1. The Discretisation Scheme

Consider a general equation:

$$k\phi+\vec{\nabla}.\left(C_y\phi\ \hat{j}+C_z\phi\ \hat{k}\right)+\vec{\nabla}.\left(D_y\frac{\partial\phi}{\partial y}\hat{j}+D_z\frac{\partial\phi}{\partial z}\hat{k}\right)=S$$

where,

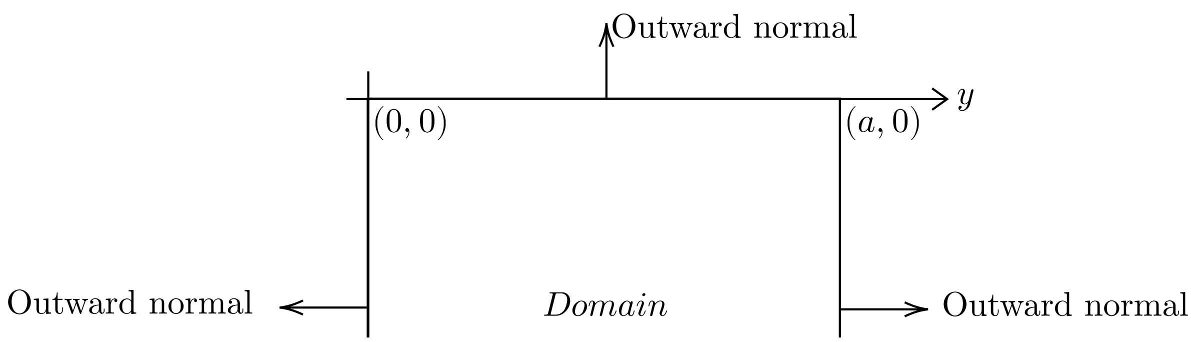
$$\vec{\nabla}=\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$$

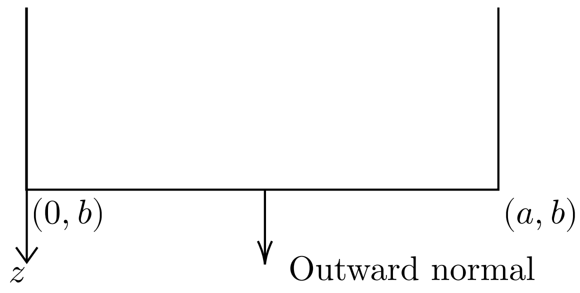
This equation is subject to boundary conditions:

$$\alpha\phi+\beta\frac{\partial\phi}{\partial n}=G$$

at the walls. The direction n the outward normal. Thus,

$$\begin{aligned}\frac{\partial\phi}{\partial n}&=\ -\frac{\partial\phi}{\partial y}\ at\ y=0\\ \frac{\partial\phi}{\partial n}&=\ +\frac{\partial\phi}{\partial y}\ at\ y=a\\ \frac{\partial\phi}{\partial n}&=\ -\frac{\partial\phi}{\partial z}\ at\ z=0\\ \frac{\partial\phi}{\partial n}&=\ +\frac{\partial\phi}{\partial z}\ at\ z=b\end{aligned}$$



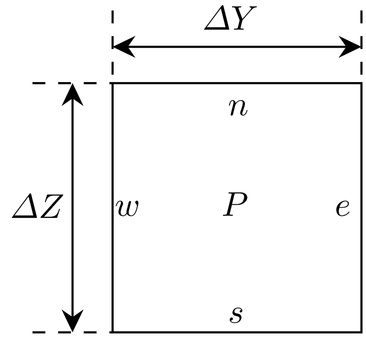


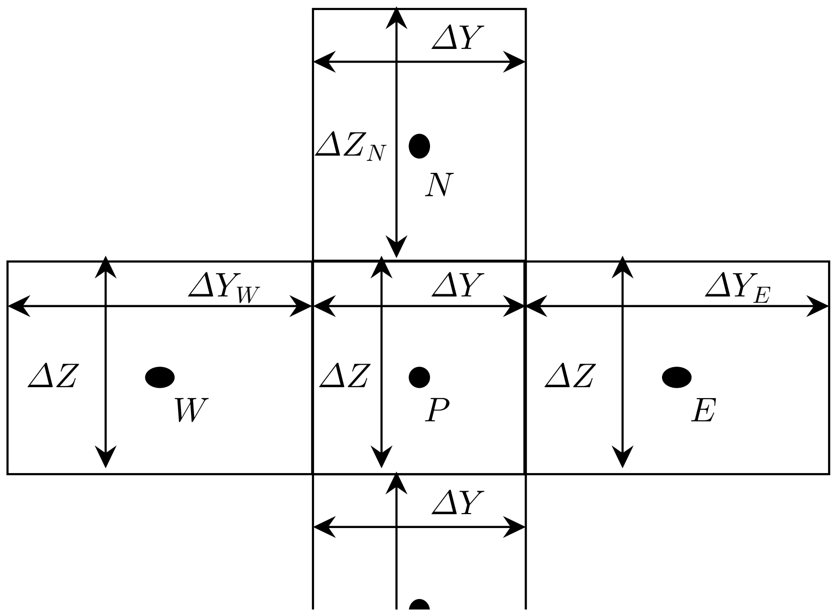
Expanding eq. (1), and setting $k = S = 0$,

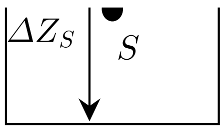
$$\left(\frac{\partial C_y\phi}{\partial y} + \frac{\partial C_z\phi}{\partial z}\right) + \left(\frac{\partial}{\partial y}D_y\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}D_z\frac{\partial \phi}{\partial z}\right) = 0$$

1.1. Discretisation in the bulk

Consider a typical cell as shown below







Integrating eq. (3) on this cell, we get

$$\left(\frac{\partial C_y\phi}{\partial y}+\frac{\partial C_z\phi}{\partial z}\right)+\left(\frac{\partial}{\partial y}D_y\frac{\partial\phi}{\partial y}+\frac{\partial}{\partial z}D_z\frac{\partial\phi}{\partial z}\right)=0$$

$$\left(\frac{\Delta Z(C_{y,e}\phi_e\ -\ C_{y,w}\phi_w)}{\Delta Y\Delta Z}+\frac{\Delta Y(C_{z,s}\phi_s\ -\ C_{z,n}\phi_n)}{\Delta Y\Delta Z}\right)+$$
$$\left(\frac{\Delta Z\left(D_{y,e}\left.\frac{\partial\phi}{\partial y}\right|_e-D_{y,w}\left.\frac{\partial\phi}{\partial y}\right|_w\right)}{\Delta Y\Delta Z}+\frac{\Delta Y\left(D_{z,s}\left.\frac{\partial\phi}{\partial z}\right|_s-D_{z,n}\left.\frac{\partial\phi}{\partial z}\right|_n\right)}{\Delta Y\Delta Z}\right)=0$$

Where,

$$\phi_e=\frac{\Delta Y_E\ \phi_P+\Delta Y\ \phi_E}{\Delta Y_E+\Delta Y}$$
$$\phi_w=\frac{\Delta Y_W\ \phi_P+\Delta Y\ \phi_W}{\Delta Y_W+\Delta Y}$$
$$\phi_s=\frac{\Delta Z_S\ \phi_P+\Delta Z\ \phi_S}{\Delta Z_S+\Delta Z}$$
$$\phi_n=\frac{\Delta Z_N\ \phi_P+\Delta Z\ \phi_N}{\Delta Z_N+\Delta Z}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_e=\frac{2(\phi_E-\phi_P)}{\Delta Y_E+\Delta Y}$$
$$\left.\frac{\partial\phi}{\partial y}\right|_w=\frac{2(\phi_P-\phi_W)}{\Delta Y_W+\Delta Y}$$
$$\left.\frac{\partial\phi}{\partial z}\right|_s=\frac{2(\phi_S-\phi_P)}{\Delta Z_S+\Delta Z}$$
$$\left.\frac{\partial\phi}{\partial z}\right|_n=\frac{2(\phi_P-\phi_N)}{\Delta Z_N+\Delta Z}$$

$$C_{y,e} = \frac{\Delta Y_E\ C_{y,P} + \Delta Y\, C_{y,E}}{\Delta Y_E + \Delta Y}$$
$$C_{y,w} = \frac{\Delta Y_W\ C_{y,P} + \Delta Y\, C_{yW}}{\Delta Y_W + \Delta Y}$$
$$C_{z,s} = \frac{\Delta Z_S\ C_{z,P} + \Delta Z\ C_{z,S}}{\Delta Z_S + \Delta Z}$$
$$C_{z,n} = \frac{\Delta Z_N\ C_{z,P} + \Delta Z\ C_{z,N}}{\Delta Z_N + \Delta Z}$$

$$\frac{\Delta Y_E + \Delta Y}{D_{y,e}} = \frac{\Delta Y_E}{D_{y,E}} + \frac{\Delta Y}{D_{y,P}}$$
$$\frac{\Delta Y_W + \Delta Y}{D_{y,w}} = \frac{\Delta Y_W}{D_{y,W}} + \frac{\Delta Y}{D_{y,P}}$$
$$\frac{\Delta Z_E + \Delta Z}{D_{y,e}} = \frac{\Delta Z_E}{D_{y,E}} + \frac{\Delta Z}{D_{y,P}}$$
$$\frac{\Delta Z_W + \Delta YZ}{D_{y,w}} = \frac{\Delta Y_W}{D_{y,W}} + \frac{\Delta Z}{D_{y,P}}$$

Expanding eq. 5,

$$\frac{\Delta Z\left(\frac{C_{y,e}\Delta Y_E}{\Delta Y_E + \Delta Y}\phi_P + \frac{C_{y,e}\Delta Y}{\Delta Y_E + \Delta Y}\phi_E - \frac{C_{y,w}\Delta Y_W}{\Delta Y_W + \Delta Y}\phi_P - \frac{C_{y,w}\Delta Y}{\Delta Y_W + \Delta Y}\phi_W\right)}{\Delta Y \Delta Z} +$$
$$\frac{\Delta Y\left(\frac{C_{z,s}\Delta Z_S}{\Delta Z_S + \Delta Z}\phi_P + \frac{C_{z,s}\Delta Z}{\Delta Z_S + \Delta Z}\phi_S - \frac{C_{z,n}\Delta Z_N}{\Delta Z_N + \Delta Z}\phi_P - \frac{C_{z,n}\Delta Z}{\Delta Z_N + \Delta Z}\phi_N\right)}{\Delta Y \Delta Z} +$$

$$\frac{\Delta Z\left(\frac{2D_{y,e}}{\Delta Y_E + \Delta Y}\phi_E - \frac{2D_{y,e}}{\Delta Y_E + \Delta Y}\phi_P - \frac{2D_{y,w}}{\Delta Y_W + \Delta Y}\phi_P + \frac{2D_{y,w}}{\Delta Y_W + \Delta Y}\phi_W\right)}{\Delta Y \Delta Z} +$$
$$\frac{\Delta Y\left(\frac{2D_{z,s}}{\Delta Z_S + \Delta Z}\phi_S - \frac{2D_{z,s}}{\Delta Z_S + \Delta Z}\phi_P - \frac{2D_{z,n}}{\Delta Z_N + \Delta Z}\phi_P + \frac{2D_{z,n}}{\Delta Z_N + \Delta Z}\phi_N\right)}{\Delta Y \Delta Z} = 0$$

This may be written as

$$N_{Bulk}\phi_N + E_{Bulk}\phi_E + W_{Bulk}\phi_W + S_{Bulk}\phi_S + P_{Bulk}\phi_P = R_{Bulk}$$

where,

$$N_{Bulk} = \frac{\Delta Y}{\Delta Y \Delta Z}\left(\frac{2D_{z,n}}{\Delta Z_N + \Delta Z} - \frac{C_{z,n}\Delta Z}{\Delta Z_N + \Delta Z}\right)$$

$$E_{Bulk} = \frac{\Delta Z}{\Delta Y \Delta Z}\left(\frac{2D_{y,e}}{\Delta Y_E + \Delta Y} + \frac{C_{y,e}\Delta Y}{\Delta Y_E + \Delta Y}\right)$$

$$W_{Bulk} = \frac{\Delta Z}{\Delta Y \Delta Z}\left(\frac{2D_{y,w}}{\Delta Y_W + \Delta Y} - \frac{C_{y,w}\Delta Y}{\Delta Y_W + \Delta Y}\right)$$

$$S_{Bulk} = \frac{\Delta Y}{\Delta Y \Delta Z}\left(\frac{2D_{z,s}}{\Delta Z_S + \Delta Z} + \frac{C_{z,s}\Delta Z}{\Delta Z_S + \Delta Z}\right)$$

$$P_{Bulk} = \frac{\Delta Y}{\Delta Y \Delta Z}\left(\frac{C_{z,s}\Delta Z_S}{\Delta Z_S + \Delta Z} - \frac{C_{z,n}\Delta Z_N}{\Delta Z_N + \Delta Z} - \frac{2D_{z,s}}{\Delta Z_S + \Delta Z} - \frac{2D_{z,n}}{\Delta Z_N + \Delta Z}\right)$$

$$+\frac{\Delta Z}{\Delta Y \Delta Z}\left(\frac{C_{y,e}\Delta Y_E}{\Delta Y_E + \Delta Y} - \frac{C_{y,w}\Delta Y_W}{\Delta Y_W + \Delta Y} - \frac{2D_{y,e}}{\Delta Y_E + \Delta Y} - \frac{2D_{y,w}}{\Delta Y_W + \Delta Y}\right)$$

$$R_{Bulk} = 0$$

1.2. Discretisation at the boundaries

1.2.1. At the Eastern boundary:

$$\alpha\phi_e + \beta \left.\frac{\partial\phi}{\partial y}\right|_e = G$$

$$But \ \left.\frac{\partial\phi}{\partial y}\right|_e = \frac{2(\phi_e - \phi_P)}{\Delta Y}$$

$$\alpha\phi_e + \beta\frac{2(\phi_e - \phi_P)}{\Delta Y} = G$$

$$Solving\ for\ \phi_e,$$

$$\phi_e = \frac{G\Delta Y + 2\beta\phi_P}{\alpha\Delta Y + 2\beta}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_e = \frac{2(G - \alpha\phi_P)}{\alpha\Delta Y + 2\beta}$$

Substituting in (5),

$$\left(\frac{\Delta Z\left(C_{y,e}\dfrac{G\Delta Y+2\beta\phi_P}{\alpha\Delta Y+2\beta}\;-\;C_{y,w}\phi_w\right)}{\Delta Y\Delta Z}+\frac{\Delta Y(C_{z,s}\phi_s\;-\;C_{z,n}\phi_n)}{\Delta Y\Delta Z}\right)+$$

$$\left(\frac{\Delta Z\left(2D_{y,e}\dfrac{2(G-\alpha\phi_P)}{\alpha\Delta Y+2\beta}-D_{y,w}\left.\dfrac{\partial\phi}{\partial y}\right|_w\right)}{\Delta Y\Delta Z}+\frac{\Delta Y\left(D_{z,s}\left.\dfrac{\partial\phi}{\partial z}\right|_s-D_{z,n}\left.\dfrac{\partial\phi}{\partial y}\right|_n\right)}{\Delta Y\Delta Z}\right)=0$$

$$Expanding,$$

$$\frac{\Delta Z\left(C_{y,e}\dfrac{G\Delta Y}{\alpha\Delta Y+2\beta}+C_{y,e}\dfrac{2\beta}{\alpha\Delta Y+2\beta}\phi_P-\dfrac{C_{y,w}\Delta Y_W}{\Delta Y_W+\Delta Y}\phi_P-\dfrac{C_{y,w}\Delta Y}{\Delta Y_W+\Delta Y}\phi_W\right)}{\Delta Y\Delta Z}+$$

$$\frac{\Delta Y\left(\dfrac{C_{z,s}\Delta Z_S}{\Delta Z_S+\Delta Z}\phi_P+\dfrac{C_{z,s}\Delta Z}{\Delta Z_S+\Delta Z}\phi_S-\dfrac{C_{z,n}\Delta Z_N}{\Delta Z_N+\Delta Z}\phi_P-\dfrac{C_{z,n}\Delta Z}{\Delta Z_N+\Delta Z}\phi_N\right)}{\Delta Y\Delta Z}+$$

$$\frac{\Delta Z\left(2D_{y,e}\dfrac{G}{\alpha\Delta Y+2\beta}-2D_{y,e}\dfrac{\alpha}{\alpha\Delta Y+2\beta}\phi_P-\dfrac{2D_{y,w}}{\Delta Y_W+\Delta Y}\phi_P+\dfrac{2D_{y,w}}{\Delta Y_W+\Delta Y}\phi_W\right)}{\Delta Y\Delta Z}+$$

$$\frac{\Delta Y\left(\dfrac{2D_{z,s}}{\Delta Z_S+\Delta Z}\phi_S-\dfrac{2D_{z,s}}{\Delta Z_S+\Delta Z}\phi_P-\dfrac{2D_{z,n}}{\Delta Z_N+\Delta Z}\phi_P+\dfrac{2D_{z,n}}{\Delta Z_N+\Delta Z}\phi_N\right)}{\Delta Y\Delta Z}=0$$

This may be written as

$$N_{EB}\phi_N+E_{EB}\phi_E+W_{EB}\phi_W+S_{EB}\phi_S+P_{EB}\phi_P=R_{EB}$$

where,

$$N_{EB}=\frac{\Delta Y}{\Delta Y\Delta Z}\left(\frac{2D_{z,n}}{\Delta Z_N+\Delta Z}-\frac{C_{z,n}\Delta Z}{\Delta Z_N+\Delta Z}\right)$$

$$E_{EB}=0$$

$$W_{EB}=\frac{\Delta Z}{\Delta Y\Delta Z}\left(\frac{2D_{y,w}}{\Delta Y_W+\Delta Y}-\frac{C_{y,w}\Delta Y}{\Delta Y_W+\Delta Y}\right)$$

$$S_{EB}=\frac{\Delta Y}{\Delta Y\Delta Z}\left(\frac{2D_{z,s}}{\Delta Z_S+\Delta Z}+\frac{C_{z,s}\Delta Z}{\Delta Z_S+\Delta Z}\right)$$

$$P_{EB}=\frac{\Delta Y}{\Delta Y\Delta Z}\left(\frac{C_{z,s}\Delta Z_S}{\Delta Z_S+\Delta Z}-\frac{C_{z,n}\Delta Z_N}{\Delta Z_N+\Delta Z}-\frac{2D_{z,s}}{\Delta Z_S+\Delta Z}-\frac{2D_{z,n}}{\Delta Z_N+\Delta Z}\right)$$

$$+\frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{2\beta}{\alpha\Delta Y+2\beta}\phi_P-\ \frac{C_{y,w}\Delta Y_W}{\Delta Y_W+\Delta Y}-2D_{y,e}\frac{\alpha}{\alpha\Delta Y+2\beta}-\frac{2D_{y,w}}{\Delta Y_W+\Delta Y}\right)$$

$$R_{EB}=\ -\frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{G\Delta Y}{\alpha\Delta Y+2\beta}+2D_{y,e}\frac{G}{\alpha\Delta Y+2\beta}\right)$$

Thus, we find that,

$$N_{EB}=N_{Bulk}$$
$$W_{EB}=W_{Bulk}$$
$$E_{EB}=0$$
$$S_{EB}=S_{Bulk}$$

$$P_{EB}=P_{Bulk}+\frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{2\beta}{\alpha\Delta Y+2\beta}+2D_{y,e}\frac{2\beta}{\Delta Y(\alpha\Delta Y+2\beta)}\right)$$
$$R_{EB}=R_{Bulk}-\frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{G\Delta Y}{\alpha\Delta Y+2\beta}+2D_{y,e}\frac{G}{\alpha\Delta Y+2\beta}\right)$$

1.2.2. At the Western Boundary

$$\alpha\phi_w-\beta\left.\frac{\partial\phi}{\partial y}\right|_w=G$$
$$But\ \left.\frac{\partial\phi}{\partial y}\right|_w=\frac{2(\phi_P-\phi_w)}{\Delta Y}$$

$$\alpha\phi_w+\beta\frac{2(\phi_w-\phi_P)}{\Delta Y}=G$$
$$Solving\ for\ \phi_w,$$
$$\phi_w=\frac{G\Delta Y+2\beta\phi_P}{\alpha\Delta Y+2\beta}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_w=\frac{2(G-\alpha\phi_P)}{\alpha\Delta Y+2\beta}$$

Substituting in (5) and expanding,

$$\frac{\Delta Z\left(\frac{C_{y,e}\Delta Y_E}{\Delta Y_E+\Delta Y}\phi_P\ +\frac{C_{y,e}\Delta Y}{\Delta Y_E+\Delta Y}\phi_E-C_{y,w}\frac{G\Delta Y}{\alpha\Delta Y+2\beta}-C_{y,w}\frac{2\beta}{\alpha\Delta Y+2\beta}\phi_P\right)}{\Delta Y\Delta Z}+$$

$$\frac{\Delta Y\left(\frac{C_{z,s}\Delta Z_S}{\Delta Z_S+\Delta Z}\phi_P+\frac{C_{z,s}\Delta Z}{\Delta Z_S+\Delta Z}\phi_S-\frac{C_{z,n}\Delta Z_N}{\Delta Z_N+\Delta Z}\phi_P-\frac{C_{z,n}\Delta Z}{\Delta Z_N+\Delta Z}\phi_N\right)}{\Delta Y\Delta Z}+$$

$$\frac{\Delta Z\left(\frac{2D_{y,e}}{\Delta Y_E+\Delta Y}\phi_E-\frac{2D_{y,e}}{\Delta Y_E+\Delta Y}\phi_P+2D_{y,w}\frac{G}{\alpha\Delta Y+2\beta}-2D_{y,w}\frac{\alpha}{\alpha\Delta Y+2\beta}\phi_P\right)}{\Delta Y\Delta Z}+$$

$$\frac{\Delta Y\left(\frac{2D_{z,s}}{\Delta Z_S+\Delta Z}\phi_S-\frac{2D_{z,s}}{\Delta Z_S+\Delta Z}\phi_P-\frac{2D_{z,n}}{\Delta Z_N+\Delta Z}\phi_P+\frac{2D_{z,n}}{\Delta Z_N+\Delta Z}\phi_N\right)}{\Delta Y\Delta Z}=0$$

This may be written as

$$N_{WB}\phi_N+E_{WB}\phi_E+W_{WB}\phi_W+S_{WB}\phi_S+P_{WB}\phi_P=R_{WB}$$

where,

$$N_{WB}=\frac{\Delta Y}{\Delta Y\Delta Z}\left(\frac{2D_{z,n}}{\Delta Z_N+\Delta Z}-\frac{C_{z,n}\Delta Z}{\Delta Z_N+\Delta Z}\right)$$

$$E_{WB}=\frac{\Delta Z}{\Delta Y\Delta Z}\left(\frac{2D_{y,e}}{\Delta Y_E+\Delta Y}+\frac{C_{y,e}\Delta Y}{\Delta Y_E+\Delta Y}\right)$$

$$W_{WB}=0$$

$$S_{WB}=\frac{\Delta Y}{\Delta Y\Delta Z}\left(\frac{2D_{z,s}}{\Delta Z_S+\Delta Z}+\frac{C_{z,s}\Delta Z}{\Delta Z_S+\Delta Z}\right)$$

$$P_{WB}=\frac{\Delta Y}{\Delta Y\Delta Z}\left(\frac{C_{z,s}\Delta Z_S}{\Delta Z_S+\Delta Z}-\frac{C_{z,n}\Delta Z_N}{\Delta Z_N+\Delta Z}-\frac{2D_{z,s}}{\Delta Z_S+\Delta Z}-\frac{2D_{z,n}}{\Delta Z_N+\Delta Z}\right)$$
$$+\frac{\Delta Z}{\Delta Y\Delta Z}\left(\frac{C_{y,e}\Delta Y_E}{\Delta Y_E+\Delta Y}-C_{y,w}\frac{2\beta}{\alpha\Delta Y+2\beta}-\frac{2D_{y,e}}{\Delta Y_E+\Delta Y}-2D_{y,w}\frac{\alpha}{\alpha\Delta Y+2\beta}\right)$$

$$R_{WB}=-\frac{\Delta Z}{\Delta Y\Delta Z}\left(2D_{y,w}\frac{G}{\alpha\Delta Y+2\beta}-C_{y,w}\frac{G\Delta Y}{\alpha\Delta Y+2\beta}\right)$$

Thus, we find that,

$$N_{WB}=N_{Bulk}$$
$$E_{WB}=E_{Bulk}$$
$$W_{WB}=0$$
$$S_{WB}=S_{Bulk}$$

$$P_{WB} = P_{Bulk} + \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{2\beta}{\Delta Y(\alpha\Delta Y + 2\beta)} - C_{y,w}\frac{2\beta}{\alpha\Delta Y + 2\beta}\right)$$
$$R_{WB} = R_{Bulk} - \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{G}{\alpha\Delta Y + 2\beta} - C_{y,w}\frac{G\Delta Y}{\alpha\Delta Y + 2\beta}\right)$$

1.2.3. At the Southern Boundary

Here, we use an anology of the discretization at the eastern boundary, to directly write

$$N_{SB} = N_{Bulk}$$
$$W_{SB} = W_{Bulk}$$
$$E_{SB} = E_{SB}$$
$$S_{SB} = 0$$

$$P_{SB} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{2\beta}{\alpha\Delta Z + 2\beta} + 2D_{z,s}\frac{2\beta}{\Delta Z(\alpha\Delta Z + 2\beta)}\right)$$
$$R_{SB} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{G\Delta Z}{\alpha\Delta Z + 2\beta} + 2D_{z,s}\frac{G}{\alpha\Delta Z + 2\beta}\right)$$

1.2.4. At the Northern Boundary

Here, we use an anology of the discretization at the western boundary, to directly write

$$N_{NB} = 0$$
$$E_{NB} = E_{Bulk}$$
$$W_{NB} = W_{Bulk}$$
$$S_{NB} = S_{Bulk}$$

$$P_{NB} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{2\beta}{\Delta Z(\alpha\Delta Z + 2\beta)} - C_{z,n}\frac{2\beta}{\alpha\Delta Z + 2\beta}\right)$$
$$R_{NB} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{G}{\alpha\Delta Z + 2\beta} - C_{z,n}\frac{G\Delta Z}{\alpha\Delta Z + 2\beta}\right)$$

2. Summary of equations

In the bulk,

$$N_{Bulk} = \frac{\Delta Y}{\Delta Y \Delta Z}\left(\frac{2D_{z,n}}{\Delta Z_N + \Delta Z} - \frac{C_{z,n}\Delta Z}{\Delta Z_N + \Delta Z}\right)$$

$$E_{Bulk} = \frac{\Delta Z}{\Delta Y \Delta Z}\left(\frac{2D_{y,e}}{\Delta Y_E + \Delta Y} + \frac{C_{y,e}\Delta Y}{\Delta Y_E + \Delta Y}\right)$$
$$W_{Bulk} = \frac{\Delta Z}{\Delta Y \Delta Z}\left(\frac{2D_{y,w}}{\Delta Y_W + \Delta Y} - \frac{C_{y,w}\Delta Y}{\Delta Y_W + \Delta Y}\right)$$
$$S_{Bulk} = \frac{\Delta Y}{\Delta Y \Delta Z}\left(\frac{2D_{z,s}}{\Delta Z_S + \Delta Z} + \frac{C_{z,s}\Delta Z}{\Delta Z_S + \Delta Z}\right)$$
$$P_{Bulk} = \frac{\Delta Y}{\Delta Y \Delta Z}\left(\frac{C_{z,s}\Delta Z_S}{\Delta Z_S + \Delta Z} - \frac{C_{z,n}\Delta Z_N}{\Delta Z_N + \Delta Z} - \frac{2D_{z,s}}{\Delta Z_S + \Delta Z} - \frac{2D_{z,n}}{\Delta Z_N + \Delta Z}\right)$$
$$+\frac{\Delta Z}{\Delta Y \Delta Z}\left(\frac{C_{y,e}\Delta Y_E}{\Delta Y_E + \Delta Y} - \frac{C_{y,w}\Delta Y_W}{\Delta Y_W + \Delta Y} - \frac{2D_{y,e}}{\Delta Y_E + \Delta Y} - \frac{2D_{y,w}}{\Delta Y_W + \Delta Y}\right)$$
$$R_{Bulk} = 0$$

Eastern Boundary:

$$N_{EB} = N_{Bulk}$$
$$W_{EB} = W_{Bulk}$$
$$E_{EB} = 0$$
$$S_{EB} = S_{Bulk}$$

$$P_{EB} = P_{Bulk} + \frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{2\beta}{\alpha \Delta Y + 2\beta} + 2D_{y,e}\frac{2\beta}{\Delta Y(\alpha \Delta Y + 2\beta)}\right)$$
$$R_{EB} = R_{Bulk} - \frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{G\Delta Y}{\alpha \Delta Y + 2\beta} + 2D_{y,e}\frac{G}{\alpha \Delta Y + 2\beta}\right)$$

Western Boundary:

$$N_{WB} = N_{Bulk}$$
$$E_{WB} = E_{Bulk}$$
$$W_{WB} = 0$$
$$S_{WB} = S_{Bulk}$$

$$P_{WB} = P_{Bulk} + \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{2\beta}{\Delta Y(\alpha \Delta Y + 2\beta)} - C_{y,w}\frac{2\beta}{\alpha \Delta Y + 2\beta}\right)$$
$$R_{WB} = R_{Bulk} - \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{G}{\alpha \Delta Y + 2\beta} - C_{y,w}\frac{G\Delta Y}{\alpha \Delta Y + 2\beta}\right)$$

Southern Boundary:

$$N_{SB} = N_{Bulk}$$
$$W_{SB} = W_{Bulk}$$

$$E_{SB} = E_{SB}$$
$$S_{SB} = 0$$

$$P_{SB} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{2\beta}{\alpha \Delta Z + 2\beta} + 2D_{z,s}\frac{2\beta}{\Delta Z(\alpha \Delta Z + 2\beta)}\right)$$
$$R_{SB} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{G\Delta Z}{\alpha \Delta Z + 2\beta} + 2D_{z,s}\frac{G}{\alpha \Delta Z + 2\beta}\right)$$

Northern Boundary

$$N_{NB} = 0$$
$$E_{NB} = E_{Bulk}$$
$$W_{NB} = W_{Bulk}$$
$$S_{NB} = S_{Bulk}$$

$$P_{NB} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{2\beta}{\Delta Z(\alpha \Delta Z + 2\beta)} - C_{z,n}\frac{2\beta}{\alpha \Delta Z + 2\beta}\right)$$
$$R_{NB} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{G}{\alpha \Delta Z + 2\beta} - C_{z,n}\frac{G\Delta Z}{\alpha \Delta Z + 2\beta}\right)$$

North-Eastern Corner

$$N_{NE} = 0$$
$$E_{NE} = 0$$
$$W_{NE} = W_{Bulk}$$
$$S_{NE} = S_{Bulk}$$

$$P_{NE} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{2\beta}{\Delta Z(\alpha \Delta Z + 2\beta)} - C_{z,n}\frac{2\beta}{\alpha \Delta Z + 2\beta}\right)$$
$$+ \frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{2\beta}{\alpha \Delta Y + 2\beta} + 2D_{y,e}\frac{2\beta}{\Delta Y(\alpha \Delta Y + 2\beta)}\right)$$

$$R_{NE} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{G}{\alpha \Delta Z + 2\beta} - C_{z,n}\frac{G\Delta Z}{\alpha \Delta Z + 2\beta}\right)$$
$$- \frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{G\Delta Y}{\alpha \Delta Y + 2\beta} + 2D_{y,e}\frac{G}{\alpha \Delta Y + 2\beta}\right)$$

North-Western Corner

$$N_{NW} = 0$$
$$E_{NW} = E_{Bulk}$$
$$W_{NW} = 0$$
$$S_{NW} = S_{Bulk}$$

$$P_{NW} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{2\beta}{\Delta Z(\alpha\Delta Z + 2\beta)} - C_{z,n}\frac{2\beta}{\alpha\Delta Z + 2\beta}\right) + \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{2\beta}{\Delta Y(\alpha\Delta Y + 2\beta)} - C_{y,w}\frac{2\beta}{\alpha\Delta Y + 2\beta}\right)$$

$$R_{NW} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(2D_{z,n}\frac{G}{\alpha\Delta Z + 2\beta} - C_{z,n}\frac{G\Delta Z}{\alpha\Delta Z + 2\beta}\right) - \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{G}{\alpha\Delta Y + 2\beta} - C_{y,w}\frac{G\Delta Y}{\alpha\Delta Y + 2\beta}\right)$$

South-Eastern Corner

$$N_{SE} = N_{Bulk}$$
$$W_{SE} = W_{Bulk}$$
$$E_{SE} = 0$$
$$S_{SE} = 0$$

$$P_{SE} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{2\beta}{\alpha\Delta Z + 2\beta} + 2D_{z,s}\frac{2\beta}{\Delta Z(\alpha\Delta Z + 2\beta)}\right) + \frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{2\beta}{\alpha\Delta Y + 2\beta} + 2D_{y,e}\frac{2\beta}{\Delta Y(\alpha\Delta Y + 2\beta)}\right)$$

$$R_{SE} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{G\Delta Z}{\alpha\Delta Z + 2\beta} + 2D_{z,s}\frac{G}{\alpha\Delta Z + 2\beta}\right) - \frac{\Delta Z}{\Delta Y \Delta Z}\left(C_{y,e}\frac{G\Delta Y}{\alpha\Delta Y + 2\beta} + 2D_{y,e}\frac{G}{\alpha\Delta Y + 2\beta}\right)$$

South-Western Corner

$$N_{SW} = N_{Bulk}$$
$$W_{SW} = 0$$
$$E_{SW} = E_{SB}$$
$$S_{SW} = 0$$

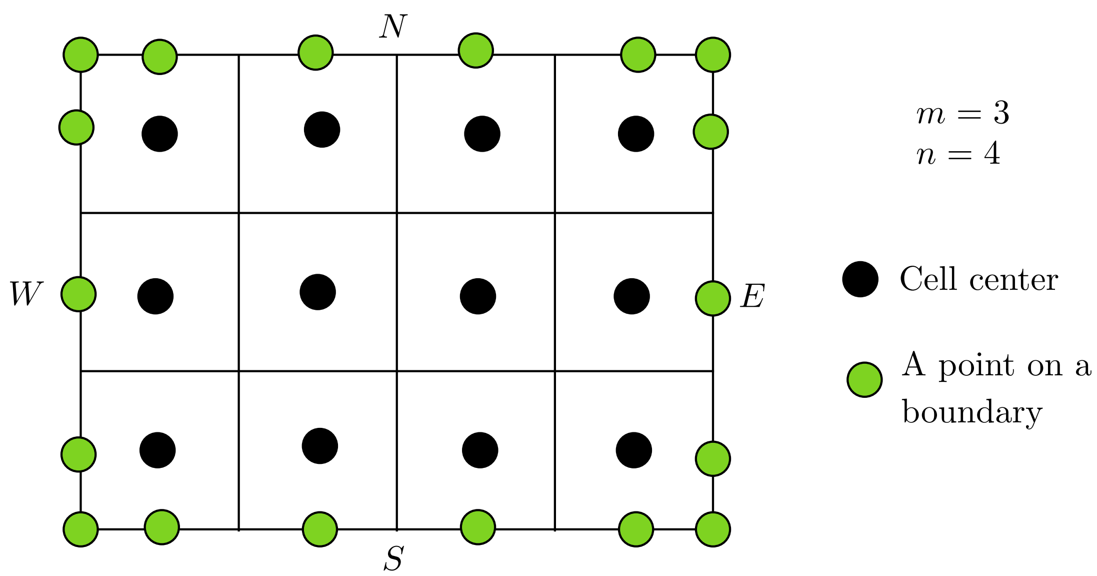
$$P_{SB} = P_{Bulk} + \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{2\beta}{\alpha\Delta Z + 2\beta} + 2D_{z,s}\frac{2\beta}{\Delta Z(\alpha\Delta Z + 2\beta)}\right) + \frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{2\beta}{\Delta Y(\alpha\Delta Y + 2\beta)} - C_{y,w}\frac{2\beta}{\alpha\Delta Y + 2\beta}\right)$$

$$R_{SB} = R_{Bulk} - \frac{\Delta Y}{\Delta Y \Delta Z}\left(C_{z,s}\frac{G\Delta Z}{\alpha\Delta Z + 2\beta} + 2D_{z,s}\frac{G}{\alpha\Delta Z + 2\beta}\right)$$

$$-\frac{\Delta Z}{\Delta Y \Delta Z}\left(2D_{y,w}\frac{G}{\alpha\Delta Y+2\beta}-C_{y,w}\frac{G\Delta Y}{\alpha\Delta Y+2\beta}\right)$$

3. Using the code

Consider a $m\times n$ grid of cells as shown below There are $(m+2)$ and $(n+2)$ points on the boundaries



To use the code, create the following matrices

NB: A $3\times n$ matrix. Row 1: Value of α, Row 2: Value of β and Row 3: Value of G, from eq. different value of $\alpha, \beta,$ and G can be specified for each cell.

SB: A $3\times n$ matrix. Row 1: Value of α, Row 2: Value of β and Row 3: Value of G, from eq. (different value of $\alpha, \beta,$ and G can be specified for each cell.

EB: A $3\times m$ matrix. Row 1: Value of α, Row 2: Value of β and Row 3: Value of G, from eq. A different value of $\alpha, \beta,$ and G can be specified for each cell.

WB: A $3 \times m$ matrix. Row 1: Value of α, Row 2: Value of β and Row 3: Value of G, from eq.
A different value of $\alpha, \beta,$ and G can be specified for each cell.

dY: A $m \times n$ matrix specifying ΔY for each cell
dZ: A $m \times n$ matrix specifying ΔZ for each cell.

Dy: A $(m+2) \times (n+2)$ matrix specifying the value of D_y at each cell center and each point
the boundary

Dz: A $(m+2) \times (n+2)$ matrix specifying the value of D_z at each cell center and each point
the boundary

Cy: A $(m+2) \times (n+2)$ matrix specifying the value of C_y at each cell center and each point
the boundary

Cz: A $(m+2) \times (n+2)$ matrix specifying the value of C_z at each cell center and each point
the boundary

Finally, call

```
[N, E, W, S, P, R] = EqnWriter(Cy,Cz,Dy,Dz,dY,dZ,NB,EB,WB,SB)
```

Note: We have still solved only eq. (3), a simplified form of eq. (1). To solve eq. (1), simply add

```
P = P + k;
R = R + S;
```

where,

k: A $m \times n$ matrix specifying k for each cell (from eq. (1))
S: A $m \times n$ matrix specifying S for each cell (from eq. (1))

Finally, once can solve the system by simply calling

```
Soln = PentDiagSol(N,E,W,S,P,R)
```

Soln: A $m \times n$ matrix giving ϕ for each cell.

To get the value of ϕ on the boundary, call

```
Soln_ext = BoundCond(NB,EB,WB,SB,dY,dZ,U)
```

Soln_ext: A $(m+2) \times (n+2)$ matrix giving ϕ for each cell center and all points on the bour