1. The Discretisation scheme

Consider the equation

$$\nabla^4\phi = S$$

Where,

$$\vec{\nabla} = \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

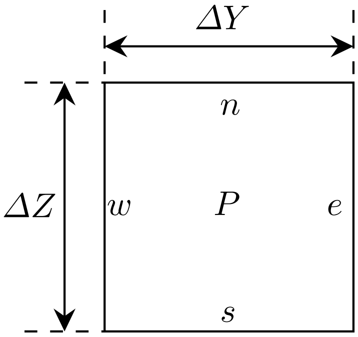
This equation is subject to boundary conditions:

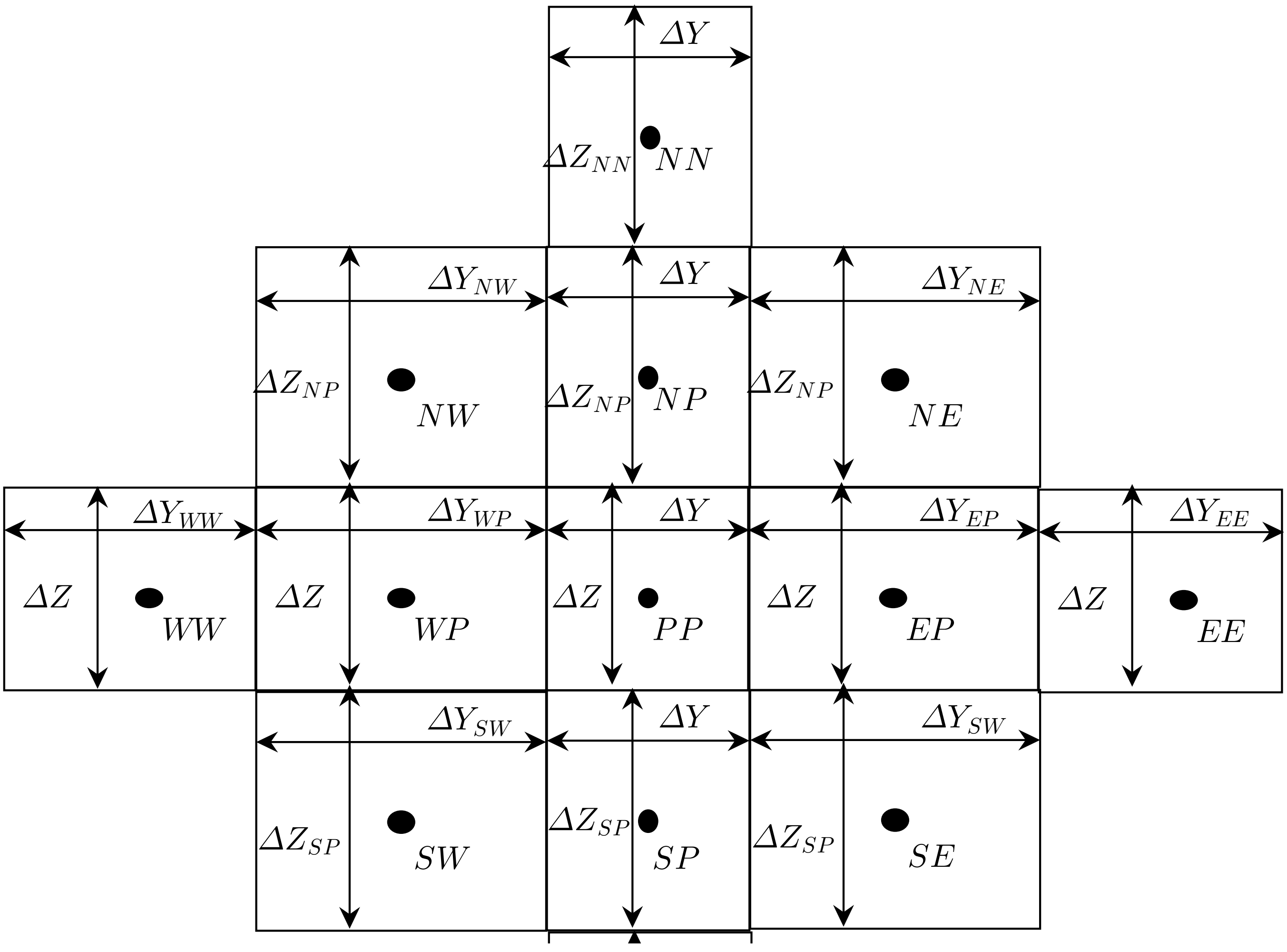
$$\phi = 0,\ \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0$$

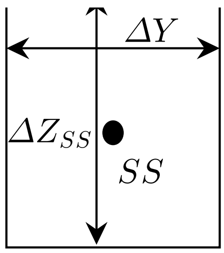
on all boundaries

1.1. Discretisation in the bulk

Consider a typical cell as shown below


ΔY
n
ΔZ
w
P
e
s


ΔY
ΔZ_{NN}
NN
ΔY_{NW}
ΔY
ΔY_{NE}
ΔZ_{NP}
NW
ΔZ_{NP}
NP
ΔZ_{NP}
NE
ΔY_{WW}
ΔY_{WP}
ΔY
ΔY_{EP}
ΔY_{EE}
ΔZ
WW
ΔZ
WP
ΔZ
PP
ΔZ
EP
ΔZ
EE
ΔY_{SW}
ΔY
ΔY_{SW}
ΔZ_{SP}
SW
ΔZ_{SP}
SP
ΔZ_{SP}
SE



Integrating eq. (2) on this cell, we get

$$\left(\frac{\Delta Z\left(\left.\frac{\partial\phi}{\partial y}\right|_{e}-\left.\frac{\partial\phi}{\partial y}\right|_{w}\right)}{\Delta Y\Delta Z}+\frac{\Delta Y\left(\left.\frac{\partial\phi}{\partial z}\right|_{s}-\left.\frac{\partial\phi}{\partial z}\right|_{n}\right)}{\Delta Y\Delta Z}\right)=0$$

Where,

$$\begin{aligned}\frac{\partial\phi}{\partial y}|_{e}&=\frac{2(\phi_E-\phi_P)}{\Delta Y_E+\Delta Y}\\ \left.\frac{\partial\phi}{\partial y}\right|_{w}&=\frac{2(\phi_P-\phi_W)}{\Delta Y_W+\Delta Y}\\ \left.\frac{\partial\phi}{\partial z}\right|_{s}&=\frac{2(\phi_S-\phi_P)}{\Delta Z_S+\Delta Z}\\ \left.\frac{\partial\phi}{\partial z}\right|_{n}&=\frac{2(\phi_P-\phi_N)}{\Delta Z_N+\Delta Z}\end{aligned}$$

Substituting,

$$\frac{1}{\Delta Y}\left(\left.\frac{\partial\phi}{\partial y}\right|_{e}-\left.\frac{\partial\phi}{\partial y}\right|_{w}\right)+\frac{1}{\Delta Z}\left(\left.\frac{\partial\phi}{\partial z}\right|_{s}-\left.\frac{\partial\phi}{\partial z}\right|_{n}\right)=0$$

$$\frac{1}{\Delta Y}\left(\frac{2(\phi_E-\phi_P)}{\Delta Y_E+\Delta Y}-\frac{2(\phi_P-\phi_W)}{\Delta Y_W+\Delta Y}\right)+\frac{1}{\Delta Z}\left(\frac{2(\phi_S-\phi_P)}{\Delta Z_S+\Delta Z}_{s}-\frac{2(\phi_P-\phi_N)}{\Delta Z_N+\Delta Z}_{n}\right)=0$$

$$\frac{1}{\Delta Y}\frac{2}{\Delta Y_E+\Delta Y}\phi_E+\frac{1}{\Delta Y}\frac{2}{\Delta Y_W+\Delta Y}\phi_W-\frac{1}{\Delta Y}\frac{2}{\Delta Y_E+\Delta Y}\phi_P-\frac{1}{\Delta Y}\frac{2}{\Delta Y_W+\Delta Y}\phi_P+$$

$$\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_S+\mathit{\Delta}Z}\phi_S-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_S+\mathit{\Delta}Z}\phi_P-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_N+\mathit{\Delta}Z}\phi_P+\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_N+\mathit{\Delta}Z}\phi_N=0$$

$$\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_E+\mathit{\Delta}Y}\phi_E+\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_W+\mathit{\Delta}Y}\phi_W+\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_S+\mathit{\Delta}Z}\phi_S+\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_N+\mathit{\Delta}Z}\phi_N$$
$$-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_S+\mathit{\Delta}Z}\phi_P-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_N+\mathit{\Delta}Z}\phi_P-\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_W+\mathit{\Delta}Y}\phi_P-\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_E+\mathit{\Delta}Y}\phi_P=0$$

$$Thus,$$
$$N\phi_N+E\phi_E+W\phi_W+S\phi_S+P\phi_P=R$$

$$Where,$$
$$N=\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_N+\mathit{\Delta}Z}$$
$$S=\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_S+\mathit{\Delta}Z}$$
$$E=\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_E+\mathit{\Delta}Y}$$
$$W=\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_W+\mathit{\Delta}Y}$$
$$P=-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_S+\mathit{\Delta}Z}-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_N+\mathit{\Delta}Z}-\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_E+\mathit{\Delta}Y}-\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_W+\mathit{\Delta}Y}$$

Applying this scheme at PP,

$$N_P\phi_{NP}+E_P\phi_{EP}+W_P\phi_{WP}+S_P\phi_{SP}+P_P\phi_{PP}=R_P$$

$$Where,$$

$$N_P=\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_{NP}+\mathit{\Delta}Z}$$
$$S_P=\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_{SP}+\mathit{\Delta}Z}$$
$$E_P=\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_{EP}+\mathit{\Delta}Y}$$
$$W_P=\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_{WP}+\mathit{\Delta}Y}$$
$$P_P=-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_{SP}+\mathit{\Delta}Z}-\frac{1}{\mathit{\Delta}Z}\frac{2}{\mathit{\Delta}Z_{NP}+\mathit{\Delta}Z}-\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_{EP}+\mathit{\Delta}Y}-\frac{1}{\mathit{\Delta}Y}\frac{2}{\mathit{\Delta}Y_{WP}+\mathit{\Delta}Y}$$
$$R_P=0$$

Applying this scheme at NP,

$$N_N\phi_{NN} + E\phi_{NE} + W\phi_{NW} + S\phi_{PP} + P\phi_{NP} = R_N$$

$$Where,$$

$$N_N = \frac{1}{\Delta Z_{NP}}\frac{2}{\Delta Z_{NP} + \Delta Z}$$

$$S_N = \frac{1}{\Delta Z_{NP}}\frac{2}{\Delta Z_{SP} + \Delta Z}$$

$$E_N = \frac{1}{\Delta Y}\frac{2}{\Delta Y_{EP} + \Delta Y}$$

$$W_N = \frac{1}{\Delta Y}\frac{2}{\Delta Y_{WP} + \Delta Y}$$

$$P_N = -\frac{1}{\Delta Z_{NP}}\frac{2}{\Delta Z + \Delta Z_{NP}} - \frac{1}{\Delta Z_{NP}}\frac{2}{\Delta Z_{NN} + \Delta Z_{NP}} - \frac{1}{\Delta Y}\frac{2}{\Delta Y_{NE} + \Delta Y} - \frac{1}{\Delta Y}\frac{2}{\Delta Y_{NW} + \Delta Y}$$

$$R_N = 0$$

Applying this scheme at SP,

$$N_S\phi_{PP} + E_S\phi_{SE} + W_P\phi_{SW} + S_P\phi_{SS} + P_P\phi_{SP} = R_S$$

$$Where,$$

$$N_P = \frac{1}{\Delta Z_{SP}}\frac{2}{\Delta Z + \Delta Z_{SP}}$$

$$S_P = \frac{1}{\Delta Z_{SP}}\frac{2}{\Delta Z_S + \Delta Z}$$

$$E_P = \frac{1}{\Delta Y}\frac{2}{\Delta Y_{SE} + \Delta Y}$$

$$W_P = \frac{1}{\Delta Y}\frac{2}{\Delta Y_{SW} + \Delta Y}$$

$$P_P = -\frac{1}{\Delta Z_{SP}}\frac{2}{\Delta Z_{SP} + \Delta Z} - \frac{1}{\Delta Z_{SP}}\frac{2}{\Delta Z_{NP} + \Delta Z} - \frac{1}{\Delta Y}\frac{2}{\Delta Y_{SE} + \Delta Y} - \frac{1}{\Delta Y}\frac{2}{\Delta Y_{SW} + \Delta Y}$$

$$R_S = 0$$

Applying this scheme at EP

$$N_E\phi_{NE} + E_E\phi_{EE} + W_E\phi_{PP} + S_E\phi_{SE} + P_E\phi_{EP} = R_E$$

$$
\begin{gathered}
Where, \\
N_E = \frac{1}{\Delta Z}\frac{2}{\Delta Z_{NE} + \Delta Z} \\
S_E = \frac{1}{\Delta Z}\frac{2}{\Delta Z_{SE} + \Delta Z} \\
E_E = \frac{1}{\Delta Y_{EP}}\frac{2}{\Delta Y_{EE} + \Delta Y_{EP}} \\
W_E = \frac{1}{\Delta Y_{EP}}\frac{2}{\Delta Y + \Delta Y_{EP}} \\
P = -\frac{1}{\Delta Z}\frac{2}{\Delta Z_{SE} + \Delta Z} - \frac{1}{\Delta Z}\frac{2}{\Delta Z_{NE} + \Delta Z} - \frac{1}{\Delta Y_{EP}}\frac{2}{\Delta Y_{EE} + \Delta Y_{EP}} - \frac{1}{\Delta Y_{EP}}\frac{2}{\Delta Y + \Delta Y_{EP}} \\
R_E = 0
\end{gathered}
$$

Applying this scheme at WP

$$N_W\phi_{NW} + E_W\phi_{PP} + W_W\phi_{WW} + S_W\phi_{SW} + P_W\phi_{WP} = R$$

$$
\begin{gathered}
Where, \\
N_W = \frac{1}{\Delta Z}\frac{2}{\Delta Z_N + \Delta Z} \\
S_W = \frac{1}{\Delta Z}\frac{2}{\Delta Z_S + \Delta Z} \\
E_W = \frac{1}{\Delta Y}\frac{2}{\Delta Y_E + \Delta Y} \\
W_W = \frac{1}{\Delta Y}\frac{2}{\Delta Y_W + \Delta Y} \\
P_W = -\frac{1}{\Delta Z}\frac{2}{\Delta Z_{SW} + \Delta Z} - \frac{1}{\Delta Z}\frac{2}{\Delta Z_{NW} + \Delta Z} - \frac{1}{\Delta Y_{WP}}\frac{2}{\Delta Y + \Delta Y_{WP}} - \frac{1}{\Delta Y_{WP}}\frac{2}{\Delta Y_{WW} + \Delta Y
\end{gathered}
$$

Consider the equation

$$
\begin{gathered}
\nabla^4\phi = 0 \\
\Longrightarrow \nabla^2\left(\nabla^2\phi\right) = 0
\end{gathered}
$$

Which can be discretised as

$$
\begin{gathered}
NN\phi_{NN} + SS\phi_{SS} + EE\phi_{EE} + WW\phi_{WW} + NE\phi_{NE} + NW\phi_{NW} + SE\phi_{SE} + SW\phi_{SW} \\
+NP\phi_{NP} + SP\phi_{SP} + EP\phi_{EP} + WP\phi_{WP} + PP\phi_{PP} = R
\end{gathered}
$$

where,

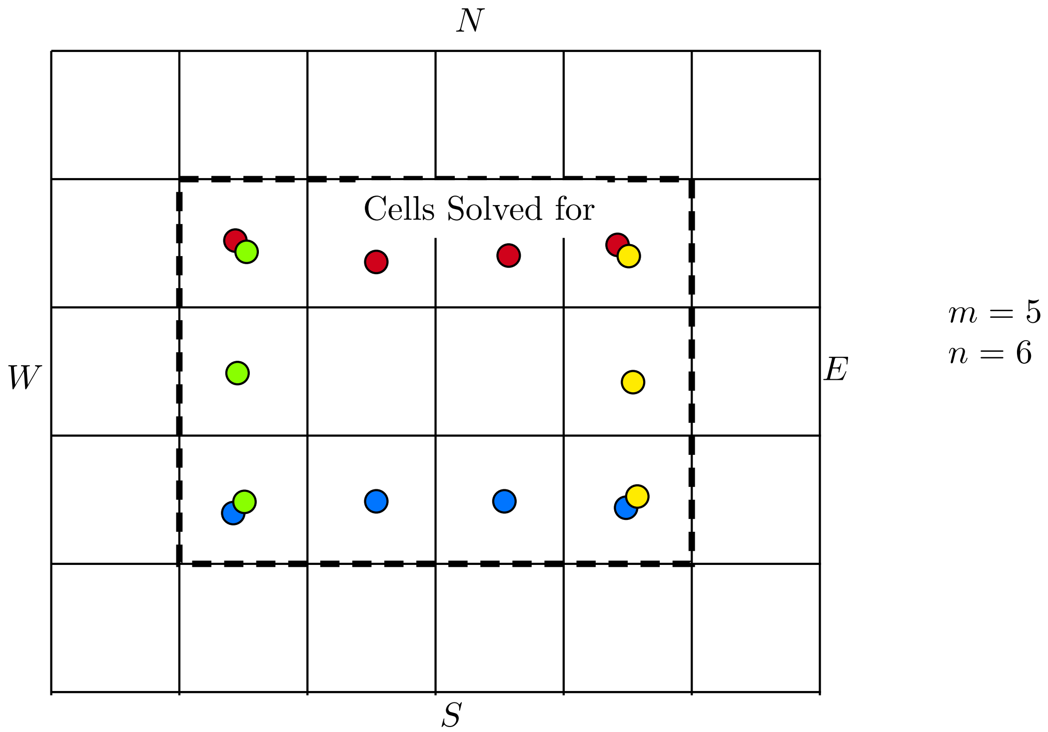
$$
\begin{aligned}
NN &= N_N N_P \\
SS &= S_S S_P \\
EE &= E_E E_P \\
WW &= W_W W_P \\
NP &= N_P P_P + P_N N_P \\
SP &= S_P P_P + P_S S_P
\end{aligned}
$$

$$\begin{aligned} EP &= E_P P_P + P_E E_P \\ WP &= W_P P_P + P_W W_P \\ NE &= N_E E_P + E_N N_P \end{aligned}$$

$$\begin{aligned} SE &= S_E E_P + E_S S_P \\ NW &= N_W W_P + W_N N_P \\ SW &= S_W W_P + W_S S_P \end{aligned}$$

$$PP = P_P P_P + W_E E_P + E_W W_P + N_S S_P + S_N N_P$$

Consider a $m \times n$ grid.



1.2. Applying the Boundary conditions

For the boundary conditions,

On the cells below the north boundary (in red), set $NP = NE = NW = NN = 0$
On the cells above the south boundary (in blue), set $SP = SE = SW = SS = 0$
On the cells beside the east boundary (in green), set $EP = EN = ES = EE = 0$
On the cells beside the west boundary (in yellow), set $WP = WN = WS = WW = 0$

2. Using the code

We solve only for $(m-2) \times (n-2)$ cells. Due to the boundary conditions imposed in eq. (2), value of ϕ remains 0 on the bundary and the cells immediately adjacent. Create 2 matrices

dY: A $m \times n$ matrix specifying ΔY for each cell
dZ: A $m \times n$ matrix specifying ΔZ for each cell.
Finally, call

```
[NN,NP,EE,EP,WW,WP,SS,SP,NE,NW,SE,SW,PP,R] = MatrixCode4(dY, dZ)
```

We have still been solving eq. (7). To solve eq. (1), add

```
R = R + S;
```

S: A $m \times n$ matrix specifying S for each cell (from eq. (1))
Finally, once can solve the system by simply calling

```
Soln = DecaTriaDiagSol(NN,NP,EE,EP,WW,WP,SS,SP,NE,NW,SE,SW,PP,R)
```

Soln: A $(m-1) \times (n-1)$ matrix giving ϕ for each cell.