7.4.12

Aditya Mishra — EE25BTECH11005

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Question

The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. Find the locus of any point $\mathbf{x}$ in the set.

Solution

For a circle,

$$\mathbf{u} = -\mathbf{c}, \quad f = \|\mathbf{u}\|^2 - r^2 \tag{1}$$

Hence, the general equation is

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2}$$

For this family,

$$\|\mathbf{u}\| = 5, \quad r = 3, \quad f = 25 - 9 = 16 \tag{3}$$

Thus,

$$\|\mathbf{x}\|^2 + 2\mathbf{u}^T\mathbf{x} + 16 \leq 0 \tag{4}$$

$$2\mathbf{u}^T\mathbf{x} \leq -\|\mathbf{x}\|^2 - 16 \tag{5}$$

Since $\|\mathbf{u}\| = 5$, by Cauchy-Schwarz inequality:

$$-5\|\mathbf{x}\| \leq \mathbf{u}^T\mathbf{x} \leq 5\|\mathbf{x}\| \tag{6}$$

The minimum value of $2\mathbf{u}^T\mathbf{x}$ is $-10\|\mathbf{x}\|$. Hence, existence requires

$$-10\|\mathbf{x}\| \leq -\|\mathbf{x}\|^2 - 16 \tag{7}$$

$$\Rightarrow \|\mathbf{x}\|^2 - 10\|\mathbf{x}\| + 16 \leq 0 \tag{8}$$

Let $t = \|\mathbf{x}\|$. Then

$$t^2 - 10t + 16 \leq 0 \tag{9}$$

$$\Rightarrow (t-2)(t-8) \leq 0 \tag{10}$$

$$\Rightarrow 2 \leq t \leq 8 \tag{11}$$

Locus

$$2 \leq \|\mathbf{x}\| \leq 8 \quad \Rightarrow \quad 4 \leq \mathbf{x}^T\mathbf{x} \leq 64 \tag{12}$$

$$\boxed{4 \leq x^2 + y^2 \leq 64}$$

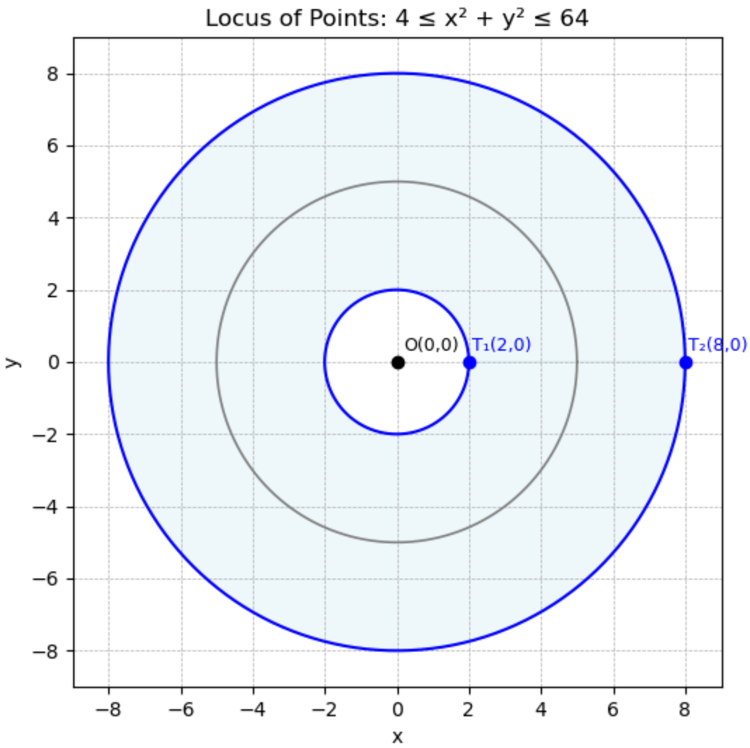

Figure 1: Locus of points for circles of radius 3 with centers on $x^2 + y^2 = 25$.