5.13.31

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Question

Let $\mathbf{A}$ and $\mathbf{B}$ be 3×3 matrices of real numbers, where $\mathbf{A}$ is symmetric, $\mathbf{B}$ is skew-symmetric and

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = (\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B}).$$

If

$$(\mathbf{AB})^\top = (-1)^k \mathbf{AB},$$

where $(\mathbf{AB})^\top$ is the transpose of the matrix $\mathbf{AB}$, find the value(s) of k.

Given:

- $\mathbf{A}$ is symmetric: $\mathbf{A}^\top = \mathbf{A}$
- $\mathbf{B}$ is skew-symmetric: $\mathbf{B}^\top = -\mathbf{B}$

Expanding the Given Equation

Expand both sides of

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = (\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})$$

Left:

$$= \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2$$

Right:

$$= \mathbf{A}^2 + \mathbf{AB} - \mathbf{BA} - \mathbf{B}^2$$

Simplifying the Equation

Set both expansions equal:

$$\mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 = \mathbf{A}^2 + \mathbf{AB} - \mathbf{BA} - \mathbf{B}^2$$

Subtract $\mathbf{A}^2$ and $-\mathbf{B}^2$ from both sides:

$$-\mathbf{AB} + \mathbf{BA} = \mathbf{AB} - \mathbf{BA}$$

$$2\mathbf{BA} = 2\mathbf{AB}$$
$$\implies \mathbf{AB} = \mathbf{BA}$$

So $\mathbf{A}$ and $\mathbf{B}$ commute.

Transpose of the Product

We have:

$$(\mathbf{AB})^\top = \mathbf{B}^\top \mathbf{A}^\top$$

Using properties,

$$= (-\mathbf{B})\mathbf{A} = -\mathbf{BA}$$

But since they commute:

$$= -\mathbf{AB}$$

Thus,

$$(\mathbf{AB})^\top = -\mathbf{AB}$$

Solving for k

Given,

$$(\mathbf{AB})^\top = (-1)^k \mathbf{AB}$$

So,

$$-\mathbf{AB} = (-1)^k \mathbf{AB}$$

If $\mathbf{AB} \neq 0$, comparing:

$$-1 = (-1)^k$$

So k must be odd.

Final Answer

$$\boxed{k \text{ is any odd integer.}}$$