t - K X_2 \sin \omega t = 0$$

$$\therefore -m_1 X_1 \omega^2 + K X_1 - K X_2 = 0$$

$$\therefore (K - m_1 \omega^2) X_1 = K X_2$$

$$\therefore \frac{X_1}{X_2} = \frac{K}{(K - m_1 \omega^2)} \quad ...(e)$$

- Substituting Equations (c) and (d) in Equation (b),

$$-m_2 X_2 \omega^2 \sin \omega t - K X_1 \sin \omega t + K X_2 \sin \omega t = 0$$

$$\therefore -m_2 X_2 \omega^2 - K X_1 + K X_2 = 0$$

$$\therefore (K - m_2 \omega^2) X_2 = K X_1$$

$$\therefore \frac{X_1}{X_2} = \frac{(K - m_2 \omega^2)}{K} \quad ...(f)$$

- **Frequency equation :**

  From Equations (e) and (f),

$$\frac{K}{(K - m_1 \omega^2)} = \frac{(K - m_2 \omega^2)}{K}$$

$$\therefore (K - m_1 \omega^2)(K - m_2 \omega^2) = K^2$$

$$\therefore K^2 - K m_2 \omega^2 - K m_1 \omega^2 + m_1 m_2 \omega^4 = K^2$$

$$\therefore m_1 m_2 \omega^4 - (m_1 + m_2) K \omega^2 = 0$$

$$\therefore \omega^4 - \frac{(m_1 + m_2) K}{m_1 m_2} \omega^2 = 0 \quad ...(g)$$

  This Equation (g) is called as **frequency equation.**

2. Two Natural Frequencies :

- Substituting $m_1 = 20$ kg, $m_2 = 35$ kg and $K = 3000$ N / m in Equation (g), we get,

$$\omega^4 - \frac{(20 + 35)\,3000}{20 \times 35} \omega^2 = 0$$

$$\therefore \quad \omega^4 - 235.71\, \omega^2 = 0$$

$$\therefore \quad \omega^2 = \frac{+235.71 \pm \sqrt{(235.71)^2 - 0}}{2}$$

$$\therefore \quad \omega^2 = \frac{235.71}{2} \pm \frac{235.71}{2}$$

$\therefore \quad \omega_{n1}^2 = 0$ and $\omega_{n2}^2 = 235.71$

$\therefore \quad \omega_{n1} = 0 \text{ rad/s}$ and $\omega_{n2} = 15.35 \text{ rad/s}$ ...Ans.

- It is seen that as if one of the natural frequencies of the system is zero, the system is not vibrating. There is no relative motion between masses m_1 and m_2 and system can be moved as a rigid body. Such systems are known as **semi-definite systems or degenerate systems.**

3. Ratio of Amplitudes :

- **For first mode shape :**

$$\frac{X_1}{X_2} = \frac{K}{K - m_1 \omega^2}$$

$$\therefore \left(\frac{X_1}{X_2}\right) = \frac{K}{K - m_1 \omega_{n1}^2} = \frac{3000}{3000 - 20 \times 0}$$

$$\text{or} \quad \left(\frac{X_1}{X_2}\right)_1 = 1 \qquad \text{...Ans.}$$

- **For second mode shape :**

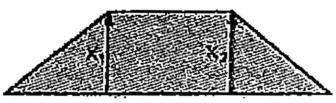
$$\left(\frac{X_1}{X_2}\right)_2 = \frac{K}{K - m_1 \omega_{n2}^2}$$

$$= \frac{3000}{3000 - 20 \times 235.71}$$

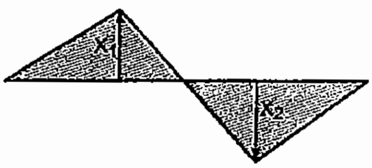
$$\text{or} \quad \left(\frac{X_1}{X_2}\right)_2 = -1.75 \qquad \text{...Ans.}$$

4. Principal Mode Shapes :

- The two principal mode shapes for given system are shown in Fig. P. 5.3.3(e) and Fig. P. 5.3.3(f).



(e) First Mode Shape at $\omega_{n1} = 0$ rad/s for $\left(\frac{X_1}{X_2}\right)_1 = 1$



(f) Second Mode Shape at $\omega_{n2} = 15.35$ rad/s for $\left(\frac{X_1}{X_2}\right)_2 = -1.75$

Fig. P. 5.3.3

Ex. 5.3.4 : Find natural frequencies and mode shapes for the system shown. Consider $m_1 = 25$ kg, $m_2 = 20$ kg and $k = 2000$ N/m. **SPPU - Oct. 19(In Sem.), 10 Marks**

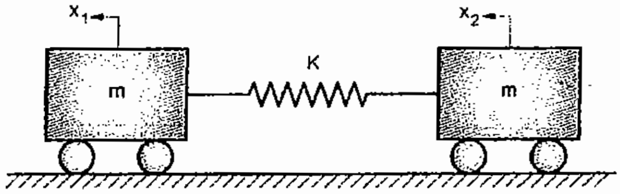
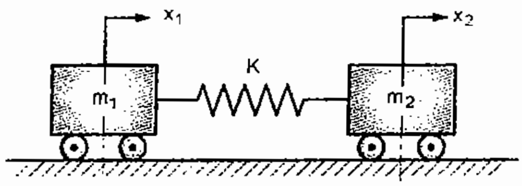


Fig. P. 5.3.4

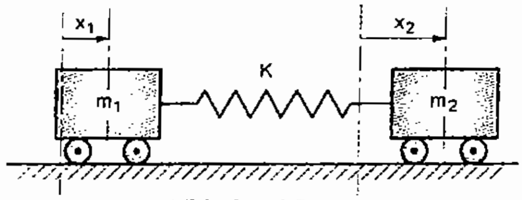
Soln. :

1. Frequency Equation :

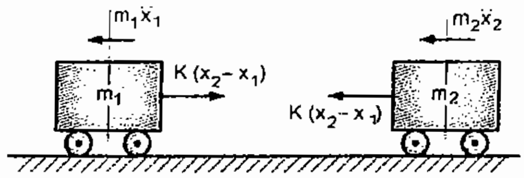
Assumption : : $x_2 > x_1$



(b) Equilibrium Position



(c) Displaced Position



(d) F.B.D. of Masses

Fig. P. 5.3.4

- **Two differential equations of motion :**

From Fig. P. 5.3.4(d),

$$m_1 \ddot{x}_1 - K(x_2 - x_1) = 0$$

$$\text{and} \quad m_2 \ddot{x}_2 + K(x_2 - x_1) = 0$$

$$\text{or} \quad m_1 \ddot{x}_1 + K x_1 - K x_2 = 0 \qquad \text{...(a)}$$

$$\text{and} \quad m_2 \ddot{x}_2 - K x_1 + K x_2 = 0 \qquad \text{...(b)}$$

- **Solutions for x_1 and x_2 under steady state conditions :**

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right\} \qquad \text{...(c)}$$

Therefore,
$$\left.\begin{aligned}\ddot{x}_1 &= -X_1\,\omega^2 \sin \omega t\\ \ddot{x}_2 &= -X_2\,\omega^2 \sin \omega t\end{aligned}\right\} \quad ...(d)$$

- Substituting Equations (c) and (d) in Equation (a),

$$-m_1 X_1 \omega^2 \sin \omega t + K X_1 \sin \omega t - K X_2 \sin \omega t = 0$$

$$\therefore\ -m_1 X_1 \omega^2 + K X_1 - K X_2 = 0$$

$$\therefore\ (K - m_1 \omega^2) X_1 = K X_2$$

$$\therefore\ \frac{X_1}{X_2} = \frac{K}{(K - m_1 \omega^2)} \quad ...(e)$$

- Substituting Equations (c) and (d) in Equation (b),

$$-m_2 X_2 \omega^2 \sin \omega t - K X_1 \sin \omega t + K X_2 \sin \omega t = 0$$

$$\therefore\ -m_2 X_2 \omega^2 - K X_1 + K X_2 = 0$$

$$\therefore\ (K - m_2 \omega^2) X_2 = K X_1$$

$$\therefore\ \frac{X_1}{X_2} = \frac{(K - m_2 \omega^2)}{K} \quad ...(f)$$

- **Frequency equation :**
- From Equations (e) and (f),

$$\frac{K}{(K - m_1 \omega^2)} = \frac{(K - m_2 \omega^2)}{K}$$

$$\therefore\ (K - m_1 \omega^2)(K - m_2 \omega^2) = K^2$$

$$\therefore\ K^2 - K m_2 \omega^2 - K m_1 \omega^2 + m_1 m_2 \omega^4 = K^2$$

$$\therefore\ m_1 m_2 \omega^4 - (m_1 + m_2) K \omega^2 = 0$$

$$\therefore\ \omega^4 - \frac{(m_1 + m_2)K}{m_1 m_2}\omega^2 = 0 \quad ...(g)$$

This Equation (g) is called as **frequency equation.**

2. Two Natural Frequencies :

- Substituting $m_1 = 25$ kg, $m_2 = 20$ kg and $K = 2000$ N / m in Equation (g), we get,

$$\omega^4 - \frac{(25 + 20)\,2000}{25 \times 20}\omega^2 = 0$$

$$\therefore\ \omega^4 - 180\,\omega^2 = 0$$

$$\therefore\ \omega^2 = \frac{+180 \pm \sqrt{(180)^2 - 0}}{2}$$

$$\therefore\ \omega^2 = \frac{180}{2} \pm \frac{180}{2}$$

$$\therefore\ \omega_{n1}^2 = 0 \text{ and } \omega_{n2}^2 = 180$$

$$\therefore\ \omega_{n1} = 0 \text{ rad/s and } \omega_{n2} = 13.41 \text{ rad/s} \quad ...\text{Ans.}$$

- It is seen that as if one of the natural frequencies of the system is zero, the system is not vibrating. There is no relative motion between masses m_1 and m_2 and system can be moved as a rigid body. Such systems are known as **semi-definite systems or degenerate systems.**

3. Ratio of Amplitudes :

- **For first mode shape :**

$$\frac{X_1}{X_2} = \frac{K}{K - m_1 \omega^2}$$

$$\therefore\ \left(\frac{X_1}{X_2}\right) = \frac{K}{K - m_1 \omega_{n1}^2} = \frac{2000}{2000 - 25 \times 0}$$

$$\text{or}\ \left(\frac{X_1}{X_2}\right)_1 = 1 \quad ...\text{Ans.}$$

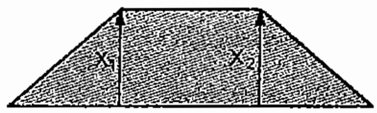
- **For second mode shape :**

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{K}{K - m_1 \omega_{n2}^2} = \frac{2000}{2000 - 25 \times 180}$$

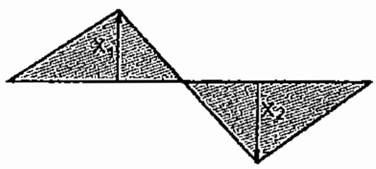
$$\text{or}\ \left(\frac{X_1}{X_2}\right)_2 = -0.8 \quad ...\text{Ans.}$$

4. Principal Mode Shapes :

- The two principal mode shapes for given system are shown in Fig. P. 5.3.4(e) and Fig. P. 5.3.4(f).



(e) First Mode Shape at $\omega_{n1} = 0$ rad/s for $\left(\frac{X_1}{X_2}\right)_2 = 1$



(f) Second Mode Shape at $\omega_{n2} = 31.41$ rad/s for $\left(\frac{X_1}{X_2}\right)_2 = -0.8$

Fig. P. 5.3.4

Examples for Practice

Refer our website for complete solution of following examples

Ex. 5.3.5 : Find natural frequencies and mode shapes for the system shown in Fig. P. 5.3.5(a). Take $m_1 = 15$ kg, $m_2 = 20$ kg and $K = 200$ N / m.

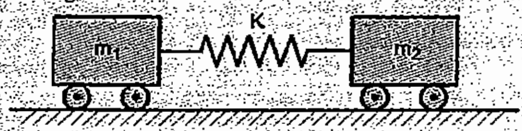


Fig. P. 5.3.5(a)

Ex. 5.3.6 : Two subway cars, as shown in following Fig. P. 5.3.6(a) have 2000 kg mass each and are connected by a coupler. The coupler can be modeled as a spring of stiffness K = 280 kN/m. Write the equations of motion and determine the natural frequencies and mode shapes. **SPPU - May 16, 12 Marks**

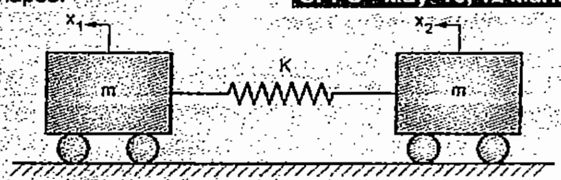


Fig. P. 5.3.6(a)

Ex. 5.3.7 : Determine the natural frequency and corresponding mode shapes of given system, shown in Fig. P. 5.3.7(a). Assume each spring stiffness as K. **SPPU - Dec. 15, 12 Marks**

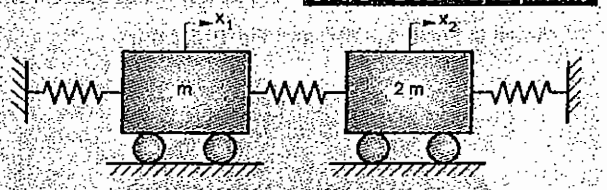


Fig. P. 5.3.7(a)

Ex. 5.3.8 : What do you mean by semi definite system ? An electric train made of two cars each of mass 2000 kg is connected by couplings of stiffness equal to 40×10^6 N/m as shown in Fig. P. 5.3.8(a). Determine the natural frequency of the system. **SPPU - Oct.18(In sem),10 Marks**

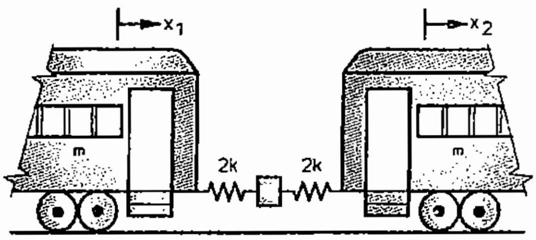


Fig. P. 5.3.8(a)

1. **Equivalent Spring :**

- The two coaches are connected by coupling where, the two springs are in series. Therefore equivalent spring stiffness is,

$$\frac{1}{K_e} = \frac{1}{2K} + \frac{1}{2K} = \frac{2}{2K}$$

$$\therefore \quad K_e = K = 40 \times 10^6$$

$$\text{or} \quad K_e = 40 \times 10^6 \text{ N / m}$$

The equivalent system is shown in Fig. P.5.3.8(b)

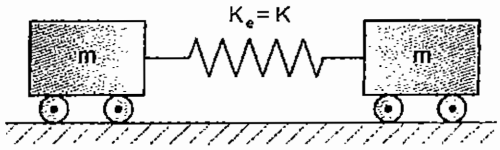


Fig. P. 5.3.8(b): Equivalent System

2. **Frequency Equation :**

- **Assumption :** $x_2 > x_1$

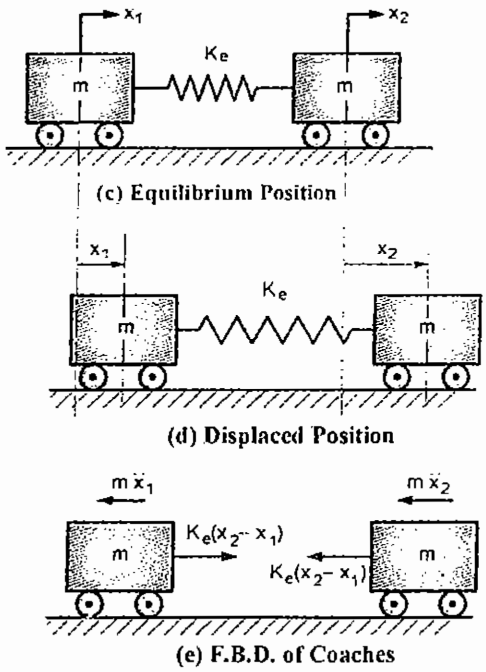


Fig. P. 5.3.8

- **Two differential equations of motion :**

From Fig. P. 5.3.8(e)

$$m\ddot{x}_1 - K_e(x_2 - x_1) = 0$$

$$\text{and} \quad m\ddot{x}_2 + K_e(x_2 - x_1) = 0$$

$$\text{or} \quad m\ddot{x}_1 + K_e x_1 - K_e x_2 = 0 \qquad \text{...(a)}$$

$$\text{and} \quad m\ddot{x}_2 - K_e x_1 + K_e x_2 = 0 \qquad \text{...(b)}$$

- **Solutions for x_1 and x_2 under steady state conditions :**

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right\} \quad ...(c)$$

Therefore,

$$\left.\begin{aligned} \ddot{x}_1 &= -X_1 \omega^2 \sin \omega t \\ \ddot{x}_2 &= -X_2 \omega^2 \sin \omega t \end{aligned}\right\} \quad ...(d)$$

- Substituting Equations (c) and (d) in Equation (a),

$$-m X_1 \omega^2 \sin \omega t + K_e X_1 \sin \omega t - K_e X_2 \sin \omega t = 0$$

$$\therefore -m X_1 \omega^2 + K_e X_1 - K_e X_2 = 0$$

$$\therefore (K_e - m\omega^2) X_1 = K_e X_2$$

$$\therefore \frac{X_1}{X_2} = \frac{K_e}{(K_e - m\omega^2)} \quad ...(e)$$

- Substituting Equations (c) and (d) in Equation (b),

$$-m X_2 \sin \omega t - K_e X_1 \sin \omega t + K_e X_2 \sin \omega t = 0$$

$$\therefore -m X_2 \omega^2 - K_e X_1 + K_e X_2 = 0$$

$$\therefore (K_e - m\omega^2) X_2 = K_e X_1$$

$$\therefore \frac{X_1}{X_2} = \frac{(K_e - m\omega^2)}{K_e} \quad ...(f)$$

- **Frequency equation :**
- From Equation (e) and (f),

$$\frac{K_e}{(K_e - m\omega^2)} = \frac{(K_e - m\omega^2)}{K_e}$$

$$\therefore (K_e - m\omega^2)(K_e - m\omega^2) = K^2$$

$$\therefore K_e^2 - 2K_e m \omega^2 + m^2\omega^4 = K_e^2$$

$$\therefore m^2\omega^4 - 2K_e m^2 \omega^2 = 0$$

$$\therefore \omega^4 - \frac{2K_e}{m}\omega^2 = 0 \quad ...(g)$$

3. Two Natural Frequencies :

Substituting m = 2000 kg and $K_e = 40 \times 10^6$ N/m in Equation (g),

$$\therefore \omega^4 - \frac{2 \times 40 \times 10^6}{2000}\omega^2 = 0$$

$$\therefore \omega^4 - 40000\omega^2 = 0$$

$$\therefore \omega^2 = \frac{+40000 \pm \sqrt{(40000)^2 - 0}}{2}$$

$$\therefore \omega^2 = \frac{40000}{2} \pm \frac{40000}{2}$$

$$\therefore \omega_{n1}^2 = 0 \text{ and } \omega_{n2}^2 = 40000$$

$$\therefore \omega_{n1} = 0 \text{ rad/s}$$

and $\omega_{n2} = 200$ rad / s ...**Ans.**

- Since one of the natural frequency is zero, system is not vibrating. There is no relative motion between two coaches, and system can be moved as a rigid body.

4. Ratio of Amplitudes :

- **First mode shape :**

$$\frac{X_1}{X_2} = \frac{K}{K - m\omega^2}$$

$$\therefore \left(\frac{X_1}{X_2}\right)_1 = \frac{K}{K - m\omega_{n1}^2} = \frac{40 \times 10^6}{40 \times 10^6 - 2000 \times 0}$$

$$\text{or } \left(\frac{X_1}{X_2}\right)_1 = 1 \quad ...\textbf{Ans.}$$

- **For second mode shape :**

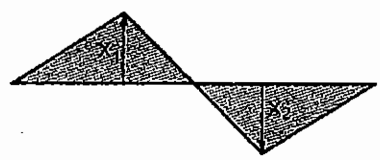
$$\left(\frac{X_1}{X_2}\right)_2 = \frac{K}{K - m\omega_{n2}^2} = \frac{40 \times 10^6}{40 \times 10^6 - 2000 \times 40000}$$

$$\text{or } \left(\frac{X_1}{X_2}\right)_2 = -1 \quad ...\textbf{Ans.}$$

5. Principal Mode Shapes :

- The two principal mode shapes for given system are shown in Fig. P. 5.3.8(f) and Fig. P. 5.3.8(g).

(f) First Mode Shape at ω_{n1} = 0 rad/s
for $\left(\frac{X_1}{X_2}\right)_1 = 1$



(g) Second Mode Shape at ω_{n2} = 200 rad/s
for $\left(\frac{X_1}{X_2}\right)_2 = -1$

Fig. P. 5.3.8

Examples for Practice

Refer our website for complete solution of following examples

Ex. 5.3.9 : The two railway coaches, each of mass 2000 kg, are connected by coupling, as shown in Fig. P. 5.3.9(a). Determine the natural frequencies and amplitude ratio for vibration of the railway coaches, when $K = 40 \times 10^6$ N / m.

Fig. P. 5.3.9(a)

Ex. 5.3.10 : Fig. P. 5.3.10(a) shows an engine connected to a compartment by a spring coupling having effective linear stiffness of 3×10^6 N/m. The engine weighs 20 tonnes where as the compartment weighs 15 tonnes. Determine the natural frequency.

SPPU - Dec. 15,12 Marks

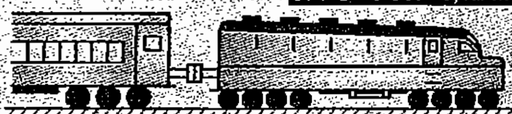


Fig. P. 5.3.10(a)

Ex. 5.3.11 : Set up the differential equations of motion for the system shown in Fig. P. 5.3.11(a) and determine :

(i) the natural frequencies ;

(ii) the ratio of amplitudes for the two modes ; and

(iii) the principal mode shapes when K = 40 N/m and m = 10 kg.

SPPU - May 07

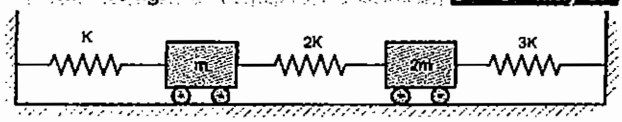
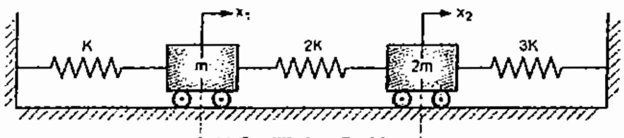


Fig. P. 5.3.11(a)

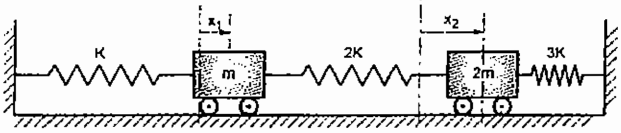
Soln. :

1. **Frequency Equation :**

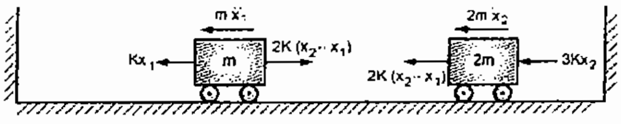
Assuming : $x_2 > x_1$



(b) Equilibrium Position



(c) Displaced Position



(d) F.B.D. of Masses

Fig. P. 5.3.11

- **Two differential equations of motion :**

From Fig. P. 5.3.11(d),

$$m\ddot{x}_1 - Kx_1 - 2K(x_2 - x_1) = 0$$

and $$2m\ddot{x}_2 + 2K(x_2 - x_1) + 3Kx_2 = 0$$

or $$m\ddot{x}_1 + 3Kx_1 - 2Kx_2 = 0 \quad \text{...(a)}$$

and $$2m\ddot{x}_2 + 5Kx_2 - 2Kx_1 = 0 \quad \text{...(b)}$$

- **Solutions for x_1 and x_2 under steady state conditions :**

$$\left.\begin{aligned} x_1 &= X_1 \sin \omega t \\ x_2 &= X_2 \sin \omega t \end{aligned}\right\} \quad \text{...(c)}$$

- Therefore,

$$\left.\begin{aligned} \ddot{x}_1 &= -X_1 \omega^2 \sin \omega t \\ \ddot{x}_2 &= -X_2 \omega^2 \sin \omega t \end{aligned}\right\} \quad \text{...(d)}$$

- Substituting Equations (c) and (d) in Equations (a),

$$-mX_1\omega^2 \sin \omega t + 3KX_1 \sin \omega t - 2KX_2 \sin \omega t = 0$$

$$\therefore \quad -mX_1\omega^2 + 3KX_1 - 2KX_2 = 0$$

$$\therefore \quad (3K - m\omega^2)X_1 = 2KX_2$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{2K}{3K - m\omega^2} \quad \text{...(e)}$$

- Substituting Equations (c) and (d) in Equations (b),

$$-2mX_2\omega^2 \sin \omega t + 5KX_2 \sin \omega t - 2KX_1 \sin \omega t = 0$$

$$\therefore \quad -2mX_2\omega^2 + 5KX_2 - 2KX_1 = 0$$

$$\therefore \quad (5K - 2m\omega^2)X_2 = 2KX_1$$

$$\therefore \quad \frac{X_1}{X_2} = \frac{5K - 2m\omega^2}{2K} \quad \text{...(f)}$$

- **Frequency Equation :** From Equations (e) and (f),

$$\frac{2K}{(3K - m\omega^2)} = \frac{(5K - 2m\omega^2)}{2K}$$

$$\therefore \quad (3K - m\omega^2)(5K - 2m\omega^2) = (2K)^2$$

$$\therefore \quad 15K^2 - 6Km\omega^2 - 5Km\omega^2 + 2m^2\omega^4 - 4K^2 = 0$$

$$\therefore \quad 2m^2\omega^4 - 11Km\omega^2 + 11K^2 = 0$$

$$\therefore \quad \omega^4 - \frac{11K}{2m}\omega^2 + \frac{11K^2}{2m^2} = 0 \quad \text{...(g)}$$

2. **Two Natural Frequencies :**

- Substituting K = 40 N/m and m = 10 kg in Equation (g),

$$\omega^4 - \frac{11 \times 40}{2 \times 10}\,\omega^2 + \frac{11 \times (40)^2}{2 \times (10)^2} = 0$$

$$\therefore \quad \omega^4 - 22\,\omega^2 + 88 = 0$$

$$\therefore \quad \omega^2 = \frac{+22 \pm \sqrt{(22)^2 - 4 \times 88}}{2} = 11 \pm 5.74$$

$$\therefore \quad \omega_{n1}^2 = 5.26$$

$$\text{and} \quad \omega_{n2}^2 = 16.74$$

$$\therefore \quad \omega_{n1} = \mathbf{2.29\ rad/s\ and\ \omega_{n2} = 4.09\ rad/s} \quad \text{...Ans.}$$

3. **Ratio of Amplitudes :**

- **For first mode shape :**

$$\frac{X_1}{X_2} = \frac{2K}{(3K - m\omega^2)}$$

$$\therefore \quad \left(\frac{X_1}{X_2}\right)_1 = \frac{2K}{3K - m\,\omega_{n1}^2} = \frac{2 \times 40}{3 \times 40 - 10 \times 5.26}$$

$$\text{or} \quad \left(\frac{X_1}{X_2}\right)_1 = \mathbf{1.186} \quad \text{...Ans.}$$

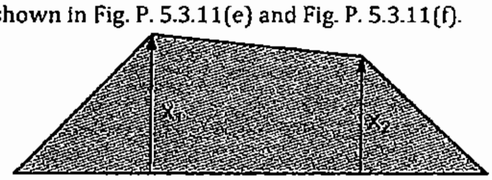
- **For second mode shape :**

$$\left(\frac{X_1}{X_2}\right)_2 = \frac{2K}{3K - m\,\omega_{n2}^2} = \frac{2 \times 40}{3 \times 40 - 10 \times 16.74}$$

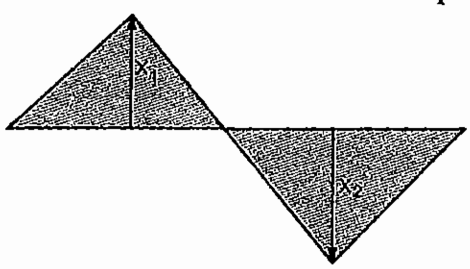
$$\therefore \quad \left(\frac{X_1}{X_2}\right)_2 = -1.6877 \quad \text{...Ans.}$$

4. **Principal Mode Shapes :**

The two principal mode shapes for given system are shown in Fig. P. 5.3.11(e) and Fig. P. 5.3.11(f).



(e) First Mode Shape at ω_{n1} = 2.29 rad/s for $\left(\frac{X_1}{X_2}\right)_1$ = 1.186



(f) Second Mode Shape at ω_{n2} = 4.09 rad/s for $\left(\frac{X_1}{X_2}\right)_2$ = – 1.6877

Fig. P. 5.3.11

Examples for Practice

Refer our website for complete solution of following examples

Ex. 5.3.12 : Determine the natural frequencies for the system, shown in Fig. P. 5.3.12(a). Use following data : K_1 = 1000 N / m ; K_2 = 500 N / m ; m_1 = 50 kg/s ; m_2 = 10 kg.

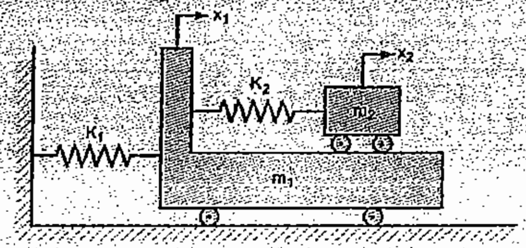


Fig. P. 5.3.12(a)

Ex.5.3.13 : Write governing differential equations for system given below. **SPPU - May 19, 5 Marks**

Fig. P. 5.3.13(a)

Ex. 5.3.14 : A double pendulum having length *l* is shown in Fig. P. 5.3.14(a). Determine the natural frequency of the double pendulum when K = 50 N / m, m_1 = 3 kg, m_2 = 5 kg, *l* = 200 mm and a = 100 mm.

SPPU - May 12, May 14, 10 Marks

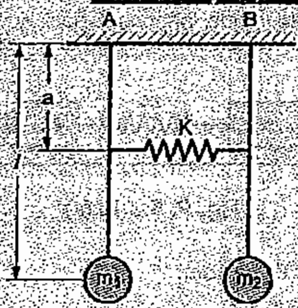


Fig. P. 5.3.14(a)

Ex. 5.3.15 : Two uniform rods AB and CD, having lengths 0.08 m and 0.1 m respectively, are pivoted at their upper ends as shown in Fig. P. 5.3.15(a). The mass of rods AB and CD are 3 kg and 5 kg respectively. The stiffness of spring is 2000 N / m. Determine :

(i) the natural frequencies, neglecting effect of gravity ;

(ii) the angular amplitude of rod CD for second mode of vibration, if rod AB moves through 1° on either side of the vertical; and

(iii) the maximum force in spring during second mode of vibration.

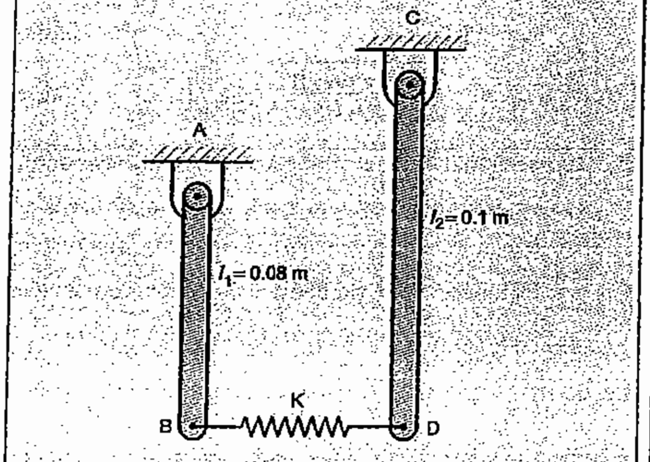


Fig. P. 5.3.15(a)

Ex. 5.3.16 : Find the natural frequency of vibration for the system shown in Fig. P. 5.3.16(a). **SPPU - Dec. 07**

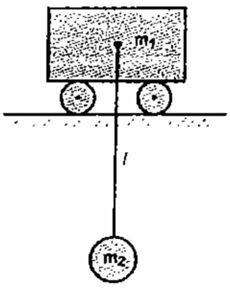
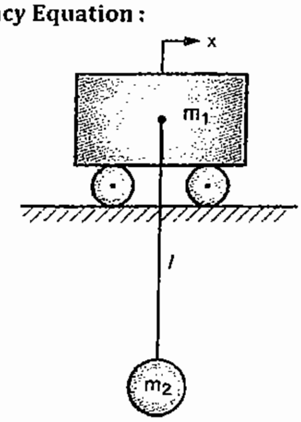


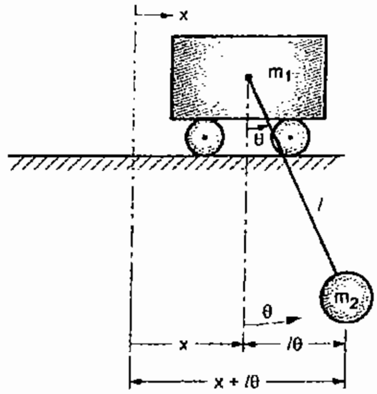
Fig. P. 5.3.16(a)

Soln. :

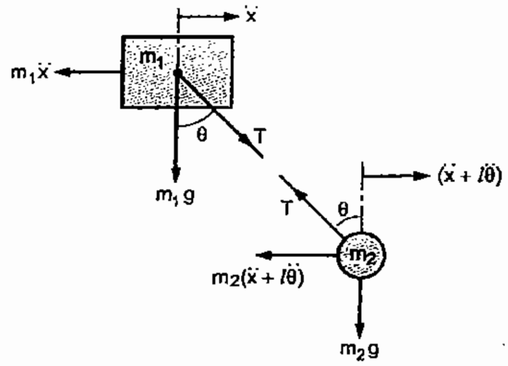
1. **Frequency Equation :**



(b) Equilibrium Position



(c) Displaced Position



(d) F.B.D of System

Fig. P. 5.3.16

- Total displacement of mass m_2 = $(x + l\,\theta)$ [Fig. P. 5.3.16(c)]
- Let T be the tension in string.
- The F.B.D of system is shown in Fig. P. 5.3.16(d).
- **F.B.D of mass m_2 :**

$$\Sigma F_V = 0$$

$$T\cos\theta - m_2 g = 0$$

$$T\cos\theta = m_2 g$$

$$\text{or} \quad T = m_2 g \ldots [\text{ as } \theta \text{ is small, } \cos\theta = 1] \qquad \ldots(a)$$

- **Differential equations of motion in horizontal direction :**

$$m_1\ddot{x} - T\sin\theta = 0$$

$$\text{and} \quad m_2(\ddot{x} + l\ddot{\theta}) + T\sin\theta = 0$$

$$\text{or} \quad m_1\ddot{x} - T\theta = 0 \qquad \ldots(b)$$

$$\text{and} \quad m_2(\ddot{x} + l\ddot{\theta}) + T\theta = 0 \qquad \ldots(c)$$

Substituting Equation (a) in Equations (b) and (c),

$$\text{or} \quad m_1\ddot{x} - m_2 g\theta = 0$$

$$\text{and} \quad m_2(\ddot{x} + l\ddot{\theta}) + m_2 g\theta = 0$$

or $$m_1 \ddot{x} - m_2 g \theta = 0 \quad ...(d)$$

and $$\ddot{x} + l \ddot{\theta} + g \theta = 0 \quad ...(e)$$

- **Solutions for x and θ under steady state conditions :**

$$\left.\begin{aligned} x &= X \sin \omega t \\ \theta &= \phi \sin \omega t \end{aligned}\right\} \quad ...(f)$$

where, X and ϕ are amplitudes of m_1 and m_2 respectively

Therefore,

$$\left.\begin{aligned} \ddot{x} &= -X \omega^2 \sin \omega t \\ \ddot{\theta} &= -\phi \omega^2 \sin \omega t \end{aligned}\right\} \quad ...(g)$$

- Substituting Equations (f) and (g) in Equation (d),

$$-m_1 X \omega^2 \sin \omega t - m_2 g \phi \sin \omega t = 0$$

$$\therefore \quad -m_1 X \omega^2 - m_2 g \phi = 0$$

$$\therefore \quad -m_1 X \omega^2 = m_2 g \phi$$

$$\therefore \quad \frac{X}{\phi} = \frac{m_2 g}{-m_1 \omega^2} \quad ...(h)$$

- Substituting Equations (f) and (g) in Equation (e),

$$-X \omega^2 \sin \omega t - l \phi \omega^2 \sin \omega t + g \phi \sin \omega t = 0$$

$$\therefore \quad -X \omega^2 - l \phi \omega^2 + g \phi = 0$$

$$\therefore \quad (g - l \omega^2) \phi = X \omega^2$$

$$\therefore \quad \frac{X}{\phi} = \frac{(g - l \omega^2)}{\omega^2} \quad ...(i)$$

- **Frequency equation :**
- From Equation (h) and (i) we can write

$$\frac{m_2 g}{-m_1 \omega^2} = \frac{g - l \omega^2}{\omega^2}$$

$$\therefore \quad -m_1 \omega^2 (g - l \omega^2) = m_2 g \omega^2$$

$$\therefore \quad -m_1 g \omega^2 + m_1 l \omega^4 = m_2 g \omega^2$$

$$\therefore \quad m_1 l \omega^4 - (m_1 + m_2) g \omega^2 = 0$$

$$\therefore \quad \omega^4 - \frac{(m_1 + m_2) g}{m_1 l} \omega^2 = 0 \quad ...(j)$$

2. Two Natural Frequencies :

From Equation (j)

$$\omega^2 = \frac{+\dfrac{(m_1 + m_2) g}{m_1 l} \pm \sqrt{\left[\dfrac{(m_1 + m_2) g}{m_1 l}\right]^2 + 0}}{2}$$

$$\therefore \quad \omega^2 = \frac{(m_1 + m_2) g}{2 m_1 l} \pm \frac{(m_1 m_2) g}{2 m_1 l}$$

$$\therefore \quad \omega_{n1}^2 = 0$$

And $$\omega_{n2}^2 = \frac{(m_1 + m_2) g}{m_1 l}$$

$$\therefore \quad \omega_{n1} = 0 \text{, rad/s} \quad \textbf{...Ans.}$$

and $$\omega_{n2} = \sqrt{\frac{(m_1 + m_2) g}{m_1 l}} \text{, rad/s} \quad \textbf{...Ans.}$$

- It is seen that as if one of the natural frequencies of the system is zero, the system is not vibrating. There is no relative motion between masses m_1 and m_2 and system can be moved as a rigid body. Such systems are known as **semi-definite systems or degenerate systems.**

Ex. 5.3.17 : Fig. P. 5.3.17(a) shows combined pendulum and spring - mass system. Determine an expression for the natural frequencies. Obtained the equation for natural frequencies when :

(i) K = 0 ; (ii) $m_2 = 0$; (iii) $l = 0$. **SPPU - Dec. 06**

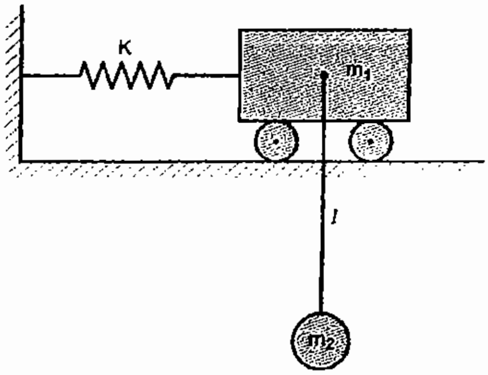
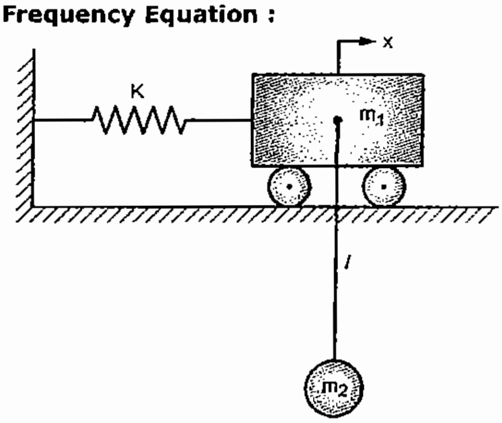


Fig. P. 5.3.17(a)

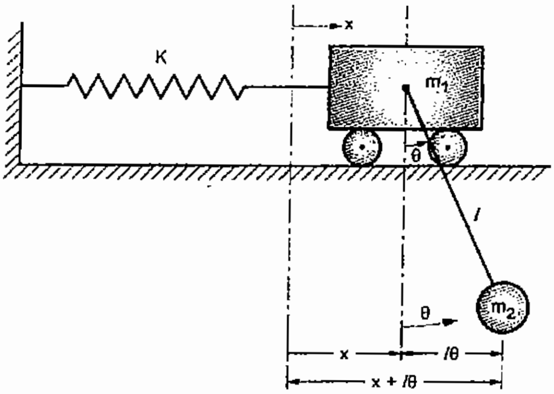
Soln. :

1. Frequency Equation :

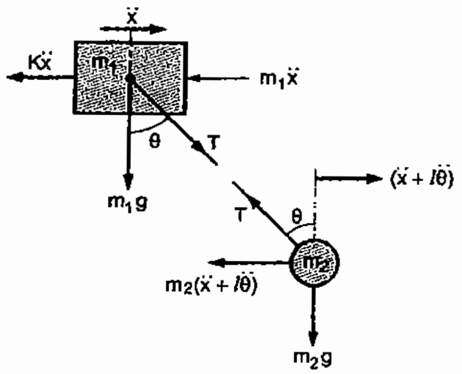


(b) Equilibrium Position

Fig. P. 5.3.17



(c) Displaced Position



(d) F.B.D of System

Fig. P. 5.3.17

- **From F.B.D of mass m_2 :**

$$\sum F_V = 0$$

$$T\cos\theta - m_2 g = 0$$

$$T\cos\theta = m_2 g$$

or $T = m_2 g$ ... [as θ is small, cos θ = 1] ...(a)

- **Differential equations of motion in horizontal direction :**

$$m_1\ddot{x} + Kx - T\sin\theta = 0$$

and $m_2(\ddot{x} + l\ddot{\theta}) + T\sin\theta = 0$

or $m_1\ddot{x} + Kx - T\theta = 0$ ...(b)

and $m_2(\ddot{x} + l\ddot{\theta}) + T\theta = 0$ ...(c)

Substituting Equation (a) in Equations (b) and (c),

or $m_1\ddot{x} + Kx - m_2 g\theta = 0$

and $m_2(\ddot{x} + l\ddot{\theta}) + m_2 g\theta = 0$

or $m_1\ddot{x} + Kx - m_2 g\theta = 0$ ...(d)

and $\ddot{x} + l\ddot{\theta} + g\theta = 0$ ...(e)

- **Solutions for x and θ under steady state conditions :**

$$\left.\begin{aligned} x &= X\sin\omega t \\ \theta &= \phi\sin\omega t \end{aligned}\right\} \quad ...(f)$$

where, X and ϕ are amplitudes of m_1 and m_2 respectively,

Therefore, $$\left.\begin{aligned} \ddot{x} &= -X\omega^2\sin\omega t \\ \ddot{\theta} &= -\phi\omega^2\sin\omega t \end{aligned}\right\} \quad ...(g)$$

- Substituting Equations (f) and (g), in Equation (d),

$$-m_1 X\omega^2\sin\omega t + KX\sin\omega t - m_2 g\phi\sin\omega t = 0$$

$$\therefore\ -m_1 X\omega^2 + KX - m_2 g\phi = 0$$

$$\therefore\ (K - m_1\omega^2)X = m_2 g\phi$$

$$\therefore\ \frac{X}{\phi} = \frac{m_2 g}{(K - m_1\omega^2)} \quad ...(h)$$

- Substituting Equations (f) and (g) in Equations (e),

$$-X\omega^2\sin\omega t - l\phi\omega^2\sin\omega t + g\phi\sin\omega t = 0$$

$$\therefore\ -X\omega^2 - l\phi\omega^2 + g\phi = 0$$

$$\therefore\ (g - l\omega^2)\phi = X\omega^2$$

$$\therefore\ \frac{X}{\phi} = \frac{(g - l\omega^2)}{\omega^2} \quad ...(i)$$

- **Frequency equations :**
- From Equations (h) and (i),

$$\frac{m_2 g}{(K - m_1\omega^2)} = \frac{(g - l\omega^2)}{\omega^2}$$

$$\therefore\ (K - m_1\omega^2)(g - l\omega^2) = m_2 g\omega^2$$

$$\therefore\ Kg - Kl\omega^2 - m_1 g\omega^2 + m_1 l\omega^4 - m_2 g\omega^2 = 0$$

$$\therefore\ m_1 l\omega^4 - (m_1 g + m_2 g + Kl)\omega^2 + Kg = 0$$

$$\therefore\ \mathbf{m_1 l\omega^4 - [(m_1 + m_2)g + Kl]\omega^2 + Kg = 0} \quad ...(j)$$

2. Two Natural Frequencies :

From Equation (j),

$$\therefore\ \omega^2 = \frac{+(m_1 + m_2)g + Kl \pm \sqrt{[(m_1 + m_2)g + Kl]^2 - 4m_1 lKg}}{2m_1 l}$$

...Ans.

This is equation for the two natural frequencies.

(i) When K = 0 :

When K = 0, the equation for natural frequencies becomes,

$$\omega^2 = \frac{+(m_1 + m_2)g}{2m_1 l} \pm \frac{\sqrt{[(m_1 + m_2)g]^2}}{2m_1 l}$$

$$\omega^2 = \frac{+(m_1 + m_2)g}{2m_1 l} \pm \frac{(m_1 + m_2)g}{2m_1 l} \quad \textbf{...Ans.}$$

(ii) When $m_2 = 0$:

When $m_2 = 0$, the equation for natural frequencies becomes,

$$\omega^2 = \frac{+ m_1 g + K l \pm \sqrt{[(m_1 g) + K l]^2 - 4 m_1 l K g}}{2 m_1 l} \quad \text{...Ans.}$$

(iii) When $l = 0$

When $l = 0$, the equation for natural frequencies becomes,

$$\omega^2 = \frac{+ (m_1 + m_2) g \pm \sqrt{[(m_1 + m_2) g + 0]^2 - 0}}{0}$$

...Ans.

$$\omega^2 = \infty \quad \text{...Ans.}$$

Examples for Practice

Refer our website for complete solution of following examples

Ex. 5.3.18 : Determine the natural frequencies of the system shown in Fig. P. 5.3.18. k = 90 N/m, l = 25 m, $m_1 = 2$ kg, $m_2 = 0.5$ kg. **SPPU - May 17, 12 Marks**

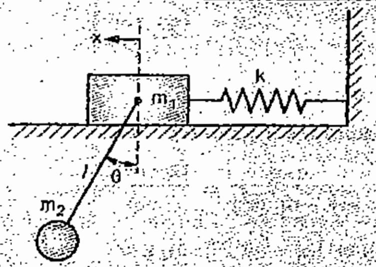


Fig. P. 5.3.18

Ex. 5.3.19 : Determine the natural frequencies of the system, shown in Fig. P. 5.3.19(a). Assume that cord is inextensible and there is no slip between cord and pulley. Take $K_1 = 30$ N / m, $K_2 = 50$ N / m, $m_1 = 2$ kg and $m_2 = 8$ kg

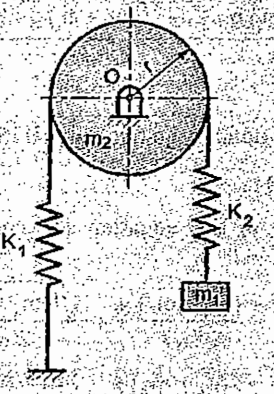


Fig. P. 5.3.19(a)

Ex. 5.3.20 : Marine engine is connected to a propeller through gears, as shown in Fig. P. 5.3.20(a). The mass moment of inertia of flywheel, engine, gear 1, gear 2 and propeller are 9000, 1000, 250, 150 and 2000 kg-m^2 respectively. Find the natural frequencies and mode shapes of the system in torsional vibration. Neglect mass moment of inertia of engine, gear 1 and gear 2. **SPPU - May 13, 12 Marks**

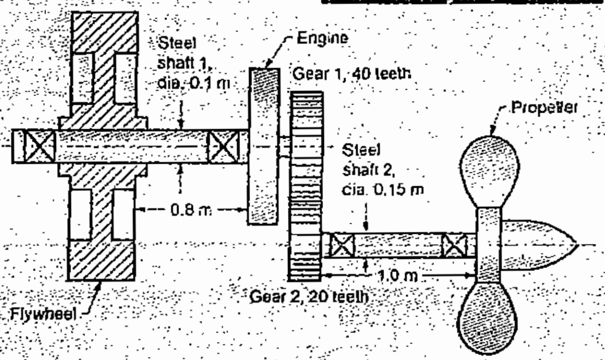


Fig. P. 5.3.20(a)

Ex. 5.3.21 : Find the natural frequencies of the system shown in Fig. P. 5.3.21.

$m_1 = 10$ kg ; $m_2 = 12$ kg ;

$r_1 = 0.10$ m ; $r_2 = 0.11$ kg ;

$K_1 = 40 \times 10^3$ N/m ; $K_2 = 50 \times 10^3$ N/m ;

$K_3 = 60 \times 10^3$ N/m ; **SPPU - Dec. 16, Dec. 17, 12 Marks**

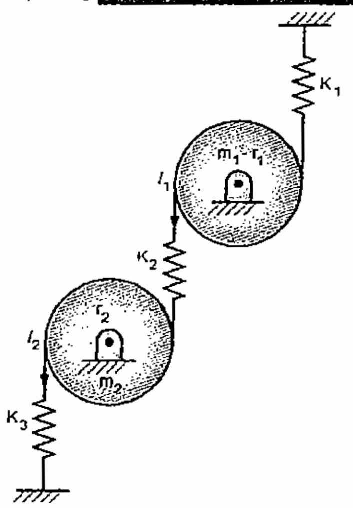


Fig. P. 5.3.21

Soln. :

$m_1 = 10$ kg ; $m_2 = 12$ kg ;

$r_1 = 0.10$ m ; $r_2 = 0.11$ m ;

$K_1 = 40 \times 10^3$ N/m ; $K_2 = 50 \times 10^3$ N/m ;

$K_3 = 60 \times 10^3$ N/m

1. **Frequency Equation :**

- **Two differential equations of motion :**

$$I_1 \ddot{\theta}_1 + \left(K_1 r_1^2 + K_2 r_1^2\right)\theta_1 - K_2 r_1 r_2 \theta_2 = 0 \quad \text{...(a)}$$

$$I_2 \ddot{\theta}_2 + \left(K_3 r_2^2 + K_2 r_2^2\right)\theta_2 - K_2 r_1 r_2 \theta_1 = 0 \quad \text{...(b)}$$

- **Solutions for θ_1 and θ_2 under steady state conditions :**

Assume,

$$\left.\begin{aligned}\theta_1 &= \phi_1 \sin \omega t\\ \theta_2 &= \phi_2 \sin \omega t\end{aligned}\right\}$$

$$\left.\begin{aligned}\ddot{\theta}_1 &= -\omega^2 \phi_1 \sin \omega t ;\\ \ddot{\theta}_2 &= -\omega \phi_2 \sin \omega t\end{aligned}\right\} \quad \text{...(d)}$$

$$\left.\begin{aligned}I_1 &= \frac{1}{2} m_1 r_1^2 ;\\ I_2 &= \frac{1}{2} m_2 r_2^2\end{aligned}\right\} \quad \text{...(e)}$$

Substituting Equations (c), (d) and (e) in Equation (a).

$$\frac{1}{2} m_1 r_1^2 [-\omega^2 \phi_1 \sin \omega t] + \left(K_1 r_1^2 + K_2 r_1^2\right)\phi_1 \sin \omega t - K_2 r_1 r_2 \phi_2 \sin \omega t = 0$$

$$\frac{1}{2} m_1 r_1^2 (-\omega^2)\phi_1 + \left(K_1 r_1^2 + K_2 r_1^2\right)\phi_1 - K_2 r_1 r_2 \theta_1 = 0$$

$$\frac{\phi_1}{\phi_2} = \frac{2 K_2 r_1 r_2}{(-m_1 r_1^2 \omega^2) + 2\left(K_1 r_1^2 + K_2 r_1^2\right)} \quad \text{...(f)}$$

- Substituting Equations (c), (d) and (e) in Equation (b).

$$\frac{1}{2} m_2 r_2^2 (-\omega^2 \phi_2 \sin \omega t) + \left(K_3 r_2^2 + K_2 r_2^2\right)\phi_2 \sin \omega t - K_2 r_1 r_2 \phi_1 \sin \omega t = 0$$

$$\frac{1}{2} m_2 r_2^2 (-\omega^2)\phi_2 + \left(K_3 r_2^2 + K_2 r_2^2\right)\phi_2 - K_2 r_1 r_2 \phi_1 = 0$$

$$\frac{\phi_1}{\phi_2} = \frac{\left(\omega^2 m_2 r_2^2\right) + 2\left(K_3 r_2^2 + K_2 r_2^2\right)}{2 K_2 r_1 r_2} \quad \text{...(g)}$$

- **Frequency equation :** From Equations (f) and (g),

$$\frac{2 K_2 r_1 r_2}{\left[-m_1 r_1^2 \omega^2 + 2\left(K_1 r_1^2 + K_2 r_1^2\right)\right]} = \frac{\left(-m_2 r_2^2 \omega^2\right) + 2\left(K_3 r_2^2 + K_2 r_2^2\right)}{2 K_2 r_1 r_2}$$

$$\omega^4 - \left(\frac{2K_3}{m_2} + \frac{2K_2}{m_2} + \frac{2K_1}{m_1} + \frac{2K_2}{m_1}\right)\omega^2 + \frac{4}{m_1 m_2}(K_1K_3 + K_2K_3 + K_1K_2) = 0$$

2. **Two natural frequencies :**

Substituting the values of m_1, m_2, K_1, K_2, and K_3

$$\omega^4 - \left(\frac{2\times 60\times 10^3}{12} + \frac{2\times 50\times 10^3}{12} + \frac{2\times 40\times 10^3}{10} + \frac{2\times 50\times 10^3}{10}\right)\omega^2 + \frac{4}{10\times 12}\times(40\times 10^3\times 60\times 10^3 + 50\times 10^3\times 60\times 10^3 + 40\times 10^3\times 50\times 10^3) = 0$$

$$\omega^4 - (18333.3 + 18000)\omega^2 + 246\times 10^6 = 0$$

$$\omega^4 - 36333.3\,\omega^2 + 246\times 10^6 = 0$$

$$\omega^2 = \frac{36333.3 \pm \sqrt{(-36333.3)^2 - 4\times 246\times 10^6}}{2\times 1}$$

$$\omega^2 = \frac{36333.3 \pm 18333.26}{2}$$

$$\omega_{n1}^2 = 9000 \text{ and } \omega_{n2}^2 = 27333.3 \quad \text{...Ans.}$$

$$\omega_{n1} = \mathbf{94.86\ rad/s} \text{ and } \omega_{n2} = \mathbf{165.2\ rad/s} \quad \text{...Ans.}$$

Ex. 5.3.22 : Find the natural frequencies and mode shapes for the torsional system shown in Fig. P. 5.3.22(a). Assume $J_1 = J_0$, $J_2 = 2J_0$ and stiffness for each spring as K_t.

SPPU - Dec. 16, Dec. 17, May 18, 12 Marks

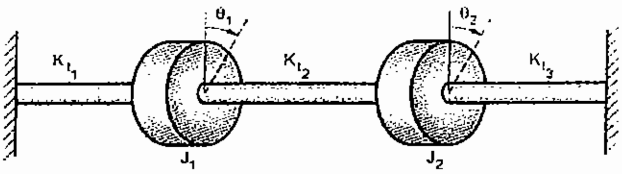


Fig. P. 5.3.22(a)

Soln. :

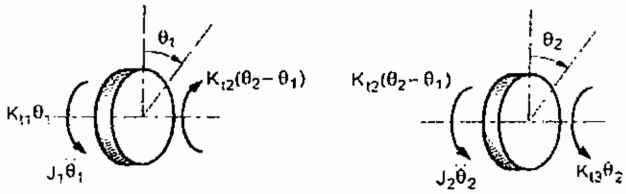


Fig. P. 5.3.22(b) : F.B.D

1. **Frequency Equation :**

- **Two differential equations of motion :**

From Fig. P. 5.3.22(b) ;

$$J_1\ddot{\theta}_1 + K_{t1}\theta_1 - K_{t2}(\theta_2 - \theta_1) = 0 \quad \text{...(a)}$$

and $$J_2\ddot{\theta}_2 + K_{t2}(\theta_2 - \theta_1) + K_{t3}\theta_2 = 0 \quad \text{...(b)}$$

- Substituting $J_1 = J_0$, $J_2 = 2J_0$ and $K_{t1} = K_{t2} = K_{t3} = K_t$ in Equations (a) and (b)

$$J_0\ddot{\theta}_1 + K_t\theta_1 - K_t(\theta_2 - \theta_1) = 0$$

And $$2J_0\ddot{\theta}_2 + K_t(\theta_2 - \theta_1) + K_t\theta_2 = 0$$

$$\therefore\ J_0\ddot{\theta}_1 + 2K_t\theta_1 - K_t\theta_2 = 0 \quad \text{...(c)}$$

And $$2J_0\ddot{\theta}_1 - K_t\theta_1 + 2K_t\theta_2 = 0 \quad \text{...(d)}$$

- **Solutions for θ_1 and θ_2 under steady state conditions**

$$\left.\begin{aligned}\theta_1 &= \phi_1 \sin \omega t\\ \theta_2 &= \phi_2 \sin \omega t\end{aligned}\right\} \quad \text{...(e)}$$

Therefore,

$$\left.\begin{aligned}\ddot{\theta}_1 &= -\phi_1\,\omega^2 \sin\omega t \\ \ddot{\theta}_2 &= -\phi_2\,\omega^2 \sin\omega t\end{aligned}\right\} \quad \text{...(f)}$$

- Substituting Equations (e) and (f) in Equations (c),

$$J_0(-\phi_1\,\omega^2 \sin\omega t) + 2K_t\,\phi_1 \sin\omega t - K_t\,\phi_2 \sin\omega t = 0$$

$$-J_0\omega^2\,\phi_1 + 2K_t\cdot\phi_1 - K_t\phi_2 = 0$$

$$(2K_t - J_0\,\omega^2)\,\phi_1 = K_t\,\phi_2$$

$$\frac{\phi_1}{\phi_2} = \frac{K_t}{(2K_t - J_0\omega^2)} \quad \text{...(g)}$$

- Substituting Equations (e) and (f) in Equation (d),

$$2J_0(-\phi_2\,\omega^2 \sin\omega t) - K_t\,\phi_1 \sin\omega t + 2K_t\,\phi_2 \sin\omega t = 0$$

$$(2K_t - 2J_0\,\omega^2)\,\phi_2 = K_t\,\phi_1$$

$$\frac{\phi_1}{\phi_2} = \frac{2K_t - 2J_0\omega^2}{K_t} \quad \text{...(h)}$$

- **Frequency equation :**

From Equations (g) and (h),

$$\frac{K_t}{(2K_t - J_0\omega^2)} = \frac{2K_t - 2J_0}{K_t}$$

$$(2K_t - J_0\,\omega^2)(2K_t - 2J_0\,\omega^2) = K_t^2$$

$$4K_t^2 - 4J_0\,K_t\,\omega^2 - 2\,J_0\,K_t\,\omega^2 + 2\,J_0^2\,\omega^4 = K_t^2$$

$$2J_0^2\,\omega^4 - 6\,J_0\,K_t\,\omega^2 + 3K_t^2 = 0$$

$$\omega^4 - 3\frac{K_t}{J_0}\omega^2 + 1.5\left(\frac{K_t}{J_0}\right)^2 = 0 \quad \text{...(i)}$$

2. **Two Natural Frequencies :**

$$\omega^2 = +\frac{\frac{3K_t}{J_0} \pm \sqrt{\left(\frac{-3K_t}{J_0}\right)^2 - 4\times 1.5\times\left(\frac{K_t}{J_0}\right)^2}}{2\times 1}$$

$$= \frac{\frac{3K_t}{J_0} \pm \sqrt{3\left(\frac{K_t}{J_0}\right)^2}}{2}$$

$$\omega^2 = \left[\frac{+3\pm\sqrt{3}}{2}\right]\frac{K_t}{J_0}$$

$$\therefore\ \omega_{n1}^2 = \left[\frac{3-\sqrt{3}}{2}\right]\frac{K_t}{J_0} \text{ and } \omega_{n2}^2 = \left[\frac{3+\sqrt{3}}{2}\right]\frac{K_t}{J_0}$$

$$\therefore\ \omega_{n1} = \sqrt{\left[\frac{3-\sqrt{3}}{2}\right]\frac{K_t}{J_0}} \text{ and } \omega_{n2} = \sqrt{\left[\frac{3+\sqrt{3}}{2}\right]\frac{K_t}{J_0}}$$

...Ans.

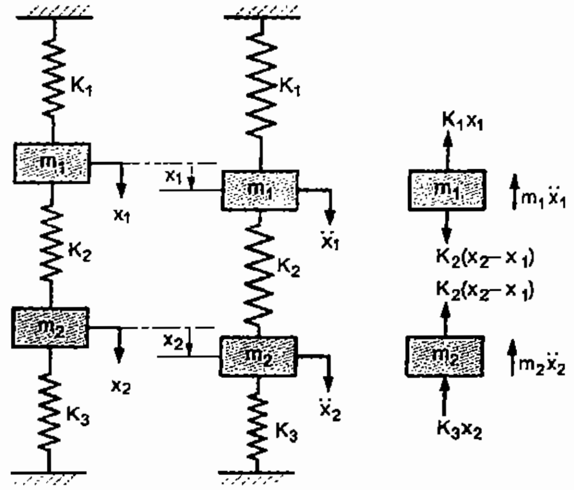
5.4 EIGEN VALUES AND EIGEN VECTORS

Two Degrees of Freedom System :

- Consider the spring mass system having two degrees of freedom, as shown in Fig. 5.4.1.

Let, x_1 = displacement of mass m_1 after time 't', m

x_2 = displacement of mass m_2 after time 't', m



(a) Equilibrium Position **(b) Displaced Position** **(c) F.B.D. of Masses**

Fig. 5.4.1 : Two Degrees of Freedom System

Differential Equations of motion :

- **Two differential equations of motion :**

$$\left.\begin{aligned}m_1\ddot{x}_1 + K_1x_1 - K_2(x_2 - x_1) &= 0 \\ m_2\ddot{x}_2 + K_2(x_2 - x_1) + K_3x_2 &= 0\end{aligned}\right\} \quad \text{...(5.4.1)}$$

Rearranging the terms of Equations (5.4.1) we get,

$$\left.\begin{aligned}m_1\ddot{x}_1 + (K_1 + K_2)\,x_1 - K_2\,x_2 &= 0 \\ m_2\ddot{x}_2 - K_2\,x_1 + (K_2 + K_3)\,x_2 &= 0\end{aligned}\right\} \quad \text{...(5.4.2)}$$

$$\left.\begin{aligned}\ddot{x}_1 + \frac{(K_1 + K_2)}{m_1}x_1 - \frac{K_2}{m_1}x_2 &= 0 \\ \ddot{x}_2 - \frac{K_2}{m_1}x_1 + \frac{(K_2 + K_3)}{m_2}x_2 &= 0\end{aligned}\right\} \quad \text{...(5.4.3)}$$

Above equations (5.4.3) may be written in matrix form as,

$$\begin{Bmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{Bmatrix} + \begin{bmatrix}\frac{(K_1 + K_2)}{m_1} & -\frac{K_2}{m_1} \\ -\frac{K_2}{m_2} & \frac{(K_2 + K_3)}{m_2}\end{bmatrix}\begin{Bmatrix}x_1 \\ x_2\end{Bmatrix} = 0 \quad \text{...(5.4.4)}$$

Solutions of Differential Equations :

- **Solutions for x_1 and x_2 :**

$$\{x\} = \{x\} \sin \omega t$$

$$\therefore \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t \qquad ...(5.4.5)$$

Where, ω = frequency of vibrations, rad/s

X_1 = amplitude of vibration of mass m_1

X_2 = amplitude of vibration of mass m_2

- Therefore,

$$\{\ddot{x}\} = -\omega^2 \{X\} \sin \omega t = -\lambda \{x\} \sin \omega t$$

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = -\omega^2 \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t = -\lambda \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \sin \omega t \qquad ...(5.4.6)$$

where, $\lambda = \omega^2$ = eigen value

$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ = eigen vector

- **Eigen value and eigen vector :**

  Substituting Equations (5.4.5) and (5.4.6) in Equation (5.4.4) and cancelling common term, we get,

$$-\lambda \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{(K_1+K_2)}{m_1} & -\dfrac{K_2}{m_1} \\ -\dfrac{K_2}{m_2} & \dfrac{(K_2+K_3)}{m_2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$-\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{(K_1+K_2)}{m_1} & -\dfrac{K_2}{m_1} \\ -\dfrac{K_2}{m_2} & \dfrac{(K_2+K_3)}{m_2} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0 \qquad ...(5.4.7)$$

$$-\lambda [I] \{X\} + [D] \{X\} = 0$$

$$\mathbf{[[D] - \lambda [I] \{X\}] = 0} \qquad \mathbf{...(5.4.8)}$$

Where, $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$[D] = \begin{bmatrix} \dfrac{K_1+K_2}{m_1} & -\dfrac{K_2}{m_1} \\ -\dfrac{K_2}{m_2} & \dfrac{K_2+K_3}{m_2} \end{bmatrix}$$

$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$ = **Eigen vector**

$\lambda = \omega^2$ = **Eigen value**

- **Solution of Equation (5.4.8) :**

  The non-zero solution for eigen vector $\{X\}$ will occur when, $[[D] - \lambda [I]]$ is a singular matrix.

i.e. $|[D] - \lambda [I]| = 0$

$$\begin{vmatrix} \dfrac{(K_1+K_2)}{m_1} - \lambda & -\dfrac{K_2}{m_1} \\ -\dfrac{K_2}{m_2} & \dfrac{(K_2+K_3)}{m_2} - \lambda \end{vmatrix} = 0 \qquad ...(5.4.9)$$

$$\therefore \left[\frac{(K_1+K_2)}{m_1} - \lambda\right]\left[\frac{(K_2+K_3)}{m_2} - \lambda\right] - \frac{K_2^2}{m_1 m_2} = 0 \qquad ...(5.4.10)$$

Solving above Equation (5.4.10), the eigen value λ i.e. ω^2 can be obtained.

Special case of Two degrees of Freedom system :

Case : $m_1 = m_2 = m$; $K_1 = K_2 = K$ and $K_3 = 0$

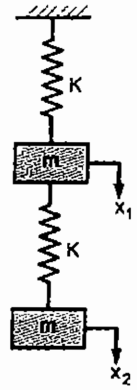


Fig. 5.4.2 : Special Case

Refer Fig. 5.4.2 ;

1. **Two Natural Frequencies :**

- Substituting $m_1 = m_2 = m$; $K_1 = K_2 = K$ and $K_3 = 0$ in Equation (5.4.7),

$$\begin{vmatrix} \dfrac{(K+K)}{m} - \lambda & -\dfrac{K}{m} \\ -\dfrac{K}{m} & \dfrac{K}{m} - \lambda \end{vmatrix} = 0$$

$$\therefore \left[\frac{2K}{m} - \lambda\right]\left[\frac{K}{m} - \lambda\right] - \frac{K^2}{m^2} = 0$$

$$\frac{2K^2}{m^2} - \frac{2K}{m}\lambda - \frac{K}{m}\lambda + \lambda^2 - \frac{K^2}{m^2} = 0$$

$$\lambda^2 - \frac{3K}{m}\lambda + \frac{K^2}{m^2} = 0$$

$$\therefore \lambda = \frac{\dfrac{3K}{m} \pm \sqrt{\left(-\dfrac{3K}{m}\right)^2 - \dfrac{4K^2}{m^2}}}{2}$$

$$= \frac{\dfrac{3K}{m} \pm \sqrt{\dfrac{9K^2}{m^2} - \dfrac{4K^2}{m^2}}}{2}$$

$$= \frac{\frac{3K}{m} \pm \sqrt{\frac{5K^2}{m^2}}}{2}$$

$$= \frac{3K}{2m} \pm \frac{\sqrt{5}\,K}{2m}$$

$$\therefore \quad \lambda = 1.5\frac{K}{m} \pm 1.12\frac{K}{m}$$

$$\lambda_1 = 0.38\frac{K}{m}$$

and $$\lambda_2 = 2.62\frac{K}{m} \qquad \text{...(5.4.11)}$$

$$\omega_{n1}^2 = 0.38\frac{K}{m}$$

and $$\omega_{n2}^2 = 2.62\frac{K}{m}$$

$$\omega_{n1} = \mathbf{0.616\sqrt{\frac{K}{m}}}$$

and $$\omega_{n2} = \mathbf{1.62\sqrt{\frac{K}{m}}} \qquad \text{...Ans.}$$

2. **Mode Shapes :**

- **First mode shape :**

Substituting $\lambda_1 = 0.38\frac{K}{m}$ in Equation (5.4.7),

$$-0.38\frac{K}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} \frac{(K+K)}{m} & -\frac{K}{m} \\ -\frac{K}{m} & \frac{K}{m} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} \left(\frac{2K}{m} - 0.38\frac{K}{m}\right) & -\frac{K}{m} \\ -\frac{K}{m} & \left(\frac{K}{m} - 0.38\frac{K}{m}\right) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\frac{K}{m}\begin{bmatrix} 1.62 & -1 \\ -1 & 0.62 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 1.62 & -1 \\ -1 & 0.62 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\therefore 1.62\,X_1 - X_2 = 0 \qquad \text{...(5.4.12)}$$

$$-X_1 + 0.62\,X_2 = 0 \qquad \text{...(5.4.13)}$$

$$\therefore \quad \mathbf{\frac{X_1}{X_2} = 0.62} \qquad \text{...Ans.}$$

- **Second mode shape :**

Substituting $\lambda_2 = 2.62\frac{K}{m}$ in Equation (5.4.7),

$$-2.62\frac{K}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} \frac{(K+K)}{m} & -\frac{K}{m} \\ -\frac{K}{m} & \frac{K}{m} \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} \left(2\frac{K}{m} - 2.62\frac{K}{m}\right) & -\frac{K}{m} \\ -\frac{K}{m} & \left(\frac{K}{m} - 2.62\frac{K}{m}\right) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\frac{K}{m}\begin{bmatrix} -0.62 & -1 \\ -1 & -1.62 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\begin{bmatrix} 0.62 & 1 \\ 1 & 1.62 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$0.62\,X_1 + X_2 = 0 \qquad \text{...(5.4.14)}$$

$$X_1 + 1.62\,X_2 = 0 \qquad \text{...(5.4.15)}$$

$$\therefore \quad \mathbf{\frac{X_1}{X_2} = -1.62} \qquad \text{...Ans.}$$

5.5 COMBINED RECTILINEAR AND ANGULAR MODES

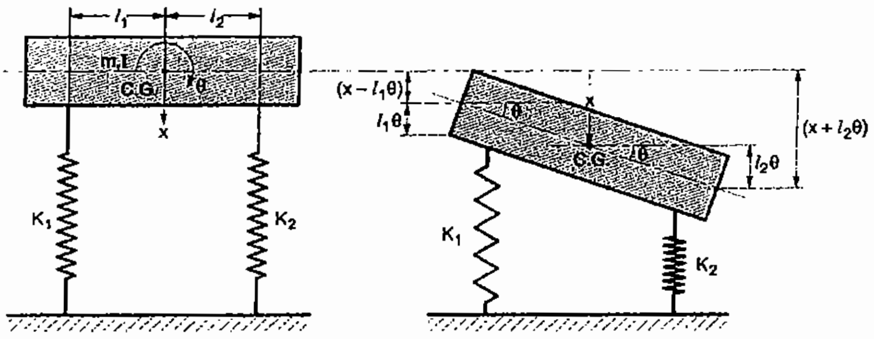
Reason for Combined Motion :

- In previous sections we discussed the systems in which the body (mass) is subjected to either rectilinear (translation) motion or angular motion.
- But in actual practice, sometimes, the body is subjected to combined rectilinear and angular motions. For example, in case of vehicle, when brakes are applied on moving vehicle, two motions of vehicle occur simultaneously:

  (i) rectilinear motion (x), and

  (ii) angular motion (θ).
- This type of combined motion in the system is due to the fact that C.G. of vehicle and centre of rotation do not coincide.

Let,
- m = mass of body, kg
- I = M.I. of body about C G, kg-m^2
- K_1, K_2 = stiffnesses of springs, N/m
- x = linear displacement of body at any instant
- θ = angular displacement of body at any instant.



(a) Equilibrium Position **(b) Displaced Position**

Fig 5.5.1 : Combined Rectilinear and Angular Modes

Differential Equations of Motion :

Compression of spring, $K_1 = x - l_1\theta$

Compression of spring, $K_2 = x + l_2\theta$

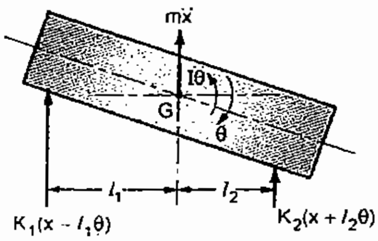


Fig. 5.5.2 : F.B.D of System

- From Fig. 5.5.2., the differential equations of motion for the system are,

$$m\ddot{x} + K_1(x - l_1\theta) + K_2(x + l_2\theta) = 0$$

and $$I\ddot{\theta} - K_1(x - l_1\theta)l_1 + K_2(x + l_2\theta)l_2 = 0$$

or $$m\ddot{x} + (K_1 + K_2)x - (K_1 l_1 - K_2 l_2)\theta = 0 \quad ...(5.5.1)$$

and $$I\ddot{\theta} - (K_1 l_1 - K_2 l_2)x + (K_1 l_1^2 + K_2 l_2^2)\theta = 0 \quad ...(5.5.2)$$

- **Coupled differential equations :** The Equations (5.5.1) and (5.4.2) have both the terms x and θ. Such equations are called the **coupled differential equations.**

Special Cases of System :

Case I - When $K_1 l_1 = K_2 l_2$

Case II - When $K_1 l_1 \neq K_2 l_2$

Case I - When $K_1 l_1 = K_2 l_2$

- When $K_1 l_1 = K_2 l_2$, the Equations (5.5.1) and (5.5.2) become,

$$m\ddot{x} + (K_1 + K_2)x = 0 \quad ...(5.5.3)$$

and $$I\ddot{\theta} + (K_1 l_1^2 + K_2 l_2^2)\theta = 0 \quad ...(5.5.4)$$

- **Uncoupled differential equations :** From Equations (5.5.3) and (5.5.4) it is seen that, rectilinear and angular motions can exist independently. Therefore, such equations are called as **uncoupled differential equations.**
- **Two Natural frequencies of system :**

From Equations (5.5.3) and (5.5.4),

$$\omega_{n1} = \sqrt{\frac{(K_1 + K_2)}{m}}, \text{ rad/s} \quad ...(5.5.5)$$

and $$\omega_{n2} = \sqrt{\frac{K_1 l_1^2 + K_2 l_2^2}{I}} \text{ rad/s} \quad ...(5.5.6)$$

Case II - When $K_1 l_1 \neq K_2 l_2$

- **Solutions for x and θ under steady state condition :**

$$\left.\begin{aligned} x &= X \sin \omega t \\ \theta &= \phi \sin \omega t \end{aligned}\right\} \quad ...(a)$$

where, X and ϕ are amplitudes of rectilinear and angular motions respectively.

- Therefore,

$$\left.\begin{aligned} \ddot{x} &= -X\omega^2 \sin \omega t \\ \ddot{\theta} &= -\phi\omega^2 \sin \omega t \end{aligned}\right\} \quad ...(b)$$

- Substituting Equations (a) and (b) in Equations (5.5.1), we get

$$-mX\omega^2 \sin \omega t + (K_1 + K_2)X \sin \omega t - (K_1 l_1 - K_2 l_2)\phi \sin \omega t = 0$$

$$\therefore \quad -mX\omega^2 + (K_1 + K_2)X - (K_1 l_1 - K_2 l_2)\phi = 0$$

$$\therefore \quad (K_1 + K_2 - m\omega^2)X = (K_1 l_1 - K_2 l_2)\phi$$

$$\therefore \quad \frac{X}{\phi} = \frac{(K_1 l_1 - K_2 l_2)}{(K_1 + K_2 - m\omega^2)} \quad ...(5.5.7)$$

- Substituting Equations (a) and (b) in Equation (5.5.2), we get,

$$-I\phi\omega^2 \sin \omega t - (K_1 l_1 - K_2 l_2)X \sin \omega t + (K_1 l_1^2 + K_2 l_2^2)\phi \sin \omega t = 0$$

$$-I\omega^2\phi - (K_1 l_1 - K_2 l_2)X + (K_1 l_1^2 + K_2 l_2^2)\phi = 0$$

$$\therefore\ (K_1 l_1^2 + K_2 l_2^2 - I\omega^2)\,\phi = (K_1 l_1 - K_2 l_2)\,X$$

$$\therefore\ \frac{X}{\phi} = \frac{\left(K_1 l_1^2 + K_2 l_2^2 - I\omega^2\right)}{(K_1 l_1 - K_2 l_2)} \quad ...(5.5.8)$$

- **Frequency equation :**

From Equations (5.5.7) and (5.5.8),

$$\frac{(K_1 l_1 - K_2 l_2)}{(K_1 + K_2 - m\omega^2)} = \frac{\left(K_1 l_1^2 + K_2 l_2^2 - I\omega^2\right)}{(K_1 l_1 - K_2 l_2)}$$

$$\therefore\ (K_1 + K_2 - m\omega^2)(K_1 l_1^2 + K_2 l_2^2 - I\omega^2) = (K_1 l_1 - K_2 l_2)^2$$

$$K_1^2 l_1^2 + K_1 K_2 l_2^2 - K_1 I\omega^2 + K_1 K_2 l_1^2 + K_2^2 l_2^2 - K_2 I\omega^2 - K_1 l_1^2$$
$$m\omega^2 - K_2 l_2^2 m\omega^2 + I m\omega^4 = K_1 l_1^2 - 2 K_1 K_2 l_1 l_2 + K_2^2 l_2^2$$

$$I m\omega^4 - (K_1 I + K_2 I + K_1 l_1^2 m + K_2 l_2^2 m)\,\omega^2$$
$$+ K_1 K_2 l_1^2 + K_1 K_2 l_2^2 + 2 K_1 K_2 l_1 l_2 = 0$$

$$I m\omega^4 - (K_1 I + K_2 I + K_1 l_1^2 m + K_2 l_2^2 m)\,\omega^2 + K_1 K_2 (l_1 + l_2)^2 = 0$$

$$\omega^4 - \left(\frac{K_1}{m} + \frac{K_2}{m} + \frac{K_1 l_1^2}{I} + \frac{K_2 l_2^2}{I}\right)\omega^2 + \frac{K_1 K_2 (l_1 + l_2)^2}{I m} = 0$$

$$\omega^4 - \left(\frac{K_1 + K_2}{m} + \frac{K_1 l_1^2 + K_2 l_2^2}{I}\right)\omega^2 + \frac{K_1 K_2 (l_1 + l_2)^2}{I m} = 0$$

$$\omega^4 - B\omega^2 + C = 0 \quad ...(5.5.9)$$

where,
$$B = \left(\frac{K_1 + K_2}{m} + \frac{K_1 l_1^2 + K_2 l_2^2}{I}\right);$$

$$C = \frac{K_1 K_2 (l_1 + l_2)^2}{I m}$$

This Equation (5.5.9) is known as **frequency Equation**

From Equation (5.5.9),

$$\therefore\ \omega^2 = \frac{+B \pm \sqrt{B^2 - 4C}}{2} = \frac{1}{2}\left[B \pm \sqrt{B^2 - 4C}\right]$$

$$\therefore\ \omega_{n1}^2 = \frac{1}{2}\left[B - \sqrt{B^2 - 4C}\right] \quad ...(5.5.10)$$

and
$$\omega_{n2}^2 = \frac{1}{2}\left[B + \sqrt{B^2 - 4C}\right] \quad ...(5.5.11)$$

Ex. 5.5.1 : An automobile weighing 250 kg, has wheel base of 4 m and its C.G. is at 2 m from front wheel axle as shown in Fig. P. 5.5.1(a). The radius of gyration about C.G. is 1.5 m and the equivalent spring stiffness for front and rear wheel is 2000 N / m each. Determine the natural frequencies of the automobile.

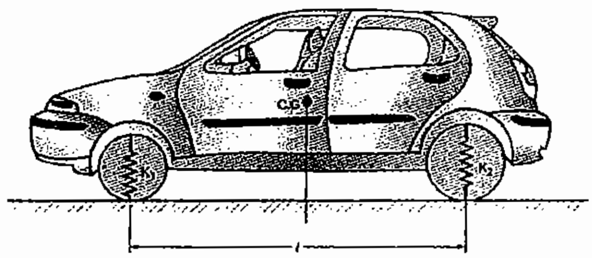
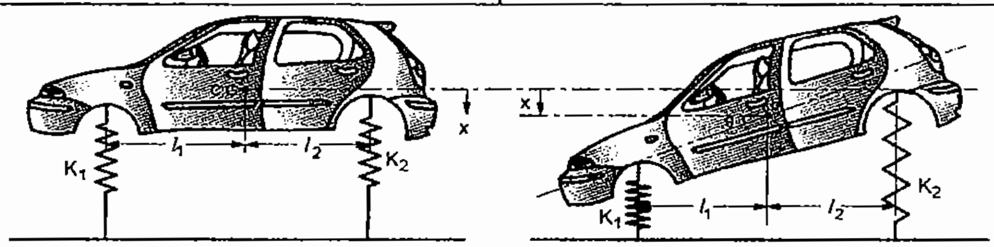


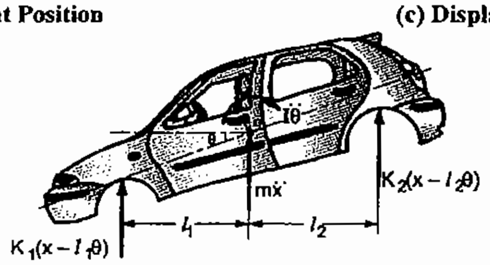
Fig. P. 5.5.1(a)

Soln. :

Given : $m = 250$ kg ; $l = l_1 + l_2 = 4$ m ;
$l_1 = 2$ m ; $l_2 = 2$ m ;
$k = 1.5$ m ; $K_1 = K_2 = 2000$ N/m.



(b) Equivalent Position **(c) Displaced Position**



(d) F.B.D. of System

Fig. P. 5.5.1

1. Differential Equations of Motion :

From Fig. P. 5.5.1(d),

$$\left.\begin{aligned} m\ddot{x} + K_1(x + l_1\theta) + K_2(x - l_2\theta) &= 0 \\ \text{and} \quad I\ddot{\theta} + K_1(x + l_1\theta)\,l_1 - K_2(x - l_2\theta)\,l_2 &= 0 \end{aligned}\right\} \quad \text{...(a)}$$

or

$$\left.\begin{aligned} m\ddot{x} + K_1(x + l_1\theta) + K_2(x - l_2\theta) &= 0 \\ \text{and} \quad I\ddot{\theta} + K_1(x + l_1\theta)\,l_1 - K_2(x - l_2\theta)\,l_2 &= 0 \end{aligned}\right\} \quad \text{...(b)}$$

- Since $K_1 = K_2$ and $l_1 = l_2$, the Equations (b) become,

$$m\ddot{x} + (K_1 + K_2)\,x = 0 \quad \text{...(c)}$$

and

$$I\ddot{\theta} + (K_1 l_1^2 + K_2 l_2^2)\,\theta = 0 \quad \text{...(d)}$$

2. Natural Frequencies :

$$\omega_{n1} = \sqrt{\frac{(K_1 + K_2)}{m}} = \sqrt{\frac{(2000 + 2000)}{250}}$$

or $$\omega_{n1} = \mathbf{4\ rad/s} \quad \text{...Ans.}$$

and $$\omega_{n2} = \sqrt{\frac{K_1 l_1^2 + K_2 l_2^2}{I}} = \sqrt{\frac{2000 \times (2)^2 + 2000\,(2)^2}{250 \times (1.5)^2}} \quad \text{...}[\because I = mk^2]$$

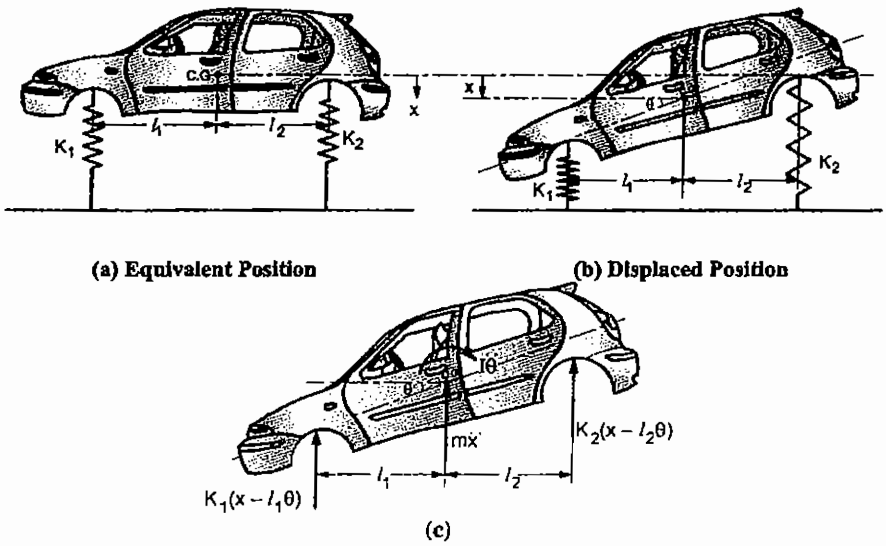
or $$\omega_{n2} = \mathbf{5.33\ rad/s} \quad \text{...Ans.}$$

Ex. 5.5.2 : An automobile weighs 2000 kg and has radius of gyration about its C.G as 1.1 meter. The C.G is located 1.4 meter behind the front wheel axis and the distance between rear wheel axis and CG is 1.6 meter. The front springs have a combined stiffness of 600 N/m and rear springs 650 N/m. Formulate the frequency equations and find out the pitch and bounce frequencies of an automobile. **SPPU - Dec. 11, 12 Marks**

Soln. : Given : Mass of automobile, $m = 2000$ kg ;

$l_1 = 1.4$ m	;	$\therefore\ l_2 = 1.6$ m	;
$l = l_1 + l_2 = 1.4 + 1.6 = 3$ m	;	$k = 1.1$ m	;
$K_1 = 600$ N/m	;	$K_2 = 650$ N/m	

$\therefore$ M.I. of automobile, $I = mk^2 = 2000 \times (1.1)^2 = 2420$ kg-m^2



(a) Equivalent Position (b) Displaced Position

(c)

Fig. P. 5.5.2 : F.B.D of System

1. Differential Equations : From Fig. P. 5.5.2(c),

$$\left.\begin{aligned} m\ddot{x} + K_1(x + l_1\theta) + K_2(x - l_2\theta) &= 0 \\ \text{and} \quad I\ddot{\theta} + K_1(x + l_1\theta)l_1 - K_2(x - l_2\theta)l_2 &= 0 \end{aligned}\right\} \quad ...(a)$$

$$\text{or} \quad \left.\begin{aligned} m\ddot{x} + (K_1 + K_2)x + (K_1 l_1 - K_2 l_2)\theta &= 0 \\ \text{and} \quad I\ddot{\theta} + (K_1 l_1 - K_2 l_2)x + (K_1 l_1^2 + K_2 l_2^2)\theta &= 0 \end{aligned}\right\} \quad ...(b)$$

- Substituting values of I, K, K_2, l_1, l_2 in Equations (b), we get,

$$2000 \times \ddot{x} + (600 + 650)x + (600 \times 1.4 - 650 \times 1.6)\theta = 0$$

$$\text{and} \quad 2420\ddot{\theta} + (600 \times 1.4 - 650 \times 1.6)x + (600 \times 1.4^2 + 650 \times 1.6^2)\theta = 0$$

$$\text{or} \quad 2000\ddot{x} + 1250x - 200\theta = 0$$

$$\text{and} \quad 2420\ddot{\theta} - 200x + 2840\theta = 0$$

$$\text{or} \quad \mathbf{\ddot{x} + 0.625x - 0.1 = 0} \quad ...(c)$$

$$\text{and} \quad \mathbf{\ddot{\theta} - 0.082x + 1.17\theta = 0} \quad ...(d)$$

2. Frequency Equation :

- **Solutions for x and θ under steady state conditions :**

$$\left.\begin{aligned} x &= X\sin\omega t \\ \text{and} \quad \theta &= \phi\sin\omega t \end{aligned}\right\} \quad ...(e)$$

$$\left.\begin{aligned} \text{Therefore,} \quad \ddot{x} &= -X\omega^2\sin\omega t \\ \text{and} \quad \ddot{\theta} &= -\phi\omega^2\sin\omega t \end{aligned}\right\} \quad ...(f)$$

- Substituting Equations (e) and (f) in Equation (c),

$$-X\omega^2\sin\omega t + 0.625X\sin\omega t - 0.1\phi\sin\omega t = 0$$

$$-X\omega^2 + 0.625X - 0.1\phi = 0$$

$$\therefore \quad (0.625 - \omega^2)X - 0.1\phi = 0$$

$$(0.625 - \omega^2)X = 0.1\phi$$

$$\therefore \quad \frac{\mathbf{X}}{\boldsymbol{\phi}} = \frac{\mathbf{0.1}}{\mathbf{(0.625 - \omega^2)}} \quad ...(g)$$

- Substituting Equations (e) and (f) in Equation (d),

$$-\phi\omega^2\sin\omega t - 0.082X\sin\omega t + 1.17\phi\sin\omega t = 0$$

$$-\phi\omega^2 - 0.082X + 1.17\phi = 0$$

$$\therefore \quad (1.17 - \omega^2)\phi - 0.082X = 0$$

$$\therefore \quad (1.17 - \omega^2)\phi = 0.082X$$

$$\therefore \quad \frac{\mathbf{X}}{\boldsymbol{\phi}} = \frac{\mathbf{(1.17 - \omega^2)}}{\mathbf{0.082}} \quad ...(h)$$

- **Frequency equation :** From Equations (g) and (h),

$$\frac{0.1}{(0.625 - \omega^2)} = \frac{(1.17 - \omega^2)}{0.082}$$

$$\therefore \quad (1.17 - \omega^2)(0.625 - \omega^2) = 0.1 \times 0.082$$

$$\therefore \quad 0.7312 - 1.17\omega^2 - 0.625\omega^2 + \omega^4 = 8.2 \times 10^{-3}$$

$$\therefore \quad \mathbf{\omega^4 - 1.795\omega^2 + 0.723 = 0} \quad ...(i)$$

- Equation (i) is known as **frequency Equation**

5. Natural Frequencies :

From Equation (i),

$$\omega^2 = \frac{1.795 \pm \sqrt{(1.795)^2 - 4 \times 1 \times 0.723}}{2}$$

$$\text{or} \quad \omega^2 = \frac{1.795 \pm 0.574}{2}$$

$$\therefore \quad \omega_{n1}^2 = \frac{1.795 + 0.574}{2} = 1.18$$

$$\text{and} \quad \omega_{n2}^2 = \frac{1.795 - 0.574}{2} = 0.61$$

$$\therefore \quad \omega_{n1} = \sqrt{1.18} = \mathbf{1.0862\ rad/s} \quad ...\text{Ans.}$$

$$\text{and} \quad \omega_{n2} = \sqrt{0.61} = \mathbf{0.7810\ rad/s} \quad ...\text{Ans.}$$

4. Ratio of Amplitudes :

- **For first mode shape :**

$$\left(\frac{X}{\phi}\right)_1 = \frac{0.1}{0.625 - \omega_{n1}^2} = \frac{0.1}{0.625 - (1.0862)^2} = \mathbf{-0.18} \quad ...\text{Ans.}$$

- **For second mode shape :**

$$\left(\frac{X}{\phi}\right)_2 = \frac{1.17 - \omega_{n1}^2}{0.082} = \frac{1.17 - (0.7810)^2}{0.082}$$

$$\text{or} \quad \left(\frac{X}{\phi}\right)_2 = \mathbf{6.82} \quad ...\text{Ans.}$$

Example for Practice

Refer our website for complete solution of following example

Ex. 5.5.3 : An automobile of mass 2000 kg has a wheel base of 2.5 m. Its C.G. is located 1.5 m behind the front wheel axle and has a radius of gyration about C.G. 1.2 m. the front springs have a combined stiffness of 4000 N/m and rear springs 4500 N/m. Refer Fig. P. 5.5.3(a).

Determine :

(i) The natural frequency

(ii) Amplitude ratio for two modes of vibration.

SPPU - May 18, 12 Marks

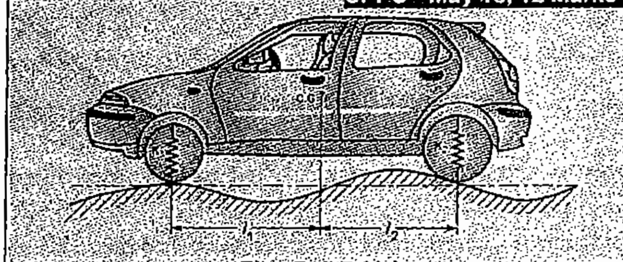


Fig. P. 5.5.3(a)

5.6 FREE TORSIONAL VIBRATIONS OF TWO ROTOR SYSTEM

University Question

Q. Define the following terms :

(i) Zero frequency (ii) Node point

SPPU : Dec. 16, Dec. 17, Dec. 18, Dec. 19

Two Rotor System :

- Two rotor system, as shown in Fig. 5.6.1. It consist of a shaft with two rotors A and B at its end.
- **Torsional vibrations :** The torsional vibration occurs only when two rotors A and B rotate in opposite direction. The amplitudes of vibration at both ends will be in opposite directions. In such case, some length of shaft is twisted in one direction, while the rest is twisted in the other direction.
- **Node point :** There is a point or a section of the shaft which remains untwisted. *This point or section where amplitude of vibration is zero is known as* **node point or nodal section.**

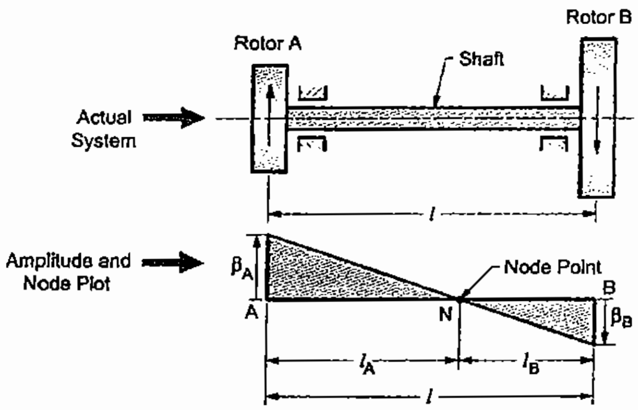


Fig. 5.6.1 : Two Rotor System

- **Equivalence of two rotor system :** Two rotor system with shafts AB carrying rotors A and B at ends can be considered as equivalent to two single rotor systems with :
  (i) shaft NA carrying rotor A ; and
  (ii) shaft NB carrying rotor B

Let,

I_A and I_B = mass moment of inertia of rotors A and B, kg-m^2

l_A and l_B = lengths of two portions of shaft from node point, m

K_{tA} and K_{tB} = torsional stiffness of shaft for length l_A and l_B respectively, N-m/rad

J = Polar moment of inertia of shaft, m^4

= $\pi d^4 / 32$

d = diameter of shaft, m

G = modulus of rigidity for shaft material, N/ m^2

β_A = amplitude of vibration of rotor A , rad

β_A = amplitude of vibration of rotor B, rad

Important Parameters for Two Rotor System :

1. Circular Natural Frequency (ω_n)
2. Position of Node (l_A)
3. Ratio of Amplitudes (β_A / β_B)

1. Circular Natural Frequency (ω_n) :

- Circular natural frequency for single rotor system :

$$\omega_n = \sqrt{\frac{K_t}{I}} = \sqrt{\frac{GJ}{l\,I}} \text{ rad/s}$$

- Circular natural frequency for shaft NA and rotor A :

$$\omega_{nA} = \sqrt{\frac{GJ}{l_A I_A}} \text{ rad/s} \qquad ...(5.6.1)$$

- Circular natural frequency for shaft NB and rotor B :

$$\omega_{nB} = \sqrt{\frac{GJ}{l_B I_B}} \text{ rad/s} \qquad ...(5.6.2)$$

2. Position of Node (l_A) :

- The circular natural frequency of shaft is same. Therefore,

$$\omega_{nA} = \omega_{nB} \qquad ...(5.6.3)$$

$$\therefore \quad \sqrt{\frac{GJ}{l_A I_A}} = \sqrt{\frac{GJ}{l_B I_B}}$$

$$\therefore \quad \frac{GJ}{l_A I_A} = \frac{GJ}{l_B I_B}$$

$$\therefore \quad \frac{1}{l_A I_A} = \frac{1}{l_B I_B}$$

$$\therefore \quad l_A I_A = l_B I_B \qquad ...(5.6.4)$$

$$\therefore \quad \frac{l_A}{l_B} = \frac{I_B}{I_A}$$

$$\therefore \quad l_A = \frac{l_B I_B}{I_A} \qquad ...(5.6.5)$$

- **Total length of shaft :**

$$l = l_A + l_B \qquad ...(5.6.6)$$

- From Equation (5.6.5) and (5.6.6), l_A and l_B can be determined.

- Once l_A and l_B are known, from Equation (5.6.1) or (5.6.2) the natural circular frequency of the total system i.e. two rotor system can be determined.

3. Ratio of Amplitudes (β_A / β_B) :

From Fig. 5.6.1,

$$\frac{\beta_A}{l_A} = \frac{-\beta_B}{l_B}$$

$$\therefore \quad \frac{\beta_A}{\beta_B} = \frac{-l_A}{l_B}$$

$$\text{or} \quad \frac{\beta_A}{\beta_B} = -\frac{I_B}{I_A} \qquad ...(5.6.7)$$

5.6.1 Zero Frequency Vibration of Two Rotor System :

- If two rotors A and B rotate in same direction, then the shaft is said to vibrate with **zero frequency**. Such behaviour is called as **zero node behaviour**, as the amplitudes of vibration at both ends will be in the same direction as shown in Fig. 5.6.2.

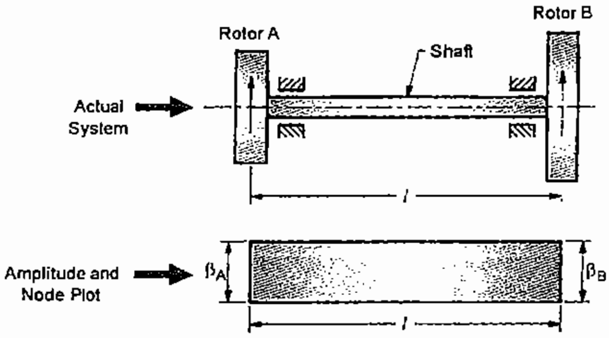


Fig. 5.6.2 : Two Rotor System (When Two Rotors Rotates in Same Direction)

5.7 FREE TORSIONAL VIBRATIONS OF THREE ROTOR SYSTEM

Three Rotor System :

- Three rotor system as shown in Fig. 5.7.1, consist of a shaft with three rotors A, B and C. The rotor A and C are attached at the end of the shaft, whereas rotor B is attached in between A and C. Out of three rotors, two rotors rotate in one direction and third rotor rotates in opposite direction with the same frequency.

Two Different Ways of Torsional Vibration :

- The torsional vibration occurs in following two ways :

1. Two Nodes Vibration
2. Single Node Vibration

1. Two Nodes Vibration :

If two rotors A and C rotate is same direction and the rotor B rotate in opposite direction, then torsional vibrations occur with **two nodes.**

2. Single Node vibration :

- If two rotors A and B rotate in same direction and the rotor C rotates in opposite direction or rotors B and C rotate in same direction and the rotor A rotates in opposite direction, then torsional vibrations occur with **single node.**
- The amplitudes of vibration at both ends will be in opposite directions. In such case some length of shaft is twisted in one direction, while the rest is twisted in the other direction.

Node point :

- There is a point or a section of the shaft which remains untwisted. *This point or section where amplitude of vibration is zero is known as* **node point or nodal section.**

5.7.1 Two Nodes Vibration of Three Rotor System :

- Consider two rotors A and C rotate in one direction and rotor B rotates in opposite direction, as shown in Fig. 5.7.1.

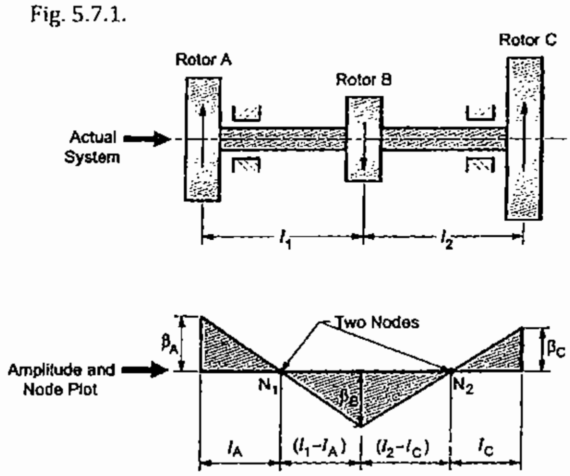


Fig. 5.7.1 : Two Nodes Vibration of Three Rotor System

Let,

I_A, I_B and I_C = mass moment of inertia of the rotors A, B and C respectively, kg-m^2

l_A = distance of node N_1 from rotor A, m

l_C = distance of node N_2 from rotor C, m

l_1 = distance between rotors A and B, m

l_2 = distance between rotors B and C, m

- **Stiffness of shaft :**

  (i) Stiffness of shaft of length $(l_1 - l_A)$, $K_{t1} = \frac{GJ}{l_1 - l_A}$

  (ii) Stiffness of shaft of length $(l_2 - l_C)$, $K_{t2} = \frac{GJ}{l_2 - l_C}$

  (iii) Total stiffness of shaft of length $(l_1 - l_A) + (l_2 - l_C)$,

$$K_{tB} = K_{t1} + K_{t2} = \frac{GJ}{l_1 - l_A} + \frac{GJ}{l_2 - l_C}$$

$$\therefore \quad K_{tB} = GJ\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right) \quad \text{...(a)}$$

- **Natural circular frequency for rotor A and shaft length 'l_A' :**

$$\omega_{nA} = \sqrt{\frac{GJ}{l_A I_A}}\ \text{, rad/s} \quad \text{...(b)}$$

- **Natural circular frequency for rotor C and shaft length 'l_C' :**

$$\text{or} \quad \omega_{nC} = \sqrt{\frac{GJ}{l_C I_C}}\text{, rad/s} \quad \text{...(c)}$$

- **Natural circular frequency for rotor B :**

$$\omega_{nB} = \sqrt{\frac{K_{tB}}{I_B}} \quad \text{...(d)}$$

Substituting Equation (c) in Equation (d), we get,

$$\omega_{nB} = \sqrt{\frac{GJ}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right)}\text{, rad/s} \quad \text{...(5.7.1)}$$

- **Determination of 'l_A' and 'l_C' :**

$$\omega_{nA} = \omega_{nB} = \omega_{nC}$$

$$\sqrt{\frac{GJ}{l_A I_A}} = \sqrt{\frac{GJ}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right)} = \sqrt{\frac{GJ}{l_C I_C}} \quad \text{...(e)}$$

From Equation (e)

$$\frac{1}{l_A I_A} = \frac{1}{l_C I_C}$$

$$\text{or} \quad l_A = \frac{l_C I_C}{I_A} \quad \text{...(5.7.2)}$$

Again from Equation (e)

$$\frac{1}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right) = \frac{1}{l_C I_C} \quad \text{...(5.7.3)}$$

- Substituting Equation (5.7.2) in Equation (5.7.3) a quadratic equation in l_C is obtained. After solving this quadratic equation, two values of l_C (l_{C1} and l_{C2}) and correspondingly two values of l_A (l_{A1} and l_{A2} are obtained).
- **Positions of nodes :** One set of l_A and l_C gives the positions of two nodes while other set of l_A and l_C gives the position of single node.
- **Condition for two nodes vibration :** If $l_A < l_1$ and $l_C < l_2$, then l_A and l_C gives the positions of two nodes.
- **Conditions for single node vibration :**

  If $l_A > l_1$ and $l_C < l_2$, then l_C gives the position of single node, as shown in Fig. 5.7.2.

  If $l_A < l_1$ and $l_C > l_2$ then l_A gives the position of single node, as shown in Fig. 5.7.3.

5.7.2 Single Node Vibration of Three Rotor System :

In a single node vibration of three rotor system, these are two possible conditions :

1. Rotors A and B Rotate in same direction and Rotor C Rotates in Opposite Direction
2. Rotors B and C Rotate in same direction and Rotor A Rotates Opposite Direction

1. Rotors A and B Rotate in Same Direction and Rotor C Rotates in Opposite Direction :

- When rotors A and B rotate in same direction and rotor C rotates in opposite direction, then the torsional vibrations occurs with single node as shown in Fig. 5.7.2.
- In this case $l_A > l_1$ and $l_C < l_2$. Therefore, l_c gives the actual position of node as shown in Fig. 5.7.2.

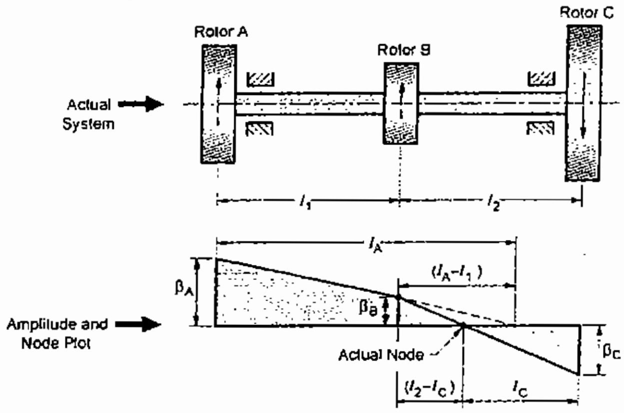


Fig. 5.7.2 : Single Node Vibration of Three Rotor System ($l_A > l_1$ and $l_C < l_2$)

$$\frac{\beta_B}{(l_A - l_1)} = \frac{\beta_A}{l_A}$$

$$\therefore \quad \beta_B = \beta_A\left[\frac{l_A - l_1}{l_A}\right] \quad \text{...(5.7.4)}$$

- Again, $\frac{\beta_B}{(l_2 - l_C)} = \frac{\beta_C}{l_C}$ ...(5.7.5)

2. Rotors B and C Rotate in Same Direction and Rotor A Rotates in Opposite Direction :

- When rotors B and C rotate in same direction and rotor A rotates in opposite direction, then the torsional vibrations occur with single node, as shown in Fig. 5.7.3.

- In this case, $l_A < l_1$ and $l_C > l_2$. Therefore, l_A gives, actual position of node, as shown in Fig. 5.7.3.

$$\frac{\beta_B}{(l_C - l_2)} = \frac{\beta_C}{l_C}$$

Again, $$\frac{\beta_B}{(l_1 - l_A)} = \frac{\beta_A}{l_A}$$

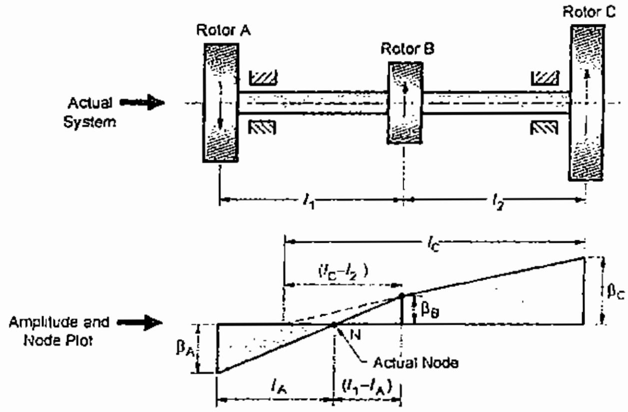


Fig. 5.7.3 : Single Node Vibration of Three Rotor System ($l_A < l_1$ and $l_C > l_2$)

5.7.3 Zero Frequency Vibration of Three Rotor System :

- When three rotors rotate in same direction, the shaft is said to vibrate with zero frequency, as shown in Fig. 5.7.4. The torsional vibrations cannot occur in such case.

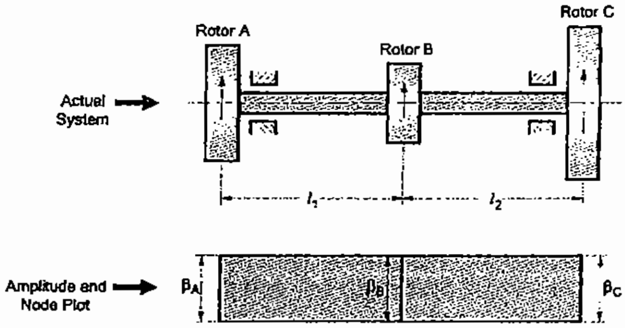


Fig. 5.7.4 : Zero Node Vibration of Three Rotor System

5.8 TORSIONALLY EQUIVALENT SHAFT

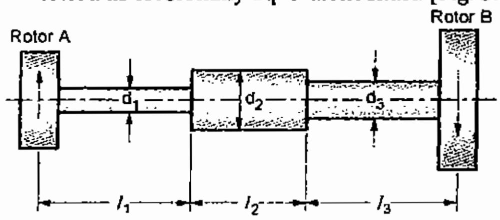
University Questions

Q. Explain the concept of torsionally equivalent shaft. **SPPU : May 15, Dec. 16, Dec. 17, May 18, Dec. 19**

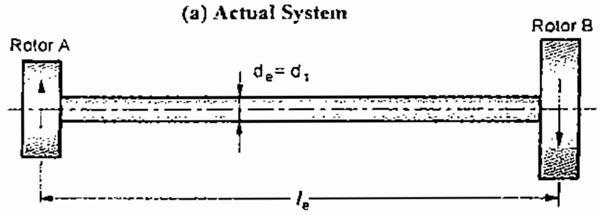
Q. Explain the concept of torsionally equivalent shaft and derive the equation for its equivalent length. **SPPU : Dec. 13, May 16**

Q. Explain Torsionally equivalent shaft with respect to 2 DOF free vibration. **SPPU : May 19**

- **Torsional equations shaft :** In two rotor and three rotor systems it is assumed that the diameter of shaft is uniform. But in actual practice, the shaft may have different diameters for different lengths. If the shafts of different diameters are replaced by a theoretically equivalent shaft of a uniform diameter, such shaft is called as torsionally equivalent shaft. [Fig. 5.8.1]



(a) Actual System



(b) Equivalent System

Fig. 5.8.1 : Torsionally Equivalent Shaft

Let,

d_1, d_2 and d_3 = diameters of shaft for the lengths l_1, l_2, and l_3 respectively, m.

θ_1, θ_2 and θ_3 = angles of twist for shaft lengths l_1, l_2 and l_3 respectively, rad

J_1, J_2 and J_3 = polar moment of inertia of shaft of diameters d_1 and d_2 and d_3 respectively, m^4

d_e = diameter of torsionally equivalent shaft, m

θ_e = angle of twist for equivalent shaft, rad

l_e = length of torsionally equivalent shaft, m

T = torque acting on shaft, N-m

G = modulus of rigidity for the shaft material, N /m^2

- **Total angle of twist for actual shaft :**

$$\theta = \theta_1 + \theta_2 + \theta_3 \quad \text{...(a)}$$

or $$\theta = \frac{Tl_1}{GJ_1} + \frac{Tl_2}{GJ_2} + \frac{Tl_3}{GJ_3} \quad \text{...(b)}$$

- **Angle of twist of equivalent shaft :**

$$\therefore \quad \theta_e = \frac{Tl_e}{GJ_e} \quad \text{...(c)}$$

- Angle of twist of equivalent shaft = Total angle of twist of actual shaft

$$\theta_e = \theta_1 + \theta_2 + \theta_3 \quad \text{...(d)}$$

- From Equations (b) and (c),

$$\frac{T\,l_e}{G\,J_e} = \frac{T\,l_1}{G\,J_1} + \frac{T\,l_2}{G\,J_2} + \frac{T\,l_3}{G\,J_3}$$

$$\therefore \quad \frac{l_e}{J_e} = \frac{l_1}{J_1} + \frac{l_2}{J_2} + \frac{l_3}{J_3}$$

$$\therefore \quad \frac{l_e}{\frac{\pi}{32}d_e^4} = \frac{l_1}{\frac{\pi}{32}d_1^4} + \frac{l_2}{\frac{\pi}{32}d_2^4} + \frac{l_3}{\frac{\pi}{32}d_3^4}$$

$$\therefore \quad \frac{l_e}{d_e^4} = \frac{l_1}{d_1^4} + \frac{l_2}{d_2^4} + \frac{l_3}{d_3^4}$$

$$\therefore \quad l_e = l_1\left(\frac{d_e}{d_1}\right)^4 + l_2\left(\frac{d_e}{d_2}\right)^4 + l_3\left(\frac{d_e}{d_3}\right)^4 \qquad ...(5.8.1)$$

In actual practice, the diameter of equivalent shaft is taken as one of the diameters of the actual shaft. Let us assume that $d_e = d_1$, then the length of equivalent shaft is given by,

$$l_e = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4 \qquad ...(5.8.2)$$

Ex. 5.8.1 : A horizontal circular disc of 400 mm diameter and 20 kg mass is supported by a vertical stepped shaft at the center. The shaft has two steps. First step is of 20 mm diameter and 200 mm length and second step is of 15 mm diameter and 250 mm length. Determine frequency of torsional oscillations of the disc if modulus of rigidity of shaft is 80×10^9 N / mm^2. **SPPU - Dec. 02, Dec. 07**

Soln. :

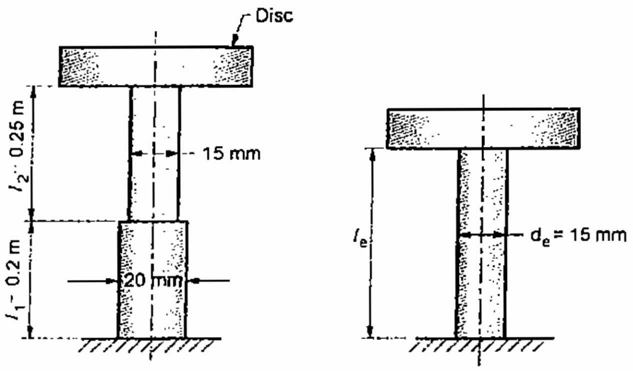
Given : $D = 0.4$ m ; $m = 20$ kg ;

$l_1 = 0.2$ m ; $d_1 = 0.02$ m;

$l_2 = 0.25$ m ; $d_2 = 0.015$ m.

$$G = 80 \times 10^9\ \text{N / mm}^2 = 8 \times 10^{10}\ \text{N / m}^2.$$

$$R = \frac{D}{2} = \frac{0.4}{2} = 0.2\ \text{m}$$



(a) Actual System (b) Equivalent System

Fig. P. 5.8.1

1. **Equivalent Length :**

1. **Equivalent Length :**

- The equivalent length of shaft having diameter d_2 (15 mm) is,

$$l_e = l_1\left(\frac{d_e}{d_1}\right)^4 + l_2\left(\frac{d_e}{d_2}\right)^4 = l_1\left(\frac{d_2}{d_1}\right)^4 + l_2\left(\frac{d_2}{d_2}\right)^4$$

$$= l_1\left(\frac{d_2}{d_1}\right)^4 + l_2 = 0.2\left(\frac{0.015}{0.020}\right)^4 + 0.25$$

or $\quad l_e = 0.313$ m

2. **Natural Frequency :**

- **Mass moment of inertia of disc :**

$$I = \frac{1}{2}mR^2 = \frac{1}{2} \times 20\,(0.2)^2 = 0.4\ \text{kg} \cdot \text{m}^2$$

- **Polar moment of inertia of the equivalent shaft :**

$$J_e = \frac{\pi}{32}(d_e)^4 = \frac{\pi}{32}\,(0.015)^4 = 4.97 \times 10^{-9}\ \text{m}^4$$

- **Natural circular frequency of system :**

$$\omega_n = \sqrt{\frac{K_{te}}{I}} = \sqrt{\frac{GJ}{l_e\,I}} = \sqrt{\frac{8 \times 10^{10} \times 4.97 \times 10^{-9}}{0.313 \times 0.4}}$$

or $\quad \omega_n = 56.35$ rad/s. ...Ans.

- Natural frequency of system :

$$f_n = \frac{\omega_n}{2\pi} = \frac{56.35}{2\pi}$$

or $\quad f_n = 8.96$ Hz ...Ans.

Ex. 5.8.2 : The flywheel of an engine driving a dynamo has mass of 200 kg and has a radius of gyration of 300 mm. The shaft at the flywheel end has an effective length of 250 mm and is 50 mm Diameter the arc mature mass is 225 kg and has a radius of gyration of 255 mm. The dynamo shaft has a diameter of 43.75 mm and a length of 255 mm. Neglecting the inertia of the shaft and coupling calculate the frequency of the torsional vibrations and position of node. Take the modulus of rigidity for shaft material as 80 GPa. **SPPU - May 16, 12 Marks**

Soln. : Mass movement of inertia for rotor A,

$$I_A = 200 \times 0.3^2 = 18\ \text{kg-m}^2$$

Mass movement of inertia for rotor B,

$$I_B = 225 \times 0.255^2 = 14.63\ \text{kg-m}^2$$

Modulus of rigidity for shaft material,

$$G = 80\ \text{GPa} = 80 \times 10^9\ \text{N/m}^2$$

The equivalent length of shaft having diameter d_1 (50 mm) is,

$$l_e = l_1\left(\frac{d_e}{d_1}\right)^4 + l_2\left(\frac{d_e}{d_2}\right)^4 = l_1\left(\frac{d_1}{d_1}\right)^4 + l_2\left(\frac{d_1}{d_2}\right)^4 = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 = 250 + 255\left(\frac{50}{43.75}\right)^4$$

$\therefore$ $\boldsymbol{l_e = 685\ \text{mm}}$ **...Ans.**

The equivalent system is shown in Fig. P.5.8.2

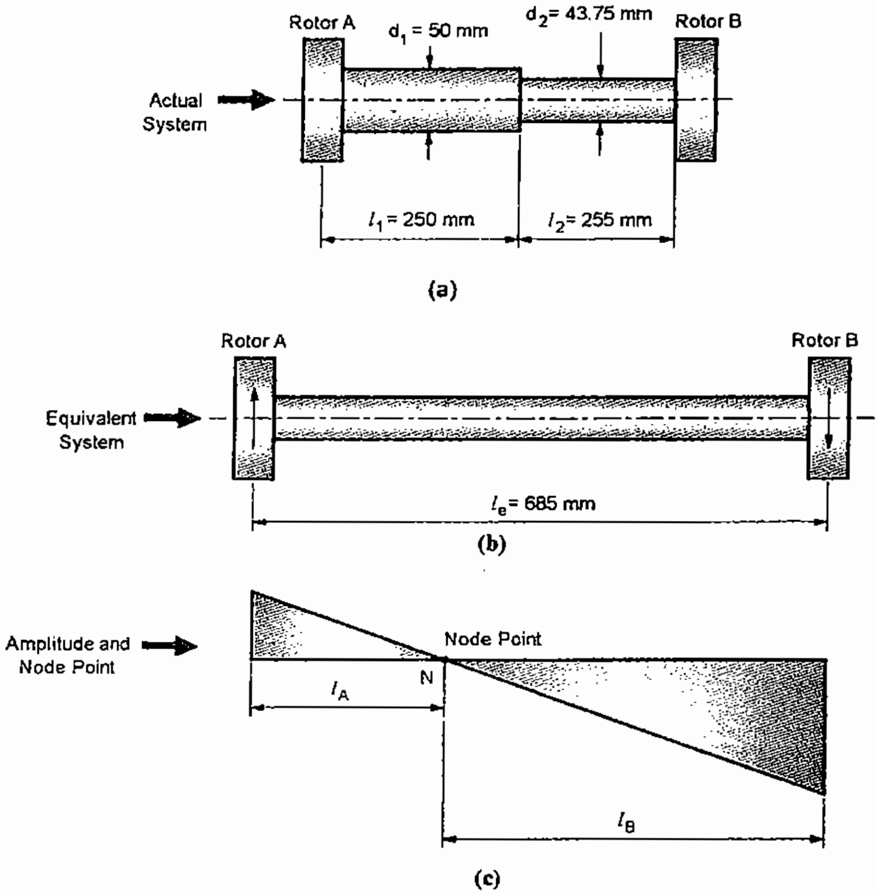


Fig. P. 5.8.2

2. Position of Node :

We know that,

$$l_A I_A = l_B I_B$$

$$\therefore \quad l_A = \frac{l_B I_B}{I_A} = \frac{l_B \times 14.63}{18}$$

or $$l_A = 0.812\, l_B$$

But $$l_e = l_A + l_B\,;$$

$$685 = 0.812\, l_B + l_B$$

$$\therefore \quad l_B = 377.87 \text{ m}$$

$$l_A = 0.812 \times 377.87$$

$$= \mathbf{306.83\ m}$$ **...Ans.**

3. Natural Frequency :

The natural circular frequency of two rotor system is,

$$\omega_n = \sqrt{\frac{GJ}{l_A I_A}}$$

or $$\sqrt{\frac{GJ}{l_B I_B}} = \sqrt{\frac{G \times \frac{\pi}{32} d_e^4}{l_A I_A}}$$

$$= \sqrt{\frac{80 \times 10^9 \times \frac{\pi}{32}(0.05)^4}{0.30683 \times 18}}$$

or $\boldsymbol{\omega_n = 94.27\ \text{rad/s}}$ **...Ans.**

The natural frequency of two rotor system is,

$$f_n = \frac{\omega_n}{2\pi} = \frac{94.27}{2\pi}$$

or $\boldsymbol{f_n = 15\ \text{Hz}}$ **...Ans.**

Ex. 5.8.3 : A shaft, shown in Fig. P. 5.8.3(a), carries two rotors A and B. The mass of rotor A is 300 kg with radius of gyration of 0.75 m, while the mass of rotor B is 500 kg with radius of gyration of 0.9 m.

(i) Find the natural frequency of torsional vibration.

(ii) It is desired to have the node at mid section of the shaft of diameter 120 mm by changing the diameter of the portion of shaft having 90 mm diameter. What would be the new diameter ?

Assume $G = 84 \times 10^9$ N/m². **SPPU - May 05**

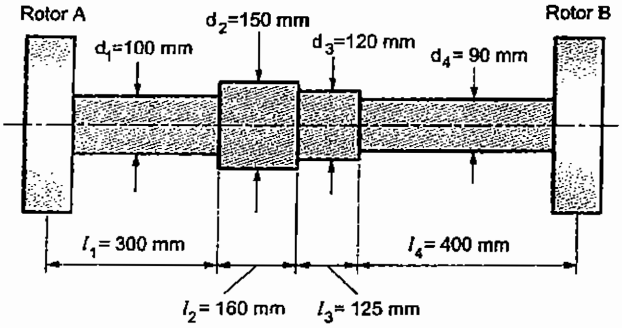


Fig. P. 5.8.3 (a)

Soln. :

$$m_A = 300 \text{ kg} \quad ; \quad k_A = 0.75 \text{ m} ;$$
$$m_B = 500 \text{ kg} \quad ; \quad k_B = 0.9 \text{ m} ;$$
$$G = 84 \times 10^9 \text{ N/m}^2.$$

1. **Equivalent Length :**

- The equivalent length of shaft having diameter d_1 (100 mm) is,

$$l_e = l_1\left(\frac{d_e}{d_1}\right)^4 + l_2\left(\frac{d_e}{d_2}\right)^4 + l_3\left(\frac{d_e}{d_3}\right)^4 + l_4\left(\frac{d_e}{d_4}\right)^4$$

$$= l_1\left(\frac{d_1}{d_1}\right)^4 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4 + l_4\left(\frac{d_1}{d_4}\right)^4$$

$$= l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4 + l_4\left(\frac{d_1}{d_4}\right)^4$$

$$= 0.3 + 0.16\left(\frac{0.1}{0.150}\right)^4 + 0.125\left(\frac{0.1}{0.120}\right)^4 + 0.4\left(\frac{0.1}{0.09}\right)^4$$

or l_e = **1002 mm**

- **Mass moment of inertia for rotor A :**

$$I_A = m_A k_A^2 = 300 \times (0.75)^2 = 168.75 \text{ kg-m}^2$$

- **Mass moment of inertia for rotor B :**

$$I_B = m_B k_B^2 = 500 \times (0.9)^2 = 405 \text{ kg-m}^2$$

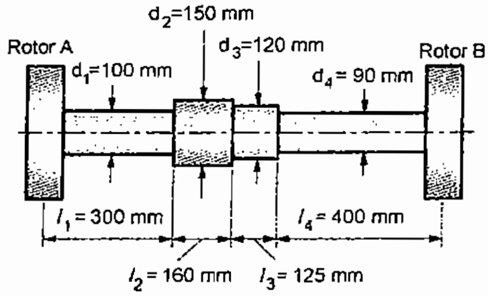
2. **Position of Node :**

- We know that,

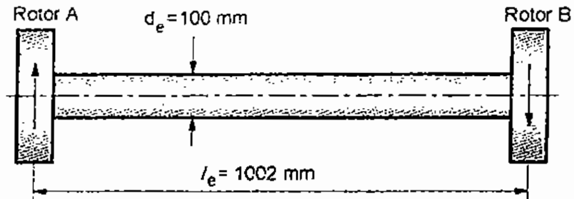
$$l_A I_A = l_B I_B$$

$$\therefore \quad l_A = \frac{l_B I_B}{I_A} = \frac{l_B \times 405}{168.75}$$

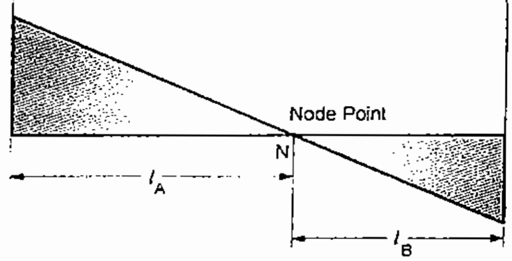
or $\quad l_A = 2.4\, l_B \quad$ ...(a)



(b) Actual System



(c) Equivalent System



(d) Amplitude and Node Plot

Fig. P. 5.8.3

Now, $\quad l_e = l_A + l_B$

$$\therefore \quad 1.002 = 2.4\, l_B + l_B$$

$$\therefore \quad 1.002 = 3.4\, l_B$$

$$\therefore \quad l_B = 0.294 \text{ m}$$

and, $\quad l_A = 2.4 \times 0.294$

or $\quad l_A$ = **0.707 m**

3. **Natural Frequency :**

- **Natural circular frequency of two rotor system :**

$$\omega_n = \sqrt{\frac{GJ}{l_A I_A}} = \sqrt{\frac{G \times \frac{\pi}{32} d_e^4}{l_A I_A}} = \sqrt{\frac{84 \times 10^9 \times \frac{\pi}{32}(0.1)^4}{0.707 \times 168.75}}$$

or $\quad \omega_n$ = **83.12 rad/s** ...**Ans.**

- **Natural frequency of two rotor system :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{83.12}{2\pi}$$

or $f_n = 13.22$ Hz ...Ans.

4. New Diameter for 90 mm Diameter Section :

- The node point is required at midway of 120 mm diameter section, by changing the diameter of 90 mm section. Let this new diameter be 'd'.
- **Equivalent length of shaft from rotor A upto midway of 120 mm diameter, (new position of node) :**

$$l'_e = l'_A = l_1\left(\frac{d_1}{d_1}\right)^4 + l_2\left(\frac{d_1}{d_2}\right)^4 + \frac{l_3}{2}\left(\frac{d_1}{d_3}\right)^4$$

$$= l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + \frac{l_3}{2}\left(\frac{d_1}{d_3}\right)^4$$

$$l'_A = 0.3 + 0.16\left(\frac{0.1}{0.150}\right)^4 + \frac{0.125}{2}\left(\frac{0.1}{0.120}\right)^4$$

$$l'_A = 0.36175 \text{ m}$$

- We know that,

$$l'_A I_A = l'_B I_B$$

$$\therefore \quad 0.36175 \times 168.75 = l'_B \times 405$$

$$\therefore \quad l'_B = 0.150 \text{ m}$$

- **Equivalent length of shaft from midway of 120 mm diameter to rotor B (l'_B) :**

$$\therefore \quad l'_B = \frac{l_3}{2}\left(\frac{d_1}{d_3}\right)^4 + l_4\left(\frac{d_1}{d}\right)^4$$

$$\therefore \quad 0.150 = \frac{0.125}{2}\left(\frac{0.1}{0.120}\right)^4 + 0.4\left(\frac{0.1}{d}\right)^4$$

$$\therefore \quad d = 0.13490 \text{ m}$$

$\therefore$ **d ≃ 134.9 mm** ...Ans.

- Therefore, new shaft diameter of section having 90 mm should be replaced by 134.9 mm in order to have the node point at midway of section having diameter 120 mm.

Ex. 5.8.4 : Determine the natural frequencies and the position of node of torsional vibration system having 2 rotors A and B attached to the ends of a shaft 1500 mm long. The mass moment of inertia of rotor A is 650 kg-m^2 and that of rotor B is 215 kg-m^2. The shaft is 95 mm diameter for the first 600 mm, 60 mm diameter for the next 500 mm length and 50 mm diameter for the remaining length. Modulus of rigidity of shaft material is 0.8×10^5 MPa. **SPPU - Dec. 11, 12 Marks**

Soln. :

Given : $l = 1.5$ m ; $I_A = 650$ kg-m^2 ;
$I_B = 215$ kg-m^2 ; $d_1 = 0.095$ m ;
$l_1 = 0.6$ m ; $d_2 = 0.06$ m ;
$l_2 = 0.5$ m ; $d_3 = 0.05$ m ;
$G = 0.8 \times 10^5$ MPa $= 80 \times 10^9$ N/m^2

1. Equivalent Length :

$$l_3 = l - (l_1 + l_2) = 1.5 - (0.6 + 0.5) = 0.4 \text{ m}$$

- **Equivalent length of shaft having diameter d_1 = 95 mm :**

$$l_e = l_1\left(\frac{d_e}{d_1}\right)^4 + l_2\left(\frac{d_e}{d_2}\right)^4 + l_3\left(\frac{d_e}{d_3}\right)^4$$

$$= l_1\left(\frac{d_1}{d_1}\right)^4 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4$$

$$= l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4$$

$$= 0.6 + 0.5\left(\frac{0.095}{0.06}\right)^4 + 0.4\left(\frac{0.095}{0.05}\right)^4$$

$\therefore$ **l_e = 8.95 m** ...Ans.

2. Position of Node :

We know that,

$$l_A I_A = l_B I_B$$

$$\therefore \quad l_A = \frac{l_B I_B}{I_A} = \frac{l_B \times 212}{650}$$

$$\text{or} \quad l_A = 0.326\, l_B \qquad \text{...(a)}$$

$$\text{But} \quad l_e = l_A + l_B$$

$$\therefore \quad 8.95 = 0.326\, l_B + l_B$$

$$\therefore \quad l_B = 6.75 \text{ m}$$

$$\text{and,} \quad l_A = 0.326 \times 6.75$$

or **l_A = 2.20 m**

3. Natural Frequency :

- **Natural circular frequency of two rotor system :**

$$\omega_n = \sqrt{\frac{GJ}{l_A I_A}} = \sqrt{\frac{G \times \frac{\pi}{32} d_e^4}{l_A I_A}}$$

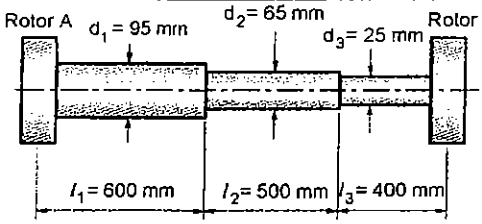
$$= \sqrt{\frac{80 \times 10^9 \times \frac{\pi}{32}(0.095)^4}{2.2 \times 650}}$$

or **ω_n = 21.17 rad/s** ...Ans.

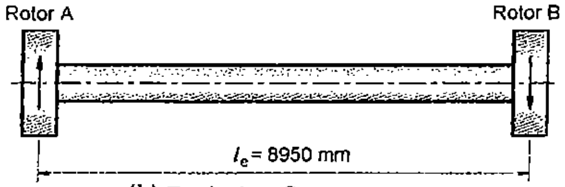
- **Natural frequency of two rotor system :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{21.17}{2\pi}$$

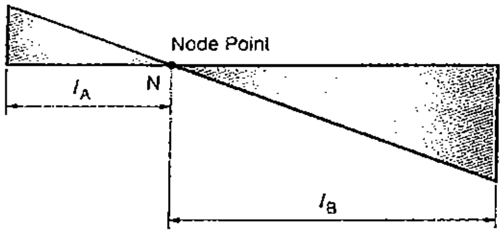
or **f_n = 3.37 Hz** ...Ans.



(a) Actual System



(b) Equivalent System



(c) Amplitude and Node Point

Fig. P. 5.8.4

Examples for Practice

Refer our website for complete solution of following examples

Ex. 5.8.5 : Two rotors A and B are attached to the end of a shaft 500 mm long. Weight of the rotor A is 300 N and its radius of gyration is 300 mm. The corresponding values of rotor B are 500 N and 450 mm respectively. The shaft is 70 mm in diameter for the first 250 mm, 120 mm diameter for next 100 mm and 100 mm diameter for the remainder of its length. Modulus of rigidity for the shaft material is 8×10^{11} N/m^2. Find (i) the position of the node and (ii) the frequency of torsional vibration. Draw the mode shapes.

SPPU - May 11, Dec. 13, 12 Marks

Ex. 5.8.6 : Two equal masses of weight 400 N each and radius of gyration 400 mm are keyed to the opposite ends of a shaft 600 mm long. The shaft is 750 mm diameter for the first 250 mm of its length, 125 mm diameter for the next 125 mm and 85 mm diameter for the remaining of its length. Find the frequency of free torsional vibrations of the system and position of node. Assume

$G = 0.84 \times 10^{11}$ N/m^2

SPPU - Dec. 14, 8 Marks, Oct. 18(In-sem), Oct. 19(In Sem.), 10 Marks

Ex. 5.8.7 : Two rotors, A and B are attached to the ends of the shaft 600 mm long. The mass and radius of gyration of rotor A is 40 kg and 400 mm respectively and that of rotor B are 50 kg and 500 mm respectively. The shaft is 80 mm diameter for first 250 mm, 120 mm for next 150 mm and 100 mm for the remaining length from the rotor A. Assume, the modulus of rigidity of the shaft material 0.8×10^5 N/mm^2 and find:

(i) Position of node on equivalent shaft of diameter 80 mm and on the actual shaft.

(ii) Natural frequency of the torsional vibrations.

Ex. 5.8.8 : A torsional system is shown in Fig. P. 5.8.8(a). Find the frequencies of torsional vibrations and the positions of the nodes. Also find the amplitude of vibrations. Take $G = 84 \times 10^9$ N/m^2.

SPPU - Dec. 8

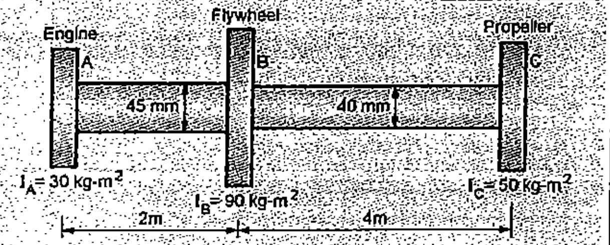


Fig. P. 5.8.8(a)

Ex. 5.8.9 : A motor generator set consists of two armatures A and C, as shown in Fig. P. 5.8.9 (a), with a flywheel B connected between them. The modulus of rigidity of the connecting shaft is 84×10^9 N / m^2. The system can torsionally vibrate with one node at x = 95 mm from armature A, fly wheel being at antinode. Find :

(i) the position of the another node;

(ii) the natural frequency of vibration;

(iii) the radius of gyration of armature C.

Use the following data :

Masses : M_A = 400 kg, M_B = 500 kg, M_C = 300 kg

Radius of gyration : k_A = 300 mm, k_B = 375 mm.

SPPU - Dec. 06

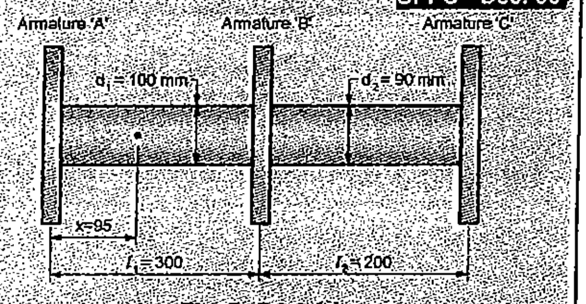


Fig. P. 5.8.9 (a)

5.9 TORSIONAL VIBRATIONS OF GEARED SYSTEM

University Questions

Q. Explain the torsional vibrations of a geared system by
(i) neglecting inertia of gears ; and
(ii) considering inertia of gears. **SPPU : Dec. 12**

Q. Write short note on : torsional vibration of geared system. **SPPU : Dec. 15**

Geared System :

Many a times torsional vibrations are observed in a geared system. This leads to high dynamic load in geared system and hence failure of the gear teeth. The geared system is shown in Fig. 5.9.1. The shaft 1 carries a rotor (prime mover) A at one end and a pinion at the other end. The shaft 2 carries a gear meshing with the pinion at one end and a rotor (machine) B at the other end.

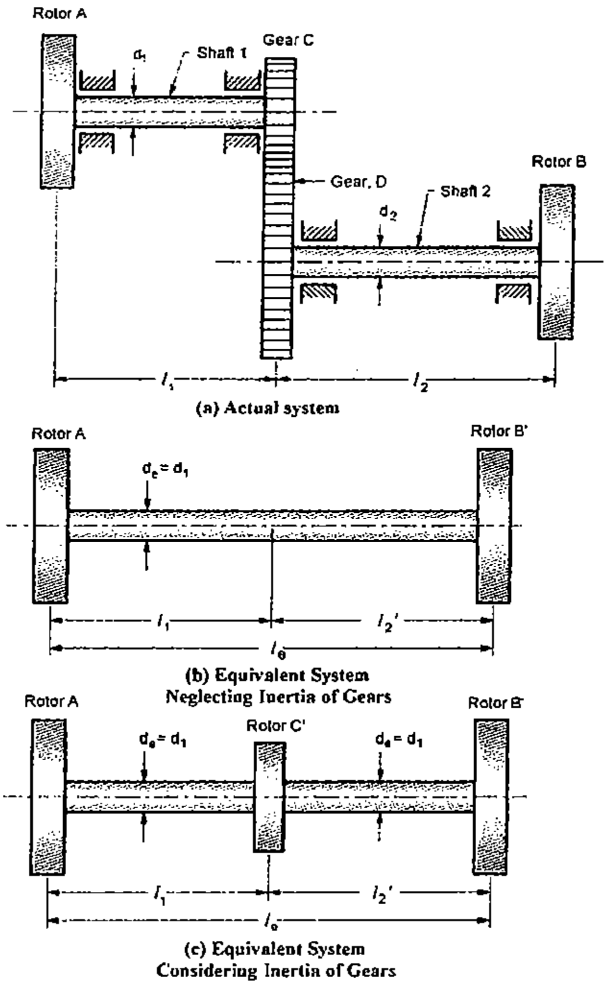


Fig. 5.9.1 : Equivalent System of Geared System

Torsionally Equivalent Two Rotor and Three Rotor Systems :

- The geared system may be replaced by an equivalent system of continuous shaft carrying a rotor A at one end and equivalent rotor B′ at the other end. If the inertia of the gears is negligible, then, the system will become equivalent to a **two rotor system** as shown in Fig. 5.9.1(b).
- If the inertia of gears are considered then, the system will become equivalent to **three rotor system,** as shown in Fig. 5.9.1(c).

Let, d_A, d_B = diameters of shafts 1 and 2, m

l_1, l_2 = lengths of shafts 1 and 2, m

I_A, I_B = mass moment of inertia of rotors A and B, kg-m^2

ω_A, ω_B = angular speeds of rotors A and B, rad / s

g = gear ratio = ω_A / ω_B

d_e = diameter of equivalent shaft, m

l_e = length of equivalent shaft, m

I_B' = mass moment of inertia of rotor B′, kg-m^2

I_C' = mass moment of inertia of rotor C′, kg-m^2

ω_B', ω_C' = angular speeds of rotors B′ and C′, rad / s

Cases of Torsional Vibrations of Geared System :

1. Torsional Vibrations of geared system by neglecting inertia of gears
2. Torsional Vibrations of geared system by considering inertia of gears

5.9.1 Torsional Vibrations of Geared System by Neglecting Inertia of Gears [Fig. 5.9.1(b)] :

- If the shafts are not strained beyond the elastic limit, each rotor in the geared system will oscillate with SHM. The two rotors A and B will reach their extreme positions at the same instant and at this instant the whole energy in the system will be strain energy (PE) due to the twisting of the two shafts. The two rotors A and B will also pass through their equilibrium positions at the same instant and at this instant the whole energy in the system will be kinetic energy (KE) of rotors.
- **Conditions for two systems to be equivalent :** The system shown in Fig. 5.9.1(b) will be equivalent to the actual geared system if :

(i) KE of Actual Geared system = KE of the Equivalent System.

(ii) Strain Energy of, Actual Geared System = Strain Energy of, the Equivalent System

(i) K.E. of Actual Geared System = K.E. of Equivalent System

$$\text{K.E. of Section } l_1 + \text{K.E. of Section } l_2 = \text{K.E. of Section } l_1 + \text{K.E. of Section } l_2'$$

$$\therefore \quad \text{K.E. of Section } l_2 = \text{K.E. of Section } l_2'$$

$$\therefore \quad \frac{1}{2} I_B (\omega_B)^2 = \frac{1}{2} I_B' (\omega_B')^2$$

$$\therefore \quad I_B (\omega_B)^2 = I_B' (\omega_B')^2 \qquad \text{...(a)}$$

$$I_B (\omega_B)^2 = I_B' (\omega_A)^2 \qquad \text{....}[\because \ \omega_B' = \omega_A]$$

$$I_B' = I_B \left(\frac{\omega_B}{\omega_A}\right)^2$$

$$I_B' = \frac{I_B}{\left(\frac{\omega_A}{\omega_B}\right)^2} \qquad \text{...(5.9.1)}$$

$$I_B' = \frac{I_B}{g^2}$$

(ii) Strain Energy of Actual Geared System = Strain Energy of Equivalent System

$$\text{Strain Energy of Section } l_1 + \text{Strain Energy of Section } l_2 = \text{Strain Energy of Section } l_1 + \text{Strain Energy of Section } l_{2'}$$

$$\text{Strain Energy of Section } l_2 = \text{Strain Energy of Section } l_{2'}$$

$$\frac{1}{2} T_2 \theta_2 = \frac{1}{2} T_2' \theta_2' \qquad \text{...(b)}$$

But,

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\therefore \quad T = \frac{GJ\theta}{l}$$

Let, J_2 = polar moment of inertia of shaft 2 $= \frac{\pi}{32}(d_2)^4$

J_2' = polar moment of inertia of equivalent shaft $= \frac{\pi}{32}(d_e)^4$

Hence Equation (b) becomes,

$$\frac{1}{2} \cdot \frac{G J_2 \cdot \theta_2 \cdot \theta_2}{l_2} = \frac{1}{2} \cdot \frac{G J_2' \cdot \theta_2' \cdot \theta_2'}{l_{2'}}$$

$$\therefore \quad \frac{J_2 (\theta_2)^2}{l_2} = \frac{J_2' (\theta_{2'})^2}{l_{2'}}$$

$$\therefore \quad \frac{J_2 (\omega_B \cdot t)^2}{l_2} = \frac{J_2' (\omega_{B'} \cdot t)^2}{l_{2'}}$$

$$\therefore \quad \frac{\pi (d_2)^4 (\omega_B \cdot t)^2}{32\, l_2} = \frac{\pi (d_e)^4 (\omega_{B'} \cdot t)^2}{32\, l_{2'}}$$

$$\therefore \quad l_2' = l_2 \left(\frac{d_e}{d_2}\right)^4 \left(\frac{\omega_{B'}}{\omega_B}\right)^2$$

$$\therefore \quad l_2' = l_2 \left(\frac{d_e}{d_2}\right)^4 \left(\frac{\omega_A}{\omega_B}\right)^2 \qquad \text{...}[\because \ \omega_{B'} = \omega_A]$$

$$\therefore \quad l_2' = g^2 l_2 \left(\frac{d_e}{d_2}\right)^4 \qquad \text{...(5.9.2)}$$

- **Total length of equivalent shaft :**

$$l_e = l_1 + l_2'$$

or

$$l_e = l_1 + g^2 l_2 \left(\frac{d_e}{d_2}\right)^4 \qquad \text{...(5.9.3)}$$

- The equivalent two rotor system can be analyzed as discussed in section 5.6.

5.9.2 Torsional Vibrations of a Geared System by Considering Inertia of Gears [Fig. 5.9.1(c)] :

- If the inertia of gears is not negligible, then an additional rotor C' must be placed at a distance l_A from the rotor A on the shaft.

K. E. of Actual Geared System = K.E. of Equivalent System

K. E. of Section l_1 + K. E. of Section l_2 = K. E. of Section l_e

$$\frac{1}{2} I_A (\omega_A)^2 + \frac{1}{2} I_C (\omega_A)^2 + \frac{1}{2} I_D (\omega_B)^2 + \frac{1}{2} I_B (\omega_B)^2$$

$$= \frac{1}{2} I_A (\omega_A)^2 + \frac{1}{2} I_{C'} (\omega_{B'})^2 + \frac{1}{2} I_{B'} (\omega_{B'})^2 \qquad \text{...(c)}$$

- Substituting Equation (a) in Equation (c),

$$\frac{1}{2} I_C (\omega_A)^2 + \frac{1}{2} I_D (\omega_B)^2 = \frac{1}{2} I_{C'} (\omega_{B'})^2$$

$$\therefore \quad I_C (\omega_A)^2 + I_D \left(\frac{\omega_A}{g}\right)^2 = I_{C'} (\omega_A)^2 \qquad [\because \ \omega_{B'} = \omega_A]$$

$$\left(I_C + \frac{I_D}{g^2}\right) \omega_A^2 = I_{C'} (\omega_A)^2$$

$$\therefore \quad I_C' = I_C + \frac{I_D}{g^2} \quad ...(5.9.4)$$

Where, I_C = Mass moment of inertia of pinion C, kg-m²

I_D = Mass moment of inertia of gear D, kg-m²

- The system will act as three rotor system with rotors, I_A, I_C' and I_B'.
- The equivalent three rotor system can be analyzed as discussed in section 5.7.

Ex. 5.9.1 : An electric motor is driving a rotor through a single stage reduction gear. The particulars are as follows :

- Gear ratio = 4.5
- M.I. of armature = 16 kg-m²
- M.I. of rotor = 40 kg-m²
- Length of pinion shaft = 0.5 m
- Diameter of pinion shaft = d_1
- Length of rotor gear shaft = 3 m
- Diameter of rotor gear shaft = 90 mm

Neglecting the inertia of gears, find the diameter 'd_1' to have the node at the gear. Also find the natural frequency of torsional vibrations and amplitude of rotor oscillations, if armature has amplitude of 1°. Take G = 80 GPa.

SPPU - May 02, May 14, 12 Marks

Soln. :

$g = 4.5$; $I_A = 16$ kg-m²;

$I_B = 40$ kg-m² ; $l_1 = 0.5$ m ;

$l_2 = 3$ m ; $d_2 = 90$ mm ;

$G = 80$ GPa $= 80 \times 10^9$ N / m²; $a_1 = 1°$.

1. Equivalent Two Rotor System :

- When inertia of the gear is neglected, the equivalent system will become two rotor system as shown in Fig. P. 5.9.1(A).
- **Condition (i) :**

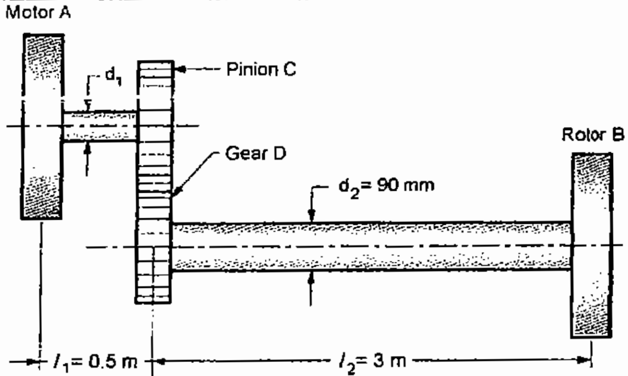
$$I_B' = \frac{I_B}{g^2} = \frac{40}{(4.5)^2}$$

$$= \mathbf{1.9753 \ kg\text{-}m^2}$$

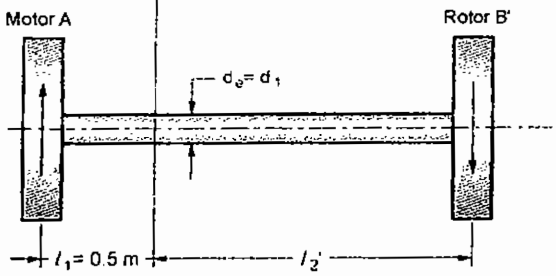
- **Condition (ii) :**

$$l_e = l_1 + g^2 l_2 \left(\frac{d_e}{d_2}\right)^4 = 0.5 + (4.5)^2 \times 3 \left(\frac{d_e}{0.09}\right)^4$$

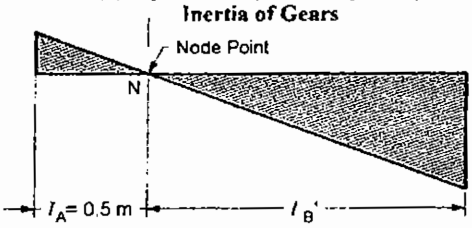
or $$l_e = 0.5 + 925925.93\,(d_e)^4 \quad ...(a)$$



(a) Actual System



(b) Equivalent System Neglecting Inertia of Gears



(c) Amplitude and Node Plot

Fig. P. 5.9.1(A)

- **For two rotor system :**

$$I_A l_A = I_B' l_B'$$

$$\therefore \quad 16 \times 0.5 = 1.9753 \times l_B'$$

$$\therefore \quad l_B' = \mathbf{4.05 \ m} \quad ...(b)$$

But, $l_e = l_A + l_B'$

$= l_1 + l_B'$ [As node is at gear, $l_1 = l_A$]

or $$l_e = 0.5 + 4.05 \quad ...(c)$$

from Equations (a) and (c),

$$\therefore \quad 0.5 + 925925.93\,(d_e)^4 = 0.5 + 4.05$$

$$\therefore \quad d_e = 0.04573 \text{ m}$$

or $$d_e = \mathbf{45.73 \ mm} \quad ...\text{Ans.}$$

2. Natural Frequency :

- **Natural circular frequency of two rotor system :**

$$\omega_n = \sqrt{\frac{GJ}{l_A I_A}} = \sqrt{\frac{G \times \frac{\pi}{32} (d_e)^4}{l_A I_A}}$$

$$= \sqrt{\frac{80 \times 10^9 \times \frac{\pi}{32}(0.04573)^4}{0.5 \times 16}}$$

or $\omega_n = \mathbf{65.52\ rad/s}$

- **Natural frequency of the system :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{65.52}{2\pi}$$

or $f_n = \mathbf{10.42\ Hz}$ **...Ans.**

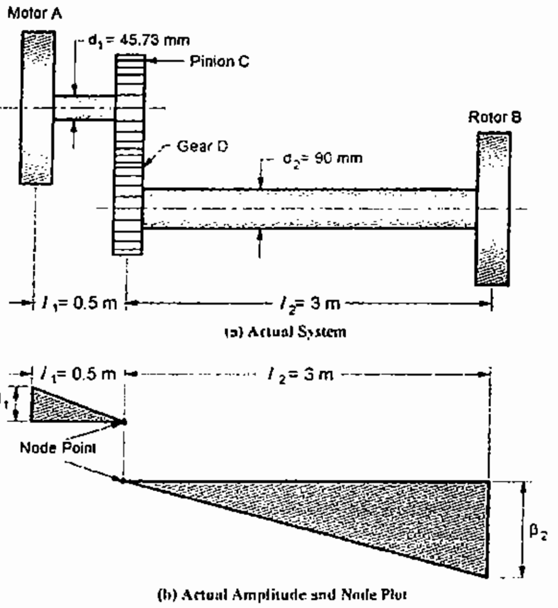


Fig. P. 5.9.1(B)

3. **Amplitude of Oscillations :**

- To determine the amplitude of vibration on the actual shaft, consider Fig. P. 5.9.1(B).

- **Torsional stiffness of motor shaft :**

$$K_{tA} = \frac{GJ}{l_1} = \frac{G \times \frac{\pi}{32}(d_1)^4}{l_1}$$

$$= \frac{80 \times 10^9 \times \frac{\pi}{32}(0.04573)^4}{0.5}$$

$\therefore$ $K_{tA} = 68694.93$ N-m/rad

- **Torsional stiffness of rotor shaft :**

$$K_{tB} = \frac{GJ}{l_2} = \frac{G \times \frac{\pi}{32}(d_2)^4}{l_2} = \frac{80 \times 10^9 \times \frac{\pi}{32}(0.09)^4}{3}$$

$\therefore$ $K_{tB} = 171766.58$ N-m/rad

But,

Torque on rotor shaft = g × Torque on motor shaft

$$T_{RB} = g\, T_{RA}$$

$$\therefore \quad K_{tB} \times \beta_B = g \times K_{tA}\,\beta_A$$

$$\therefore \quad 171766.58 \times \beta_B = 4.5 \times 68694.93 \times \beta_A$$

$$\therefore \quad \frac{\beta_B}{\beta_A} = 1.79$$

$$\therefore \quad \frac{\beta_B}{1^\circ} = 1.79$$

$\therefore$ $\boldsymbol{\beta_B = 1.79^\circ}$ **...Ans.**

Therefore, amplitude of oscillation of rotor is 1.79°.

Ex. 5.9.2 : In a geared system, shown in Fig. P. 5.9.2(a), the mass of moment of inertia of rotor A and B are 2 kg-m^2 and 0.3 kg-m^2 respectively. The gear ratio between rotor B and A is 3. Calculate the node position and natural frequency of torsional oscillations. Ignore the inertia of the gears and shafts. Take modulus of rigidity of shaft material as 80×10^9 N/m^2

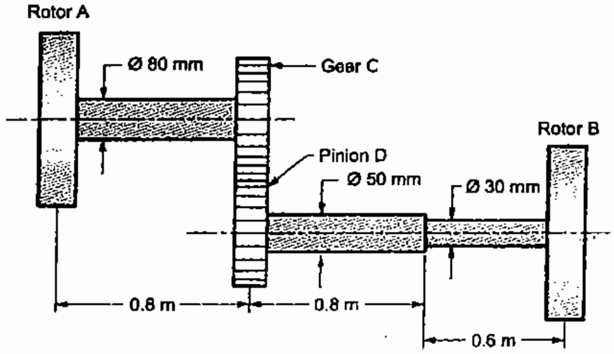


Fig. P. 5.9.2(a)

Soln. :

Given : $I_A = 2$ kg-m^2 ; $I_B = 0.3$ kg-m^2;

$g = \frac{\omega_A}{\omega_B} = \frac{1}{3} = 0.3333$: $G = 80 \times 10^9$ N/m^2;

$d_1 = 0.08$ mm ; $l_1 = 0.8$ m ;

$d_2 = 0.05$ mm ; $l_2 = 0.8$ m ;

$d_3 = 0.03$ mm ; $l_3 = 0.6$ m.

1. **Equivalent Two Rotor System :**

- When inertia of the gears is neglected, then the equivalent system will be two rotor system, as shown in Fig. P. 5.9.2.

- **Condition (i) :**

$$I_{B'} = \frac{I_B}{g^2} = \frac{0.3}{(0.3333)^2}$$

$$= \mathbf{2.7\ kg\text{-}m^2}$$

- **Condition (ii) :**

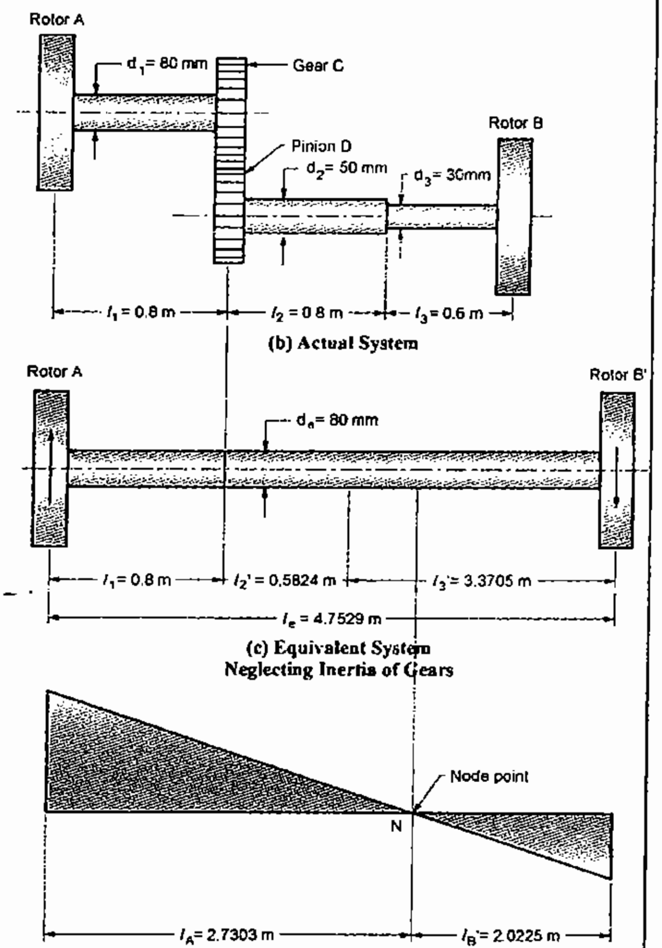
Take d_e = 80 mm or 0.08 m

$$l_2' = g^2 l_2 \left(\frac{d_e}{d_2}\right)^4 = (0.3333)^2 \times 0.8 \times \left(\frac{0.080}{0.050}\right)^4$$

$$= 0.5824 \text{ m}$$

Again, $$l_3' = g^2 l_3 \left(\frac{d_e}{d_3}\right)^4 = (0.3333)^2 \times 0.6 \left(\frac{0.080}{0.030}\right)^4$$

$$= \mathbf{3.3705 \text{ m}}$$



(b) Actual System

(c) Equivalent System Neglecting Inertia of Gears

(d) Amplitude and Node Plot

Fig. P. 5.9.2

- **Total equivalent length of shaft :**

$$l_e = l_1 + l_2' + l_3' = 0.8 + 0.5824 + 3.3705$$

$$= 4.7529 \text{ m}$$

- **For two rotor system :**

$$I_A l_A = I_{B'} l_{B'}$$

$$2 \times l_A = 2.7 \, l_{B'}$$

$$l_A = 1.35 \, l_{B'} \quad \text{...(a)}$$

But, $$l_e = l_A + l_{B'} \quad \text{...(b)}$$

Substituting Equation (a) in Equation (b),

$$\therefore \; 4.7529 = 1.35 \, l_{B'} + l_{B'}$$

$$\therefore \; 4.7529 = 1.35 \, l_{u'}$$

$$\therefore \; \mathbf{l_{B'} = 2.0225 \text{ m}}$$

and, $$l_A = 1.35 \, l_{B'}$$

$$\therefore \; l_A = 1.35 \times 2.0225$$

$$\therefore \; \mathbf{l_A = 2.7303 \text{ m}}$$

- **Actual position of node :** Since $l_{B'}$ is less than l_3, the actual position of the node from rotor B is determined as follows :

$$l_{B'} = g^2 \times l_{B\,Actual} \times \left(\frac{d_e}{d_3}\right)^4$$

$$2.0225 = (0.3333)^2 \times l_{B\,Actual} \times \left(\frac{0.08}{0.03}\right)^4$$

$$l_{B\,Actual} = \mathbf{0.36 \text{ m}}$$ **...Ans.**

- Therefore, actual node position is 0.36 m from rotor B.

2. **Natural Frequency :**

- **Natural circular frequency of two rotor system :**

$$\omega_n = \sqrt{\frac{GJ}{l_A I_A}} = \sqrt{\frac{G \times \frac{\pi}{32} (d_e)^4}{l_A I_A}}$$

$$= \sqrt{\frac{80 \times 10^9 \times \frac{\pi}{32} (0.08)^4}{2.7303 \times 2}}$$

or ω_n = 242.71 rad/s

- **Natural frequency of two rotor system :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{242.71}{2\pi}$$

or f_n = **38.63 Hz** **...Ans.**

Examples for Practice

Refer our website for complete solution of following examples

Ex. 5.9.3 : A twelve cylinder aero-engine drives an air screw through gearing. The air screw runs at 0.6 times the speed of the engine. The shaft from the engine to the pinion is of 1000 mm length and of 70 mm diameter. The screw shaft is 650 mm long and 90 mm in diameter. The mass moment of inertia of engine and air screw are 0.5 kg-m^2 and 15 kg-m^2 respectively. Neglecting inertia of gears and shafts, determine the frequency of torsional vibrations. Also suggest suitable location of the gears to avoid adverse effect of torsional vibrations. Assume modulus of elasticity = 80 GN / m^2. **SPPU - Dec. 05**

Ex. 5.9.4 : An electric motor, running at 1500 r.p.m., drives a pump through gearing. The pump runs at 500 r.p.m. The motor armature has mass moment of inertia of 400 kg-m^2 and the pump impeller has a mass of moment of inertia of 1400 kg-m^2. The motor shaft is 45 mm diameter and 180 mm long. The pump shaft is having 90 mm diameter and 450 mm length.

(i) Find an equivalent system having a uniform shaft diameter of 45 mm and running at 1500 r.p.m.; and

(ii) Find the natural frequency of the system neglecting inertia of gears. Take $G = 84 \times 10^9$ N / m^2

SPPU - Dec. 03, May 07

Ex. 5.9.5 : A reciprocating IC engine is coupled to a centrifugal pump through gearing. The shaft from the flywheel of the engine to the gear wheel is 60 mm diameter and 950 mm long. Shaft from pinion to pump is 40 mm diameter and 300 mm long. Engine speed is 1/4th the pump speed. The moment of inertia of the flywheel and gear wheel are 800 kg-m^2 and 15 kg-m^2 respectively. The moment of inertia of the pinion ad pump impeller are 4 kg-m^2 and 17 kg-m^2 respectively. Determine the natural frequency and position of node of the torsional oscillation of the system. Modulus of rigidity of shaft material is G = 84 GPa. **SPPU - May 12, 12 Marks**

Ex. 5.9.6 : A motor drives a pump through gearing so that the pump runs at three times the motor speed. The mass of moment of inertia of a rotor of motor is 100 kg-m^2, that of the gear is 10 kg-m^2 and pinion is 0.6 kg-m^2. If one of the natural frequency of the system is 60 Hz, find the moment of inertia of the pump impeller assuming modulus of rigidity of the shaft material as 0.785×10^5 N/mm^2. The diameter of shaft from motor to gear is 70 mm and length is 1000 mm. The diameter of shaft from pinion to pump impeller is 50 mm and length is 630 mm. **SPPU - Dec. 04**

5.10 LIST OF FORMULAE

1. Two Rotor System :

(i) Natural circular frequency :

- $\omega_n = \sqrt{\frac{GJ}{I_A l_A}}$, rad /s
- $\omega_n = \sqrt{\frac{GJ}{I_B l_B}}$, rad /s

(ii) Position of node :

- $l_A = \frac{I_B l_B}{I_A}$
- $l = l_A + l_B$

2. Three Rotor System :

(i) Natural circular frequency :

- $\omega_n = \sqrt{\frac{K_{tA}}{I_A}} = \sqrt{\frac{GJ}{I_A l_A}}$
- $\omega_n = \sqrt{\frac{K_{tC}}{I_C}} = \sqrt{\frac{GJ}{I_C l_C}}$
- $\omega_n = \sqrt{\frac{K_{tB}}{I_B}} = \sqrt{\frac{GJ}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right)}$

(ii) Positions of nodes :

- $\frac{1}{I_B}\left(\frac{1}{l_1 - l_A} + \frac{1}{l_2 - l_C}\right) = \frac{1}{I_C l_C}$
- $l_A = \frac{I_C l_C}{I_A}$

(iii) Condition for two nodes vibration :

- If $l_A < l_1$ and $l_C < l_2$: l_A and l_C gives the positions of two nodes.

(iv) Conditions for single node vibration :

- If $l_A > l_1$ and $l_C < l_2$: l_C gives the position of single node.
- If $l_A < l_1$ and $l_C > l_2$: l_A gives the position of single node.

3. Torsionally Equivalent Shaft :

(i) Diameter of equivalent shaft : d_e

(ii) Length of equivalent shaft :

- $l_e = l_1\left(\frac{d_e}{d_1}\right)^4 + l_2\left(\frac{d_e}{d_2}\right)^4 + l_3\left(\frac{d_e}{d_3}\right)^4 + l_4\left(\frac{d_e}{d_4}\right)^4$

4. Torsional Vibrations of Geared System (Neglecting Inertia of Gears) :

- $I_B' = \frac{I_B}{g^2}$
- $l_2' = g^2 l_2 \left(\frac{d_e}{d_2}\right)^4$
- $l_e = l_1 + g^2 l_2 \left(\frac{d_e}{d_2}\right)^4$

5. Torsional Vibrations of Geared System (Considering Inertia of Gears) :

- $I_B' = \dfrac{I_B}{g^2}$
- $I_C' = I_C + \dfrac{I_D}{g^2}$
- $l_2' = g^2 l_2 \left(\dfrac{d_e}{d_2}\right)^4$
- $l_e = l_1 + g^2 l_2 \left(\dfrac{d_e}{d_2}\right)^4$

Exercise

1. Find the two natural frequencies and mode shapes for the system shown in Fig. 1.

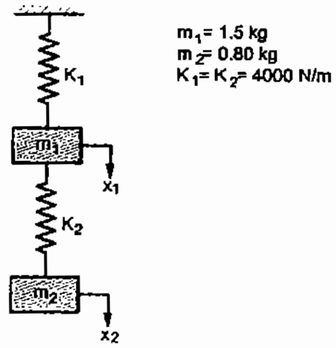


Fig. 1

[**Ans. :** $\omega_{n1} = 3.87$ rad / s; $\omega_{n2} = 9.38$ rad / s
$\left(\dfrac{X_1}{X_2}\right)_1 = 0.69$; $\left(\dfrac{X_1}{X_2}\right)_2 = -0.76$]

2. Find the two natural frequencies of the system shown in Fig. 2.

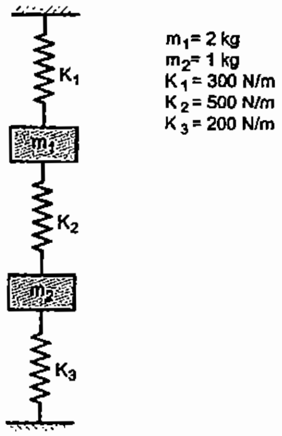


Fig. 2

[**Ans. :** $\omega_{n1} = 12.88$ rad / s; $\omega_{n2} = 30.56$ rad / s]

3. Set up the differential equations of motion for the system shown in Fig. 3 and determine :
   (i) The natural frequencies ; and
   (ii) The principal mode shapes

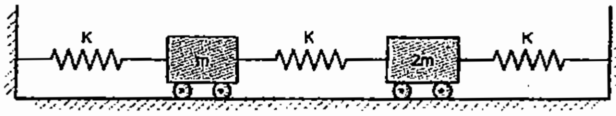


Fig. 3

[**Ans. :** $\omega_{n1} = 1.5\sqrt{K/m}$;
$\omega_{n2} = 0.8\sqrt{K/m}$;
$\left(\dfrac{X_1}{X_2}\right)_1 = 0.72$; $\left(\dfrac{X_1}{X_2}\right)_2 = -4$]

4. Determine the two natural frequencies and mode shapes for the system shown in Fig. 4.

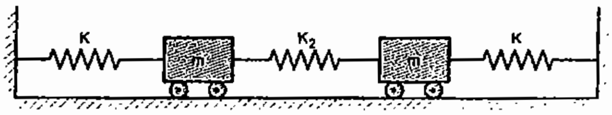


Fig. 4

[**Ans. :** $\omega_{n1} = \sqrt{K_1/m}$;
$\omega_{n2} = \sqrt{\dfrac{K_1 + 2K_2}{m}}$;
$\left(\dfrac{X_1}{X_2}\right)_1 = 1$, $\left(\dfrac{X_1}{X_2}\right)_2 = -1$]

5. Determine the two natural frequencies and mode shapes for the system shown in Fig. 5.

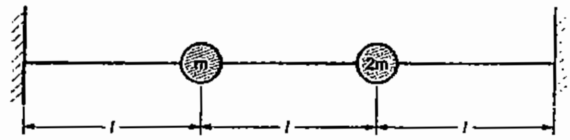


Fig. 5

[**Ans. :** $\omega_{n1} = \dfrac{T}{m\,l}\left(\dfrac{3+\sqrt{3}}{2}\right)$; $\omega_{n2} = \dfrac{T}{m\,l}\left(\dfrac{3-\sqrt{3}}{2}\right)$ $\left(\dfrac{X_1}{X_2}\right)_1 = -1-\sqrt{3}$, $\left(\dfrac{X_1}{X_2}\right)_2 = -1+\sqrt{3}$]

6. A shift carries two identical rotors A and B each having mass of 400 kg and radius of gyration of 0.4 m. They are keyed to the shaft of 600 mm length at two ends. The diameter of shaft is 75 mm for the first 250 mm of its length, 125 mm for the next 100 mm length and 85 mm for the remaining of its length. Find the natural frequency of torsional vibration. Assume $G = 84 \times 10^9$ N/m^2.

[**Ans. :** $f_n = 22.10$ Hz]

7. An electric motor drives a pump through gear pair. The speed of motor is 4 times that of pump. The motor has mass moment of inertia of 3 kg-m^2 and is at a distance of 900 mm from the gear. The mass moment of inertia of pump impeller has 16 kg-m^2 and is at a distance of 600 mm from the gear. The motor shaft is 50 mm and pump shaft is 70 mm in diameters. Determine the natural frequency of the system neglecting the inertia of gears. Assume $G = 84 \times 10^9$ N/m^2.
[**Ans.** : f_n = 22.5 Hz]

❑❑❑

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UNIT 6

6 MEASUREMENT AND CONTROL OF VIBRATIONS AND INTRODUCTION TO NOISE

Syllabus

A) Measurement : Vibration Measuring Instruments, Accelerometers, Impact Hammer, Vibration Shakers, Vibration Analyzer, Vibration Based Condition Monitoring, Analysis of Vibration Spectrum, Standards Related to Measurement of Vibration.

B) Control : Vibration Control Methods- Passive, Semi Active and Active Vibration Control, Control of Excitation at the Source, Control of Natural Frequency, Vibration Isolators, Tuned Dynamic Vibration Absorbers.

C) Noise : Fundamentals of Noise, Sound Concepts, Decibel Level, Logarithmic Addition, Subtraction and Averaging, Sound Intensity, Noise Measurement, Noise Control at Source, Along the Path and at the Receiver, Pass-by-Noise, Reverberation Chamber, Anechoic Chamber, Noise Standards. (Unit VI – Only theoretical treatment)

TOPICS

Part I : Vibration Measurement

6.1 INTRODUCTION TO VIBRATION MEASUREMENT

University Question

Q. Explain general vibration measurement process, with neat sketch. **SPPU : May 15**

Now a days, the measurement of vibrations becomes necessary due to the following reasons :

1. The measurement of natural frequencies of machine or structure, which is vibrating, is useful for selecting the operational speeds for the machine or machine mounted on the structure. The operational speeds are selected such that they should be far off from the natural frequency so as to avoid the resonance.
2. To know the actual values of vibration characteristics of a machine or structure. This is because, the theoretically calculated values may be different from the actual values due to assumptions made in the analysis.
3. In design and development of vibration isolation system, it is necessary to know the frequencies of vibrations and the forces developed due to vibrations.
4. To know the information of ground vibration due to earthquake, fluctuating wind velocities on structure or buildings, random vibrations due to ocean waves, etc.

Fig. 6.1.1 shows the steps in vibration measurement process.

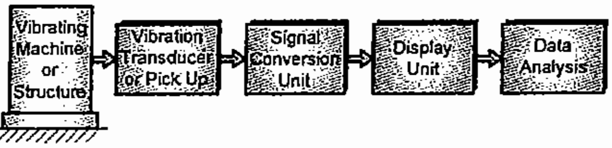


Fig. 6.1.1 : Steps in Vibration Measurement Process

1. **Vibration transducer or pickup :** The vibration measurement process starts by sensing the vibratory motion from vibrating machine or structure and converting it into an electrical signal with the help of **transducer** or **pickup**.
2. **Signal conversion unit :** The output signal from transducer, which is in the form of voltage or current, is too small to record it. Therefore the **signal conversion unit** is used to amplify the signals from the transducer to the required value.
3. **Display unit :** The output signals from the signal conversion unit is displayed on **display unit** for visual inspection. This information is recorded and stored in computer for later use.
4. **Data analysis :** Finally, the data can be analysed to know the vibration characteristics of machine or structure.

6.2 CLASSIFICATION OF VIBRATION MEASURING INSTRUMENTS

- The vibration measuring instruments can be classified as follows [Fig. 6.2.1] :

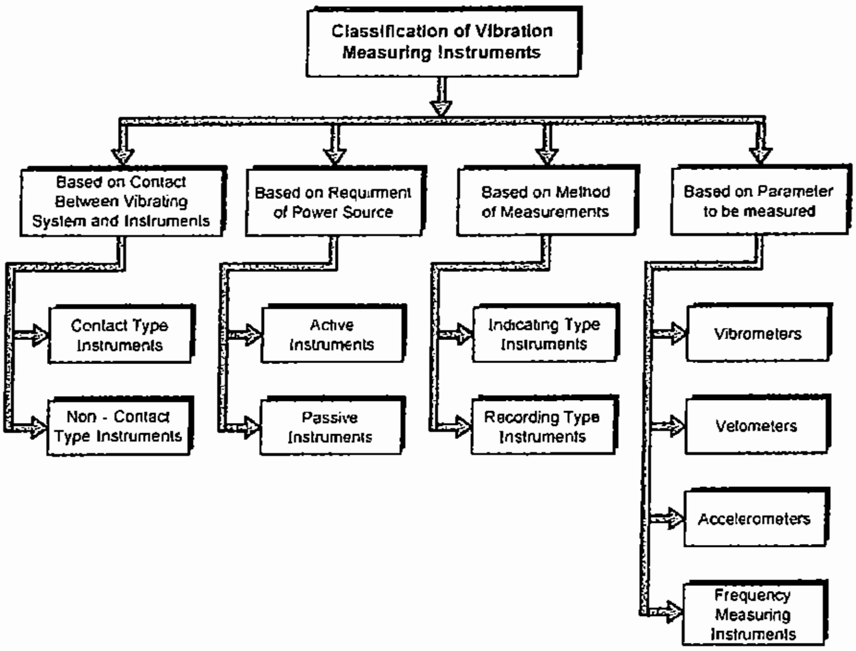


Fig. 6.2.1 : Classification of Vibration Measuring Instruments

1. Classification Based on Contact Between Vibrating System and Measuring Instrument

(i) Contacting type instruments : Generally the contacting type measuring instruments are used for a system whose characteristics remain unaffected when the instrument is connected to it.

(ii) Non-contacting type instruments : In a very light and flexible vibrating system, having a small vibratory response, if contacting type vibration measuring instruments are used, the characteristics of the vibratory system will alter. Therefore, in such cases, non-contact type measuring instruments are used.

2. Classification Based on Requirement of Power Source :

(i) Active instruments : In active measuring instruments, a separate power source is required for measuring the vibratory response of a vibratory machine or a system.

(ii) Passive instruments : In passive measuring instruments, no separate power source is required for measuring the vibratory response of a vibratory machine or a system.

3. Classification Based on Method of Measurements :

(i) Indicating type instruments : The indicating instruments indicate the reading on a dial or scale at any given instant.

(ii) Recording type instruments : The recording instruments record the readings over a period of time and store them for later use.

4. Classification Based on Parameter to be Measured :

(i) Vibrometers : Vibrometer is an instrument which measures the displacement (amplitude) of vibration.

(ii) Velometers : The velometer is an instrument which measures the velocity of vibration.

(iii) Accelerometers : The accelerometer is an instrument which measures an acceleration of vibration.

(iv) Frequency measuring instruments : The frequency measuring instruments are used for measuring the frequency of vibration.

6.3 VIBRATION MEASURING DEVICES

The various vibration measuring devices are :

1. Vibrometers
2. Velometers
3. Acclerometers
4. Frequency Measuring Instruments

6.3.1 Vibrometers (Amplitude Measuring Instruments) :

- ***Vibrometers or seismometer*** *is an instrument which measures the displacement (i.e. amplitude) of a vibrating machine or structure.*
- The various types of amplitude measuring instruments (i.e. vibrometers) are as follows :

Types of Amplitude Measuring Instruments

1. Stylus Recording Instrument
2. Seismic Instrument or Seismometer or Vibration Pickup
3. Optical Recording Instrument
4. Simple Potentiometer
5. Capacitance Pickup
6. Mutual Inductance Pickup

1. Stylus Recording Instrument :

- Fig. 6.3.1 shows a stylus recording instrument which is used to measure the amplitude of a vibrating machine or a structure.
- It consist of a drum, which is rotating about Y-Y axis and a stylus, which is pivoted at a fulcrum 'O'. To the other end of the stylus, a link is attached which pickups the vibratory motion (i.e. amplitude of vibration) from vibrating machine or structure.
- The motion between the rotating drum and linear movement of stylus, plot an amplitude of vibratory motion on paper which is attached on drum.
- From two successive amplitudes, the logarithmic decrement is obtained which further gives damping factor and damped circular frequency of vibrations.

- Though this instrument is very cheap and simple, it cannot be used for high frequency and high acceleration vibratory system. This is due to the fact that, elasticity and mass of stylus recording instrument constitutes vibration of its own. When the frequency of vibration comes closer to the natural frequency of vibration of stylus recording instrument, the resonance occurs.

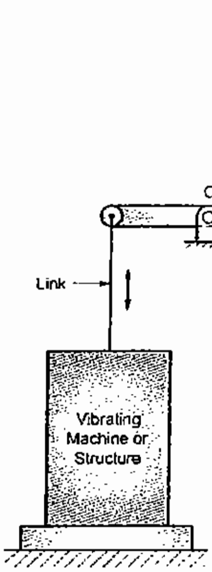


Fig. 6.3.1 : Stylus Recording Instrument

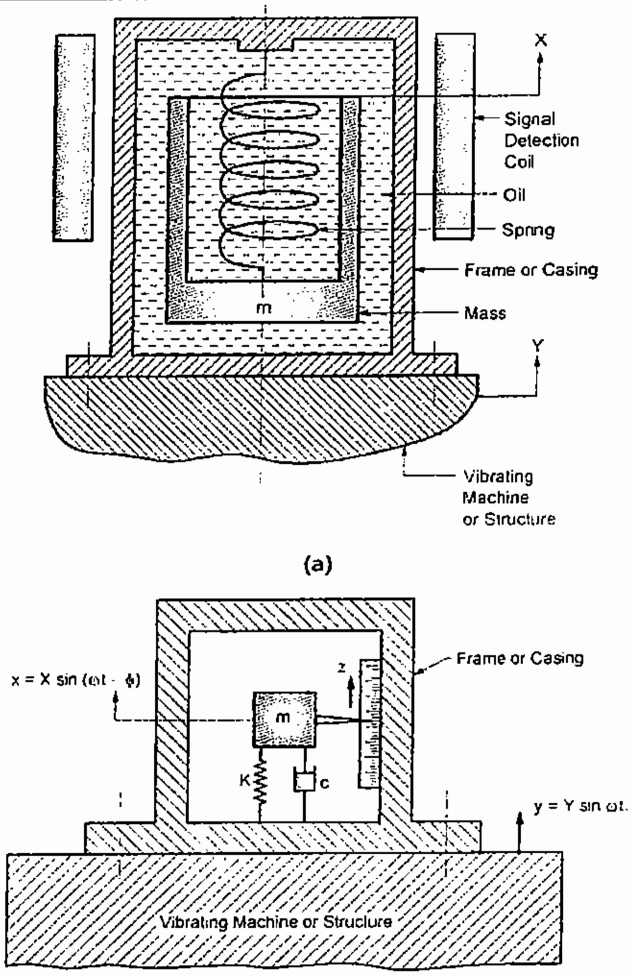
2. Seismic Instrument or Seismometer or Vibration Pick-up :

University Question

Q. Write short note on : Seismic instruments. **SPPU : Dec. 15**

Q. Explain with diagram the working principle of Seismic instrument. **SPPU : May 16, May 19**

- A seismic instrument consists of a spring-mass-damper system in a frame or casing, which is mounted (fastened) on the vibrating machine or structure to measure the displacement or amplitude of vibratory motion, as shown in Fig. 6.3.2.
- The mass 'm' is supported in a frame or casing by means of spring having stiffness 'K' and dashpot of damping coefficient 'c'.
- The seismometer, shown in Fig. 6.3.2 (a), is equivalent to a spring-mass-damper system shown in Fig. 6.3.2 (b), having base or support excitation.



(b)

Fig. 6.3.2 : Seismic Instrument or Seismometer

- The relative amplitude Z can be measured by strain sensing transducer which is rigidly fixed to the seismic mass.
- The output voltage from the transducer is proportional to the displacement y of vibrating body. Hence, such instrument is called as **vibration pick-ups** [Fig. 6.3.3]

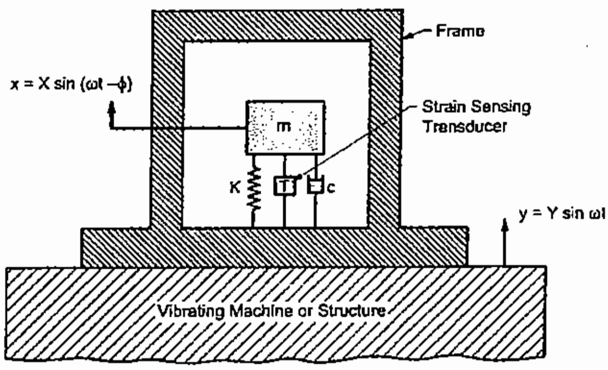


Fig. 6.3.3 : Vibration Pick-up

3. Optical Recording Instrument :

- The optical recording instrument is shown in Fig. 6.3.4.

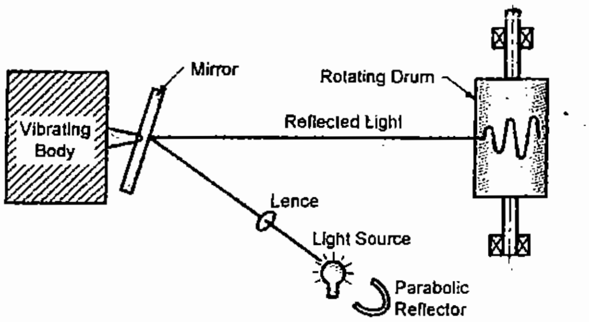


Fig. 6.3.4 : Optical Recording Instrument

- A light source sends the light signal through a lens to a mirror. The mirror is attached to a vibrating body by means of some linkage.
- The light which is reflected from the mirror falls on a sensitized film on the revolving drum and plots the displacement of vibratory motion.
- Such instruments have the advantage of wide range of frequency because of the less mass and negligible inertia.

4. Simple Potentiometer :

- A simple potentiometer is shown in Fig. 6.3.5. It consists of a voltmeter, a battery and a resistance.

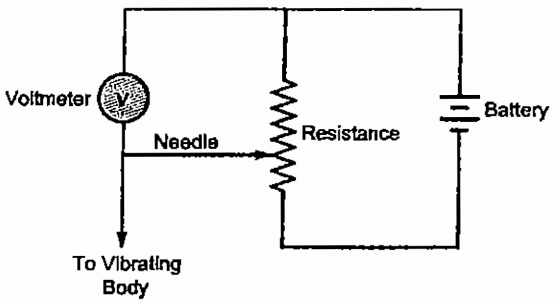


Fig. 6.3.5 : Simple Potentiometer

- A needle is connected to the vibrating body and it is allowed to slide on the resistance.
- The change in voltage due to movement of needle on the resistance is recorded. This voltage is proportional to the amplitude of vibrations.

5. Capacitance Pick-Up :

- The capacitance pick-up is a non-contacting active type vibration measuring instrument, which generates an output proportional to the displacement of the vibratory body.
- It has one plate of the capacitor attached to the vibratory body and other being the probe kept at some distance from the vibratory surface, as shown in Fig. 6.3.6.
- The change in capacitance due to variation in the air gap is utilized in an RC circuit to indicate the amount of the vibratory displacement.
- The size of the probe to be used depends upon the range of amplitude to be measured. The range of amplitude covered by such a pick-up is 0.025 to 10 mm

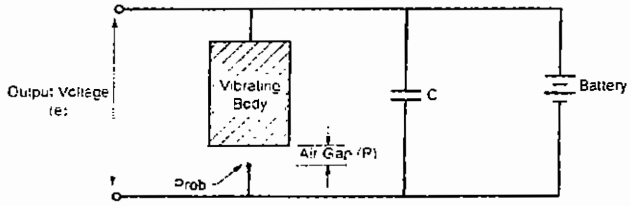


Fig. 6.3.6 : Capacitance Pick-Up

6.3.2 Velometers (Velocity Pick-Ups):

Velometer (Velocity Pick-up) :

- **Velometer or Velocity Pick-Up** *is an instrument which measures the velocity of a vibrating body.*
- We know that, for seismic instrument the steady state relative amplitude is given by,

$$z = x - y$$

or $$\dot{z} = \dot{x} - \dot{y} \quad \text{...(6.3.1)}$$

- When, the ratio (ω / ω_n) is very large or in other words, the natural frequency of the instrument 'ω_n' is very small as compare to the frequency of vibration of the body 'ω', Equation (6.3.1) becomes

$$\dot{z} \simeq -\dot{y} \quad \text{....[} \because \omega_n \text{ is small, } \dot{x} \simeq 0 \text{]}$$

$$\therefore \; \dot{z} \propto \dot{y}$$

Relative amplitude ∝ Velocity of body.

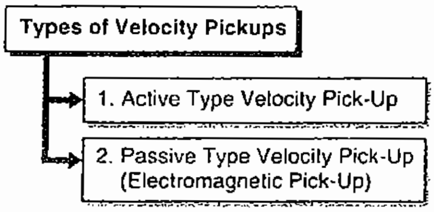
Conditions for Using Seismometer as Velocity Measuring Instrument :

The seismometer can be used as a **velocity measuring instrument** or **velocity pick-up** or **velometer**, if it satisfies the following conditions :

(i) It should have very small natural frequency (ω_n); and

(ii) It should generate the output signal proportional to the relative velocity of vibrating body.

Types of Velocity Pickups :



1. Active Type Velocity Pick-Up :

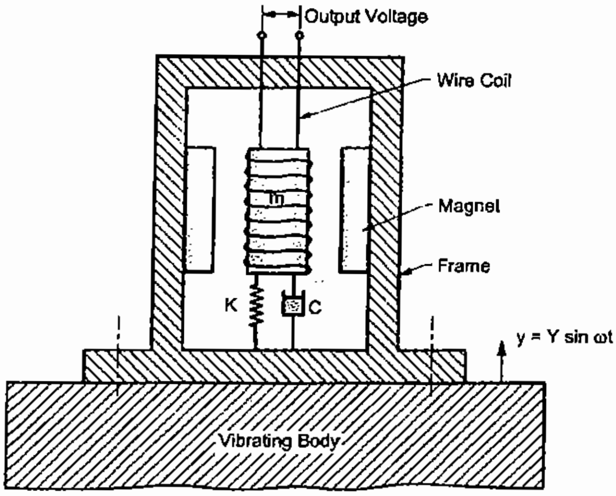


Fig. 6.3.7 : Active Type Velocity Pick-Up

- Fig. 6.3.7 shows the active type velocity pick-up which is similar to seismometer in construction, with slight modification.
- It consist of seismic mass 'm' with wire coils, supported by spring 'K' and dashpot 'c'. The magnet is fixed to the frame and frame is fitted on the vibrating body.
- Due to the relative motion between coil and magnet, a voltage is induced across the coil. The output voltage across the coil is proportional to the relative velocity. The output voltage signal is calibrated to give the velocity of vibration.
- Just like a seismometer, in velometer also the damping factor is kept about 0.7 or less than 0.7.

2. Passive Type Velocity Pick-Up (Electromagnetic Pick-Up) :

- The electromagnetic pick-up, shown in Fig. 6.3.8, is fixed at some distance from the vibrating body.

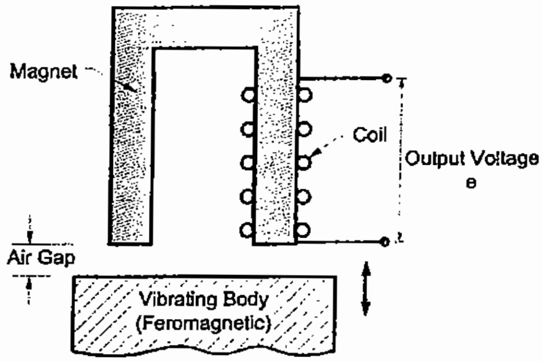


Fig. 6.3.8 : Electromagnetic Pick-up

- The change in air gap between the vibrating body and the pick-up, resulting due to vibration of the body, changes the magnetic reluctance of the path.
- As a result, the magnetic flux cutting across the coil changes. This results in inducing the voltage across the coil which is given by,

$$e = -N\left[\frac{d\phi}{dt}\right]$$

or $$e = -N \cdot \frac{d\phi}{dx} \cdot \frac{dx}{dt}$$

or $$e = -N \cdot \frac{d\phi}{dx}\dot{x} \quad ...(6.3.2)$$

Where, N=Number of coils

$\dot{x}$ = Velocity of oscillation or vibration

$\frac{d\phi}{dt}$ = Rate of change of flux

$\frac{d\phi}{dx}$ = Change of flux with respect to distance

- Now, for a given range of air gap, $d\phi / dx$ remains constant and if number of coils is fixed then,

$$e \propto \dot{x} \quad ...(6.3.3)$$

6.3.3 Accelerometers (Acceleration Pick-Ups) :

University Questions

Q. Write short note on : Piezoelectric accelerometer. **SPPU : Dec. 11, Dec. 13, May 15**

Q. Explain Piezoelectric accelerometer, with neat sketch. **SPPU : May 14**

Q. Explain the working principle of Accelerometer. **SPPU : May 19**

Accelerometer (Acceleration Pick-Up) :

- ***Acceleration pick-up*** *or* ***accelerometer*** *is an instrument that measures the acceleration of a vibrating body.*
- For seismic instrument, steady state relative amplitude is given by,

$$Z = \frac{Y(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad ...(6.3.4)$$

- When, the frequency ratio (ω/ω_n) is very small or in other words the natural frequency of instrument 'ω_n' is very high as compare to the frequency of vibration of the body 'ω', Equation (6.3.4) becomes,

$$Z \simeq Y(\omega/\omega_n)^2 \quad ...(6.3.5)$$

or $$Z \simeq Y\omega^2$$

....[$\because \omega_n$ is constant for a given instrument]

or $$Z \simeq -\ddot{y} \text{ (i.e. acceleration of the vibrating body)}$$

Relative amplitude('Z') $\propto$ acceleration of the vibrating body.

- Once the acceleration is recorded, the velocity and displacement is obtained by integrating.

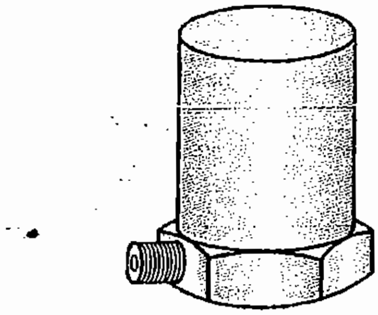
Conditions for Using Seismometer as Acceleration Measuring Instrument :

(i) It should have very high natural frequency (ω_n); and

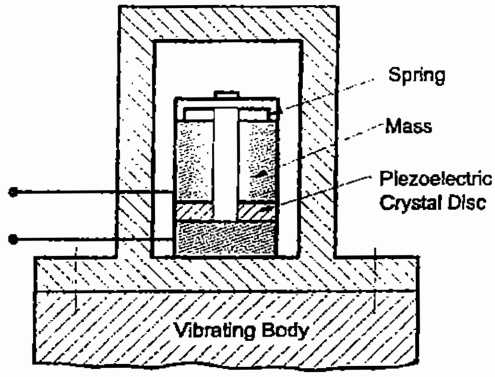
(ii) It should generate output signal proportional to the relative acceleration of the vibrating body.

Piezoelectric Accelerometer :

- The natural frequency of the instrument should be very large as compared to the frequency of vibrating body. This is possible only when the seismic mass 'm' is small and the spring has large value of stiffness 'K' is large (i.e. short spring).
- Fig. 6.3.9 shows piezoelectric crystal accelerometer, which is used to measure the acceleration of vibrating body with higher frequencies.
- However a piezoelectric crystal is capable of generating a measurable signal even for a small deformation. If the signals are very weak, amplifier is used to amplify them.



(a) Actual Accelerometer



(b) Schematic of Accelerometer

Fig. 6.3.9 : Accelerometer

6.3.4 Frequency Measuring Instruments :

University Question

Q. Write short note on : Frequency measuring instrument.

SPPU : Dec. 11, Dec. 12

- Most of the frequency measuring instruments are of the mechanical type and based on the principle of resonance.
- The following instruments are used to measure the frequency of vibrating body :

Frequency Measuring Instruments

1. Single-Reed Frequency Meter or Fullarton Tachometer
2. Multi-Reed Frequency Meter or Frahm Tachometer
3. Stroboscope

1. Single-Reed Frequency Meter or Fullarton Tachometer :

- **Single-reed frequency meter or Fullarton tachometer**, shown in Fig. 6.3.10, consists of a thin strip or reed with a mass attached at one of its ends.
- The other end of the strip or reed is pressed over the vibrating body whose natural frequency is to be measured. The free or cantilever length of strip or reed 'l' can be changed by means of screw mechanism. Therefore, each free length of strip or reed corresponds to a different natural frequency of strip. The different free lengths of the strip or reed are calibrated for the natural frequencies.

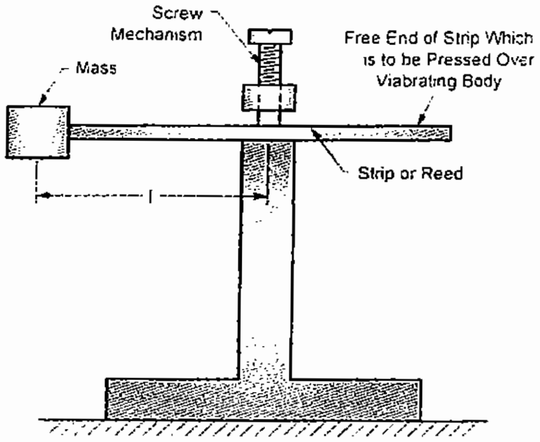


Fig. 6.3.10 : Single-Reed Frequency Meter or Fullarton Tachometer

- The screw mechanism is adjusted to change the length of strip till the amplitude of vibration becomes maximum. At that instant, the excitation frequency of vibrating body is equal to the natural frequency of strip i.e. resonance condition occurs.
- Since the reed is calibrated along its length, in terms of its natural frequency, the frequency of vibrating body is read directly from the strip.

2. Multi-Reed Frequency Meter or Frahm's Tachometer :

University Question

Q. Explain Frahm's reed tachometer, with neat sketch. **SPPU : May 13)**

- **Multi-reed frequency meter or Frahm's tachometer**, shown in Fig. 6.3.11, consists of number of strips or reeds in the form of cantilevers, carrying small masses at their free ends.

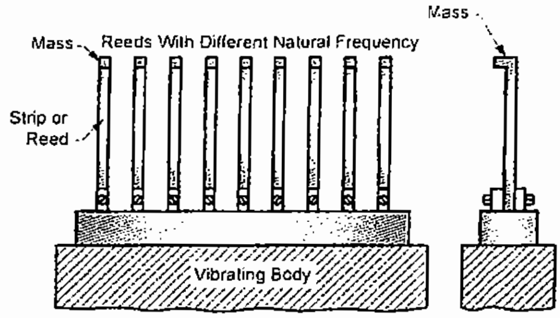


Fig. 6.3.11 : Multi-Reed Instrument or Frahm's Tachometer

- Each reed has a different natural frequency and accordingly marked on it.
- When this instrument is mounted over the vibrating body whose natural frequency is to be measured, the reed whose natural frequency is nearest to the excitation frequency of the vibrating body vibrates with maximum amplitude due to resonance condition.
- Therefore, the frequency of the vibrating body can be found from the known frequency of the vibrating reed.

3. Stroboscope :

University Question

Q. Write short notes on : Stroboscope. **SPPU : Dec. 13**

- **Stroboscope** *is an instrument which produces the light pulses at equal interval of time which can be varied.*
- The frequency of light pulses, which are produced, can be changed and read easily from control panel instrument.
- When a reference point on a vibrating body is viewed with the pulsating light, which is generated by stroboscope, it will appear to be stationary when frequency of the pulsating light is equal to the frequency of vibrating body. Therefore, frequency of vibrating body is read directly from control panel.
- The advantage of stroboscope is that it does not have any contact with vibrating body. Hence, it is non-contact type frequency measuring instrument.
- The lowest frequency upto 15 Hz can be measured easily with the help of stroboscope. Fig. 6.3.12 shows a typical stroboscope.

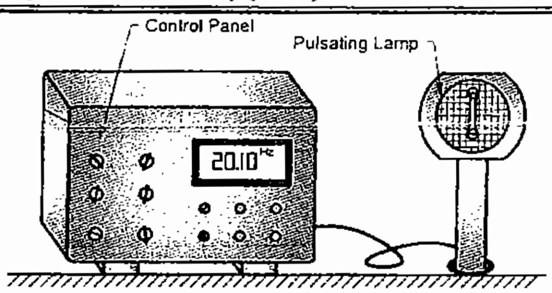


Fig. 6.3.12 : Stroboscope

6.3.5 Vibration Exciters :

University Questions

Q. Write short note on : Vibration exciter **SPPU : May 15**

Q. What are vibration exciters? **SPPU : May 19**

- ***Vibration exciters** or **shakers** are used to produce the required cyclic excitation force at a required frequency.* The cyclic excitation force produced by the exciter can be applied to the machine or structure so as to study its dynamic characteristics
- The following three types of vibration exciters are commonly used :

Types of Vibration Exciters
- 1. Mechanical Exciters
- 2. Electrodynamic Exciters
- 3. Hydraulic and Pneumatic Exciters

1. Mechanical Exciters :

(i) Scotch yoke mechanism :

University Question

Q. Explain Mechanical Exciter. **SPPU : Dec. 14**

- In mechanical exciters, a scotch yoke mechanism can be used to produce the harmonic excitation force, as shown in Fig. 6.3.13.
- In this system, the crank of mechanism is driven with constant speed or variable speed by the motor. The other end of the crank slides in a slotted rod which reciprocates in the vertical guide.
- The harmonic force that can be applied on the structure by the exciter, to which the exciter is attached, is given by,

$$F = m\omega^2 r \sin \omega t \quad ...(6.3.6)$$

Where, m = mass attached to the scotch yoke mechanism, kg

ω = crank speed, rad/s

r = radius of crank, m

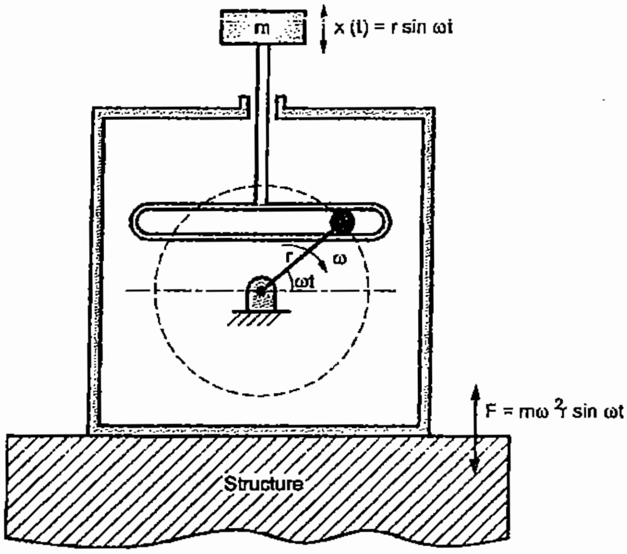


Fig. 6.3.13 : Mechanical Exciter (Scotch Yoke Mechanism)

(ii) Two rotating masses :

- The harmonic force can also be created by means of two masses rotating at the same speed but in opposite directions as shown in Fig. 6.3.14.

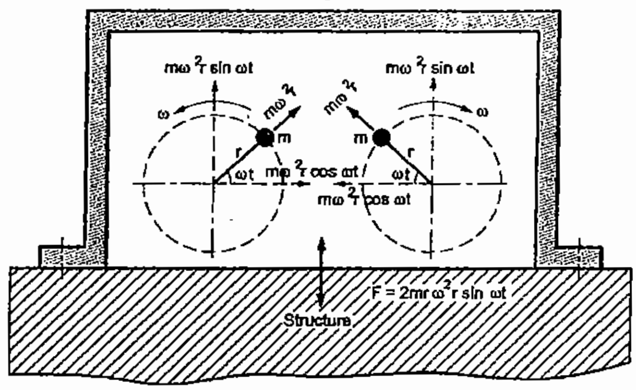


Fig. 6.3.14 : Mechanical Exciter (Exciting Force Due to Two Rotating Masses)

- When two masses of equal magnitude, rotate at an angular velocity 'ω' at radius 'r', then the vertical sinusoidal force generated is given by,

$$F = 2m\omega^2 r \sin \omega t \quad ...(6.3.7)$$

Where, m = rotating unbalanced mass, kg

ω = crank speed, rad/s ,

r = Radius of crank, m

- The horizontal components of two unbalanced forces act opposite to each other, hence the resultant horizontal force will be zero.

2. Electrodynamic Exciters :

University Question

Q. Explain Electrodynamics Exciter. **SPPU : Dec. 14**

- Fig. 6.3.15 shows an **electrodynamic exciter or shaker** which is also known as **electromagnetic exciter**. It consists of : heavy magnet, coil, moving element, flexible support and exciter table.
- When the current passes through the coil placed in magnetic field, a force is developed which is proportional to current and magnetic flux. This force accelerates the component placed on the exciter table.
- The magnitude of force developed is given by,

$$F = BIl \quad ...(6.3.8)$$

where, B = magnetic flux intensity in Wb / m^2

I = current in coil, A

l = length of coil, m

- The magnitude of accelerating force produced depends upon the maximum current, and the mass of moving element.
- If the current passing through the coil varies harmonically with time (i.e. A.C. current), the force produced also varies harmonically.
- The electrodynamic exciters are used to generate the forces upto 30 kN and displacements upto 25 mm and frequencies in the range of 5 Hz to 20 Hz.

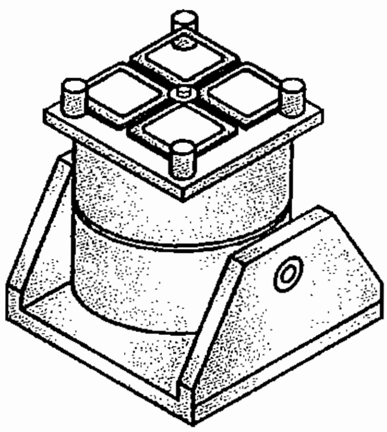
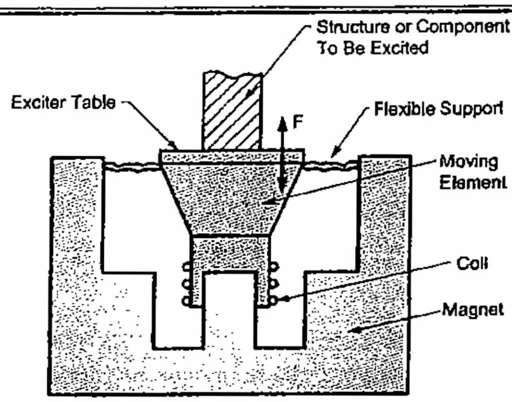


Fig. 6.3.15(a) : Actual Electrodynamic Exciter



(b) Schematic of Electrodynamic Exciter

Fig. 6.3.15: Electrodynamic Exciter

3. Hydraulic and Pneumatic Exciters :

University Question

Q. Explain any one Exciter. **SPPU : May 19**

- The mechanical and electrodynamic exciters are generally used for limited force and limited frequency range. When exciters are to be used for larger force capacity and wider frequency range, the **hydraulic** or **pneumatic exciters** are preferred.
- Fig. 6.3.16 shows the block diagram of a hydraulic or pneumatic exciter.
- In this arrangement, an electrically actuated servo valve operates a main control valve to regulate the flow of fluid (i.e. oil in case of hydraulic exciter or air in case of pneumatic exciter) to each end of the cylinder.
- The exciting force of high magnitude and frequency (upto 400 Hz) can be obtained by using such exciters.

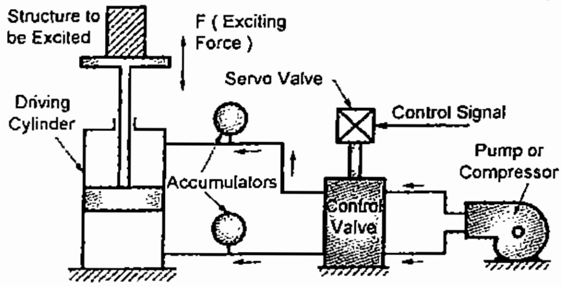


Fig. 6.3.16 : Hydraulic and Pneumatic Exciter

Ex. 6.3.1 : A seismic instrument having vibrating mass of 5 kg is supported by a spring having stiffness 1000 N/m with negligible damping. A pen is attached to the mass which draws a curve on paper which is mounted on a rotating drum. The instrument is placed on a vibrating surface vibrating at 5 rad/s having amplitude of 10 mm. Determine the amplitude of curve traced on paper if the vibrations are sinusoidal. **SPPU : Dec. 02**

Soln. :

Given : $m = 5\ kg$; $K = 1000\ N/m$;

$\xi = 0$; $\omega = 5\ rad/s$;

$Y = 10\ mm = 0.01\ m$.

- **Natural circular frequency of vibration :**

$$\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{1000}{5}} = 14.142\ rad/s.$$

- **Amplitude of curve traced on paper :**

The seismic instrument is placed on the vibrating surface. Therefore, the amplitude of curve traced on paper is the steady state relative amplitude. It is given by,

$$Z = \frac{mY\omega^2 / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

when $\xi = 0$,

$$Z = \frac{mY\omega^2 / K}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2}} = \frac{mY\omega^2 / K}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

$$= \frac{5 \times 0.01 \times (5)^2 / 1000}{1-\left(\frac{5}{14.14}\right)^2}$$

or $Z = 0.001428\ m$

$\therefore$ $\mathbf{Z = 1.4286\ mm}$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 6.3.2 : In a seismic instrument if mass $m = 0.1$ kg, stiffness of spring, $K = 1$ N/mm and damping ratio, $\xi = 0.5$, determine the amplitude of recorded motion if the motion of vibrating body is $3 \sin 200\, t$ (mm).

SPPU - May 19, 6 Marks

Ex. 6.3.3 : A vibrometer consists of a seismic mass of 1 kg, spring of stiffness 50 N/m and a damping factor of 0.7. The amplitude of displacement shown on vibrometer scale is 10 mm. If the vibrometer is mounted on a machine vibrating at 30 rad/s, determine the amplitude of vibration of a machine.

Ex. 6.3.4 : A fullarton tachometer is used to measure the frequency of vibration system. A mass of 0.02 kg is attached at the end of the reed so that its resonance is at a frequency of 50Hz. If the reed is of 50 mm long and 5 mm wide, determine the thickness of reed. Take modulus of elasticity for reed material as $200 \times 10^9\ N/m^2$.

Ex. 6.3.5 : A device used to measure tensional acceleration consist of a ring having a moment of inertia of 0.049 kg-m^2 connected to a shaft by a spiral spring having a scale of 0.98 N-m/rad and a viscous damper having a constant of 0.11 N-m sec/rad. When the shaft vibrates with a frequency of 15 cpm, the relative amplitude between the ring and the shaft is found to be 2°. What is the maximum acceleration of the shaft ?

SPPU - Dec. 10, May 19, 6 Marks

Ex. 6.3.6 : A seismic instrument is used to find the displacement, velocity and acceleration of a machine running at 250 r.p.m. If the natural frequency of the instrument is 5 Hz and it records the displacement 5 mm, determine the displacement, velocity and acceleration of the vibrating machine assuming no damping.

SPPU - May 14, 8 Marks, Dec. 18, 6 Marks

Soln. :

Given : $N = 250$ r.p.m. ; $f_n = 5$ Hz ;

$Z = 5$ mm ; $\xi = 0$.

- **Excitation frequency :**

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 250}{60} = 26.17\ rad/s.$$

- **Natural circular frequency of instrument :**

$$\omega_n = f_n \times 2\pi = 5 \times 2\pi = 31.41\ rad/s.$$

- **Steady-state relative of vibrating machine :**

$$Z = \frac{Y(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\therefore \quad Z = \frac{Y(\omega/\omega_n)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2} \quad \text{... [} \therefore \xi = 0 \text{]}$$

$$\therefore \quad 5 = \frac{Y\left(\frac{26.17}{31.41}\right)^2}{1-\left(\frac{26.17}{31.41}\right)^2}$$

$\therefore$ $\mathbf{Y = 2.20\ mm}$ ...Ans.

$\therefore$ Displacement of vibrating machine = 2.20 mm

- **Velocity of vibrating machine :**

$\dot{y} = \omega \cdot Y = 26.17 \times 2.20$

$= \mathbf{57\ mm/s}$ ...Ans.

- **Acceleration of vibrating machine :**

$\ddot{y} = \omega^2 Y = (26.17)^2 \times 2.20$

$= \mathbf{1506.7\ mm/s^2}$ ...Ans.

Ex. 6.3.7 : An undamped seismic instrument is used to find the magnitude of vibration of a machine tool structure. It gives a reading of relative displacement of 0.8 μm. The natural frequency of the instrument is 5 Hz. The machine tool structure is subjected to an excitation at a frequency of 2 Hz. Find the magnitude of the acceleration of the vibrating machine tool structure. **SPPU - May 11, 6 Marks**

Soln. :

Given : $Z = 0.8\,\mu m = 0.8 \times 10^{-3}$ m ; $f_n = 5$ Hz ; $f = 2$ Hz.

- **Natural frequency of system :**

$$\therefore \quad \omega_n = 2\pi \times f_n = 2\pi \times 5$$

$$\omega_n = 31.41 \text{ rad/s}$$

- **Frequency ratio :**

and, $\omega = 2\pi \times f = 2\pi \times 2 = 12.56$ rad/s

$$\text{But, } \frac{\omega}{\omega_n} = \frac{12.56}{31.41} = 0.399$$

- **Steady-state relative amplitude of vibration :**

$$Z = \frac{Y(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\xi\frac{\omega}{\omega_n}\right]^2}}$$

$$\therefore \quad Z = \frac{Y(\omega/\omega_n)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2} \qquad \ldots [\because \xi = 0]$$

$$\therefore \quad 0.8 \times 10^{-3} = \frac{Y(0.399)^2}{1-(0.399)^2}$$

$$Y = 4.22 \times 10^{-3} \text{ m} = 4.22 \text{ mm}$$

- **Acceleration of vibrating machine :**

$$\ddot{y} = \omega^2 Y = (12.56)^2 \times (4.22 \times 10^{-3})$$

$$\ddot{y} = \mathbf{0.665\ m/s^2} \qquad \textbf{...Ans.}$$

Ex. 6.3.8 : A commercial type vibration pick up has a natural frequency of 5.75 Hz and a damping factor of 0.65, what is lowest frequency beyond which the amplitude can be measured within 2 percent errors.

Soln. :

Given : $f_n = 5.75$ Hz ; $\xi = 0.65$;

$$\frac{Z}{Y} = 1.02 \text{ or } 0.98.$$

- **Steady-state relative amplitude of vibration :**

$$Z = \frac{Y(\omega/\omega_n)^2}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)\right]^2 + \left[2\xi\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$\therefore \quad \frac{Z}{Y} = \frac{r^2}{\sqrt{[1-r^2]^2 + [2\xi r]^2}} \qquad \ldots [\text{put } \omega/\omega_n = r]$$

From response curve of vibration measuring instrument, it is seen that the curve for $\xi = 0.65$ does not go beyond Z/Y = 1. Therefore, to get frequency for 2% error, take Z/Y = 0.98.

$$\therefore \quad 0.98 = \frac{r^2}{\sqrt{[1-r^2]^2 + [2 \times 0.65 \times r]^2}}$$

$$\therefore \quad 0.04\,r^4 + 0.31\,r^2 - 1 = 0$$

Solving above equation and taking positive value, we get,

$$\therefore \quad r = 1.55$$

$$\therefore \quad \frac{f}{f_n} = \frac{\omega}{\omega_n} = 1.55$$

$$\therefore \quad f = 1.55 \times 5.75 = \mathbf{8.90\ Hz} \qquad \textbf{...Ans.}$$

Examples for Practice

Refer our website for complete solution of following examples

Ex. 6.3.9 : A commercial type vibration pickup has a natural frequency of 6 Hz and a damping factor of 0.65. What is the lowest frequency beyond which the amplitude can be measured within : (i) one percent error, (ii) two percent error **SPPU - Dec. 15, 6 Marks**

Ex. 6.3.10 : A seismic instrument is mounted on machine at 1000 r.p.m. The natural frequency of seismic instrument is 20 rad/s. The instrument records relative amplitude of 0.5 mm. Compute displacement, velocity and acceleration of machine. Damping of seismic instrument is neglected.

Ex. 6.3.11 : The static deflection of the vibrometer mass is 20 mm. The instrument when attached to a machine vibrating with a frequency of 125 cpm, records relative amplitude of 0.03 cm. Find for the machine :

(i) the amplitude of vibration ,

(ii) the maximum velocity of vibration and ;

(iii) the maximum acceleration. **SPPU - Dec. 14, 6 Marks**

Ex. 6.3.12 : A vibration measuring device is used to find the displacement, velocity and acceleration of a machine running at 120 rpm. If the natural frequency of the instrument is 5 Hz and it shows 0.04 mm. What are the 3 readings ? Assume no damping. **SPPU - Dec. 13, 6 Marks**

Ex. 6.3.13 : A vibrometer, having the amplitude of vibration of the machine part as 4 mm and damping factor $\xi = 0.2$, performs harmonic motion. If the difference between the maximum and minimum recorded values is 10 mm, determine the natural frequency of vibrometer if the frequency of vibrating part is 12 rad/sec. **SPPU - Dec. 12, 6 Marks**

Ex. 6.3.14 : A seismic instrument with a natural frequency of 6 Hz is used to measure the vibration of a machine running at 120 rpm. The instrument gives the reading for relative displacement of the seismic mass as 0.05 mm. Determine the amplitudes of displacements, velocity and acceleration of the vibrating machine. Neglect damping.

SPPU - May 12, 6 Marks

Ex. 6.3.15 : A vibrometer has a period of free vibration of 2 sec. It is attached to a machine with a vertical harmonic frequency of 1 Hz. If the vibrometer mass has an amplitude of 2.5 mm relative to the vibrometer frame, what is the amplitude of vibration of machine.

SPPU - Dec. 11, May 13, 6 Marks

Ex. 6.3.16 : An accelerometer has a suspended mass of 0.01 kg with a damped natural frequency of vibration of 150 Hz. It is mounted on an engine running at 6000 rpm and undergoes an acceleration of 1g. The instrument records an acceleration of 9.5 m/s^2. Find the damping constant and the spring stiffness of the accelerometer.

SPPU - Dec. 16, Dec. 17, 8 Marks

Ex. 6.3.17 : Vibrations of a machine tool structure subjected to an excitation at 2 Hz is measured using a seismic instrument whose natural frequency is 5 Hz. The relative displacement shown is 0.4 μm. Determine the acceleration of the machine tool structure.

SPPU - May 17, 8 Marks

Ex. 6.3.18 : For finding vibration parameters of a machine running at 260 rpm, a seismic instrument is used. The natural frequency of the instrument is 7 Hz and the recorded displacement is 6 mm. Determine the displacement, velocity and acceleration of the vibrating machine assuming no damping.

SPPU - Dec. 16, Dec. 17, 8 Marks

Ex. 6.3.19 : It is required to measure the maximum acceleration of a machine, which vibrates violently with the frequency of 700 cycles per min. Accelerometer with negligible damping is attached to it and the indicator travels by 8.2 mm. If the accelerometer weighs 0.5 kg and has a spacing rate of 17500 N/m, what is the maximum amplitude and maximum acceleration of the part ?

SPPU - Dec. 19, 6 Marks

Ex. 6.3.20 : A vibrometer with a natural frequency of 2 Hz and with negligible damping is attached to a vibrating system which performs a harmonic excitation. Assuming the difference between the maximum and minimum recorded values are 0.6 mm. Determine the amplitude of motion of the vibrating system when its frequency is 20 Hz and 4 Hz.

SPPU - Dec. 18, Dec. 19, 6 Marks

Ex. 6.3.21 : A vibrometer has a natural frequency of 5 rad/sec and a damping factor of 0.2. An instrument is used to measure a vibrations of a body having a harmonic frequency of 45 rad/sec. The difference between the maximum and minimum reading is 7 mm. Find the amplitude of motion of vibrating body.

SPPU - Dec. 19, 6 Marks

6.4 FFT SPECTRUM ANALYZER

University Questions

Q. What is FFT analyser ? Explain how frequency spectrum can be used to detect vibration related faults in a system. **SPPU : Dec. 11, May 14**

Q. Write short note on : FFT spectrum analyzer

SPPU : Dec. 12, May 13, Dec. 13, May 15, Dec. 15, Dec. 17, May 18

Q. What is FFT? With the help of block diagram, explain the working of FFT analyzer. State the applications of FFT analyzer with reference to vibration & noise.

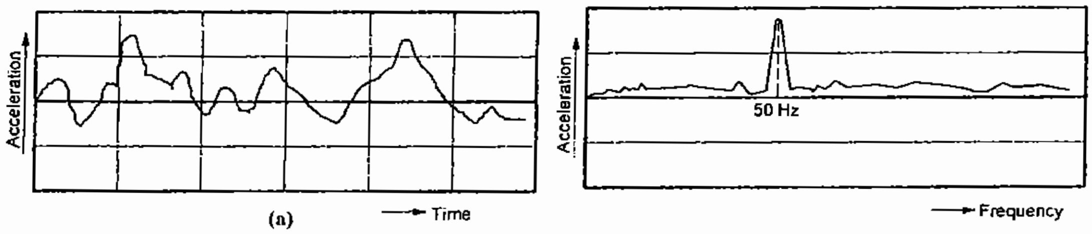
SPPU : Dec. 14

Q. Explain the working of FFT Analyzer.

SPPU : Dec. 18, Dec. 19

- **Fourier transform :** *Fourier transform is a mathematical procedure to obtain the spectrum of a given input signal.* A signal which is represented by an equation or a graph or a set of data points with time as an independent variable is transformed into another equation or graph or a set of data points where frequency is the independent variable, by using fourier transform.
- **Fast Fourier Transform (FFT) :** *The method to obtain the spectrum using a computer is called as fast fourier transform (FFT).*
- **FFT analyzer :** *The instrument which converts the input signal, with time as an independent variable, into frequency spectrum and displays it in graphical form is called as **spectrum analyzer or FFT analyzer**.*
- **Analysis using FFT Analyzer :** Consider the acceleration time plot of a machine structure that is subjected to the excessive vibrations, shown in Fig. 6.4.1(a). From this plot, it is very difficult to identify the cause of vibrations.

- If the acceleration-time plot is transformed into the acceleration-frequency plot, the resulting frequency spectrum is shown in Fig. 6.4.1(b). From frequency spectrum it can be seen that, the peak response of the system is at 50 Hz. This frequency can easily be related, for example, to the rotational speed of the motor. Thus, the frequency spectrum shows a strong evidence that, the motor might be the cause of vibrations.
- Hence, by changing the motor or by changing its speed of operation, the resonance condition can be avoided.



(a) Signals in Time Domain **(b) Signals in Frequency Domain**

Fig. 6.4.1 : Representation of Signals In Different Form

- **Working of FFT Analyzer :** Fig. 6.4.2 shows the block diagram of FFT analyzer. The various elements of FFT analyzer are as follows :

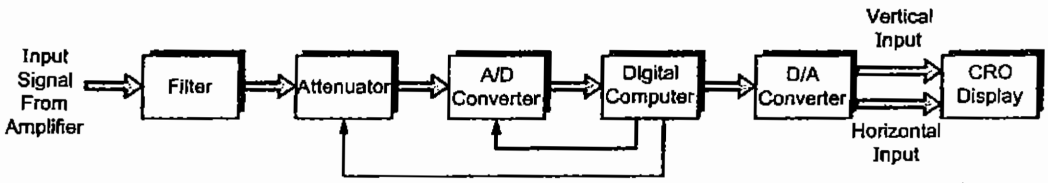


Fig. 6.4.2 : Block Diagram of FFT Analyzer

(i) Filter : The input signals are supplied through a filter. The filter is used to reject unwanted signals.

(ii) Attenuator : The next part is attenuator which sets the level of the signals to be fed to the analog to digital (A/D) converter. This is necessary to prevent the over loading of the converter.

(iii) A/D Converter : The digital computer requires the data in the digital form. Therefore, analog signals from attenuator are converted into digital signals in A/D converter.

(iv) Digital Computer : The converted data is stored in the memory of the computer. The input signals are sampled for a specific period of time called window. The sample rate, the window time and starting time are determined from the front panel controls of FFT analyzer. Once the sampling period is over and all the samples are digitised, the computer starts the calculations. The computer has a programme for the calculation and using this programme, all the spectral components are calculated and all the values are stored in the computer memory.

(v) D/A Converter : All digitised values stored in the computer memory are given to the digital to analog (D/A) converter, which converts the digital signals from computer into analog signals and send to the cathode ray oscilloscope (CRO) to display the spectrum.

(vi) CRO display : CRO displays the spectrum.

6.4.1 Types of FFT Analyzer :

- The various types of FFT analyzer, commonly used, are as follows :

Types of FFT Analyzer

1. Hand-Held FFT Analyzer
2. Computer (PC) Based FFT Analyzer
3. Single Channel FFT Analyzer
4. Two Channel FFT Analyzer
5. Multi-Channel FFT Analyzer

1. Hand - Held FFT Analyzer :

- In Hand-held type FFT analyzer, both measurement and analysis is done in same instrument and allows users to collect and store machine vibration measurement data on actual field. Such FFT analyzer is used for moderate applications only.
- A typical hand-held FFT analyzer is shown in Fig. 6.4.3.

Fig. 6.4.3 : Hand - Held FFT Analyzer.

2. Computer (PC) Based FFT Analyzer :

- Now a days, PC based FFT analyzer is commonly used, which is very compact and portable. In such FFT analyzer, only vibration measurement is done and stored in it at actual site and analysis is done with the help of software on computer off the field. Such FFT analyzer is used for wide range of applications.
- A typical PC based FFT analyzer is shown in Fig. 6.4.4.

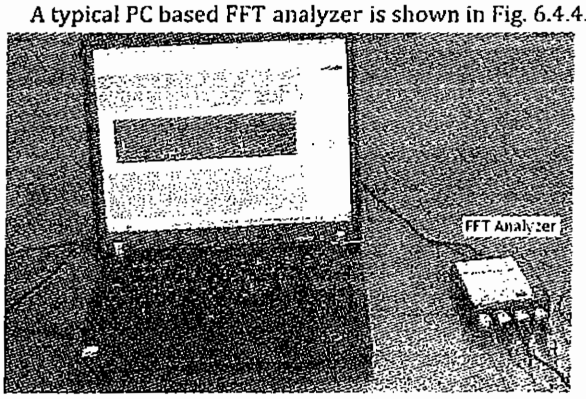


Fig. 6.4.4 : Computer (PC) Based FFT Analyzer

3. Single Channel FFT Analyzer :

- Single channel FFT analyzer is suitable for measuring continuous and transient signals (for both sound and vibration) in environmental and industrial applications. It allows user to connect either a microphone or an accelerometer so as to measure either a sound or a vibration, one at a time. In other words, single data can be measured from single-channel FFT analyzer.
- A typical single channel FFT analyzer is shown in Fig. 6.4.5.

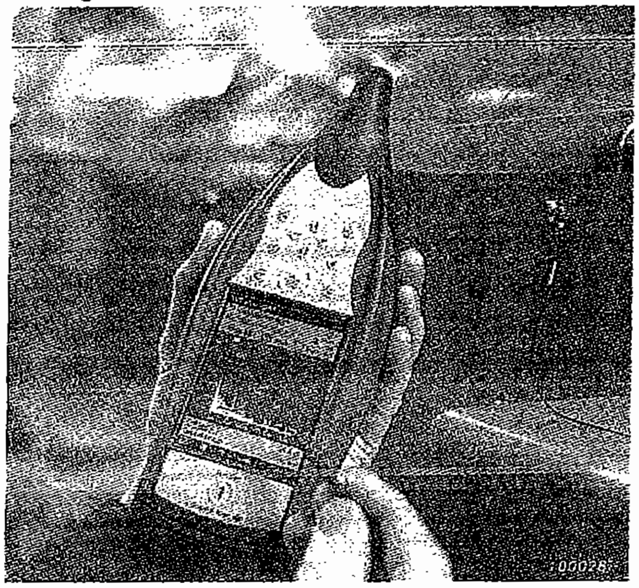


Fig. 6.4.5 : Single Channel FFT Analyzer

4. Two Channel FFT Analyzer :

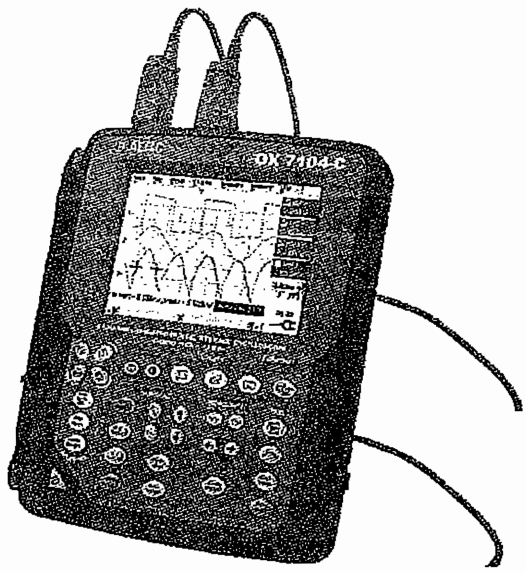


Fig. 6.4.6 : Two Channel FFT Analyzer

- Two channel FFT analyzer is used to connect a microphone as well as an accelerometer simultaneously so that both the sound and vibration can be measured at a time. Even a same type of data can be measured from two different points of machine or structure.
- A typical two channel hand-held FFT analyzer is shown in Fig. 6.4.6.

5. Multi Channel FFT Analyzer :

- When it is necessary to measure the data from more than two points from a machine or structure, multi channel FFT analyzer is used. This type of FFT analyzer is multipurpose and can be used in combination with all relevant vibration and sound transducers.
- A typical multi channel FFT analyzer is shown in Fig. 6.4.7.

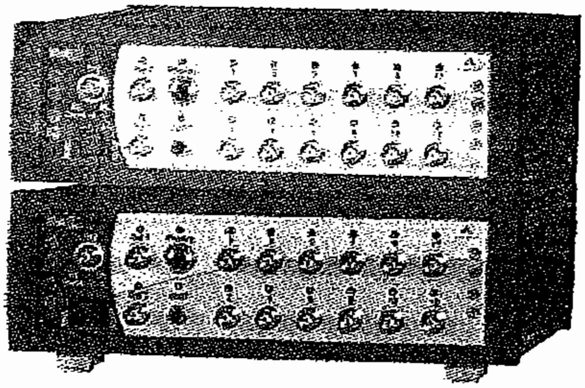


Fig. 6.4.7

6.4.2 Applications of FFT Analyzer :

1. FFT analyzer can be used for obtaining the frequency response characteristics of a vibrating structure or body.
2. FFT analyzer can also be used in experimental modal analysis or model testing for determination of natural frequency, damping ratio and mode shapes etc. through vibration testing.
3. FFT analyzer can be used for extracting the useful information from sampled signals.
4. FFT analyzer is widely used in vibration and noise monitoring systems.

6.5 VIBRATION BASED CONDITION MONITORING AND DIAGNOSIS OF MACHINES

University Question

Q. Write short note on Condition monitoring of machines

SPPU : Dec. 17, Dec. 18, Dec. 19

Condition Monitoring :

- ***Condition monitoring*** *of machine implies the determination of condition of a machine and its change with time.*
- The condition of the machines may be determined by measuring the physical parameters like : vibration, noise, temperature, wear debris, oil contamination etc. The changes in these parameters are called as signatures. The signatures indicate the change in condition or health of a machine. The analysis of signatures helps in predicting and preventing the failure of the machine.

Vibration Monitoring :

- The vibration monitoring is most commonly used for machine condition monitoring. The vibration signature of a machine are seen to be very much related to the health of a machine.
- Thus the measurement of vibration levels of machine component can provide useful information regarding the faults like : unbalance, misalignment, lack of oil, wear, etc.
- Fig. 6.5.1 shows the frequency spectrum of vibration in ball bearings for original new and old ball bearings. The increased level of vibrations and additional peaks indicate the bearing is defected.

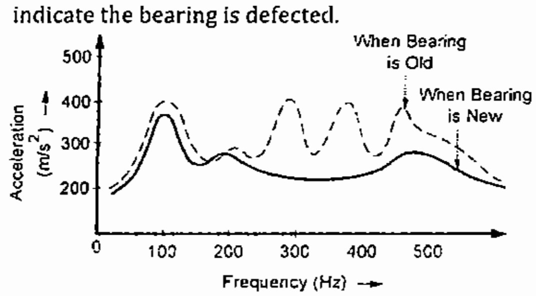


Fig. 6.5.1 : Vibration Spectrum For Ball Bearings

6.5.1 Vibration Monitoring Techniques:

University Questions

Q. What is meant by time domain and frequency domain analysis ? Explain how frequency spectrum can be used to detect vibration related faults in a system.

SPPU : May 12, May 15

Q. Write short notes on : Time domain and Frequency domain. **SPPU : May 13**

Q. Explain different techniques for Vibration Monitoring.

SPPU : Dec. 18, Dec. 19

- The vibration monitoring techniques are classified as shown in Fig. 6.5.2.

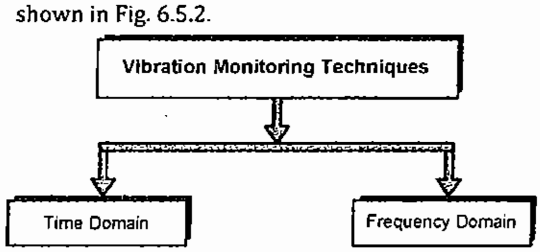
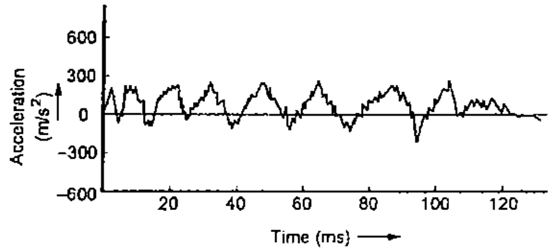


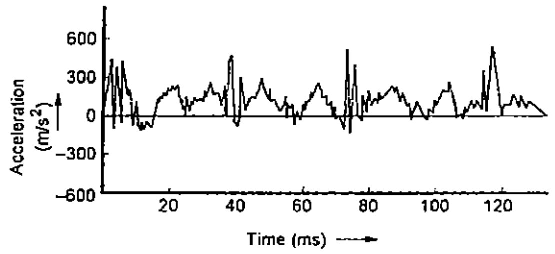
Fig. 6.5.2 : Vibration Monitoring Techniques

1. Time Domain Analysis :

- The time domain analysis uses the acceleration-time plot of the signal.
- The damages such as broken teeth in gears can be identified easily from the acceleration-time plot of the casing of a gear box.
- For example, Fig. 6.5.3 shows the acceleration-time signal of a single stage gear box.



(a) Time Domain Spectrum of a Good Gear Box

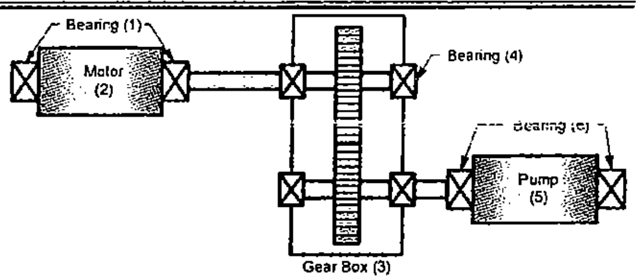


(b) Time Domain Spectrum of a Bad Gear Box

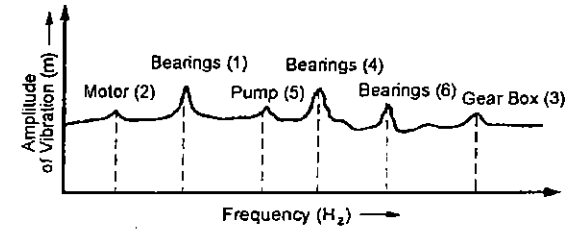
Fig. 6.5.3 : Time Domain Analysis

2. Frequency Domain Analysis :

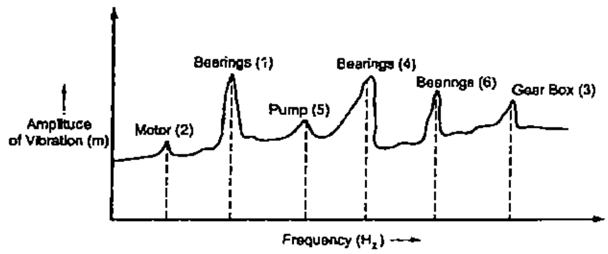
- The frequency domain analysis or frequency spectrum is the plot of amplitude of vibrations versus frequency, which is converted from time domain signals by using FFT spectrum analyzer.
- When machine starts developing faults, shape of frequency spectrum changes accordingly. By comparing frequency spectrum of machine when it is in good condition, with the frequency spectrum when it is in damaged condition, the nature and location of fault can easily be detected.
- **Advantage of frequency spectrum :** Each element in machine has identifiable frequency, therefore the change in the spectrum at given frequency can be attributed directly to that corresponding machine component, as shown in Fig. 6.5.4.
- The peaks in frequency spectrum are related to various components, as shown in Fig. 6.5.4(b).
- The analytical equations are available to find the fault frequencies of standard components like : bearings, gear boxes, motors, pumps, fans, etc.



(a) Machine Elements or Components



(b) Frequency Spectrum of Machine When it is in Good Condition



(c) Frequency Spectrum of Machine When it is in Bad Condition

Fig. 6.5.4

6.6 VIBRATION TESTS

- *The purpose of vibration tests in general is to know the system characteristics of machine or structure.* The system characteristics are : natural frequencies, the corresponding mode shapes and the damping.

6.6.1 Types of Vibration Tests :

University Question

Q. Explain the experimental setup for free and force vibration test, with instruments required.

SPPU : May 12

There are two types of vibration tests [Fig. 6.6.1] :

1. Free Vibration Tests :

- In free vibration tests, the system is subjected to free vibrations using impact hammer. The resulting free vibrations are recorded using accelerometers and FFT analyzer, from which the information regarding the natural frequency and damping can be obtained.

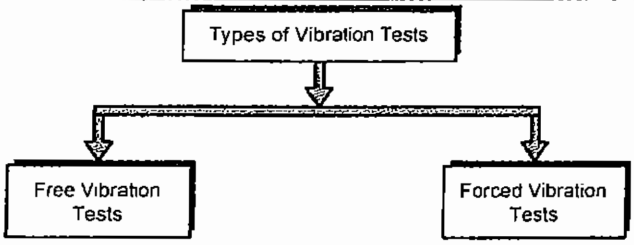


Fig. 6.6.1 : Types of Vibration Tests

- A typical experimental set up used for free vibration test is shown in Fig. 6.6.2.

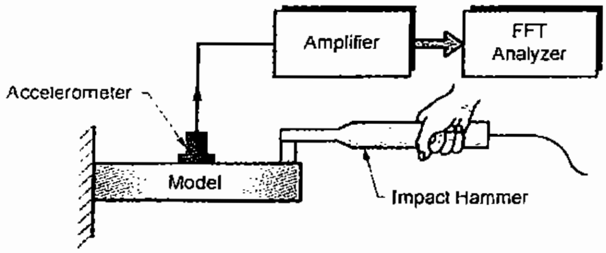


Fig. 6.6.2 : Experimental Setup for Free Vibration Test of Model

6.6.2 Forced Vibration Tests :

- In forced vibration test, the system is subjected to a known unidirectional harmonic force at a desired frequency with the help of a suitable shaker, such as an electrodynamic shaker, as shown in Fig. 6.6.3. The steady state response of the system is recorded with the help of a suitable accelerometer and vibration response is stored in FFT analyzer for further analysis.
- The frequency of excitation is varied at suitable intervals in a given range of interest and the steady state response thus obtained is plotted as a function frequency for a constant excitation force. The data thus obtained can be used to study the system performance and identify its vibration parameters.

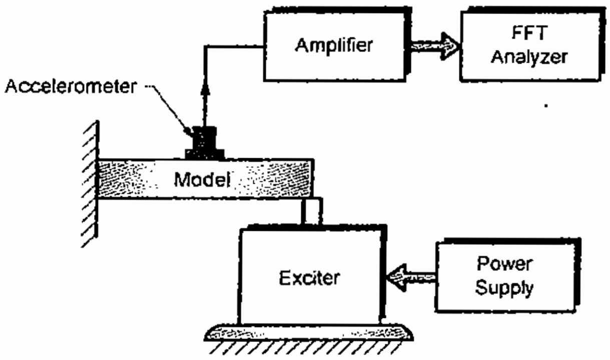


Fig. 6.6.3 : Experimental setup for forced vibration test of a model

Part II : Vibration Control

6.7 INTRODUCTION TO VIBRATION CONTROL

- Whenever the natural frequency of a machine or a structure coincides with the frequency of the external excitation, the resonance occurs which leads to the excessive deflection and failure of system. Therefore the vibrational behaviour of a structure is an important consideration in many design applications.
- **Problems arising out of uncontrolled vibrations :**
  (i) Fatigue failure of the structures/ machines;
  (ii) Excessive deflection / deformation;
  (iii) Operator and passenger discomfort in vehicles;
  (iv) Adverse effect on accuracy of process;
  (v) Earthquake damage to buildings and bridges; and
  (vi) Damage to sensitive electronic equipment.
- By vibration control the unwanted harmful vibration can be reduced to an acceptable limit.

6.8 METHODS OF VIBRATION CONTROL

University Question

Q. What are the various methods of vibration control ? Explain any one. **SPPU : May 16, Dec. 19**

- The various methods of vibration control aim at modifying the source or the system or the transmission path from source to the system. The various methods of vibration control are broadly classified, as shown in Fig. 6.8.1.

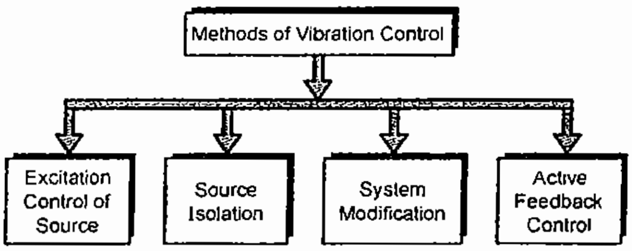


Fig. 6.8.1 : Methods of Vibration Control

- The various methods of vibration control at various phases are shown in Fig. 6.8.2.

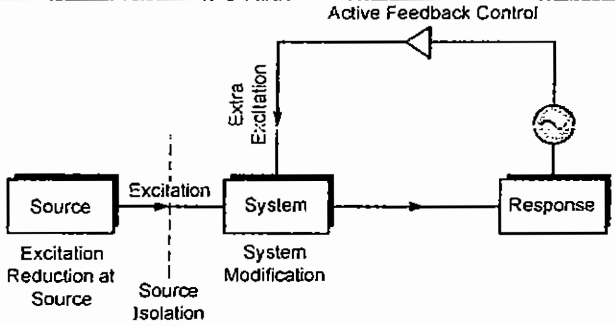


Fig. 6.8.2 : Methods of Vibration Control at Various Phases

1. Control of Excitation at Source :

The various methods to reduce the excitation level at the source are :

(i) Balancing of inertia forces.

(ii) Proper lubrication of joints.

(iii) Smoothening of fluid flows in case of flow-induced vibrations.

(iv) Modification in surface finish in case of self excited vibrations.

(v) The chatter of cutting tool due to self excited vibration may be reduced by providing coolant which reduces the friction at the tool-work piece interface.

2. Source Isolation :

- The second method to control the vibration is known as source isolation. In such method, an appropriate suspension system or an anti-vibration mountings are inserted in the path of vibration transmission from the source to the system.
- Sometimes, a machine which creates significant vibration during its normal operation may be supported upon isolators to protect other machinery and workers from shock and vibrations.

3. System Modification :

- A large number of methods exist in system modification. In this method, the system parameters namely inertia, stiffness and damping are suitably chosen or modified to reduce the vibration.
- The system parameters are effectively controlled by both geometry and material of a vibrating member.
- When designing a system, care should be taken that, its natural frequencies lie outside the excitation frequency range. This can be done by suitably adjusting the mass and stiffness parameters. This process of avoiding the resonance is called **detuning**.

4. Active Feedback Control :

- In this method, a signal generated by the response is suitably processed to produce a desired amplification of the phase change in the signal. This processed signal drives an actuator which in turn provides an additional excitation.
- This excitation is then fed back to the vibratory system which shifts the excitation frequency away from its natural frequencies, and hence, resonance is avoided. Sometime, both vibration absorber and isolators are also used to minimise the excitation level.

6.9 CONTROL OF NATURAL FREQUENCIES

- **Resonance :** When the frequency of external excitation coincides with one of the natural frequencies of the system, the resonance occurs. In resonance condition, amplitude of vibration is large which is undesirable and can lead to the failure of system.
- **Methods to avoid resonance :** In most of the cases, the excitation frequency cannot be changed or controlled, because it is imposed by the functional requirements of the system or machine. Therefore, in such cases, it is necessary to control the natural frequencies of the system to avoid the resonance condition.
- **Methods to change natural frequency :** The natural frequency of the system depends upon mass 'm' and stiffness 'K' of the system. Therefore, by changing one of these parameter, the natural frequency of system can be changed. However, in many practical cases, the mass cannot be changed easily, since its value is determined by the functional requirements of the system. For example, the mass of the flywheel on a shaft is determined by the amount of energy to be stored in one cycle. Therefore, the stiffness of the system is the parameter that is most often changed to change its natural frequencies. The stiffness of a rotating shaft can be altered by varying one or more of its parameters like : material or number and location of support points (i.e. bearings)

6.10 VIBRATION ABSORBER

University Question

Q. What do you mean by vibration absorber ? Explain its principle of operation.

SPPU : Dec. 13, May 14, Dec. 14

Vibration Absorber :

- When a structure externally excited has undesirable vibrations, this vibration can be eliminated by coupling a properly designed auxiliary spring-mass system to the main system. This auxiliary spring-mass system is known as **vibration absorber** or **dynamic vibration absorber.** After coupling the absorber to the system, the mass (machine or structure) which is excited before attaching absorber can now have zero amplitude of vibration and the spring-mass system (absorber) which is coupled to it is now vibrated freely.

Types of Vibration Absorbers :

- The various types of absorbers used are as shown in Fig. 6.10.1.

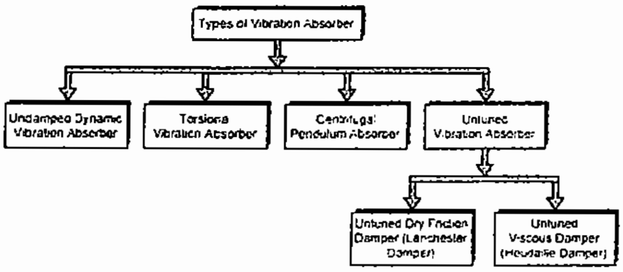


Fig. 6.10.1 : Types of Vibration Absorbers

6.10.1 Undamped Dynamic Vibration Absorber (Frahm Vibration Absorber) :

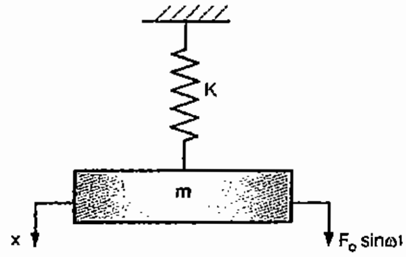
University Questions

Q. Explain undamped dynamic vibration absorber with frequency response curve. **SPPU : May 12**

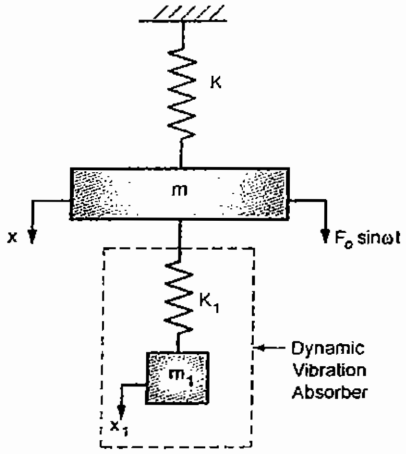
Q. Explain working principle of Frahm vibration absorber. **SPPU : May 13**

- The undamped dynamic vibration absorber is extremely effective at one speed only and thus it is suitable only for constant speed machines.
- The damped dynamic vibration absorber can be used for machines with varying speed. However, for the damped dynamic vibration absorber to be effective, they have to be operated in a very narrow range of natural frequencies.
- Consider a system shown in Fig. 6.10.2(a), having mass 'm' and stiffness 'K', which is vibrating with excitation frequency ω. When this excitation frequency ω is nearly close to the natural frequency 'ω_n' of the system, the amplitude of vibration becomes very large due to resonance.

$$\omega = \omega_n = \sqrt{\frac{K}{m}} .$$



(a) Main System



(b) Vibration Absorber Attached

Fig. 6.10.2 : Undamped Dynamic Vibration Absorber

- In such cases, the vibration of the system can be reduced by using a dynamic vibration absorber, which is simply another spring mass system, as shown in Fig. 6.10.2(b).
- The dynamic vibration absorber is designed such that the natural frequency of the resulting system is away from the excitation frequency.
- The spring mass system (m_1, K_1) acts as vibration absorber having natural frequency $\omega_{n1} = \sqrt{\frac{K_1}{m_1}}$ and reduces the amplitude of 'm' to zero, if its natural frequency is equal to the excitation frequency i.e.

$$\omega = \omega_{n1} = \sqrt{\frac{K_1}{m_1}}$$

Thus,

$$\omega = \omega_{n1} = \omega_{n1} = \frac{K}{m} = \frac{K_1}{m_1}$$

- When this condition is fulfilled, the absorber is called as **tuned absorber.** Fig. 6.10.2(a) is a single degree of freedom and after coupling the additional spring mass-system, it becomes two degrees of freedom system, as shown in Fig. 6.10.2(b).

6.10.2 Torsional Vibration Absorber :

University Question

Q. Explain torsional vibration absorber, with neat diagram. **SPPU : Dec. 11**

- A torsional vibration absorber can be used to reduce or completely eliminate torsional oscillations of a system.
- Consider a torsional vibration system having torsional stiffness K_t and mass moment of inertia J, as shown in Fig. 6.10.3(a). It is subjected to a periodic torque $T_0 \sin \omega t$. The torsional vibration absorber is represented by K_{t1} and J_1.
- The natural frequency of the main system is given by,

$$\omega_n = \sqrt{\frac{K_t}{J}} \quad \text{...(6.10.1)}$$

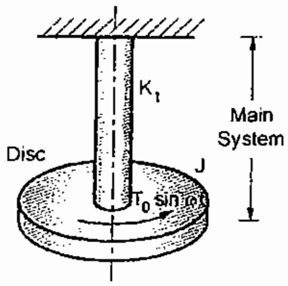


Fig. 6.10.3(a) : Main System

- When this natural frequency of the system coincides with the impressed torque frequency, resonance occur.
- The vibration of the system can be reduced by using a dynamic vibration absorber, which is simply another shaft and disk system, as shown in Fig. 6.10.3(b).

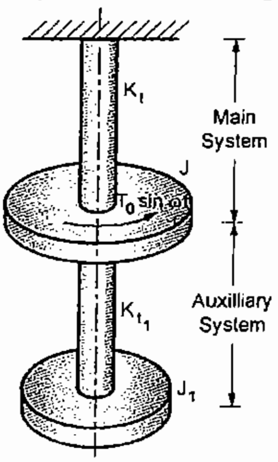


Fig. 6.10.3(b) : Torsional Absorber System

- The shaft and disk system (K_{t_1} and J_1) acts as vibration absorber having natural frequency,

$$\omega_{n1} = \omega = \sqrt{\frac{K_{t_1}}{J_1}} \quad \text{...(6.10.2)}$$

- It reduces amplitude of J to zero, if natural frequency ω_{n1} is equal to excitation frequency ω.

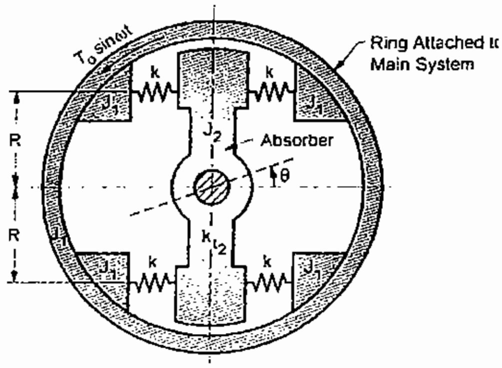


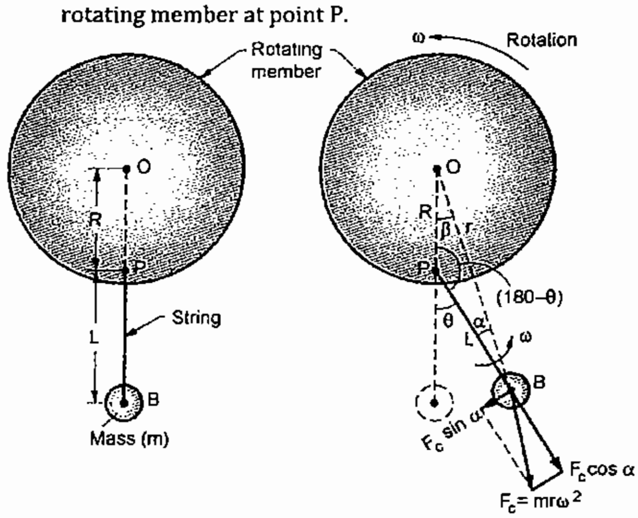
Fig. 6.10.4 : Torsional Vibration Absorber

6.10.3 Centrifugal Pendulum Absorber (Self-Tuned) :

University Question

Q. Explain with neat sketch the working principle of a centrifugal pendulum absorber. **SPPU : Dec. 12**

- The undamped dynamic torsional vibration absorber is fully effective only at a particular frequency for which it has been designed.
- The centrifugal pendulum type dynamic vibration absorber can be effectively used over a wide range of speeds of rotation, in torsional system.
- Fig. 6.10.5 shows a centrifugal pendulum absorber. A pendulum with string PB of length 'L' is attached to a rotating member at point P.



(a) When it is stationary **(b) When it is rotating**

Fig. 6.10.5 : Centrifugal Pendulum Absorber

Let, L = Length of weightless string PB, m

R = l(OP), m

r = l(OB), m

m = mass of pendulum bob, kg

ω = angular velocity of rotating body, rad/sec

θ = angular displacement of pendulum from radial line, rad

α = angle between pendulum and radial line OB.

- **Differential equation of motion for oscillation of pendulum about P :**

$$I\ddot{\theta} + F_c \sin\alpha \times L = 0 \quad \text{...(a)}$$

$$mL^2\ddot{\theta} + (mr\,\omega^2 \sin\alpha) \times L = 0 \quad \text{...(b)}$$

$[\because I = mL^2]$

Note : If mass of string is also considered then $I = \left(m + \frac{M_s}{3}\right)L^2$, where M_s = mass of string.

$$\therefore \quad \ddot{\theta} + \frac{r}{L}\,\omega^2 \sin\alpha = 0 \quad \text{...(c)}$$

From ΔOPB,
$$\frac{R}{\sin\alpha} = \frac{r}{\sin(180° - \theta)}$$

$$\text{or} \quad r\sin\alpha = R\sin\theta \quad \text{...(d)}$$

Substituting the value of $r \sin\alpha$ from Equation (d) in Equation (c), we get,

$$\ddot{\theta} + \frac{R}{L}\,\omega^2 \sin\theta = 0 \quad \text{...(e)}$$

For small values of θ; $\sin\theta \approx \theta$

Therefore, Equation (e) becomes,

$$\ddot{\theta} + \frac{R}{L}\,\omega^2 \theta = 0 \quad \text{...(f)}$$

- **Circular natural frequency :**

$$\omega_n = \sqrt{\frac{R\omega^2}{L}} = \omega\sqrt{\frac{R}{L}} \quad \text{...(g)}$$

- **Natural frequency of pendulum absorber :**

$$f_n = \frac{\omega_n}{2\pi} = \frac{\omega}{2\pi}\sqrt{\frac{R}{L}} = N\sqrt{\frac{R}{L}} \quad \text{...(h)}$$

where N = revolutions per second of rotating body = $\frac{\omega}{2\pi}$

$$\therefore \quad f_n \propto N \quad \text{...(i)}$$

- Hence, the natural frequency of the pendulum absorber is always proportional to the speed of the rotating body.
- **Order number :** The usual torsional system receives a certain number of disturbing torques per revolution. The number of these disturbing torques per revolution is known as **order number** of the system. A two cylinder engine working on a four stroke cycle has one disturbing torque per revolution and its order number is one.
- A four cylinder engine working on a four stroke cycle has two disturbing torques per revolution and its order number is two.
- A pendulum absorber is designed to eliminate or reduce the torsional disturbances of a particular order number. If several order numbers are present in a system, several pendulum absorbers are required.
- **Application of centrifugal pendulum vibration absorber :** It is used in I.C. engines, where the frequency of torque contains the harmonics of speed 'ω'.
- **Design of centrifugal pendulum vibration absorber :**

(i) For the pendulum absorber to be effective, its natural frequency 'f_n' should be equal to the excitation frequency 'f' or frequency of disturbing torque.

(ii) Torque on I.C. engines, $T = T_0 \sin(n\,\omega\,t)$.

(iii) Natural frequency of pendulum absorber,

$$f_n = N\sqrt{\frac{R}{L}}, \text{ Hz} \quad \text{...(j)}$$

(iv) Excitation frequency, $f = \frac{n\omega}{2\pi}$, Hz ...(k)

$$f_n = f \quad \text{...}(l)$$

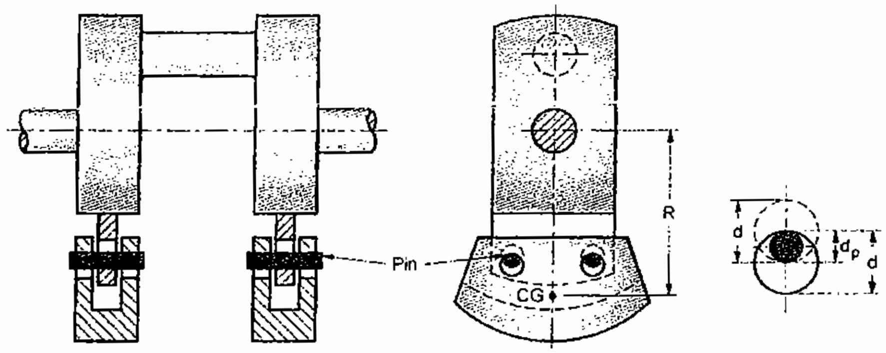
Substituting equations (j) and (k) in Equation (l),

$$N\sqrt{\frac{R}{L}} = \frac{n\omega}{2\pi}$$

$$\therefore \quad \frac{\omega}{2\pi}\cdot\sqrt{\frac{R}{L}} = \frac{n\omega}{2\pi} \quad \text{...(m)}$$

$$\therefore \quad \mathbf{n} = \sqrt{\frac{R}{L}} = \textbf{Order Number} \quad \text{...(n)}$$

- The size of the pendulum mass is a function of the magnitude of disturbing torque. For a certain disturbing torque amplitude, larger the mass of the pendulum, smaller is its amplitude of vibration. Thus the pendulum mass is made as large as possible so that it will then absorb the greatest amount of energy with the minimum amplitude and the amplitudes of vibrations are kept small.
- When used on an I.C. engine, the pendulum is usually attached to a crank web, so that R is usually about equal to the crank throw '**r**'.
- In bifilar type of centrifugal pendulum absorber, the pendulum weight is divided into two equal parts, as shown in Fig. 6.10.6.



(a) Front View (b) Side View (c) Enlarged View of Hole and Pin

Fig. 6.10.6 : Bifilar Type Centrifugal Pendulum Absorber

6.10.4 Untuned Vibration Absorbers :

There are two types of untuned vibration absorbers :

1. Untuned Dry Friction Damper (Lanchester Damper)
2. Untuned Viscous Damper (Houdaille Damper)

1. Untuned Dry Friction Damper (Lanchester Damper) :

- The unturned dry friction damper consists of two flywheels mounted freely over a hub, as shown in Fig. 6.10.7. The hub is rigidly fixed to the shaft undergoing vibrations. The friction plates are attached to the extension of the hub. These friction plates apply pressure on the flywheels and are responsible for the driving of the flywheels. The pressure between the friction plates and the flywheels can be adjusted through the spring loaded bolts which hold both the flywheels together.

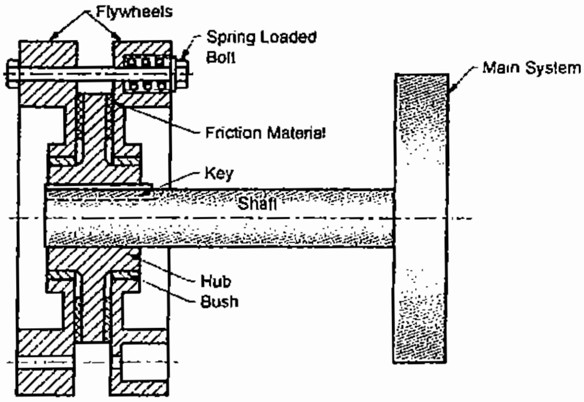


Fig. 6.10.7 : Dry Friction Torsional Vibration Absorber (Lanchester Damper)

Energy Dissipated α Frictional torque × Relative velocity

- The reduction of amplitude of vibration will be more, if greater amount of energy is dissipated.
- The untuned dry friction damper is very advantageous to use for torsional vibrations near resonance conditions. It effectively reduces the amplitudes of torsional vibrations near resonance conditions.

2. Untuned Viscous Damper (Houdaille Damper) :

- In the untuned viscous damper (Houdaille damper), instead of using friction plates, the system uses a viscous damping, as shown in Fig. 6.10.8.

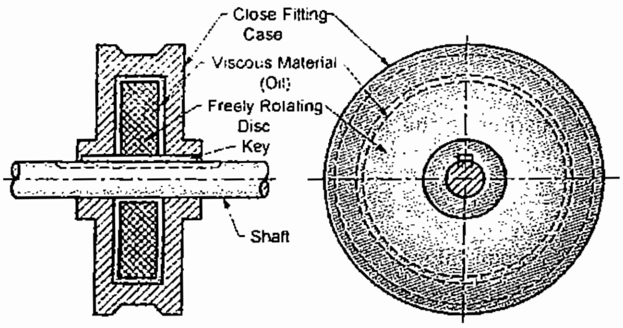


Fig. 6.10.8 : Untuned Viscous Damper (Houdaille Damper)

- It consists of a freely rotating disc enclosed in a close-fitting case which is keyed to the shaft. Normally, the disc rotates at the shaft speed owing to viscous drag of the oil between the disc and the case. However, if the shaft vibrates torsionally, viscous action of the oil between the disc and casing gives a damping action.
- When the main system rotates at such a speed that torsional vibrations are produced, the energy is dissipated due to the viscous drag of the viscous material filled between the disc and the casing.
- Thus the energy of the main system is reduced thereby reducing the amplitude of torsional vibrations of the main system.

Two special cases of unturned viscous damper :

1. When damping is zero ($\xi = 0$) : The system becomes ineffective and corresponds to a single degree of freedom.
2. When damping is infinite ($\varepsilon = \infty$) : The damper mass becomes integral with the shaft or the main mass.

6.11 VIBRATION ISOLATION

University Questions

Q. What is vibration isolation ? Discuss the various methods of vibration isolation. **SPPU : May 12, Dec. 13**

Q. Explain the term : Vibration Isolation . **SPPU : Dec. 12, May 15, May 16, May 18**

Q. Write a short note on : Vibration Isolation . **SPPU : May 13, Dec. 15**

Q. Write short note on : Different types of vibration isolators. **SPPU : May 14**

Q. What do you mean by vibration isolation? Explain any 4 isolating materials along with their industrial applications. **SPPU : Dec. 14**

- **Vibration isolation :** Vibrations are produced in machines having unbalanced masses or forces. These vibrations are transmitted to the foundation upon which the machines are mounted, which is undesirable.
- Therefore, it is essential to isolate the machines from foundations so that the adjoining structure is not set into heavy vibrations. This process of isolating the machines from the foundations is known as **vibration isolation.**
- **Objectives of vibration isolation :**

The basic objectives of vibration isolation are as follows :

(i) To protect the delicate machine (e.g. measuring instruments) from excessive vibrations transmitted to it from its supporting structure.

(ii) To prevent vibratory forces generated by machine from being transmitted to its supporting structure.

- **Force isolation and motion isolation :** The effectiveness of isolation may be measured in terms of the force or motion transmitted to that in existence. Accordingly it is known as **force isolation or motion isolation.** The lesser the force or motion transmitted the greater is the isolation.

6.12 METHODS OF VIBRATION ISOLATION

The various methods of vibration isolations are classified into three categories, as shown in Fig. 6.12.1.

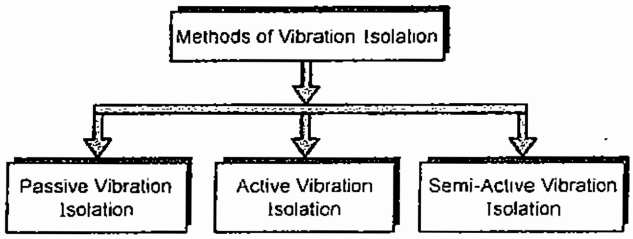
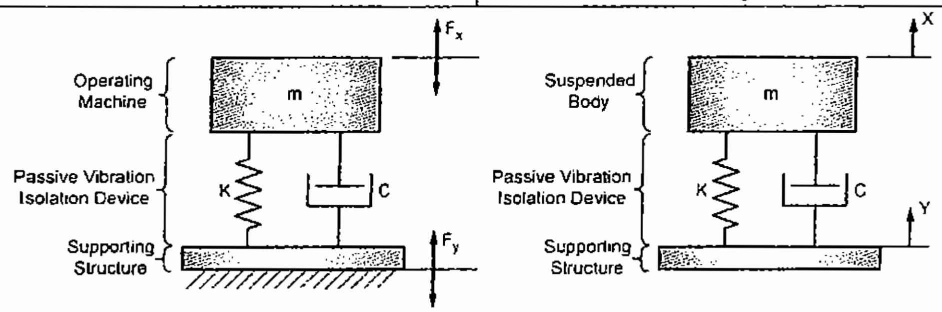


Fig. 6.12.1 : Methods of Vibration Isolation

6.13 PASSIVE VIBRATION ISOLATION

- **Passive vibration isolation : Passive vibration** isolation system does not require external power to perform its function.
- In passive vibration isolation, the isolation is achieved by reducing the amount of transmitted forces or displacements transmitted using the passive vibration isolation devices like : rubber, felt, cork and springs etc., as shown in Fig. 6.13.1.



(a) Model showing excitation force 'F_x' on mass 'm' supported by spring 'K' and damper 'c'

(b) Model Showing Base Displacement 'Y' and Response 'X' of Mass 'm' Supported by Spring 'K' and Damper 'c'

Fig. 6.13.1 : Passive Vibration Isolation

- **Transmissibility (T_r) :** *Transmissibility is defined as the ratio of the force transmitted to the supporting structure to that force impressed upon the system (i.e. F_y / F_x) or the ratio of displacement amplitude of the mass to the displacement amplitude of the supporting structure* (i.e. X / Y). It is given by,

  Force transmissibility,

$$T_r = \frac{F_y}{F_x}$$

  Motion transmissibility,

$$T_r = \frac{X}{Y}$$

$$T_r = \frac{\sqrt{1+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left[2\xi\frac{\omega}{\omega_n}\right]^2}} \quad ...(6.13.1)$$

- Fig. 6.13.2 shows the plot of transmissibility (T_r) versus frequency ratio $\left(\frac{\omega}{\omega_n}\right)$ for passive vibration isolation systems is the curves are plotted using Equation (6.13.1), as shown in Fig. 6.13.2.

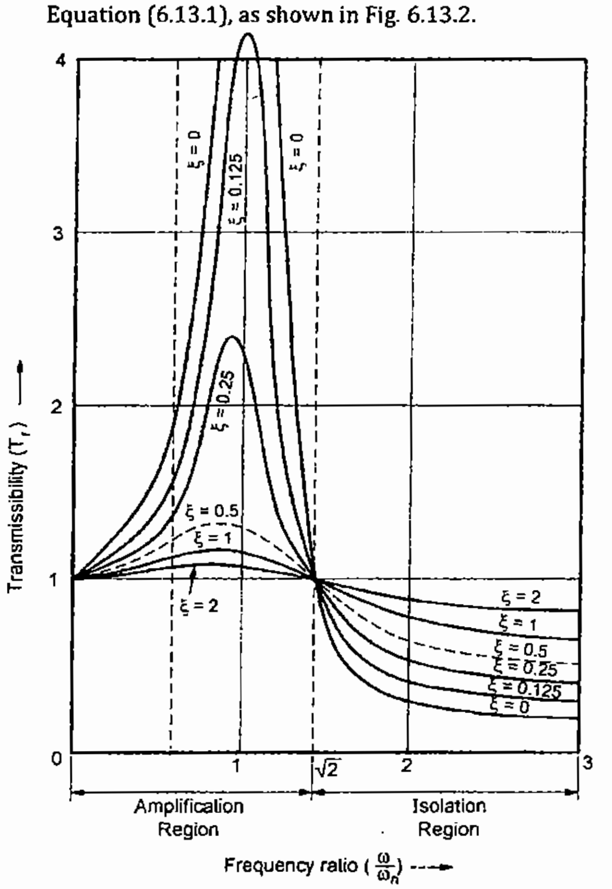


Fig. 6.13.2 : Transmissibility Verses Frequency Ratio of Passive Vibration Isolation System

- **Two regions of plot :**

(i) Isolation region
(ii) Amplification region

(i) Isolation region : It is the region with $\frac{\omega}{\omega_n} > \sqrt{2}$. In this region : (a) Transmissibility 'T_r' is less than one. (b) Increasing ξ reduces effectiveness of isolation.

(ii) Amplification region : It is the region with $\frac{\omega}{\omega_n} > \sqrt{2}$. In this region : (a) Transmissibility 'T_r' is greater than one. (b) increasing ξ increases effectiveness of isolation.

6.13.1 Types of Passive Vibration Isolators :

- The passive vibration isolation is obtained by placing isolating materials, called passive vibration isolators, in between the vibrating body and the supporting foundation or structure.
- All the isolating materials are elastic and posses damping properties.
- The various passive vibration isolators are :

(i) Springs
(ii) Elastomers or Rubbers
(iii) Ribbed Elastomers
(iv) Pneumatic Isolators
(v) Other materials

(i) Springs :

- The steel coil springs are most commonly used as passive vibration isolators. They are highly efficient and durable mechanical vibration absorbers. Springs are capable of providing natural frequencies down to about 8 Hz. The steel springs are virtually undamped.
- **Disadvantage of springs :** At high frequencies, vibration can travel along the wire of the coil, causing transmission into the structure. This is can be overcomed by incorporating a elastomer pad in the spring assembly so that there is no metal-to-metal contact. Many commercially available springs contain such a pad.

- Fig. 6.13.3 shows the typical passive vibration isolation of machine with spring as isolators.

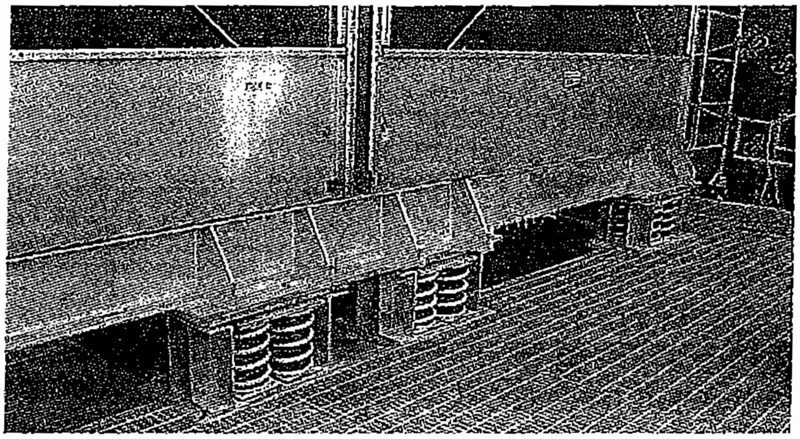


Fig. 6.13.3 : Passive Vibration Isolators (Springs)

(ii) Elastomers or Rubbers :

- The elastomers such as natural rubber, and neoprene isolators are probably the most versatile of all isolators. They can be used either in compression or shear.
- The mounts of this type can be moulded in a variety of shapes and sizes to give the desired stiffness characteristics in both vertical and lateral directions. They can be used for static deflections up to about 10 or 12 mm. They are capable of providing natural frequencies down to about 5 Hz. Fig. 6.13.4 shows the passive vibration isolation using rubbers.

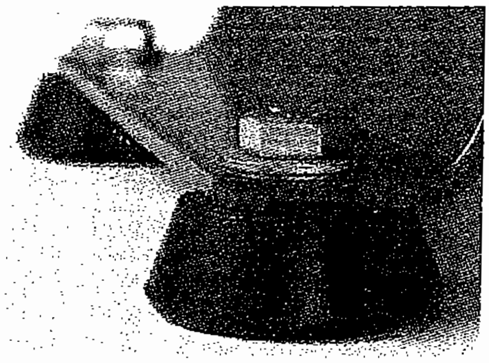


Fig. 6.13.4 : Elastomers or Rubbers

- **Disadvantages of elastomers** : They may creep excessively over a period of time. For this reason, they should not be subjected to continuous strains exceeding 10% in compression or 25% in shear.

(iii) Ribbed Elastomers :

- For static deflections of 6 mm or less, ribbed rubber, rubber-and-cork sandwich pads may be used [Fig. 6.13.5].
- However, it is not safe to depend upon them to provide natural frequencies below 10 Hz.
- They are inexpensive forms of resilient pads for machines of high speed with negligible imbalance.

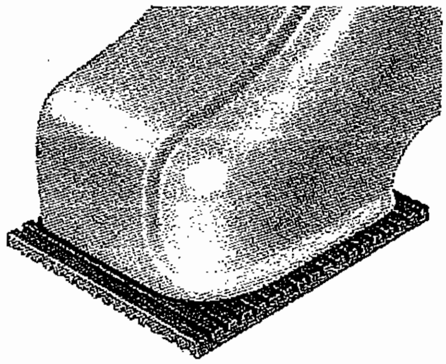


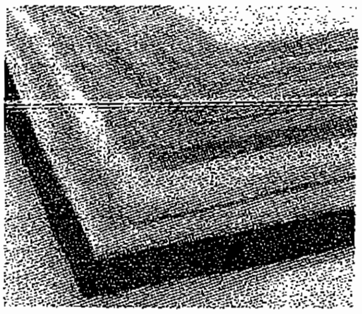
Fig. 6.13.5 : Ribbed Elastomer Isolation Pads

(iv) Pneumatic Isolators :

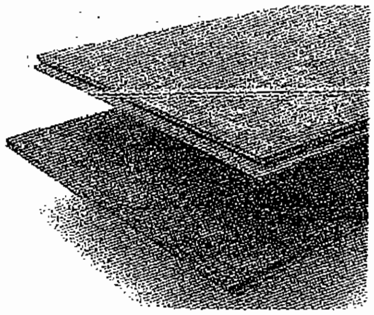
- The pneumatic isolators or air springs or air mounts are useful when low driving frequencies (e.g. below 10 Hz) are present.
- **Advantages of pneumatic isolators** : Excellent lateral stability, adequate internal damping, self-leveling by air volume adjustment and shock protection.
- **Disadvantages of pneumatic isolators** : High cost, limited load carrying capacity, necessity of periodic inspection and require a source of clean gas with pressures from 4 to 8 bar.

(v) Other materials :

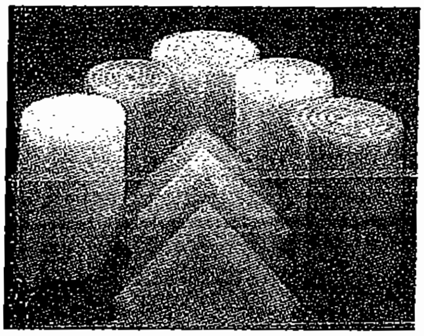
Several other materials, such as : wool felt, cork, glass fibers, foam, wire mesh, etc., shown in Fig. 6.13.6, are used as vibration isolators. These are often used in pad or blanket form. However, their properties are not as well documented as the spring, elastomer and pneumatic isolators.



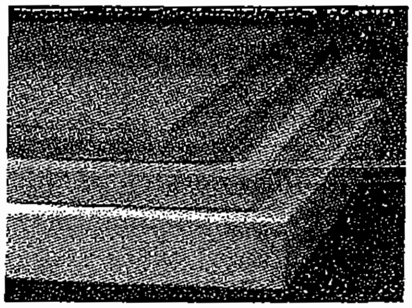
(a) Wool Felt



(b) Cork



(c) Glass Fibers



(d) Foams

Fig. 6.13.6 : Vibration Isolators Such as : Wool Felt, Cork, Glass Fibers and Foams etc.

6.14 ACTIVE VIBRATION ISOLATION

University Questions

Q. Explain in detail Active vibration control.

SPPU : May 18

- **Active vibration isolations :** Active vibration isolation system is requires an external power source to perform its function.
- A typical active vibration isolation system, shown in Fig. 6.14.1, uses sensors, controller and actuators to achieve the vibration isolation. The sensor is used to detect the vibrations to be controlled, the controller is used to interpret the vibrations detected by the sensor and to execute commands on the actuators, where as the actuators are used to reposition the mass.
- **Advantages of passive vibration isolation :**

(i) It can adapt to different vibrating bodies. If can be programmed to a targeted specifications and vibrating body.

(ii) It is more effective to suppress low-frequency vibrations (< 70 Hz).

(iii) Typical actuators, sensors, and controller used in active vibration isolation can provide quick isolation response within milliseconds.

(iv) The repositioning accuracy can be controlled within 10 μm.

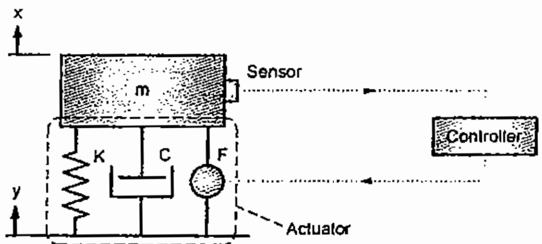


Fig. 6.14.1 : Schematic Diagram of Typical Active Vibration Isolation System

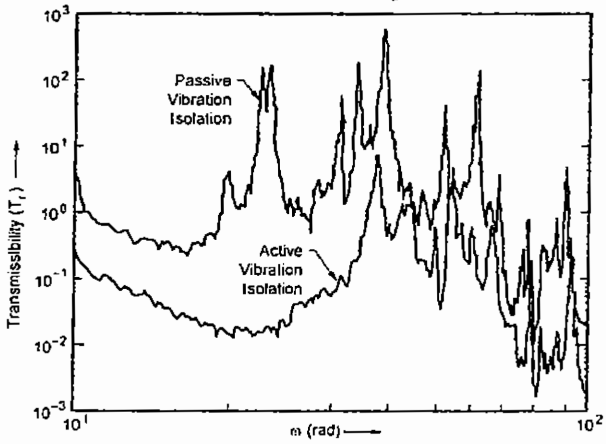


Fig. 6.14.2 : Comparison Between Transmissibility of Passive and Active Vibration Isolations

- The comparison between the transmissibility (T_r) of passive isolation system and active isolation system is shown in Fig. 6.14.2.

6.15 SEMI-ACTIVE VIBRATION ISOLATION (ELECTRO-RHEOLOGICAL AND MAGNETO-RHEOLOGICAL FLUID BASED DAMPERS)

University Question

Q. Explain working of Magneto - Rheological dampers with neat sketch and application.

SPPU : Dec. 18, May.19, Dec. 19

- Such types of dampers are used in semi-active vehicle suspensions which are monitored through sensor and adjust according to road conditions. Also such dampers are used in heavy industry to damp out the vibrations of heavy motors.

Electro-Rheological Fluids (ER Fluids) :

- *ER Fluids are smart fluids which change their viscosity by application of electric field.*
- ER fluids are suspensions of non conducting fine particles of size less than 50 μm in an electrically insulating carrier fluid. When an electric field is applied, its viscosity increases.
- The viscosity of ER fluid is proportional to the intensity of electric field. As the intensity of electric field increases, its viscosity increases and as the intensity of electric field decreases, its viscosity decreases.
- ER fluid is a smart fluid which changes from liquid to gel to viscoelastic solid.
- The ER fluid exhibits a change in viscosity upon application of external electric field due to polarization of suspended particles.

Magneto-Rheological Fluids (MR Fluids) :

- ***MR fluids** are smart fluids which change their viscosity by application of magnetic field.*
- MR fluid is a type of fluid, when subjected to magnetic field, the fluid greatly increases its viscosity. The viscosity depends upon the strength of magnetic field.
- The fluid basically consists of particles that are held in suspension by a non-conducting fluid. The suspension or carrier fluid is hydrocarbon or silicone oil. The particles dispersed in this fluid are commonly metal oxides, alumino silicates, silica, organics or polymers. In particular, the particles are very small and have sufficiently low concentration to allow the fluid to maintain a relatively low viscosity in the absence of an applied electric field.

Use of ER and MR Fluids in Vibration Isolation :

- A typical use of ER fluid for vibration isolation is shown in Fig. 6.15.1.

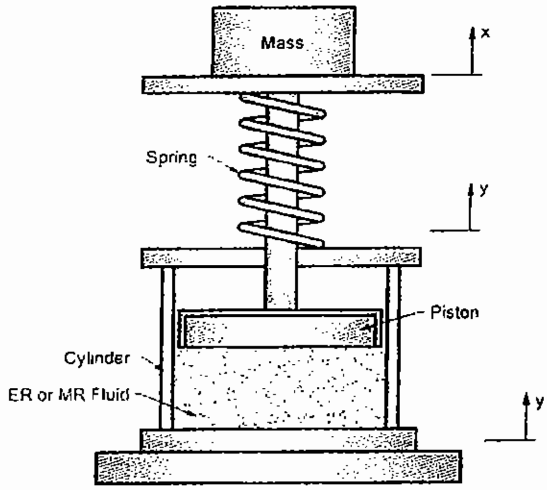


Fig. 6.15.1 : Isolator with ER fluid

- The vibration isolator consists of a parallel combination of a helical spring and a piston-cylinder with ER fluid or MR fluid. It has been observed that with suitable combination of particle concentration, carrier fluid and the strength of the electric/magnetic field using high voltage amplifier, the damping characteristics of the dashpot can be varied over a wide range. At high field strength even Coulomb damping characteristic also achieved.
- ERF can be used for both passive and active vibration isolation. In case of passive isolation, a fixed field could be applied to achieve Coulomb damping characteristics.
- However, for optimal use over a broad frequency range, one can activity control the electric field to the ERF mount for vibration control.
- Such types of dampers are used in semi-active vehicle suspensions which are monitored through sensor and adjust according to road conditions. Also such dampers are used in heavy industry to damp out the vibrations of heavy motors.

6.16 REFERENCE STANDARDS FOR VIBRATION MONITORING AND ANALYSIS

- The various standards used in vibration monitoring and analysis are :

  1. ISO Standards for Vibration Monitoring and Analysis
  2. ISO Standards for Vibration Measurements
  3. ISO Standards for Evaluation of Vibration Severity
  4. ISO Standards for Training and Certification
  5. Other standards for vibration Monitoring and Analysis

- Some of the important ISO standards for vibrations are explained below :

6.16.1 ISO Standards for Vibration Monitoring and Analysis :

University Question

Q. Explain ISO standards used in vibration.

SPPU : Dec. 18

- In the field of machinery vibration monitoring and analysis, a variety of relevant standards are developed and published by ISO (International Organization for Standardization). ISO is a worldwide federation of national standards bodies from 145 countries, and considers itself a bridge between the public and private sectors.
- The scope of standardization in the field of mechanical vibration and shock, monitoring and analysis of machines include
  (i) terminology;
  (ii) excitation by sources, such as machines and vibration / shock testing devices;
  (iii) elimination, reduction and control of vibration and shock, especially by balancing, isolation and damping;
  (iv) measurement and evaluation of human exposure to vibration and shock;
  (v) methods and means of measurement and calibration;
  (vi) methods of testing;
  (vii) methods of measurement, handling and processing of the data required to perform condition monitoring and diagnostics of machines.

6.16.2 ISO Standards for Vibration Measurements :

- **ISO 13373-1** : 2001 Condition monitoring and diagnostics of machines - Vibration condition monitoring - Part 1 : General procedures provides general guidelines for the measurement of machinery vibration for condition monitoring. Recommendations are provided for the following :
  (i) Measurement methods and parameters
  (ii) Transducer selection, location, and attachment
  (iii) Data collection
  (iv) Machine operating conditions
  (v) Vibration monitoring systems
  (vi) Signal conditioning systems
  (vii) Interfaces with data processing systems
  (viii) Continuous and periodic monitoring
- Due to the wide variety of approaches to condition monitoring, specific topics will be addressed in more detail in additional parts of 13373.
- **ISO 17359 : 2003 Condition monitoring and diagnostics of machines - General guidelines** sets out guidelines for the general procedures to be considered when setting up a condition monitoring program.

Part III : Noise

6.17 INTRODUCTION TO SOUND

Sound :

Sound *is described as a propagating disturbance through a physical or elastic medium (like air, water or gas). It is created by a vibrating object or body.*

Theory of Sound Wave :

- The sound propagates as longitudinal waves in which the particles of the medium oscillate parallel to the direction of propagation. A wave motion transfers the energy and momentum from one point to other point without involving any physical transfer of matter.
- Generally a sound wave is created by a vibrating object. Whenever a portion of a deformable medium is disturbed from its normal position by a vibrating object, the particles transmitting the wave oscillates back and forth about its equilibrium position in the direction of propagation of the sound wave. This leads to production of alternating regions of compression (region of crowding) and rarefaction (region of scarcity) in the particles of the medium, as shown in Fig. 6.17.1.

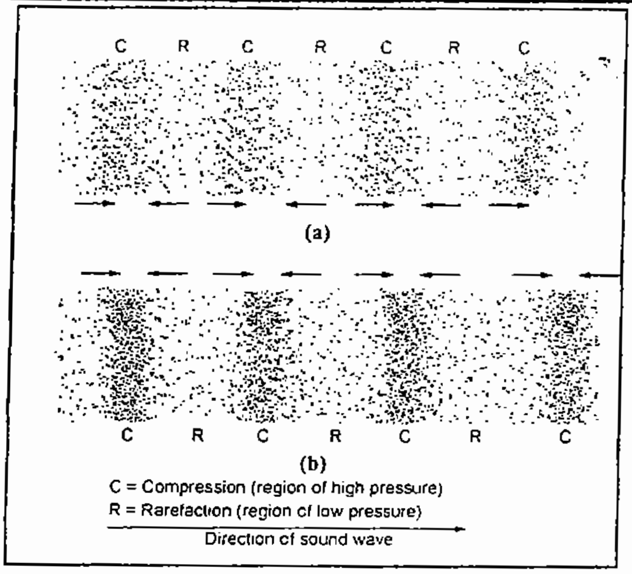


Fig. 6.17.1 : Sound Wave Propagation

- **Compression and rarefaction** : The particles crowded together represent areas of compression in which the air pressure is slightly greater than the atmospheric pressure. The spread area represent rarefaction in which the pressure is slightly less than atmospheric pressure. The sound wave causes the air particles to be pressed together in some regions and spread out in other regions.

6.18 CHARACTERISTICS OF SOUND WAVE

Pure Tone :

Pure tone *is a sinusoidal pressure wave of a specific frequency and amplitude, propagating at a velocity determined by the temperature and pressure of the medium (air or water).*

Generation of Sound Wave :

- Consider a duct with constant cross-sectional area, as shown in Fig. 6.18.1(a), with reciprocating piston at the left hand side which produces the sound wave and it propagates towards right hand side end along the longitudinal axis.
- The sound wave is detected by microphone at the right end and expressed as oscillatory electric output signal from audio measurement instrument. Fig. 6.18.1(b) shows the instantaneous pressure distribution in duct and Fig. 6.18.1(c) shows the pressure variation with respect to time, detected by the microphone.

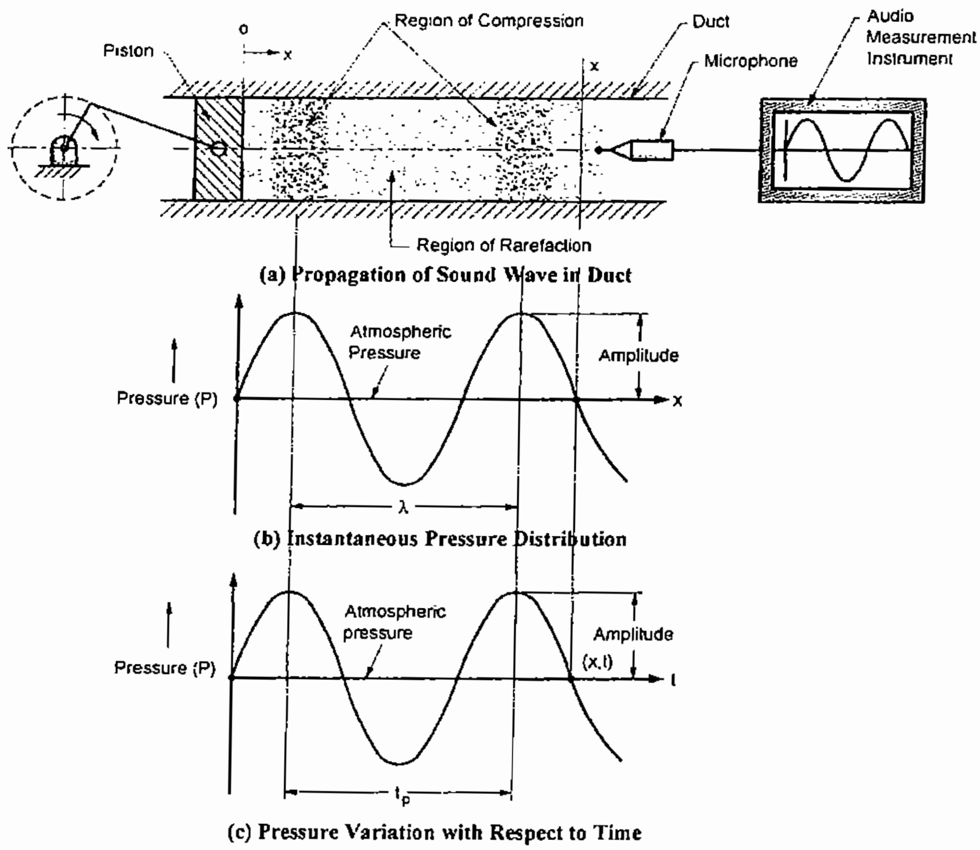


Fig. 6.18.1

Parameters to Define Sound Waves :

- Sound can be described in terms of four variables :

1.	Amplitude	2.	Frequency
3.	Time period	4.	Velocity of sound
5.	Wavelength		

1. Amplitude (a) :

- Amplitude of the sound wave is the maximum variation of the pressure above the atmospheric pressure.
- The loudness of a sound depends upon the amplitude of the fluctuations of a sound wave, above and below the atmospheric pressure.

2. Frequency (f) :

University Question

Q. Explain the term : Frequency range of sound sources.

SPPU : Dec. 15

- The frequency of sound wave is equal to the number of times per second that disturbance passes through both its positive and negative peaks. The unit of frequency is Hertz (Hz) or cycles per second (cps).
- **Pitch** : *Frequency of simple pure tone sound wave is recognized as the* ***pitch*** *of the tone.*
- The human ear responds to a range of frequencies from approximately 20 Hz to 16,000 Hz.
- **Frequency range** : The frequency range of a sound source is the difference between the maximum frequency and minimum frequency of sound generated.

3. Time period (t_p) :

Time period *of the sinusoidal wave is the time interval required for one complete cycle, as depicted in Fig. 6.18.1(c).*

The time period is given by,

$$t_p = \frac{1}{f}\text{, second} \quad \text{...(6.18.1)}$$

4. Velocity of sound (c) :

University Question

Q. Explain the term : Velocity of sound.

SPPU : Dec. 16, Dec. 17

- ***Velocity of sound*** *is identical to the velocity of wave propagation (c).*
- **Velocity or speed of sound in air :**

$$c = \sqrt{\frac{\gamma p_a}{\rho}}\text{, m/sec.} \quad \text{...(6.18.2)}$$

where, $\gamma = \dfrac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}}$

$= 1.4$ for air

p_a = ambient or equilibrium pressure, N/m^2

ρ = ambient or equilibrium density of the medium, kg/m^3

5. Wavelength (λ) :

University Question

Q. Explain the term : Wavelength.

SPPU : Dec. 16, Dec. 17

Wavelength *is the distance between two successive peaks in the waveform [Fig. 6.18.1(b)].*

$$\lambda = \frac{c}{f} \text{ or } ct_p\text{, meter} \quad \text{...(6.18.3)}$$

where, c = velocity of sound, m/s

f = frequency of sound wave, Hz

Ex. 6.18.1 : Determine the approximate wavelength of 3.5 kHz sound wave propagating at room temperature in : (i) Water, (ii) Glass, (iii) Lead, and (iv) Steel.

Soln. :

Given : f = 3.5 kHz = 3500 Hz.

At room temperature the velocity of sound in water, glass, lead and steel are :

(i) $c_{(water)} = 1372$ m /s ; (ii) $c_{(glass)} = 3658$ m/s

(iii) $c_{(lead)} = 1219$ m/s ; (iv) $c_{(steel)} = 5182$ m/s

(i) Wavelength in water :

$$\lambda_{(water)} = \frac{c_{(water)}}{f} = \frac{1372}{3500} = \mathbf{0.392\ m} \quad \text{...Ans.}$$

(ii) Wavelength in glass :

$$\lambda_{(glass)} = \frac{c_{(glass)}}{f} = \frac{3658}{3500} = \mathbf{1.045\ m} \quad \text{...Ans.}$$

(iii) Wavelength in lead :

$$\lambda_{(lead)} = \frac{c_{(lead)}}{f} = \frac{1219}{3500} = \mathbf{0.3482\ m} \quad \text{...Ans.}$$

(iv) Wavelength in steel :

$$\lambda_{(steel)} = \frac{c_{(steel)}}{f} = \frac{5182}{3500} = \mathbf{1.480\ m} \quad \text{...Ans.}$$

6.19 MEASUREMENT OF SOUND - DECIBEL SCALE

University Question

Q. Explain the term : Decibel scale.

SPPU : Dec. 13, Dec. 14, Dec. 15, Dec. 16, May 17, Dec. 17

Need of Decibel Scale :

The range of audible sound pressure to which human ear is likely to be subjected is between 2×10^{-5} N/m^2 to 200 N/m^2. Because of the wide range of sound pressure, it is convenient to describe the sound level through the use of logarithmic scales known as **decibel scale**. The use of logarithmic scale helps in covering the entire sound level range by a small scale of numbers rather than the extremely large scale of numbers.

Decibel :

Decibel is the ten times the logarithm to the base 10 of the ratio of the quantity of sound measured to an arbitrarily chosen reference quantity.

$$\text{Decibel level} = 10 \log_{10}\left(\frac{\text{quantity measured}}{\text{reference quantity}}\right), \text{dB} \quad ...(6.19.1)$$

Parameters Used for Measurement of Sound :

The various parameters used for measurement of sound by using decibel scale are as follows :

Parameters used for measurement of sound

1. Sound Pressure Level (L_p)
2. Sound Power Level (L_W)
3. Sound (Acoustic) Intensity (L_I)

1. Sound Pressure Level (L_p) :

University Question

Q. Explain the term : Sound pressure level.

SPPU : Dec. 15, Dec. 16, May 17, Dec. 17, May 19

- ***Sound pressure level (SPL)*** *is the most common decibel scale used for measurement of sound and is measured directly on sound level meter.*
- The sound pressure level is expressed in decibels as,

$$L_p = 10 \log_{10}\left(\frac{p_{rms}^2}{p_{ref}^2}\right), \text{dB} \quad ...(6.19.2)$$

$$\text{or} \quad L_p = 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)^2, \text{dB} \quad ...(6.19.3)$$

$$\text{or} \quad L_p = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right), \text{dB} \quad ...(6.19.4)$$

where, p_{rms} = root mean square (rms) sound pressure in given source, N/m^2

p_{ref} = reference sound pressure(usually 2×10^{-5} N/m^2)

2. Sound Power Level (L_W) :

University Question

Q. Explain the term : Sound power level.

SPPU : Dec. 13, Dec. 15, Dec. 16, Dec. 17, May 19

- **Sound power level** *describes the acoustical power radiated by a given source with respect to the international reference sound power of 10^{-12} W.*
- The sound power level is given by,

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right), \text{dB} \quad ...(6.19.5)$$

where, W = Sound power of the given source ;

W_{ref} = 10^{-12} W (reference sound power)

3. Sound (Acoustic) Intensity (I) :

University Questions

Q. Explain the term : Acoustic intensity.

SPPU : Dec. 11, May 12

Q. Explain the term : Sound intensity.

SPPU : Dec. 12, Dec. 13, Dec. 15, May 17

- **Sound intensity (I) :** *Sound intensity at a given point in a sound field, in a given direction is defined as the average sound power, W passing through a unit area perpendicular to the given direction at that point.*

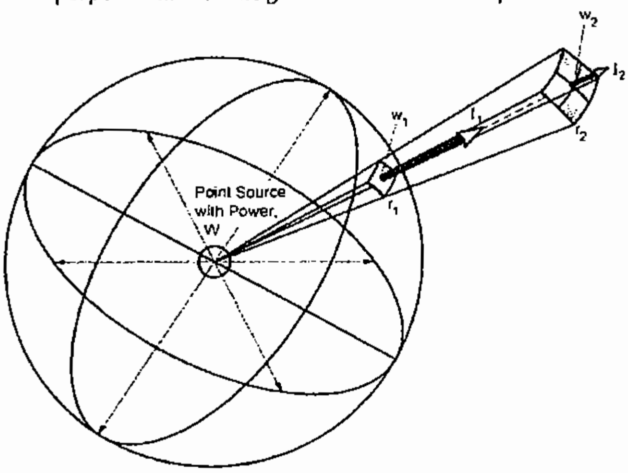


Fig. 6.19.1 : Basic Parameters of Sound

- The sound intensity is given by,

$$\therefore \quad I = \frac{W}{S}, \text{W/m}^2$$

where, W = sound power, W

S = area perpendicular to sound wave, m^2

$$\text{Again,} \quad I = \frac{p_{rms}^2}{\rho c}, \text{W/m}^2 \quad ...(6.19.6)$$

where, p = root mean square (rms) sound pressure, N/m^2;

ρ = density of the medium, kg/m^3, and

c = velocity of sound in medium, m/s

4. Sound Intensity level (L_I) :

University Question

Q. Explain the term : Sound Intensity level.

SPPU : Dec. 12, Dec. 15

$$L_I = 10 \log_{10}\left(\frac{I}{I_{ref}}\right), \text{ dB} \quad ...(6.19.7)$$

where, I = Sound intensity, W/m^2

I_{ref} = reference sound intensity (usually 10^{-12} W/m^2)

Ex. 6.19.1 : Determine the sound power level of a source that generate sound power of :

(i) 0.5 W ; (ii) 1.5 W ;

(iii) 2.2 W; and (iv) 3 W of sound power.

SPPU - Dec. 16, Dec. 17, 8 Marks

Soln. :

$$W_{ref} = 10^{-12} \text{ W}$$

(i) Sound power level at 0.5 W :

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right) = 10 \log_{10}\left(\frac{0.5}{10^{-12}}\right)$$

$$= 10 \log_{10}(5 \times 10^{11})$$

or L_W = **116.98 dB** ...Ans.

(ii) Sound power level at 1.5 W :

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right) = 10 \log_{10}\left(\frac{1.5}{10^{-12}}\right)$$

$$= 10 \log_{10}(1.5 \times 10^{12})$$

or L_W = **121.76 dB** ...Ans.

(iii) Sound power level at 2.2 W :

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right) = 10 \log_{10}\left(\frac{2.2}{10^{-12}}\right)$$

$$= 10 \log_{10}(2.2 \times 10^{12})$$

or L_W = **123.42 dB** ...Ans.

(iv) Sound power level at 3 W :

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right) = 10 \log_{10}\left(\frac{3}{10^{-12}}\right)$$

$$= 10 \log_{10}(3 \times 10^{12})$$

or L_W = **124.77 dB** ...Ans.

Ex. 6.19.2 : Determine the sound power of a machine whose specified sound power level is 125 dB.

Soln. :

Given : $L_W = 125$ dB ; $W_{ref} = 10^{-12}$ W (reference).

The sound power level is,

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right)$$

$$125 = 10 \log_{10}\left(\frac{W}{10^{-12}}\right)$$

$$\frac{125}{10} = \log_{10}\left(\frac{W}{10^{-12}}\right)$$

$$3.16 \times 10^{-12} = \frac{W}{10^{-12}}$$

$\therefore$ **W = 3.16 W** ...Ans.

Ex. 6.19.3 : Determine the sound pressure level for a following rms sound pressure :

(i) 0.33 N/m^2 ; (ii) 2.97 N/m^2 ;

(iii) 1.5 N/m^2 ; and (iv) 2 N/m^2.

Soln. : $p_{ref} = 2 \times 10^{-5}$ N/m^2 (reference)

(i) Sound pressure level at 0.33 N/m^2 :

$$L_p = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right) = 20 \log_{10}\left(\frac{0.33}{2 \times 10^{-5}}\right)$$

$$= 20 \log_{10}(16500)$$

or L_p = **84.34 dB** ...Ans.

(ii) Sound pressure level at 2.97 N/m^2 :

$$L_p = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right) = 20 \log_{10}\left(\frac{2.97}{2 \times 10^{-5}}\right)$$

$$= 20 \log_{10}(148500)$$

or L_p = **103.43 dB** ...Ans.

(iii) Sound pressure level at 1.5 N/m^2 :

$$L_p = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right) = 20 \log_{10}\left(\frac{1.5}{2 \times 10^{-5}}\right)$$

$$= 20 \log_{10}(75000)$$

or L_p = **97.50 dB** ...Ans.

(iv) Sound pressure level at 2 N/m^2 :

$$L_p = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right) = 20 \log_{10}\left(\frac{2}{2 \times 10^{-5}}\right)$$

$$= 20 \log_{10}(100 \times 10^3)$$

or $L_p = 100$ dB ...Ans.

Ex. 6.19.4 : Determine the maximum pressure of a noise with a sound pressure level of 112 dB.

SPPU - Dec. 12, 4 Marks

Soln. :

Given : $L_p = 112$ dB ;

$p_{ref} = 2 \times 10^{-5}$ N/m² (reference).

- **rms pressure of noise :**

$$L_p = 20 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)$$

$$112 = 20 \log_{10}\left(\frac{p_{rms}}{2 \times 10^{-5}}\right)$$

$$\frac{112}{20} = \log_{10}\left(\frac{p_{rms}}{2 \times 10^{-5}}\right)$$

$$10^{5.6} = \frac{p_{rms}}{2 \times 10^{-5}}$$

$$\therefore \quad p_{rms} = 7.96 \text{ N/m}^2 \quad \text{...Ans.}$$

- **Maximum pressure of noise :**

$$p = \sqrt{2} \times p_{rms}$$

$$p = \sqrt{2} \times 7.96 = 11.25 \text{ dB} \quad \text{...Ans.}$$

Examples for Practice

Refer our website for complete solution of following examples

Ex. 6.19.5 : Determine the sound pressure level for a sound with rms sound pressure of 2 N/m² and 0.4 N/m².

SPPU - May 17, 6 Marks

Ex. 6.19.6 : Determine different levels

(i) Sound pressure level, if rms sound pressure is 1 Pa,

(ii) Sound intensity level, if sound intensity is 1 W / m² and

iii) Sound Power level of a source generating 1 W of Sound Power. **SPPU - May 18, 6 Marks**

Ex. 6.19.7 : Determine the sound pressure level of a source that generate a following rms sound pressure.

(i) 1.7 N/m² (ii) 0.7 Pa **SPPU - Dec. 18, 4 Marks**

Ex.6.19.8 : Determine the sound power level of a source that generate a sound power of (i) 1.0 W (ii) 3.0 W

SPPU - May 19, 4 Marks

6.20 RELATIONSHIP OF SOUND POWER LEVEL AND SOUND INTENSITY LEVEL WITH SOUND PRESSURE LEVEL

6.20.1 Relation between Sound Power Level (L_w) and Sound Intensity Level (L_i) :

- The sound source is assumed to be a point source and its wave to be propagating from it in the form of spherical waves, as shown in Fig. 6.20.1.

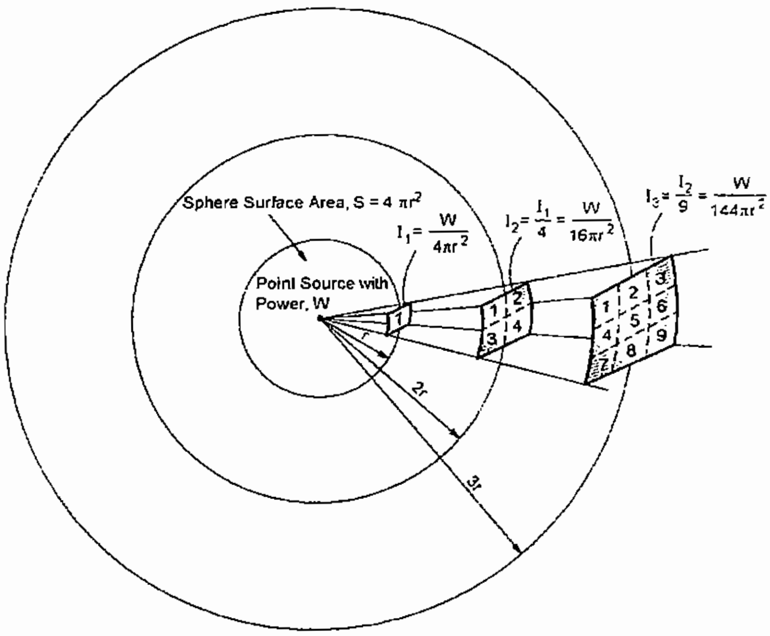


Fig. 6.20.1 : Sound Energy Distributed Over Spherical Surface in Free Space

- The intensity of sound decreases as the distance from the source of point increases. The sound energy can be assumed to spread out equally in all radial directions and considered to be distributed uniformly over a sphere of radius r and surface area, $S = 4\pi r^2$.
- **Intensity of sound at distance 'r' from a point of source with sound power 'W' :**

$$I = \frac{W}{S} = \frac{W}{4\pi r^2}, W/m^2 \quad ...(6.20.1)$$

or $$I \propto \frac{1}{r^2}$$

- **Inverse Square Law :** ***Inverse square law*** *states that the intensity of sound in a free space is inversely proportional to the square of the distance from the source of sound.*
- **Relation between sound power level and sound intensity level :**

$$L_W = 10 \log_{10}\left(\frac{W}{W_{ref}}\right) \text{ dB} \quad ...(a)$$

where, W = sound power, W

$= $ Surface area × Intensity of sound

$$= S \cdot I \quad .....(b)$$

W_{ref} = reference sound power, W

$$= S_{ref} \cdot I_{ref} \quad ...(c)$$

Substituting equations (b) and (c) in Equation (a),

$$L_W = 10 \log_{10}\left(\frac{S \cdot I}{S_{ref} \cdot I_{ref}}\right)$$

or $$L_W = 10 \log_{10}\left(\frac{I}{I_{ref}}\right) + 10 \log_{10}\left(\frac{S}{S_{ref}}\right) \quad ...(d)$$

Taking $S_{ref} = 1m^2$, Equation (d) can be written as,

$$L_W = 10 \log_{10}\left(\frac{I}{I_{ref}}\right) + 10 \log_{10}(S) \quad ...(e)$$

Substituting Equation (6.19.7) in Equation (c)

$$\mathbf{L_W = L_I + 10 \log_{10}(S)} \quad ...(6.20.2)$$

where, $S = 4\pi r^2$

6.20.2 Relation between Sound Pressure Level (L_p) and Sound Intensity Level (L_I) :

University Question

Q. Derive a relation between sound intensity level and sound pressure level.

SPPU : Dec. 12, May 13, May 16

- **Sound intensity level of sound wave :**

$$L_I = 10 \log_{10}\left(\frac{I}{I_{ref}}\right), \text{dB} \quad ...(f)$$

where, I = sound intensity, W/m^2

$$= \frac{p_{rms}^2}{\rho c} \quad ...(g)$$

I_{ref} = reference sound intensity, W/m^2

$$= \frac{p_{ref}^2}{\rho c} \quad ...(h)$$

Substituting Equation (g) in Equation (f), we get,

$$L_I = 10 \log_{10}\left(\frac{p_{rms}^2}{\rho c I_{ref}}\right)$$

$$= 10 \log_{10}\left[\left(\frac{p_{rms}^2}{p_{ref}^2}\right) \cdot \left(\frac{p_{ref}^2}{\rho c I_{ref}}\right)\right]$$

or $$L_I = 10 \log_{10}\left(\frac{p_{rms}^2}{p_{ref}^2}\right) + 10 \log_{10}\left(\frac{p_{ref}^2}{\rho c I_{ref}}\right) \quad ...(i)$$

- **Relationship between sound pressure level and sound intensity level :**

Substituting Equation (6.19.2) in Equation (i), we get,

$$L_I = L_p + 10 \log_{10}\left(\frac{p_{ref}^2}{\rho c I_{ref}}\right)$$

Taking, $p_{ref} = 2 \times 10^{-5}$ N/m², $I_{ref} = 10^{-12}$ W/m²,

$\rho = 1.21$ kg/m²; and $c = 344$ kg/m², we get,

$$L_I = L_p + 10 \log_{10}\left[\frac{(2 \times 10^{-5})^2}{1.21 \times 344 \times 10^{-12}}\right]$$

or $$L_I = L_p - 0.167 \quad ...(6.20.3)$$

For all practical purpose, the sound intensity level is taken to be numerically equal to the sound pressure level. Therefore,

$$\therefore \quad L_I \approx L_p$$

$$\textbf{or} \quad \mathbf{L_p = L_I} \quad ...(6.20.4)$$

Ex. 6.20.1 : The sound pressure level is 90 dB at a distance 5 m from a point source. Assuming a free progressive spherical wave and standard atmospheric conditions, determine the sound power level of the source.

SPPU - Dec. 14, 4 Marks

Soln. :

Given : $L_p = 92$ dB; $r = 5$m.

- **Surface area of spherical wave :**

$$S = 4\pi r^2$$

$$= 4\pi (5)^2 = 314.15 \text{ m}^2$$

- **Sound intensity level :**

$$L_I \approx L_p = 90 \text{ dB}$$

- **Sound power level :**

$$L_W = L_i + 10 \log_{10}(S) = 90 + 10 \log_{10}(314.15)$$

or $$L_W = \mathbf{114.97\ dB}$$ ...Ans.

Ex. 6.20.2 : Show that when the distance from point of source is doubled the sound intensity level decreases by 6 dB. **SPPU - Dec. 12, May 14, Dec. 15, May 18, 6 Marks**

Soln. :

Let, L_{i1} = Sound intensity level at radius r

L_{i2} = Sound intensity level at radius 2r

- **Surface area of spherical wave at distances r :**

$$S_1 = 4\pi r^2$$

- **Surface area of spherical wave at distance of 2r :**

$$S_2 = 4\pi (2r^2) = 16\pi r^2$$

- **Sound power level at distance r and 2r from same source :**

$$L_W = L_{i1} + 10 \log_{10}(S_1) = L_{i1} + 10 \log_{10}(4\pi r^2) \quad ...(a)$$

and, $$L_W = L_{i2} + 10 \log_{10}(S_2) = L_{i2} + 10 \log_{10}(16\pi r^2) \quad ...(b)$$

- **Difference of sound intensity levels :**

From Equations (a) and (b),

$$L_{i1} + 10 \log_{10}(4\pi r^2) = L_{i2} + 10 \log_{10}(16\pi r^2)$$

$$\therefore \quad L_{i2} - L_{i1} = 10 \log_{10}(4\pi r^2) - 10 \log_{10}(16\pi r^2)$$

$$= 10 \log_{10}\left(\frac{4\pi r^2}{16\pi r^2}\right)$$

$$L_{i2} - L_{i1} = -6.02 \text{ dB}$$

or $$\mathbf{L_{i2} - L_{i1} = -6\ dB}$$ ...Ans.

Examples for Practice

Refer our website for complete solution of following examples

Ex. 6.20.3 : Show that if sound pressure is doubled, the sound pressure level increases by six decibels. **SPPU - Dec. 16, Dec. 17, Dec.18, 6 Marks**

Ex. 6.20.4 : Show that if the sound power is doubled, then the sound power level increases by approximately 3 dB. **SPPU - Dec.19, 4 Marks**

6.21 ADDITION OF DECIBELS

- Generally, the noise in a workplace is affected by more than one sound source. Therefore, it is general practice to determine individually the sound pressure levels from each sound source and finally determine the total sound pressure level by adding the sound pressure levels of different sound sources.

- **Methods for adding decibels :**

There are three methods for adding decibels :

1. Exact Method for Adding Decibels
2. Approximate Method for Adding Decibels
3. Standard Chart Method for Adding Decibels

6.21.1 Exact Method for Adding Decibels :

The procedure for adding decibels is as follows :

- **Sound pressure levels, $L_{p1}, L_{p2}, ... L_{pn}$:**

Let $L_{p1}, L_{p2}, ... L_{pn}$ etc. = sound pressure levels to be added

$$L_{p1} = 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_1^2, \text{ dB}$$

$$L_{p2} = 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_2^2, \text{ dB}$$

and $$L_{pn} = 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_n^2, \text{ dB}$$

$$\therefore \frac{L_{pn}}{10} = \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_n^2, \text{ dB}$$

$$\therefore \left(\frac{p_{rms}}{p_{ref}}\right)_n^2 = 10^{(L_{pn}/10)} \quad ...(6.21.1)$$

- **Total sound pressure level :**

$$L_p = 10 \log_{10}\left[\left(\frac{p_{rms}}{p_{ref}}\right)_1^2 + \left(\frac{p_{rms}}{p_{ref}}\right)_2^2 + ... \left(\frac{p_{rms}}{p_{ref}}\right)_n^2\right]$$

$$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} \left(\frac{p_{rms}}{p_{ref}}\right)_n^2\right], \text{ dB} \quad ...(6.21.2)$$

Substituting Equation (6.21.1) in Equation (6.21.2), the total sound pressure level is,

or $$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{(L_{pn}/10)}\right], \text{ dB} \quad ...(6.21.3)$$

- **Total sound power level :**

$$L_W = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{(L_{wn}/10)}\right], \text{ dB} \quad ...(6.21.4)$$

where, L_W = total sound power level in dB

L_{Wn} = n^{th} sound power level in dB.

Ex. 6.21.1 : Determine the total sound pressure level due to 90, 95, 88 dB sound pressure level, arising due to three different sound sources. **SPPU - May 14, 4 Marks**

Soln. :

Given : L_{p1} = 90 dB ; L_{p2} = 95 dB ;

L_{p3} = 88 dB ; n = 3.

Thus total sound pressure level is,

$$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{(L_{pn}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(L_{p1}/10)} + 10^{(L_{p2}/10)} + 10^{(L_{p3}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(90/10)} + 10^{(95/10)} + 10^{(88/10)}\right]$$

$$= 10 \log_{10}\left[10^{9} + 10^{9.5} + 10^{8.8}\right]$$

or L_p = **96.80 dB** ...Ans.

Ex.6.21.2 : An operator in machine shop are operating five machines having their sound pressure levels at the position being 95, 90, 92, 88 and 83 dB respectively. Determine the total sound pressure level when :

(i) all five machines are turned ON;

(ii) when machine 1 is turned OFF;

(iii) when machine 2 and 3 are turned OFF; and

(iv) when machine 4 and 5 are turned OFF;

SPPU - Dec. 13, Dec. 14, May 15, 6 Marks

Soln. :

Given : L_{p1} = 95 dB ; L_{p2} = 90 dB ; L_{p3} = 92 dB ; L_{p4} = 88 dB ; L_{p5} = 83 dB ; n = 5.

(i) When all five machines are turned ON :

$$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{(L_{pn}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(L_{p1}/10)} + 10^{(L_{p2}/10)} + 10^{(L_{p3}/10)} + 10^{(L_{p4}/10)} + 10^{(L_{p5}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(95/10)} + 10^{(90/10)} + 10^{(92/10)} + 10^{(88/10)} + 10^{(83/10)}\right]$$

$$= 10 \log_{10}\left[10^{9.5} + 10^{9} + 10^{9.2} + 10^{8.8} + 10^{8.3}\right]$$

or L_p = **96.98 dB** ...Ans.

(ii) When machine 1 is turned OFF (i.e. $L_{p1} = 0$) :

$$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{L_{pn}/10}\right]$$

$$= 10 \log_{10}\left[10^{(L_{p2}/10)} + 10^{(L_{p3}/10)} + 10^{(L_{p4}/10)} + 10^{(L_{p5}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(90/10)} + 10^{(92/10)} + 10^{(88/10)} + 10^{(83/10)}\right]$$

$$= 10 \log_{10}\left[10^{9} + 10^{9.2} + 10^{8.8} + 10^{8.3}\right]$$

or L_p = **95.33 dB** ...Ans.

(iii) When machine 2 and 3 are turned OFF :

(i.e. $L_{p2} = L_{p3} = 0$) :

$$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{(L_{pn}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(L_{p1}/10)} + 10^{(L_{p4}/10)} + 10^{(L_{p5}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(95/10)} + 10^{(88/10)} + 10^{(83/10)}\right]$$

$$= 10 \log_{10}\left[10^{9.5} + 10^{8.8} + 10^{8.3}\right]$$

or L_p = **96.01 dB** ...Ans.

(iv) When machine 4 and 5 are turned OFF :

(i.e. $L_{p4} = L_{p5} = 0$) :

$$L_p = 10 \log_{10}\left[\sum_{n=1}^{n} 10^{(L_{pn}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(L_{p1}/10)} + 10^{(L_{p2}/10)} + 10^{(L_{p3}/10)}\right]$$

$$= 10 \log_{10}\left[10^{(95/10)} + 10^{(90/10)} + 10^{(92/10)}\right]$$

$$= 10 \log_{10}\left[10^{9.5} + 10^{9} + 10^{9.2}\right]$$

or L_p = **97.59 dB**

Examples for Practice

Refer our website for complete solution of following examples

Ex.6.21.3 : Determine the total sound power level due to three different sound powers 100 dB, 103 dB and 105 dB respectively.

Ex. 6.21.4: Calculate the total noise, if there are 4 sourcing of noise having magnitudes 45 dB, 54 dB, 53 dB, and 52 dB. What would be effect on total noise, if 45 dB noise is switched off ? **SPPU - Dec. 19, 4 Marks**

Ex.6.21.5 : A customer care centre containing six offices, individually makes noise level of 60, 56, 62, 53, 51 and 54 dB respectively. Add the noise levels when :

(i) All officers are working ; and

(ii) When first and second officers are not working.

SPPU - May 13, May 19, 6 Marks

Ex.6.21.6 : Calculate the total noise if there are four sources of noise with 45 dB, 54 dB, 48 dB and 50 dB magnitudes. **SPPU - May 12**

Ex.6.21.7 : A home theatre installation has 5 full range speakers. The 3 front ones are each capable of producing a sound pressure level of 90 dB at the listening position. The 2 rear ones are each capable of producing a sound pressure level of 85 dB at the listening position. What is the total sound pressure level that the whole installation of 5 speakers is capable of producing at the listening position ?

SPPU - May 16, 6 Marks

Ex.6.21.8 : Noise at the construction site is contributed by a few construction activities such as piling work : 104 dB, Scraper : 93 dB, Bulldozer : 94 dB, Mobile compressor : 73 dB and Mechanical Shovel : 76 dB on a weighing network. What is the overall sound pressure level?
SPPU - Dec. 18, 4 Marks

Ex.6.21.9 : Give four machines producing 100 dB, 91 dB, 90 dB and 89 dB. What is the total sound pressure level ? **SPPU - May 17, 4 Marks**

Ex. 6.21.10 : If two machines are producing 80 dB each what will be the overall sound pressure level? Derive the equation you use. **SPPU - May 18, 6 Marks**

6.21.2 Standard Chart Method for Adding Decibels :

- The another alternative method for adding decibels is by means of using standard chart.
- The graphical chart with the values of x plotted along x-axis and values of y' plotted along y-axis is shown in Fig. 6.21.1.

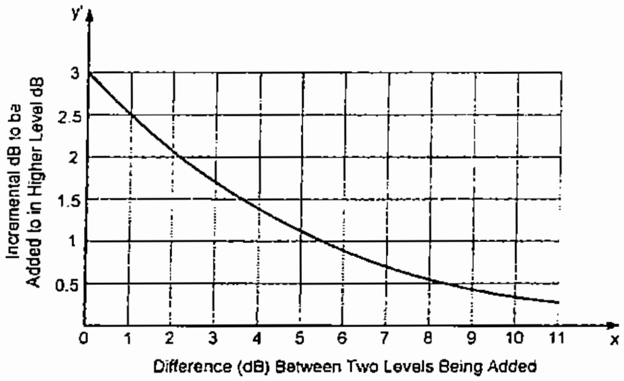


Fig. 6.21.1 : Standard Chart for Adding Decibels

- The procedure for adding decibels by using charts is explained in following Example.

Ex.6.21.11 : Determine the total sound pressure level due to individual sound pressure levels of 90 dB, 95 dB and 83 dB by using :
(i) Exact Method; and
(ii) Chart Method.
Compare the results obtained.

Soln. :

Given : $L_{p1} = 90$ dB ; $L_{p2} = 95$ dB ; $L_{p3} = 83$ dB ; n = 3.

1. Exact Method :

$$L_p = 10 \log_{10} \left[\sum_{n=1}^{n} 10^{(L_{pn}/10)} \right]$$

$$= 10 \log_{10} \left[10^{(L_{p1}/10)} + 10^{(p2/10)} + 10^{(L_{p3}/10)} \right]$$

$$= 10 \log_{10} \left[10^{(90/10)} + 10^{(95/10)} + 10^{(83/10)} \right]$$

$$= 10 \log_{10} \left[10^{9} + 10^{9.5} + 10^{8.3} \right]$$

or $L_p = 96.39$ **dB** ...**Ans.**

2. By Using Chart :

- In this method, there is no particular order of addition is required. Therefore, we will begin with two lowest dB levels and proceed to higher dB levels.

(i) The difference between two lowest dB level is,

$$x_1 = 90 - 83 = 7 \text{ dB}$$

(ii) For this value of $x_1 = 7$dB, the corresponding value of y_1 from chart shown in Fig. P. 6.21.11 is,

$$y_1 = 0.7 \text{ dB.}$$

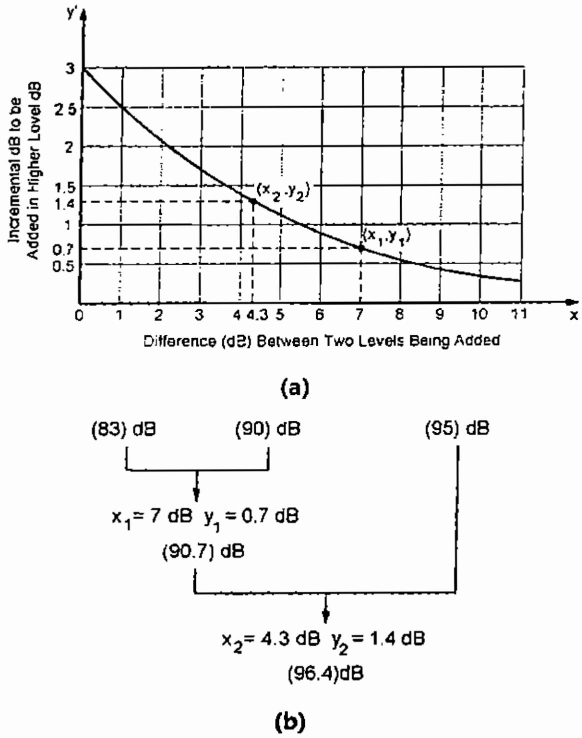


Fig. P. 6.21.11

(iii) Adding this value of $y_1 = 0.7$dB in higher level dB. i.e. 90 dB,

new value of decibel = 90 dB + 0.7 dB = 90.7 dB

(iv) Now the difference between new value of dB and next higher value of dB is,

$$x_2 = 95 \text{ dB} - 90.7 \text{ dB} = 4.3 \text{ dB}$$

(v) For this value of $x_2 = 4.3$ dB, the corresponding value from chart is,

$$y_2 = 1.4 \text{ dB.}$$

(vi) Adding this value of y_2 = 1.4 dB in higher level dB i.e. 95 dB, the total value of sound pressure level is,

$$L_p = 95 \text{ dB} + 1.4 \text{ dB} = \mathbf{96.4 \text{ dB}} \quad \text{...Ans.}$$

- The result obtained by exact method and by using chart method is approximately same. Some error may be their due to error in reading of chart.

6.22 SUBTRACTION OF DECIBELS

- In certain cases, it is desirable to subtract an ambient or background sound pressure level from a total measured value of sound level. This will allow to determine the sound pressure level produced by a particular source.
- The procedure for subtracting decibels can be established similar to that for decibel addition explained in earlier section 6.21.
- **Sound pressure levels L_{p1} and L_{p2} :**

Let L_{p1} and L_{p2} = two sound pressure levels under consideration for decibel subtractions.

L_p = resultant sound pressure level

$$L_{p1} = 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_1^2, \text{ dB}$$

$$L_{p2} = 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_2^2, \text{ dB}$$

- **Resultant sound pressure level (L_p) :**

$$L_p = L_{p1} - L_{p2}$$

$$= \left[10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_1^2 - 10 \log_{10}\left(\frac{p_{rms}}{p_{ref}}\right)_2^2\right]$$

$$= 10 \log_{10}\left[\left(\frac{p_{rms}}{p_{ref}}\right)_1^2 - \left(\frac{p_{rms}}{p_{ref}}\right)_2^2\right]$$

$$= 10 \log_{10}\left[\text{antilog}\left(\frac{L_{p1}}{10}\right) - \text{antilog}\left(\frac{L_{p2}}{10}\right)\right]$$

or $$L_p = 10 \log_{10}\left[10^{(L_{p1}/10)} - 10^{(L_{p2}/10)}\right], \text{ dB} \quad \text{...(6.22.1)}$$

Note : Similar to decibel addition, the approximate method and standard chart is also available for decibel subtraction.

Ex.6.22.1 : Determine the sound pressure level at a point due to particular machine if, at that point the sound pressure level is 92 dB when machine is operating and 85 dB when machine is OFF.

Soln. :

Given : L_{p1} = 92 dB ; L_{p2} = 85 dB, (ambient or background noise level)

Sound pressure level due to particular machine is,

$$L_p = L_{p1} - L_{p2}$$

$$= 10 \log_{10}\left[10^{(L_{p1}/10)} - 10^{(L_{p2}/10)}\right], \text{ dB}$$

$$= 10 \log_{10}\left[10^{(92/10)} - 10^{(85/10)}\right]$$

$$= 10 \log_{10}\left[10^{9.2} - 10^{8.5}\right]$$

or $$L_p = \mathbf{91.03 \text{ dB}} \quad \text{...Ans.}$$

Ex. 6.22.2 : The sound pressure level measured at a machine floor with a noisy machine operating nearby is 89.0 dB. When machine is turned off, the sound pressure level measured at the same location is 81.0 dB. What is sound pressure level due to machine alone ?

SPPU - May 12, 4 Marks

Soln. :

Given : L_{p1} = 89 dB,

L_{p2} = 81 dB, (ambient or background noise level)

Sound pressure level due to particular machine is,

$$L_p = L_{p1} - L_{p2}$$

$$= 10 \log_{10}\left[10^{(L_{p1}/10)} - 10^{(L_{p2}/10)}\right], \text{ dB}$$

$$= 10 \log_{10}\left[10^{(89/10)} - 10^{(81/10)}\right]$$

$$= 10 \log_{10}\left[10^{8.9} - 10^{8.1}\right]$$

or $$L_p = \mathbf{88.25 \text{ dB}} \quad \text{...Ans.}$$

Ex.6.22.3 : The sound pressure level measured for machines 1, 2 and 3 are 86 dB, 88 dB and 85 dB respectively, when operating independently in the presence of background noise. When the machines are turned OFF, the sound pressure level at the same point is 84 dB. Determine the overall sound pressure level of three machines independent of the background noise.

SPPU - Dec. 14, 6 Marks

Soln. :

Given : L_{p1} for machine 1 = 86 dB ;

L_{p1} for machine 2 = 88 dB ;

L_{p1} for machine 3 = 85 dB ;

L_{p2} i.e. ambient or background noise = 84 dB.

- **Sound pressure level for machine 1 independent of background noise :**

$$L_p = 10 \log_{10}\left[10^{86/10} - 10^{84/10}\right]$$

$$= 10 \log_{10}\left[10^{8.6} - 10^{8.4}\right]$$

or $$L_p = 81.67 \text{ dB}$$

- **Sound pressure level for machine 2 independent of background noise :**

$$L_p = 10 \log_{10}\left[10^{88/10} - 10^{84/10}\right]$$

$$= 10 \log_{10}\left[10^{8.8} - 10^{8.4}\right]$$

or $$L_p = 85.79 \text{ dB}$$

- **Sound pressure level for machine 3 independent of background noise:**

$$L_p = 10 \log_{10} \left[10^{85/10} - 10^{84/10} \right]$$

$$= 10 \log_{10} \left[10^{8.5} - 10^{8.4} \right]$$

or $L_p = 78.13$ dB

- **Total sound pressure level of three machine independent of background noise:**

$$L_p = 10 \log_{10} \left[10^{81.67/10} + 10^{85.79/10} + 10^{78.13/10} \right]$$

$$= 10 \log_{10} \left[10^{8.167} + 10^{8.579} + 10^{7.813} \right]$$

or $L_p = \mathbf{87.71\ dB}$ ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex. 6.22.4 : When operating independently in the presence of back ground noise, measurement at a given location of the sound pressure level for machines 1, 2 and 3 are respectively 88 dB, 90 dB and 87 dB. When the machines are turned off, the sound pressure level at the same point is 86 dB. Determine the overall sound pressure level (SPL) of the three machines independent of the background noise. **SPPU - Dec. 12, May 16, 6 Marks**

6.23 AVERAGING OF DECIBELS

- In actual practice, sometimes situation may arises to find out the average decibels at a particular location.
- In other words, it may be required to measure the sound pressure level at a single location several times and then take an average value for calculation purposes for getting accurate results.
- **Methods of averaging decibels** : There are two methods for averaging decibels :

1. Exact Method for Averaging of Decibels
2. Approximate Method for Averaging decibels

6.23.1 Exact Method for Averaging Decibels :

- The procedure for averaging decibels by exact method is carried out in similar manner to that of the exact method for adding decibels.

Let $L_{p1}, L_{p2}, \ldots, L_{pn}$ = sound pressure level measured at single location several times for which the average sound power level is to be determined.

L_p = total sound pressure level

L_{pavg} = average sound pressure level

- **Total sound pressure level :**

$$L_p = L_{p1} + L_{p2} + \ldots + L_{pn}$$

$$L_p = 10 \log_{10} \left[\sum_{n=1}^{n} 10^{(L_{pn}/10)} \right], \text{dB}$$

(From Equation 6.21.3)

- **Average sound pressure level (L_{pavg}) :**

$$L_{pavg} = 10 \log_{10} \left[\frac{1}{n} \sum_{n=1}^{n} 10^{(L_{pn}/10)} \right], \text{dB} \quad \ldots(6.23.1)$$

Ex.6.23.1 : The set of measured values of sound pressure level at a point, for a machine are : 96 dB, 100 dB, 90 dB and 96 dB. Determine the average sound pressure level by using exact method.

Soln. :

Given : $L_{p1} = 96$ dB; $L_{p2} = 100$ dB; $L_{p3} = 90$ dB; $L_{p4} = 96$ dB; $n = 4$.

The average sound pressure level is,

$$L_{pavg} = 10 \log_{10} \left\{ \frac{1}{n} \sum_{n=1}^{n} 10^{(L_{pn}/10)} \right\}$$

$$= 10 \log_{10} \left\{ \frac{1}{4} \left[10^{(L_{p1}/10)} + 10^{(L_{p2}/10)} + 10^{(L_{p3}/10)} + 10^{(L_{p4}/10)} \right] \right\}$$

.....[$\because$ n = 4]

$$= 10 \log_{10} \left\{ \frac{1}{4} \left[10^{(96/10)} + 10^{(100/10)} + 10^{(90/10)} + 10^{(96/10)} \right] \right\}$$

$$= 10 \log_{10} \left\{ \frac{1}{4} \left[10^{9.6} + 10^{10} + 10^{9} + 10^{9.6} \right] \right\}$$

or $L_{pavg} = \mathbf{96.75\ dB}$ ...Ans.

Example for Practice

Refer our website for complete solution of following example

Ex.6.23.2 : The set of measured values of the sound pressure level for a machine are : 92 dB, 91 dB, 88 dB and 90 dB. Determine the average sound pressure level.

6.24 SOUND FIELDS

University Questions

Q. Define sound fields. **SPPU : May 12, May 14, May 15**

Q. Write short note on : Sound Fields. **SPPU : May 13, Dec. 13**

Sound Field *is the region in a medium in which sound waves are being propagated from the source of sounds.*

6.24.1 Types of Sound Fields :

University Questions

Q. Explain the various types of sound fields in the vicinity of a sound source. **SPPU : May 12, Dec. 12, Dec. 13, May 14, May 15, Dec. 18, Dec. 19**

Q. Explain different sound field for sound measurements. **SPPU : May 18**

- The various types of sound fields are as shown in Fig. 6.24.1.

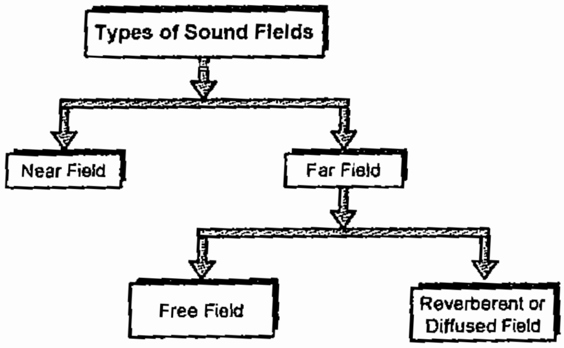


Fig. 6.24.1 : Types of Sound Fields

- Fig. 6.24.2 shows the various sound fields regions.

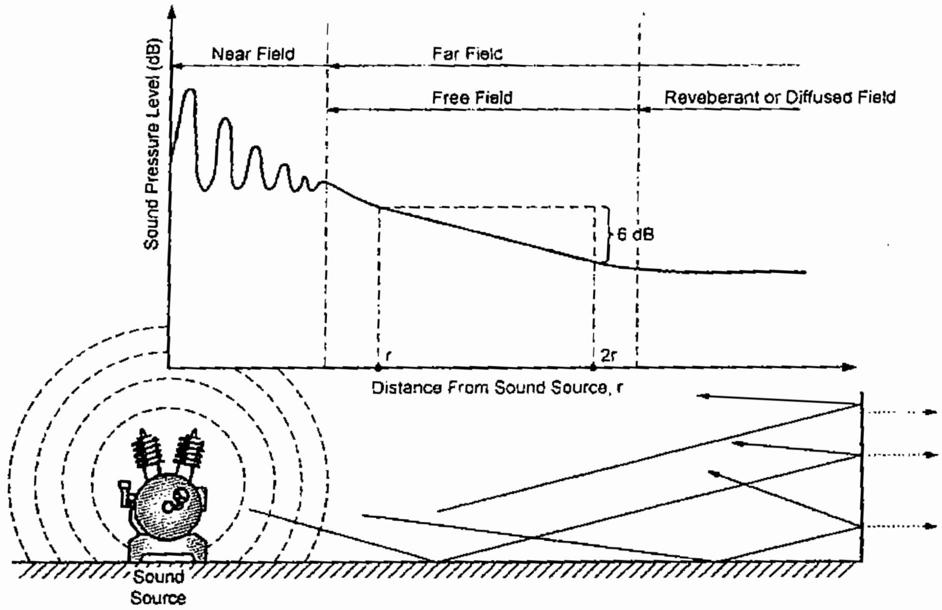


Fig. 6.24.2 : Sound Fields in Vicinity of a Sound Source

1. Near Field :

- **Near field :** *The **near field** region is located within a few wavelengths of the source of sound or within a few diameters of the source of sound.*
- In near field, the sound pressure level may be either more or less than that predicted by the inverse-square law. Therefore, the sound pressure level measured within the near field cannot be used to predict the sound pressure level.

2. Far Field :

- **Far field :** *The region beyond the near field is called as **far field**.* The far field is the region suitable for recording sound pressure level of sound source.
- When one moves out of the near field and enter into the far field, the sound pressure levels drop off at the rate of 6 dB per doubling the distance from a point source.
- According to the inverse-square law, the sound pressure level in far field is given by,

$$L_{p2} = L_{p1} - \left[20 \log_{10} \left(\frac{r_2}{r_1} \right) \right] \quad ...(6.24.1)$$

where, L_{p1} = Sound pressure level at the distance r_1 from sound source, dB

L_{p2} = Sound pressure level at the distance r_2 from sound source, dB

- **Types of far field :**

(i) Free field ; and
(ii) Reverberant or diffused field.

(i) Free field :

- *The region in which Equation (6.24.1) is valid is known as* **free field.**
- The free field conditions exist in large open outdoor spaces or in rooms having highly absorptive surfaces in which there are no obstructers or barriers in the sound travel path between source and listener.

(ii) Reverberant or diffused field :

- *The sound field which is formed by multiple reflections from obstacles or barriers like wall, floor and ceiling surfaces etc. in the room is called as* **reverberant or diffused field.**
- As sound pressure levels decay from sources within a room, they eventually drop to a relatively constant level which is determined by the amount of reflected sound within a room.

6.25 SOUND REFLECTION, ABSORPTION AND TRANSMISSION

- When sound waves fall on a boundary or material, their energy is partially reflected, partially absorbed by the boundary and partially transmitted through the boundary, as shown in Fig. 6.25.1.

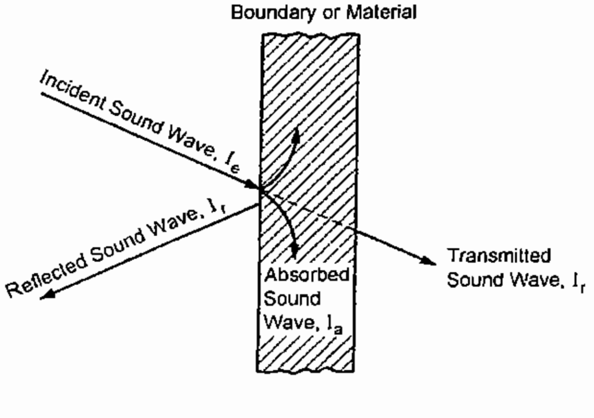


Fig. 6.25.1 : Incident Sound Wave

- The proportions of reflection, absorption and transmission depend upon the type of material and the frequency of the sound wave .

6.25.1 Reflection of Sound Wave :

University Questions

Q. Explain the term : Sound reflection. **SPPU : May 12, May 15**

Q. Explain, in brief, the term : Sound reflection coefficient. **SPPU : Dec. 12, May 19**

Q. Define the term : Reflection coefficient. **SPPU : Dec. 16, Dec. 17**

- If a sound wave in air encounters a large heavy and rigid wall, the sound wave will be reflected back. A sound wave from a given source, reflected by a plane surface, appears to come from the image of the source in that surface, as shown in Fig. 6.25.2.

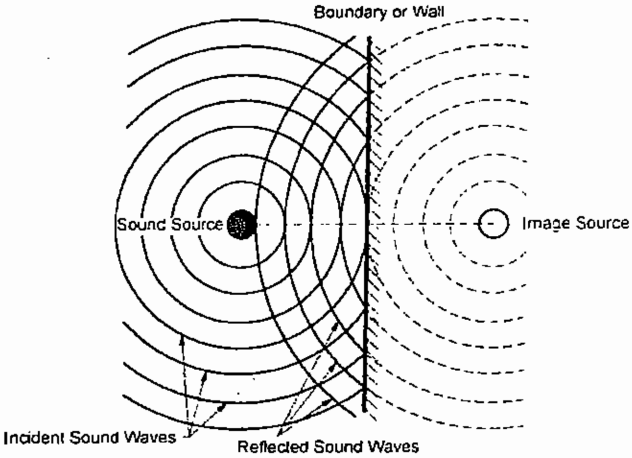


Fig. 6.25.2 : Reflection of Sound Wave from Plane Surface

☞ **Reflection Coefficient :**

Reflection coefficient *'γ' is defined as the ratio of the intensity of sound that is reflected from the boundary material to the intensity of sound incident on it.* Therefore,

$$\gamma = \frac{I_r}{I_e} = \frac{\text{Intensity of sound reflected}}{\text{Intensity of sound incident}}$$

6.25.2 Absorption of Sound Wave :

University Questions

Q. Explain the term : Sound absorption. **SPPU : May 12, May 15**

Q. Define the term : Absorption coefficient. **SPPU : Dec. 14, Dec. 16, Dec. 17, May 18, May 19**

- The absorption of sound is the process by which sound energy is diminished in passing through a medium or in striking on a surface. In the absorption of sound wave, the mechanism is usually the conversion of sound into other form of energy.

Absorption Coefficient :

Absorption coefficient *'α' is defined as the ratio of the intensity of sound absorbed from the boundary or material to the intensity of sound incident on it.* Therefore,

$$\alpha = \frac{I_a}{I_c} = \frac{\text{Intensity of sound absorbed}}{\text{Intensity of incident}}$$

Sound Absorbers :

The materials which absorb the sound energy is called as **sound absorbers.**

Types of Sound Absorber Materials :

(i) Porous materials
(ii) Non-porous panels or membrane
(iii) Perforated materials or resonators

(i) Porous materials :

- When sound waves impinge on a porous material, part of the sound energy is converted into heat energy due to viscous flow losses caused by wave propagation in the material and internal frictional losses caused by motion of the material fibres.
- **Examples of porous absorption materials :** Materials made from vegetable and mineral fibers, elastomeric foams, glass blankets, fiber boards, etc.

(ii) Non-porous panels and membrane :

- When sound waves impinge on a panels, the sound energy is converted into vibrational (i.e. mechanical) energy.
- Usually the panel or membrane is mounted at some distance from a rigid wall and the combination behaves like a spring -mass system.

(iii) Perforated materials or resonators :

- A perforated material or resonator consists of a cavity within massive solid. The air within the cavity acts as a spring which is forced in and out of the cavity through the neck which acts as a mass.

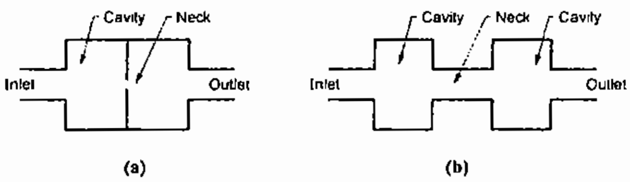


Fig. 6.25.3 : Schematic of Typical Resonators (Cavity Resonators)

- **Cavity resonators :** A typical resonator called cavity resonators is shown in Fig. 6.25.3.

6.25.3 Transmission of a Sound Wave :

University Questions

Q. Explain the term : Sound transmission. **SPPU : May 12, May 15**

Q. Define the term : Transmission coefficient **SPPU : Dec. 12, Dec. 16, Dec. 17, May 18, May 19**

Q. Write short note on : Transmission of sound wave. **SPPU : May 13**

Sound transmission occurs when sound energy passes through partition or boundary. In practically all reflectors or sound absorbers, some of the sound energy is transmitted through the wall.

Transmission Coefficient :

Transmission coefficient *'τ' is defined as the ratio of the intensity of sound transmitted from boundary wall to the intensity of sound incident on it.* Therefore,

$$\tau = \frac{I_T}{I_c} = \frac{\text{Intensity of sound transmitted}}{\text{Intensity of sound incident}}$$

Sound transmission is significantly reduced when the dimensions of the partition or boundary are larger than the largest wavelength of incident sound wave.

6.26 ACOUSTIC MATERIAL AND ITS CHARACTERISTICS

University Question

Q. Explain acoustic material and its characteristics. **SPPU : Dec. 18, May 19, Dec. 19**

Acoustic Materials :

- Acoustic or Sound absorbing materials are the materials which absorb and transmit the maximum sound waves and reflect the minimum sound waves.
- A material that could absorb and transmit more sound waves than it reflects, is considered a good sound absorbing material.

Factors Influencing Acoustic Properties of Material :

The following factors influence the acoustic properties of materials :

1. Material thickness	2. Material density
3. Material porosity	

1. Material thickness :

- One of the factors that influence a sound absorption by a material is the thickness of the material.
- The thickness of the materials is relevant or has direct relationship at low frequency range (100-2000 Hz) and is insignificant at high frequency (> 2000 Hz).
- Increase in the thickness provides better absorption of the wave and reflect less energy. As the thickness of the samples increases, the sound absorption coefficient increases. The reason is, at low frequency waves have higher wavelength, which means the thicker material contributes in better absorption.

2. Material density :

- The density of the material is another factor that influences sound absorption by a material.
- A high density material increases the sound absorption.

3. Material porosity :

- The presence of pores of voids plays a crucial part as they act as the medium of sound wave dissipation.
- Open pores with continuous channels provides better sound absorption because of the multiple reactions between the sound wave and the walls of the pores.

6.27 HUMAN HEARING MECHANISM

University Questions

Q. Explain, with neat diagram, the working of human hearing mechanism.

SPPU : Dec. 11, May 12, Dec.13, Dec. 14, May 15, Dec. 16, Dec. 17

Q. Write short note on : Mechanism of hearing.

SPPU : May 14, Dec.15

Function of Human Ear :

The function of the ear is to convert the physical vibration into an encoded nervous impulse signal which reaches the brain.

Components of Human Ear :

The human ear is commonly divided into three main components [Fig. 6.27.1] :

1. Outer Ear	2. Middle Ear
3. Inner Ear	

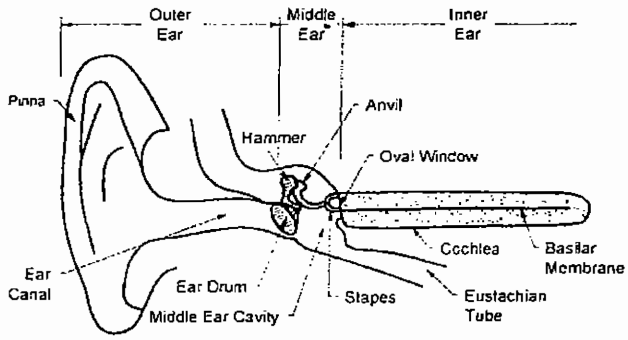


Fig. 6.27.1 : Schematic of Human Hearing Mechanism

1. Outer Ear :

The outer ear composed of (i) pinna, (ii) ear canal, and (iii) ear drum

(i) Pinna : The pinna projects from the side of the head skin. The pinna collects sound and channels it into the ear canal. The pinna is angled so that it catches sounds that come from front more than that from behind.

(ii) Ear canal : The ear canal is a tubular passage of about 25 mm long and 5-7 mm in diameter through which sound waves pass to the ear drum.

(ii) Ear drum : The ear drum is a very shallow cone of about 7 mm in diameter with its apex directed inwards. The ear drum separates the middle ear from the outer ear. It's purpose is to vibrate according to the frequency and amplitude of sounds that strike on it.

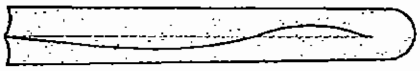
2. Middle Ear :

- The purpose of the middle ear is to transmit and amplify sounds from the ear drum to the oval window.
- The middle ear is an air filled space connected to the back of the nose by a long, thin tube called the **eustachian tube**. It transmits the sound from the ear drum to the inner ear.
- The ear drum is 13 times larger than the oval window, giving an amplification of about 1:13 compared to the oval window.

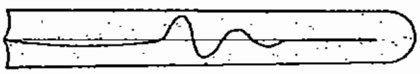
3. Inner Ear :

- In the inner ear, the cochlea is the main component where the actual reception of sound takes place. [Fig. 6.27.1]
- The cochlea which is located in extremely hard temporal bone, is divided almost its entire length by the **basilar membrane.** At the end of the cochlea, the two canals are connected by the **helicontrema**, which allows for the flow of the lymphatic fluid between the two sections. The basilar membrane which is 30 mm long and 0.2 mm wide has about 24,000 nerve ends.
- Low frequency sound results in maximum amplitude near the distant end of the basilar membrane and high frequency sound produces peaks near the oval window. For a complex signal such as music or speech, many momentary peaks are produced, constantly shifting in amplitude and position along the basilar membrane.

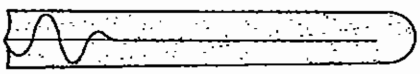
Fig. 6.27.2 shows schematic representation of sound waves travelling along the basilar membrane.



(a) Low Frequency



(b) Medium Frequency



(c) High Frequency

Fig. 6.27.2 : Schematic Representation of Sound Waves Travelling Along Basilar Membrane

PART II : NOISE MEASUREMENTS AND CONTROL

6.28 INTRODUCTION TO NOISE

Noise : In a modern city life, sound is as irritating as it is pleasant or useful. Such unpleasant or unwanted sound is called as **noise.** In other words, ***noise** is the wrong sound in the wrong place and at wrong time* [Fig. 6.28.2]. Noise is harmful and interferes with normal activities, it affects the communication and the efficiency of human being.

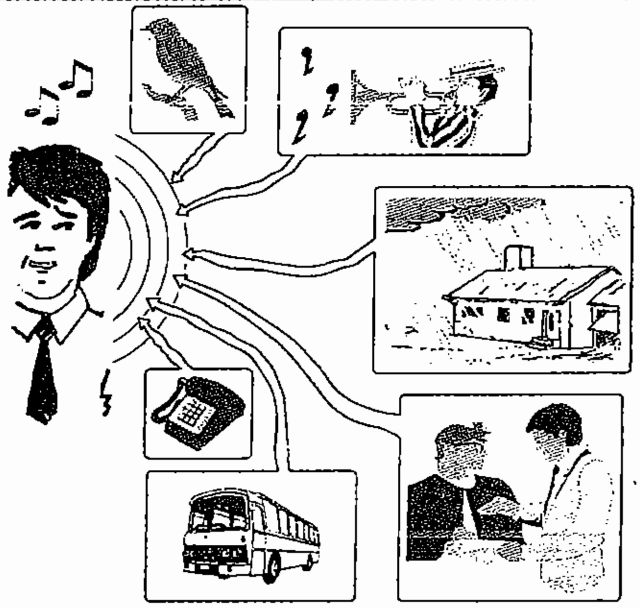


Fig. 6.28.1 : Sound

- Like air pollution and water pollution, noise has been recognized today as major pollutant of environment in the developed countries.

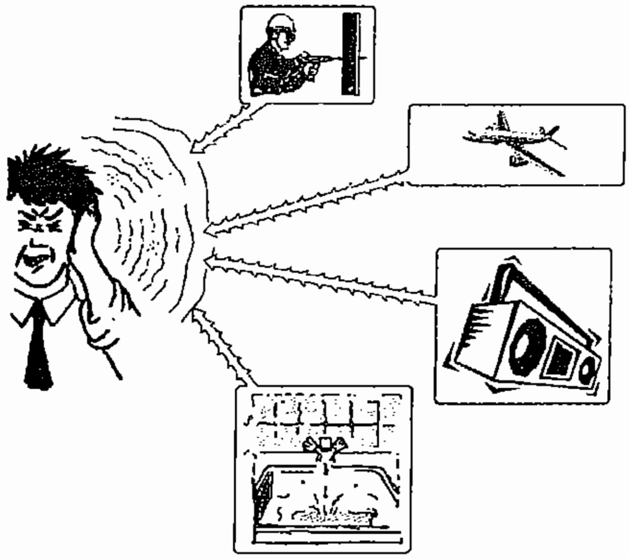


Fig. 6.28.2 : Noise

- **Effects of noise on human :**

  (i) The physical effects like loss of hearing, loss of memory and interference during communications.

  (ii) The psychological effect of annoyance, irritation, etc.

  (iii) The effect on working efficiency.

  (iv) The effect on the quality of life in cities.

6.29 SOURCES OF NOISE

University Questions

Q. Explain in brief : Various sources of noise and how to control same.
SPPU : May 12, Dec. 12, Dec. 18, Dec. 19

Q. Write short note on : Noise sources and control in industries. **SPPU : Dec. 15**

- **Categories of sources of noise :**

The sources of noise are categorized in the following major groups :

1. Industrial Noise
2. Home Appliances
3. Construction Equipment
4. Road, Air and Rail and Air Transportation

- **Noise levels in industrial machinery :**

The following tables gives typical noise levels in dB for industrial machinery, home appliances and construction equipment.

Table 6.29.1 : Machinery Noise Level in Industry

Industrial Machinery	Range of Noise Level, dB
Rivetting	105 to 125
Presses	110 to 120
Pneumatic power tool	90 to 115
Hydraulic machine tool	85 to 105
Grinding machine	99 to 105
Air compressor	90 to 100
Fluid pumps	80 to 90
Lathes and milling machine	80 to 90

Table 6.29.2 : Noise Level for Home Appliances

Hone Appliances	Range of Noise Level, dB
Food blander	62 to 88
Vacuum cleaner	62 to 85
Food mixer	50 to 80
Dish washer	55 to 75
Fan	38 to 70
Electric shaver	50 to 65
Hair drier	55 to 62
Air conditioner	50 to 60
Clothes dryer	50 to 60
Refrigerator	35 to 52

Table 6.29.3 : Noise Level for Construction Equipment (measured at 15m distance from equipment)

Construction Equipment	Range of Noise Level, dB
Rock drill	80 to 100
Scrapper and graders	80 to 95
Pneumatic wrenches	83 to 93
Concrete mixer	75 to 88
Concrete pump	80 to 83
Movable cranes	75 to 85
Front loader	72 to 84
Generators	70 to 83
Pumps	68 to 70
Compressors	65 to 70

- The major source of the noise discussed in the chapter is industrial noise (both within and around the industrial promises).

6.30 INDUSTRIAL NOISE CONTROL

University Question

Q. Write short note on : Noise control in industries. **SPPU : May 13, May 16**

6.30.1 Basic Elements in Noise Control System (S - P - R) :

1. **Source** of noise,
2. **Path** of noise, and
3. **Receiver** of noise

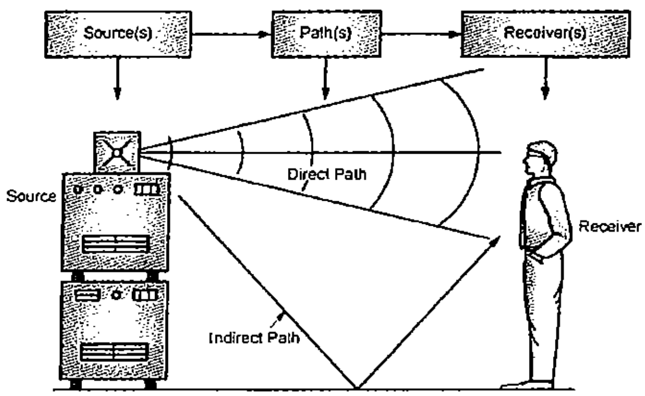


Fig. 6.30.1 : Elements of Noise Control System (S-P-R)

1. **Source of Noise :**

   *The **source** of noise may be a machine or any number of mechanical devices, vibrating surface, mechanical shock, mechanical friction, fluid flow, flame burst or an explosion.* Identification of source of noise usually helps to reduce or eliminate the noise. Therefore, acoustic engineer should identify all possible noise sources when considering a solution for a noise problem.

2. **Path of Noise :**

   *The **path** implies course or direction taken by the noise to reach the listener.* The path may be direct or indirect. The direct path for the sound may be the air between the source and receiver, as is the case for machinery noise transmitted directly to the operator's ears. The indirect path is the noise which is being reflected by a wall or flooring to a person in the room.

3. **Receiver of Noise :**

   *The **receiver** in the noise control system is usually the human.*

6.30.2 Factors to be Considered in Noise Control :

(i) Type of noise ;

(ii) Noise level and it's pattern ;

(iii) Frequency distribution ;

(iv) Noise sources (location, power) ;

(v) Noise propagation pathways, through air or through structure ; and

(vi) Number of workers exposed

6.31 METHODS OF INDUSTRIAL NOISE CONTROL

University Question

Q. Explain the various methods of industrial noise control.

SPPU : May 16, Dec.18

There are three basic methods of industrial noise control :

1. Noise Control at Source
2. Noise Control in Path
3. Noise Control at Receiver

6.31.1 Noise Control at Source :

University Question

Q. Write short note on : Noise control at the source.

SPPU : Dec. 12

☞ It is more economical and efficient to control the noise at the source. Modifications at the source of sound are usually considered to be the best solution for an industrial noise control problem.

☞ **Steps in Noise Control at Source :**

(i) Maintenance
(ii) Substitution of manufacturing processes
(iii) Substitution of equipment
(iv) Modification of parts of equipment
(v) Substitution of materials
(vi) Change of work methods
(vii) Noise enclosure

(i) Maintenance :

- Balancing of unbalanced equipment.
- Lubrication of moving parts.
- Replacement or adjustment of worn or loose parts.
- Tensioning driven belts.
- Tightening loose and vibrating screws or bolts.
- Replacement of worn out bearings.

(ii) Substitution of manufacturing processes :

- Using mechanical ejectors rather than pneumatic ejectors.
- Hot working rather than cold working.
- Pressing operation rather than rolling or forging.
- Using welding rather than stoppers.
- Using welding or squeeze stoppers rather than impact stoppers.
- Using cutting fluid in machining processes.
- Change from impact action (e.g. hammering a metal bar) to progressive pressure action (e.g. bending metal bar with pliers) or increase of time during which a force is applied.
- Replacing circular saw blades with damped blades.
- Replacing mechanical limit a stoppers with micro-switches.

(iii) Substitution of equipment :

- Use electric equipment instead of pneumatic equipment (e.g. hand tools)
- Using stepped dies instead of single-operation dies.
- Using rotating shears instead of square shears.
- Using hydraulic presses instead of mechanical presses.

- Using presses instead of hammers.
- Using electric motors instead of internal combustion engines or gas turbines.
- Using belt conveyors instead of roller conveyors.
- Using belts or hydraulic power transmissions instead of gear boxes.

(iv) Modification of parts of equipment :

- Replacing gear drives with belt drives.
- Modification of gear teeth, by replacing spur gears with helical gears.
- Replacing straight edged cutters with spiral cutters.
- Using properly shaped and sharpened cutting tools.
- Use of proper dampers for vibrating machines.
- Use of complete or partial enclosure around noisy machines.
- Use of proper mountings and isolation of vibratory machines or equipments.
- Providing machines with adequate cooling fins so that noisy fans are no longer needed.
- Using centrifugal rather than propeller fans.
- Locating fans in smooth, undisturbed air flow, or using optimized fan blades.

(v) Substitution of materials :

- Replacing metal gears with plastic gears. However, it requires additional maintenance.
- Replacing steel or solid wheels with pneumatic tyres.
- Replacement of steel sprockets in chain drives with polyamide plastics sprockets.

(vi) Change of work methods :

- In building demolition, replace use of ball machine with selective demolition.
- By changing manufacturing methods, such as moulding holes in concrete rather than cutting after production of concrete component.
- Instead of using an air jet to remove debris from a manufactured part, rotating cleaning brushes may be used.
- Keep noisy operations in the same area and separated from non-noisy processes.
- Select slowest machine speed appropriate for a job. Also select large slow machines rather than smaller faster ones.
- Minimise width of tools in contact with workpiece.

(vii) Noise enclosure :

An effective noise control procedure is to enclose the sound source in an acoustic enclosure.

6.31.2 Noise Control in Path :

Modifying the path through which the noise is propagated is often used when modification of the noise source is not possible, not practical, or not economically feasible.

☞ **Steps in Noise Control in Path :**

(i) For noise sources located outdoors, one simple approach for noise control would be to move the sound source farther away from the receiver. Keep noisy machines in the farthest corner of the industrial premises.

(ii) For noise sources located outdoors or indoors, the transmission path may be modified by placing a wall or barrier between the source and the receiver.

(iii) If the sound transmitted indirectly to the receiver through reflections from the room surfaces is significant, the noise may be reduced by applying acoustic absorbing materials on the walls of the room or by placing additional acoustic absorbing surfaces in the room.

(iv) The noise from metal cut-off saws can be reduced to acceptable levels by enclosing the saw in an acoustically treated box.

(v) The exhaust noise from engines, fans, and turbines is often controlled by using mufflers or silencers in the exhaust line for the device.

6.31.3 Noise Control at Receiver :

University Question

Q. Write short note on : Noise control at receiver.

SPPU : Dec. 11

☞ In some cases, when first two methods fail, it may be necessary to apply noise control to the receiver. The human ear is the usual receiver for noise, and there can be a limited amount of modification possible.

☞ **Steps in Noise Control at Receiver :**

(i) Limiting the time during which the person is exposed to high noise levels.

(ii) Rotating the job between the workers working at a particular extreme noise source.

(iii) Generally, the noise level of 90 dB for more than 8 hr continuous exposure is prohibited. The schedule of the workers should be planned in such a way that, they should not be over exposed to the high noise levels.

(iv) Using protectors such as earplugs, acoustic muffs, or active noise cancelling headphones.

(v) A very effective, although sometimes expensive, noise control procedure is to enclose the receiver in a personnel booth. If an equipment or process can be remotely operated, a personnel booth is usually an effective solution in reducing the workers noise exposure.

6.32 METHODS OF PROTECTING EMPLOYEES FROM NOISE

The following two methods are widely used for protecting the workers form the industrial noise :

1. Personal Protection Equipment
2. Noise Enclosures

6.32.1 Personal Protection Equipment :

- It is responsibility of industry to protect employees against hearing loss due to excessive noise exposure in the industry.
- The most sensitive part of the human ear that is prone to damage due to excessive noise is the inner ear. The hearing protection devices are designed to reduce the noise level that enters the outer and middle ear before it reaches the inner ear. When an industrial worker is exposed to a noise level of 85 dB or more, during the normal working hours, he must be advised to wear some ear defender.

Types of Personal Ear Protection Equipment Commonly Used in Industries :

(i) Ear defenders
(ii) Earplugs (aural or inserts)

(i) Ear defenders :

- Whenever it is not practical to reduce noise in an industry to an acceptable level, it is necessary to protect the hearing of the employee through Ear **defenders. Ear defenders** are capable of reducing the noise level at the ear by 35-45 dB.
- The vibrations from the incident sound waves are transmitted to the inner ear through the passage of ear defender. Due to this additional pathway of sound, the maximum attenuation is obtained with this hearing protector.

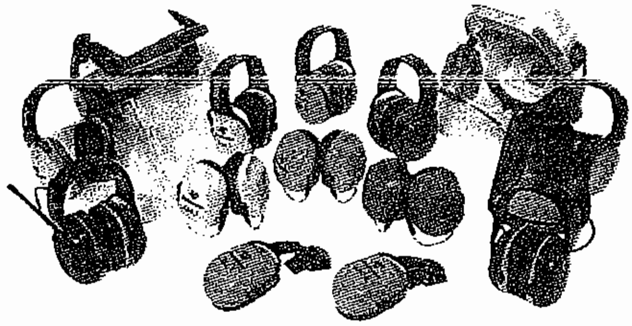


Fig. 6.32.1 : Ear Defenders

- The various types of ear defenders, commonly used in industries, are shown in Fig. 6.32.1.

(ii) Earplugs (aural or inserts) :

- The **earplugs** or **inserts** fit directly into the ear canal. They are made of : rubber, plastic or wax-impregnated cotton. The insert will be effective only if they fit properly in the ear canal, ensuring contact along the entire circumference.
- For this purpose, they should be so shaped as to exert outward pressure for proper sealing action.
- The disadvantage of ear plugs is that, they require tight seal and therefore, they may cause discomfort. These plugs can be lost easily and cannot be monitored easily by safety personnel. They become hard or shrink if not replaced at required intervals.
- **Various types of commonly used earplugs :**

(a) Pre-fabricated (pre-moulded) earplugs
(b) Disposable and malleable earplugs
(c) Custom moulded earplugs
(d) Semi-insert protectors
(e) Earmuffs

(a) Pre-fabricated (pre-moulded) earplugs :

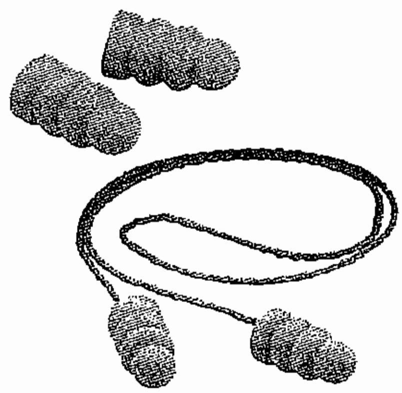


Fig. 6.32.2 : Pre-Fabricated Earplugs

The **pre-fabricated (pre-moulded) earplugs** [Fig. 6.32.2] are made from soft, flexible material, so as to conform readily to various shapes of ear canal and provide airtight and comfortable fit.

They are made of non-toxic material, have smooth surface and permit easy cleaning with water. The material of earplug is such that it retain its shape and flexibility over long period of use.

(b) Disposable and malleable earplugs :

Disposable and malleable earplugs [Fig. 6.32.3] are made from non-porous, easily formed, low-cost materials such as: cotton, glass wool or combination. These earplugs are reasonably comfortable and, if made correctly, can provide attenuation comparable to that obtained from the pre-fabricated earplugs.

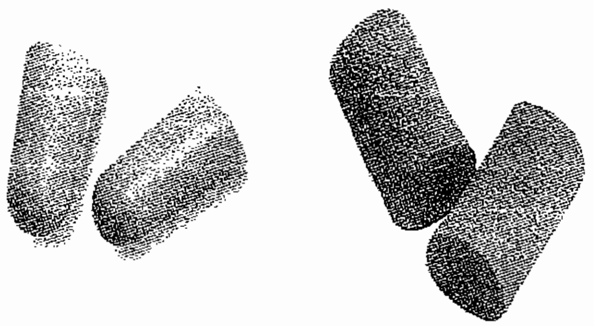


Fig. 6.32.3 : Disposable and Malleable Earplugs

(c) Custom moulded earplugs :

Custom moulded earplugs [Fig. 6.32.4] are generally available in form of kits containing some of silicon rubber and a fixative agent. The two are mixed together before inserting into the ear canal and the outer ear of an individual. The ear plug can be inserted and removed number of times without affecting their performance.

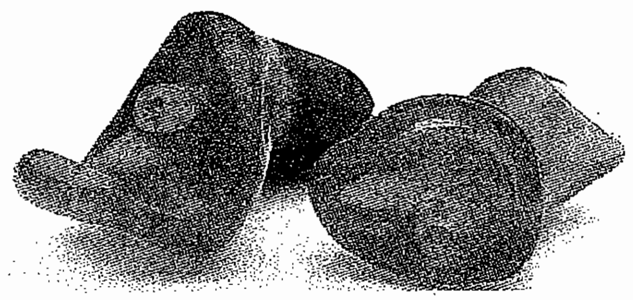


Fig. 6.32.4 : Custom Moulded Earplugs

6.32.2 Noise Enclosure :

University Questions

Q. What is sound enclosure ? Describe the two types of sound enclosures.

SPPU : Dec. 11, May 12, Dec. 13, May 15, May 17

Q. Write short note on : Sound enclosures.

SPPU : May 13

The noise level of any machines is reduced by providing a suitable enclosures. There are two types of enclosures, used for reducing the noise level :

1. Complete Enclosures
2. Partial Enclosures

1. Complete Enclosures :

- Fig. 6.32.5 shows a typical complete enclosure. Such enclosure is essentially a sealed box with stiffened walls. The panels of the walls have damping treatment applied to them to control their resonant vibration.
- The interior of the box is covered with absorptive treatment such as open cell foam or glass-fiber mat, to prevent the buildup of reverberant sound in the interior. All the openings are carefully sealed or provided with sound absorbing ducts of mufflers.
- The machine is kept on vibration isolation mount so that the machine does not excite the wall of the enclosure.

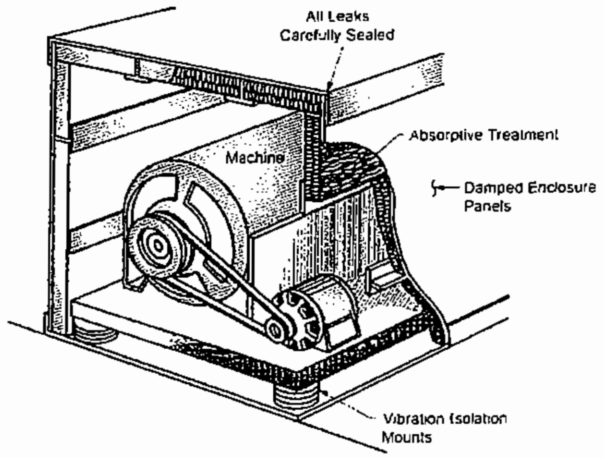


Fig. 6.32.5 : Simple Complete Enclosure

2. Partial Enclosures :

- The complete enclosure cannot be used, where opening for cooling air is provided. The mufflers can be employed to prevent noise from escaping from these opening, as shown in Fig. 6.32.6.

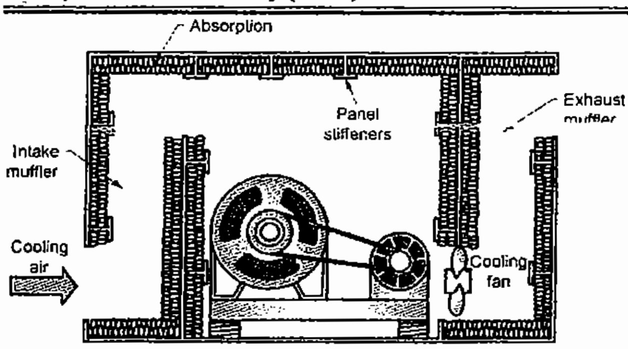


Fig. 6.32.6 : Schematic Sectional View of a Partial Enclosure with Mufflers

- With properly designed mufflers, a partial enclosure can be used as effective as complete enclosure.

6.33 WHITE NOISE

- **White noise** : *White noise is a type of noise that is produced by combining sounds of all different frequencies together within the range of human hearing (generally from 20 Hz to 20 kHz)*
- If all imaginable tones that a human can hear are taken and combined them together, it is a **white noise**.
- The word "white" is used to describe this type of noise because of the way white light works. White light is light that is made up of all colors (frequencies) of light combined together. In the similar way, the white noise is a combination of different frequencies of sound.
- Because white noise contains all frequencies, it is frequently used to mask other sounds. If you are in a room and voices from the room next-door are leaking into room, you might turn on a fan or TV to tone out the voices. The fan or TV produces a good approximation of white noise.
- If two people are talking at the same time, the human brain can normally pick up one of the two voices and understand it. If three people are talking simultaneously, human brain can probably still pick up one voice. However, if 1,000 people are talking simultaneously, there is no way that human brain can pick up one voice. Thus, 1,000 people talking together sounds like a white noise.

6.34 NOISE MEASURING INSTRUMENTS

The noise measuring instruments determine the intensity of the noise at particular distance from the source and frequency of noise.

Various Elements in Noise Measuring System :

- The various elements in a noise measuring system, shown in Fig. 6.34.1, are :

1.	Microphone
2.	Sound Level Meter
3.	Data Storage Facilities

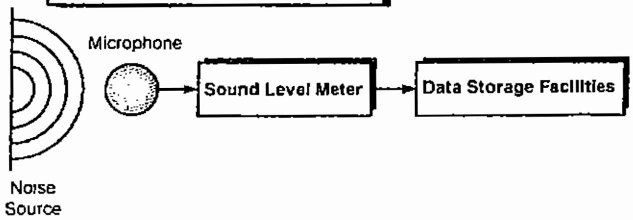


Fig. 6.34.1 : Noise Measuring System

- Not all elements are used in every measuring system. The microphone can be connected to a sound level meter or directly to a magnetic tape recorder for data storage.

6.34.1 Microphones :

University Questions

Q. What is the function of microphone ? Explain the working of anyone type of microphone.

SPPU : Dec. 11, May 14

Q. Explain the working of a microphone.

SPPU : May 12, Dec. 13, May 15, May 16

Microphone

Microphone *is a device which converts acoustical energy (sound waves) into electrical energy (the audio signal).*

Working Principle of Microphone :

- The microphone is the interface between the sound field and the measuring system. It responds to sound pressure and transforms it into an electric signal, which can be interpreted by the measuring instrument (e.g. the sound level meter).
- The microphone (Fig. 6.34.2) contains a diaphragm. The diaphragm (membrane) is a thin piece of material such as paper, plastic or aluminum which vibrates when it is struck by sound waves. When the diaphragm vibrates, it causes other components in the microphone to vibrate. These vibrations are converted into an electrical current.

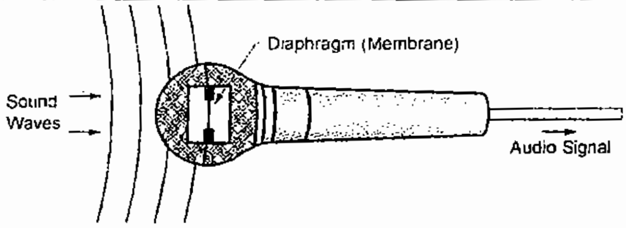


Fig. 6.34.2 : Microphone

Types of Microphones :

- The different types of microphones are as shown in Fig. 6.34.3.

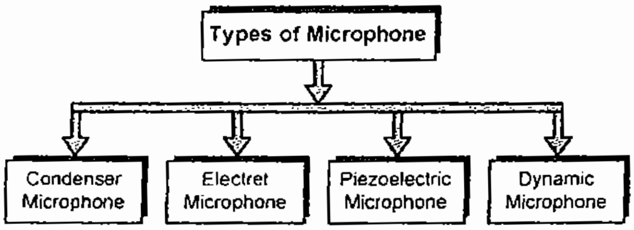


Fig. 6.34.3 : Types of Microphones

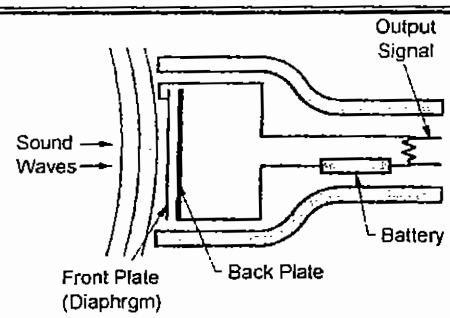
1. Condenser microphone :

University Question

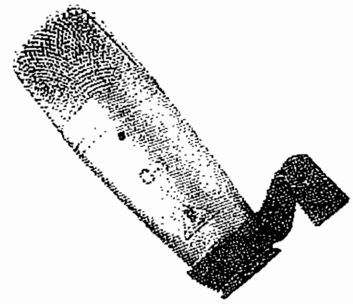
Q. Write short note on : Condenser microphone.

SPPU : Dec. 12

- Condenser or capacitor, stores energy in the form of an electrostatic field.
- A condenser microphone has a capacitor. A capacitor has two plates with a voltage between them. In the condenser mic, one of these plates is made of very light material and acts as the diaphragm (membrane). The diaphragm vibrates when struck by sound waves, changing the distance between the two plates and thereby changing the capacitance. Specifically, when the plates are closer, capacitance increases and a charge current occurs. When the plates are further apart, capacitance decreases and a discharge current occurs.
- A battery is required across the capacitor for the power source. The voltage is supplied by a battery in the mic.
- Figs. 6.34.4(a) and (b) shows the configuration of condenser microphone and actual condenser microphone respectively.
- The condenser microphones are more accurate than the other types and are mostly used in precision sound level meters.



(a) Configuration of Condenser Microphone

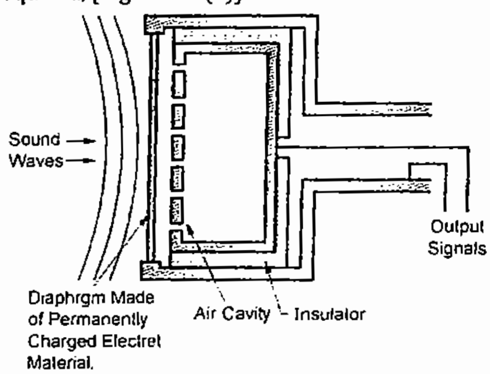


(b) Actual Condenser Microphone

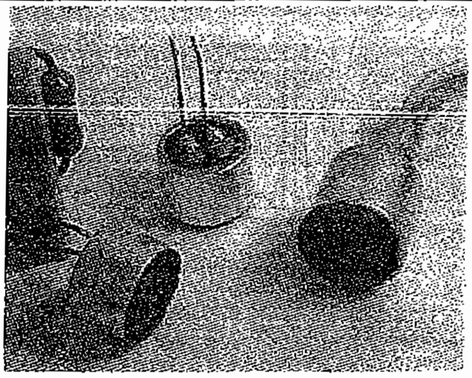
Fig. 6.34.4 : Condenser Microphone

2. Electret microphone :

- Electret microphone is exactly based on the technique of condenser microphone. However, it has a special type of capacitor with a permanent voltage built in it during the manufacturing. This is somewhat like a permanent battery and it does not require any battery or external power for operation.
- In electret microphone the potential difference is provided by a permanent electrostatic charge on the condenser plates and no external polarizing voltage is required, [Fig. 6.34.5(a)].



(a) Configuration of Electret Microphone



(b) Actual Electret Microphone

Fig. 6.34.5 : Electret Microphone

6.34.2 Sound Level Meter :

University Questions

Q. Explain, with neat sketch the working of sound level meter

SPPU : Dec. 11, May 13, Dec. 14, May 17, May 19

Q. Write short note on : Sound level meter.

SPPU : Dec. 12, May 14

- **Sound level meter** : One of the most useful and important instrument in the analysis of noise is the sound level meter, shown in Fig. 6.34.6. ***Sound level meter*** *amplifies the very small output signal from the microphone and make it available for processing and for visual display by a meter contained within the unit.*
- **Elements of sound level meter :**

  The various elements of sound level meter (Fig. 6.34.6) are as follows :

  (i) Pre-amplifier (ii) Attenuator

  (iii) Amplifier (iv) Weighting network or filter

  (v) Rectifier
- The electrical signal from the microphone is fed to the pre-amplifier of the sound level meter and it is further fed to a weighted filter over a specified range of frequencies. From filler, the signals are fed to amplifier for amplification. From amplifier the signals enter :

  (a) Rectifier and then digital display meter; or

  (b) Analog meter
- The output of the rectifier is given to other instruments such as a tape recorder, graphic level recorder or for rectification and direct reading on the meter.
- In some cases, the sound level meter does not include a logarithmic converter. The scale on the indicating device is then exponential. In this case, the dynamic range of the display is usually restricted to 10 to 16 dB. When a log converter is used, the display scale is linear in dB and its dynamic range is usually much greater.

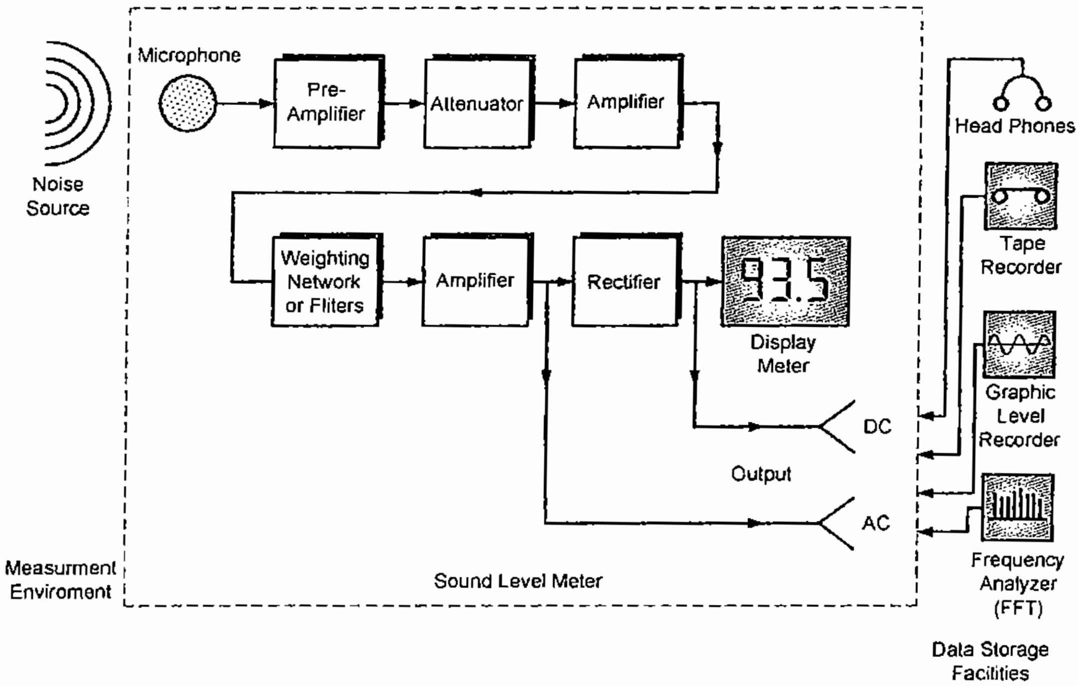


Fig. 6.34.6 : Sound Level Meter

Fig. 6.34.7 : Sound Level Meter

6.34.3 Sound Frequency Analyzer :

- ***Frequency analyzer** is a device that provides the capability for analysis of a noise signal in the frequency domain by electronically separating the signal energy into various frequency band.* This separation is performed by means of a set of filters.
- The objective of frequency analysis is to determine how the overall sound level is distributed over a range of frequencies. The simplest type of sound frequency analyzer contains sets of passive filters (octave or one third octave) that can be inserted between the two amplifiers.
- These are sequential instruments making measurements in one band at a time. This strongly restricts their use, as the noise must be constant both in amplitude and in frequency during the 5 to 10 minutes of the analysis.
- More sophisticated analyzers have the possibility to make the frequency analysis in all desired bands at the same time. These are analyzers using a set of parallel filters or using the fast fourier transform (FFT) of the input signal before recombining the data into the desired bands [Fig. 6.34.8].

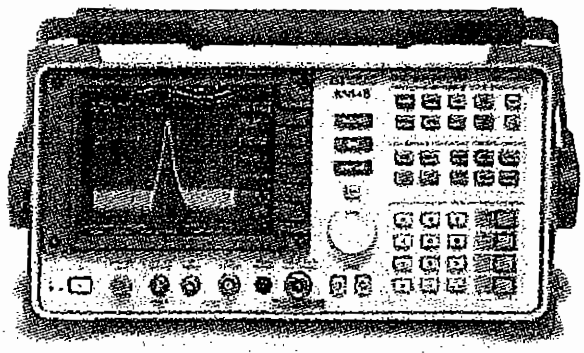


Fig. 6.34.8 : Sound Frequency Analyzer

6.34.4 Recorders :

Recorders :

The recorders are extremely important in analysis and control of industrial noise because they provide a permanent record of the noise data, which is then available for the subsequent processing in a laboratory.

Types of Recorders :

1. Graphic level recorder
2. Magnetic tape recorder

1. Graphic level recorder :

University Question

Q. Write short note with neat sketch on : Graphical level recorder. **SPPU : May 14**

- If the sound level meter has a logarithmic DC output facility, common graphic recorders can be used to obtain a permanent record of the evolution of the sound level.
- If there is no DC output or if this output is not proportional to the dB level but proportional only to the RMS pressure, then a special recorder must be used.
- The graph allows only the determination of maximum and minimum levels and cannot be used to define any average level.
- The actual graphic level recorder is shown in Fig. 6.34.9.

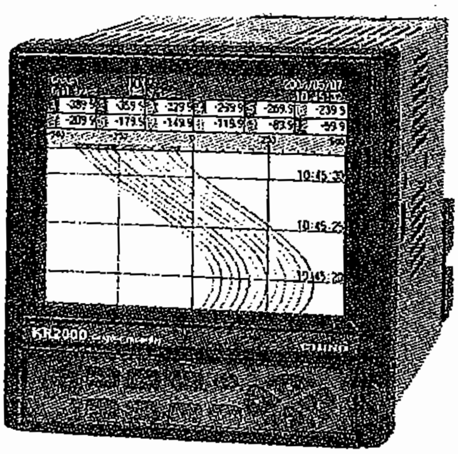


Fig. 6.34.9 : Graphic Level Recorder

(ii) Magnetic tape recorders :

University Question

Q. Write short note with neat sketch on : magnetic tape recorder. **SPPU : May 14**

- The tape recorders may be either open-reel or cassette machines.
- Magnetic tape recorders are used to make a permanent recording of the noise for future analysis or reference. Some HIFI audio recorders can be used, provided their frequency response and dynamic range are suitable.
- The actual magnetic tape recorder is shown in Fig. 6.34.10.

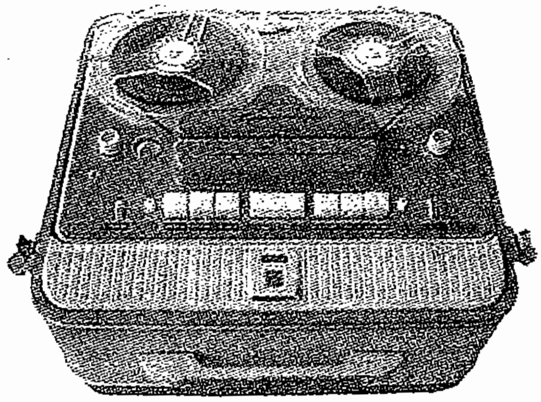


Fig. 6.34.10 : Magnetic Tape Recorders

6.35 NOISE MEASUREMENT ENVIRONMENT

University Question

Q. Explain anechoic chamber and reverberant chamber. **SPPU : May 16, Dec. 19**

- Measurement of noise can be made in following environments :

(i) Anechoic Chamber
(ii) Semi-Anechoic Chamber
(iii) Reverberant Chamber
(iv) Out of Doors Environment

(i) Anechoic Chamber :

- This is special chamber or room in which all side walls, ceiling and floor are covered by the sound absorbing materials for maximum absorption of sound during measurement.
- The ideal anechoic chamber will have no reflecting surface.

- The anechoic chamber measurement can yield the information like, sound radiation pattern and narrow band frequency data.

(ii) Semi-anechoic Chamber :

- In this chamber or room all side walls and ceilings have sound absorption materials but the floors is having reflecting material.
- The microphone response varies with the angle of incidence of the sound wave on the microphone it is called as ***"directivity"***. Due to reflection of the floor some directivity is caused in semi-anechoic chamber.

(iii) Reverberant Chamber :

- In this chamber all the walls, ceiling and floor are acoustically hard reflected surface. There is reflections and re-reflections which result in diffuse pressure field.
- Reverberant chambers are used to take average of energy radiated from different machines. In reverberant chamber, the effects of all sound sources are averaged out and therefore no information is obtained about phase relationship, directionality and location of different sources of sound.
- To increase diffusion of sound, the reverberant rooms are usually made of slightly irregular shape and paddle like turning vanes are provided for the same purpose. To minimize the effect of the vibration of the nearby structures, the foundations of such room are isolated from these structures.
- For sound power measurement, the reverberant chamber volume should be 200 m^3 minimum.

(iv) Out of Doors Environment :

- The out of doors environment can be treated as Semi-anechoic chamber if there are no reflecting surfaces within 35 meters from the microphone or from the machine under test.
- Building, publicity board or banners, and hillsides are reflecting surfaces, which should be avoided during test.

6.36 PASS-BY-NOISE MEASUREMENT

University Question

Q. Write short note on : Pass-by-noise. **SPPU : May 18**

Pass-By Noise Measurement :

- A pass-by noise measurement is defined as the method of measuring the noise emission of a road vehicle under acceleration conditions, with various gear positions in a certain measurement range.
- The pass - by noise measurement is mandatory for automotive manufacturers in terms of product certification. The, ISO (Internal Organization for Standardization) has norms for measurement, analysis and reporting format of pass-by-noise measurements test.
- The pass-by noise measurement should be performed in a large open space for type approval of commercial vehicles, or measured during the official test station's manufacturing stage. Therefore, it is very important that the certification of emission noise measurement is performed before mass production starts.

Two Types of Techniques for Pass-By-Noise Measurement :

1. Field Pass-By-Noise Measurement.
2. Indoor Pass-By-Noise measurement

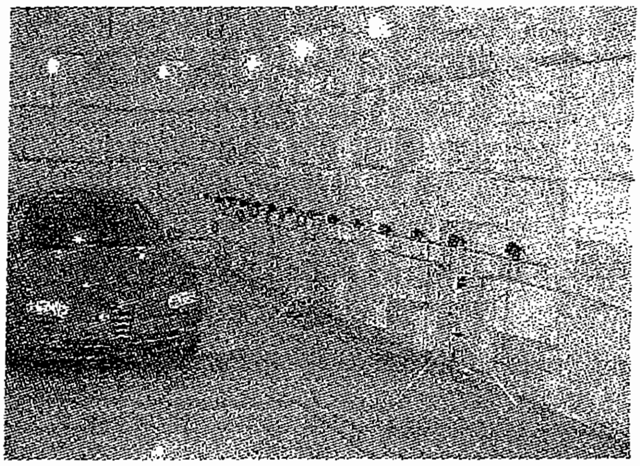
1. Field Pass-By-Noise Measurement :

In field pass-by-noise measurement, the test vehicle is made to pass by two stationary microphones. The sound is measured by two stationary microphones.

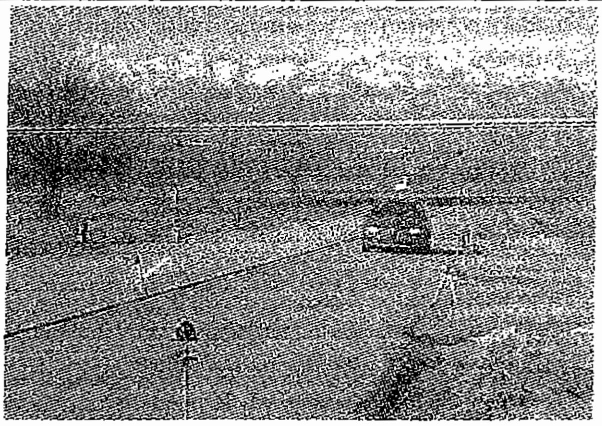
2. Indoor Pass-By-Noise Measurement :

In some cases, pass-by-noise measurement cannot be taken in the field because of bad weather or bad test-track conditions. In such cases, the indoor simulated pass-by-noise measurement is done. The indoor pass-by-noise measurement is considered as the conformance test, together with the field pass-by noise test.

- Fig. 6.36.1 shows indoor pass-by noise and field pass-by noise measurement technique.



(a) Indoor Pass By Noise Testing



(b) Outdoor Pass By Noise Testing

Fig. 6.36.1 : Pass-by noise measurement techniques

- In indoor pass-by setups, two rows of microphones placed alongside the vehicle. The vehicle runs on a chassis dynamometer and is accelerated in the same way it would be for a filed pass-by measurement. Time histories are measured by the microphones along with vehicle parameters and dynamometer drum speed. A sophisticated algorithm uses information from the dynamometer to calculate a vehicle's position relative to the microphones as a function of time. This is then used extract the contributing portions of the time histories that correspond to when the vehicle would have passed the standard microphone positions had it been moving.

6.37 BIS NOISE STANDARDS

- The rapid industrialization in developed and developing countries resulted in increased noise pollution. Most of these countries, including India, have introduced regulations about noise control to protect from the hazards of excessive noise.
- The excessive noise induces hearing loss which is either temporary or permanent in nature, depending on the length and severity of noise exposure.

6.37.1 BIS Permissible Noise Levels for Various Zones :

- According to the Government of India notification, the permissible noise levels for various zones are listed in Table 6.37.1.

Table 6.37.1 : Permissible Noise Levels for Various Zones in India

Sr. No.	Type of Zone	Exposure Noise Level (Leq) in dB (A)	
		Day Time (6.00 am to 21.00 pm)	Night Time (21.00 pm to 6.00 am)
1	Silence Zone	50	45
2	Residential Zone	55	45
3	Commercial Zone	65	55
4	Industrial Zone	75	70

- **Silence zone :** It is the area upto 100 m around certain premises like : hospitals, educational institutes, courts, etc.

6.37.2 BIS Acceptable Outdoor and Indoor Noise Levels :

According to Bureau of Indian Standards (BIS) (IS : 4954-1968), the acceptable outdoor noise levels in residential area are is given in Table 6.37.2.

Table 6.37.2 : Acceptable Outdoor Noise Levels in Residential Area as per Noise Standards (IS : 4954-1968)

Sr. No.	Type of Residential Area	Acceptable Noise Level in dB (A)
1	Rural Area	25 to 35
2	Suburban Area	30 to 40
3	Residential Urban Area	35 to 45
4	City Area	45 to 55
5	Industrial Area	50 to 65

6.37.3 BIS Permissible Noise Exposure Limits for Factories :

- According to factories noise regulation act, the permissible exposure limits for noise are given in Table 6.37.3. From this table it is seen that, for every 3 dB increase in sound pressure level, the exposure time is reduced by half.
- No person shall be exposed to an equivalent sound pressure level of 85 dB over an 8-hrs per day.

Table 6.37.3 : Permissible Noise Exposure Limits

Sound Pressure Level (SPL), dBA	Maximum Duration (T) per day
82	16 hrs
84	10 hrs, 5 mins
85	8 hrs
88	4 hrs
91	2 hrs
94	1 hrs
97	30 mins
100	15 mins
110	1.5 mins
111	1 min

Ex.6.37.1 :

- A worker is exposed to noise according to the following schedule :

Exposure level [dB]	92	95	97	102
Period of exposure [hrs]	3	2	2	1

- Does the daily noise dose is exceeded as per OSHA standards. **SPPU : Dec. 15, 6 Marks**

Soln. :

1. Permissible Time Duration for Noise Exposure at Four Given SPL :

Exposure Level (dBA)	92	95	97	102
Period of Exposure (t)	3 Hrs.	2 Hrs.	2 Hrs.	1 Hrs.
Permissible Duration from OSHA Standard (T) (from Table 6.21.3)	1 Hr.35 min i.e. 1.58 Hrs.	48 min i.e. 0.8 Hrs.	30 min i.e. 0.5 Hrs.	9 min i.e. 0.15 Hrs.

2. Daily Noise Dose (D) :

The daily does 'D' is given by,

$$D = \frac{t_1}{T_1} + \frac{t_1}{T_2} + \frac{t_3}{T_3} + \ldots + \frac{t_n}{T_n}$$

$$D = \sum_n \frac{t_n}{T_n}$$

where, t_n = duration in hours for which a man is exposed to a given SPL.

T_n = maximum exposure time limit in hours permitted at that SPL

$$\therefore \quad D = \frac{3}{1.58} + \frac{2}{0.8} + \frac{2}{0.5} + \frac{1}{0.15}$$

or $\quad D = 1.89 + 2.5 + 4 + 6.66 = 15 \geq 1$

As per OSHA standard, the daily noise dose (D) must be less than or equal to 1.

Thus, the daily dose is exceeded. ...Ans.

6.38 OCTAVE BANDS

University Question

Q. Write short note on : octave bands. **SPPU : May 18**

- In noise analysis, the frequency analysis in made by using spectrum analysers.
- The frequency range is divided into sets of frequencies known as frequency bands. Each frequency band contains a specific range of frequencies. The frequency bands are defined by scale called octave band.
- **Octave band** *is defined as a range of frequencies in which the highest frequency (f_h) is twice the lowest frequency (f_l) within the range. Thus, for one octave band,*

$$f_h = 2f_l \quad \ldots(a)$$

- For noise analysis, fixed octave bands have been internationally agreed.
- **Central frequency of octave band** : Since central frequency f_c is the arithmetic mean of f_h and f_l on logarithmic scale, we can write

$$\log f_c = \frac{1}{2} (\log f_h + \log f_l)$$

or $\quad \log f_c = \log (f_h f_l)^{1/2}$

$$\therefore \quad f_c = (f_h f_l)^{1/2} = \sqrt{f_h f_l} \quad \ldots(b)$$

From Equation (a) and (b) we can write,

$$f_c = \sqrt{2}\, f_l \text{ and } f_c = \frac{f_h}{\sqrt{2}} \quad \ldots(c)$$

Using above equations, the octave band frequencies can be determined. (Table 6.38.1)

In general, if f_h and f_l are separated by n octaves, then

$$f_h = 2^n f_l \quad \ldots(d)$$

If n = 1, the band is termed as one octave band.

If $n = \frac{1}{2}$, the band is termed as one – half octave band

If $n = \frac{1}{3}$, the band is termed as one third octave band.

Table 6.38.1 : The Centre, Lower Cut-Off and Upper Cut-Off Frequencies of All the Octave Bands

One-octave bands		
f_c	f_l	f_h
16	11	22
31.5	22	44
63	44	88
125	88	177
250	177	355
500	355	710
1000	710	1420
2000	1420	2840
4000	2840	5680
8000	5680	11360
16,000	11,360	22,720

Exercise

1. Why the measurement of vibrations is necessary?
2. How vibration measuring instruments are classified?
3. Explain the working principles of :
   (i) Vibrometers
   (ii) Velocity pick-ups
   (iii) Accelerometers
   (iv) Frequency measuring instruments
4. What are the various types of amplitude measuring instruments ?
5. Explain the working principle of seismic instrument. Discuss the response curve and variation of phase angle with frequency ratio ?
6. Explain the working of active type velocity pick-up and passive type velocity pick-up.
7. Explain the working of accelerometer.
8. What are the various types of frequency measuring instruments ? Explain the working of fullarton tachometer.
9. Write short notes on :
   (i) Fraham tachometer
   (ii) Stroboscope.
10. What are vibration exciters? Explain the working of various types of vibration exciters.
11. What is FFT? With the help of block diagram, explain the working of FFT analyzer. State the applications of FFT analyzer with reference to vibrations and noise. **[Sections 10.10 and 10.10.1]**
12. What do you mean by condition monitoring of machines? What are the various condition monitoring techniques ?
13. What do you mean by vibration monitoring of machines? Explain various types of vibration monitoring techniques.
14. What are the various instrumentation systems used for condition monitoring?
15. What are the various types of vibration tests used to know the system characteristics?

□□□