10.3.3

EE25BTECH11005 - Aditya Mishra

Question:
Find the equation of the normal lines to the curve

$$3x^2 - y^2 = 8$$

which are parallel to the line

$$x + 3y = 4.$$

Solution:

The general equation of a conic is

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{1}$$

Comparing with $3x^2 - y^2 = 8$, we get

$$\mathbf{V} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad f = -8 \tag{2}$$

The slope of the line $x + 3y = 4$ is $-\frac{1}{3}$, so the normal vector of the required normals is

$$\mathbf{n} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{3}$$

The points of contact of normals with a given normal vector are

$$\mathbf{q}_i = \mathbf{V}^{-1}(k_i \mathbf{n} - \mathbf{u}), \quad i = 1, 2 \tag{4}$$

where k_i satisfy

$$k^2 \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} - 2k\mathbf{u}^\top \mathbf{V}^{-1} \mathbf{n} + \mathbf{u}^\top \mathbf{V}^{-1} \mathbf{u} + f = 0 \tag{5}$$

Here,

$$\mathbf{V}^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{n}^\top \mathbf{V}^{-1} \mathbf{n} = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\frac{26}{3} \tag{6}$$

Substituting in the quadratic equation:

$$k^2 \left(-\frac{26}{3} \right) - 8 = 0 \quad \Rightarrow \quad k = \pm 3\sqrt{3} \tag{7}$$

The points of contact are

$$\mathbf{q}_i = \mathbf{V}^{-1}(k_i \mathbf{n}) = \begin{pmatrix} \frac{k_i}{3} \\ -3k_i \end{pmatrix} \tag{8}$$

Substituting $k_1 = 3\sqrt{3},\ k_2 = -3\sqrt{3}$, we get

$$\mathbf{q}_1 = \begin{pmatrix} \sqrt{3} \\ -9\sqrt{3} \end{pmatrix}, \quad \mathbf{q}_2 = \begin{pmatrix} -\sqrt{3} \\ 9\sqrt{3} \end{pmatrix} \tag{9}$$

The equation of the normal at $\mathbf{q}_i$ is

$$\mathbf{n}^\top (\mathbf{x} - \mathbf{q}_i) = 0 \quad \Rightarrow \quad \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{q}_i \tag{10}$$

Hence, the required normals are

$$x + 3y = \mathbf{n}^\top \mathbf{q}_1 = -26\sqrt{3} \tag{11}$$

$$x + 3y = \mathbf{n}^\top \mathbf{q}_2 = 26\sqrt{3} \tag{12}$$

$$\boxed{x + 3y = \pm 26\sqrt{3}}$$

See Figure,