QUIZ 3

purdue university · cs 51500

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matrix computations

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*The remainder of this exam represents my own work.*

(Your Name) Aditya Patel (PUID) 0035573514

Aditya Patel

(Signature)

# **Problem 1** (15 points)

Recall the stochastic gradient descent iteration for a least squares problem.

for iter=1:maxiter

s = rand(1:m)

as = A[s,:]

bs = b[s]

g = 2(as’\*x - bs).\*as

x -= step\*g

end

Justify that stochastic gradient descent for least squares takes *O*(*n*) work per step where the length of **x** is *n*.

One execution of the for-loop in the SGD is comprised of three stages, random sampling, gradient calculations, and gradient step.

**RANDOM SAMPLING:**

The random sampling stage is where the step value, as vector, and bs value are defined.

* s = rand(1:m): This is a single operation time, as it is generating a single integer.
* as = A[s,:]: Taking s = n in the worst case, defining as, the 1xn row vector, would take n operations.
* bs = b[s]: Identifying a single element from b would take a single operation.

Therefore the total time taken in the worst case for random sampling is O(n + 2) = O(n) work.

**GRADIENT CALCULATION:**

The gradient calculation step is where the gradient is calculated via three operations. Working from inside out:

* as’ \* x: This is the matrix dot product operation between a 1 x n vector and n x n matrix. This takes n operations in the worst case.
* 2 (as’ \* x – bs): This has two scalar operations, scalar multiplication by 2 and a scalar subtraction by bs. This takes 2 operations in the worst case.
* 2(as’ \* x – bs) .\* as: this is elementwise multiplication of a 1xn vector by a 1xn vector. This takes n operations to complete.

Therefore, the total time taken in the worst case for the gradient calculation is O(2n+2) = O(n) work.

**GRADIENT STEP:**

The final step adjusts the matrix x by performing the gradient descent.

* x -= step \* g: scalar multiplication on a 1xn matrix and updates each element of the matrix x by this gradient, which takes n operations.

Therefore, the total time taken in the worst case for the gradient step is O(n) work.

**CONCLUSION:**For all steps, we see that the work performed in the worst case is:

* Random Sampling: O(n)
* Gradient Calculation: O(n)
* Gradient Step: O(n)

**This means that one step for the gradient descent takes O(3n) = O(n) work overall.**

# **Problem 2** (20 points)

Recall that a kernel matrix is constructed as follows:

*Kij* = *f*(**x***i,***x***j*)

for some function *f* such as the inner-product *f*(**x***i,***x***j*) = **x***Ti* **x***j* or *f*(**x***i,***x***j*) = *e*−∥**x***i*−**x***j*∥2*/*(2*σ*). Suppose we have computed *K* for a set of *N* points: **x**1*,...,***x***N*. In this example, the result is a rank *N* matrix. Now, suppose we change two data points. Without losing any generation, let’s just suppose it’s the first and second one. So the new set of points is **x** (where **x**3*,...,***x***N* are all the same). Let *K*′ be the kernel matrix of the new set of points.

(10 points) Give an algorithm to compute *K*′ given the values of *K*.

using LinearAlgebra

# Define the kernel function

function kernel(x, y)

return dot(x, y)

end

# Function to compute the new kernel matrix K' given K

function update\_kernel\_matrix(K, x\_new1, x\_new2, x\_old)

N = size(K, 1)

K\_prime = copy(K)

# Compute new kernel values involving the new points

for i in 1:N

if i > 2

K\_prime[1, i] = kernel(x\_new1, x\_old[i, :])

K\_prime[2, i] = kernel(x\_new2, x\_old[i, :])

K\_prime[i, 1] = kernel(x\_old[i, :], x\_new1)

K\_prime[i, 2] = kernel(x\_old[i, :], x\_new2)

end

end

# Compute new kernel values for the updated points

K\_prime[1, 1] = kernel(x\_new1, x\_new1)

K\_prime[1, 2] = kernel(x\_new1, x\_new2)

K\_prime[2, 1] = kernel(x\_new2, x\_new1)

K\_prime[2, 2] = kernel(x\_new2, x\_new2)

return K\_prime

end

(10 points) Produce the tightest bound you can on the rank of the difference *K*′ − *K*. (Note: There is an extremely trivial answer that will be worth 4 points.)

**The trivial bound is 4**, since the changes are based out of a 2x2 matrix.

**The tightest bound is 2**, since the difference in between x1 and x2, given by K’ – K, can be represented by at most 2 linearly independent vectors.

# **Problem 3** (15 points)

Fill in the remainder of this julia function.

"""

‘solve\_with\_svd‘ solves a linear system of equations using the result of a singular value decomposition. -----------

The input matrices U, V are given exactly by the output of svd(A).U and svd(A).V and the input s is given by the output of svd(A).s for a square n-by-n matrix A. This function returns a vector x that will solve a linear system of equations Ax = b but does not call Julia’s built-in "\" solver at all.

"""

function solve\_with\_svd(U, s, V, b)

# Compute pseudo inverse of sigma

sigma\_inv = Diagonal(1 ./ s)

# Solve for y in U \* y = B

y = U' \* b

# Solve for z in sigma \* z = y

z = sigma\_inv \* y

# solve for x in V' \* x = z

x = V \* z

return x

end