Uncertainty and Operations Research

Zhongfeng Qin

Uncertain Portfolio Optimization

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Preface

Portfolio optimization aims at choosing the proportions of various securities to be held in a portfolio, in such a way as to making the portfolio better than any other ones according to some criterion. The criterion will combine the considerations of expected value of the portfolio return as well as risk measure associated with the portfolio. By measuring the risk by variance of the portfolio return, Markowitz (1952) proposed the well-known mean-variance model by maximizing the expected return contingent on any given amount of risk or minimizing the risk contingent on any given expected return. The philosophy of mean-variance model is to trade off between risk and expected return since achieving a higher return requires taking on more risk. After Markowitz, variance is widely accepted as a risk measure and, most of the research is devoted to the extensions of mean-variance model.

A main research extension is to add more arguments in the framework of mean-variance analysis. Another research extension is to change risk measure and propose new criteria. A typical extension is Konno and Yamazaki (1991), which employed absolute deviation to measure the risk of the portfolio return and formulated a mean-absolute deviation model. This model can cope with large-scale portfolio optimization because it can remove most of the difficulties associated with the classical Markowitz's model while maintaining its advantages. When all the returns are normally distributed random variables, the authors showed that the mean-absolute deviation model gave essentially the same results as the mean-variance model.

Although variance and absolute deviation are appropriate risk measures in many cases, they make no distinction between gains and losses, which implies they regard high returns that investors like as equally undesirable as low returns that investors dislike. Therefore, mean-variance and mean-absolute deviation models only lead to optimal decisions when the investment returns are jointly elliptically (or spherically) distributed. This assumption sounds to be unrealistic because it rules out possible asymmetry in return distributions. Actually the asymmetric returns of securities are more ordinary, especially in the stock and bond markets. In case of asymmetric return distribution, variance and absolute deviation maybe have to sacrifice too much expected return in eliminating both low and high return extremes. The asymmetric return distributions make variance and absolute deviation deficient measures of risk. Therefore, downside risk measures were suggested since they only had taken the negative deviations from the expected level into account. One of the commonly used downside risk measures is semivariance originally introduced in Markowitz (1959) and developed by several scholars such as Markowitz (1993) and Grootveld and Hallerbach (1999). Another one is semiabsolute deviation proposed by Speranza (1993), which can be easily evaluated since it could be determined using linear programming models (Papahristodoulou and Dotzauer 2004; Simaan 1997).

In Markowitz's theoretical framework, an implicit assumption is that future returns of securities can be correctly reflected by past performance. In other words, security returns should be represented by random variables whose characteristics such as expected value and variance may accurately be calculated based on the sample of available historical data. It keeps valid when large amounts of data are available such as in the developed financial market. However, since the security market is so complex as well as the occurrence of new security, investors often encounter the situation that there is a lack of data about security returns just like in an emerging market. In many cases, security returns are beset with ambiguity and vagueness. In such a situation, some researchers suggest estimating security returns by the domain experts and describe such estimates by fuzzy set or fuzzy variable. This is the so-called fuzzy portfolio optimization which has been widely investigated in different aspects, such as fuzzy goal programming, fuzzy compromise programming, fuzzy decision theory, interval programming (Lai et al. 2002), and other fuzzy formulations. By the underlying theory, the related literature falls into three categories: fuzzy set theory (Arenas-Parra et al. 2001; Bilbao-Terol et al. 2006; Gupta et al. 2008; Vercher et al. 2007), possibility theory (Tanaka and Guo 1999; Tanaka et al. 2000; Zhang et al. 2007), and credibility theory (Huang 2008a; Qin et al. 2009).

Compared with stochastic portfolio optimization, one of the advantages of the fuzzy approach is that it is more tractable and allows inclusion of experts' knowledge. In fact, fuzzy portfolio optimization is based on a possibility distribution, which is identified by using predicted values of security returns associated with possibilities estimated by experts. Based on credibility theory, Chaps. 2, 3, and 4 focus on the formulation and analysis of credibilistic mean-variance-skewness model, credibilistic mean-absolute deviation model, and credibilistic cross-entropy minimization model, respectively.

Another alternative way to describe the subjective imprecise quantity is uncertainty theory proposed by Liu (2007) by estimating indeterministic quantities subject to experts' estimations. Based on this framework, much work is undertaken to develop the theory and related practical applications. In particular, uncertainty theory is also applied to model the portfolio selection. Qin et al. (2009) first considered a mean-variance model in an uncertain environment. After that, Huang (2012) established a risk index model for uncertain portfolio selection, and Huang and Ying (2013) further employed the criterion to consider a portfolio adjusting problem. Different from a risk index model, Liu and Qin (2012) presented a semiabsolute deviation of an uncertain variable to measure risk and formulated a mean-semiabsolute deviation criterion, and Qin et al. (2016) followed the criterion to study a portfolio adjusting problem. In addition to these single-period optimization models, Huang and Qiao (2012) also modeled the multi-period problem, and Zhu (2010) founded a continuous-time uncertain portfolio selection model. Based on uncertainty theory, Chaps. 5 and 6 focus on the formulation and analysis of uncertain mean-semiabsolute deviation model and mean-LPMs model, respectively, and Chap. 7 also studies interval mean-semiabsolute deviation model.

Whether the classical portfolio selection or fuzzy/uncertain one, security returns are considered as the same type of variables. In other words, security returns are assumed to be either random variables or fuzzy/uncertain variables. As stated above, the former makes use of the historical data, and the latter makes use of the experiences of experts. However, the actual situation is that the securities having been listed for a long time have yielded a great deal of transaction data. For these "existing" securities, statistical methods are employed to estimate their returns, which implies that it is reasonable to assume that security returns are random variables. For some newly listed securities, there are lack of data or there are only insufficient data which cannot be used to effectively estimate the returns. Therefore, the returns of these newly listed securities need to be estimated by experts and thus are considered as uncertain variables. If an investor faces as such complex situation with simultaneous appearance of random and uncertain returns, how should he/she select a desirable portfolio to achieve some objectives? Chapter 8 attempts to consider the hybrid portfolio optimization and establish mathematical models by means of uncertain random variable which was proposed by Liu (2013a) for modeling complex systems with not only uncertainty but also randomness.

A financial market is always affected simultaneously by randomness and fuzziness. Historical data and subjective experiences may play equally important roles in making investment decisions. This also motivates the researchers to investigate the portfolio selection problem with mixture of randomness and fuzziness. The earlier tools to integrate these two kinds of uncertainties are fuzzy random variable (Kwakernaak 1978) and random fuzzy variable (Liu 2002). Fuzzy random variable has actually been applied to many financial optimization problems such as risk model (Huang et al. 2009), risk assessment (Shen and Zhao 2010), life annuity (de Andres-Sanchez and Puchades 2012; Shapiro 2013), and so forth. In particular, several authors have applied fuzzy random variables to characterize security returns and studied fuzzy random portfolio selection problems. Chapter 9 focuses on the construction of optimal strategy of adjusting an existing portfolio under the assumption of fuzzy random returns. In this case, not only transaction costs but also minimum transaction lots are taken into account to follow the rules of stock exchange.

The last chapter is also devoted to portfolio optimization with mixture of objective uncertainty and subjective uncertainty, which are still regarded as randomness and fuzziness, respectively. However, random fuzzy variable, instead of fuzzy random variable, is used to describe the return on individual security with ambiguous information. Chapter 10 mainly focuses on the formulation and analysis of random fuzzy mean-absolute deviation model and random fuzzy mean-semivariance model.

The contents of this book are mainly based on the research conducted by the author and his collaborators in recent years. The purpose of the book is to provide new modeling approaches for portfolio optimization when subjective uncertainty appears. The proposed models can also be applied to other optimization problems when risk constraints have to be considered. This book is suitable for the researchers, practitioners, and students in the area of portfolio optimization and related fields. The readers will find this book a stimulating and useful reference on portfolio optimization.

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Beijing, China
May 22, 2016

Zhongfeng Qin

Contents

Chapter 1
Preliminaries

This chapter presents the preliminaries for the rest of this book. On the one hand, credibility theory and uncertainty theory are outlined, respectively, which provide necessary knowledge for uncertain portfolio optimization. On the other hand, genetic algorithm is reviewed, which is used to solve the portfolio optimization models.

1.1 Credibility Theory

Credibility theory, founded by Liu (2004) and refined by Liu (2007), is a branch of mathematics for studying the behavior of fuzzy phenomena.

1.1.1 Credibility Measure

Let Θ be a nonempty set, and $\mathcal{P}$ the power set of Θ. Each element in $\mathcal{P}$ is called an event. Zadeh (1978) defined possibility measure as a set function satisfying (i) $\text{Pos}\{\Theta\} = 1$; (ii) $\text{Pos}\{A\} \leq \text{Pos}\{B\}$ whenever $A \subseteq B$; (iii) $\text{Pos}\{\cup_i A_i\} = \sup_i \text{Pos}\{A_i\}$ for any collection of events $\{A_i\}$. However, possibility measure is not self-dual, which is important in theory and intuitive in practice. In order to overcome the shortcoming, Liu and Liu (2002) defined a credibility measure as the average of possibility measure and its dual part. Then Li and Liu (2006) gave the following axiomatic definition for credibility measure,

Axiom 1. (Normality) $\text{Cr}\{\Theta\} = 1$.
Axiom 2. (Monotonicity) $\text{Cr}\{A\} \leq \text{Cr}\{B\}$ whenever $A \subset B$.
Axiom 3. (Self-Duality) $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ for any event A.

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Axiom 4. (Maximality) $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any sequence of events $\{A_i\}$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

If the set function Cr satisfies the above four axioms, then Cr is called a credibility measure, and the triplet $(\Theta, \mathcal{P}, \mathrm{Cr})$ is called a credibility space.

For any $A \in \mathcal{P}$, we have $0 \leq \mathrm{Cr}\{A\} \leq 1$. Moreover, Liu (2007) proved the subadditivity of credibility measure, i.e., $\mathrm{Cr}\{A \cup B\} \leq \mathrm{Cr}\{A\} + \mathrm{Cr}\{B\}$ for any $A, B \in \mathcal{P}$.

1.1.2 Fuzzy Variable

Definition 1.1 (Liu 2007). A fuzzy variable $\tilde{a}$ is defined as a measurable function from a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.

Example 1.1. Assume that the universal set $\Theta = \{\theta_1, \theta_2\}$ with $\mathrm{Cr}\{\theta_1\} = \mathrm{Cr}\{\theta_2\} = 0.5$. It is evident that Cr is a credibility measure on Θ. Define the function

$$\tilde{a}(\theta) = \begin{cases} 1, & \text{if } \theta = \theta_1 \\ -1, & \text{if } \theta = \theta_2. \end{cases}$$

According to Definition 1.1, $\tilde{a}$ is a fuzzy variable on $(\Theta, \mathcal{P}, \mathrm{Cr})$.

Remark 1.1. A crisp number x can be regarded as a special fuzzy variable. Actually, it is the constant function $\tilde{a}(\theta) \equiv x$ on a given credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$.

Definition 1.2 (Liu 2007). A fuzzy variable $\tilde{a}$ is called to be

(i) nonnegative if $\mathrm{Cr}\{\tilde{a} < 0\} = 0$;
(ii) positive if $\mathrm{Cr}\{\tilde{a} \leq 0\} = 0$;
(iii) continuous if $\mathrm{Cr}\{\tilde{a} = x\}$ is a continuous function of x;
(iv) discrete if there exists a countable sequence $\{x_1, x_2, \cdots\}$ such that

$$\mathrm{Cr}\{\tilde{a} \neq x_1, \tilde{a} \neq x_2, \cdots,\} = 0.$$

Assume that $\tilde{a}$ and $\tilde{b}$ are two fuzzy variables defined on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. We say $\tilde{a} = \tilde{b}$ if there exists a set $A \subset \Theta$ with $\mathrm{Cr}\{A\} = 1$ such that $\tilde{a}(\theta) = \tilde{b}(\theta)$ for each $\theta \in A$.

An n-dimensional fuzzy vector is defined by Liu (2007) as a function from a credibility space to a set of n-dimensional real vectors. It is easy to verify that $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ is a fuzzy vector if and only if $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ are fuzzy variables.

Definition 1.3 (Fuzzy Arithmetic, Liu 2007). Let $f : \Re^n \to \Re$ be a function, and $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ fuzzy variables on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. Then $\tilde{a} = f(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ is a fuzzy variable defined as

$$\tilde{a}(\theta) = f(\tilde{a}_1(\theta), \tilde{a}_2(\theta), \cdots, \tilde{a}_n(\theta)), \quad \theta \in \Theta. \tag{1.1}$$

For example, if $\tilde{a}$ and $\tilde{b}$ are two fuzzy variables on the credibility space $(\Theta, \mathcal{P}, \text{Cr})$, then their sum is

$$(\tilde{a} + \tilde{b})(\theta) = \tilde{a}(\theta) + \tilde{b}(\theta), \quad \forall \theta \in \Theta.$$

1.1.3 Membership Function

In probability theory, a random variable is described by its cumulative probability distribution function. Similarly, a fuzzy variable $\tilde{a}$ is descried by its membership function μ which represents the degree of possibility that the fuzzy variable takes some prescribed value. For instance, the membership degree $\mu(x) = 0$ if x is an impossible point, and $\mu(x) = 1$ is x is the most possible point that $\tilde{a}$ takes.

Definition 1.4 (Membership Function, Liu 2007). Let $\tilde{a}$ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}, \text{Cr})$. Then its membership function is derived from the credibility measure by

$$\mu(x) = (2\text{Cr}\{\tilde{a} = x\}) \wedge 1, \quad x \in \Re. \tag{1.2}$$

A fuzzy variable is uniquely described by a membership function. By the relationship $\mu(x) = (2\text{Cr}\{\tilde{a} = x\}) \wedge 1$, the following conclusions are immediately obtained.

Theorem 1.1 (Liu 2007). *A fuzzy variable $\tilde{a}$ with membership function μ is*

(i) nonnegative if and only if $\mu(x) = 0$ for all $x < 0$;
(ii) positive if and only if $\mu(x) = 0$ for all $x \leq 0$;
(iii) continuous if and only is μ is a continuous function;
(iv) discrete if and only if μ takes nonzero values at a countable set of real numbers.

Theorem 1.2 (Credibility Inversion Theorem, Liu 2007). *Let $\tilde{a}$ be a fuzzy variable with membership function μ. Then for any set B of real numbers, we have*

$$\text{Cr}\{\tilde{a} \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right). \tag{1.3}$$

It follows from Theorem 1.2 that if the membership function μ is continuous, then

$$\text{Cr}\{\tilde{a} = x\} = \frac{1}{2}\left(\mu(x) + 1 - \sup_{y \neq x} \mu(y)\right) = \frac{\mu(x)}{2}, \quad \forall x \in \Re.$$

Furthermore, a function $\mu : \Re \to [0, 1]$ is a membership function if and only if $\sup \mu(x) = 1$.

Definition 1.5 (Joint Membership Function, Liu 2007). Suppose that $\tilde{\boldsymbol{a}} = (\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ is a fuzzy vector on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. Then its joint membership function is derived from the credibility measure by

$$\mu(\boldsymbol{x}) = (2\mathrm{Cr}\{\tilde{\boldsymbol{a}} = \boldsymbol{x}\}) \wedge 1, \quad \forall \boldsymbol{x} \in \Re^n.$$

Next we introduce several special fuzzy variables which are commonly used in credibility theory.

1.1.3.1 Simple Fuzzy Variable

Assume that $x_1, x_2, \cdots, x_m$ are distinct real numbers and $\mu_1, \mu_2, \cdots, \mu_m$ are nonnegative real numbers with $\max\{\mu_1, \mu_2, \cdots, \mu_m\} = 1$. A fuzzy variable $\tilde{a}$ is called simple if it has a simple membership function

$$\mu(x) = \begin{cases} \mu_1, \text{ if } x = x_1 \\ \mu_2, \text{ if } x = x_2 \\ \cdots \\ \mu_m, \text{ if } x = x_m \end{cases}$$

denoted by $\tilde{a} = \mathcal{S}(x_i, \mu_i; m)$.

1.1.3.2 Equipossible Fuzzy Variable

A fuzzy variable $\tilde{a}$ is called equipossible if it has an equipossible membership function

$$\mu(x) = \begin{cases} 1, \text{ if } a - \alpha \le x \le a + \alpha \\ 0, \text{ otherwise} \end{cases} \tag{1.4}$$

denoted by $\tilde{a} = \mathcal{E}(a - \alpha, a + \alpha)$ where a and α are real numbers with $\alpha \ge 0$.

1.1.3.3 Triangular Fuzzy Variable

A fuzzy variable $\tilde{a}$ is called triangular if it has a triangular membership function

$$\mu(x) = \begin{cases} (x - a + \alpha)/\alpha, \text{ if } a - \alpha \le x \le a \\ (a + \beta - x)/\beta, \text{ if } a \le x \le a + \beta \\ 0, \quad \text{otherwise} \end{cases} \tag{1.5}$$

denoted by $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$ where a, α and β are real numbers with $\alpha, \beta > 0$. If $\alpha = \beta$, we call $\tilde{a}$ a symmetrical triangular fuzzy variable (Fig. 1.1).

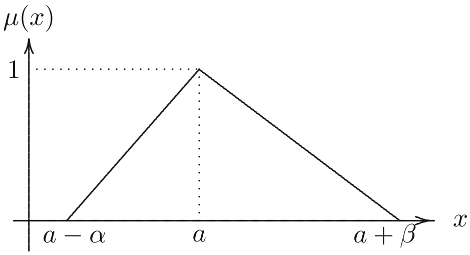

Fig. 1.1 Membership function of triangular fuzzy variable $\mathcal{T}(a-\alpha, a, a+\beta)$

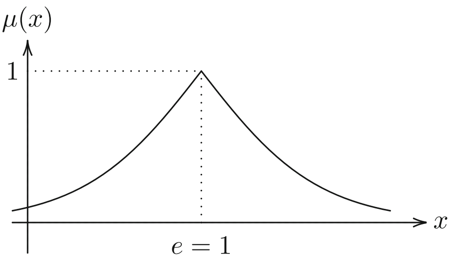

Fig. 1.2 Membership function of normal fuzzy variable $\mathcal{N}(1,1)$

1.1.3.4 Trapezoidal Fuzzy Variable

A fuzzy variable $\tilde{a}$ is called trapezoidal if it has a trapezoidal membership function

$$\mu(x) = \begin{cases} (x-a+\alpha)/\alpha, & \text{if } a-\alpha \le x \le a \\ 1, & \text{if } a \le x \le b \\ (b+\beta-x)/\beta, & \text{if } b \le x \le b+\beta \\ 0, & \text{otherwise} \end{cases} \tag{1.6}$$

denoted by $\tilde{a} = \mathcal{TP}(a-\alpha, a, b, b+\beta)$ where a, b, α and β are real numbers with $\alpha, \beta > 0$.

If $\alpha = \beta$, we call $\tilde{a}$ a symmetrical trapezoidal fuzzy variable. In addition, if $a = b$, then a trapezoidal fuzzy variable $\tilde{a} = \mathcal{TP}(a-\alpha, a, b, b+\beta)$ degenerates into a triangular one $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$.

1.1.3.5 Normal Fuzzy Variable

A fuzzy variable $\tilde{a}$ is called normal if it has a normal membership function

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x-a|}{\sqrt{6}\delta}\right)\right)^{-1}, \quad x \in \Re \tag{1.7}$$

denoted by $\tilde{a} = \mathcal{N}(a, \delta)$ where a and δ are real numbers with $\delta > 0$ (Fig. 1.2).

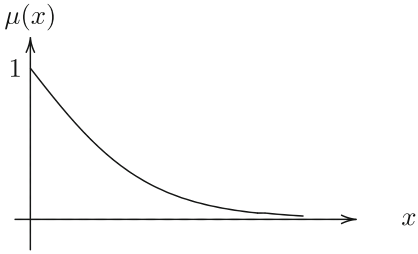

Fig. 1.3 Membership function of exponential fuzzy variable $\mathcal{EXP}(1)$

1.1.3.6 Exponential Fuzzy Variable

A fuzzy variable $\tilde{a}$ is called exponential if it has an exponential membership function

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi x}{\sqrt{6}m}\right)\right)^{-1}, \quad x \geq 0 \tag{1.8}$$

denoted by $\tilde{a} = \mathcal{EXP}(m)$ where m is a real number with $m > 0$ (Fig. 1.3).

1.1.4 Credibility Distribution

Definition 1.6 (Liu 2002). The credibility distribution $\Gamma : \Re \to [0, 1]$ of a fuzzy variable $\tilde{a}$ is defined by

$$\Gamma(x) = \mathrm{Cr}\{\theta \in \Theta \mid \tilde{a}(\theta) \leq x\}. \tag{1.9}$$

Analogous to probability distribution, $\Gamma(x)$ is the credibility measure that the fuzzy variable $\tilde{a}$ takes a value less than or equal to x. But in general the credibility distribution is neither left-continuous nor right-continuous.

Theorem 1.3 (Liu 2004). *A function* $\Gamma : \Re \to [0, 1]$ *is a credibility distribution if and only if it is an increasing function with*

$$\lim_{x \to -\infty} \Gamma(x) \leq 0.5 < \lim_{x \to \infty} \Gamma(x),$$

$$\lim_{y \downarrow x} \Gamma(y) = \Gamma(x) \text{ if } \lim_{y \downarrow x} \Gamma(y) > 0.5 \text{ or } \Gamma(x) \geq 0.5.$$

Remark 1.2. If the membership function is given, then it follows from Theorem 1.2 that

$$\Gamma(x) = \frac{1}{2}\left(\sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right). \tag{1.10}$$

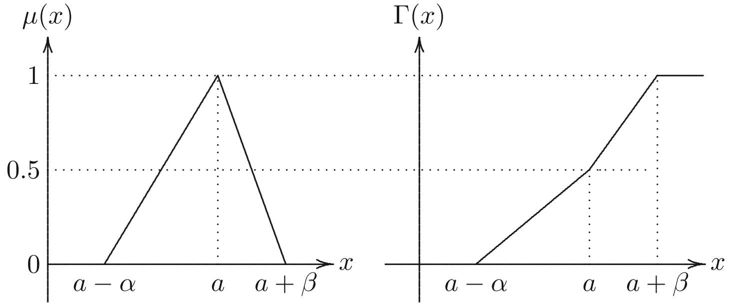

Fig. 1.4 Functions $\mu(x)$ and $\Gamma(x)$

Example 1.2. The credibility distribution of an equipossible fuzzy variable $\tilde{a} = \mathcal{E}(a-\alpha, a+\alpha)$ is

$$\Gamma(x) = \begin{cases} 0, & \text{if } x < a-\alpha \\ 1/2, & \text{if } a-\alpha \le x < a+\alpha \\ 1, & \text{if } x \ge a+\alpha. \end{cases}$$

Example 1.3. The credibility distribution of a triangular fuzzy variable $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$ is

$$\Gamma(x) = \begin{cases} 0, & \text{if } x \le a-\alpha \\ (x-a+\alpha)/2\alpha, & \text{if } a-\alpha \le x \le a \\ (x-a+\beta)/2\beta, & \text{if } a \le x \le a+\beta \\ 1, & \text{if } x \ge a+\beta, \end{cases} \tag{1.11}$$

which is shown in Fig. 1.4.

Example 1.4. The credibility distribution of a trapezoidal fuzzy variable $\tilde{a} = \mathcal{TP}(a-\alpha, a, b, b+\beta)$ is

$$\Gamma(x) = \begin{cases} 0, & \text{if } x \le a-\alpha \\ (x-a+\alpha)/2\alpha, & \text{if } a-\alpha \le x \le a \\ 1/2, & \text{if } a \le x \le b \\ (x-b+\beta)/2\beta, & \text{if } b \le x \le b+\beta \\ 1, & \text{if } x \ge b+\beta. \end{cases}$$

Example 1.5. The credibility distribution of a normal fuzzy variable $\tilde{a} = \mathcal{N}(a, \delta)$ is

$$\Gamma(x) = \left(1 + \exp\left(\frac{\pi(a-x)}{\sqrt{6}\delta}\right)\right)^{-1}, \quad x \in \Re.$$

Example 1.6. The credibility distribution of an exponential fuzzy variable $\tilde{a} = \mathcal{EXP}(m)$ is

$$\Gamma(x) = \begin{cases} 1 - \left(1 + \exp\left(\dfrac{\pi x}{\sqrt{6}m}\right)\right)^{-1}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0. \end{cases}$$

Definition 1.7 (Joint Credibility Distribution, Liu 2007). Assume that $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ is a fuzzy vector on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. Then the joint credibility distribution $\Gamma : \Re^n \to [0, 1]$ is defined by

$$\Gamma(x_1, x_2, \cdots, x_n) = \mathrm{Cr}\{\theta \in \Theta \mid \tilde{a}_1(\theta) \leq x_1, \tilde{a}_2(\theta) \leq x_2, \cdots, \tilde{a}_n(\theta) \leq x_n\}.$$

1.1.5 Independence

The independence of fuzzy variables is an important concept, which has been discussed by many researchers from various angles. In this book, we adopt the definition proposed by Liu and Gao (2007).

Definition 1.8 (Liu and Gao 2007). The fuzzy variables $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_m$ are said to be independent if

$$\mathrm{Cr}\left\{\bigcap_{i=1}^{m}\{\tilde{a}_i \in B_i\}\right\} = \min_{1 \leq i \leq m} \mathrm{Cr}\{\tilde{a}_i \in B_i\} \tag{1.12}$$

for any sets $B_1, B_2, \cdots, B_m$ of $\Re$.

Theorem 1.4 (Liu 2007). *Let μ_i be membership functions of fuzzy variables $\tilde{a}_i$, $i = 1, 2, \cdots, m$, respectively, and μ the joint membership function of fuzzy vector $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_m)$. Then the fuzzy variables $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_m$ are independent if and only if*

$$\mu(x_1, x_2, \cdots, x_m) = \min_{1 \leq i \leq m} \mu_i(x_i)$$

for any real numbers $x_1, x_2, \cdots, x_m$.

Theorem 1.5 (Liu 2007). *Let Γ_i be credibility distributions of fuzzy variables $\tilde{a}_i$, $i = 1, 2, \cdots, m$, respectively, and Γ the joint credibility distribution of fuzzy vector $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_m)$. If the fuzzy variables $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_m$ are independent, then we have*

$$\Gamma(x_1, x_2, \cdots, x_m) = \min_{1 \leq i \leq m} \Gamma_i(x_i)$$

for any real numbers $x_1, x_2, \cdots, x_m$.

Theorem 1.6 (Liu 2007). *Let $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_m$ be independent fuzzy variables, and $f_1, f_2, \cdots, f_n$ are real-valued functions. Then $f_1(\tilde{a}_1), f_2(\tilde{a}_2), \cdots, f_m(\tilde{a}_m)$ are independent fuzzy variables.*

Theorem 1.7 (Extension Principle of Zadeh, Liu 2007). *Let $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ be independent fuzzy variables with membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively, and $f : \Re^n \to \Re$ a function. Then the membership function μ of $\tilde{a} = f(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ is derived from $\mu_1, \mu_2, \cdots, \mu_n$ by*

$$\mu(x) = \sup_{x=f(x_1,x_2,\cdots,x_n)} \min_{1\le i\le n} \mu_i(x_i), \quad \forall x \in \Re.$$

Here we set $\mu(x) = 0$ if there are not real numbers $x_1, x_2, \cdots, x_n$ such that $x = f(x_1, x_2, \cdots, x_n)$.

The extension principle of Zadeh is originally used as a postulate. However, Liu (2007) treated it as a theorem in credibility theory and it is only applicable to the operations on independent fuzzy variables. The principle is relatively important since it provides an operational law for independent fuzzy variables.

Example 1.7. Let $\tilde{a}_1 = \mathcal{E}(a_1 - \alpha_1, a_1 + \alpha_1)$ and $\tilde{a}_2 = \mathcal{E}(a_2 - \alpha_2, a_2 + \alpha_2)$ be two independent equipossible fuzzy variables. For any two real numbers λ_1 and λ_2, the extension principle of Zadeh yields that the membership function of $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is

$$\mu(x) = \begin{cases} 1, \text{ if } \lambda_1(a_1 - \alpha_1) + \lambda_2(a_2 - \alpha_2) \le x \le \lambda_1(a_1 + \alpha_1) + \lambda_2(a_2 + \alpha_2) \\ 0, \text{ otherwise} \end{cases}$$

which implies $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is also an equipossible fuzzy variable. For simplicity, we also write

$$\begin{aligned} \lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2 &= \lambda_1 \cdot \mathcal{E}(a_1 - \alpha_1, a_1 + \alpha_1) + \lambda_2 \cdot \mathcal{E}(a_2 - \alpha_2, a_2 + \alpha_2) \\ &= \mathcal{E}(\lambda_1(a_1 - \alpha_1) + \lambda_2(a_2 - \alpha_2), \lambda_1(a_1 + \alpha_1) + \lambda_2(a_2 + \alpha_2)). \end{aligned}$$

Example 1.8. Let $\tilde{a}_1 = \mathcal{T}(a_1 - \alpha_1, a_1, a_1 + \beta_1)$ and $\tilde{a}_2 = \mathcal{T}(a_2 - \alpha_2, a_2, a_2 + \beta_2)$ be two independent triangular fuzzy variables. For any two real numbers λ_1 and λ_2, the extension principle of Zadeh yields that the membership function of $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is

$$\mu(x) = \begin{cases} \dfrac{x - \lambda_1(a_1 - \alpha_1) - \lambda_2(a_2 - \alpha_2)}{\lambda_1\alpha_1 + \lambda_2\alpha_2}, \\ \qquad \text{if } \lambda_1(a_1 - \alpha_1) + \lambda_2(a_2 - \alpha_2) \le x \le \lambda_1 a_1 + \lambda_2 a_2 \\ \dfrac{\lambda_1(a_1 + \beta_1) + \lambda_2(a_2 + \beta_2) - x}{\lambda_1\beta_1 + \lambda_2\beta_2}, \\ \qquad \text{if } \lambda_1 a_1 + \lambda_2 a_2 \le x \le \lambda_1(a_1 + \beta_1) + \lambda_2(a_2 + \beta_2) \\ 0, \quad \text{otherwise} \end{cases}$$

which implies $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is also a triangular fuzzy variable. For simplicity, we also write

$$\begin{aligned}
&\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2 \\
&= \lambda_1 \cdot \mathcal{T}(a_1 - \alpha_1, a_1, a_1 + \alpha_1) + \lambda_2 \cdot \mathcal{T}(a_2 - \beta_2, a_2, a_2 + \beta_2) \\
&= \mathcal{T}(\lambda_1(a_1 - \alpha_1) + \lambda_2(a_2 - \alpha_2), \lambda_1 a_1 + \lambda_2 a_2, \lambda_1(a_1 + \beta_1) + \lambda_2(a_2 + \beta_2)).
\end{aligned}$$

Similar to the triangular case, we can also derive that the linear combination of two independent trapezoidal fuzzy variables is also a trapezoidal fuzzy variable.

Example 1.9. Let $\tilde{a}_1 = \mathcal{N}(a_1, \delta_1)$ and $\tilde{a}_2 = \mathcal{N}(a_2, \delta_2)$ be two independent normal fuzzy variables. For any two real numbers λ_1 and λ_2, the extension principle of Zadeh yields that the membership function of $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is

$$\mu(x) = \left(1 + \exp\left(\frac{\pi|x - (\lambda_1 a_1 + \lambda_2 a_2)|}{\sqrt{6}(|\lambda_1|\delta_1 + |\lambda_2|\delta_2)}\right)\right)^{-1}, \ x \in \Re$$

which implies $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is also a normal fuzzy variable. For simplicity, we also write

$$\begin{aligned}
\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2 &= \lambda_1 \cdot \mathcal{N}(a_1, \delta_1) + \lambda_2 \cdot \mathcal{N}(a_2, \delta_2) \\
&= \mathcal{N}(\lambda_1 a_1 + \lambda_2 a_2, |\lambda_1|\delta_1 + |\lambda_2|\delta_2).
\end{aligned}$$

Example 1.10. Let $\tilde{a}_1 = \mathcal{EXP}(m_1)$ and $\tilde{a}_2 = \mathcal{EXP}(m_2)$ be two independent exponential fuzzy variables. For any two real numbers $\lambda_1 > 0$ and $\lambda_2 > 0$, the extension principle of Zadeh yields that the membership function of $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi x}{\sqrt{6}(\lambda_1 m_1 + \lambda_2 m_2)}\right)\right)^{-1}, \quad x \geq 0$$

which implies $\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2$ is also an exponential fuzzy variable. For simplicity, we also write

$$\begin{aligned}
\lambda_1\tilde{a}_1 + \lambda_2\tilde{a}_2 &= \lambda_1 \cdot \mathcal{EXP}(m_1) + \lambda_2 \cdot \mathcal{EXP}(m_2) \\
&= \mathcal{EXP}(\lambda_1 m_1 + \lambda_2 m_2).
\end{aligned}$$

1.1.6 Expected Value

Expected value represents the average value of a fuzzy variable in the sense of credibility measure. In credibility theory, Liu and Liu (2002) gave the following general definition, which is not only applicable to continuous fuzzy variables but also discrete ones.

Definition 1.9 (Liu and Liu 2002). Let $\tilde{a}$ be a fuzzy variable. Then the expected value of $\tilde{a}$ is defined by

$$E[\tilde{a}] = \int_0^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\tilde{a} \leq r\}\mathrm{d}r \tag{1.13}$$

provided that at least one of the two integrals is finite.

If $\tilde{a}$ is a continuous fuzzy variable with credibility distribution Γ, then we have

$$E[\tilde{a}] = \int_0^{+\infty} (1 - \Gamma(r))\mathrm{d}r - \int_{-\infty}^{0} \Gamma(r)\mathrm{d}r$$

since $\mathrm{Cr}\{\tilde{a} \geq r\} = 1 - \mathrm{Cr}\{\tilde{a} < r\} = 1 - \Gamma(r)$. If $\tilde{a}$ is nonnegative, then

$$E[\tilde{a}] = \int_0^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r\}\mathrm{d}r.$$

If $\tilde{a}$ is nonpositive, then

$$E[\tilde{a}] = -\int_{-\infty}^{0} \mathrm{Cr}\{\tilde{a} \leq r\}\mathrm{d}r.$$

Theorem 1.8. *Let $\tilde{a}$ be a fuzzy variable with finite expected value. Then for any two real numbers x and y, we have*

$$E[x\tilde{a} + y] = xE[\tilde{a}] + y.$$

Proof. If $x = 0$, then the equality holds trivially. If $x > 0$ and $y > 0$, then it follows from Definition 1.9 that

$$\begin{aligned} E[x\tilde{a} + y] &= \int_0^{+\infty} \mathrm{Cr}\{x\tilde{a} + y \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{x\tilde{a} + y \leq r\}\mathrm{d}r \\ &= \int_0^{+\infty} \mathrm{Cr}\{\tilde{a} \geq (r - y)/x\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\tilde{a} \leq (r - y)/x\}\mathrm{d}r. \end{aligned}$$

Substituting $(r - y)/x$ with u and r with $xu + y$, the change of variables yields

$$\begin{aligned} E[x\tilde{a} + y] &= x\int_{-y/x}^{+\infty} \mathrm{Cr}\{\tilde{a} \geq u\}\mathrm{d}u - x\int_{-\infty}^{-y/x} \mathrm{Cr}\{\tilde{a} \leq u\}\mathrm{d}u \\ &= x\left(E[\tilde{a}] + \int_{-y/x}^{0} \mathrm{Cr}\{\tilde{a} \geq u\}\mathrm{d}u + \int_{-y/x}^{0} \mathrm{Cr}\{\tilde{a} \leq u\}\mathrm{d}u\right) \\ &= x\left(E[\tilde{a}] + \int_{-y/x}^{0} 1 \cdot \mathrm{d}u\right) \\ &= xE[\tilde{a}] + y. \end{aligned}$$

The other cases are similarly to be proved. The theorem is proved.

Example 1.11. The triangular fuzzy variable $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$ has an expected value

$$E[\tilde{a}] = a + (\beta - \alpha)/4.$$

If $\alpha = \beta$, i.e., $\tilde{a}$ is a symmetrical triangular fuzzy variable, then $E[\tilde{a}] = a$. By the self-duality of credibility measure, we have

$$\begin{aligned}
\mathrm{Cr}\{\tilde{a} \geq r\} &= 1 - \mathrm{Cr}\{\tilde{a} < r\} \\
&= 1 - \mathrm{Cr}\{\tilde{a} \leq r\} \\
&= \begin{cases} 1, & \text{if } r \leq a - \alpha \\ (a + \alpha - r)/2\alpha, & \text{if } a - \alpha \leq r \leq a \\ (a + \beta - r)/2\beta, & \text{if } a \leq r \leq a + \beta \\ 0, & \text{if } r \geq a + \beta. \end{cases}
\end{aligned}$$

We first consider the case of $a \geq \alpha$, i.e., $a - \alpha \geq 0$ which implies that $\tilde{a}$ is nonnegative. It follows from Definition 1.9 that

$$\begin{aligned}
E[\tilde{a}] &= \int_0^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r\}\mathrm{d}r \\
&= \int_0^{a-\alpha} 1 \cdot \mathrm{d}r + \int_{a-\alpha}^{a} \frac{a + \alpha - x}{2\alpha}\mathrm{d}r + \int_a^{a+\beta} \frac{a + \beta - r}{2\beta}\mathrm{d}r \\
&= a - \alpha + \frac{2(a + \alpha)r - r^2}{4\alpha}\Big|_{a-\alpha}^{a} + \frac{2(a + \beta)r - r^2}{4\beta}\Big|_{a}^{a+\beta} \\
&= a - \alpha + \frac{3\alpha}{4} + \frac{\beta}{4} \\
&= a + \frac{\beta - \alpha}{4}.
\end{aligned}$$

We next consider the case of $a < \alpha$, i.e., $a - \alpha < 0$. According to extension principle of Zadeh (Theorem 1.7), $\tilde{a} - (a - \alpha) = \mathcal{T}(0, \alpha, \alpha + \beta)$ is a nonnegative triangular fuzzy variable whose expected value is $(3\alpha + \beta)/4$. It follows Theorem 1.8 that

$$\begin{aligned}
E[\tilde{a}] &= E[\tilde{a} - (a - \alpha) + (a - \alpha)] \\
&= E[\tilde{a} - (a - \alpha)] + (a - \alpha) \\
&= (3\alpha + \beta)/4 + a - \alpha \\
&= a + (\beta - \alpha)/4.
\end{aligned}$$

Example 1.12. The normal fuzzy variable $\tilde{a} = \mathcal{N}(a, \delta)$ has an expected value $E[\tilde{a}] = a$. We first verify it for the case of $a = 0$, in which the credibility distribution of $\tilde{a}$ is

$$\Gamma(r) = \left(1 + \exp\left(-\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1}$$

and

$$\mathrm{Cr}\{\tilde{a} \geq r\} = 1 - \Gamma(r) = \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1}.$$

It follows from Definition 1.9 that

$$E[\tilde{a}] = \int_0^{+\infty} \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1} \mathrm{d}r - \int_{-\infty}^{0} \left(1 + \exp\left(-\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1} \mathrm{d}r.$$

Substituting $\pi r/\sqrt{6}\delta$ with u and r with $\sqrt{6}\delta u/\pi$, the change of variables yields

$$E[\tilde{a}] = \frac{\sqrt{6}\delta}{\pi}\left(\int_0^{+\infty} \frac{1}{1+e^u}\mathrm{d}u - \int_{-\infty}^{0} \frac{1}{1+e^{-u}}\mathrm{d}u\right) = 0$$

since

$$\int_0^{+\infty} (1+e^u)^{-1}\mathrm{d}u = \int_{-\infty}^{0} (1+e^{-u})^{-1}\mathrm{d}u = \ln 2.$$

If $a \neq 0$, then $\tilde{a} - a = \mathcal{N}(0, \delta)$ is a normal fuzzy variable. It follows Theorem 1.8 that

$$E[\tilde{a}] = E[\tilde{a} - a + a] = E[\tilde{a} - a] + a = 0 + a = a.$$

Example 1.13. The exponential fuzzy variable $\tilde{a} = \mathcal{EXP}(m)$ has an expected value

$$E[\tilde{a}] = m\sqrt{6}\ln 2/\pi.$$

Note that $\tilde{a}$ is a nonnegative fuzzy variable. For each $r \geq 0$, we have

$$\mathrm{Cr}\{\tilde{a} \geq r\} = 1 - \mathrm{Cr}\{\tilde{a} \leq r\} = \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}m}\right)\right)^{-1}.$$

It follows from Definition 1.9 that

$$E[\tilde{a}] = \int_0^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r\}\mathrm{d}r = \int_0^{+\infty} \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}m}\right)\right)^{-1} \mathrm{d}r.$$

Substituting $\pi r/\sqrt{6}m$ with u and r with $\sqrt{6}mu/\pi$, the change of variables yields

$$E[\tilde{a}] = \frac{\sqrt{6}m}{\pi}\int_0^{+\infty}\frac{1}{1+e^u}\mathrm{d}u = \frac{\sqrt{6}\ln 2}{\pi}m.$$

Similar to the above processes, we can get the expected values of equipossible, trapezoidal and simple fuzzy variables.

Example 1.14. The equipossible fuzzy variable $\tilde{a} = \mathcal{E}(a-\alpha, a+\alpha)$ has an expected value $E[\tilde{a}] = (a-\alpha+a+\alpha)/2 = a$.

Example 1.15. The trapezoidal fuzzy variable $\tilde{a} = \mathcal{TP}(a-\alpha, a, b, b+\beta)$ has an expected value

$$E[\tilde{a}] = \frac{a+b}{2} + \frac{\beta-\alpha}{4}$$

which is consistent with that of triangular fuzzy variable.

Example 1.16. The simple fuzzy variable $\tilde{a} = \mathcal{S}(x_i, \mu_i; m)$ has an expected value

$$E[\tilde{a}] = \frac{1}{2}\sum_{i=1}^{m} w_i x_i \tag{1.14}$$

in which the weights are calculated by

$$\begin{aligned} w_i \ &= \max_{1\le j\le m}\{\mu_j | x_j \le x_i\} - \max_{1\le j\le m}\{\mu_j | x_j < x_i\} \\ &\quad + \max_{1\le j\le m}\{\mu_j | x_j \ge x_i\} - \max_{1\le j\le m}\{\mu_j | x_j > x_i\} \end{aligned}$$

for $i = 1, 2, \cdots, m$. It is easy to verify that $w_i \ge 0$ and $\sum_{i=1}^m w_i = 1$. It is evident that the expected value of fuzzy variable is completely different from that of random variable.

Further, if we know more information about the membership degrees μ_i, then we can simplify Eq. (1.14). For example, if $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_m \equiv 1$, then $E[\tilde{a}] = \sum_{i=1}^m y_i\mu_i/2$ in which

$$y_i = \max_{j\ge i} x_j - \max_{j>i} x_j + \min_{j\ge i} x_j - \min_{j>i} x_j$$

for $i = 1, 2, \cdots, m$. However, if $1 \equiv \mu_1 \ge \mu_2 \ge \cdots \ge \mu_m \ge 0$, then, for $i = 1, 2, \cdots, m$, the weights become

$$y_i = \max_{j\le i} x_j - \max_{j<i} x_j + \min_{j\le i} x_j - \min_{j<i} x_j.$$

For this case of $1 \equiv \mu_1 > \mu_2 > \cdots > \mu_m > 0$, Li et al. (2012) proved the following formula,

$$E[\tilde{a}] = \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{j\le i} x_j + \min_{j\le i} x_j\right) \tag{1.15}$$

where $\mu_{m+1} = 0$.

Theorem 1.9 (Linearity of Expected Value Operator, Liu and Liu 2002). *Let $\tilde{a}$ and $\tilde{b}$ be independent fuzzy variables with finite expected values. Then for any numbers x and y, we have*

$$E[x\tilde{a} + y\tilde{b}] = xE[\tilde{a}] + yE[\tilde{b}]. \tag{1.16}$$

Example 1.17. This property does not hold if $\tilde{a}$ and $\tilde{b}$ are not independent. For example, let $\Theta = (\theta_1, \theta_2, \theta_3)$ and define $\mathrm{Cr}\{\theta_1\} = 0.6, \mathrm{Cr}\{\theta_2\} = 0.4, \mathrm{Cr}\{\theta_3\} = 0.2$. Then Cr is a credibility measure on Θ and $(\Theta, 2^{\Theta}, \mathrm{Cr})$ is a credibility space. Define three fuzzy variables

$$\tilde{a}_1(\theta) = \begin{cases} 0, \text{ if } \theta = \theta_1 \\ 1, \text{ if } \theta = \theta_2 \\ 2, \text{ if } \theta = \theta_3, \end{cases} \quad \tilde{a}_2(\theta) = \begin{cases} 0, \text{ if } \theta = \theta_1 \\ 3, \text{ if } \theta = \theta_2 \\ 1, \text{ if } \theta = \theta_3, \end{cases} \quad \tilde{a}_3(\theta) = \begin{cases} 2, \text{ if } \theta = \theta_1 \\ 0, \text{ if } \theta = \theta_2 \\ 3, \text{ if } \theta = \theta_3 \end{cases}$$

Then we have

$$(\tilde{a}_1 + \tilde{a}_2)(\theta) = \begin{cases} 0, \text{ if } \theta = \theta_1 \\ 4, \text{ if } \theta = \theta_2 \\ 3, \text{ if } \theta = \theta_3, \end{cases} \quad (\tilde{a}_1 + \tilde{a}_3)(\theta) = \begin{cases} 2, \text{ if } \theta = \theta_1 \\ 1, \text{ if } \theta = \theta_2 \\ 5, \text{ if } \theta = \theta_3, \end{cases}$$

It is easy to get that

$$E[\tilde{a}_1] = 0.6, E[\tilde{a}_2] = 1.2, E[\tilde{a}_3] = 1.4, E[\tilde{a}_1 + \tilde{a}_2] = 1.6, E[\tilde{a}_1 + \tilde{a}_3] = 2.2$$

which implies that

$$E[\tilde{a}_1 + \tilde{a}_2] < E[\tilde{a}_1] + E[\tilde{a}_2],$$
$$E[\tilde{a}_1 + \tilde{a}_3] > E[\tilde{a}_1] + E[\tilde{a}_3].$$

If $\tilde{a}$ is a fuzzy variable and $f : \Re \to \Re$ is a function, then $f(\tilde{a})$ is also a fuzzy variable, whose expected value can be derived from

$$E[f(\tilde{a})] = \int_0^{+\infty} \mathrm{Cr}\{f(\tilde{a}) \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{f(\tilde{a}) \le r\}\mathrm{d}r$$

provided that at least one of the two integrals is finite.

1.1.7 Variance

The variance of a random variable provides a measure of the spread of the probability distribution around its expected value. In order to measure the spread of fuzzy variable around its mean, Liu and Liu (2002) defined the variance of fuzzy variable. Similar to stochastic case, a small value of variance indicates that the fuzzy variable is tightly close to its expected value. However, a large value of variance indicates that the fuzzy variable has a wide spread around its expected value.

Definition 1.10 (Liu and Liu 2002). Let $\tilde{a}$ be a fuzzy variable with finite expected value a. Then the variance of $\tilde{a}$ is defined by

$$V[\tilde{a}] = E\left[(\tilde{a} - a)^2\right].$$

Definition 1.10 shows that the variance of a fuzzy variable $\tilde{a}$ is the expected value of the squared deviation from its expected value $a = E[\tilde{a}]$. Therefore, the variance is always nonnegative. Let μ be the membership function of $\tilde{a}$. It follows from Definition 1.9 that

$$\begin{aligned} V[\tilde{a}] &= \int_0^{+\infty} \mathrm{Cr}\{(\tilde{a} - a)^2 \geq r\}\mathrm{d}r \\ &= \int_0^{+\infty} \mathrm{Cr}\{|\tilde{a} - a| \geq \sqrt{r}\}\mathrm{d}r \\ &= 2\int_0^{+\infty} r\mathrm{Cr}\{|\tilde{a} - a| \geq r\}\mathrm{d}r \\ &= 2\int_0^{+\infty} r\mathrm{Cr}\{\{\tilde{a} \geq a + r\} \cup \{\tilde{a} \leq a - r\}\}\mathrm{d}r \end{aligned}$$

which implies that the key of calculating $V[\tilde{a}]$ is to calculate the credibility measure

$$\mathrm{Cr}\{\{\tilde{a} \geq a + r\} \cup \{\tilde{a} \leq a - r\}\}.$$

Once the membership function of $\tilde{a}$ is given, this measure is equal to

$$\frac{1}{2}\left(\sup_{x \geq a+r \text{ or } x \leq a-r} \mu(x) + 1 - \sup_{a-r<x<a+r} \mu(x)\right)$$

by credibility inversion theorem.

Theorem 1.10 (Liu 2007). *If $\tilde{a}$ is a fuzzy variable with finite variance, x and y are real numbers, then we have*

$$V[x\tilde{a} + y] = x^2 V[\tilde{a}].$$

Theorem 1.10 tells us that $V[-\tilde{a}] = V[\tilde{a}]$ for any given fuzzy variable $\tilde{a}$.

Example 1.18. Assume that $\tilde{a} = \mathcal{E}(a-\alpha, a+\alpha)$ is an equipossible fuzzy variable. Then its expected value is $E[\tilde{a}] = a$, and for any $r \geq 0$, we have

$$\mathrm{Cr}\{\{\tilde{a} \geq a+r\} \cup \{\tilde{a} \leq a-r\}\} = \begin{cases} 0.5, & \text{if } r \leq \alpha \\ 0, & \text{if } r > \alpha. \end{cases}$$

Thus its variance is

$$V[\tilde{a}] = 2\int_0^{\alpha} 0.5r\mathrm{d}r = \frac{\alpha^2}{2}.$$

Example 1.19. Assume that $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$ is a triangular fuzzy variable. Then its expected value is $E[\tilde{a}] = a + (\beta-\alpha)/4$. We first consider the case $\alpha \geq \beta$, in which $E[\tilde{a}] \leq a$. Further, for any $r \geq 0$, we can get

$$\begin{aligned}
&\mathrm{Cr}\{\{\xi \geq a+r\} \cup \{\xi \leq a-r\}\} \\
&= \begin{cases} \dfrac{11\alpha+\beta+4r}{8\alpha}, & \text{if } 0 \leq r \leq \dfrac{\alpha-\beta}{4} \\ \dfrac{\alpha+3\beta-4r}{8\beta}, & \text{if } \dfrac{\alpha-\beta}{4} \leq r \leq \dfrac{\alpha+\beta}{4} \\ \dfrac{3\alpha+\beta-4r}{8\alpha}, & \text{if } \dfrac{\alpha+\beta}{4} \leq r \leq \dfrac{3\alpha+\beta}{4} \\ 0, & \text{if } r \geq \dfrac{3\alpha+\beta}{4} \end{cases}
\end{aligned}$$

which implies that

$$\begin{aligned}
V[\xi] &= 2\int_0^{\frac{\alpha-\beta}{4}} \frac{r(11\alpha+\beta+4r)}{8\alpha}\mathrm{d}r + \int_{\frac{\alpha-\beta}{4}}^{\frac{\alpha+\beta}{4}} \frac{r(\alpha+3\beta-4r)}{8\beta}\mathrm{d}r \\
&\quad + \int_{\frac{\alpha+\beta}{4}}^{\frac{3\alpha+\beta}{4}} \frac{r(3\alpha+\beta-4r)}{8\alpha}\mathrm{d}r \\
&= \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}.
\end{aligned}$$

We next assume that $\alpha \leq \beta$. Note that $-\tilde{a} = (-a-\beta, -a, -a+\alpha)$ is also a triangular fuzzy variable. Moreover, its left spread β is greater than or equal to its right spread α. Following the above process, we have

$$V[-\tilde{a}] = V[\tilde{a}] = \frac{33\beta^3 + 21\beta^2\alpha + 11\beta\alpha^2 - \alpha^3}{384\beta}.$$

Thus the variance of a triangular fuzzy variable $\mathcal{T}(a-\alpha, a, a+\beta)$ is

$$V[\tilde{a}] = \begin{cases} \dfrac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}, \text{ if } \alpha \geq \beta \\ \dfrac{33\beta^3 + 21\alpha\beta^2 + 11\alpha^2\beta - \alpha^3}{384\beta}, \text{ if } \alpha \leq \beta. \end{cases}$$

For symmetrical triangular fuzzy variable, i.e., $\alpha = \beta$, we have

$$E[\tilde{a}] = a \quad \text{and} \quad V[\tilde{a}] = (\alpha+\beta)^2/24 = \alpha^2/6.$$

Example 1.20. Let $\tilde{a} = \mathcal{N}(a, \delta)$ be a normal fuzzy variable. Without loss of generality, we assume that $a = 0$. Then for any $r \geq 0$, it is easy to see that

$$\mathrm{Cr}\{\tilde{a} \geq r\} = \mathrm{Cr}\{\tilde{a} \leq -r\} = \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1} \leq \frac{1}{2}.$$

According to the Axiom 4 of credibility measure, we have

$$\begin{aligned} &\mathrm{Cr}\{\{\tilde{a} \geq a + r\} \cup \{\tilde{a} \leq a - r\}\} \\ &= \mathrm{Cr}\{\tilde{a} \geq a + r\} \vee \mathrm{Cr}\{\tilde{a} \leq a - r\} \\ &= \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1}. \end{aligned}$$

Thus the variance is

$$V[\tilde{a}] = 2\int_0^{+\infty} r\left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1} \mathrm{d}r.$$

Substituting $\pi r/\sqrt{6}\delta$ with u and r with $\sqrt{6}\delta u/\pi$, the change of variables yields

$$\begin{aligned} V[\tilde{a}] &= 2\int_0^{+\infty} r\left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\delta}\right)\right)^{-1} \mathrm{d}r \\ &= \frac{12\delta^2}{\pi^2}\int_0^{+\infty} \frac{u}{1+e^u}\mathrm{d}u \\ &= \delta^2 \end{aligned}$$

where the last equality holds since the integral

$$\int_0^{+\infty} u/(1+e^u)\mathrm{d}u = \pi^2/12.$$

If $a \neq 0$, then $\tilde{a} - a$ is a normal fuzzy variable with expected value 0 and variance δ^2. Further, it follows from Theorem 1.10 that

$$V[\tilde{a}] = V[(\tilde{a} - a) + a] = V[\tilde{a} - a] = \delta^2.$$

Let $\tilde{a}$ be a fuzzy variable with expected value a. Liu (2007) proved that $V[\tilde{a}] = 0$ if and only if $\mathrm{Cr}\{\tilde{a} = a\} = 1$, which indicates that the fuzzy variable with variance of zero degenerates into a real number.

1.2 Uncertainty Theory

Uncertainty theory was founded by Liu (2007) and further developed by Liu (2010) as a branch of mathematics for modelling subjective uncertain phenomena. This section recalls the basic contents in uncertainty theory.

1.2.1 Uncertain Measure

Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. In order to indicate the chance that Λ will happen, Liu (2007) proposed the following three axioms to ensure that the number $\mathcal{M}\{\Lambda\}$ satisfying certain mathematical properties,

Axiom 1. *(Normality)* $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2. *(Self-Duality)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *(Countable Subadditivity) For every countable sequence of events* $\Lambda_1, \Lambda_2, \cdots$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1.17}$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space. Following these axioms, it is easy to verify that the empty set $\emptyset$ has an uncertain measure zero, i.e., $\mathcal{M}\{\emptyset\} = 0$ and $0 \leq \mathcal{M}\{\Lambda\} \leq 1$ for any event Λ. In addition, Liu (2010) proved the monotonicity theorem of uncertain measure. That is, for any events Λ_1 and Λ_2 with $\Lambda_1 \subset \Lambda_2$, we have $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$.

Further, by defining product uncertain measure, Liu (2010) provided the following product axiom. If $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ are uncertainty spaces for $k = 1, 2, \cdots$, then the product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{1.18}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Note that (1.18) defines a product uncertain measure only for rectangles. The uncertain measure $\mathcal{M}$ could be extended from the class of rectangles to the product σ-algebra $\mathcal{L}$ via

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c} \min\limits_{1\le k<\infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise} \end{cases} \tag{1.19}$$

in which $\Lambda \in \mathcal{L}$ is an event.

1.2.2 Uncertain Variable

Definition 1.11 (Liu 2010). An uncertain variable ξ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event for any Borel set B of real numbers.

A crisp number x may be regarded as a special uncertain variable since it is a constant function $\xi(\gamma) \equiv x$ on a given uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Assume that ξ_1 and ξ_2 are two uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We say $\xi_1 = \xi_2$ if there exists a set $\Lambda \subset \Gamma$ with $\mathcal{M}\{\Lambda\} = 1$ such that $\xi_1(\gamma) = \xi_2(\gamma)$ for each $\gamma \in \Lambda$.

Definition 1.12 (Liu 2010). An uncertain variable ξ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be (a) nonnegative if $\mathcal{M}\{\xi < 0\} = 0$; and (b) positive if $\mathcal{M}\{\xi \le 0\} = 0$.

Let $\xi_1, \xi_2, \cdots, \xi_m$ be uncertain variables, and let f be a real valued measurable function. Then Liu (2010) proves that the function defined by

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_m(\gamma)), \quad \forall \gamma \in \Gamma$$

is an uncertain variable. For example, if ξ_1 and ξ_2 are two uncertain variables, then the sum $\xi = \xi_1 + \xi_2$ defined by $\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma)$ is an uncertain variable. Similarly, the product $\eta = \xi_1 \cdot \xi_2$ is also an uncertain variable.

As an extension of uncertain variable, Liu (2007) introduced a concept of uncertain vector.

Definition 1.13 (Liu 2007). An k-uncertain vector $\boldsymbol{\xi}$ is a function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k-dimensional real vectors such that $\{\boldsymbol{\xi} \in B\}$ is an event for any Borel set B of k-dimensional real vectors.

Theorem 1.11 (Liu 2007). *The vector $(\xi_1, \xi_2, \cdots, \xi_k)$ is an uncertain vector if and only if $\xi_1, \xi_2, \cdots, \xi_k$ are uncertain variables.*

1.2.2.1 Independence

Definition 1.14 (Liu 2010). The uncertain variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\} \tag{1.20}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Theorem 1.12 (Liu 2010). *Suppose that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables, and let $f_1, f_2, \cdots, f_n$ be measurable functions. Then $f_1(\xi_1), f_x(\xi_2), \cdots, f_n(\xi_n)$ are independent uncertain variables.*

1.2.3 Uncertainty Distribution

Definition 1.15 (Liu 2010). The uncertainty distribution Φ of an uncertain variable ξ is defined by

$$\Phi(r) = \mathcal{M}\{\xi \le r\} \tag{1.21}$$

for any real number r.

Peng and Iwamura (2010) proved that a function $\Phi(r) : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function except $\Phi(r) \equiv 0$ and $\Phi(r) \equiv 1$. Next we introduce some of commonly used uncertainty distributions.

1.2.3.1 Linear Uncertainty Distribution

An uncertain variable ξ is called linear if it has a linear uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \le a, \\ (r-a)/(b-a), & \text{if } a \le r \le b, \\ 1, & \text{if } r \ge b. \end{cases}$$

The linear uncertain variable is denoted by $\mathcal{L}(a, b)$ where a and b are real numbers with $a < b$.

1.2.3.2 Zigzag Uncertainty Distribution

An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution

$$\Phi(r) = \begin{cases} 0, & \text{if } r \leq a, \\ (r-a)/2(b-a), & \text{if } a \leq r \leq b, \\ (x+c-2b)/2(c-b), & \text{if } b \leq r \leq c, \\ 1, & \text{if } r \geq c. \end{cases}$$

The zigzag uncertain variable is denoted by $\mathcal{Z}(a, b, c)$ where a, b, c are real numbers with $a < b < c$.

1.2.3.3 Normal Uncertainty Distribution

An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(r) = \left(1 + \exp\left(\frac{\pi(e-r)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ r \in \Re.$$

The normal uncertain variable is denoted by $\mathcal{N}(e, \sigma)$ where e and σ are real numbers with $\sigma > 0$.

1.2.3.4 Regular Uncertainty Distribution

Definition 1.16 (Liu 2010). An uncertainty distribution $\Phi(r)$ is said to be regular if it is a continuous and strictly increasing function with respect to r at which $0 < \Phi(r) < 1$, and

$$\lim_{r \to -\infty} \Phi(r) = 0, \qquad \lim_{r \to +\infty} \Phi(r) = 1.$$

It is evident that a regular uncertainty distribution $\Phi(r)$ has an inverse function on the range of r with $0 < \Phi(r) < 1$, and the inverse function $\Phi^{-1}(\alpha)$ exists on the open interval $(0, 1)$. The commonly used uncertainty distributions are all regular such as linear uncertainty distribution, zigzag uncertainty distribution and normal uncertainty distribution.

1.2.3.5 Inverse Uncertainty Distribution

Definition 1.17 (Liu 2010). Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.

Example 1.21. The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a,b)$ is $\Phi^{-1}(\alpha) = (1-\alpha)a + \alpha b$.

Example 1.22. The inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a,b,c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1-2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2-2\alpha)b + (2\alpha-1)c, & \text{if } \alpha \geq 0.5. \end{cases}$$

Example 1.23. The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e,\sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Theorem 1.13 (Liu 2010). *A function $\Phi^{-1}(\alpha) : (0,1) \to \Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to α.*

1.2.4 Operational Law

Theorem 1.14 (Liu 2010). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an uncertainty distribution*

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\leq i\leq n} \Phi_i(x_i), \tag{1.22}$$

and has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{1.23}$$

Example 1.24. Let ξ_1 and ξ_2 be independent uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Then $\xi_1 \vee \xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{x_1 \vee x_2 \leq x} \Phi_1(x_1) \wedge \Phi_2(x_2) = \Phi_1(x) \wedge \Phi_2(x),$$

and $\xi_1 \wedge \xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{x_1 \wedge x_2 \le x} \Phi_1(x_1) \wedge \Phi_2(x_2) = \Phi_1(x) \vee \Phi_2(x),$$

Example 1.25. Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. It follows from Theorem 1.14 that $\xi = \xi_1 + \xi_2 + \cdots + \xi_n$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha).$$

1.2.5 Expected Value

Expected value of an uncertain variable is essentially its average value in the sense of uncertain measure, and represents its size.

Definition 1.18 (Liu 2007). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathcal{M}\{\xi \le r\}\mathrm{d}r \tag{1.24}$$

provided that at least one of the two integrals is finite.

Example 1.26. The linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an expected value $E[\xi] = (a + b)/4$; the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an expected value $E[\xi] = (a + 2b + c)/4$; the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an expected value e, i.e., $E[\xi] = e$.

Theorem 1.15 (Liu 2007). *Let ξ be an uncertain variable with an uncertainty distribution Φ. Then we have*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(r))\mathrm{d}r - \int_{-\infty}^0 \Phi(r)dr. \tag{1.25}$$

Theorem 1.16 (Liu 2010). *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. Then we have*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{1.26}$$

Theorem 1.17 (Liu 2010). *Let a and b be two real numbers, and ξ and η two uncertain variables. Then we have*

$$E[a\xi + b] = aE[\xi] + b.$$

Further, if ξ and η are independent, then

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Theorem 1.18 (Liu and Ha 2010). *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_n$, then the expected value of uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is*

$$E[\xi] = \int_0^1 f\left(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)\right) \mathrm{d}\alpha.$$

Generally speaking, the expected value operator of uncertain variable is not necessarily linear if the independence is not assumed. The detailed example can be found in Liu (2010).

The variance of uncertain variable represents the degree of the spread of its distribution around its expected value.

Definition 1.19 (Liu 2007). Let ξ be an uncertain variable with a finite expected value e. Then its variance $V[\xi]$ is defined by

$$V[\xi] = E[(\xi - e)^2]. \tag{1.27}$$

Let ξ be an uncertain variable with finite expected value e and uncertainty distribution Φ. In general, we can only obtain an upper bound of its variance $V[\xi]$ as follows,

$$V[\xi] \leq \int_0^{\infty} [1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})]\mathrm{d}x.$$

1.3 Genetic Algorithm

Genetic algorithm (GA) was initiated by Holland (1975) as an adaptive heuristic search algorithm premised on the evolutionary ideas of natural selection and natural genetics. During the past four decades, GA has demonstrated considerable success in providing good solutions to many complex optimization problems. The advantage of GA is just able to obtain the global optimal solution when the objective functions are multimodal or the search spaces are irregular. Moreover, GA does not require the specific mathematical properties of optimization problems, which makes GA easily be used in reality.

For the optimization problems with fuzzy parameters or other uncertain parameters, simulation methods are required to approximately calculate the corresponding expressions, and GA provides good solutions to these uncertain optimization problems. Liu (2002) first applied genetic algorithm to solve the optimization problems with fuzzy parameters. After that, genetic algorithm is also applied to solve portfolio optimization problems such as Huang (2006, 2007a), Qin et al. (2009) and so on. Next we briefly review the basic components of GA. For detailed expositions, the interested readers may consult Liu (2002).

1.3.1 Representation Structure

In general, the representation structure is problem-dependent. Since the decision variables are required to be [0, 1], real encoding is used for portfolio optimization problems. That is, a solution $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ is encoded by a chromosome $\boldsymbol{c} = (c_1, c_2, \cdots, c_n)$ where the genes $c_1, c_2, \cdots, c_n$ are restricted as nonnegative numbers. The decoding process is determined by the link $x_i = c_i/(c_1+c_2+\cdots+c_n)$, which ensures that $x_1 + x_2 + \cdots + x_n = 1$ always holds. A chromosome is called feasible if it satisfies the corresponding constraint conditions.

1.3.2 Initialization Process

Let an integer *pop_size* be the population size. The purpose of the initialization is to obtain *pop_size* feasible chromosomes as the first population. Here, a feasible chromosome is one which satisfies the corresponding constraint conditions. The procedure of initialization process is summarized as follows,

Algorithm 1.1 (Initialization process)

Step 1. Set i=1;
Step 2. Randomly generate n nonnegative numbers $c_1, c_2, \cdots, c_n$ such that $\boldsymbol{c}_i = (c_1, c_2, \cdots, c_n)$ is a feasible chromosome;
Step 3. If $i = pop\_size$, stop; otherwise, set $i = i + 1$ and go to Step 2.

1.3.3 Evaluation Function

Evaluation function is to measure the likelihood of reproduction for each chromosome. In this book, we employ the more popular rank-based evaluation function. Let $v \in (0, 1)$ be a parameter in the genetic system (for example, $v = 0.05$). The rank-based evaluation function is defined as:

$$Eval(\mathbf{c}_i) = v(1-v)^{i-1}, \quad i = 1, 2, \cdots, pop\_size.$$

It is easy to see that $i = 1$ indicates the best individual and $i = pop\_size$ indicates the worst one.

1.3.4 Selection Process

The selection of chromosomes is done by spinning the roulette wheel which is a fitness-proportional selection. Each time one chromosome is selected for a new child population. Continuing this process for *pop_size* times, we can obtain the next population. Define $p_0 = 0$ and

$$p_i = \sum_{j=1}^{i} Eval(\boldsymbol{c}_j), i = 1, 2, \cdots, pop\_size.$$

The algorithm for selection process is summarized as follows,

Algorithm 1.2 (Selection process)

Step 1. Set j=1;
Step 2. Randomly generate a number r from the interval $(0, p_{pop\_size}]$;
Step 3. Select the chromosome $\boldsymbol{c}_i$ if $r \in (p_{i-1}, p_i]$;
Step 4. If $j > pop\_size$, stop; otherwise, set $j = j + 1$ and go to Step 2.

1.3.5 Crossover Operation

Let P_c be the probability of crossover operation, which gives us the expected number $P_c \cdot pop\_size$ of chromosomes undergoing the crossover operation. Randomly generate a number $r \in [0, 1]$. If $r < P_c$, randomly select two parent chromosomes denoted by $\mathbf{c}_1$ and $\mathbf{c}_2$, and then produce two offsprings through crossover operator

$$\boldsymbol{x} = r \cdot \mathbf{c}_1 + (1 - r) \cdot \mathbf{c}_2, \quad \boldsymbol{y} = (1 - r) \cdot \mathbf{c}_1 + r \cdot \mathbf{c}_2.$$

If both $\boldsymbol{x}$ and $\boldsymbol{y}$ are feasible, take them as children to replace their parents. If at least one of them is infeasible, then redo the crossover operation until two feasible children are obtained or a given number of iterations is finished. The crossover operation is finished by repeating the above process for *pop_size* times.

1.3.6 Mutation Operation

Let P_m be the probability of mutation operation, which gives us the expected number of $P_m \cdot pop\_size$ of chromosomes undergoing the mutation operations. Randomly generate a real number $s \in (0, 1)$. If $s < P_m$, then choose a chromosome **c** as the parent one for mutation operation. Next, randomly generate n numbers $d_1, d_2, \cdots, d_n$ which belong to the interval $[-D, D]$ where $D > 0$ is an appropriate number. A new chromosome is created by $\boldsymbol{x} = \boldsymbol{c} + (d_1, d_2, \cdots, d_n)$. If $\boldsymbol{x}$ is feasible, take it as the child. Otherwise, redo the mutation operation until one feasible child is obtained or a given number of iterations is finished. The mutation operation is finished by repeating the above process for *pop_size* times.

1.3.7 Genetic Algorithm

After selection, crossover and mutation, a new population can be produced, and the next evaluation is continued. GA is finished after a given number of iterations of the above process. The procedure of GA is summarized as follows:

Algorithm 1.3 (Genetic algorithm)

Step 1. Input the parameters of GA: pop_size, P_c, P_m and v;
Step 2. Initialize *pop_size* feasible chromosomes at random;
Step 3. Update the chromosomes by crossover and mutation operations;
Step 4. Compute the objective value for each chromosome;
Step 5. Calculate the fitness of each chromosome by the rank-based-evaluation function according to their objective values;
Step 6. Select the chromosomes by using spinning the roulette wheel;
Step 7. Repeat the second to fifth step for a given number of generations;
Step 8. Take the best chromosome as the solution.

Chapter 2
Credibilistic Mean-Variance-Skewness Model

2.1 Introduction

Most of the existing works on portfolio optimization have been done based on only the first two moments of return distributions. However, there is a controversy over the issue of whether higher moments should be considered in portfolio selection. Many researchers (e.g. Arditti 1967; Konno et al. 1993; Konno and Suzuki 1995; Kraus and Litzenberger 1976; Liu et al. 2003; Prakash et al. 2003; Samuelson 1970) argued that the higher moments cannot be neglected unless there are reasons to trust that the returns are symmetrically distributed (e.g. normal) or that higher moments are irrelevant to the investors' decisions. Samuelson (1970) also showed that higher moments are relevant for investors to make decisions in portfolio optimization and almost all investors would prefer a portfolio with a larger third order moment if first and second moments are same. Moreover, numerous empirical studies show that portfolio returns are generally asymmetric. All the above discussions motivated the researchers to add the third moment of return distribution of a portfolio into a general mean-variance model.

Analogous to stochastic cases, investors also face to construct an optimal portfolio from the candidate securities with asymmetrical fuzzy returns. In the framework of credibility theory, Huang (2008a) first employed semivariance to describe the asymmetry of fuzzy returns. However, semivariance of fuzzy variable is difficult to be calculated. Different from the previous works, Li et al. (2010) defined skewness of fuzzy returns to characterize the asymmetry as an alternative approach and formulated mean-variance-skewness model. Then Kamdema et al. (2012) extended credibilitic mean-variance-skewness model by adding kurtosis, i.e., the fourth moment.

This chapter focuses on the fuzzy portfolio optimization with asymmetrical returns and introduces credibilistic mean-variance-skewness model. The main contents include the definition of skewness of fuzzy variable, the formulation and

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variants of mean-variance-skewness model, fuzzy simulation for skewness, and numerical analysis.

2.2 Skewness of Fuzzy Variable

In this section, we review the concept of skewness for fuzzy variables and its basic mathematical properties.

Definition 2.1 (Li et al. 2010). Let $\tilde{a}$ be a fuzzy variable with finite expected value. Then the skewness of $\tilde{a}$ is defined as

$$S_k[\tilde{a}] = E\left[(\tilde{a} - E[\tilde{a}])^3\right]. \tag{2.1}$$

Example 2.1. The equipossible fuzzy variable $\tilde{a} = \mathcal{E}(a-\alpha, a+\alpha)$ has a skewness

$$S_k[\tilde{a}] = 0. \tag{2.2}$$

In Example 1.14, it has been verified that $E[\tilde{a}] = a$. In addition, we have

$$\mathrm{Cr}\{\tilde{a} \le r + a\} = \begin{cases} 0, & \text{if } r < -\alpha \\ 1/2, & \text{if } -\alpha \le r < \alpha \\ 1, & \text{if } r \ge \alpha, \end{cases}$$

$$\mathrm{Cr}\{\tilde{a} \ge r + a\} = \begin{cases} 1, & \text{if } r < -\alpha \\ 1/2, & \text{if } -\alpha \le r < \alpha \\ 0, & \text{if } r \ge \alpha. \end{cases}$$

It follows from Definition 2.1 that

$$\begin{aligned} S_k[\tilde{a}] &= E\left[(\tilde{a} - a)^3\right] \\ &= \int_0^{+\infty} \mathrm{Cr}\{(\tilde{a} - a)^3 \ge r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{(\tilde{a} - a)^3 \le r\}\mathrm{d}r \\ &= 3\int_0^{+\infty} r^2 \mathrm{Cr}\{\tilde{a} \ge r + a\}\mathrm{d}r - 3\int_{-\infty}^0 r^2 \mathrm{Cr}\{\tilde{a} \le r + a\}\mathrm{d}r \\ &= 3\int_0^{\alpha} \frac{r^2}{2}\mathrm{d}r - 3\int_{-\alpha}^0 \frac{r^2}{2}\mathrm{d}r \\ &= 0. \end{aligned}$$

Example 2.2. The triangular fuzzy variable $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$ has a skewness

$$S_k[\tilde{a}] = \frac{(\beta+\alpha)^2(\beta-\alpha)}{32}. \tag{2.3}$$

In Example 1.11, it has been verified that $E[\tilde{a}] = a + (\beta - \alpha)/4$. It follows from Eq. (1.11) that

$$\mathrm{Cr}\{\tilde{a} \le r + E[\tilde{a}]\} = \begin{cases} 0, & \text{if } r \le -\dfrac{3\alpha+\beta}{4} \\ \dfrac{4r+3\alpha+\beta}{8\alpha}, & \text{if } -\dfrac{3\alpha+\beta}{4} \le r \le \dfrac{\alpha-\beta}{4} \\ \dfrac{4r-\alpha+5\beta}{8\beta}, & \text{if } \dfrac{\alpha-\beta}{4} \le r \le \dfrac{\alpha+3\beta}{4} \\ 1, & \text{if } r \ge \dfrac{\alpha+3\beta}{4} \end{cases}$$

and

$$\mathrm{Cr}\{\tilde{a} \ge r + E[\tilde{a}]\} = \begin{cases} 1, & \text{if } r \le -\dfrac{3\alpha+\beta}{4} \\ \dfrac{-4r+5\alpha-\beta}{8\alpha}, & \text{if } -\dfrac{3\alpha+\beta}{4} \le r \le \dfrac{\alpha-\beta}{4} \\ \dfrac{-4r+\alpha+3\beta}{8\beta}, & \text{if } \dfrac{\alpha-\beta}{4} \le r \le \dfrac{\alpha+3\beta}{4} \\ 0, & \text{if } r \ge \dfrac{\alpha+3\beta}{4}. \end{cases}$$

Without loss of generality, we assume that $\alpha \ge \beta$. Further, it follows from Definition 2.1 that

$$\begin{aligned} S_k[\tilde{a}] &= 3\int_0^{+\infty} r^2\mathrm{Cr}\{\tilde{a} \ge r + E[\tilde{a}]\}\,\mathrm{d}r - 3\int_{-\infty}^0 r^2\mathrm{Cr}\{\tilde{a} \le r + E[\tilde{a}]\}\,\mathrm{d}r \\ &= 3\int_0^{\frac{\alpha-\beta}{4}} \frac{-4r^3+(5\alpha-\beta)r^2}{8\alpha}\mathrm{d}r + 3\int_{\frac{\alpha-\beta}{4}}^{\frac{\alpha+3\beta}{4}} \frac{-4r^3+(\alpha+3\beta)r^2}{8\beta}\mathrm{d}r \\ &\quad -3\int_{-\frac{3\alpha+\beta}{4}}^0 \frac{4r^3+(3\alpha+\beta)r^2}{8\alpha}\mathrm{d}r \\ &= \frac{(\alpha-\beta)^3(17\alpha-\beta)-(3\alpha+\beta)^4}{2048\alpha} + \frac{(\alpha+3\beta)^4-(\alpha-\beta)^3(\alpha+15\beta)}{2048\beta} \\ &= \frac{\alpha[(\alpha+3\beta)^4-(\alpha-\beta)^4]-\beta[(3\alpha+\beta)^4-(\alpha-\beta)^4]}{2048\alpha\beta} \\ &= \frac{16\alpha\beta(\alpha+\beta)(\alpha^2+2\alpha\beta+5\beta^2)-16\alpha\beta(\alpha+\beta)(5\alpha^2+2\alpha\beta+\beta^2)}{2048\alpha\beta} \\ &= \frac{(\beta+\alpha)^2(\beta-\alpha)}{32}. \end{aligned}$$

This formula obviously implies that if $\beta \ge \alpha$, then $S_k[\tilde{a}] \ge 0$ and if $\beta \le \alpha$, then $S_k[\tilde{a}] \le 0$. Especially, if $\tilde{a}$ is symmetric, then we have $\beta = \alpha$ and $S_k[\tilde{a}] = 0$.

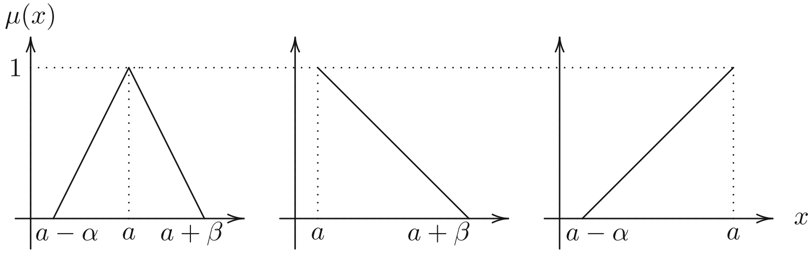

Fig. 2.1 Several membership functions of triangular fuzzy variables

Furthermore, if $\alpha = 0$, then $S_k[\tilde{a}]$ obtains its maximum value $\beta^3/32$; and if $\beta = 0$, then $S_k[\tilde{a}]$ obtains its minimum value $-\alpha^3/32$ (Fig. 2.1).

Example 2.3. The normal fuzzy variable $\tilde{a} = \mathcal{N}(a, \delta)$ has a skewness

$$S_k[\tilde{a}] = 0. \tag{2.4}$$

It follows from Examples 1.5 and 1.12 that $E[\tilde{a}] = a$ and

$$\mathrm{Cr}\{\tilde{a} \le r\} = \Gamma(r) = \left(1 + \exp\left(\frac{\pi(a-r)}{\sqrt{6}\sigma}\right)\right)^{-1},$$
$$\mathrm{Cr}\{\tilde{a} \ge r\} = 1 - \Gamma(r) = \left(1 + \exp\left(\frac{\pi(r-a)}{\sqrt{6}\sigma}\right)\right)^{-1}.$$

Similar to Example 2.2, further, it follows from Definition 2.1 that

$$\begin{aligned} S_k[\tilde{a}] &= 3\int_0^{+\infty} r^2 \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1} \mathrm{d}r \\ &\quad -3\int_{-\infty}^{0} r^2 \left(1 + \exp\left(-\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1} \mathrm{d}r \\ &= 3\int_0^{+\infty} \left(r^2 \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1} - r^2 \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1}\right) \mathrm{d}r \\ &= 0. \end{aligned}$$

Example 2.4. The exponential fuzzy variable $\tilde{a} = \mathcal{EXP}(m)$ has a skewness

$$S_k[\tilde{a}] = \kappa m^3. \tag{2.5}$$

where $\kappa = 3\sqrt{6}\left(9\zeta(3) + 12\ln 2 - \pi^2 \ln 2\right)/\pi^3 \approx 2.914$, and $\zeta(w) = \sum_{k=1}^{\infty} k^{-w}$. For any $r \geq 0$, it follows from Example 1.13 that

$$\mathrm{Cr}\{\tilde{a} \leq r\} = 1 - \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}m}\right)\right)^{-1},$$

and from the credibility inversion theorem that

$$\mathrm{Cr}\{\tilde{a} \geq r\} = \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}m}\right)\right)^{-1}.$$

Note that $E[\tilde{a}] = (\sqrt{6}\ln 2)m/\pi$. It follows from Definition 2.1 that

$$\begin{aligned}
S_k[\tilde{a}] &= E\left[(\tilde{a} - E[\tilde{a}])^3\right] \\
&= 3\int_{E[\tilde{a}]}^{+\infty} (r - E[\tilde{a}])^2\, \mathrm{Cr}\{\tilde{a} \geq r\}\mathrm{d}r - 3\int_0^{E[\tilde{a}]} (r - E[\tilde{a}])^2\, \mathrm{Cr}\{\tilde{a} \leq r\}\mathrm{d}r \\
&= \frac{18\sqrt{6}m^3}{\pi^3}\left(\int_{\ln 2}^{+\infty} \frac{(r - \ln 2)^2}{1 + \exp(r)}\mathrm{d}r - \int_0^{\ln 2} \frac{(r - \ln 2)^2 \exp(r)}{1 + \exp(r)}\mathrm{d}r\right) \\
&= \frac{18\sqrt{6}m^3}{\pi^3}\int_0^{+\infty} \frac{(r - \ln 2)^2}{1 + \exp(r)}\mathrm{d}r - \frac{6\sqrt{6}m^3 \ln 8}{\pi^3} \\
&= \frac{3\sqrt{6}\left(9\sum_{k=1}^{\infty} 3^{-w} + 12\ln 2 - \pi^2 \ln 2\right)}{\pi^3} m^3.
\end{aligned}$$

Theorem 2.1 (Li et al. 2010). *Let $\tilde{a}$ be a fuzzy variable with finite expected value. For any real numbers x and y, we have*

$$S_k[x\tilde{a} + y] = x^3 S_k[\tilde{a}].$$

Proof. It follows from Theorem 1.8 that $E[x\tilde{a} + y] = xE[\tilde{a}] + y$. According to Definition 2.1, we have

$$\begin{aligned}
S_k[x\tilde{a} + y] &= E[(x\tilde{a} + y - (xE[\tilde{a}] + y))^3] \\
&= E\left[x^3(\tilde{a} - E[\tilde{a}])^3\right] \\
&= x^3 E\left[(\tilde{a} - E[\tilde{a}])^3\right] \\
&= x^3 S_k[\tilde{a}].
\end{aligned}$$

The theorem is proved.

Theorem 2.2 (Li et al. 2010). *Let $\tilde{a}$ be a symmetric fuzzy variable with finite expected value. Then we have*

$$S_k[\tilde{a}] = 0.$$

Proof. Let μ be the membership function of $\tilde{a}$. Since $\tilde{a}$ is symmetric, there is a real number e such that

$$\mu(e + r) = \mu(e - r), \quad \forall r \in \Re.$$

Furthermore, it can be obtained that

$$\sup_{s \geq r+e} \mu(s) = \sup_{s \geq r} \mu(s + e) = \sup_{s \geq r} \mu(e - s) = \sup_{s \leq e-r} \mu(s).$$

It follows from the credibility inversion theorem (Theorem 1.2) that

$$\begin{aligned}
\mathrm{Cr}\{\tilde{a} \geq r + e\} &= \frac{1}{2}\left(\sup_{s \geq r+e} \mu(s) + 1 - \sup_{s < r+e} \mu(s)\right) \\
&= \frac{1}{2}\left(\sup_{r \leq e-x} \mu(r) + 1 - \sup_{r > e-x} \mu(r)\right) \\
&= \mathrm{Cr}\{\tilde{a} \leq e - x\}.
\end{aligned}$$

According to Definition 1.9, it is obtained that

$$\begin{aligned}
E[\tilde{a}] &= \int_0^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\tilde{a} \leq r\}\mathrm{d}r \\
&= \int_{-e}^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r + e\}\mathrm{d}r - \int_{e}^{+\infty} \mathrm{Cr}\{\tilde{a} \leq e - r\}\mathrm{d}r \\
&= \int_{-e}^{0} \mathrm{Cr}\{\tilde{a} \geq r + e\}\mathrm{d}r + \int_{0}^{+\infty} \mathrm{Cr}\{\tilde{a} \geq r + e\}\mathrm{d}r \\
&\quad - \int_{0}^{+\infty} \mathrm{Cr}\{\tilde{a} \leq e - r\}\mathrm{d}r + \int_{0}^{e} \mathrm{Cr}\{\tilde{a} \leq e - r\}\mathrm{d}r \\
&= \int_{-e}^{0} \mathrm{Cr}\{\tilde{a} \geq r + e\}\mathrm{d}r + \int_{0}^{e} \mathrm{Cr}\{\tilde{a} \leq e - r\}\mathrm{d}r \\
&= \int_{0}^{e} \left(\mathrm{Cr}\{\tilde{a} \geq e - r\} + \mathrm{Cr}\{\tilde{a} \leq e - r\}\right)\mathrm{d}r \\
&= e.
\end{aligned}$$

Further, it follows from the definition of skewness (Definition 2.1) that

$$\begin{aligned}
S_k[\tilde{a}] &= \int_0^{+\infty} \mathrm{Cr}\{(\tilde{a}-e)^3 \geq r\}\mathrm{d}r \\
&= \int_0^{+\infty} 3r^2\mathrm{Cr}\{\tilde{a}-e \geq r\}\mathrm{d}r - \int_{-\infty}^{0} 3r^2\mathrm{Cr}\{\tilde{a}-e \leq r\}\mathrm{d}r \\
&= \int_0^{+\infty} 3r^2\mathrm{Cr}\{\tilde{a}-e \leq -r\}\mathrm{d}r - \int_0^{+\infty} 3r^2\mathrm{Cr}\{\tilde{a}-e \leq -r\}\mathrm{d}r \\
&= 0.
\end{aligned}$$

The theorem is proved.

2.3 Mean-Variance-Skewness Model

We consider a capital market with n risky securities. An investor allocates his/her money among these n risky securities to achieve some objective. Let $\tilde{a}_i$ be the return of the ith security, $i = 1, 2, \cdots, n$. In general, $\tilde{a}_i$ is given as $(p_i' + d_i - p_i)/p_i$ in which p_i is the opening price of the ith security in one period, p_i' its estimated closing price in the next period and d_i its estimated dividends during the coming period. When lack of sufficient data, p_i' and d_i can be given by domain experts via membership function. Thus, we regard $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ as fuzzy variables defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$.

Let x_i be the proportion of total fund invested in security i, $i = 1, 2, \cdots, n$. Then $(x_1, x_2, \cdots, x_n)$ is a portfolio which represents a possible investment strategy. The total return on the portfolio $(x_1, x_2, \cdots, x_n)$ is $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$. It follows from Definition 1.3 that $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$ is also a fuzzy variable. Following Markowitz's mean-variance analysis, we may employ the expected value and variance of $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$ as the investment return and risk of the given portfolio $(x_1, x_2, \cdots, x_n)$, respectively.

Following Markowitz's idea, in the case of minimizing the risk for a given level of return, Huang (2006) formulated a credibilistic mean-variance model for fuzzy portfolio optimization as follows,

$$\begin{cases}
\min\ V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\
\text{s.t.}\ \ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \geq r \\
\qquad x_1 + x_2 + \cdots + x_n = 1, \\
\qquad x_i \geq 0, \quad i = 1, 2, \cdots, n
\end{cases} \tag{2.6}$$

in which E denotes the expected value operator, V the variance operator, and r is the minimum expected return level the investors can accept. Model (2.6) is valid when the returns of securities are symmetrical.

When minimal expected return and maximal risk levels are given, the investors interested in the use of skewness prefer a portfolio with larger skewness. For this case, Li et al. (2010) proposed the following mean-variance-skewness model for portfolio optimization,

$$\begin{cases} \max\ S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\ \text{s.t.}\quad E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \ge r \\ \qquad V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \le d \\ \qquad x_1 + x_2 + \cdots + x_n = 1, \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.7}$$

The first constraint ensures the expected return is no less than some target value r, and the second one assures that risk does not exceed some given level d the investor can bear. The last two constraints imply that all the capital will be invested to n securities and short-selling is not allowed.

If security returns are symmetrical fuzzy variables, then it follows from Theorem 2.2 that $S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] = 0$, which implies that model (2.7) is invalid for the symmetrical case. In other words, model (2.7) is only suitable for the case with asymmetrical fuzzy returns.

The first variation proposed by Li et al. (2010) of model (2.7) is the following,

$$\begin{cases} \min\ V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\ \text{s.t.}\quad E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \ge r \\ \qquad S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \ge s \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.8}$$

The aim of this model is to minimize risk when expected return and skewness are both no less than some given target values r and s, respectively. If the second constraint is removed or in the symmetrical case, then the above model degenerates to mean-variance model proposed by Huang (2007b).

The second variation proposed by Li et al. (2010) of model (2.7) is the following,

$$\begin{cases} \max\ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\ \text{s.t.}\quad S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \ge s \\ \qquad V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \le d \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.9}$$

The aim of this model is to maximize the expected return. Similarly, if the first constraint is removed or in the symmetrical case, then it degenerates to the other mean-variance model considered by Huang (2007b).

The final variation proposed by Li et al. (2010) of model (2.7) is the following multi-objective nonlinear programming,

$$\begin{cases} \max\ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\ \min\ \ V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\ \max\ S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \\ \text{s.t.}\quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\ \ x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.10}$$

When the membership functions of $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ are symmetric, the third objective vanishes and model (2.10) degenerates into a bi-objective mean-variance model.

Theorem 2.3 (Li et al. 2010). *Suppose that the returns $\tilde{a}_i = (a_i - \alpha_i, a_i, a_i + \beta_i)$ are independent triangular fuzzy variables for $i = 1, 2, \cdots, n$. Then model (2.7) is equivalent to the following crisp programming,*

$$\begin{cases} \max\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(\beta_i + \alpha_i)(\beta_j + \alpha_j)(\beta_k - \alpha_k)x_ix_jx_k \\ \text{s.t.}\quad \displaystyle\sum_{i=1}^{n} x_i(4a_i + \beta_i - \alpha_i) \geq 4r \\ \qquad\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}(17\alpha_i\alpha_j + 17\beta_i\beta_j + 11\alpha_i\beta_j + 11\alpha_j\beta_i)x_ix_j\left|\sum_{k=1}^{n}(\alpha_k - \beta_k)x_k\right| \\ \qquad\ \displaystyle + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}16(\alpha_i\alpha_j + \beta_i\beta_j)(\alpha_k + \beta_k)x_ix_jx_k \\ \qquad\ \displaystyle\leq 192d\left(\sum_{i=1}^{n}x_i(\alpha_i + \beta_i) + \left|\sum_{i=1}^{n}x_i(\alpha_i - \beta_i)\right|\right) \\ \qquad\ x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\ x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{2.11}$$

Proof. Since $\tilde{a}_i = (a_i - \alpha_i, a_i, a_i + \beta_i)$ is triangular fuzzy variable, it follows from Theorem 1.7 (Extension Principle of Zadeh) that

$$\sum_{i=1}^{n}\tilde{a}_ix_i = \left(\sum_{i=1}^{n}x_i(a_i - \alpha_i), \sum_{i=1}^{n}x_ia_i, \sum_{i=1}^{n}x_i(a_i + \beta_i)\right),$$

which is also a triangular fuzzy variable. In Examples 1.11 and 1.19, we have given the expected value and variance of a triangular fuzzy variable. Further, according to Example 2.2, it is easy to obtain the expressions of expected value, variance and skewness of the portfolio return $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$. Substituting these expressions into model (2.7) and rearranging, the theorem is proved.

Theorem 2.4. *Suppose that the vector of returns* $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ *is a fuzzy vector defined by*

$$(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \begin{cases} (a_{11}, a_{21}, \cdots, a_{n1}), & \text{with membership degree } \mu_1 \\ (a_{12}, a_{22}, \cdots, a_{n2}), & \text{with membership degree } \mu_2 \\ \cdots & \cdots \\ (a_{1m}, a_{2m}, \cdots, a_{nm}), & \text{with membership degree } \mu_m \end{cases}$$

where $1 = \mu_1 > \mu_2 > \cdots > \mu_{m+1} = 0$. *Then model* (2.7) *is equivalent to the following crisp model,*

$$\begin{cases} \min \ \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i}(u_j - e)^3 + \min_{1\le j\le i}(u_j - e)^3\right) \\ s.t.\ \ x_1a_{1j} + x_2a_{2j} + \cdots + x_na_{nj} = u_j,\ j = 1, 2, \cdots, m \\ \quad \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i} u_j + \min_{1\le j\le i} u_j\right) \ge r \\ \quad \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i}(u_j - e) + \min_{1\le j\le i}(u_j - e)\right) = 0 \\ \quad \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i}(u_j - e)^2 + \min_{1\le j\le i}(u_j - e)^2\right) \le d \\ \quad x_1 + x_2 + \cdots + x_n = 1 \\ \quad x_i \ge 0, \quad i = 1, 2, \cdots, n. \end{cases}$$

Proof. Note that the portfolio return $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$ is a simple fuzzy variable taking the value $u_j = a_{1j}x_1 + a_{2j}x_2 + \cdots + a_{nj}x_n$ with membership degree μ_j for $j = 1, 2, \cdots, m$. It follows from Eq. (1.15) that

$$E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] = \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i} u_j + \min_{1\le j\le i} u_j\right).$$

Denote the expected value $E[\sum_{i=1}^{n} \tilde{a}_jx_j]$ by e. Since e is a crisp number, it follows from Definition 1.3 that $(\sum_{i=1}^{n} \tilde{a}_jx_j - e)^2$ is also a simple fuzzy variable taking the value $(\sum_{i=1}^{n} a_{ij}x_j - e)^2$ with membership degree μ_j for $j = 1, 2, \cdots, n$. Therefore, the expected value of $(\sum_{i=1}^{n} \tilde{a}_jx_j - e)^2$ is

$$E\left[\left(\sum_{i=1}^{n}\tilde{a}_j x_j - e\right)^2\right] = \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i}(u_j - e)^2 + \min_{1\le j\le i}(u_j - e)^2\right)$$

which is just the variance of $\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n$ by Definition 1.10. Similar to the above process, it is easy to obtain the skewness of $\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n$ as

$$S_k\left[\sum_{i=1}^{n}\tilde{a}_j x_j\right] = \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i}(u_j - e)^3 + \min_{1\le j\le i}(u_j - e)^3\right).$$

Substituting these equations and simplifying, the theorem is proved.

Theorem 2.4 does not require the assumption of independence. It assigns different weights to the historical data to differentiate their importance when constructing an optimal portfolio.

Similar to Theorems 2.3 and 2.4, we can also obtain the equivalent crisp forms of models (2.8), (2.9), and (2.10).

2.4 Fuzzy Simulation

If the security returns are fuzzy variables with different types, then it is difficult to obtain the exact values of expected value, variance and skewness of the portfolio return. Therefore, an alternative is to employ fuzzy simulation to calculate these values. Fuzzy simulation was proposed by Liu and Iwamura (1998) approximately calculate the fuzzy parameters involved in the fuzzy systems. Numerous numerical experiments have shown that fuzzy simulation is a feasible approach to numerically approximate the fuzzy parameters. Liu (2006) also provided the proof of the convergence for fuzzy simulation under some conditions. The detailed procedures on fuzzy simulation may also be found in Liu (2002).

Fuzzy simulation techniques used in most of the existing literature are based on the scheme provided by Liu and Liu (2002). However, their method produces a relatively larger error especially for computing expected value and other moments. Qin et al. (2011) improved fuzzy simulation by applying numerical integration technique. Here we restate the process in detail.

Definition 1.9 tells us expected value of a fuzzy variable $\tilde{a}$ is composed of two parts: $\int_0^{+\infty}\mathrm{Cr}\{\tilde{a}\ge r\}\mathrm{d}r$ and $\int_{-\infty}^{0}\mathrm{Cr}\{\tilde{a}\le r\}\mathrm{d}r$, which are the generalized integrals of the functions $\mathrm{Cr}\{\tilde{a}\ge r\}$ and $\mathrm{Cr}\{\tilde{a}\le r\}$ with respect to r, respectively. Therefore, the key to fuzzy simulation is how to calculate these two functions. Once these functions are given, numerical integration is used to compute the expected value. For this, we first introduce fuzzy simulation for credibility.

2.4.1 Fuzzy Simulation for Credibility

Assume that fuzzy variable $\tilde{a}_i$ has a membership function μ_i for $i = 1, 2, \cdots, n$. For any $r \geq 0$, the detailed algorithm for computing $\text{Cr}\{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n \geq r\}$ is as follows,

---

Algorithm 2.1 (Fuzzy simulation for credibility)

Step 1. Set $m = 1$ and take M as a sufficiently large integer;
Step 2. Randomly generate n numbers w_{im} such that the membership degrees $\mu_i(w_{im}) \geq \varepsilon$ for $i = 1, 2, \cdots, n$, respectively, where ε is a sufficiently small number;
Step 3. Compute $x_1 w_{1m} + x_2 w_{2m} + \cdots + x_n w_{nm}$. If the value is no less than r, let $u_m = \min_{1 \leq i \leq n} \mu_i(w_{im})$ and $v_m = 0$; otherwise, let $u_m = 0$ and $v_m = \min_{1 \leq i \leq n} \mu_i(w_{im})$;
Step 4. If $m > M$, then go to next step; otherwise, set $m = m + 1$ and go to Step 2;
Step 5. Return

$$\frac{1}{2}\left(\max_{1 \leq m \leq M} u_m + 1 - \max_{1 \leq m \leq M} v_m\right)$$

as the value of $\text{Cr}\{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n \geq r\}$.

Algorithm 2.1 is based on the Monte Carlo method and M is called the cycle of fuzzy simulation. In general, the larger M will increase the accuracy of fuzzy simulation, and meanwhile it also will take a longer computation time. We take two examples to illustrate the performance of fuzzy simulation and the influence of M on the accuracy.

Example 2.5. We consider a triangular fuzzy variable $\tilde{a} = \mathcal{T}(-0.5, 0.3, 1.5)$. For this case, the credibility measure of event $\{\tilde{a} \geq r\}$ is given as follows,

$$\text{Cr}\{\tilde{a} \geq r\} = \begin{cases} (11 - 10r)/16, & \text{if } -0.5 \leq r \leq 0.3 \\ (15 - 10r)/24, & \text{if } 0.3 \leq r \leq 1.5 \\ 0, & \text{otherwise.} \end{cases}$$

For any given r, we can get the real credibility measure of the event $\{\tilde{a} \geq r\}$. For example, we have

$$\text{Cr}\{\tilde{a} \geq -0.5\} = 1, \ \ \text{Cr}\{\tilde{a} \geq -0.1\} = 0.75, \ \ \text{Cr}\{\tilde{a} \geq 0.3\} = 0.5,$$
$$\text{Cr}\{\tilde{a} \geq 0.7\} = 0.333, \ \ \text{Cr}\{\tilde{a} \geq 1.1\} = 0.167.$$

Next we use Algorithm 2.1 to compute $\text{Cr}\{\tilde{a} \geq r\}$ for $r = -0.5, -0.1, 0.3, 0.7$ and 1.1, respectively, by changing M from 100 to 10,000. To compare the simulated value with the real one, we define relative error (RE) as follows,

$$\text{RE} = \frac{|\text{Real Value} - \text{Simulated Value}|}{|\text{Real Value}|} \times 100\,\%.$$

Table 2.1 Triangular case: performance of fuzzy simulation to calculate credibility measure

M	$r = -0.5$	$r = -0.1$	$r = 0.3$	$r = 0.7$	$r = 1.1$	Maximum
100	0.945	0.033	4.720	0.966	0.369	4.720
200	0.293	1.407	0.458	1.218	3.026	3.026
300	0.293	0.150	1.834	0.213	0.699	1.834
500	0.163	0.056	0.257	0.687	0.134	0.687
1000	0.090	0.002	0.053	0.247	0.026	0.247
2000	0.053	0.044	0.023	0.043	0.073	0.073
3000	0.053	0.007	0.008	0.071	0.012	0.071
4000	0.053	0.007	0.008	0.264	0.522	0.522
5000	0.022	0.007	0.010	0.005	0.179	0.179
6000	0.022	0.007	0.015	0.007	0.156	0.156
7000	0.022	0.010	0.031	0.085	0.027	0.085
8000	0.022	0.010	0.008	0.020	0.027	0.027
9000	0.018	0.018	0.018	0.003	0.027	0.027
10,000	0.018	0.018	0.003	0.003	0.079	0.079
15,000	0.006	0.002	0.005	0.005	0.012	0.012

Table 2.1 presents the relative errors of simulated credibility measures to real ones for different r. The last column shows the maximum relative error in the five cases for the given cycle M. From Table 2.1, we see that the relative error has a downward trend as M increases. It does not exceed 1.0 % when M ranges from 500 to 15,000. Moreover, it will also not exceed 0.2 % if M is greater than 5000. This implies that fuzzy simulation can perform very well in the calculation of credibility measure. An acceptable value of M can be determined according to the restriction of computation times and the tolerance of inaccuracy.

Example 2.6. Similar to the above example, we consider a normal fuzzy variable $\tilde{b} = \mathcal{N}(0.4, 0.5)$. For this case, the credibility measure of the event $\{\tilde{b} \geq r\}$ is given as follows,

$$\mathrm{Cr}\{\tilde{b} \geq r\} = \left(1 + \exp\left(\frac{\pi(r - 0.4)}{0.5\sqrt{6}}\right)\right)^{-1}, \quad r \in \Re.$$

We calculate the real credibility measure of the event $\{\tilde{b} \geq r\}$ for $r = -0.6, -0.1, 0.4, 0.9$ and 1.4 as follows,

$$\mathrm{Cr}\{\tilde{b} \geq -0.6\} = 0.9285, \quad \mathrm{Cr}\{\tilde{b} \geq -0.1\} = 0.7829, \quad \mathrm{Cr}\{\tilde{b} \geq 0.4\} = 0.5,$$
$$\mathrm{Cr}\{\tilde{b} \geq 0.9\} = 0.2171, \quad \mathrm{Cr}\{\tilde{b} \geq 1.4\} = 0.0714.$$

Table 2.2 Normal case: performance of fuzzy simulation to calculate credibility measure

M	$r = -0.6$	$r = -0.1$	$r = 0.4$	$r = 0.9$	$r = 1.4$	Maximum
100	0.319	0.291	2.222	0.436	76.091	76.091
200	0.759	0.228	12.863	6.330	3.417	12.863
300	0.258	0.848	1.581	0.664	3.665	3.665
500	0.216	0.543	0.407	0.261	0.682	0.682
1000	0.216	0.012	1.143	0.703	2.009	2.009
2000	0.089	0.066	0.266	0.239	0.102	0.266
3000	0.095	0.163	0.344	0.696	0.436	0.696
4000	0.095	0.073	0.297	0.109	0.322	0.322
5000	0.095	0.017	0.031	0.035	0.020	0.095
6000	0.095	0.010	0.063	0.050	0.252	0.252
7000	0.097	0.010	0.141	0.050	0.143	0.143
8000	0.097	0.010	0.141	0.108	0.114	0.141
9000	0.097	0.030	0.078	0.010	0.004	0.097
10,000	0.021	0.004	0.078	0.010	0.257	0.257
15,000	0.026	0.004	0.016	0.003	0.033	0.033

We use Algorithm 2.1 to compute $\mathrm{Cr}\{\tilde{b} \geq r\}$ for $r = -0.6, -0.1, 0.4, 0.9$ and 1.4, respectively, by changing M from 100 to 10,000. Table 2.2 presents the relative errors of simulated credibility measures to real ones for different r. The last column shows the maximum relative error in the five cases for the given cycle M. From Table 2.2, we see that the relative error has a downward trend as M increases. It does not exceed 2.01 % when M ranges from 500 to 15,000. Moreover, it will also not exceed 0.3 % if M is greater than 5000. These results also imply that fuzzy simulation can perform very well to calculate the credibility measure.

Example 2.7. We next consider the sum of $\tilde{a} = \mathcal{T}(-0.5, 0.3, 1.5)$ and $\tilde{b} = \mathcal{N}(0.4, 0.5)$. According to the extension principle of Zadeh (1.7), the membership function of the sum $\tilde{a} + \tilde{b}$ is given by

$$\mu(x) = \sup_{x_1+x_2=x} \mu_1(x_1) \wedge \mu_2(x_2) = \nu_1(x) \vee \nu_2(x)$$

where

$$\nu_1(x) = \sup_{-0.5 \leq x_1 \leq 0.3} \frac{10x_1 + 5}{8} \wedge \frac{2\exp(\sqrt{6})}{\exp(\sqrt{6}) + \exp(2\pi|x - x_1 - 0.4|)}$$

and

$$\nu_2(x) = \sup_{0.3 \leq x_1 \leq 1.5} \frac{15 - 10x_1}{12} \wedge \frac{2\exp(\sqrt{6})}{\exp(\sqrt{6}) + \exp(2\pi|x - x_1 - 0.4|)}.$$

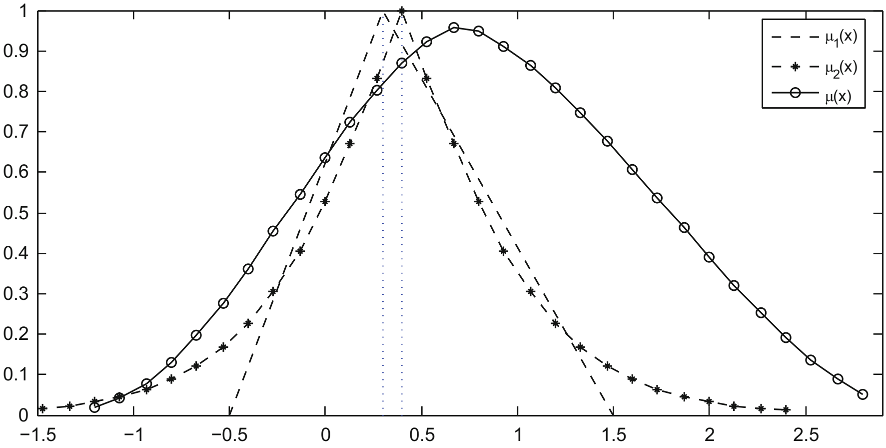

Fig. 2.2 The membership function of the sum of a triangular fuzzy variable and a normal fuzzy variable

Generally speaking, it is difficult to obtain a closed-form expression of $\mu(x)$. However, it is easy to numerically calculate $\mu(x)$ for each given x. For example, we can estimate the value of $\nu_1(x)$ at the point x by solving

$$\max_{1\le k\le K} \frac{10x_1^k+5}{8} \wedge \frac{2\exp(\sqrt{6})}{\exp(\sqrt{6})+\exp\left(2\pi|x-x_1^k-0.4|\right)}$$

in which K is a positive integer and $x_1^k = -0.5 + 0.8k/K$. The parameter K will have an influence on the accuracy of estimation. The numerical graph of $\mu(x)$ is shown in Fig. 2.2 in which $\mu_1(x)$ and $\mu_2(x)$ represents the membership functions of $\tilde{a}$ and $\tilde{b}$, respectively. The theoretical maximum of $\mu(x)$ should be 1. However here the obtained maximum is smaller than 1 due to the calculation error. The results can be improved by improving the numerical accuracy. From Fig. 2.2, we find that the membership function $\mu(x)$ is smoother than $\mu_1(x)$ and $\mu_2(x)$.

Further, we can numerically calculate $\text{Cr}\{\tilde{a}+\tilde{b}\ge r\}$ for each given r by performing Algorithm 2.1. We first set $M = 2000$ and then change the parameter K from 10 to 1000 to test the sensitivity of the simulated credibility to K. The computational results are plotted in Fig. 2.3 which shows that the fluctuation does not exceed 0.2 % when K is larger than 200. Therefore, fuzzy simulation for credibility is still stable after introducing the numerical estimation of the membership function of the sum.

We may also test the stability of fuzzy simulation by repeatedly performing Algorithm 2.1. Specifically, we set $M = 2000$ and $K = 500$, and perform the algorithm thirty times to compute $\text{Cr}\{\tilde{a}+\tilde{b}\ge r\}$ for $r = -0.5, 0, 0.5, 1, 1.5$ and 2, respectively. The computational results are shown in Table 2.3 in which the last two

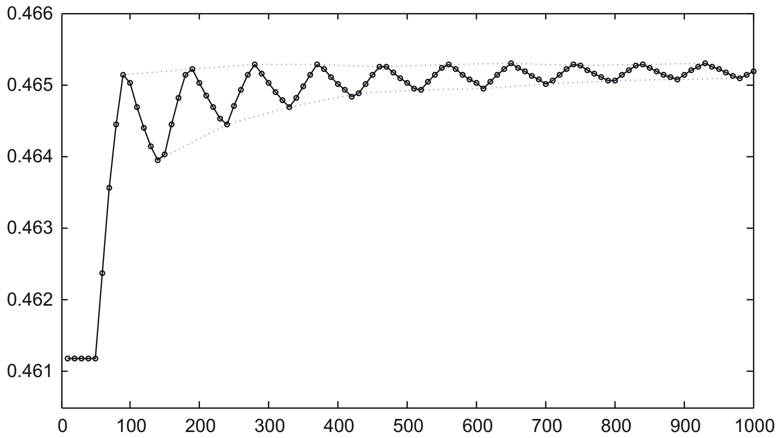

Fig. 2.3 The simulated value of $\mathrm{Cr}\{\tilde{a}+\tilde{b}\geq 1\}$ with the change of K

rows list the maximum and minimum of the simulated values for each case. From Table 2.3 we also see that fuzzy simulation is more stable.

2.4.2 Fuzzy Simulation for Expected Value

According to Definition 1.9, we have

$$E\left[\sum_{i=1}^{n}\tilde{a}_i x_i\right]=\int_0^{+\infty}\mathrm{Cr}\left\{\sum_{i=1}^{n}\tilde{a}_i x_i\geq r\right\}\mathrm{d}r-\int_{-\infty}^{0}\mathrm{Cr}\left\{\sum_{i=1}^{n}\tilde{a}_i x_i\leq r\right\}\mathrm{d}r$$

provided that at least one term is finite. We assume that these two terms are both finite. Then the former is the integral of the function $\mathrm{Cr}\{\tilde{a}_1x_1+\tilde{a}_2x_2+\cdots+\tilde{a}_nx_n\geq r\}$ with regard to r over the positive real axis, and the latter is the integral of the function $\mathrm{Cr}\{\tilde{a}_1x_1+\tilde{a}_2x_2+\cdots+\tilde{a}_nx_n\leq r\}$ with regard to r over the negative real axis. Thus we can use numerical integration technique to design fuzzy simulation for the calculation of expected value.

For simplicity, we write

$$f(r)=\mathrm{Cr}\{x_1\tilde{a}_1+\cdots+x_n\tilde{a}_n\geq r\},\ r\in\Re.$$

Then

$$\mathrm{Cr}\{x_1\tilde{a}_1+\cdots+x_n\tilde{a}_n\leq r\}=1-f(r).$$

Table 2.3 Performance of fuzzy simulation to calculate credibility measure

No.	$r = -0.5$	$r = 0.0$	$r = 0.5$	$r = 1.0$	$r = 1.5$	$r = 2.0$
1	0.8313	0.6613	0.5247	0.4650	0.3504	0.2157
2	0.8319	0.6623	0.5238	0.4655	0.3497	0.2151
3	0.8313	0.6613	0.5242	0.4654	0.3507	0.2157
4	0.8313	0.6609	0.5241	0.4647	0.3507	0.2157
5	0.8313	0.6613	0.5240	0.4654	0.3497	0.2162
6	0.8322	0.6613	0.5237	0.4648	0.3490	0.2160
7	0.8323	0.6609	0.5239	0.4651	0.3497	0.2159
8	0.8328	0.6613	0.5240	0.4654	0.3508	0.2163
9	0.8317	0.6612	0.5240	0.4649	0.3507	0.2147
10	0.8313	0.6607	0.5243	0.4653	0.3507	0.2150
11	0.8313	0.6620	0.5240	0.4642	0.3497	0.2157
12	0.8313	0.6613	0.5240	0.4647	0.3507	0.2164
13	0.8323	0.6620	0.5237	0.4653	0.3507	0.2141
14	0.8314	0.6616	0.5243	0.4647	0.3507	0.2157
15	0.8322	0.6613	0.5239	0.4654	0.3496	0.2157
16	0.8313	0.6612	0.5243	0.4647	0.3507	0.2157
17	0.8314	0.6613	0.5238	0.4655	0.3507	0.2163
18	0.8313	0.6613	0.5238	0.4651	0.3504	0.2157
19	0.8323	0.6613	0.5241	0.4651	0.3504	0.2163
20	0.8313	0.6610	0.5237	0.4651	0.3507	0.2159
21	0.8313	0.6613	0.5238	0.4655	0.3507	0.2147
22	0.8316	0.6623	0.5243	0.4654	0.3507	0.2157
23	0.8315	0.6607	0.5243	0.4647	0.3488	0.2157
24	0.8314	0.6613	0.5239	0.4655	0.3507	0.2159
25	0.8323	0.6613	0.5240	0.4649	0.3507	0.2157
26	0.8313	0.6612	0.5243	0.4654	0.3507	0.2162
27	0.8313	0.6613	0.5239	0.4647	0.3507	0.2163
28	0.8314	0.6612	0.5243	0.4649	0.3506	0.2164
29	0.8313	0.6633	0.5239	0.4655	0.3507	0.2127
30	0.8318	0.6606	0.5238	0.4655	0.3507	0.2157
min	0.8313	0.6606	0.5237	0.4642	0.3488	0.2141
max	0.8328	0.6633	0.5247	0.4655	0.3508	0.2164

If we denote by U a sufficiently large positive number and by L a sufficiently large negative number, then the expected value $E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n]$ may be approximated by the definite integral

$$\int_0^U f(r)\mathrm{d}r - \int_L^0 (1 - f(r))\mathrm{d}r.$$

The detailed procedure for the above integral is summarized as follows,

Algorithm 2.2 (Fuzzy simulation for expected value)

Step 1. Set $e_1 = 0$, $e_2 = 0$, and let $h_1 = U/N$ and $h_2 = L/N$ where N is a sufficiently large integer;

Step 2. Partition the intervals $[0, U]$ and $[L, 0]$ into N small subintervals $[r_i^1, r_{i+1}^1]$ and $[r_i^2, r_{i+1}^2]$, respectively, in which $r_i^1 = ih_1$ and $r_i^2 = L + ih_2$ for $i = 0, 1, \cdots, N-1$;

Step 3. Calculate $e_1 = h_1 \sum_{i=0}^{N-1} f(r_i^1)$;

Step 4. Calculate $e_2 = h_2 \sum_{i=0}^{N-1} [1 - f(r_i^2)]$;

Step 5. Return $e_1 - e_2$ as the simulated value of $E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n]$.

Algorithm 2.2 is based on the rectangle method in numerical analysis and N is called the number of integration points. We may improve the accuracy of fuzzy simulation by applying composite trapezoidal rule or composite Simpson's rule when N is an even number. In the composite trapezoidal rule,

$$e_1 = \frac{h_1}{2} \sum_{i=0}^{N-1} [f(x_i^1) + f(x_{i+1}^1)] \quad \text{and} \quad e_2 = \frac{h_2}{2} \sum_{i=0}^{N-1} [2 - f(x_i^2) - f(x_{i+1}^2)].$$

In the composite Simpson's rule,

$$e_1 = \frac{h_1}{3} \sum_{i=0}^{N/2-1} [f(x_{2i}^1) + 4f(x_{2i+1}^1) + f(x_{2i+2)}^1)],$$

$$e_2 = \frac{h_2}{3} \sum_{i=0}^{N/2-1} [6 - f(x_{2i}^2) - 4f(x_{2i+1}^2) - f(x_{2i+2)}^2)].$$

In general, a larger N will increase the accuracy of numerical integration or fuzzy simulation, but meanwhile it also takes a longer computation time. We next show the performance of fuzzy simulation to compute expected value by presenting an example.

Example 2.8. We use Algorithm 2.2 to simulate the expected value of the triangular fuzzy variable $\tilde{a} = \mathcal{T}(-0.5, 0.3, 1.5)$. Note that its real expected value is

$$E[\tilde{a}] = 0.3 + [(1.5 - 0.3) - (0.3 - (-0.5))]/4 = 0.4.$$

We first choose an appropriate method for future use by comparing these three numerical integration methods involved in fuzzy simulation. Specifically, we perform Algorithm 2.2 by fixing $M = 2000$, setting $L = -10, U = 10$, and changing N from 10 to 200. The computational results are summarized in Table 2.4 in which $E^R[\tilde{a}]$, $E^T[\tilde{a}]$ and $E^S[\tilde{a}]$ are the simulated expected value by using rectangle method, composite trapezoidal rule and composite Simpson's rule, respectively. The last row is the average relative error (ARE) for each method. From Table 2.4, we can see that

Table 2.4 Comparisons of numerical integration methods in fuzzy simulation

	Rectangle method		Trapezoidal rule		Simpson's rule	
N	$E^R[\tilde{a}]$	RE (%)	$E^T[\tilde{a}]$	RE (%)	$E^S[\tilde{a}]$	RE (%)
10	0.89638	124.095	0.39570	1.076	0.40308	0.769
20	0.65574	63.935	0.40600	1.500	0.41012	2.529
30	0.56562	41.405	0.39753	0.618	0.39763	0.592
40	0.52553	31.381	0.40056	0.139	0.39861	0.347
50	0.49988	24.969	0.39992	0.020	0.39970	0.076
60	0.48281	20.703	0.40004	0.010	0.40062	0.155
70	0.47116	17.791	0.39929	0.177	0.39980	0.051
80	0.46252	15.630	0.40000	0.000	0.39985	0.037
90	0.45488	13.720	0.39955	0.113	0.39906	0.236
100	0.44991	12.479	0.39964	0.090	0.39964	0.091
110	0.44485	11.211	0.39978	0.056	0.39939	0.153
120	0.44163	10.407	0.39985	0.039	0.39960	0.099
130	0.43799	9.497	0.39985	0.037	0.39983	0.043
140	0.43572	8.930	0.40041	0.101	0.39998	0.004
150	0.43261	8.152	0.40011	0.037	0.39945	0.137
160	0.43122	7.806	0.40036	0.091	0.39984	0.039
170	0.42917	7.293	0.40013	0.033	0.39959	0.103
180	0.42744	6.859	0.40012	0.029	0.39974	0.064
190	0.42592	6.480	0.40022	0.055	0.40216	0.054
200	0.42452	6.131	0.40017	0.042	0.40019	0.047
ARE		22.444		0.213		0.281

composite trapezoidal rule and Simpson's rule provide better approximations to the expected value. By trapezoidal rule, the relative error hardly exceeds 0.1 % when N is larger than 100. The rectangle method has a slower convergence rate than the last two methods. Composite trapezoidal rule is preferred since it has a smaller average relative error and requires less computational cost than composite Simpson's rule. In what follows, we can select $N = 200$.

2.4.3 Fuzzy Simulation for Skewness

The skewness of fuzzy variable $\tilde{a}$ is essentially the expected value of $(\tilde{a} - E[\tilde{a}])^3$. Once the expected value $E[\tilde{a}]$ is obtained, we can apply Algorithm 2.2 to calculate $S_k[\tilde{a}]$ by changing $f(r)$. Here we define

$$g(r) = 3r^2 \cdot \mathrm{Cr}\{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n \geq r\}.$$

Fuzzy simulation for skewness is obtained by replacing $f(r)$ with $g(r)$ in Algorithm 2.2.

For simplicity, write $\rho = E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n]$. Different from the above method, Li et al. (2010) used the following algorithm to calculate the skewness,

Algorithm 2.3 (Fuzzy simulation for Skewness)

Step 1. Set $\upsilon = 0$;
Step 2. Randomly generate υ_{jk} such that $\mu_j(\upsilon_{jk}) \geq \varepsilon$ for $j = 1, 2, \cdots, n,\ k = 1, 2, \cdots, K$, where ε is a sufficiently small number and K is sufficiently large integer;
Step 3. Set two numbers $a = \min\limits_{1\leq k\leq K} (\upsilon_{1k}x_1 + \upsilon_{2k}x_2 + \cdots + \upsilon_{nk}x_n - \rho)^3$ and $b = \max\limits_{1\leq k\leq K} (\upsilon_{1k}x_1 + \upsilon_{2k}x_2 + \cdots + \upsilon_{nk}x_n - \rho)^3$;
Step 4. Randomly generate a real number r from $[a, b]$;
Step 5. If $r \geq 0$, then $\upsilon \leftarrow \upsilon + \text{Cr}\{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n \geq r\}$;
Step 6. If $r < 0$, then $\upsilon \leftarrow \upsilon - \text{Cr}\{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n \leq r\}$;
Step 7. Repeat the fourth to sixth steps for K times;
Step 8. Return $a \vee 0 + b \wedge 0 + \upsilon(b - a)/K$ as the target value.

Here, the value $\text{Cr}\{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n \geq r\}$ is estimated by the formula

$$\frac{1}{2}\left(\max_{1\leq i\leq N}\left\{\min_{1\leq j\leq n} \mu_j(w_{ji}) \mid \sum_{j=1}^{n} w_{ji}x_j \geq r\right\} + 1 - \max_{1\leq i\leq N}\left\{\min_{1\leq j\leq n} \mu_j(w_{ji}) \mid \sum_{j=1}^{n} w_{ji}x_j < r\right\}\right).$$

2.5 Numerical Examples

In this section, we illustrate the application of credibilistic mean-variance-skewness model by numerical examples in Li et al. (2010). The data is composed of membership functions of 10 security returns, which is shown in Table 2.5 and from Huang (2008a). The returns of first seven securities are triangular fuzzy variables, and the others are fuzzy variables with membership functions μ_i, $i = 8, 9, 10$. For example, the return of the first security is fuzzy variable $(-0.3, 1.8, 2.3)$ which represents about 1.8 units per stock.

Example 2.9. Assume that the candidate set is composed of the first seven securities. In order to apply model (2.7) to search for optimal portfolio, the investor needs to set two parameters: the minimum expected return r and the bearable maximum risk d. Without loss of generality, let $r = 1.6$ and $d = 0.8$. Since the first seven returns are all triangular fuzzy variables, model (2.11) is actually used.

Since the returns are asymmetric, the investor may also employ mean-semivariance model to construct an optimal portfolio. In order to compare the

Table 2.5 Fuzzy returns of 10 securities (units per stock)

Security i	Fuzzy return	Security i	Fuzzy return
1	$(-0.3, 1.8, 2.3)$	6	$(-0.8, 2.5, 3.0)$
2	$(-0.4, 2.0, 2.2)$	7	$(-0.6, 1.8, 3.0)$
3	$(-0.5, 1.9, 2.7)$	8	$(1 + (r - 1.6)^4)^{-1}$
4	$(-0.6, 2.2, 2.8)$	9	$(1 + (5r - 7.4)^2))^{-1}$
5	$(-0.7, 2.4, 2.7)$	10	$\exp(-(r - 1.6)^2)$

Table 2.6 Comparison of results in Example 2.9

Model	Allocation (%)	V	S_v	Skewness
(2.11)	$x_1 = 20.00,\ x_4 = 80.00$	0.7019	0.6141	–0.6823
(2.12)	$x_2 = 47.06, x_4 = 35.28, x_5 = 17.66$	0.7232	0.6124	–0.7532

results of credibilistic mean-variance-skewness model and mean-semivariance model, we consider model (8) of Huang (2008a).

$$\begin{cases} \min S_v[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_7x_7] \\ \text{s.t.} \quad E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_7x_7] \geq 1.6 \\ \qquad x_1 + x_2 + \cdots + x_7 = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 7 \end{cases} \tag{2.12}$$

where S_v is the semivariance operator of fuzzy variable.

We use MATLAB to solve models (2.11) and (2.12) and the computational results are shown in Table 2.6 in which V and S_v represent variance and semivariance, respectively. The two models obtain different optimal portfolios which have the same mean 1.60 and almost the same semivariance. However, the first portfolio has lower variance, and higher skewness than the second one, which is desired by the investor.

Example 2.10. If an investor chooses securities from the whole 10 securities, then all the mean-variance-skewness models cannot be converted into crisp models. Therefore, we use genetic algorithm in Sect. 1.3 to solve these models. Assume that the minimum expected return the investor can accept is 1.5 and the bearable maximum risk is 1.2. Then model (2.7) is reformulated as follows,

$$\begin{cases} \max S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \\ \text{s.t.} \quad E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \geq 1.5 \\ \qquad V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \leq 1.2 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{2.13}$$

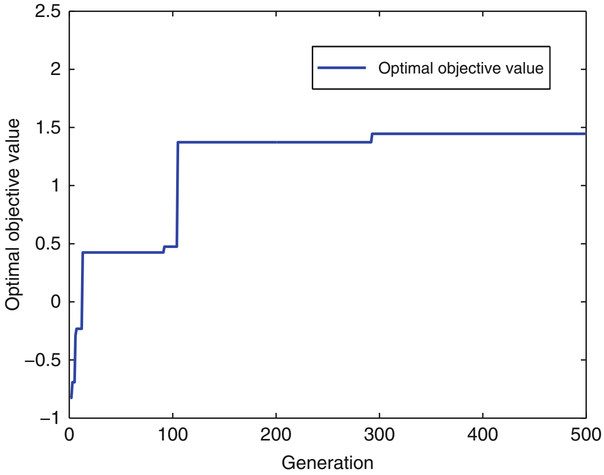

Fig. 2.4 The convergence of objective value of Example 2.10

We choose the following parameters in the GA: $P_c = 0.3, P_m = 0.2$, *pop_size* = 50. A run of genetic algorithm (500 generations and 3000 cycles for fuzzy simulation) shows that the optimal allocation (%) of money is $x_1 = 4.04$, $x_2 = 5.52$, $x_3 = 8.22$, $x_4 = 9.47$, $x_5 = 8.17$, $x_6 = 0.20$, $x_7 = 16.55$, $x_8 = 17.47$, $x_9 = 21.22$, $x_{10} = 9.14$. The corresponding maximum skewness is 1.72, and the convergence of maximum skewness of portfolio return is shown in Fig. 2.4 which indicates genetic algorithm is effective to solve the proposed model.

Example 2.11. Suppose that an investor wishes the skewness of his portfolio is at least −1.0, and the minimal expected return is 1.5. If he accepts variance as risk, then the model is formulated as follows,

$$\begin{cases} \min\ V[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_{10} x_{10}] \\ \text{s.t.}\ \ E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_{10} x_{10}] \geq 1.5 \\ \qquad S_k[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_{10} x_{10}] \geq -1.0 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{2.14}$$

First note that if we do not consider skewness of the portfolio, then model (2.14) is a mean-variance model. The parameters of GA are chosen as follows: $P_c = 0.4, P_m = 0.3$, *pop_size* = 60. Here, we compare the allocation of capital between this model and mean-variance model by Fig. 2.5. The minimum risk of mean-variance-skewness model is 0.383, and the minimum risk of mean-variance model is 0.285, respectively.

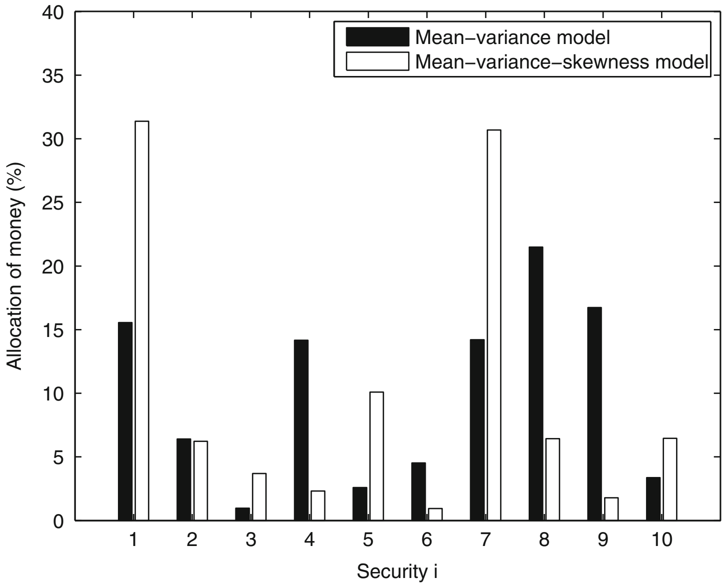

Fig. 2.5 Allocations of capital for credibilisitc mean-variance model and mean-variance-skewness model

Table 2.7 Comparison of solutions in Example 2.12

No.	1	2	3	4	5	6	7
pop_size	50	30	80	90	110	130	70
P_c	0.3	0.4	0.8	0.6	0.7	0.5	0.2
P_m	0.2	0.3	0.3	0.7	0.8	0.2	0.1
N	2500	2500	2300	2500	2500	2200	2500
Gen	200	200	150	100	100	100	200
Mean	1.6855	1.6856	1.6879	1.6905	1.6812	1.6919	1.6871
RE (%)	0.38	0.37	0.24	0.08	0.63	0.00	0.28

Example 2.12. Assume that an investor wishes that the skewness of his portfolio is at least −1.0, and the maximal risk does not exceed 1.2. Meanwhile, if the investor wants to maximize the expected return, then the model is reformulated as follows,

$$\begin{cases} \max\ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \\ \text{s.t.}\quad S_k[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \geq -1.0 \\ \qquad V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \leq 1.2 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{2.15}$$

In order to test the robustness of genetic algorithm, we solve model (2.15) by setting the different parameters in the GA. In order to compare the results, we employ the relative error (RE) which is defined by (Maximal objective $-$ Actual objective)/Maximal objective $\times$ 100 %, where the maximal objective is the maximum of all the computational results obtained. The detailed results are shown in Table 2.7 in which N is the cycle of fuzzy simulation. Obviously, the relative errors do not exceed 1 %. That is, genetic algorithm is robust to set parameters and effective for solving the credibilistic mean-variance-skewness models.

Chapter 3
Credibilistic Mean-Absolute Deviation Model

3.1 Introduction

Mean-absolute deviation model was first proposed by Konno and Yamazaki (1991) for stochastic portfolio optimization by using absolute deviation risk function to replace variance. It removes most of the difficulties associated with Markowitz's mean-variance model. This model can cope with large-scale portfolio optimization problem because it leads to a linear programming. Furthermore, the authors showed that this model gave essentially the same results as the mean-variance model if all the returns are normally distributed random variables. Since then, absolute deviation has been accepted as a risk measure. As extensions, Konno et al. (1993) presented mean-absolute deviation-skewness model for the case when the distributions of returns are asymmetrical around their means, and Yu et al. (2010) presented a multiperiod portfolio optimization model with risk control for absolute deviation.

Traditionally the returns of individual securities were considered as random variables in the framework of mean-absolute deviation analysis. When new securities are listed in the market, or the real market has changed, the future returns do not correctly reflected by past data. For the case with lack of sufficient data, we invite domain experts to estimate security returns based on their evaluations and consider them as fuzzy variables. Following Konno and Yamazaki's idea, Qin et al. (2011) defined absolute deviation of fuzzy variable and used it as a risk measure. They formulated credibilistic mean-absolute deviation model for portfolio optimization with fuzzy returns. In some situations, the model is equivalent to a linear programming. In addition, based on possibility measure, Zhang and Zhang (2014) also introduced possibilistic absolute deviation as the risk control of portfolio and presented a multiperiod mean-absolute deviation model by maximizing the terminal wealth.

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This chapter will focuses on credibilistic mean-absolute deviation model. The main contents include the definition of absolute deviation of fuzzy variable, model formulation and its crisp equivalents, fuzzy simulation for absolute deviation, and numerical analysis.

3.2 Absolute Deviation of Fuzzy Variable

Definition 3.1 (Qin et al. 2011). Let $\tilde{a}$ be a fuzzy variable with finite expected value. Then its absolute deviation is defined as

$$A[\tilde{a}] = E[|\tilde{a} - E[\tilde{a}]|]. \tag{3.1}$$

Remark 3.1. Let $\tilde{a}$ be a fuzzy variable with membership function μ and finite expected value. Then it follows from Definition 3.1 that

$$\begin{aligned} A[\tilde{a}] &= E\left[|\tilde{a} - E[\tilde{a}]|\right] \\ &= \int_0^{+\infty} \mathrm{Cr}\{|\tilde{a} - E[\tilde{a}]| \geq r\}\mathrm{d}r \\ &= \int_0^{+\infty} \mathrm{Cr}\left\{\{\tilde{a} \geq E[\tilde{a}] + r\} \cup \{\tilde{a} \leq E[\tilde{a}] - r\}\right\}\mathrm{d}r \\ &= \frac{1}{2}\int_0^{+\infty} f(r, E[\tilde{a}])\mathrm{d}r \end{aligned}$$

where

$$f(r, E[\tilde{a}]) = \sup_{x \geq E[\tilde{a}]+r \text{ or } x \leq E[\tilde{a}]-r} \mu(x) + 1 - \sup_{E[\tilde{a}]-r<x<E[\tilde{a}]+r} \mu(x)$$

by credibility inversion theorem (Theorem 1.2).

Example 3.1. Let $\tilde{a} = \mathcal{E}(a - \alpha, a + \alpha)$ be an equipossible fuzzy variable. Since $E[\tilde{a}] = a$, for any $r \geq 0$, we can get

$$\begin{aligned} f(r, E[\tilde{a}]) &= \sup_{x \geq a+r \text{ or } x \leq a-r} \mu(x) + 1 - \sup_{a-r<x<a+r} \mu(x) \\ &= \begin{cases} 1, \text{ if } r \leq \alpha \\ 0, \text{ if } r > \alpha. \end{cases} \end{aligned}$$

It follows from Remark 3.1 that the absolute deviation of $\tilde{a}$ is

$$A[\tilde{a}] = \frac{1}{2}\int_0^{\alpha} 1\mathrm{d}r + \frac{1}{2}\int_{\alpha}^{+\infty} 0\mathrm{d}r = \frac{\alpha}{2}$$

which implies that

$$A[\tilde{a}] = \frac{\sqrt{2}}{2}\sqrt{V[\tilde{a}]}.$$

Example 3.2. Let $\tilde{a} = (a-\alpha, a, a+\beta)$ be a triangular fuzzy variable. By the expression of triangular membership function, we have

$$\begin{aligned} f(r, E[\tilde{a}]) &= f\left(r, a+\frac{\beta-\alpha}{4}\right) \\ &= \sup_{x \geq a+\frac{\beta-\alpha}{4}+r \text{ or } x \leq a+\frac{\beta-\alpha}{4}-r} \mu(x) + 1 - \sup_{a+\frac{\beta-\alpha}{4}-r<x<a+\frac{\beta-\alpha}{4}+r} \mu(x). \end{aligned}$$

If $\alpha \geq \beta$, then for any $r \geq 0$, we have

$$f(r, E[\tilde{a}]) = \begin{cases} \dfrac{5\alpha-\beta-4r}{4\alpha}, \text{ if } r \leq \dfrac{\alpha-\beta}{4} \\ \dfrac{\alpha+3\beta-4r}{4\beta}, \text{ if } \dfrac{\alpha-\beta}{4} \leq r \leq \dfrac{\alpha+\beta}{4} \\ \dfrac{3\alpha+\beta-4r}{4\alpha}, \text{ if } \dfrac{\alpha+\beta}{4} \leq r \leq \dfrac{3\alpha+\beta}{4} \\ \quad 0, \qquad \text{if } r \geq \dfrac{3\alpha+\beta}{4}. \end{cases}$$

Thus the absolute deviation is

$$\begin{aligned} A[\tilde{a}] &= \frac{1}{2}\int_0^{\frac{\alpha-\beta}{4}} \frac{5\alpha-\beta-4r}{4\alpha}\mathrm{d}r + \frac{1}{2}\int_{\frac{\alpha-\beta}{4}}^{\frac{\alpha+\beta}{4}} \frac{\alpha+3\beta-4r}{4\beta}\mathrm{d}r \\ &\quad + \frac{1}{2}\int_{\frac{\alpha+\beta}{4}}^{\frac{3\alpha+\beta}{4}} \frac{3\alpha+\beta-4r}{4\alpha}\mathrm{d}r + \frac{1}{2}\int_{\frac{3\alpha+\beta}{4}}^{+\infty} 0 \cdot \mathrm{d}r \\ &= \frac{(\alpha+\beta)^2+12\alpha^2}{64\alpha}. \end{aligned}$$

If $\alpha \leq \beta$, then $-\tilde{a} = (-a-\beta, -a, -a+\alpha)$ has the following absolute deviation

$$A[-\tilde{a}] = \frac{(\alpha+\beta)^2+12\beta^2}{64\beta} = A[\tilde{a}].$$

In a word, the absolute deviation of a triangular fuzzy variable $\tilde{a} = \mathcal{T}(a-\alpha, a, a+\beta)$ is

$$A[\tilde{a}] = \begin{cases} \dfrac{(\alpha+\beta)^2 + 12\alpha^2}{64\alpha}, & \text{if } \alpha \geq \beta \\ \dfrac{(\alpha+\beta)^2 + 12\beta^2}{64\beta}, & \text{if } \alpha \leq \beta \end{cases}$$

or the equivalent expression

$$A[\tilde{a}] = \frac{(\alpha+\beta)^2 + 3(\alpha+\beta+|\alpha-\beta|)^2}{32(\alpha+\beta+|\alpha-\beta|)}.$$

Especially, if $\tilde{a}$ is symmetric, i.e., $\alpha = \beta$, then we have $A[\tilde{a}] = \alpha/4$ which implies that

$$A[\tilde{a}] = \frac{\sqrt{6}}{4}\sqrt{V[\tilde{a}]}.$$

However, the relationship does not hold for asymmetric triangular fuzzy variable. For example, we compare the absolute deviation and the square root of variance by fixing $\beta = 1$ and allowing α to change from 1 to 10. The computational results are shown in Fig. 3.1. It is easy to see that the absolute deviation is only not equivalent to the square root of variance when $\alpha > 1$.

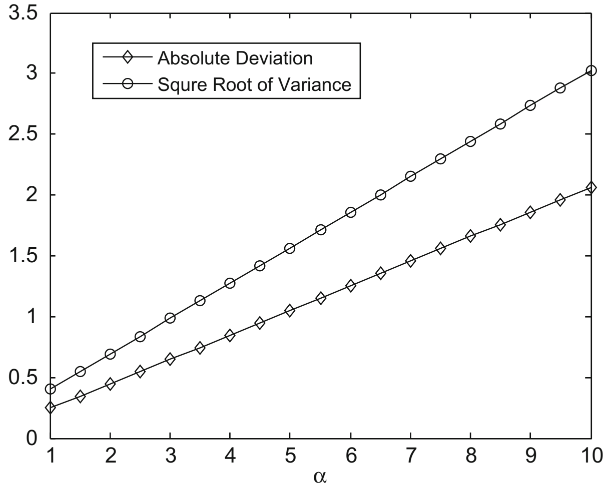

Fig. 3.1 Comparison of absolute deviation and square root of variance of triangular fuzzy variable

Example 3.3. Let $\tilde{a} = \mathcal{S}(x_i, \mu_i; m)$ be a simple fuzzy variable. Then its expected value can be obtained in Example 1.16 and denoted by $E[\tilde{a}] = a$. Similarly, its absolute deviation is

$$A[\tilde{a}] = \frac{1}{2}\sum_{i=1}^{m} w_i|x_i - a|$$

in which the weights are calculated by

$$\begin{aligned} w_i \; = & \max_{1\le j\le m}\{\mu_j \,|\, |x_j - a| \le |x_i - a|\} - \max_{1\le j\le m}\{\mu_j \,|\, |x_j - a| < |x_i - a|\} \\ & + \max_{1\le j\le m}\{\mu_j \,|\, |x_j - a| \ge |x_i - a|\} - \max_{1\le j\le m}\{\mu_j \,|\, |x_j - a| > |x_i - a|\} \end{aligned}$$

for $i = 1, 2, \cdots, m$. Further, if $0 \le \mu_1 \le \mu_2 \le \cdots \le \mu_m \equiv 1$, then

$$A[\tilde{a}] = \frac{1}{2}\sum_{i=1}^{m} y_i \mu_i$$

in which the weights are given by

$$y_i = \max_{j\ge i}|x_j - a| - \max_{j>i}|x_j - a| + \min_{j\ge i}|x_j - a| - \min_{j>i}|x_j - a|$$

for $i = 1, 2, \cdots, m$. If $1 \equiv \mu_1 \ge \mu_2 \ge \cdots \ge \mu_m \ge 0$, then the weights are

$$y_i = \max_{j\le i}|x_j - a| - \max_{j<i}|x_j - a| + \min_{j\le i}|x_j - a| - \min_{j<i}|x_j - a|$$

for $i = 1, 2, \cdots, m$. If $1 \equiv \mu_1 > \mu_2 > \cdots > \mu_m > 0 = \mu_{m+1}$, then we have

$$A[\tilde{a}] = \frac{1}{2}\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\le j\le i}|x_j - a| + \min_{1\le j\le i}|x_j - a|\right).$$

Theorem 3.1 (Qin et al. 2011). *Let $\tilde{a}$ be a fuzzy variable with finite expected value. Then we have*

$$A[x\tilde{a} + y] = |x| \cdot A[\tilde{a}] \tag{3.2}$$

for any real numbers x and y.

Proof. For any $x, y \in \Re$, we have $E[x\tilde{a} + y] = xE[\tilde{a}] + y$. Further, it follows from Definition 3.1 that

$$\begin{aligned} A[x\tilde{a} + y] &= E\left[|x\tilde{a} + y - xE[\tilde{a}] - y|\right] \\ &= |x| \cdot E[|\tilde{a} - E[\tilde{a}]|] \\ &= |x| \cdot A[\tilde{a}]. \end{aligned}$$

The theorem is proved.

Theorem 3.2 (Qin et al. 2011). *Let $\tilde{a}$ be a fuzzy variable with finite expected value a. Then $A[\tilde{a}] = 0$ if and only if $\mathrm{Cr}\{\tilde{a} = a\} = 1$.*

Proof. If $A[\tilde{a}] = 0$, then by Definition 3.1, we have

$$A[\tilde{a}] = E[|\tilde{a} - a|] = \int_0^{+\infty} \mathrm{Cr}\{|\tilde{a} - a| \geq r\}\mathrm{d}r = 0$$

which implies that $\mathrm{Cr}\{|\tilde{a} - a| \geq r\} = 0$ for any real number $r > 0$. By self-duality of credibility measure, we obtain $\mathrm{Cr}\{|\tilde{a} - a| = 0\} = 1$, i.e., $\mathrm{Cr}\{\tilde{a} = a\} = 1$. Conversely, if $\mathrm{Cr}\{\tilde{a} = a\} = 1$, then we get $\mathrm{Cr}\{|\tilde{a} - a| = 0\} = 1$ which means that $\mathrm{Cr}\{|\tilde{a} - a| \geq r\} = 0$ for any $r > 0$. It follows again from Definition 3.1 that

$$A[\tilde{a}] = \int_0^{+\infty} \mathrm{Cr}\{|\tilde{a} - a| \geq r\}\mathrm{d}r = 0.$$

The theorem is proved.

Theorem 3.3 (Qin et al. 2011). *Let $\tilde{a}$ be a fuzzy variable taking values in the closed interval $[a_1, a_2]$. If the expected value of $\tilde{a}$ is a, then we have*

$$A[\tilde{a}] \leq \frac{2(a_2 - a)(a - a_1)}{a_2 - a_1}, \tag{3.3}$$

and the fuzzy variable with the maximum absolute deviation is

$$\tilde{a} = \begin{cases} a_1, \text{ with membership degree } \dfrac{a_2 - a}{a_2 - a_1} \wedge 1 \\ a_2, \text{ with membership degree } \dfrac{a - a_1}{a_2 - a_1} \wedge 1. \end{cases} \tag{3.4}$$

Proof. Assume that $\tilde{a}$ is defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. For each $\theta \in \Theta$, we have $a_1 \leq \tilde{a}(\theta) \leq a_2$ and

$$\tilde{a}(\theta) = \frac{a_2 - \tilde{a}(\theta)}{a_2 - a_1}a_1 + \frac{\tilde{a}(\theta) - a_1}{a_2 - a_1}a_2.$$

Define $f(x) = |x - a|$. It follows from the convexity of f that

$$|\tilde{a}(\theta) - a| \leq \frac{a_2 - \tilde{a}(\theta)}{a_2 - a_1}|a_1 - a| + \frac{\tilde{a}(\theta) - a_1}{a_2 - a_1}|a_2 - a|.$$

Taking expected values on both sides, we obtain

$$A[\tilde{a}] \leq \frac{a_2 - a}{a_2 - a_1}(a - a_1) + \frac{a - a_1}{a_2 - a_1}(a_2 - a) = \frac{2(a_2 - a)(a - a_1)}{a_2 - a_1}.$$

The second conclusion is easy to be verified by calculating the absolute deviation. The theorem is proved.

Theorem 3.4 (Qin et al. 2011). *Let $\tilde{a} = \mathcal{N}(a, \delta)$ be a normally distributed fuzzy variable. Then we have*

$$A[\tilde{a}] = \frac{\sqrt{6}\ln 2}{\pi}\delta. \tag{3.5}$$

Proof. It is easy to prove that the membership function of $|\tilde{a} - a|$ is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\delta}\right)\right)^{-1}, \ x \geq 0.$$

For any $x \geq 0$, it follows from the credibility inversion theorem that

$$\mathrm{Cr}\{|\tilde{a} - a| \geq x\} = \left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\delta}\right)\right)^{-1}.$$

It follows from Remark 3.1 that

$$\begin{aligned} A[\tilde{a}] &= \int_0^{+\infty} \mathrm{Cr}\{|\tilde{a} - a| \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\delta}\right)\right)^{-1} \mathrm{d}x \\ &= \frac{\sqrt{6}\ln 2}{\pi}\delta. \end{aligned}$$

The theorem is proved.

Theorem 3.4 shows that $A[\tilde{a}] = \sqrt{6}\ln 2\sqrt{V[\tilde{a}]}/\pi$ which indicates that the absolute deviation of a normal fuzzy variable is equivalent to its variance.

3.3 Mean-Absolute Deviation Model

We consider to construct an optimal portfolio from the candidate set of n risky securities. Let $\tilde{a}_i$ be the return of the ith security and x_i the proportion of total fund invested in security i, $i = 1, 2, \cdots, n$. In this chapter, we assume that $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ are fuzzy variables defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. When the investors choose absolute deviation as a risk measure, Qin et al. (2011) proposed credibilistic mean-absolute deviation model for portfolio optimization with fuzzy returns.

Let r be the expected return level preset by an investor. If he/she wants to minimize the absolute deviation, then Qin et al. (2011) formulated the following credibilistic mean-absolute deviation model,

$$\begin{cases} \min A[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \\ \text{s.t.} \quad E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \geq r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{3.6}$$

where E denotes the expected value operator and A the absolute deviation operator of fuzzy variable. The constraint $E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \geq r$ requires that the optimal portfolio will be chosen only from the feasible portfolios whose expected return is no less than the target return r. The second constraint ensures that all the money will be invested. In addition, $x_i \geq 0$ implies that short-selling or borrowing of security i is not allowed.

As an alternative of model (3.6), one investor maybe choose to maximize the expected return while limiting the absolute deviation of the portfolio return. In this case, Qin et al. (2011) proposed the following credibilistic mean-absolute deviation model,

$$\begin{cases} \max E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \\ \text{s.t.} \quad A[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \leq d \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{3.7}$$

where d denotes the maximum risk level the investor can tolerate. The constraint $A[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \leq d$ ensures that the optimal portfolio will be chosen only from the feasible portfolios whose absolute deviation does not exceed d.

From models (3.6) and (3.7), the choice of parameters r and d may change the obtained optimal portfolio. However, the investors sometimes do not know how to set the reasonable values of these parameters in practice. In this case, we may use the following bi-objective credibilistic mean-absolute deviation model,

$$\begin{cases} \min \ A[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \\ \max \ E[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] \\ \text{s.t.} \quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{3.8}$$

If the absolute deviation A is replaced by variance V or semivariance S_v, then models (3.6), (3.7), and (3.8) become the corresponding credibilistic mean-variance model or mean-semivariance model (Huang 2010) in fuzzy environment. Next we discuss the crisp equivalents of models (3.6) and (3.7) when the membership functions of fuzzy returns are given.

Theorem 3.5. *Suppose that $\tilde{a}_i = \mathcal{N}(a_i, \delta_i)$ are independent normal fuzzy variables for $1 \leq i \leq n$. Then models (3.6) and (3.7) are respectively equivalent to the following linear programmings,*

$$\begin{cases} \min\ \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n \\ \text{s.t.}\ \ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{3.9}$$

and

$$\begin{cases} \max\ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \\ \text{s.t.}\ \ \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n \leq \pi d/(\sqrt{6}\ln 2) \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.10}$$

Proof. By the assumption of independence, it follows from the extension principle of Zadeh (1.7) that $\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n$ is also a normal fuzzy variable with expected value $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$ and variance $(\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n)^2$. According to Theorem 3.4, we find

$$A[\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n] = \frac{\sqrt{6}\ln 2}{\pi}\left(\delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_n x_n\right).$$

Substituting these expressions into models (3.6) and (3.7), the desired results are obtained. The theorem is proved.

Example 3.4. Suppose that we are to construct an optimal portfolio from two risky securities. Let $\tilde{a}_1$ and $\tilde{a}_2$ denote the fuzzy returns on the two risky securities. Especially, we assume that $\tilde{a}_i$ is a normal fuzzy variables with expected value a_i and variance δ_i^2 for $i = 1, 2$, and $\tilde{a}_1$ and $\tilde{a}_2$ are independent. Without loss of generality, we assume $a_1 > a_2$. If $r \in [a_2, a_1]$, then model (3.9) becomes

$$\begin{cases} \min\ \delta_1 x_1 + \delta_2 x_2 \\ \text{s.t.}\ \ a_1 x_1 + a_2 x_2 \geq r, \\ \qquad x_1 + x_2 = 1, \\ \qquad x_1, x_2 \geq 0. \end{cases}$$

The argument breaks down into two cases.

Case 1: $\delta_1 > \delta_2$. In this case, it is easy to calculate that the portion of the optimal portfolio that should be invested in securities 1 and 2 respectively are

$$x_1^* = \frac{r - a_2}{a_1 - a_2}, \quad x_2^* = \frac{a_1 - r}{a_1 - a_2}.$$

Note that x_1^* is increasing about r, which implies that the higher the expected return level is, the larger the investment proportion on security 1 is.
Case 2: $\delta_1 \leq \delta_2$. In this case, the minimum risk portfolio is $x_1^* = 0$ and $x_2^* = 1$. Since security 2 has higher return and lower risk, the investor prefers security 2 to security 1. Thus, the result is clearly consistent with the actual.

If $r < e_2$, then the first constraint holds for all portfolios (x_1, x_2). Hence, it is clear that the minimum risk portfolio should be

$$x_1^* = \begin{cases} 1, & \delta_1 \leq \delta_2, \\ 0, & \text{otherwise}, \end{cases} \qquad x_2^* = \begin{cases} 1, & \delta_1 \geq \delta_2, \\ 0, & \text{otherwise}. \end{cases} \tag{3.11}$$

If $r > a_1$, then the first constraint does not hold for all portfolios (x_1, x_2) since $a_1x_1 + a_2x_2 < rx_1 + rx_2 = r$.

The similar result can be obtained when we use model (3.10) to determine the optimal portfolio.

Theorem 3.6. *Suppose that $\tilde{a}_i = \mathcal{T}(a_i - \alpha_i, a_i, a_i + \beta_i)$ are independent triangular fuzzy variables for $i = 1, 2, \cdots, n$. Then models (3.6) and (3.7) are respectively equivalent to the following crisp forms,*

$$\begin{cases} \max \ \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{n}[4(\alpha_i+\beta_i)(\alpha_j+\beta_j)+(\alpha_i-\beta_i)(\alpha_j-\beta_j)]x_ix_j}{32\sum_{i=1}^{n}(\alpha_i+\beta_i)x_i + 32\left|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i\right|} \\ \qquad + \dfrac{6\left|\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i+\beta_i)(\alpha_j-\beta_j)x_ix_j\right|}{32\sum_{i=1}^{n}(\alpha_i+\beta_i)x_i + 32\left|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i\right|} \\ \text{s.t.} \ \sum_{i=1}^{n}(4a_i+\beta_i-\alpha_i)x_i \geq 4r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{3.12}$$

and

$$\begin{cases} \max \sum_{i=1}^{n}(4a_i + \beta_i - \alpha_i)x_i \\ \text{s.t.} \quad 4\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i + \beta_i)(\alpha_j + \beta_j)x_i x_j + \sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i - \beta_i)(\alpha_j - \beta_j)x_i x_j \\ \qquad +6\left|\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i + \beta_i)(\alpha_j - \beta_j)x_i x_j\right| \\ \qquad \leq 32d\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i + 32d\left|\sum_{i=1}^{n}(\alpha_i - \beta_i)x_i\right| \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.13}$$

Proof. Since $\tilde{a}_1, \cdots, \tilde{a}_n$ are independent triangular fuzzy variable, Theorem 1.7, i.e., the extension principle of Zadeh, tells us that

$$\sum_{i=1}^{n}\tilde{a}_i x_i = \mathcal{T}\left(\sum_{i=1}^{n}(a_i - \alpha_i)x_i, \sum_{i=1}^{n}a_i x_i, \sum_{i=1}^{n}(a_i + \beta_i)x_i\right)$$

is also a triangular fuzzy variable. Thus the expected value and absolute deviation of $\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n$ are

$$\sum_{i=1}^{n}a_i x_i + \frac{1}{4}\sum_{i=1}^{n}\beta_i x_i - \frac{1}{4}\sum_{i=1}^{n}\alpha_i x_i$$

and

$$\frac{\left(\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i\right)^2 + 3\left(\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i + |\sum_{i=1}^{n}(\alpha_i - \beta_i)x_i|\right)^2}{32\left(\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i + |\sum_{i=1}^{n}(\alpha_i - \beta_i)x_i|\right)}.$$

After rearranging these two expressions, the desired results are easy to be obtained. The theorem is proved.

Corollary 3.1. *If $\alpha_i = \beta_i$ for $i = 1, 2, \cdots, n$, i.e., each $\tilde{a}_i$ is a symmetrical triangular fuzzy variable, then models (3.6) and (3.7) are respectively equivalent to the following linear programmings,*

$$\begin{cases} \min\ \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \\ s.t.\quad a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq r \\ \qquad\quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\quad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases}$$

and

$$\begin{cases} \max\ a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \\ s.t.\quad \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \leq 4d \\ \qquad\quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\quad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases}$$

Theorem 3.7. *Suppose that $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$ be a fuzzy vector defined by*

$$(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \begin{cases} (a_{11}, a_{21}, \cdots, a_{n1}), & \text{with membership degree } \mu_1 \\ (a_{12}, a_{22}, \cdots, a_{n2}), & \text{with membership degree } \mu_2 \\ \cdots & \cdots \\ (a_{1m}, a_{2m}, \cdots, a_{nm}), & \text{with membership degree } \mu_m \end{cases}$$

where $1 = \mu_1 > \mu_2 > \cdots > \mu_{m+1} = 0$. Then models (3.6) and (3.7) are respectively transformed into the following forms,

$$\begin{cases} \min\ \dfrac{1}{2}\displaystyle\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\leq j\leq i}|v_j| + \min_{1\leq j\leq i}|v_j|\right) \\ s.t.\quad x_1 a_{1j} + x_2 a_{2j} + \cdots + x_n a_{nj} = u_j,\ j = 1, 2, \cdots, m \\ \qquad\quad \dfrac{1}{2}\displaystyle\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(\max_{1\leq j\leq i} u_j + \min_{1\leq j\leq i} u_j\right) \geq r \\ \qquad\quad \dfrac{1}{2}\displaystyle\sum_{i=1}^{m}(\mu_i - \mu_{i+1})\left(2u_j - \max_{1\leq j\leq i} u_j - \min_{1\leq j\leq i} u_j\right) = v_j \\ \qquad\quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\quad x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases}$$

and

$$
\begin{cases}
\max \ \dfrac{1}{2}\sum_{i=1}^{m}(\mu_i-\mu_{i+1})\left(\max\limits_{1\le j\le i} u_j + \min\limits_{1\le j\le i} u_j\right) \\
\text{s.t.} \quad x_1a_{1j}+x_2a_{2j}+\cdots+x_na_{nj}=u_j,\ j=1,2,\cdots,m \\
\qquad \dfrac{1}{2}\sum_{i=1}^{m}(\mu_i-\mu_{i+1})\left(2u_j-\max\limits_{1\le j\le i} u_j - \min\limits_{1\le j\le i} u_j\right)=v_j \\
\qquad \dfrac{1}{2}\sum_{i=1}^{m}(\mu_i-\mu_{i+1})\left(\max\limits_{1\le j\le i} |v_j| + \min\limits_{1\le j\le i} |v_j|\right)\le d \\
\qquad x_1+x_2+\cdots+x_n=1 \\
\qquad x_i\ge 0, \quad i=1,2,\cdots,n.
\end{cases}
$$

Proof. The theorem is proved.

It can be seen that credibilistic mean-absolute deviation model is equivalent to linear programming only in the symmetric case such as normal or symmetric triangular fuzzy returns. In the independent situation, if all the fuzzy returns have one common membership function, then the model can also be simplified similar to Theorem 3.6. If the fuzzy returns have different membership functions, then it is difficult or impossible to obtain the analytical expression of the absolute deviation of the return on a portfolio so that credibilistic mean-absolute deviation model can not be converted into a crisp form. In this case, we may adopt fuzzy simulation technique to approximately calculate the absolute deviation and use genetic algorithm to solve the corresponding models.

3.3.1 Credibility Mean-Absolute Deviation-Skewness Model

We assume that an investor prefers a portfolio with a larger skewness if expected value and absolute deviation are same. For the portfolio optimization problem with asymmetric fuzzy returns, we may formulate the credibilistic mean-absolute deviation-skewness model as follows,

$$
\begin{cases}
\max \ S_k[\tilde{a}_1x_1+\tilde{a}_2x_2+\cdots+\tilde{a}_nx_n] \\
\text{s.t.} \quad E[\tilde{a}_1x_1+\tilde{a}_2x_2+\cdots+\tilde{a}_nx_n]\ge r \\
\qquad A[\tilde{a}_1x_1+\tilde{a}_2x_2+\cdots+\tilde{a}_nx_n]\le d \\
\qquad x_1+x_2+\cdots+x_n=1, \\
\qquad x_i\ge 0, \quad i=1,2,\cdots,n.
\end{cases}
\tag{3.14}
$$

The difference between model (2.7) and model (3.14) is the use of risk measure. Similarly, models (2.8), (2.9), and (2.10) may be reformulated by replacing variance with absolute deviation. Similar to Theorems 3.6 and 3.7, we may convert credibilistic mean-absolute deviation-skewness model into crisp programming when $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ have membership functions with the same type.

3.4 Fuzzy Simulation for Absolute Deviation

Let $\tilde{a}$ be a fuzzy variable with finite expected value e. According to Definition 3.1 and Remark 3.1, we have

$$A[\tilde{a}] = \frac{1}{2}\int_0^{\infty} f(r,e)\mathrm{d}r$$

in which

$$f(r,e) = \sup_{x \ge e+r \text{ or } x \le e-r} \mu(x) + 1 - \sup_{e-r<x<e+r} \mu(x).$$

Similar to Algorithm 2.1, we use the following algorithm to calculate $f(r,e)$.

Algorithm 3.1 (Fuzzy simulation for $f(r,e)$)

Step 1. Set $m = 1$ and take M as a sufficiently large integer;
Step 2. Randomly generate a real number w_m such that the membership degree $\mu(w_m) \ge \varepsilon$ where ε is a sufficiently small number;
Step 3. If $w_m \ge e + r$ or $w_m \le e - r$, let $u_m = w_m,\ v_m = 0$; otherwise, let $u_m = 0,\ v_m = w_m$;
Step 4. If $m > M$, then go to next step; otherwise, set $m = m + 1$ and go to Step 2;
Step 5. Return $\max_{1\le m\le M} u_m + 1 - \max_{1\le m\le M} v_m$ as the value of $f(r,e)$.

The detailed procedure for calculating $A[\tilde{a}]$ is summarized as follows,

---

Algorithm 3.2 (Fuzzy simulation for absolute deviation)

Step 1. Set $h = U/N$ where U a sufficiently large positive number and N is a sufficiently large integer;
Step 2. Partition the intervals $[0, U]$ into N small subintervals $[r_i, r_{i+1}]$, in which $r_i = ih$ for $i = 0, 1, \cdots, N-1$;
Step 3. Calculate $0.25h\sum_{i=0}^{N-1}(f(r_i,e) + f(r_{i+1},e))$ and return it as the simulated value of $A[\tilde{a}]$.

3.5 Numerical Examples

In this section, we illustrate the application of credibilistic mean-absolute deviation model by numerical examples in Qin et al. (2011). Assume that the investors employ absolute deviation of fuzzy variable as risk measure. We continue discussing the example in Sect. 2.5, i.e., constructing an optimal portfolio from a candidate set of 10 risky securities.

Example 3.5. Suppose that the minimum expected return level the investor can accept is 1.5. Then we reformulate the credibilistic mean-absolute deviation model as follows,

$$\begin{cases} \min A[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \\ \text{s.t.} \quad E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \geq 1.5 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.15}$$

A run of the hybrid intelligent algorithm (2500 cycles in simulation, 500 generations in GA) shows that the optimal portfolio is

$$\boldsymbol{x}^* = (0.121, 0.140, 0.114, 0.100, 0.116, 0.086, 0.102, 0.069, 0.081, 0.071)$$

which shows the allocation of the capital to each security. The corresponding minimal absolute deviation is 0.827.

Next, we test the effectiveness of genetic algorithm by changing the parameters in the GA. In order to compare the results, we employ the relative error (RE) defined as

$$\frac{\text{Actual absolute derivation} - \text{Minimal absolte derivation}}{\text{Minimal absolute deviation}} \times 100\,\%.$$

Here, the minimal absolute derivation is the minimum of all the obtained results. As summarized in Table 3.1, the relative errors do not exceed 2 %, which shows that genetic algorithm is robust to set parameters and effective for solving credibilistic mean-absolute deviation model.

Example 3.6. If an investor requires that the maximum absolute deviation (the risk) cannot exceed the level 1.1, then he/she may employ the following model,

$$\begin{cases} \max E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \\ \text{s.t.} \quad A[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_{10}x_{10}] \leq 1.1, \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1, \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{3.16}$$

Table 3.1 Computational solutions of Example 3.5

No.	*pop_size*	P_c	P_m	Times	Generation	AD	RE (%)
1	50	0.3	0.8	1500	600	0.8353	1.02
2	90	0.4	0.7	1300	400	0.8331	0.75
3	75	0.3	0.5	1000	300	0.8330	0.74
4	120	0.6	0.4	1500	400	0.8336	0.81
5	150	0.7	0.6	2500	200	0.8270	0.01
6	300	0.5	0.7	2000	220	0.8377	1.31
7	200	0.2	0.1	2200	250	0.8412	1.72
8	350	0.6	0.5	1500	250	0.8290	0.25
9	100	0.8	0.3	2500	500	0.8269	0
10	400	0.4	0.3	2500	100	0.8375	1.28

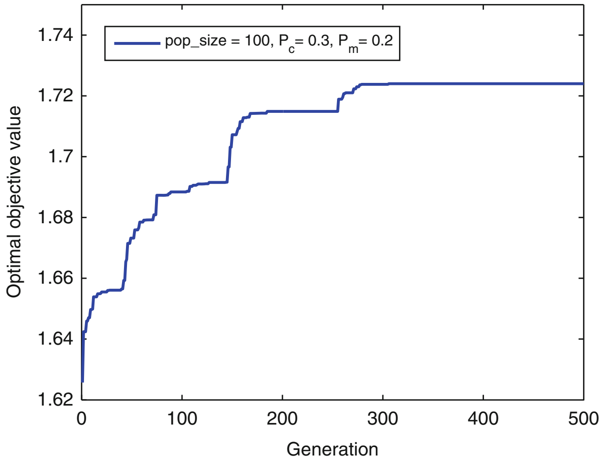

Fig. 3.2 The convergence of objective value in Example 3.6

The parameters of GA are chosen as *pop_size* $= 100$, $P_c = 0.3$ and $P_m = 0.2$. A run of the hybrid intelligent algorithm (2000 cycles in simulation, 500 generations in GA) shows that the optimal portfolio is $x_1 = 0,\ x_2 = 0.0043,\ x_3 = 0.0226,\ x_4 = 0.064,\ x_5 = 0.0906,\ x_6 = 0.5875,\ x_7 = 0.0283,\ x_8 = 0.0193,\ x_9 = 0,\ x_{10} = 0.1834$, which provides a maximum expected return 1.72. In addition, the evolution of optimal objective value at each generation is shown in Fig. 3.2. It can be seen that the result keep stable after 300 generations.

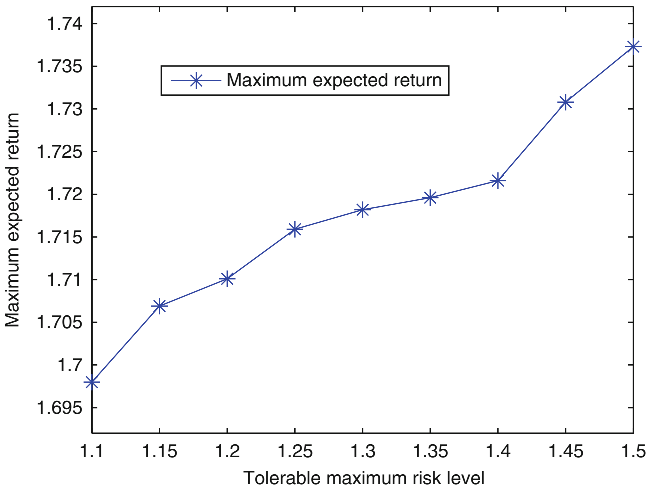

Fig. 3.3 The sensitivity to changes of tolerable maximum risk

In order to examine the sensitivity of the tolerable risk to total expected return, we experimented this model by changing the value of the tolerable risk. Computational results are summarized in Fig. 3.3 which is almost consistent with the actual expectation. Generally speaking, genetic algorithm only offers the approximate solution for the problem not the exact optimal solution. Therefore, there maybe exist larger computational errors in the results, which is the reason why the curve is not smooth.

Chapter 4
Credibilistic Cross-Entropy Minimization Model

4.1 Introduction

Kapur and Kesavan (1992) respectively proposed an entropy maximization model and a cross-entropy minimization model for portfolio optimization. The objective of the first model is to maximize the uncertainty of the random investment return and the second one is to minimize the divergence of the random investment return from a priori one. From then on, many researchers accepted the criterion and investigated these entropy optimization models (Cherny and Maslov 2003; Fang et al. 1997; Rubinstein 2008; Simonelli 2005).

In order to measure the difficulty degree of the prediction, Li and Liu (2008) proposed a fuzzy entropy for measuring the uncertainty of fuzzy variables. Following this concept, Huang (2008b) extended Kapur's entropy maximization model to fuzzy environment and formulated mean-entropy model. Further, Li (2015) defined fuzzy cross-entropy to measure the divergence of fuzzy variable from a priori one. Based on the fuzzy cross-entropy, Qin et al. (2009) established credibilistic cross-entropy minimization models for portfolio optimization with fuzzy returns in the framework of credibility theory.

This chapter focuses on credibilistic cross-entropy minimization model. The main contents include the definition of cross-entropy of fuzzy variable, the formulation and equivalents of cross-entropy minimization model, fuzzy simulation for cross-entropy, and numerical examples.

4.2 Cross-Entropy of Fuzzy Variable

This section reviews the definition of cross-entropy of fuzzy variable.

Z. Qin, *Uncertain Portfolio Optimization*, Uncertainty and Operations Research,
DOI 10.1007/978-981-10-1810-7_4

Definition 4.1 (Li 2015). Let $\tilde{a}$ and $\tilde{b}$ be two discrete fuzzy variables taking values in $\{y_1, y_2, \cdots, y_m\}$. Then the fuzzy cross-entropy of $\tilde{a}$ from $\tilde{b}$ is defined as

$$D[\tilde{a}; \tilde{b}] = \sum_{i=1}^{m} T\left(\mathrm{Cr}\{\tilde{a} = y_i, \mathrm{Cr}\{\tilde{b} = y_i\}\right) \tag{4.1}$$

where $T(s, t) = s \ln(s/t) + (1 - s) \ln((1 - s)/(1 - t))$.

Remark 4.1. Note that $T(s, t)$ is a function from $[0, 1] \times [0, 1]$ to $[0, +\infty)$ satisfying

$$T(s, 0) = \begin{cases} 0, & \text{if } s = 0 \\ +\infty & \text{if } s > 0, \end{cases} \qquad T(s, 1) = \begin{cases} 0, & \text{if } s = 1 \\ +\infty & \text{if } s < 1. \end{cases}$$

It is easy to verify that (a) $T(s, t)$ is strictly convex with respect to (s, t) and attains its minimum value 0 on the line $s = t$; and (b) for any $0 \leq s \leq 1$ and $0 \leq t \leq 1$, we have $T(s, t) = T(1 - s, 1 - t)$.

Definition 4.2 (Li 2015). Let $\tilde{a}$ and $\tilde{b}$ be two continuous fuzzy variables taking values in $[y_1, y_2]$. Then the cross-entropy of $\tilde{a}$ from $\tilde{b}$ is defined as

$$D[\tilde{a}; \tilde{b}] = \int_{y_1}^{y_2} T\left(\mathrm{Cr}\{\tilde{a} = y\}, \mathrm{Cr}\{\tilde{b} = y\}\right) \mathrm{d}y. \tag{4.2}$$

Let μ and ν be the membership functions of $\tilde{a}$ and $\tilde{b}$, respectively. Since $\mathrm{Cr}\{\tilde{a} = y\} = \mu(y)/2$ and $\mathrm{Cr}\{\tilde{b} = y\} = \nu(y)/2$, the cross-entropy of $\tilde{a}$ from $\tilde{b}$ can be rewritten as

$$D[\tilde{a}; \tilde{b}] = \int_{y_1}^{y_2} \left(\frac{\mu(y)}{2} \ln\left(\frac{\mu(y)}{\nu(y)}\right) + \left(1 - \frac{\mu(y)}{2}\right) \ln\left(\frac{2 - \mu(y)}{2 - \nu(y)}\right)\right) \mathrm{d}y.$$

If $\tilde{b}$ is an equipossible fuzzy variable on $[y_1, y_2]$, i.e., $\upsilon(y) \equiv 1$ for any $y \in [y_1, y_2]$, then we have

$$D[\tilde{a}; \tilde{b}] = \frac{1}{2} \int_{y_1}^{y_2} [\mu(y) \ln \mu(y) + (2 - \mu(y)) \ln(2 - \mu(y))] \,\mathrm{d}y. \tag{4.3}$$

Theorem 4.1 (Li 2015). *For any fuzzy variables $\tilde{a}$ and $\tilde{b}$, we have $D[\tilde{a}; \tilde{b}] \geq 0$. The equality holds if and only if $\tilde{a}$ and $\tilde{b}$ have the same membership function.*

Example 4.1. Let $\tilde{a} = \mathcal{T}(a - \alpha, a, a + \beta)$ be a triangular fuzzy variable, and $\tilde{b}$ is an equipossible fuzzy variable on $[a - \alpha, a + \beta]$. Then we have

$$D[\tilde{a}; \tilde{b}] = (\ln 2 - 0.5)(\alpha + \beta).$$

It follows from Eq. (1.5) that the right integral in Eq. (4.3) is equivalent to

$$\int_{a-\alpha}^{a} \frac{y-a+\alpha}{\alpha} \ln \frac{y-a+\alpha}{\alpha} \mathrm{d}y + \int_{a}^{a+\beta} \frac{a+\beta-y}{\beta} \ln \frac{a+\beta-y}{\beta} \mathrm{d}y$$
$$+ \int_{a-\alpha}^{a} \left(2 - \frac{y-a+\alpha}{\alpha}\right) \ln \left(2 - \frac{y-a+\alpha}{\alpha}\right) \mathrm{d}y$$
$$+ \int_{a}^{a+\beta} \left(2 - \frac{a+\beta-y}{\beta}\right) \ln \left(2 - \frac{a+\beta-y}{\beta}\right) \mathrm{d}y.$$

By the changes of variables, we have

$$D[\tilde{a};\tilde{b}] = \frac{\alpha+\beta}{2} \int_{0}^{2} y \ln y \mathrm{d}y = (\ln 2 - 0.5)(\alpha+\beta).$$

Example 4.2. Suppose that $\tilde{a} = \mathcal{TP}(a-\alpha, a, b, b+\beta)$ is a trapezoidal fuzzy variable, and $\tilde{b}$ is an equipossible fuzzy variable on $[a-\alpha, a+\beta]$. Then we also have

$$D[\tilde{a};\tilde{b}] = (\ln 2 - 0.5)\,(\alpha+\beta).$$

4.3 Credibilistic Cross-Entropy Minimization Model

In this section, Kapur cross-entropy minimization model is extended to the portfolio optimization with fuzzy returns. Let $\tilde{a}_i$ be the return of the ith security and x_i the proportion of total fund invested in security i, $i = 1, 2, \cdots, n$. We assume that $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ are fuzzy variables defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$.

Suppose there is a priori fuzzy investment return $\tilde{b}$ for an investor whose objective is to minimize the divergence of the return on his/her portfolio from $\tilde{b}$. Meanwhile, the investor requires that the return is above a given level and the risk remains below a given level. Generally speaking, we use the cross-entropy to measure the degree of divergence and use the expected value to measure the return. By choosing appropriate risk measure, we may formulate different credibilistic cross-entropy minimization model for portfolio optimization. When the fuzzy return $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$ is symmetric, Qin et al. (2009) used variance as risk measure and proposed the following credibilistic cross-entropy minimization model,

$$\begin{cases} \min\ D[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n; \tilde{b}] \\ \text{s.t.}\ \ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \geq r \\ \qquad V[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \leq d \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0,\ i = 1, 2, \cdots, n. \end{cases} \tag{4.4}$$

In the case with symmetric returns, we may also employ absolute deviation to measure the risk. The corresponding credibilistic cross-entropy minimization model is as follows,

$$\begin{cases} \min\ D[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n; \tilde{b}] \\ \text{s.t.}\ \ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \geq r \\ \qquad A[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \leq d \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0,\ i = 1, 2, \cdots, n. \end{cases} \tag{4.5}$$

In models (4.4) and (4.5), r and d are the predetermined confidence levels. Their values depend on the degree of risk aversion of the investor, and are preset by the investor based on the specific circumstances. For fixed d, the objective will decrease with r increasing since decision variables satisfying the constraint decrease. Similarly, the objective will increase with d increasing for fixed r.

If the membership function of $\tilde{a} = \tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$ is asymmetric, then we can not use variance or absolute deviation to measure the risk because it punishes not only the undesirable part ($\tilde{a} \leq E[\tilde{a}]$), but also the desirable part ($\tilde{a} > E[\tilde{a}]$). In this case, semivariance (Huang 2007a) is more suitable to measure risk because it only punishes the investment return below the expected value. Qin et al. (2009) used semivariance to measure the risk and formulated the following credibilistic cross-entropy minimization model,

$$\begin{cases} \min\ D[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n; \tilde{b}] \\ \text{s.t.}\ \ E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \geq r \\ \qquad S_v[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \leq d \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0,\ i = 1, 2, \cdots, n. \end{cases} \tag{4.6}$$

Here, S_v represents the semivariance operator of fuzzy variable, and r and d are also predetermined confidence levels by the investor and depended on the specific circumstances of the investor.

If there is an acceptable investment return C, then the risk may be measured by the chance of bad outcome. That is, the investment return is less than C. In this sense, Qin et al. (2009) proposed the following credibilistic cross-entropy minimization model,

$$\begin{cases} \min\ D[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n; \tilde{b}] \\ \text{s.t.}\ \ \mathrm{Cr}\{\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n \leq C\} \leq \alpha \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0,\ i = 1, 2, \cdots, n \end{cases} \tag{4.7}$$

where d is a given confidence level. The first constraint may control the chance of bad outcome, which ensures the optimal portfolio satisfies that the credibility measure of its return no less than C is no less than α. We can add other constraint such as $E[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] \geq r$ to satisfy the requirements of the investor.

Remark 4.2. If there is no priori return $\tilde{b}$, then we may regard $\tilde{b}$ as an equipossible fuzzy variable. That is, $\tilde{b}$ has a membership function $\upsilon(x) \equiv 1$ on its support set. If the membership function of $\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n$ is μ with finite support set $[y_1, y_2]$, we have

$$
\begin{aligned}
&D[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n; \tilde{b}]\\
&= \int_{y_1}^{y_2} \left(\frac{\mu(x)}{2} \ln \mu(x) + \left(1 - \frac{\mu(x)}{2}\right) \ln (2 - \mu(x)) \right) \mathrm{d}x\\
&= \int_{y_1}^{y_2} \left(\frac{\mu(x)}{2} \ln \frac{\mu(x)}{2} + \left(1 - \frac{\mu(x)}{2}\right) \ln \frac{2 - \mu(x)}{2} \right) \mathrm{d}x + (y_2 - y_1) \ln 2\\
&= -H[\tilde{a}_1x_1 + \tilde{a}_2x_2 + \cdots + \tilde{a}_nx_n] + (y_2 - y_1) \ln 2
\end{aligned}
$$

in which H is the entropy operator of fuzzy variable. Hence, the credibilistic cross-entropy minimization models degenerate to the corresponding entropy maximization models. Next we consider the equivalents of credibilistic cross-entropy minimization model in the case without priori return.

Theorem 4.2. *Suppose that $\tilde{a}_i = \mathcal{T}(a_i - \alpha_i, a_i, a_i + \beta_i)$ are independent triangular fuzzy variables for $i = 1, 2, \cdots, n$. Then model (4.5) is equivalent to the crisp model,*

$$
\begin{cases}
\max \displaystyle\sum_{i=1}^{n} (\alpha_i + \beta_i)x_i\\
\text{s.t.} \quad \displaystyle\sum_{i=1}^{n} (4a_i + \beta_i - \alpha_i) \geq 4r\\
\qquad 4\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_i + \beta_i)(\alpha_j + \beta_j)x_ix_j + \sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_i - \beta_i)(\alpha_j - \beta_j)x_ix_j\\
\qquad\quad +6\left|\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} (\alpha_i + \beta_i)(\alpha_j - \beta_j)x_ix_j\right|\\
\qquad \leq 32d\displaystyle\sum_{i=1}^{n} (\alpha_i + \beta_i)x_i + 32d\left|\sum_{i=1}^{n} (\alpha_i - \beta_i)x_i\right|\\
\qquad x_1 + x_2 + \cdots + x_n = 1\\
\qquad x_i \geq 0, \quad i = 1, 2, \cdots, n.
\end{cases}
\tag{4.8}
$$

Proof. Similar to the proof of Theorem 3.13, we have

$$\sum_{i=1}^{n} \tilde{a}_i x_i = \mathcal{T}\left(\sum_{i=1}^{n}(a_i - \alpha_i)x_i, \sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n}(a_i + \beta_i)x_i\right).$$

When there is no priori return, i.e., $\tilde{b}$ is an equipossible fuzzy variable, the cross-entropy of $\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_n x_n$ from $\tilde{b}$ is

$$D\left[\sum_{i=1}^{n} \tilde{a}_i x_i; \tilde{b}\right] = (\ln 2 - 0.5)\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i.$$

Since $\ln 2 - 0.5 < 0$, the objective of model (4.5) is equivalent to the maximization of $\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i$. The theorem is proved.

Based on the assumption of Theorem 4.2, we may obtain the equivalent crisp models of other credibilistic cross-entropy minimization models. Moreover, Theorem 4.2 may also be extended to other cases such as trapezoidal fuzzy returns and simple fuzzy returns. However, if the priori return $\tilde{b}$ is not an equipossible fuzzy variable, then we cannot obtain the analytical expression of cross-entropy. In this case, we have to design fuzzy simulation to calculate cross-entropy.

4.4 Fuzzy Simulation for Cross-Entropy

In the discrete case, we may directly compute the cross-entropy according to Eq. (4.1) even though we do not know its analytical expression. In the continuous case, when the membership functions $\mu(y)$ and $\nu(y)$ are given, the cross-entropy can be calculated by

$$D[\tilde{a}; \tilde{b}] = \int_{y_1}^{y_2}\left(\frac{\mu(y)}{2}\ln\left(\frac{\mu(y)}{\nu(y)}\right) + \left(1 - \frac{\mu(y)}{2}\right)\ln\left(\frac{2-\mu(y)}{2-\nu(y)}\right)\right)\mathrm{d}y.$$

Considering the complexity of the integrand, we design fuzzy simulation for cross-entropy based on the numerical integration techniques. Define

$$f(y) = \frac{\mu(y)}{2}\ln\left(\frac{\mu(y)}{\nu(y)}\right) + \left(1 - \frac{\mu(y)}{2}\right)\ln\left(\frac{2-\mu(y)}{2-\nu(y)}\right).$$

Detailed procedure for calculating $D[\tilde{a}; \tilde{b}]$ are summarized as follows,

Algorithm 4.1 (Fuzzy simulation for cross-entropy)

Step 1. Set $h = (y_2 - y_1)/N$ where N is a sufficiently large integer;

Step 2. Partition the intervals $[y_1, y_2]$ into N small subintervals $[r_i, r_{i+1}]$, in which $r_i = ih$ for $i = 0, 1, \cdots, N-1$;

Step 3. Calculate $h\sum_{i=0}^{N/2-1}[f(r_{2i}) + 4f(x_{2i+1}) + f(r_{2i+2})]/3$ and return it as the simulated value of $D[\tilde{a}; \tilde{b}]$.

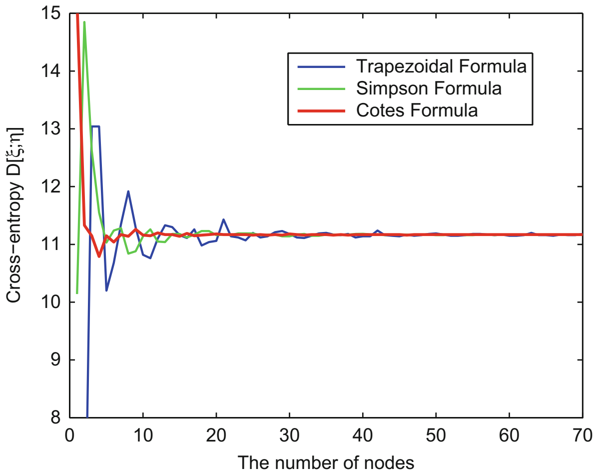

Fig. 4.1 Different numerical integration formulas for cross-entropy $D[\tilde{a}; \tilde{b}]$

These are a group of formulas for numerical integration based on evaluating the integrand at equally-spaced nodes. Algorithm 4.1 is based on the composite Simpson's rule in numerical integration and N is called the number of integration points. We may apply other rules to design the fuzzy simulation. Next we take an example to illustrate the use of Algorithm 4.1 and compare with different rules. Suppose that $\tilde{a} = (0, 50, 180)$ and $\tilde{b} = (0, 100, 200)$ are two triangular fuzzy variables. The cross-entropy of $\tilde{a}$ from $\tilde{b}$ are respectively calculated by the trapezoidal, Simpson and Cotes formulas which are special instances of Newton-Cotes formulas. The computational result is shown in Fig. 4.1, which implies that the Simpson formula is more efficient for our problem.

4.5 Numerical Examples

In this section, we illustrate the application of credibilistic cross-entropy minimization model by numerical examples in Qin et al. (2009). The first three examples consider the case with 10 risky securities with triangular fuzzy returns, denoted by

Table 4.1 Fuzzy returns of 10 securities (units per stock)

Security no.	Fuzzy return $\tilde{a}_i$
1	(−0.4, 2.7, 3.4)
2	(−0.1, 1.9, 2.6)
3	(−0.2, 3.0, 4.0)
4	(−0.5, 2.0, 2.9)
5	(−0.6, 2.2, 3.3)
6	(−0.1, 2.5, 3.6)
7	(−0.3, 2.4, 3.5)
8	(−0.1, 3.3, 4.5)
9	(−0.7, 1.1, 2.7)
10	(−0.2, 2.1, 3.8)

$\tilde{a}_i = \mathcal{T}(a_i - \alpha_i, a_i, a_i + \beta_i)$, $i = 1, 2, \cdots, 10$, in which parameters a_i, α_i and β_i are determined based on the real historical data and the estimated values of stock experts. The data set is given in Table 4.1.

In addition, these three examples are all performed on a personal computer and the parameters in hybrid intelligent algorithm are set as follows: 3000 cycles in fuzzy simulation, 1000 cycles in numerical integration, 1100 generations in genetic algorithm, the probability of crossover $P_c = 0.4$, the probability of mutation $P_m = 0.3$ and the parameter in the rank-based evaluation function $v = 0.05$. In addition, the prior fuzzy investment return is a triangular fuzzy variable $\tilde{b} = (-0.2, 2.3, 4)$.

Example 4.3. Since Markowitz quantified the risk of portfolio, variance has been widely accepted as a popular risk measure. Assume that an investor considers variance as the risk measure of a portfolio when he/she chooses these 10 securities to invest. Then the following credibilistic cross-entropy minimization model is employed,

$$\begin{cases} \min\ D[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}; \tilde{b}] \\ \text{s.t.}\ \ E[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}] \geq 2.25 \\ \qquad V[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}] \leq 1.0 \\ \qquad x_1 + x_2 + \cdots + x_{10} = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 10, \end{cases}$$

where the tolerable risk level does not exceed 1.0 and the minimum expected return is no less than 2.25.

A run of genetic algorithm shows that among 10 securities, satisfying the constraint, in order to minimize the cross-entropy of the return from the prior $\tilde{b}$, the investor should assign his money according to Table 4.2. The corresponding minimal cross-entropy is 0.016, the expected return and variance of the portfolio are 2.255 and 0.936, respectively. Here the total fuzzy return is $(-0.18, 2.61, 3.98)$. The graphic comparison of the obtained investment return and the prior one is shown in Fig. 4.2.

Table 4.2 Allocation of money to 10 securities (%)

Security i	1	2	3	4	5	6	7	8	9	10
Allocation	1.8	1.1	1.9	2.7	1.0	5.6	5.3	37.7	0.9	42.0

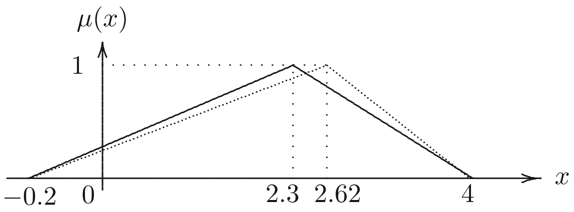

Fig. 4.2 Comparison of investment return and the prior one in Example 4.3

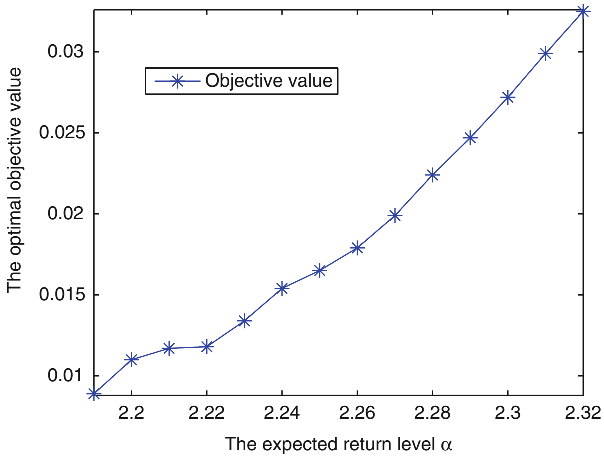

Fig. 4.3 The sensitivity to changes of expected return level r in Example 4.3

In order to examine the sensitivity of the predetermined confidence level, i.e., expected return level r, to the cross-entropy, we experimented on this example by changing the value of r. The computational results are summarized in Fig. 4.3. The result indicates that as expected return level increases, the minimal cross-entropy or the optimal objective will approximately linearly increase. For given expected return level, we also examine the sensitivity of the acceptable risk level d to the optimal objective in the same way.

Remark 4.3. If there is no priori investment $\tilde{b}$, it follows from Theorem 4.2 that model (4.5) becomes a crisp model. Using the same parameters and solving the crisp model, the obtained optimal portfolio is $x_1 = 1.2$, $x_2 = 0$, $x_3 = 7.7$, $x_4 = 1.4$, $x_5 = 0.3$, $x_6 = 0.5$, $x_7 = 3.4$, $x_8 = 46.5$, $x_9 = 0$, $x_{10} = 39.0$, which corresponds to the fuzzy return $(-0.16, 2.75, 4.02)$. It can be seen that the allocations of money to 10 securities are different.

Example 4.4. In the asymmetric case, if the investor considers semivariance as risk measure, then the following credibilistic cross-entropy minimization model is used to construct the optimal portfolio,

$$\begin{cases} \min\ D[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}; \tilde{b}] \\ \text{s.t.}\ \ E[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}] \geq 2.25 \\ \quad\ \ S_v[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}] \leq 0.70 \\ \quad\ \ x_1 + \cdots + x_{10} = 1 \\ \quad\ \ x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases}$$

in which 0.70 is accepted as the maximal risk level and 2.25 as the expected return level.

A run of genetic algorithm shows that the investor should assign his money according to Table 4.3. The corresponding minimal cross-entropy is 0.017, the expected return and semivariance of the portfolio are 2.252 and 0.70, respectively. Here, the total fuzzy return is $(-0.20, 2.63, 3.95)$. The computational results are shown in Table 4.3 and Fig. 4.4.

In Examples 4.3 and 4.4, the confidence levels 2.25, 1.0 and 0.7 are predetermined by the investors. In real life, the investors may consider different confidence levels to reflect their risk aversion and the pursuit of profit. From the computational results of these two examples, we know that the allocation of money will vary with the risk measure and confidence levels.

Example 4.5. Suppose the investors want to control the chance of bad outcome event, and set the predetermined investment return as 0.8 and accept 0.2 as the risk level. Then the following credibilistic cross-entropy minimization model is used,

$$\begin{cases} \min\ D[\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10}; \tilde{b}] \\ \text{s.t.}\ \ \mathrm{Cr}\{\tilde{a}_1 x_1 + \cdots + \tilde{a}_{10} x_{10} \leq 0.8\} \leq 0.2 \\ \quad\ \ x_1 + \cdots + x_{10} = 1 \\ \quad\ \ x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases}$$

Table 4.3 Allocation of money to 10 securities (%)

Security i	1	2	3	4	5	6	7	8	9	10
Allocation	2.3	0.9	3.0	0.0	3.6	5.8	8.7	39.8	3.7	32.2

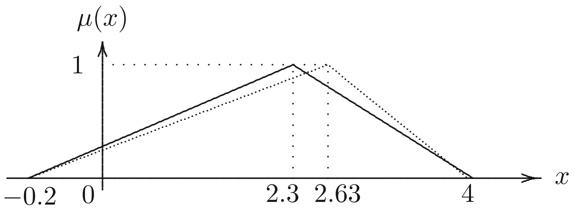

Fig. 4.4 Comparison of investment return and the prior one

Table 4.4 Allocation of money to 10 securities (%)

Security i	1	2	3	4	5	6	7	8	9	10
Allocation	8.9	0.0	14.9	1.2	1.1	13.3	2.8	18.3	0.9	38.6

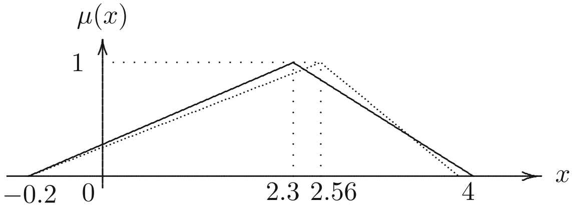

Fig. 4.5 Comparison of investment return and the prior one

Table 4.5 Comparison of optimal objectives of Example 4.6

No.	*pop_size*	P_c	P_m	Gen.	Times	Nodes	Obj.	RE (%)
1	100	0.2	0.1	1000	3000	1000	1.9631	0.28
2	100	0.3	0.2	3000	3000	1000	1.9577	0
3	100	0.4	0.3	1000	3000	1000	1.9612	0.18
4	100	0.5	0.4	2000	3000	1000	1.9588	0.06
5	100	0.7	0.3	1000	3000	1000	1.9630	0.27
6	80	0.3	0.4	3000	2000	100	1.9604	0.14
7	60	0.4	0.6	2000	2000	100	1.9611	0.17
8	40	0.7	0.3	2000	2000	100	1.9615	0.19
9	30	0.1	0.3	1000	2000	100	1.9610	0.17
10	20	0.9	0.5	1000	2000	100	1.9622	0.23
11	15	0.2	0.1	1000	2000	100	1.9636	0.30

A run of genetic algorithm shows that the investors should assign his money according to Table 4.4. The corresponding minimal cross-entropy is 0.015, and the credibility of the portfolio below 0.80 is 0.182. The total fuzzy return is $(-0.20, 2.56, 3.86)$. The computational results are shown in Table 4.4 and Fig. 4.5.

The genetic algorithm is the same in solving models (4.4), (4.5), (4.6), and (4.7) except for different constraint computed by fuzzy simulation. Thus, next we only solve model (4.4) to test the robust of the algorithm in the following example.

Example 4.6. In order to test the robustness of genetic algorithm for large portfolio optimization problem, we solve model (4.4) with 1000 securities using different parameters in the GA. Suppose that confidence levels $r = 2.15$ and $d = 1.75$, and the returns of all the securities are triangular fuzzy variables denoted by $\xi = \mathcal{T}(a_i - \alpha_i, a_i, a_i + \beta_i)$ for $i = 1, 2, \cdots, 1000$ with $a_i > \alpha_i - 2$ and $a_i < 8 - \beta_i$. Here, a_i, α_i and β_i are generated randomly. Some computational results are shown in Table 4.5. To compare the objective values, we use relative error (RE) as the index, i.e., (actual value − minimum)/minimum $\times$ 100 %, where the minimum is the minimal value of all the objective values calculated.

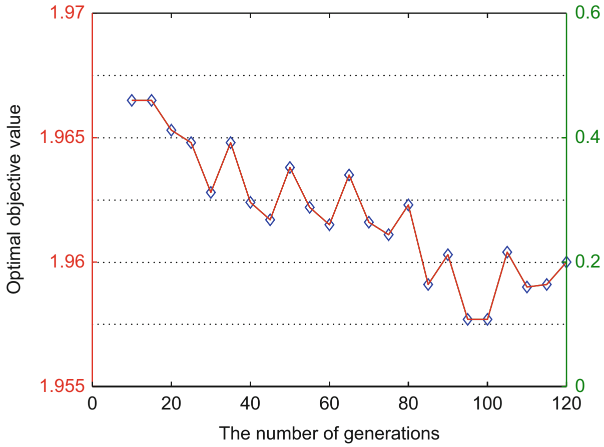

Fig. 4.6 A sensitivity to changes of population size to objective values

It can be seen from Table 4.5 that the relative errors do not exceed 0.5 % by choosing different parameters in the GA, which means that genetic algorithm is effective and robust to set parameters. In addition, we examine the impact of population size on the optimal objective value when other parameters are fixed in GA. For example, we set $P_c = 0.3$, $P_m = 0.2$ and use 3000 generations, 3000 cycles in fuzzy simulation for expected value and variance and 1000 nodes in fuzzy simulation for cross-entropy. It follows from Fig. 4.6 that the suitable population size is between 95 and 100.

Chapter 5
Uncertain Mean-Semiabsolute Deviation Model

5.1 Introduction

In the case with lack of historical data, another feasible way is to estimate returns by experts based on their subjective evaluations in the framework of uncertainty theory (Liu 2007). In particular, several researchers have studied portfolio optimization in which security returns are assumed to be uncertain variables. The first attempt is Qin et al. (2009) who formulated the uncertain counterpart of mean-variance model. As extensions, Liu and Qin (2012) proposed an uncertain mean-semiabsolute deviation model for the asymmetric case and Huang and Qiao (2012) presented a risk index model for multi-period case. Different from these, Zhu (2010) applied uncertain optimal control to model continuous-time problem, but Yao and Ji (2014) considered the problem by using uncertain decision making.

For the analogous reasons in stochastic environment, variance and absolute deviation become unreasonable in the case that uncertain returns are asymmetric. More importantly, in the framework of uncertainty theory, the variance of an uncertain variable cannot be exactly calculated from its uncertain distribution. Actually Liu (2010) have to use a stipulation to handle this situation. To overcome the shortcoming, Liu and Qin (2012) defined the semiabsolute deviation of uncertain variable to measure downside risk in the case of asymmetric uncertain returns. The main advantage is that the semiabsolute deviation is exactly determined by the uncertainty distribution. This implies that semiabsolute deviation provides an exact measurement of risk or downside risk in uncertain environment.

Due to the rapidly changing situations in the financial markets, an existing portfolio may not be efficient after a certain period of time. Again changing the financial data in the market has a great impact on the investor's holdings. Therefore, portfolio adjustment is necessary in response to the changing situation in financial markets and investor's capital. The cost associated with buying or selling of a risky asset, known as transaction cost is one of the main concerns for portfolio managers. Arnott and Wagner (1990) first suggested that ignoring transaction costs would

Z. Qin, *Uncertain Portfolio Optimization*, Uncertainty and Operations Research,
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result in an inefficient portfolio; whereas, adding transaction costs would assist decision makers to understand better an efficient frontier. Some researchers such as Patel and Subrahmanyam (1982), Morton and Pliska (1995), Yoshimoto (1996), Choi et al. (2007), Lobo et al. (2007), Bertsimas and Pachamanova (2008), Baule (2010), and Woodside-Oriakhi et al. (2013) etc. extended the works on portfolio selection problems with transaction costs. In the portfolio adjusting problem, investors always update their existing portfolios by buying or selling risk assets to hedge the fluctuations of financial markets. Some researchers such as Fang et al. (2006), Glen (2011), Yu and Lee (2011), and Zhang et al. (2011) studied the portfolio adjusting problems in the framework of return-risk trade-off. Huang and Ying (2013) also studied the portfolio adjusting problem. Qin et al. (2016) first focused on the problem in the assumption of uncertain variable returns by using semiabsolute deviation to measure risk.

This chapter focuses on the portfolio optimization with uncertain returns subject to experts' evaluations. The main contents of this chapter include the definition of semiabsolute deviation of uncertain variable, formulation of uncertain mean-semiabsolute deviation model, formulation of uncertain mean-semiabsolute deviation adjusting model and its equivalents, and numerical examples.

5.2 Semiabsolute Deviation of Uncertain Variable

In this section, we review the definition of semiabsolute deviation of uncertain variable and its properties.

Definition 5.1 (Li et al. 2012). Let ξ be an uncertain variable with finite expected value e. Then the semiabsolute deviation of ξ is defined as

$$S_a[\xi] = E\left[|(\xi - e) \wedge 0|\right]. \tag{5.1}$$

Remark 5.1. Definition 5.1 tells us that the semiabsolute deviation is the expected value of $|(\xi - e) \wedge 0|$. Since $|(\xi - e) \wedge 0|$ is a nonnegative uncertain variable, we also have

$$\begin{aligned} S_a[\xi] &= \int_0^{\infty} \mathcal{M}\{|(\xi - e)^-| \geq r\}\,\mathrm{d}r \\ &= \int_0^{\infty} \mathcal{M}\{e - \xi \geq r\}\,\mathrm{d}r \\ &= \int_{-\infty}^{e} \mathcal{M}\{\xi \leq r\}\,\mathrm{d}r \\ &= \int_{-\infty}^{e} \Phi(r)\mathrm{d}r \end{aligned} \tag{5.2}$$

where $\Phi(\cdot)$ is the uncertainty distribution of ξ. This formula will facilitate the calculation of semiabsolute deviation in many cases.

Example 5.1. Let $\xi \sim \mathcal{L}(a, b)$ be a linear uncertain variable. Then its semiabsolute deviation is

$$S_a[\xi] = \int_{-\infty}^{E[\xi]} \Phi(r)\mathrm{d}r = \int_a^{(a+b)/2} \frac{r-a}{b-a}\mathrm{d}r = \frac{b-a}{8}.$$

Example 5.2. Let $\xi \sim \mathcal{Z}(a, b, c)$ be a zigzag uncertain variable. The semiabsolute deviation of ξ is

$$S_a[\xi] = \int_{-\infty}^{(a+2b+c)/4} \Phi(r)\mathrm{d}r = \begin{cases} \dfrac{(3(b-a)+(c-b))^2}{64(b-a)}, & \text{if } b-a \ge c-b, \\ \dfrac{((b-a)+3(c-b))^2}{64(c-b)}, & \text{if } b-a \le c-b, \end{cases}$$

which has alternative expression

$$S_a[\xi] = \frac{(2c-2a+|2b-a-c|)^2}{32(c-a+|2b-a-c|)}.$$

Example 5.3. Let $\xi \sim \mathcal{N}(e, \sigma)$ be a normal uncertain variable. Then the semiabsolute deviation of ξ is

$$S_a[\xi] = \int_{-\infty}^{e} \Phi(r)\mathrm{d}r = \int_0^{e} \left(1 + \exp\left(\frac{\pi(e-r)}{\sqrt{3}\sigma}\right)\right)^{-1} \mathrm{d}r = \frac{\sqrt{3}\sigma \ln 2}{\pi}.$$

Theorem 5.1 (Liu and Qin 2012). *Let ξ be an uncertain variable with finite expected value. Then for any real numbers a and b, we have*

$$S_a[a\xi + b] = |a| \cdot S_a[\xi]. \tag{5.3}$$

Proof. It follows from Definition 5.1 that

$$\begin{aligned} S_a[a\xi + b] &= E[|(a\xi + b - aE[\xi] - b) \wedge 0|] \\ &= E[|a(\xi - E[\xi]) \wedge 0|] \\ &= |a| \cdot E[|(\xi - E[\xi]) \wedge 0|] \\ &= |a| \cdot S_a[\xi]. \end{aligned}$$

The theorem is proved.

Theorem 5.2 (Liu and Qin 2012). *Let ξ be an uncertain variable with expected value e, and $S_a[\xi]$ the semiabsolute deviation of ξ. Then we have*

$$0 \leq S_a[\xi] \leq E[|\xi - e|]. \tag{5.4}$$

Proof. Since $S_a[\xi]$ is nonnegative, for any real number r, we have

$$\{\gamma \mid |\xi(\gamma) - e| \geq r\} \supset \{\gamma \mid |(\xi(\gamma) - e) \wedge 0| \geq r\}.$$

It follows from the monotonicity of uncertain measure that

$$\mathcal{M}\{|\xi - e| \geq r\} \geq \mathcal{M}\{|(\xi - e) \wedge 0| \geq r\}, \forall\, r.$$

It immediately follows from Definition 5.1 that

$$S_a[\xi] = \int_0^{+\infty} \mathcal{M}\{|(\xi - e) \wedge 0| \geq r\}\mathrm{d}r \leq \int_0^{+\infty} \mathcal{M}\{|\xi - e| \geq r\}\mathrm{d}r = E[|\xi - e|].$$

The theorem is proved.

Theorem 5.3 (Liu and Qin 2012). *Let ξ be an uncertain variable with finite expected value e. Then $S_a[\xi] = 0$ if and only if $\mathcal{M}\{\xi = e\} = 1$.*

Proof. If $S_a[\xi] = 0$, then $E[|(\xi - e) \wedge 0|] = 0$. From Definition 1.18, we have

$$E[|(\xi - e) \wedge 0|] = \int_0^{+\infty} \mathcal{M}\{|(\xi - e) \wedge 0| \geq r\}\mathrm{d}r,$$

which implies $\mathcal{M}\{|(\xi - e) \wedge 0| \geq r\} = 0$ for any $r > 0$. Further, it is obtained that

$$\mathcal{M}\{|(\xi - e) \wedge 0| = 0\} = 1, \tag{5.5}$$

which implies $\xi - e = (\xi - e) \vee 0 + (\xi - e) \wedge 0 = (\xi - e) \vee 0$ almost everywhere. Thus, we have

$$\int_0^{+\infty} \mathcal{M}\{(\xi - e) \vee 0 \geq r\}\mathrm{d}r = E[(\xi - e) \vee 0] = E[\xi - e] = 0.$$

This indicates that $\mathcal{M}\{(\xi - e) \vee 0 \geq r\} = 0$ for any $r > 0$. Considering the self-duality of uncertainty measure, we have

$$\mathcal{M}\{(\xi - e) \vee 0 = 0\} = 1. \tag{5.6}$$

If follows from Eqs. (5.5) and (5.6) that $\mathcal{M}\{\xi - e = 0\} = 1$, i.e., $\mathcal{M}\{\xi = e\} = 1$. Conversely, if $\mathcal{M}\{\xi = e\} = 1$, then we have $\mathcal{M}\{|\xi - e| = 0\} = 1$ and $\mathcal{M}\{|\xi - e| \geq r\} = 0$ for any real number $r > 0$, which implies that $E[|\xi - e|] = 0$. Further, it follows from Theorem 5.2 that $S_a[\xi] = 0$. The theorem is proved.

5.3 Mean-Semiabsolute Deviation Model

We consider the portfolio optimization problem aiming at constructing an optimal portfolio from a candidate set of n risky securities. Let ξ_i be the return of the ith security, and x_i be the proportion of total amount of funds invested in security i for $i = 1, 2, \cdots, n$. Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables defined on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. By uncertain arithmetic, the total return of the portfolio is $\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ which is also an uncertain variable. This means that the portfolio is risky, and semiabsolute deviation is employed as risk measure.

If an investor wants to maximize the expected return at the given risk level, then Liu and Qin (2012) formulated a single-objective uncertain mean-semiabsolute deviation model as follows,

$$\begin{cases} \max\ E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{s.t.}\quad S_a[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \le d \\ \qquad\ \ x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\ \ x_i \ge 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{5.7}$$

where d denotes the maximum risk level the investors can take. The constraints ensure that all the capital will be invested to n securities and short sales are not allowed. It is worth pointing out that the proposed model can deal with both the case with symmetric returns and the one with asymmetric returns.

When an investor wants to minimize the risk of investment with an acceptable expected return level, another uncertain mean-semiabsolute deviation model is proposed by Liu and Qin (2012) as follows,

$$\begin{cases} \min\ S_a[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{s.t.}\quad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \ge r \\ \qquad\ \ x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\ \ x_i \ge 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{5.8}$$

where r is the minimum expected return level accepted by the investor.

A risk-averse investor always wants to maximize the return and minimize the risk of the portfolio. However, these two objects are inconsistent. To determine an optimal portfolio with a given degree of risk aversion, Liu and Qin (2012) formalized the following uncertain mean-semiabsolute deviation model,

$$\begin{cases} \max\ E[\xi_1 x_1 + \cdots + \xi_n x_n] - \phi \cdot S_a[\xi_1 x_1 + \cdots + \xi_n x_n] \\ \text{s.t.}\quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad\ \ x_i \ge 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{5.9}$$

where $\phi \in [0, +\infty)$ representing the degree of absolute risk aversion. Here, the greater the value of ϕ is, the more risk-averse the investors are. Note that $\phi = 0$ means that the investor does not consider risk, and ϕ approaching infinity means that the investor will allocate all the money to risk-less securities.

Note that in uncertain environment, $E[\xi_1 x_1 + \xi_2 x_2 \cdots + \xi_n x_n] \neq x_1 E[\xi_1] + x_2 E[\xi_2] + \cdots + x_n E[\xi_n]$ for uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ of different types. However, the inequality will become equality when $\xi_1, \xi_2, \cdots, \xi_n$ are independent. In particular, if security returns are all linear uncertain variables, denoting the return of security i by $\xi_i = \mathcal{L}(a_i, b_i)$, it follows that the portfolio return

$$\sum_{i=1}^{n} \xi_i x_i = \mathcal{L}\left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i\right)$$

is also a linear uncertain variable. Based on this, model (5.8) is equivalent to the crisp model,

$$\begin{cases} \min \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{n} (b_i - a_i)(b_j - a_j) x_i x_j \\ \text{s.t. } \displaystyle\sum_{i=1}^{n} (a_i + b_i) x_i \geq 2r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{5.10}$$

Further, we assume that security returns are all zigzag uncertain variables, denoting the return of security i by $\xi_i = \mathcal{Z}(a_i, b_i, c_i)$. It follows that the portfolio return

$$\sum_{i=1}^{n} \xi_i x_i = \mathcal{Z}\left(\sum_{i=1}^{n} a_i x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n} c_i x_i\right)$$

is also a zigzag uncertain variable. For simplicity, we write $\alpha_i = c_i - a_i$ and $\beta_i = 2b_i - a_i - c_i$ for $i = 1, 2, \cdots, n$. Then the model (5.8) is equivalent to the crisp model,

$$\begin{cases} \min \dfrac{\sum_{i=1}^{n} \sum_{j=1}^{n} (4\alpha_i \alpha_j + \beta_i \beta_j) x_i x_j + \left| 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j x_i x_j \right|}{\sum_{i=1}^{n} \alpha_i x_i + \left| \sum_{j=1}^{n} \beta_j x_j \right|} \\ \text{s.t. } \sum_{i=1}^{n} (4b_i - \beta_i) x_i \geq 4r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{5.11}$$

Assume that security returns are all normal uncertain variables, denoting the return of security i by $\xi_i = \mathcal{N}(e_i, \sigma_i)$. It follows that the portfolio return

$$\sum_{i=1}^{n} \xi_i x_i = \mathcal{N}\left(\sum_{i=1}^{n} e_i x_i, \sum_{i=1}^{n} \sigma_i x_i\right)$$

is also a normal uncertain variable. Then the model (5.8) is equivalent to the crisp model,

$$\begin{cases} \min \sigma_1 x_1 + \sigma_2 x_2 + \cdots + \sigma_n x_n \\ \text{s.t.} \quad e_1 x_1 + e_2 x_2 + \cdots + e_n x_n \geq r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{5.12}$$

When the security returns are all linear or zigzag uncertain variables, both models (5.7) and (5.9) can be converted into crisp mathematical programming in a similar way.

Example 5.4. This numerical example is from Liu and Qin (2012) to show the application of uncertain mean-semiabsolute deviation model. Assume that an investor plans to invest his fund among 20 securities. It is worth pointing out that the experiment process may be repeated for more and/or less securities. Further, all the future returns of securities are assumed to be zigzag uncertain variables denoted by $\xi_i = \mathcal{Z}(a_i, b_i, c_i)$ for $i = 1, 2, \cdots, 20$. For the practical investment problem, the key is to accurately estimate the values of parameters a_i, b_i and c_i for $i = 1, 2, \cdots, 20$. The detailed method may be found in the chapter of Uncertain Statistics in Liu (2010). Once these parameters are obtained, we can apply uncertain semiabsolute deviation to construct an optimal portfolio for the investor according to his/her requirements. The assumed returns of securities are shown in Table 5.1.

Table 5.1 Zigzag uncertain returns of 20 securities

No.	Uncertain return	No.	Uncertain return
1	(−0.12, 0.05, 0.21)	11	(−0.16, −0.01, 0.14)
2	(−0.19, 0.02, 0.22)	12	(−0.13, 0.01, 0.19)
3	(−0.11, −0.01, 0.13)	13	(−0.14, 0.03, 0.21)
4	(−0.17, 0.02, 0.16)	14	(−0.18, 0.02, 0.16)
5	(−0.19, 0.03, 0.19)	15	(−0.14, 0.00, 0.18)
6	(−0.13, 0.01, 0.20)	16	(−0.17, 0.00, 0.33)
7	(−0.18, 0.01, 0.25)	17	(−0.24, 0.02, 0.45)
8	(−0.15, 0.05, 0.18)	18	(−0.14, 0.03, 0.21)
9	(−0.21, −0.01, 0.18)	19	(−0.12, 0.02, 0.14)
10	(−0.09, 0.02, 0.13)	20	(−0.20, 0.01, 0.24)

Table 5.2 Investment proportion (%) to 20 securities

r	SAD	$(x_1,\ x_3,\ x_8,\ x_{10},\ x_{17},\ x_{19})$
0.005	0.922	(0.00, 50.00, 0.00, 50.00, 0.00, 0.00)
0.010	0.922	(0.00, 50.00, 0.00, 50.00, 0.00, 0.00)
0.015	0.947	(0.00, 16.67, 0.00, 50.00, 0.00, 33.33)
0.020	0.982	(7.69, 0.00, 0.00, 50.00, 0.00, 42.31)
0.025	1.025	(23.08, 0.00, 0.00, 50.00, 0.00, 26.92)
0.030	1.068	(38.46, 0.00, 0.00, 50.00, 0.00, 11.54)
0.035	1.145	(50.00, 0.00, 10.00, 40.00, 0.00, 0.00)
0.040	1.324	(50.00, 0.00, 50.00, 0.00, 0.00, 0.00)
0.045	1.560	(50.00, 0.00, 33.33, 0.00, 16.67, 0.00)
0.050	1.803	(50.00, 0.00, 16.67, 0.00, 33.33, 0.00)
0.055	2.051	(50.00, 0.00, 0.00, 0.00, 50.00, 0.00)

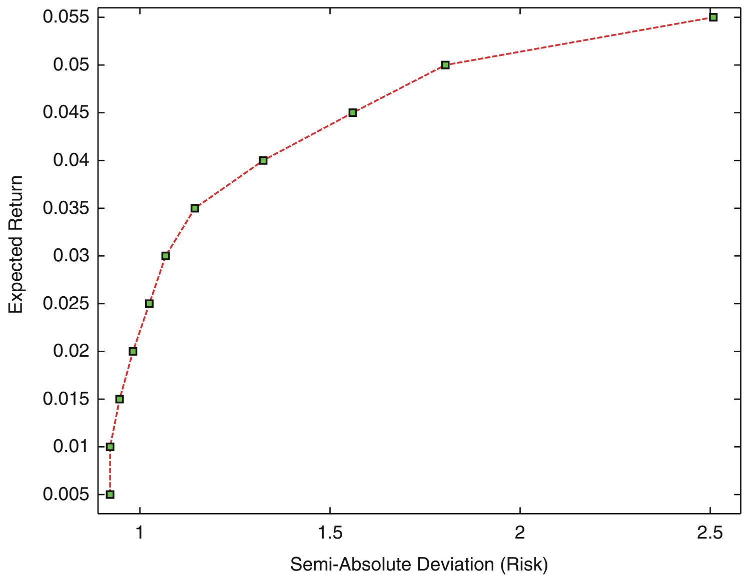

Fig. 5.1 Efficient frontier of mean semi-absolute deviation model (5.11)

Since all the security returns are zigzag uncertain variables, the proposed mean semi-absolute deviation models can be converted into equivalent crisp models. Here, we apply model (5.11) to determine the optimal portfolio and employ the function “fmincon” in MATLAB 7.1 to solve it. For given minimal return level r, we may obtain a series of optimal investment strategies. The computational results are shown in Table 5.2 in which the first column is the preset minimal return level and the second column SAD is semiabsolute deviation of the optimal portfolio. Further, the efficient frontier of model (5.11) is shown in Fig. 5.1.

5.4 Mean-Semiabsolute Deviation Adjusting Model

In this section, we consider portfolio adjusting problem which aims at the desirable portfolio by rebalancing the existing portfolio. Suppose that an investor has an existing portfolio $x^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ in which x_i^0 is the current holding of risk security i for $i = 1, 2, \cdots, n$. Due to the changes of situation in financial market, the investor decides to adjust his/her portfolio to maximizing the return and/or minimizing the risk.

Let $x^+ = (x_1^+, x_2^+, \cdots, x_n^+)$ and $x^- = (x_1^-, x_2^-, \cdots, x_n^-)$, where x_i^+ and x_i^- are respectively the proportion of the i-th security brought and sold by the investor. It is evident that x_i^+ and x_i^- are both nonnegative. After adjusting, the holding amount of the i-th risk security can be expressed as

$$x_i = x_i^0 + x_i^+ - x_i^-,\ i = 1, 2, \cdots, n.$$

Let b_i and s_i are respectively the unit transaction cost for purchasing and selling the risk security i. Without loss of generality, we assume that $b_i, s_i > 0$ for $i = 1, 2, \cdots, n$. Then the total transaction cost incurred by adjusting the existing portfolio is $\sum_{i=1}^n (b_i x_i^+ + s_i x_i^-)$. Let ξ_i be the future return of security i. Then the expected net return of the portfolio $x = (x_1, x_2, \cdots, x_n)$ after rebalancing is

$$r(x_1, x_2, \cdots, x_n) = \sum_{i=1}^n \xi_i x_i - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-). \tag{5.13}$$

Analogous to the framework of mean-variance model, the expected value of $r(x_1, x_2, \cdots, x_n)$ is regarded as the investment return and its semiabsolute deviation is regarded as the investment risk of the portfolio $(x_1, x_2, \cdots, x_n)$. By trading off return and risk, Qin et al. (2016) established the following uncertain mean-semiabsolute deviation adjusting model

$$\begin{cases} \min\ S_a\left[r(x_1, x_2, \cdots, x_n)\right] \\ \max\ E\left[r(x_1, x_2, \cdots, x_n)\right] \\ \text{s.t.}\quad x_i = x_i^0 + x_i^+ - x_i^-,\ i = 1, 2, \cdots, n \\ \qquad\ x_i, x_i^+, x_i^- \ge 0,\ i = 1, 2, \cdots, n. \end{cases} \tag{5.14}$$

Definition 5.2 (Qin et al. 2016). A feasible solution $(\hat{x}_1^+, \cdots, \hat{x}_n^+, \hat{x}_1^-, \cdots, \hat{x}_n^-)$ is said to be a Pareto optimal solution of model (5.14) if there is no feasible solution $(x_1^+, \cdots, x_n^+, x_1^-, \cdots, x_n^-)$ such that

$$E[r(\hat{x}_1^+, \cdots, \hat{x}_n^+, \hat{x}_1^-, \cdots, \hat{x}_n^-)] \le E[r(x_1^+, \cdots, x_n^+, x_1^-, \cdots, x_n^-)],$$
$$S_a[r(\hat{x}_1^+, \cdots, \hat{x}_n^+, \hat{x}_1^-, \cdots, \hat{x}_n^-)] \ge S_a[r(x_1^+, \cdots, x_n^+, x_1^-, \cdots, x_n^-)]$$

and at least one of these two inequalities strictly holds.

Theorem 5.4 (Qin et al. 2016). *Model (5.14) is equivalent to the following model,*

$$\begin{cases} \min\ S_a\left[\sum_{i=1}^n \xi_i x_i\right] \\ \max\ E\left[\sum_{i=1}^n \xi_i x_i\right] - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) \\ s.t.\ \ x_i = x_i^0 + x_i^+ - x_i^-,\ i = 1, 2, \cdots, n \\ \qquad x_i, x_i^+, x_i^- \ge 0,\ i = 1, 2, \cdots, n, \end{cases} \tag{5.15}$$

Proof. Since $\sum_{i=1}^n \xi_i x_i$ is also an uncertain variable, it immediately follows from Theorems 1.17 and 5.1 that the required result holds.

Theorem 5.5 (Qin et al. 2016). *Suppose that $(\hat{x}_1^+, \cdots, \hat{x}_n^+, \hat{x}_1^-, \cdots, \hat{x}_n^-)$ is the Pareto optimal solution of model (5.15). Then we have $\hat{x}_i^+ \cdot \hat{x}_i^- = 0$ for $i = 1, 2, \cdots, n$.*

Proof. Assume that there exists $k \in \{1, 2, \cdots, n\}$ such that $\hat{x}_k^+ > 0$ and $\hat{x}_k^- > 0$. Without loss of generality, it is assumed that $\hat{x}_k^+ > \hat{x}_k^-$. The optimal holding quantity of security i after adjusting is $\hat{x}_k = x_k^0 + \hat{x}_k^+ - \hat{x}_k^-$. We set $\tilde{x}_k^+ = \hat{x}_k^+ - \hat{x}_k^-$ and $\tilde{x}_k^- = 0$. It is evident that $\tilde{x}_k^+ \cdot \tilde{x}_k^- = 0$, $\tilde{x}_k^+, \tilde{x}_k^- \ge 0$ and $\tilde{x}_k = x_k^0 + \tilde{x}_k^+ - \tilde{x}_k^- = \hat{x}_k$ which implies that $(\hat{x}_1^+, \cdots, \hat{x}_{k-1}^+, \tilde{x}_k^+, \hat{x}_{k+1}^+, \cdots, \hat{x}_n^+, \hat{x}_1^-, \cdots, \hat{x}_{k-1}^-, \tilde{x}_k^-, \hat{x}_{k+1}^-, \cdots, \hat{x}_n^-)$ is a feasible solution of model (5.15). Note that

$$\begin{aligned} & r(\hat{x}_1, \cdots, \hat{x}_{k-1}, \tilde{x}_k, \hat{x}_{k+1}, \cdots, \hat{x}_n) - r(\hat{x}_1, \cdots, \hat{x}_{k-1}, \hat{x}_k, \hat{x}_{k+1}, \cdots, \hat{x}_n) \\ & = (b_k + s_k)\hat{x}_k^- > 0 \end{aligned}$$

which means that

$$E[r(\hat{x}_1, \cdots, \hat{x}_{k-1}, \tilde{x}_k, \hat{x}_{k+1}, \cdots, \hat{x}_n)] > E[r(\hat{x}_1, \cdots, \hat{x}_{k-1}, \hat{x}_k, \hat{x}_{k+1}, \cdots, \hat{x}_n)].$$

In addition, since $\tilde{x}_k = \hat{x}_k$, the return on the portfolio $(\hat{x}_1, \cdots, \hat{x}_{k-1}, \tilde{x}_k, \hat{x}_{k+1}, \cdots, \hat{x}_n)$ has the same semiabsolute deviation as that on the portfolio $(\hat{x}_1, \cdots, \hat{x}_{k-1}, \hat{x}_k, \hat{x}_{k+1}, \cdots, \hat{x}_n)$. Therefore, it is in contradiction to that $(\hat{x}_1^+, \cdots, \hat{x}_n^+, \hat{x}_1^-, \cdots, \hat{x}_n^-)$ is Pareto optimal. The theorem is proved.

If the investment is self-financing, i.e., no new fund is added and no fund is taken out of the existing portfolio, then we have

$$\sum_{i=1}^n x_i^0 - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^n x_i.$$

Let l_i and u_i be the lower bound and the upper bound of holding on risk security i after adjusting for $i = 1, 2, \cdots, n$. Then Qin et al. (2016) reformulated the uncertain mean-semiabsolute deviation model as follows,

$$\begin{cases} \min\ S_a\left[\sum_{i=1}^n \xi_i x_i\right] \\ \max\ E\left[\sum_{i=1}^n \xi_i x_i\right] - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) \\ \text{s.t.}\ \ \sum_{i=1}^n x_i^0 - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^n x_i \\ \qquad x_i = x_i^0 + x_i^+ - x_i^-,\ i = 1, 2, \cdots, n \\ \qquad x_i^+ \cdot x_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad x_i^+, x_i^- \geq 0,\ i = 1, 2, \cdots, n \\ \qquad l_i \leq x_i \leq u_i,\ i = 1, 2, \cdots, n. \end{cases} \tag{5.16}$$

If a risk-averse factor ϕ is converted into the model (5.16), then Qin et al. (2016) formulated a single-objective programming as follows,

$$\begin{cases} \max\ E\left[\sum_{i=1}^n \xi_i x_i\right] - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) - \phi S_a\left[\sum_{i=1}^n \xi_i x_i\right] \\ \text{s.t.}\ \ \sum_{i=1}^n x_i^0 - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^n x_i \\ \qquad x_i = x_i^0 + x_i^+ - x_i^-,\ i = 1, 2, \cdots, n \\ \qquad x_i^+ \cdot x_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad x_i^+, x_i^- \geq 0,\ i = 1, 2, \cdots, n \\ \qquad l_i \leq x_i \leq u_i,\ i = 1, 2, \cdots, n. \end{cases} \tag{5.17}$$

The greater the factor ϕ is, the more conservative the investor is.

In order to simplify uncertain mean-semiabsolute deviation adjusting models, we consider several special situations to deduce the equivalent crisp forms. Next assume that uncertain returns $\xi_1, \xi_2, \cdots, \xi_n$ are independent in the sense of uncertain measure, which implies that $E[\xi_1 x_1 + \xi_2 x_2 \cdots + \xi_n x_n] = x_1 E[\xi_1] + x_2 E[\xi_2] + \cdots + x_n E[\xi_n]$.

Theorem 5.6 (Qin et al. 2016). *Suppose that security returns $\xi_1, \xi_2, \cdots, \xi_n$ are all linear uncertain variables, denoted by $\xi_i = \mathcal{L}(c_i, d_i)$ for $i = 1, 2, \cdots, n$. Then model (5.16) is equivalent to the crisp model,*

$$\begin{cases} \min\ \sum_{i=1}^n (d_i - c_i) x_i \\ \max\ \sum_{i=1}^n (d_i + c_i) x_i - 2\sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) \\ s.t.\ \ \sum_{i=1}^n x_i^0 - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^n x_i \\ \qquad x_i = x_i^0 + x_i^+ - x_i^-,\ i = 1, 2, \cdots, n \\ \qquad x_i^+ \cdot x_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad x_i^+, x_i^- \geq 0,\ i = 1, 2, \cdots, n, \\ \qquad l_i \leq x_i \leq u_i,\ i = 1, 2, \cdots, n \end{cases} \tag{5.18}$$

which is a bi-objective linear programming model.

Proof. It follows from the operational law of uncertain variables that the portfolio return

$$\sum_{i=1}^{n} \xi_i x_i = \mathcal{L}\left(\sum_{i=1}^{n} c_i x_i, \sum_{i=1}^{n} d_i x_i\right)$$

is also a linear uncertain variable with expected value $\sum_{i=1}^{n}(d_i + c_i)x_i/2$. Further, we have

$$\begin{aligned} &E\left[\sum_{i=1}^{n} \xi_i x_i\right] - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n}(d_i + c_i)x_i - 2\sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-)\right). \end{aligned}$$

Therefore, the second objective is equivalent to maximize the term in parentheses on the right-hand side of the above equation. In addition, it follows from Example 5.1 that

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = \frac{1}{8}\sum_{i=1}^{n}(d_i - c_i)x_i.$$

Note that since $x_i \ge 0$ and $d_i > c_i$ for $i = 1, 2, \cdots, n$, we have $\sum_{i=1}^{n}(d_i - c_i)x_i \ge 0$ which implies that the first objective is equivalent to minimize it. The theorem is proved.

Theorem 5.7 (Qin et al. 2016). *Suppose that security returns $\xi_1, \xi_2, \cdots, \xi_n$ are all zigzag uncertain variables, denoted by*

$$\xi_i = \mathcal{Z}\left(b_i - \frac{\alpha_i + \beta_i}{2}, b_i, b_i + \frac{\alpha_i - \beta_i}{2}\right)$$

with $\alpha_i > \beta_i$ and $\alpha_i > 0$ for $i = 1, 2, \cdots, n$. Then model (5.16) is equivalent to the following crisp model,

$$\begin{cases} \min \ \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{n}(4\alpha_i\alpha_j + \beta_i\beta_j)x_i x_j + \left|2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\beta_j x_i x_j\right|}{\sum_{i=1}^{n}\alpha_i x_i + \left|\sum_{i=1}^{n}\beta_i x_i\right|} \\ \max \ \sum_{i=1}^{n}(4b_i - \beta_i)x_i - 4\sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-) \\ \text{s.t.} \ \ \sum_{i=1}^{n} x_i^0 - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^{n} x_i \\ \qquad x_i = x_i^0 + x_i^+ - x_i^-, \ i = 1, 2, \cdots, n \\ \qquad x_i^+ \cdot x_i^- = 0, \ i = 1, 2, \cdots, n \\ \qquad x_i^+, x_i^- \ge 0, \ i = 1, 2, \cdots, n, \\ \qquad l_i \le x_i \le u_i, \ i = 1, 2, \cdots, n. \end{cases} \tag{5.19}$$

Proof. It follows that the portfolio return

$$\sum_{i=1}^{n} \xi_i x_i = \mathcal{Z}\left(\sum_{i=1}^{n}(b_i - (\alpha_i + \beta_i)/2)x_i, \sum_{i=1}^{n} b_i x_i, \sum_{i=1}^{n}(b_i + (\alpha_i - \beta_i)/2)x_i\right)$$

is also a zigzag uncertain variable. According to the Definition 1.18, we have

$$E\left[\sum_{i=1}^{n} \xi_i x_i\right] = \frac{1}{4}\sum_{i=1}^{n} x_i(4b_i - \beta_i)$$

which implies that the second objective is equivalent to maximize $\sum_{i=1}^{n} x_i(4b_i - \beta_i) - 4\sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-)$. Further, by Definition 5.1, it is obtained that

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}(4\alpha_i\alpha_j + \beta_i\beta_j)x_i x_j + \left|2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\beta_j x_i x_j\right|}{\sum_{i=1}^{n}\alpha_i x_i + \left|\sum_{i=1}^{n}\beta_i x_i\right|}.$$

Substituting the semiabsolute deviation of the portfolio return into the first objective, the theorem is proved.

Corollary 5.1. *If $\beta_i < 0$ for $i = 1, 2, \cdots, n$, then we have*

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}(2\alpha_i - \beta_i)(2\alpha_j - \beta_j)x_i x_j}{\sum_{i=1}^{n}(\alpha_i - \beta_i)x_i}.$$

If $\beta_i = 0$ for $i = 1, 2, \cdots, n$, then we have

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = 4\sum_{i=1}^{n}\alpha_i x_i.$$

If $\beta_i > 0$ for $i = 1, 2, \cdots, n$, then we have

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}(2\alpha_i + \beta_i)(2\alpha_j + \beta_j)x_i x_j}{\sum_{i=1}^{n}(\alpha_i + \beta_i)x_i}.$$

In these special situations, the first objective function of model (5.19) has a simpler expression.

Theorem 5.8 (Qin et al. 2016). *Suppose that security returns $\xi_1, \xi_2, \cdots, \xi_n$ are normal uncertain variables, denoted by $\xi_i = \mathcal{N}(e_i, \sigma_i)$ for $i = 1, 2, \cdots, n$. Then model (5.16) is equivalent to the following crisp model,*

$$\begin{cases} \min \ \sum_{i=1}^{n} \sigma_i x_i \\ \max \ \sum_{i=1}^{n} e_i x_i - \sum_{i=1}^{n} (b_i x_i^+ + s_i x_i^-) \\ s.t. \quad \sum_{i=1}^{n} x_i^0 - \sum_{i=1}^{n} (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^{n} x_i \\ \qquad x_i = x_i^0 + x_i^+ - x_i^-, \ i = 1, 2, \cdots, n \\ \qquad x_i^+ \cdot x_i^- = 0, \ i = 1, 2, \cdots, n \\ \qquad x_i^+, x_i^- \geq 0, \ i = 1, 2, \cdots, n \\ \qquad l_i \leq x_i \leq u_i, \ i = 1, 2, \cdots, n \end{cases} \tag{5.20}$$

which is also a bi-objective linear programming.

Proof. It follows from the operational law of normal uncertain variables that the portfolio return

$$\sum_{i=1}^{n} \xi_i x_i = \mathcal{N}\left(\sum_{i=1}^{n} x_i e_i, \sum_{i=1}^{n} x_i \sigma_i\right)$$

is also a normal uncertain variable. Further, it follows from Definitions 1.18 and 5.1 that

$$E\left[\sum_{i=1}^{n} \xi_i x_i\right] = \sum_{i=1}^{n} e_i x_i \geq 0,$$

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = \frac{\sqrt{3}\ln 2}{\pi} \sum_{i=1}^{n} \sigma_i x_i \geq 0$$

in which nonnegativity holds due to nonnegativity of x_i, e_i and σ_i for $i = 1, 2, \cdots, n$. Substituting them into the two objective functions in model (5.16), the theorem is proved.

Theorem 5.9 (Qin et al. 2016). *Suppose that security returns $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$. Denote by $\Phi_1^{-1}, \Phi_2^{-1}, \cdots, \Phi_n^{-1}$ the inverse uncertainty distributions of $\Phi_1, \Phi_2, \cdots, \Phi_n$. Then model (5.16) is equivalent to the following crisp model,*

$$\begin{cases} \min \ \displaystyle\int_{-\infty}^{\sum_{i=1}^{n} x_i \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha} \alpha(r)\mathrm{d}r \\ \max \ \sum_{i=1}^{n} x_i \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha - \sum_{i=1}^{n} (b_i x_i^+ + s_i x_i^-) \\ s.t. \quad \sum_{i=1}^{n} x_i^0 - \sum_{i=1}^{n} (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^{n} x_i \\ \qquad x_i = x_i^0 + x_i^+ - x_i^-, \ i = 1, 2, \cdots, n \\ \qquad x_i^+ \cdot x_i^- = 0, \ i = 1, 2, \cdots, n \\ \qquad x_i^+, x_i^- \geq 0, \ i = 1, 2, \cdots, n \\ \qquad l_i \leq x_i \leq u_i, \ i = 1, 2, \cdots, n \end{cases} \tag{5.21}$$

in which $\alpha(r)$ is just the root of the equation $x_1\Phi_1^{-1}(\alpha) + \cdots + x_n\Phi_n^{-1}(\alpha) = r$.

Proof. The second objective holds since $E[\xi_i] = \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha$ for $i = 1, 2, \cdots, n$. According to Definition 5.1, we have

$$S_a\left[\sum_{i=1}^{n} \xi_i x_i\right] = \int_0^{+\infty} \mathcal{M}\left\{\min\left\{\sum_{i=1}^{n} \xi_i x_i - \sum_{i=1}^{n} x_i \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha, 0\right\} \geq r\right\} \mathrm{d}r$$
$$= \int_{-\infty}^{\sum_{i=1}^{n} x_i \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha} \mathcal{M}\left\{\sum_{i=1}^{n} \xi_i x_i \leq r\right\} \mathrm{d}r.$$

Note that since $x_i \geq 0$ for $i = 1, 2, \cdots, n$, it follows from the operational law of uncertain variables that $\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = x_1 \Phi_1^{-1}(\alpha) + x_2 \Phi_2^{-1}(\alpha) + \cdots + x_n \Phi_n^{-1}(\alpha).$$

For any given r, the value of $\Psi(r) = \mathcal{M}\{\xi_1 x_1 + \cdots + \xi_n x_n \leq r\}$ is just the root of the equation $\Psi^{-1}(\alpha) = r$, i.e., $x_1 \Phi_1^{-1}(\alpha) + \cdots + x_n \Phi_n^{-1}(\alpha) = r$. Substituting it into the expression of $S_a[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$, the first objective function is obtained. The theorem is proved.

Similar to Theorems 5.6, 5.7, 5.8, and 5.9, we can also translate model (5.17) into a crisp one respectively when the security returns have the uncertainty distributions of same type.

5.5 Numerical Examples

In this section, we illustrate the application of uncertain mean-semiabsolute deviation model by two numerical examples in Qin et al. (2016).

Example 5.5. We consider a problem with 10 securities with zigzag uncertain returns $\xi_i = \mathcal{Z}(b_i - (\alpha_i + \beta_i)/2, b_i, b_i + (\alpha_i - \beta_i)/2)$ for $i = 1, 2, \cdots, 10$. The values of parameters b_i, α_i and β_i are shown in Table 5.3.

Assume that unit purchasing cost b_i is 0.01 and unit selling cost s_i is 0.02 for $i = 1, 2, \cdots, 10$. Further, assume that the holding quantity after adjusting is no

Table 5.3 Zigzag uncertain returns of 10 securities in Example 5.5

Security No.	Uncertain return			Security no.	Uncertain return		
	b_i	α_i	β_i		b_i	α_i	β_i
1	0.1	0.9	0.1	6	−0.1	0.8	−0.2
2	−0.2	1.0	0.2	7	0.1	1.7	0.1
3	−0.1	0.8	0	8	0.3	1.2	0.4
4	0.1	1.3	0.3	9	0.2	1.3	0.5
5	0.3	1.5	0.3	10	0.0	1.4	−0.2

Table 5.4 The optimal adjusting strategies for 10 securities for different values of ϕ

	$\phi = 0.2$			$\phi = 1.0$			$\phi = 3.0$		
No.	x_i^+	x_i^-	x_i	x_i^+	x_i^-	x_i	x_i^+	x_i^-	x_i
1	0.150	0	0.250	0.200	0	0.300	0.200	0	0.300
2	0	0.100	0	0	0.100	0	0	0.100	0
3	0	0.100	0	0	0.100	0	0	0.100	0
4	0	0.100	0	0	0.100	0	0	0.100	0
5	0.200	0	0.300	0.200	0	0.300	0.099	0	0.199
6	0	0.100	0	0	0.100	0	0.084	0	0.184
7	0	0.067	0.033	0	0.100	0	0	0.100	0
8	0.200	0	0.300	0.200	0	0.300	0.200	0	0.300
9	0	0	0.100	0	0.018	0.072	0	0.100	0
10	0	0.100	0	0	0.100	0	0	0.100	0

more than 0.3 and short selling is not allowed. That is to say, $l_i = 0$ and $u_i = 0.3$ for $i = 1, 2, \cdots, 10$. Assume that the investor's current existing portfolio before adjusting is

$$
\begin{aligned}
&(x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0, x_7^0, x_8^0, x_9^0, x_{10}^0)\\
&= (0.10, 0.10, 0.10, 0.10, 0.10, 0.10, 0.10, 0.10, 0.10, 0.10).
\end{aligned}
$$

The function "fmincon" in Matlab is employed to solve model (5.19), and the computational results are shown in Table 5.4 with different risk-averse factors ϕ.

Example 5.6. We consider another problem with 10 securities with uncertain returns of different types. The first five securities have zigzag uncertain returns denoted by $\xi_1 = \mathcal{Z}(-0.4, 0.1, 0.6)$, $\xi_2 = \mathcal{Z}(-0.8, -0.2, 0.4)$, $\xi_3 = \mathcal{Z}(-0.5, -0.1, 0.3)$, $\xi_4 = \mathcal{Z}(-0.6, 0.1, 0.8)$ and $\xi_5 = \mathcal{Z}(-0.6, 0.2, 1.0)$, respectively, and the other five securities have normal uncertain returns denoted by $\xi_6 = \mathcal{N}(-0.10, 0.05), \xi_7 = \mathcal{N}(0.10, 0.24), \xi_8 = \mathcal{N}(-0.05, 0.12), \xi_9 = \mathcal{N}(0.15, 0.15), \xi_{10} = \mathcal{N}(0.05, 0.16)$, respectively. First we deduce their inverse uncertainty distributions as follows,

$$
\begin{aligned}
\Phi_1^{-1}(\alpha) &= \alpha - 0.4\\
\Phi_2^{-1}(\alpha) &= 1.2\alpha - 0.8\\
\Phi_3^{-1}(\alpha) &= 0.8\alpha - 0.5\\
\Phi_4^{-1}(\alpha) &= 1.4\alpha - 0.6\\
\Phi_5^{-1}(\alpha) &= 1.6\alpha - 0.6\\
\Phi_6^{-1}(\alpha) &= -0.10 + 0.123 \ln(\alpha/(1-\alpha))\\
\Phi_7^{-1}(\alpha) &= 0.10 + 0.270 \ln(\alpha/(1-\alpha))\\
\Phi_8^{-1}(\alpha) &= -0.05 + 0.191 \ln(\alpha/(1-\alpha))
\end{aligned}
$$

$$\Phi_9^{-1}(\alpha) = 0.15 + 0.214\ln(\alpha/(1-\alpha))$$
$$\Phi_{10}^{-1}(\alpha) = 0.05 + 0.221\ln(\alpha/(1-\alpha))$$

Let $(x_1, x_2, \cdots, x_{10})$ be the portfolio after adjusting. Further, we have

$$\begin{aligned}
&x_1\Phi_1^{-1}(\alpha) + x_2\Phi_2^{-1}(\alpha) + \cdots + x_{10}\Phi_{10}^{-1}(\alpha)\\
&= (x_1 + 1.2x_2 + 0.8x_3 + 1.4x_4 + 1.6x_5)\alpha\\
&\quad +(0.123x_6 + 0.270x_7 + 0.191x_8 + 0.214x_9 + 0.221x_{10})\ln(\alpha/(1-\alpha))\\
&\quad -0.4x_1 - 0.8x_2 - 0.5x_3 - 0.6x_4 - 0.6x_5\\
&\quad -0.10x_6 + 0.10x_7 - 0.05x_8 + 0.15x_9 + 0.05x_{10}.
\end{aligned}$$

If the short selling is not allowed, then $x_1, \cdots, x_{10}$ are all nonnegative. Consequently, $x_1\Phi_1^{-1}(\alpha)+x_2\Phi_2^{-1}(\alpha)+\cdots+x_{10}\Phi_{10}^{-1}(\alpha)$ is an increasing function of α which implies that the equation

$$x_1\Phi_1^{-1}(\alpha) + x_2\Phi_2^{-1}(\alpha) + \cdots + x_{10}\Phi_{10}^{-1}(\alpha) = r$$

has only one root, denoted by $\alpha(r)$. The function "fzero" in Matlab can be used to find the root of the above equation.

Assume that the investor's current holding on security i is $x_i^0 = 0.10$, and $b_i = 0.01, s_i = 0.02$ for $i = 1, 2, \cdots, 10$. In addition, we assume that the holding quantity is no more than 0.3 after adjusting. If the investor wishes to minimize the semiabsolute deviation (risk) when the expected return is no less than the return level r_0, then the corresponding uncertain mean-semiabsolute deviation adjusting model is reformulated as follows,

$$\begin{cases}
\min \displaystyle\int_{-\infty}^{\tau} \alpha(r)\mathrm{d}r\\
\text{s.t.}\ \ 0.025 + 0.1x_1^+ - 0.2x_2^+ - 0.1x_3^+ + 0.1x_4^+ + 0.2x_5^+\\
\qquad -0.1x_6^+ + 0.1x_7^+ - 0.05x_8^+ + 0.15x_9^+ + 0.05x_{10}^+\\
\qquad -0.1x_1^- + 0.2x_2^- + 0.1x_3^- - 0.1x_4^- - 0.2x_5^-\\
\qquad +0.1x_6^- - 0.1x_7^- + 0.05x_8^- - 0.15x_9^- - 0.05x_{10}^- = \tau\\
\qquad 0.025 + 0.09x_1^+ - 0.21x_2^+ - 0.11x_3^+ + 0.09x_4^+ + 0.19x_5^+\\
\qquad -0.11x_6^+ + 0.09x_7^+ - 0.06x_8^+ + 0.14x_9^+ + 0.04x_{10}^+\\
\qquad -0.12x_1^- + 0.18x_2^- + 0.08x_3^- - 0.12x_4^- - 0.22x_5^-\\
\qquad +0.08x_6^- - 0.12x_7^- + 0.03x_8^- - 0.17x_9^- - 0.07x_{10}^- \ge r_0\\
\qquad \sum_{i=1}^{10} x_i^+ - 0.9703\sum_{i=1}^{10} x_i^- = 0\\
\qquad x_i^+ \cdot x_i^- = 0,\ i = 1, 2, \cdots, 10\\
\qquad 0 \le x_i^+ \le 0.2,\ i = 1, 2, \cdots, 10\\
\qquad 0 \le x_i^- \le 0.1,\ i = 1, 2, \cdots, 10
\end{cases} \tag{5.22}$$

Table 5.5 The optimal adjusting strategies for 10 securities for different return levels r_0

No.	$r_0 = 0.04$			$r_0 = 0.08$			$r_0 = 0.12$		
	x_i^+	x_i^-	x_i	x_i^+	x_i^-	x_i	x_i^+	x_i^-	x_i
1	0.200	0	0.300	0.200	0	0.300	0.200	0	0.300
2	0	0.100	0	0	0.100	0	0	0.100	0
3	0	0.090	0.010	0	0.100	0	0	0.100	0
4	0	0.100	0	0	0.100	0	0	0.059	0.041
5	0	0.028	0.072	0.100	0	0.200	0.200	0	0.300
6	0.200	0	0.300	0.080	0	0.180	0	0.082	0.018
7	0	0.100	0	0	0.100	0	0	0.100	0
8	0	0.100	0	0	0.100	0	0	0.100	0.300
9	0.200	0	0.300	0.200	0	0.300	0.200	0.100	0
10	0	0.100	0	0	0.100	0	0	0.077	0.023
SAD			0.1254			0.1404			0.1566

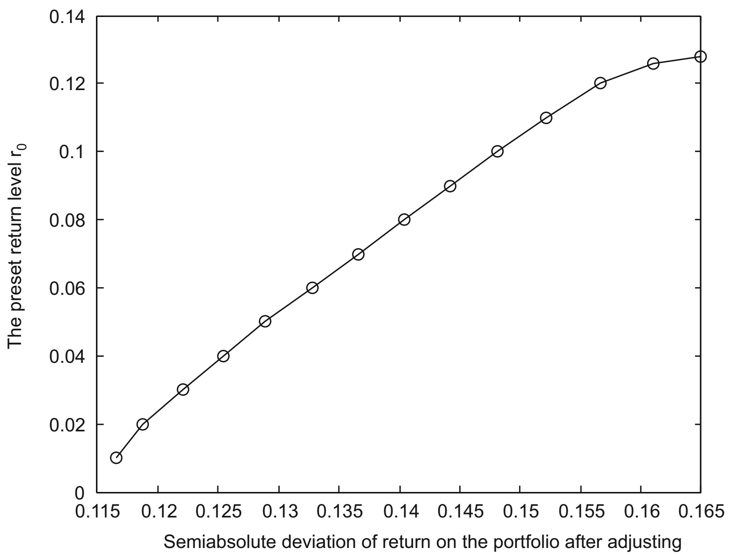

Fig. 5.2 The efficient frontier in Example 5.6

where $\alpha(r)$ is the root of

$$
\begin{aligned}
&\big(0.6 + x_1^+ + 1.2x_2^+ + 0.8x_3^+ + 1.4x_4^+ + 1.6x_5^+ \\
&\quad -x_1^- - 1.2x_2^- - 0.8x_3^- - 1.4x_4^- - 1.6x_5^-)\alpha \\
&+(0.1019 + 0.123x_6^+ + 0.270x_7^+ + 0.191x_8^+ + 0.214x_9^+ + 0.221x_{10}^+ \\
&\quad -0.123x_6^- - 0.270x_7^- - 0.191x_8^- - 0.214x_9^- - 0.221x_{10}^-) \ln(\alpha/(1-\alpha))
\end{aligned}
$$

$$
\begin{aligned}
&-0.275 - 0.4x_1^+ - 0.8x_2^+ - 0.5x_3^+ - 0.6x_4^+ - 0.6x_5^+ \\
&-0.10x_6^+ + 0.10x_7^+ - 0.05x_8^+ + 0.15x_9^+ + 0.05x_{10}^+ \\
&+0.4x_1^- + 0.8x_2^- + 0.5x_3^- + 0.6x_4^- + 0.6x_5^- \\
&+0.10x_6^- - 0.10x_7^- + 0.05x_8^- - 0.15x_9^- - 0.05x_{10}^- = r
\end{aligned}
$$

in which $r \leq \tau$.

The function "fmincon" in Matlab is again employed to solve the above model, in which the objective is calculated based on numerical integral and the function "fzero" is called. The obtained results are shown in Table 5.5. The last row shows the corresponding semiabsolute deviations of optimal portfolios. It can be observed that the semiabsolute deviation will increase with the desired return levels r_0 increasing.

To see intuitively this point, the efficient frontier is shown in Fig. 5.2 in which the vertical axis is the return level and the horizontal axis is the corresponding minimum semiabsolute deviation.

Chapter 6
Uncertain Mean-LPMs Model

6.1 Introduction

Downside risk is a class of risk measures which focuses on the asymmetry of returns about some target level of return (Harlow 1991). It has gradually attracted more and more attentions since investors are often sensitive to downside losses, relative to upside gains. Moreover, it requires simpler theoretical assumptions to justify its application. In portfolio management, investors always prefer securities with smaller downside risk. In the situation with symmetrically distributed returns, some downside risks are consistent with general risk measures. For example, semivariance is exactly proportional to variance for normal distribution, which implies they are equivalent in measuring risk.

Shortfall probability is the earliest downside risk measure initialized by Roy (1952) based on the safety-first criterion. In fact, in order to handle asymmetrical returns, Markowitz (1959) also proposed semivariance to replace variance as a risk measure to extend his mean-variance analysis. Afterwards, Speranza (1993) introduced semiabsolute deviation as a simpler downside risk measure in the literature. These are commonly used downside risk measures based on lower partial moments (LPMs), which are of interest in portfolio optimization. LPMs only consider the tail of the relevant distribution of returns below some specific threshold level or target rate (Harlow 1991). The earliest rationale was provided by Bawa (1975) for using LPMs to measure portfolio risk. Then LPMs were further applied to study capital asset pricing model by Bawa and Lindenberg (1977) and to measure systematic risk by Price et al. (1982) when return distributions are lognormal. Since then, LPMs have been widely employed as risk measures especially in the asymmetrical cases. For example, Zhu et al. (2009) presented worst-case LPMs and applied them to study portfolio optimization problem with parameter uncertainty.

In above works, the security returns are described by random variables and dealt with in the framework of probability theory. However, in some situations, there are no sufficient data to accurately estimate the stochastic parameters. An alternative

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way is to estimate these parameters by experts' evaluations and information. Thus, the parameters involve subjective impression instead of objective uncertainty. In order to describe this case, we may use uncertainty theory provided by Liu (2007) to handle the uncertainty associated with security returns. As an attempt of downside risk measure, Liu and Qin (2012) defined semiabsolute deviation of uncertain return and Qin et al. (2016) extended it to portfolio adjusting problem. Afterwards, Qin and Yao (2014) comprehensively studied LPMs of uncertain return in the framework of uncertainty theory and applied them to model portfolio optimization.

This chapter focuses on uncertain mean-LPMs model for portfolio optimization. The main contents of this chapter include the definition of LPMs of uncertain variable, formulation and equivalents of uncertain mean-LPMs model, and numerical examples.

6.2 Lower Partial Moments of Uncertain Variable

In this section, we review the LPMs of uncertain variable and their mathematical properties.

Definition 6.1 (Qin and Yao 2014). Let ξ be an uncertain variable and τ a chosen reference level. Then its LPMs of degree n are defined as

$$LPM_n^\tau(\xi) = E\left[((\tau - \xi) \vee 0)^n\right]. \tag{6.1}$$

In finance, LPMs can be considered as a family of risk measures specified by n and τ. In general, τ is often set at the risk-free rate or the expected value. The value of n represents the risk aversion since larger n will mean to penalize large deviations more than lower values. The semiabsolute deviation (Li et al. 2012) and semivariance are two special cases of LPMs for which the degree of the moment are set to 1 and 2, respectively, and the reference level is set as the expected value.

Theorem 6.1 (Qin and Yao 2014). *Assume that ξ is an uncertain variable with an uncertainty distribution $\Phi(x)$. Then its LPM of degree n with reference level τ is*

$$LPM_n^\tau(\xi) = n\int_{-\infty}^{\tau} (\tau - r)^{n-1}\Phi(r)\mathrm{d}r. \tag{6.2}$$

Proof. Let $\Psi(x)$ denote the uncertainty distribution of the uncertain variable $((\tau - \xi) \vee 0)^n$. Then for any $x < 0$, we have

$$\Psi(x) = \mathcal{M}\{((\tau - \xi) \vee 0)^n \le x\} \le \mathcal{M}\{0 \le x\} = 0.$$

For any $x \ge 0$, we have

$$\begin{aligned}\Psi(x) &= \mathcal{M}\{((\tau - \xi) \vee 0)^n \le x\}\\ &= \mathcal{M}\left\{(\tau - \xi) \vee 0 \le x^{1/n}\right\}\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{M}\left\{\tau - \xi \le x^{1/n}\right\} \\
&= \mathcal{M}\left\{\xi \ge \tau - x^{1/n}\right\} \\
&= 1 - \Phi\left(\tau - x^{1/n}\right).
\end{aligned}
$$

In other words, the uncertain variable $((\tau - \xi) \vee 0)^n$ has an uncertainty distribution

$$
\Psi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - \Phi\left(\tau - x^{1/n}\right), & \text{if } x \ge 0. \end{cases}
$$

It follows from Theorem 1.15 that the LPMs of ξ with degree n is

$$
\begin{aligned}
LPM_n^\tau(\xi) &= \int_0^{+\infty} (1 - \Psi(x))\,\mathrm{d}x - \int_{-\infty}^0 \Psi(x)\mathrm{d}x \\
&= \int_0^{+\infty} \Phi\left(\tau - x^{1/n}\right)\mathrm{d}x.
\end{aligned}
$$

Substituting $\tau - x^{1/n}$ with r, it is obtained that

$$
LPM_n^\tau(\xi) = n\int_{-\infty}^{\tau} (\tau - r)^{n-1}\Phi(r)\mathrm{d}r.
$$

The theorem is proved.

Theorem 6.1 means that the LPMs of an uncertain variable can be easily calculated if its uncertainty distribution is known. Next we present two commonly used examples in uncertainty theory.

Example 6.1. Let $\xi = \mathcal{L}(a, b)$ be a linear uncertain variable. Then its LPM of degree n with preference level τ is

$$
LPM_n^\tau(\xi) = \begin{cases} 0, & \text{if } \tau \le a \\ \dfrac{(\tau - a)^{n+1}}{(n+1)(b-a)}, & \text{if } a \le \tau \le b \\ \dfrac{(b-a)^n}{n+1} + (\tau - b)^n, & \text{if } \tau \ge b. \end{cases} \tag{6.3}
$$

Especially, if the preference level $\tau = E[\xi] = (a + b)/2$, then we have

$$
LPM_n^{E[\xi]}(\xi) = \frac{(b-a)^n}{2^{n+1}(n+1)}
$$

which implies that the semiabsolute deviation is $(b - a)/8$ and the semivariance is $(b - a)^2/24$.

Example 6.2. Let $\mathcal{Z}(a, b, c)$ be a zigzag uncertain variable. Then its LPM of degree n with preference level τ is

$$LPM_n^\tau(\xi) = \begin{cases} 0, & \text{if } \tau \le a \\ \dfrac{(\tau-a)^{n+1}}{2(n+1)(b-a)}, & \text{if } a \le \tau \le b \\ \dfrac{(b-a)^n}{2(n+1)} + \dfrac{(\tau-b)^{n+1}}{2(n+1)(c-b)} + \dfrac{(\tau-b)^n}{2}, & \text{if } b \le \tau \le c \\ \dfrac{(b-a)^n + (n+2)(c-b)^n}{2(n+1)} + (\tau-c)^n, & \text{if } \tau \ge c. \end{cases} \tag{6.4}$$

It follows from Definition 1.18 that the expected value of ξ is $E[\xi] = (a+2b+c)/4$. If $b-a \ge c-b$, then $E[\xi] \le b$ and we have

$$LPM_n^{E[\xi]}(\xi) = \frac{[3(b-a)+(c-b)]^{n+1}}{2^{2n+3}(n+1)(b-a)}$$

which implies that

$$S_a[\xi] = LPM_1^{E[\xi]}(\xi) = \frac{[3(b-a)+(c-b)]^2}{64(b-a)}$$

$$S_v[\xi] = LPM_2^{E[\xi]}(\xi) = \frac{[3(b-a)+(c-b)]^3}{384(b-a)}.$$

On the other hand, if $b-a \le c-b$, then $E[\xi] \ge b$ and we have

$$LPM_n^{E[\xi]}(\xi) = \frac{(b-a)^n}{2(n+1)} + \frac{[(c-b)-(b-a)]^n[(4n+5)(c-b)-(b-a)]}{2^{2n+3}(n+1)(c-b)}$$

which implies that

$$S_a[\xi] = LPM_1^{E[\xi]}(\xi) = \frac{[(b-a)+3(c-b)]^2}{64(b-a)}.$$

Theorem 6.2 (Qin and Yao 2014). *Let ξ be an uncertain variable with a regular uncertainty distribution Φ. Then its LPM of degree n with reference level τ are*

$$LPM_n^\tau(\xi) = \int_{1-\Phi(\tau)}^{1} \left(\tau - \Phi^{-1}(\alpha)\right)^n \mathrm{d}\alpha.$$

Proof. Since $f(x) = (\max\{\tau - x, 0\})^n$ is a decreasing function with respect to x, it follows from Theorem 1.16 that the uncertain variable $((\tau-\xi)\vee 0)^n$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \left(\max\{\tau - \Phi^{-1}(1-\alpha), 0\}\right)^n.$$

Further, according to Theorem 1.16, we have

$$\begin{aligned} E[((\tau - \xi) \vee 0\})^n] &= \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha \\ &= \int_0^1 \left(\max\{\tau - \Phi^{-1}(1-\alpha), 0\}\right)^n \mathrm{d}\alpha \\ &= \int_{1-\Phi(\tau)}^1 \left(\tau - \Phi^{-1}(\alpha)\right)^n \mathrm{d}\alpha. \end{aligned}$$

The theorem is proved.

Theorem 6.3 (Qin and Yao 2014). *Let $\xi_1, \xi_2, \cdots, \xi_l$ be independent uncertain variables with regular uncertainty distributions Φ_1, Φ_2, $\cdots$, Φ_l, respectively, and $h(x_1, x_2, \cdots, x_l)$ be a strictly monotone function which is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+1}, \cdots, x_l$. Then we have*

$$\begin{aligned} &LPM_n^\tau(h(\xi_1, \xi_2, \cdots, \xi_l)) \\ &= \int_0^1 \Big(\max\Big\{\tau - h\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha),\right. \\ &\qquad \left.\Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_l^{-1}(1-\alpha)\right), 0\Big\}\Big)^n \mathrm{d}\alpha. \end{aligned} \tag{6.5}$$

Proof. By the operational law of uncertain variables, the uncertain variable $h(\xi_1, \xi_2, \cdots, \xi_l)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = h\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_l^{-1}(1-\alpha)\right).$$

Then it follows from Theorem 6.2 that

$$\begin{aligned} &LPM_n^\tau(h(\xi_1, \xi_2, \cdots, \xi_l)) \\ &= \int_0^1 \left(\max\{\tau - \Psi^{-1}(\alpha), 0\}\right)^n \mathrm{d}\alpha \\ &= \int_0^1 \Big(\max\Big\{\tau - h\left(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha),\right. \\ &\qquad \left.\Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_l^{-1}(1-\alpha)\right), 0\Big\}\Big)^n \mathrm{d}\alpha. \end{aligned}$$

The theorem is proved.

Theorem 6.4 (Qin and Yao 2014). *Let ξ be an uncertain variable with a finite expected value. Then we have*

$$0 \leq LPM_n^\tau(\xi) \leq E[|\xi - \tau|^n].$$

Proof. First note that the identity

$$|\xi - \tau| = (\tau - \xi) \vee 0 + (\xi - \tau) \vee 0.$$

Due to the nonnegativity of $(\tau - \xi) \vee 0$ and $(\xi - \tau) \vee 0$, we have

$$0 \leq ((\xi - \tau) \vee 0)^n \leq |\xi - \tau|^n.$$

Taking expected values on both sides, it immediately follows from Definition 6.1 that the required result will be obtained.

Theorem 6.5 (Qin and Yao 2014). *Let ξ be an uncertain variable with continuous uncertainty distribution $\Phi(x)$. Then $LPM_n^\tau(\xi) = 0$ if and only if $\Phi(\tau) = 0$.*

Proof. If $LPM_n^\tau(\xi) = 0$, then by Definition 6.1, we have

$$E[((\xi - \tau) \vee 0)^n] = \int_0^{+\infty} \mathcal{M}\{((\xi - \tau) \vee 0)^n \geq r\}\mathrm{d}r = 0$$

which implies that $\mathcal{M}\{((\xi - \tau) \vee 0)^n \geq r\} = 0$ for any $r > 0$. That is to say, $\mathcal{M}\{((\xi - \tau) \vee 0)^n = 0\} = 1$, i.e., $\mathcal{M}\{\xi \geq \tau\} = 1$. According to the duality of uncertain measure, we have $\Phi(\tau) = 1 - \mathcal{M}\{\xi \geq \tau\} = 0$. Conversely, if $\Phi(\tau) = 0$, then it follows from Eq. (6.2) that $LPM_n^\tau(\xi) = 0$ since $\Phi(x) \leq \Phi(\tau) = 0$ for any $x \leq \tau$. The theorem is proved.

Theorem 6.6 (Qin and Yao 2014). *Let ξ be an uncertain variable with a finite expected value e. Then $LPM_n^e(\xi) = 0$ if and only if $\mathcal{M}\{\xi = e\} = 1$.*

Proof. If $LPM_n^e(\xi) = 0$, then it follows from the proof of Theorem 6.5 that

$$\mathcal{M}\{(e - \xi) \vee 0 = 0\} = \mathcal{M}\{\xi \geq e\} = 1$$

which implies that

$$\xi - e = (\xi - e) \vee 0 - (e - \xi) \vee 0 = (\xi - e) \vee 0$$

almost everywhere. Further, we have

$$\int_0^{+\infty} \mathcal{M}\{(\xi - e) \vee 0 \geq r\}\mathrm{d}r = E[(\xi - e) \vee 0] = E[\xi - e] = 0$$

which indicates that $\mathcal{M}\{(\xi - e) \vee 0 \geq r\} = 0$ for any $r > 0$. By the self-duality of uncertainty measure, we have $\mathcal{M}\{(\xi - e) \vee 0 = 0\} = 1$, i.e., $\mathcal{M}\{\xi \leq e\} = 1$. Thus,

$\mathcal{M}\{\xi = e\} = 1$ holds. Conversely, if $\mathcal{M}\{\xi = e\} = 1$, then we have $\mathcal{M}\{|\xi - e|^n = 0\} = 1$ and $\mathcal{M}\{|\xi - e|^n \geq r\} = 0$ for any real number $r > 0$, which implies that $E[|\xi - e|^n] = 0$. It follows from Theorem 6.4 that $LPM_n^e(\xi) = 0$. The theorem is proved.

6.3 Uncertain Mean-LPMs Model

In this section, Mean-LPMs models are formulated for portfolio optimization problem by considering LPMs as risk measures. These models can be used to deal with the case with asymmetric uncertain returns.

Assume that ξ_i and x_i are the future return on security i and the proportion of total number of funds invested in it, respectively, for $i = 1, 2, \cdots, m$. Assume that $\xi_1, \xi_2, \cdots, \xi_m$ are uncertain variables defined on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. The total return on portfolio $(x_1, x_2, \cdots, x_m)$ is $\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m$ which is also an uncertain variable. Let r be the acceptable return level by the investor. If he/she accepts LPMs as risk measures, then Qin and Yao (2014) formulated the following uncertain mean-LPMs portfolio optimization model

$$\begin{cases} \min \ LPM_n^\tau(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m) \\ \text{s.t.} \quad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m] \geq r \\ \qquad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, m. \end{cases} \tag{6.6}$$

The constraints ensure that all the capital will be invested to m securities and short sales are not allowed.

If an investor wishes to maximize the investment return with a tolerable risk level d, then the corresponding uncertain mean-LPMs portfolio optimization model is formulated by Qin and Yao (2014) as follows,

$$\begin{cases} \max \ E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m] \\ \text{s.t.} \quad LPM_n^\tau(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m) \leq d \\ \qquad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, m, \end{cases} \tag{6.7}$$

where d denotes the maximum risk level the investor can bear.

For the risk-averse investors, Qin and Yao (2014) established the following uncertain mean-LPMs models for portfolio optimization,

$$\begin{cases} \max \ E[\xi_1 x_1 + \cdots + \xi_m x_m] - \phi \cdot LPM_n^\tau(\xi_1 x_1 + \cdots + \xi_m x_m) \\ \text{s.t.} \quad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, m, \end{cases} \tag{6.8}$$

where $\phi \in [0, +\infty)$ representing the degree of absolute risk aversion. In general, the greater the value of ϕ is, the more risk-averse the investors are. In particular, $\phi = 0$ indicates that the investor does not consider risk, and ϕ approaching infinity implies that the investor will allocate all the money to risk-less securities.

Remark 6.1. Each of models (6.6), (6.7), and (6.8) is actually a class of uncertain portfolio optimization models. For example, the solution to model (6.6) will change with different n and τ. If the expected value is taken as the reference level τ, then model (6.6) will become uncertain mean-semiabsolute deviation model proposed in Chap. 5 when $n = 1$ or become mean-semivariance model when $n = 2$.

Theorem 6.7 (Qin and Yao 2014). *Assume that uncertain returns $\xi_1, \xi_2, \cdots, \xi_m$ have regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_m$, respectively. If they are mutually independent in the sense of uncertain measure, then model (6.6) can be transformed into the following form,*

$$\begin{cases} \min \displaystyle\int_0^1 \left(\max\{\tau - (x_1\Phi_1^{-1}(\alpha) + \cdots + x_m\Phi_m^{-1}(\alpha)), 0\}\right)^n \mathrm{d}\alpha \\ \text{s.t. } \ x_1 \displaystyle\int_0^1 \Phi_1^{-1}(\alpha)\mathrm{d}\alpha + \cdots + x_m \displaystyle\int_0^1 \Phi_m^{-1}(\alpha)\mathrm{d}\alpha \geq r \\ \qquad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, m. \end{cases} \tag{6.9}$$

Proof. Since $\xi_1, \xi_2, \cdots, \xi_m$ are mutually independent, it follows from Theorem 1.18 that

$$\begin{aligned} & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m] \\ &= \int_0^1 (x_1\Phi_1^{-1}(\alpha) + \cdots + x_m\Phi_m^{-1}(\alpha))\mathrm{d}\alpha \\ &= x_1 \int_0^1 \Phi_1^{-1}(\alpha)\mathrm{d}\alpha + \cdots + x_m \int_0^1 \Phi_m^{-1}(\alpha)\mathrm{d}\alpha. \end{aligned}$$

For any portfolio $(x_1, x_2, \cdots, x_m)$ with $\sum_{i=1}^m x_i = 1$ and $x_i \geq 0$ for $i = 1, 2, \cdots, m$, without loss of generality, we assume that $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$ are all positive and the others are zeros, in which $1 \leq k \leq m$. Then function $\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m$ is obviously a strictly increasing function with respect to $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_k}$. It follows from Theorem 6.3 that

$$\begin{aligned} & LPM_n^\tau(\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_m x_m) \\ &= \int_0^1 \left(\max\{\tau - (x_1\Phi_1^{-1}(\alpha) + \cdots + x_m\Phi_m^{-1}(\alpha)), 0\}\right)^n \mathrm{d}\alpha. \end{aligned}$$

The theorem is proved.

Remark 6.2. Model (6.9) is a crisp mathematical programming model. That is to say, Theorem 6.7 implies that model (6.6) can be converted into a crisp one as long as that the uncertainty distribution of each security return is provided. In the practical application, uncertainty distribution can be estimated by the experts' data according to uncertain statistics (Liu 2010). In addition, models (6.7) and (6.8) are also converted into equivalent crisp forms based on the same conditions of Theorem 6.7.

Corollary 6.1. *Assume that security returns $\xi_i = \mathcal{L}(a_i, b_i)$ are linear uncertain variables for $i = 1, 2, \cdots, m$. If they are independent in the sense of uncertain measure, then model (6.6) is equivalent to the crisp model,*

$$\begin{cases} \min \; \dfrac{(\tau-a)^{n+1}}{(n+1)(b-a)} I_{\{a\le\tau\le b\}} + \left[\dfrac{(b-a)^n}{n+1} + (\tau-b)^n\right] I_{\{\tau\ge b\}} \\ \text{s.t.} \;\; (b_1+a_1)x_1 + \cdots + (b_m+a_m)x_m \ge 2r \\ \qquad a_1x_1 + a_2x_2 + \cdots + a_mx_m = a \\ \qquad b_1x_1 + b_2x_2 + \cdots + b_mx_m = b \\ \qquad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, m. \end{cases} \tag{6.10}$$

Remark 6.3. If the expected value is chosen as the reference level, i.e., $\tau = [(a_1 + b_1)x_1 + (a_2 + b_2)x_2 + \cdots + (a_m + b_m)x_m]/2$, then model (6.10) may be simplified as the following linear programming

$$\begin{cases} \min \sum_{i=1}^n (b_i - a_i)x_i \\ \text{s.t.} \;\; \sum_{i=1}^n (b_i + a_i)x_i \ge 2r \\ \qquad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, m. \end{cases}$$

Corollary 6.2. *Assume that security returns $\xi_i = \mathcal{Z}(a_i, b_i, c_i)$ are zigzag uncertain variables for $i = 1, 2, \cdots, m$. If they are independent in the sense of uncertain measure, then model (6.6) is equivalent to the following crisp model,*

$$\begin{cases} \min \; \dfrac{(\tau-a)^{n+1}}{2(n+1)(b-a)} I_{\{a\le\tau\le b\}} + \left[\dfrac{(\tau-b)^{n+1}}{2(n+1)(c-b)} + \dfrac{(\tau-b)^n}{2}\right] I_{\{b\le\tau\le c\}} \\ \qquad + \dfrac{(b-a)^n}{2(n+1)} I_{\{\tau\ge b\}} + \left[\dfrac{(n+2)(c-b)^n}{2(n+1)} + (\tau-c)^n\right] I_{\{\tau\ge c\}} \\ \text{s.t.} \;\; (a_1+2b_1+c_1)x_1 + \cdots + (a_m+2b_m+c_m)x_m \ge 4r \\ \qquad a_1x_1 + a_2x_2 + \cdots + a_mx_m = a \\ \qquad b_1x_1 + b_2x_2 + \cdots + b_mx_m = b \\ \qquad c_1x_1 + c_2x_2 + \cdots + c_mx_m = c \\ \qquad x_1 + x_2 + \cdots + x_m = 1 \\ \qquad x_i \ge 0, \quad i = 1, 2, \cdots, m. \end{cases} \tag{6.11}$$

Different from Corollaries 6.1 and 6.2, security returns may be of different types. For instance, some are zigzag uncertain variables and the others are normal ones. In this case, we have to employ model (6.9) to search for optimal portfolios. Moreover, the objective function (i.e. LPMs) may not be analytically obtained. Therefore, numerical integral technique is be adopted to approximately calculate the objective function.

6.4 Numerical Examples

In this section, we illustrate the application of uncertain mean-LPMs model by numerical experiments in Qin and Yao (2014).

Example 6.3. We consider a simple example with five securities whose returns are all zigzag and shown in Table 6.1. We take $\xi_1 = \mathcal{Z}(-0.2, 0.5, 0.9)$ as an example to give an explanation. Here, 0.5 represents the most likely return of first security, -0.2 and 0.9 are its smallest and largest returns, respectively. It follows Definition 1.18 where their expected values (i.e., means) and variances are obtained, which are also shown in Table 6.1. In addition, by Definition 6.1, we may calculate the LPMs for each security return and the results are listed in Table 6.2. It is easy to see that the order relations of these five LPMs are consistent in the different cases. In addition, it is worth noting that security 5 has a larger LPMs than security 4 even though it has a smaller variance than security 4.

First we consider the reference level τ as an exogenous parameter. Set the return level $r = 0.40$. We employ the function 'fmincon' in Matlab to seek the optimal solution to model (6.11) and the computational results are shown in Table 6.3. It can

Table 6.1 Zigzag uncertain returns of 5 securities and their expected values

No.	Return	Mean	Variance
ξ_1	$(-0.2, 0.5, 0.9)$	0.425	0.1027
ξ_2	$(-0.3, 0.6, 1.0)$	0.475	0.1460
ξ_3	$(-0.1, 0.3, 0.8)$	0.325	0.0677
ξ_4	$(-0.2, 0.3, 1.0)$	0.350	0.1208
ξ_5	$(-0.3, 0.5, 0.7)$	0.350	0.0908

Table 6.2 LPMs of individual security returns with different degrees n and preference levels τ

	$\tau = 0$				
$\mathrm{LPM}_n^\tau(\cdot)$	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
$n = 1$	0.0143	0.0250	0.0063	0.0200	0.0281
$n = 2$	0.0019	0.0050	0.0004	0.0027	0.0056
	$\tau = 0.15$				
$\mathrm{LPM}_n^\tau(\cdot)$	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
$n = 1$	0.0438	0.0563	0.0391	0.0613	0.0633
$n = 2$	0.0102	0.0169	0.0065	0.0143	0.0190

Table 6.3 Computational results of model (6.11) when $r = 0.40$

	$\tau = 0$	
	Allocation	LPM_n^τ
$n = 1$	$x_1 = 0.75, x_3 = 0.25$	0.0123
$n = 2$	$x_1 = 0.42, x_2 = 0.22, x_3 = 0.36$	0.0017
$n = 3$	$x_1 = 0.24, x_2 = 0.29, x_3 = 0.14$ $x_4 = 0.17, x_5 = 0.16$	0.0005
	$\tau = 0.15$	
	Allocation	LPM_n^τ
$n = 1$	$x_1 = 0.75, x_3 = 0.25$	0.0423
$n = 2$	$x_1 = 0.75, x_2 = 0.25$	0.0092
$n = 3$	$x_1 = 0.42, x_2 = 0.22, x_3 = 0.36$	0.0025
	$\tau = 0.30$	
	Allocation	LPM_n^τ
$n = 1$	$x_1 = 1.00$	0.0893
$n = 2$	$x_1 = 0.75, x_2 = 0.25$	0.0286
$n = 3$	$x_1 = 0.75, x_2 = 0.25$	0.0102

be seen that the larger the value of n is, the more diversified the obtained portfolio is. Since n represents the risk aversion, an investor will prefer investing more securities to reduce risk with n increasing. For each fixed n, the relatively smaller τ will yield more diversified portfolios.

In order further to see that the effect of n on the optimal portfolio, we create a set of portfolios by solving model (6.11) associated with various return levels r for $n = 1, 2, 3$, respectively. Then we depict three return-risk efficient frontiers in which risk is measured by LPMs with $n = 1, 2, 3$, respectively. The results are shown in Fig. 6.1 where the reference level $\tau = 0$. It can be observed that the curve is zigzag for the case of $n = 1$. This is because when the return level $r \in [0.35, 0.425]$, the optimal portfolio is composed of securities 1 and 3 and when $r \in [0.425, 0.475]$, it is composed of securities 1 and 2. The curve associated with $n = 2$ is relatively smooth. The curve associated with $n = 3$ has an obviously piecewise property. When $r > 0.455$, the curve is linear since the optimal portfolio is only composed of securities 1 and 2. However, when r is set in the interval $[0.435, 0.455]$, the risk (i.e., LPMs with $n = 3$) does not significantly increase.

Next we consider the reference level τ as the expected value of corresponding return on portfolio, i.e.,

$$\begin{aligned}\tau &= E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_5 x_5] \\ &= 0.425x_1 + 0.475x_2 + 0.325x_3 + 0.35x_4 + 0.35x_5.\end{aligned}$$

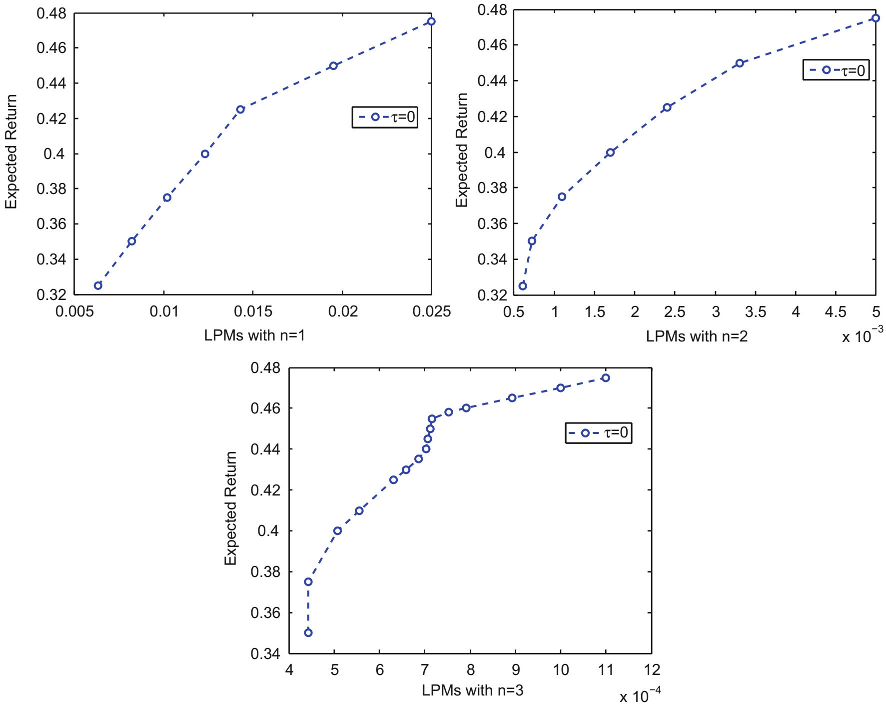

Fig. 6.1 Comparisons of efficient frontiers obtained by various values of n when $\tau = 0$

Table 6.4 Computational results of model (6.11) when τ is chosen as expected value

Return level	Allocation	LPM_1	LPM_2	LPM_3
$r = 0.325$	$x_3 = 1.00$	0.1128	0.0270	0.008
$r = 0.350$	$x_1 = 0.25, x_3 = 0.75$	0.1187	0.0376	0.0134
$r = 0.375$	$x_1 = 0.50, x_3 = 0.50$	0.1253	0.0438	0.0173
$r = 0.400$	$x_1 = 0.75, x_3 = 0.25$	0.1322	0.0507	0.0219
$r = 0.425$	$x_1 = 1.00$	0.1395	0.0581	0.0272
$r = 0.450$	$x_1 = 0.50, x_2 = 0.50$	0.1531	0.0715	0.0375
$r = 0.475$	$x_2 = 1.00$	0.1668	0.0862	0.0501

In this case, the optimal portfolios are same for given r regardless of whatever value of n. The reason may be that model (6.11) is essentially a linear programming since all the security returns are zigzag uncertain variables. The computational results are shown in Table 6.4 in which LPM_1 and LPM_2 are semiabsolute deviation and semivariance, respectively.

Chapter 7
Interval Mean-Semiabsolute Deviation Model

7.1 Introduction

Most existing researches deal with the expected returns of securities as crisp values, which may be estimated by the experts based on their experience and the historical data. However, since the improper estimations may bring on an unsuccessful investment decision, portfolio experts generally prefer to offer interval estimations instead of crisp point estimations. As early as 1980, Bitran (1980) proposed a linear multiobjective portfolio selection model with interval expected returns. From then on, the interval portfolio selection models were widely studied. For example, Lai et al. (2002) extended the classical mean-semiabsolute deviation model to interval mean-semiabsolute deviation model, Ida (2003) proposed a quadratic interval mean-variance model, Li and Xu (2009) proposed an interval goal programming model on the assumption that security returns are fuzzy variables, and Wu et al. (2013) revisited interval mean-variance analysis by assuming that the expected returns and covariances of assets are both intervals.

The existing works on interval portfolio optimization are undertaken based on the operational law of interval numbers. Different from them, Li and Qin (2014) studied the problem within the framework of uncertainty theory by considering the security returns with interval expected returns as uncertain variables. They respectively used expected value and semiabsolute deviation to measure the investment return and risk, and then they formulated interval mean-semiabsolute deviation model.

This chapter focuses on the interval portfolio optimization within the framework of uncertainty theory. The main contents of this chapter include uncertainty generation theorem, formulation of mean-semiabsolute deviation model, and numerical examples.

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7.2 Uncertainty Generation Theorem

In this section, we review the maximum uncertainty principle and the uncertainty generation theorem. Suppose that ξ is a random variable which is characterized by density function $\phi(x, \lambda)$. Traditionally λ is assumed to be a crisp number, which corresponds to stochastic portfolio optimization. In this chapter, we assume that λ takes its values from an interval $[a, b]$ and prove that such a random variable is an uncertain variable.

7.2.1 Maximum Uncertainty Principle

Let A be an event with known measure μ. If $\mu > 0.5$, then we prefer to believe that A will occur, and if $\mu < 0.5$, then we prefer to believe that its complement will occur. However, if $\mu = 0.5$, then there is no preference information for us because in this case event A and its complement are equally likely. Hence, an event has the highest uncertainty if its measure is 0.5. In practice, if there is no information about the measure of an event, we should assign 0.5 to it. If there is partial information, which implies that the measure should take value in some range, we should assign its value as close to 0.5 as possible. This is the maximum uncertainty principle provided by Liu (2007): for any event, if there are multiple reasonable values that a measure may take, then the value as close to 0.5 as possible will be assigned to the event.

Example 7.1. Let A be an event with uncertainty measure taking value from interval $[a, b]$. Then according to the maximum uncertainty principle, we have

$$\mathcal{M}\{A\} = \begin{cases} a, & \text{if } 0.5 < a \le b \\ 0.5, & \text{if } a \le 0.5 \le b \\ b, & \text{if } a \le b < 0.5. \end{cases}$$

For example, (i) if it is known that the uncertainty of event A lies in $[0.2, 0.4]$, then we should assign the maximum 0.4 to $\mathcal{M}\{A\}$, (ii) if it is known that the uncertainty of event A lies in $[0.6, 0.7]$, then we should assign the minimum 0.6 to $\mathcal{M}\{A\}$, and (iii) if it is known that the uncertainty of event A lies in $[0.4, 0.6]$, then we should assign 0.5 to $\mathcal{M}\{A\}$.

Theorem 7.1 (Uncertainty Generation Theorem, Li and Qin 2014). *Let Γ be a nonempty set, and let $\mathcal{A}$ be a σ-algebra over it. If L and U are two set functions from $\mathcal{A}$ to $[0, 1]$ satisfying $L\{A\} \le U\{A\}$ for each event A as well as the following conditions:*

(a) $L\{\Gamma\} = 1$;
(b) $L\{A\} \le L\{B\}$ *whenever* $A \subseteq B$;

(c) $L\{A\} + U\{A^c\} = 1$ *for any event* A*;*
(d) $U\{\cup_i A_i\} \le \sum_{i=1}^{\infty} U\{A_i\}$ *for any countable sequence of events* $\{A_i\}$*,*

then the set function induced by the maximum uncertainty principle

$$\mathcal{M}\{A\} = \begin{cases} U\{A\}, \text{ if } U\{A\} < 0.5 \\ L\{A\}, \text{ if } L\{A\} > 0.5 \\ 0.5, \quad \text{otherwise} \end{cases} \tag{7.1}$$

is an uncertain measure.

Proof. Since $L\{\Gamma\} = 1$, it follows from Eq. (7.1) that $\mathcal{M}\{\Gamma\} = 1$, which implies that the normality holds. The proof of self-duality breaks down into two cases. For any event A with $L\{A\} > 0.5$, it follows from condition (c) that $U\{A^c\} < 0.5$, which implies that

$$\mathcal{M}\{A\} + \mathcal{M}\{A^c\} = L\{A\} + U\{A^c\} = 1.$$

On the other hand, for any event A with $L\{A\} \le 0.5$, we have $U\{A^c\} \ge 0.5$ and meanwhile

$$\begin{aligned} \mathcal{M}\{A\} + \mathcal{M}\{A^c\} &= U\{A\} \wedge 0.5 + L\{A^c\} \vee 0.5 \\ &= U\{A\} \wedge 0.5 + (1 - U\{A\}) \vee 0.5 \\ &= 1. \end{aligned}$$

Finally we prove the countable subadditivity. Let $\{A_i\}$ be a sequence of events. If $U\{A_i\} < 0.5$ for all i, it follows immediately from condition (d) that

$$\mathcal{M}\{\cup_i A_i\} \le U\{\cup_i A_i\} \le \sum_{i=1}^{\infty} U\{A_i\} = \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}.$$

Otherwise, there is at least one event with uncertainty measure larger than 0.5. Without loss of generality, we assume $U\{A_1\} \ge 0.5$. Then we have $U\{\cup_i A_i\} \ge 0.5$. It follows from Eq. (7.1) that $\mathcal{M}\{A_1\} = L\{A_1\} \vee 0.5$, and $\mathcal{M}\{\cup_i A_i\} = L\{\cup_i A_i\} \vee 0.5$, and it follows from conditions (c) and (d) that

$$\begin{aligned} L\{\cup_i A_i\} &= 1 - U\{\cap_i A_i^c\} \\ &\le 1 - U\{A_1^c\} + U\{\cup_{i \ge 2} A_i\} \\ &\le L\{A_1\} + \sum_{i=2}^{\infty} U\{A_i\}, \end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{M}\{\cup_i A_i\} &= L\{\cup_i A_i\} \vee 0.5 \\ &\leq L\{A_1\} \vee 0.5 + \sum_{i=2}^{\infty} U\{A_i\} \\ &= \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}.\end{aligned}$$

Hence, set function defined by Eq. (7.1) is an uncertain measure. The theorem is proved.

Theorem 7.2 (Li and Qin 2014). *Let ξ be a random variable with a density function $\phi(x, \lambda)$. If λ is an unknown parameter belonging to $[a, b]$, then ξ is an uncertain variable on uncertainty space $(\Re, \mathcal{B}, \mathcal{M})$ defined as $\xi(x) = x$ for all $x \in \Re$, where $\mathcal{B}$ is the Borel algebra over $\Re$ and*

$$\mathcal{M}\{B\} = \begin{cases} \sup\limits_{\lambda\in[a,b]} \int_B \phi(y, \lambda)\mathrm{d}y, & \text{if } \sup\limits_{\lambda\in[a,b]} \int_B \phi(y, \lambda)\mathrm{d}y < 0.5 \\ \inf\limits_{\lambda\in[a,b]} \int_B \phi(y, \lambda)\mathrm{d}y, & \text{if } \inf\limits_{\lambda\in[a,b]} \int_B \phi(y, \lambda)\mathrm{d}y > 0.5 \\ 0.5, & \text{otherwise}. \end{cases} \tag{7.2}$$

Proof. What we need to prove is that $\mathcal{M}$ is an uncertain measure. For simplicity, define two set functions

$$L\{B\} = \inf_{\lambda\in[a,b]} \int_B \phi(x, \lambda)\mathrm{d}x,$$

$$U\{B\} = \sup_{\lambda\in[a,b]} \int_B \phi(x, \lambda)\mathrm{d}x.$$

It is clear that $L\{B\} \leq U\{B\}$ for each event B. We will prove that L and U satisfy the conditions of uncertainty generation theorem. First, it is easy to prove that

$$L\{\Re\} = \inf_{\lambda\in[a,b]} \int_{-\infty}^{\infty} \phi(x, \lambda)\mathrm{d}x = 1.$$

Furthermore, for any Borel sets A and B with $A \subseteq B$, we have

$$L\{A\} = \inf_{\lambda\in[a,b]} \int_A \phi(x, \lambda)\mathrm{d}x \leq \inf_{\lambda\in[a,b]} \int_B \phi(x, \lambda)\mathrm{d}x = L\{B\}.$$

In addition, for any Borel set A, it is easy to prove that

$$\begin{aligned}
U\{A^c\} &= \sup_{\lambda\in[a,b]} \int_{A^c} \phi(x,\lambda)\mathrm{d}x \\
&\le \sup_{\lambda\in[a,b]} \left(1 - \int_A \phi(x,\lambda)\mathrm{d}x\right) \\
&= 1 - \inf_{\lambda\in[a,b]} \int_A \phi(x,\lambda)\mathrm{d}x \\
&= 1 - L\{A\},
\end{aligned}$$

which implies that $L\{A\} + U\{A^c\} = 1$. Finally, let us prove the countable subadditivity. For any countable sequence of Borel sets $A_1, A_2, \cdots$, since

$$\begin{aligned}
U\{\cup_i A_i\} &= \sup_{\lambda\in[a,b]} \int_{\cup_i A_i} \phi(x,\lambda)\mathrm{d}x \\
&= \sup_{\lambda\in[a,b]} \sum_{i=1}^{\infty} \int_{A_i} \phi(x,\lambda)\mathrm{d}x \\
&\le \sum_{i=1}^{\infty} \sup_{\lambda\in[a,b]} \int_{A_i} \phi(x,\lambda)\mathrm{d}x \\
&= \sum_{i=1}^{\infty} U\{A_i\},
\end{aligned}$$

the subadditivity is proved. Above all, the theorem is proved.

Theorem 7.2 implies that a random variable with interval parameters can be considered as an uncertain variable. Therefore, we can apply uncertainty theory to model interval portfolio optimization.

7.3 Mean-Semiabsolute Deviation Model

Let ξ_i be the return of the ith security, and let x_i be the investment proportion on security i for all $i = 1, 2, \cdots, n$. Suppose that $(\xi_1, \xi_2, \cdots, \xi_n)$ is a normal random vector with mean value $(e_1, e_2, \cdots, e_n)$ and covariance matrix Σ, where e_i is an unknown parameter belonging to $[a_i, b_i]$ for all $i = 1, 2, \cdots, n$. For simplicity, we denote the following notations:

$$\begin{aligned}
\xi &: \text{portfolio return } \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n;\\
e &: \text{expected value of portfolio return } e_1 x_1 + e_2 x_2 + \cdots + e_n x_n;\\
\sigma^2 &: \text{variance of portfolio return } \textstyle\sum_{i=1}^n \sum_{j=1}^n \sigma_{ij} x_i x_j;\\
a &: \text{lower bound for expected value } a_1 x_1 + a_2 x_2 + \cdots + a_n x_n;\\
b &: \text{upper bound for expected value } b_1 x_1 + b_2 x_2 + \cdots + b_n x_n;\\
\phi(x, e, \sigma^2) &: \text{density function for normal random variable with}\\
&\ \text{expected value } e \text{ and variance } \sigma^2.
\end{aligned}$$

For each portfolio $(x_1, x_2, \cdots, x_n)$, it is easy to prove that ξ has density function $\phi(x, e, \sigma^2)$ where $e \in [a, b]$. Since ξ contains not only the objective randomness but also the subjective uncertainty for expected value e, probability theory is powerless to deal with it. Here, it will be considered as an uncertain variable defined in Theorem 7.2.

7.3.1 Distribution Function

For any $x \in \Re$, it is easy to prove that

$$\int_{-\infty}^{x} \phi(y, e, \sigma^2)\mathrm{d}y = \int_{-\infty}^{(x-e)/\sigma} \phi(y, 0, 1)\mathrm{d}y,$$

which implies that

$$\sup_{e\in[a,b]} \int_{-\infty}^{x} \phi(y, e, \sigma^2)\mathrm{d}y = \int_{-\infty}^{x} \phi(y, a, \sigma^2)\mathrm{d}y,$$

$$\inf_{e\in[a,b]} \int_{-\infty}^{x} \phi(y, e, \sigma^2)\mathrm{d}y = \int_{-\infty}^{x} \phi(y, b, \sigma^2)\mathrm{d}y.$$

Furthermore, it follows from the fact

$$\int_{-\infty}^{a} \phi(x, a, \sigma^2)\mathrm{d}x = \int_{-\infty}^{b} \phi(x, b, \sigma^2)\mathrm{d}x = 0.5$$

that

$$\mathcal{M}\{\xi \le x\} = \begin{cases} \displaystyle\int_{-\infty}^{x} \phi(y, a, \sigma^2)\mathrm{d}y, & \text{if } x < a \\ \displaystyle\int_{-\infty}^{x} \phi(y, b, \sigma^2)\mathrm{d}y, & \text{if } x > b \\ 0.5, & \text{otherwise.} \end{cases} \tag{7.3}$$

In addition, it follows from the self-duality of uncertain measure that

$$\mathcal{M}\{\xi \ge x\} = \begin{cases} \displaystyle\int_x^{+\infty} \phi(y, a, \sigma^2)\mathrm{d}y, & \text{if } x > a \\ \displaystyle\int_x^{+\infty} \phi(y, b, \sigma^2)\mathrm{d}y, & \text{if } x < b \\ 0.5, & \text{otherwise.} \end{cases} \tag{7.4}$$

7.3.2 Expected Value

In order to measure the investment return for each portfolio $(x_1, x_2, \cdots, x_n)$, let us consider the expected value of portfolio return ξ. The argument breaks down into two cases. First, we assume $a \ge 0$. It follows from Definition 1.18 and the Fubini's Theorem that

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}\mathrm{d}x \\ &= \int_0^a \int_x^{+\infty} \phi(y, a, \sigma^2)\mathrm{d}y\mathrm{d}x + \int_a^b 0.5\mathrm{d}x \\ &\quad + \int_b^{+\infty} \int_x^{+\infty} \phi(y, b, \sigma^2)\mathrm{d}y\mathrm{d}x - \int_{-\infty}^0 \int_{-\infty}^x \phi(y, a, \sigma^2)\mathrm{d}y\mathrm{d}x \\ &= \frac{a+b}{2} + \int_b^{+\infty} \int_x^{+\infty} \phi(y, b, \sigma^2)\mathrm{d}y\mathrm{d}x - \int_{-\infty}^a \int_{-\infty}^x \phi(y, a, \sigma^2)\mathrm{d}y\mathrm{d}x \\ &= \frac{a+b}{2} + \int_b^{+\infty} (x-b)\phi(x, b, \sigma^2)\mathrm{d}x - \int_{-\infty}^a (a-x)\phi(x, a, \sigma^2)\mathrm{d}x \\ &= \frac{a+b}{2} + \int_0^{+\infty} x\phi(x, 0, \sigma^2)\mathrm{d}x - \int_0^{+\infty} x\phi(x, 0, \sigma^2)\mathrm{d}x \\ &= \frac{a+b}{2}. \end{aligned}$$

Next, we consider the case of $a < 0$. Let m be a real number such that $a + m \ge 0$. Then we have

$$\begin{aligned} E[\xi + m] &= (a + m + b + m)/2 \\ &= (a + b)/2 + m. \end{aligned}$$

It follows from the linearity of expected value (Theorem 1.17) that $E[\xi + m] = E[\xi] + m$, which implies that $E[\xi] = (a + b)/2$.

7.3.3 Semiabsolute Deviation

In order to measure the investment risk for each portfolio $(x_1, x_2, \cdots, x_n)$, let us consider the semiabsolute deviation of portfolio return ξ. It follows from Remark 5.1 and Fubini's Theorem that

$$
\begin{aligned}
S_a[\xi] &= \int_{-\infty}^{(a+b)/2} \mathcal{M}\{\xi \le x\}\,\mathrm{d}x \\
&= \int_{-\infty}^{a}\int_{-\infty}^{x} \phi(y, a, \sigma^2)\mathrm{d}y\mathrm{d}x + \int_{a}^{(a+b)/2} 0.5\mathrm{d}x \\
&= \frac{b-a}{4} + \int_{-\infty}^{a}\int_{y}^{a} \phi(y, a, \sigma^2)\mathrm{d}x\mathrm{d}y \\
&= \frac{b-a}{4} + \int_{-\infty}^{a} (a-y)\phi(y, a, \sigma^2)\mathrm{d}y \\
&= \frac{b-a}{4} + \frac{\sigma}{\sqrt{2\pi}}.
\end{aligned}
$$

7.3.4 Uncertain Mean-Semiabsolute Deviation Model

In spirit with the classical mean-semiabsolute deviation portfolio selection model, we use expected value and semiabsolute deviation to measure the investment return and risk, respectively, and assume that a rational investor would minimize the investment risk under certain return level. If there is a given minimum return level r, then Li and Qin (2014) proposed the following interval mean-semiabsolute deviation model,

$$
\begin{cases}
\min \displaystyle\sum_{i=1}^{n} \frac{b_i - a_i}{4} x_i + \left(\frac{1}{2\pi} \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \right)^{1/2} \\
\text{s.t.} \displaystyle\sum_{i=1}^{n} \frac{a_i + b_i}{2} x_i \ge r \\
\qquad \displaystyle\sum_{i=1}^{n} x_i = 1 \\
\qquad x_i \ge 0, i = 1, 2, \cdots, n,
\end{cases}
\tag{7.5}
$$

where the objective function is to minimize the investment risk, the first constraint is to ensure that the investment return is larger than the given minimum return level r, the others imply that all the capital will be invested to the n securities, and short sale and borrowing are not allowed.

Remark 7.1. If there are no uncertainty about parameter e_i, i.e., $a_i = b_i = e_i$ for all $i = 1, 2, \cdots, n$, then the interval mean-semiabsolute deviation model (7.5) degenerates to

$$\begin{cases} \min \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \right)^{1/2} \\ \text{s.t.} \ \sum_{i=1}^{n} e_i x_i \geq r \\ \quad \sum_{i=1}^{n} x_i = 1 \\ \quad x_i \geq 0, i = 1, 2, \cdots, n, \end{cases}$$

which is just the classical mean-semiabsolute deviation model. Hence, the interval mean-semiabsolute deviation model and the classical one are consistent.

Example 7.2. Let us consider a simplified portfolio selection problem with one security and one national bond. Let $\mathcal{N}(e_1, \sigma^2)$ be the security return and let e_2 be the fixed return for the national bond where $e_1 \in [a_1, b_1]$ and $e_2 \in [a_2, b_2]$. If we use x_1 and x_2 to represent the proportions invested to the security and the national bond, respectively, then the uncertain mean-semiabsolute deviation model is

$$\begin{cases} \min \dfrac{b_1 - a_1}{4} x_1 + \dfrac{b_2 - a_2}{4} x_2 + \dfrac{\sigma}{\sqrt{2\pi}} x_1 \\ \text{s.t.} \ \dfrac{a_1 + b_1}{2} x_1 + \dfrac{a_2 + b_2}{2} x_2 \geq r \\ \quad x_1 + x_2 = 1 \\ \quad x_1, x_2 \geq 0, \end{cases} \tag{7.6}$$

which is a linear programming model. If $r > \max((a_1 + b_1)/2, (a_2 + b_2)/2)$, then it is clear that there is no feasible portfolio because the minimum return level is larger than the expected return for both the security and the bond. Otherwise, the optimal portfolio is shown in Table 7.1.

Remark 7.2. If there is a given maximum risk level β, then the interval mean-semiabsolute deviation model has the following variation

Table 7.1 The optimal proportion invested in the first security

	$\frac{b_1-a_1}{2}+\frac{\sigma}{\sqrt{2\pi}}\le\frac{b_2-a_2}{2}$	$\frac{b_1-a_1}{2}+\frac{\sigma}{\sqrt{2\pi}}>\frac{b_2-a_2}{2}$
$a_1+b_1>a_2+b_2$	1	$\frac{2r-(a_2+b_2)}{a_1+b_1-(a_2+b_2)}\wedge 1$
$a_1+b_1=a_2+b_2$	1	0
$a_1+b_1<a_2+b_2$	$\frac{2r-(a_2+b_2)}{a_1+b_1-(a_2+b_2)}\vee 0$	0

$$
\begin{cases}
\max \displaystyle\sum_{i=1}^{n}\frac{a_i+b_i}{2}x_i \\
\text{s.t.} \ \displaystyle\sum_{i=1}^{n}\frac{b_i-a_i}{4}x_i+\left(\frac{1}{2\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j\right)^{1/2}\le\beta \\
\qquad x_1+x_2+\cdots+x_n=1 \\
\qquad x_i\ge 0, i=1,2,\cdots,n,
\end{cases}
$$

where the objective function is to maximize the investment return, the first constraint is to ensure that the investment risk is less than the maximum risk level β, the other ones imply that all the capital will be invested to the n securities, and short sales and borrowing are not allowed.

Remark 7.3. A risk-averse investor always wants to maximize the investment return and minimize the investment risk simultaneously. However, these two objects are inconsistent. If there is a given degree of risk aversion, we can formalize the following compromise model

$$
\begin{cases}
\min \displaystyle\sum_{i=1}^{n}\frac{a_i+b_i}{2}x_i-\gamma\sum_{i=1}^{n}\frac{b_i-a_i}{4}x_i-\gamma\left(\frac{1}{2\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j\right)^{1/2} \\
\text{s.t.} \ x_1+x_2+\cdots+x_n=1 \\
\qquad x_i\ge 0, i=1,2,\cdots,n
\end{cases}
\tag{7.7}
$$

where $\gamma\in[0,+\infty)$ representing the degree of risk aversion. Here, the greater the value of γ is, the more risk-averse the investor is. Especially, $\gamma=0$ means that the investor does not consider risk, and γ approaching infinity means that the investor will allocate all the money to the risk-less securities.

Based on risk aversion degree, Lai et al. (2002) proposed a similar interval portfolio optimization model

$$\begin{cases} \min \displaystyle\sum_{i=1}^{n} [\lambda a_i + (1-\lambda) b_i]\, x_i - \gamma \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ \text{s.t. } \displaystyle\sum_{i=1}^{n} x_i = 1 \\ \qquad x_i \geq 0, i = 1, 2, \cdots, n, \end{cases} \tag{7.8}$$

where $\lambda \in [0, 1]$. As comparisons, the advantages of compromise model (7.7) are concluded as follows: (i) compromise model considers the risk arising from both the randomness

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

and the uncertainty of the expected value

$$\sum_{i=1}^{n} \frac{b_i - a_i}{2} x_i = \sum_{i=1}^{n} \frac{a_i + b_i}{2} x_i - \sum_{i=1}^{n} a_i x_i;$$

(ii) compromise model defines the investment return as the Choquet integral

$$\int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq x\}\mathrm{d}x = \sum_{i=1}^{n} \frac{a_i + b_i}{2} x_i,$$

which is more reasonable than the linear combination

$$\sum_{i=1}^{n} (\lambda a_i + (1-\lambda) b_i) x_i.$$

7.4 Numerical Examples

In this section, we illustrate the application of interval mean-semiabsolute deviation model by numerical examples in Li and Qin (2014).

Example 7.3. To examine the performance of the proposed models, we employ the historical data introduced by Markowitz (1959), which covers the period from 1937 to 1954 with 18 years observations for nine securities. Let x_{ij} be the return of the jth year for the ith security for all $i = 1, 2, \cdots, 9$ and $j = 1, 2, \cdots, 18$. Based on these historical data, the crisp covariance matrix obtained by the point estimation method is

0.0565	0.0228	0.0303	0.0518	0.0172	0.0341	0.0257	0.0464	0.0383
0.0228	0.0155	0.0199	0.0259	0.0085	0.0106	0.0153	0.0265	0.0221
0.0303	0.0199	0.0905	0.0663	0.0470	0.0141	0.0111	0.0836	0.0445
0.0518	0.0259	0.0663	0.1011	0.0546	0.0307	0.0220	0.0775	0.0388
0.0172	0.0085	0.0470	0.0546	0.1354	0.0136	0.0221	0.0683	0.0476
0.0341	0.0106	0.0141	0.0307	0.0136	0.0437	0.0119	0.0254	0.0229
0.0257	0.0153	0.0111	0.0220	0.0221	0.0119	0.0305	0.0229	0.0184
0.0464	0.0265	0.0836	0.0775	0.0683	0.0254	0.0229	0.1024	0.0553
0.0383	0.0221	0.0445	0.0388	0.0476	0.0229	0.0184	0.0553	0.0839.

Furthermore, the interval expected returns may be obtained by the following interval estimation formulas

$$[a_i, b_i] = \left[\bar{X}_i - T_{(1+\alpha)/2}(17)S/\sqrt{18},\ \bar{X}_i + T_{(1+\alpha)/2}(17)S/\sqrt{18}\right],$$

where α is the confidence level, T expresses the student distribution,

$$\bar{X}_i = \frac{1}{18}\sum_{j=1}^{18} x_{ij}, \quad S^2 = \frac{1}{17}\sum_{j=1}^{18}(x_{ij} - \bar{X}_i)^2$$

which are the unbiased estimations for the expected value and variance of the ith security return, respectively. The computational results are shown in Table 7.2. Please note that if $\alpha = 0$, then the confidence interval will degenerate to the point estimation $\bar{X}_i$ and the interval mean-semiabsolute deviation model will degenerate to the classical mean-semiabsolute deviation model.

Assume that the minimum return level r is 0.12. In order to find the optimal portfolio, the investor may use the interval mean-semiabsolute deviation model (7.5). Here, we employ the MATLAB function *fmincon* to search the optimal solution, the computational results are shown in Table 7.3.

In this table, the seventh column is the semiabsolute deviation in interval case, and the eighth column is the classical semiabsolute deviation corresponding to each optimal portfolio where $\xi_1, \xi_2, \cdots, \xi_n$ are considered as random variables with crisp expected values in the second column of Table 7.2. It is easy to conclude that (a) the interval semiabsolute deviations gradually increase with the increasing of confidence levels. The reason is that in order to estimate the expected returns with high confidence degree, we have to make the investment decision with a wider confidence interval, which will increase the risk clearly; (b) as the confidence level improves significantly from zero to one, the classical semiabsolute deviation is almost unchanged, which implies that the interval semiabsolute deviation model is effective because it improves the confidence degree for the estimation of expected returns significantly but scarcely increase the risk. In order to show the conclusions more clearly, we illustrate the efficient frontiers in Fig. 7.1. The efficient frontiers

Table 7.2 Interval expected returns with different confidence levels

	$\alpha = 0$	$\alpha = 0.90$		$\alpha = 0.95$	
No.	e_i	a_i	b_i	a_i	b_i
1	0.0659	−0.0315	0.1634	−0.0523	0.1842
2	0.0616	0.0104	0.1127	−0.0005	0.1236
3	0.1461	0.0227	0.2694	−0.0036	0.2957
4	0.1734	0.0431	0.3038	0.0153	0.3315
5	0.1981	0.0472	0.3490	0.0151	0.3811
6	0.0551	−0.0306	0.1408	−0.0488	0.1590
7	0.1276	0.0560	0.1993	0.0407	0.2145
8	0.1348	0.0036	0.2660	−0.0243	0.2939
9	0.1156	−0.0032	0.2344	−0.0285	0.2597
	$\alpha = 0.98$		$\alpha = 0.99$		
No.	a_i	b_i	a_i	b_i	
1	−0.0779	0.2098	−0.0965	0.2284	
2	−0.0139	0.1370	−0.0236	0.1467	
3	−0.0360	0.3281	−0.0595	0.3516	
4	−0.0189	0.3658	−0.0437	0.3906	
5	−0.0245	0.4208	−0.0533	0.4495	
6	−0.0713	0.1816	−0.0877	0.1979	
7	0.0219	0.2333	0.0083	0.2470	
8	−0.0588	0.3284	−0.0838	0.3533	
9	−0.0597	0.2909	−0.0823	0.3135	

Table 7.3 Optimal portfolios with minimum return level 0.12

α	0.00	0.90	0.95	0.98	0.99
x_2	0.1785	0.1795	0.1633	0.1425	0.1280
x_3	0.1223	0.0741	0.0651	0.0532	0.0441
x_5	0.0909	0.0408	0.0280	0.0117	0.0005
x_6	0.0618	0.0000	0.0000	0.0000	0.0000
x_7	0.5466	0.7056	0.7436	0.7926	0.8274
U-semiabsolute	0.0598	0.0989	0.1068	0.1164	0.1233
C-semiabsolute	0.0598	0.0614	0.0621	0.0631	0.0638

with different confidence levels are nearly identical, and the efficient frontier with higher confidence level is always dominated by the one with lower confidence level.

In order to test the sensitivity of optimal portfolio with respect to the minimum return level, we perform further experiments using this example by changing the value of r. In Fig. 7.2, the investment proportions on securities 5 and 7 are illustrated by regarding them as the functions of r. Furthermore, as the variation of minimum return level, the investment proportions of securities which enter all the optimal portfolios are presented by Fig. 7.3, where for each given minimum return level, the value of each line represents the optimal investment proportion to the corresponding security.

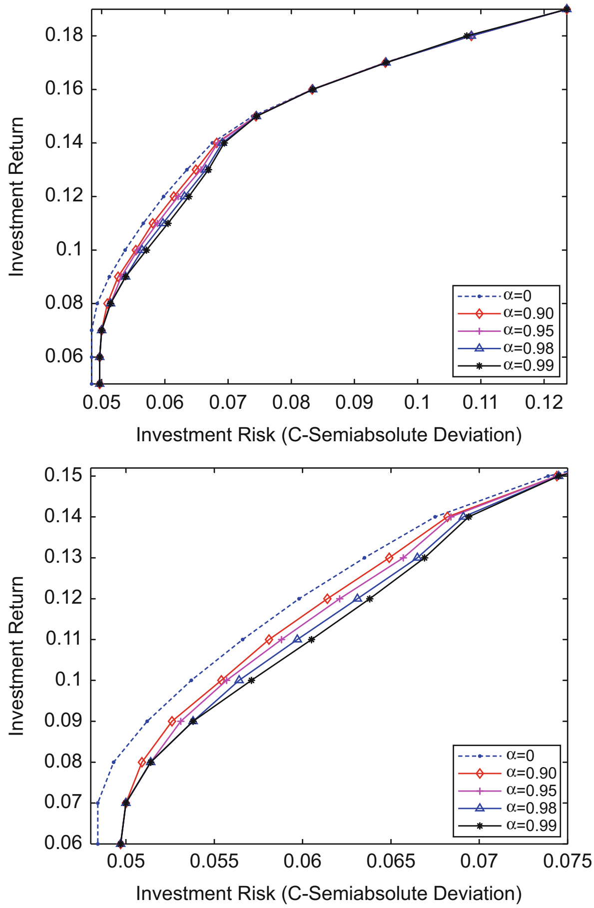

Fig. 7.1 Efficient frontiers with different confidence levels

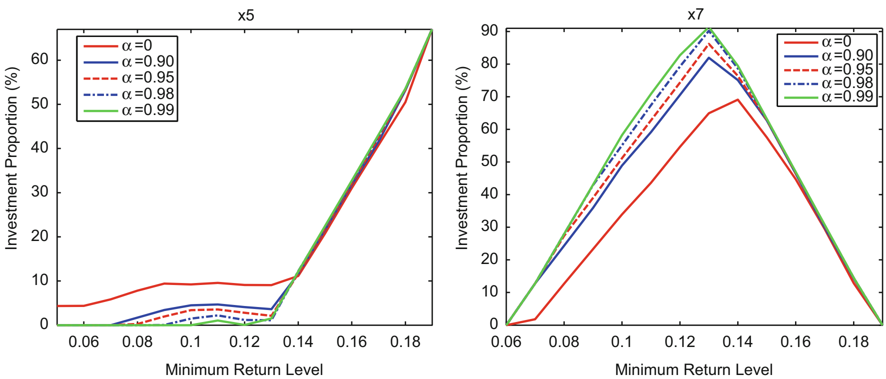

Fig. 7.2 Sensitivity of optimal proportions on securities 5 and 7 with respect to minimum return levels

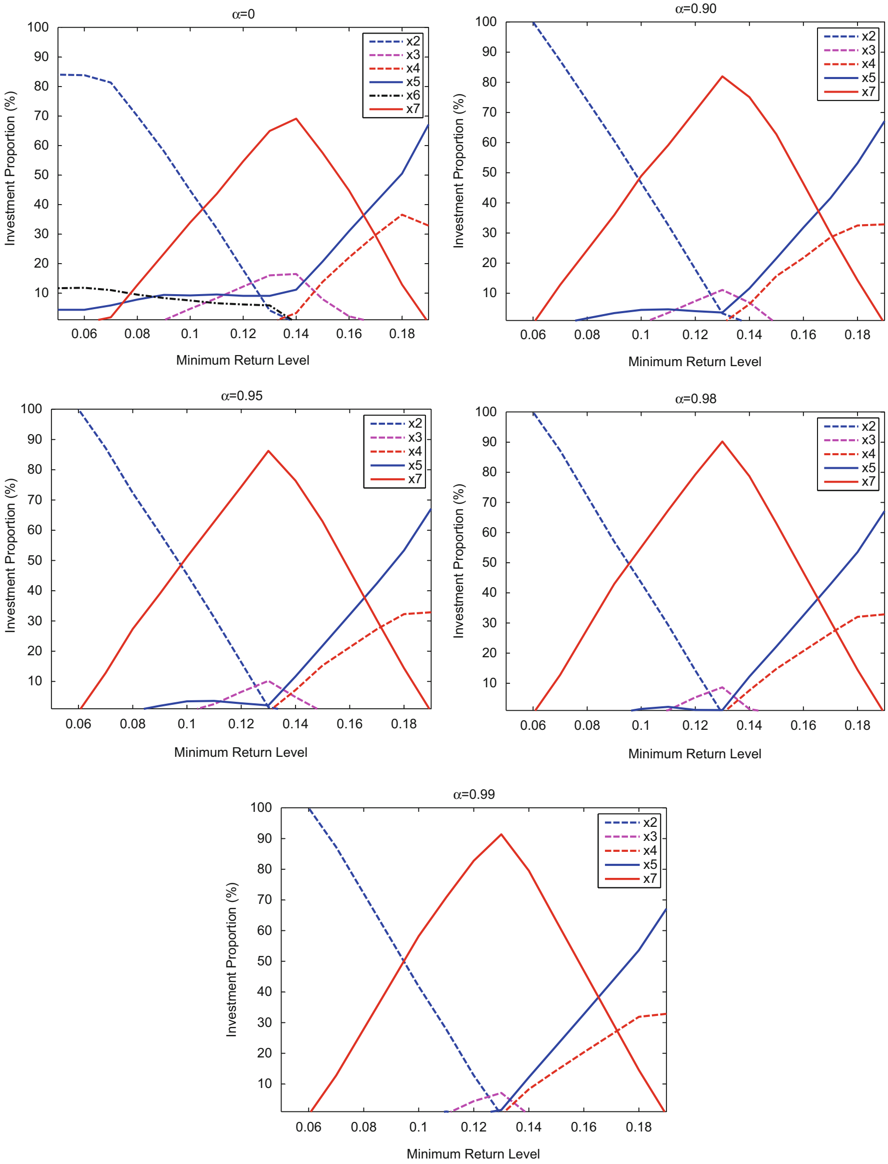

Fig. 7.3 Sensitivity of optimal portfolios with respect to minimum return levels

Chapter 8
Uncertain Random Mean-Variance Model

8.1 Introduction

Whether in the classical portfolio optimization or the fuzzy/uncertain one, security returns are considered as the same type of variables. In other words, security returns are assumed to be either random variables or fuzzy/uncertain ones. As stated above, the former makes use of the historical data and the latter makes use of the experiences of experts. However, the actual situation is that the securities having been listed for a long time have yielded a great deal of transaction data. For these "existing" securities, statistical methods are employed to estimate their returns, which implies that it is reasonable to assume that security returns are random variables. For some newly listed securities, there are lack of data or there are only insufficient data which can not be used to estimate the returns effectively. Therefore, the returns of these newly listed securities need to be estimated by experts and thus are considered as uncertain variables.

If an investor faces as such a complex situation with simultaneous appearance of random and uncertain returns, how should he/she select a desirable portfolio to achieve some objectives? Qin (2015) studied the hybrid portfolio optimization problem and established mathematical models by means of uncertain random variable which was proposed by Liu (2013a) for modeling complex systems with not only uncertainty but also randomness. Following this concept, several works have been done such as Liu (2013b), Wang et al. (2015), and so on.

This chapter focuses on uncertain random mean-variance model in the simultaneous presence of random and uncertain returns. The main contents include reviews of uncertain random variable, variance of uncertain random return, model formulation and analysis, and numerical examples.

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DOI 10.1007/978-981-10-1810-7_8

8.2 Uncertain Random Variable

Uncertain random variable is employed to describe a complex system with not only uncertainty but also randomness. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space, and $(\Omega, \mathcal{P}, \Pr)$ a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$ is called a chance space.

Definition 8.1 (Liu 2013a). Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{P}$. Then the chance measure of uncertain random event Θ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | (\gamma, \omega) \in \Theta\} \ge r\} \mathrm{d}r.$$

The chance measure Ch is proved by Liu (2013a) to be monotone increasing and self-dual, i.e., for any event Θ, we have $\mathrm{Ch}\{\Theta\} + \mathrm{Ch}\{\Theta^c\} = 1$. For any $\Lambda \in \mathcal{L}$ and $A \in \mathcal{P}$, we have

$$\mathrm{Ch}\{\Lambda \times A\} = \mathcal{M}\{\Lambda\} \times \Pr\{A\}.$$

Especially, $\mathrm{Ch}\{\emptyset\} = 0$ and $\mathrm{Ch}\{\Gamma \times \Omega\} = 1$. Moreover, the chance measure is also subadditive. That is, for any countable sequence of events $\Theta_1, \Theta_2, \cdots$, we have

$$\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty} \Theta_i\right\} \le \sum_{i=1}^{\infty} \mathrm{Ch}\{\Theta_i\}.$$

Definition 8.2 (Liu 2013a). An uncertain random variable is a function ξ from the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$ to the set of real numbers, i.e., $\xi \in B$ is an event in $\mathcal{L} \times \mathcal{P}$ for any Borel set B. Its chance distribution is defined by $\Phi(x) = \mathrm{Ch}\{\xi \le x\}$ for any $x \in \Re$.

It follows from Definition 8.2 that random variables and uncertain variables are special cases of uncertain random variables. If η is a random variable and τ is an uncertain variable, then the sum $\eta + \tau$ and the product $\eta\tau$ are both uncertain random variables.

Theorem 8.1 (Liu 2013a). *Let $f : \Re^n \to \Re$ be a measurable function, and $\xi_1, \xi_2, \cdots, \xi_n$ uncertain random variables on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{P}, \Pr)$. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain random variable determined by*

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega)) \tag{8.1}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Theorem 8.2 (Liu 2013b). *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables (not necessarily independent). Then the uncertain random variables $\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ has a chance distribution*

$$\Phi(x) = \int_{\Re^m} F(x; y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)$$

where $F(x; y_1, \cdots, y_m)$ is the uncertainty distribution of uncertain variable $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ for any real numbers $y_1, \cdots, y_m$.

Definition 8.3 (Liu 2013a). Let ξ be an uncertain random variable. Then its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq r\} \mathrm{d}r - \int_{-\infty}^0 \mathrm{Ch}\{\xi \leq r\} \mathrm{d}r$$

provided that at least one of the two integrals is finite.

Theorem 8.3 (Liu 2013b). *Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function with respect to $\tau_1, \cdots, \tau_n$, then the uncertain random variable $\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ has an expected value*

$$E[\xi] = \int_{\Re^m} \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m).$$

Let ξ be an uncertain random variable with finite expected value e. The variance of ξ is defined by Liu (2013a) as $V[\xi] = E[(\xi - e)^2]$. Suppose that Φ is its chance distribution. Guo and Wang (2014) proposed a stipulation to calculate the variance by

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x})) \mathrm{d}x.$$

Based on the stipulation, we have the following theorem.

Theorem 8.4 (Sheng and Yao 2014). *Let ξ be an uncertain random variable with chance distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

8.3 Problem Statement

In this section, we will state the problem of finding out the optimal portfolio in the simultaneous presence of random and uncertain returns and then discuss the variance of uncertain random returns. For sake of convenience, we list the notations used in the rest as follows,

m the number of existing risky securities with enough historical data
n the number of newly listed risky securities with lack of historical data
ξ_i the future return of the ith existing risky security, which is a random variable, $i = 1, 2, \cdots, m$
μ_i the expected value of random return ξ_i of the ith existing security, $i = 1, 2, \cdots, m$
σ_{ij} the covariance of random returns ξ_i and ξ_j, $i, j = 1, 2, \cdots, m$
Ψ_i the probability distribution of random return ξ_i, $i = 1, 2, \cdots, m$
η_j the future return of the jth newly listed risky security, which is an uncertain variable, $j = 1, 2, \cdots, n$
ν_j the expected value of uncertain return η_j of the jth newly listed security, $j = 1, 2, \cdots, n$
δ_j^2 the variance of uncertain return η_j of the jth newly listed security, $j = 1, 2, \cdots, n$
Υ_j the uncertainty distribution of uncertain return η_j, $j = 1, 2, \cdots, n$
x_i the holding proportion of the existing security i, $i = 1, 2, \cdots, m$
y_j the holding proportion of the newly listed security j, $j = 1, 2, \cdots, n$.

Let $'$ be the matrix transpose symbol. Next vectors are represented by bold letters. Denote by $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)'$ the portfolio vector of the existing securities, by $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_m)'$ the random vector of returns of the existing securities, by $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)'$ the portfolio vector of the newly listed securities, by $\boldsymbol{\eta} = (\eta_1, \eta_2, \cdots, \eta_n)'$ the uncertain vector of returns of the newly listed securities. Then $(\boldsymbol{x}', \boldsymbol{y}')' = (x_1, x_2, \cdots, x_m, y_1, \cdots, y_n)'$ represents the portfolio vector of all the candidate securities, which yields the total return

$$\begin{aligned} r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\eta}) &= \boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta} \\ &= x_1\xi_1 + x_2\xi_2 + \cdots + x_m\xi_m + y_1\eta_1 + \cdots + y_n\eta_n \end{aligned}$$

where $\boldsymbol{x}'\boldsymbol{\xi}$ and $\boldsymbol{y}'\boldsymbol{\eta}$ are the total returns yielded by the portfolios of the existing securities and the newly listed securities, respectively. Obviously, $\boldsymbol{x}'\boldsymbol{\xi}$ is a random variable and $\boldsymbol{y}'\boldsymbol{\eta}$ is an uncertain variable. It follows from Theorem 5 of Liu (2013b) that

$$E[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\eta})] = E[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] = E[\boldsymbol{x}'\boldsymbol{\xi}] + E[\boldsymbol{y}'\boldsymbol{\eta}]. \tag{8.2}$$

Denote by $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_m)'$ the expected vector of $\boldsymbol{\xi}$, by $\Sigma = (\sigma_{ij})_{m\times m}$ the covariance matrix of $\boldsymbol{\xi}$. Qin (2015) made use of the following assumptions.

Assumption 1. The random vector $\boldsymbol{\xi}$ has a multivariate normal distribution with the following probability density function

$$\psi_{\boldsymbol{\xi}}(\boldsymbol{z}) = \frac{1}{\sqrt{(2\pi)^m|\Sigma|}} \exp\left(-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{z}-\boldsymbol{\mu})\right).$$

Assumption 2. Uncertain returns $\eta_1, \eta_2, \cdots, \eta_n$ are independent in the sense of uncertain measure, i.e.,

$$\mathcal{M}\left\{\bigcap_{j=1}^{n}\{\eta_j \in B_j\}\right\} = \bigwedge_{j=1}^{n}\mathcal{M}\left\{\eta_j \in B_j\right\}$$

for any Borel set $B_1, B_2, \cdots, B_n$ of real numbers, in which $\wedge$ is the minimum operator.

According to Assumption 1, $\boldsymbol{\mu}$ and Σ are the mean and covariance matrix of $\boldsymbol{\xi}$, respectively, and $\boldsymbol{x}'\boldsymbol{\xi}$ is a normally distributed random variable with the expected value $\boldsymbol{\mu}'\boldsymbol{x} = x_1\mu_1 + x_2\mu_2 + \cdots + x_m\mu_m$, the variance $\boldsymbol{x}'\Sigma\boldsymbol{x} = \sum_{i=1}^{m}\sum_{j=1}^{m} x_i x_j \sigma_{ij}$ and the following probability density function

$$\psi(w) = \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right) \tag{8.3}$$

where $\sigma(\boldsymbol{x}) = \sqrt{\boldsymbol{x}'\Sigma\boldsymbol{x}}$ and $\sigma^2(\boldsymbol{x}) = (\sigma(\boldsymbol{x}))^2$. Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \cdots, \nu_n)'$ the expected vector of $\boldsymbol{\eta}$. Further, based on Assumption 2, it follows from the linearity of expected value of uncertain variables that

$$E[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] = E[\boldsymbol{x}'\boldsymbol{\xi}] + E[\boldsymbol{y}'\boldsymbol{\eta}] = \boldsymbol{x}'\boldsymbol{\mu} + \boldsymbol{y}'\boldsymbol{\nu}. \tag{8.4}$$

Next, we consider the variance of the portfolio return $r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})$ which is an uncertain random variable. It follows from Theorem 8.4 that the variance is

$$V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] = \int_{-\infty}^{+\infty} \left[u - (\boldsymbol{x}'\boldsymbol{\mu} + \boldsymbol{y}'\boldsymbol{\nu})\right]^2 \mathrm{d}\Phi(u), \tag{8.5}$$

where $\Phi(u)$ is the chance distribution of $r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})$ determined by

$$\Phi(u) = \int_{-\infty}^{+\infty} \Upsilon(u-w)\mathrm{d}\Psi(w) \tag{8.6}$$

in which $\Psi(\cdot)$ is the probability distribution of $\boldsymbol{x}'\boldsymbol{\xi}$ and $\Upsilon(\cdot)$ is the uncertainty distribution of $\boldsymbol{y}'\boldsymbol{\eta}$.

Similar to stochastic situation, the uncertain returns $\eta_1, \eta_2, \cdots, \eta_n$ are always assumed to follow the same type of uncertainty distribution, such as linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and so forth. In these cases, the analytical expressions of variances $V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\eta})]$ can be obtained. For simplicity, we write $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)'$, $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)'$, $\boldsymbol{c} = (c_1, c_2, \cdots, c_n)'$, $\boldsymbol{e} = (e_1, e_2, \cdots, e_n)'$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \cdots, \delta_n)'$. The results are given in the form of theorems stated below.

Theorem 8.5 (Qin 2015). *Assume that $\eta_j \sim \mathcal{L}(a_j, b_j)$ is a linear uncertain variable for $j = 1, 2, \cdots, n$. Then*

$$V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\eta})] = \sigma^2(\boldsymbol{x}) + \frac{(\boldsymbol{b}'\boldsymbol{y} - \boldsymbol{a}'\boldsymbol{y})^2}{12}. \tag{8.7}$$

Proof. It follows from the operational law of uncertain variables that $\boldsymbol{y}'\boldsymbol{\eta}$ is also a linear uncertain variable with expected value $(\boldsymbol{a}'\boldsymbol{y} + \boldsymbol{b}'\boldsymbol{y})/2$ and the following uncertainty distribution

$$\Upsilon(z) = \frac{z - \boldsymbol{a}'\boldsymbol{y}}{\boldsymbol{b}'\boldsymbol{y} - \boldsymbol{a}'\boldsymbol{y}} \cdot I_{\{\boldsymbol{a}'\boldsymbol{y} \le z \le \boldsymbol{b}'\boldsymbol{y}\}} + I_{\{z \ge \boldsymbol{b}'\boldsymbol{y}\}}$$

where $I_{\{\cdot\}}$ is the indicator function of the set $\{\cdot\}$. It follows from Eq. (8.3) and (8.6) that

$$\begin{aligned}
\Phi(u) &= \int_{-\infty}^{+\infty} \left(\frac{u - w - \boldsymbol{a}'\boldsymbol{y}}{\boldsymbol{b}'\boldsymbol{y} - \boldsymbol{a}'\boldsymbol{y}} \cdot I_{\{\boldsymbol{a}'\boldsymbol{y} \le u - w \le \boldsymbol{b}'\boldsymbol{y}\}} + I_{\{u - w \ge \boldsymbol{b}'\boldsymbol{y}\}} \right) \psi(w) \mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{-\infty}^{u - \boldsymbol{b}'\boldsymbol{y}} \exp\left(-\frac{(w - \boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})} \right) \mathrm{d}w \\
&\quad - \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{u - \boldsymbol{b}'\boldsymbol{y}}^{u - \boldsymbol{a}'\boldsymbol{y}} \frac{w - u + \boldsymbol{a}'\boldsymbol{y}}{\boldsymbol{b}'\boldsymbol{y} - \boldsymbol{a}'\boldsymbol{y}} \exp\left(-\frac{(w - \boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})} \right) \mathrm{d}w.
\end{aligned}$$

Taking the derivative with respect to u on both sides of the above equation, we have

$$\frac{\mathrm{d}\Phi(u)}{\mathrm{d}u} = \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{b}'\boldsymbol{y} - \boldsymbol{a}'\boldsymbol{y})} \int_{u - \boldsymbol{b}'\boldsymbol{y}}^{u - \boldsymbol{a}'\boldsymbol{y}} \exp\left(-\frac{(w - \boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})} \right) \mathrm{d}w. \tag{8.8}$$

It follows from Eq. (8.5) that the variance is

$$V[r(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\xi}, \boldsymbol{\eta})] = \int_{-\infty}^{+\infty} \left[u - \left(\boldsymbol{\mu}'\boldsymbol{x} + \frac{\boldsymbol{a}'\boldsymbol{y} + \boldsymbol{b}'\boldsymbol{y}}{2} \right) \right]^2 \mathrm{d}\Phi(u).$$

Writing $g(u) = [u-(\boldsymbol{\mu}'\boldsymbol{x}+(\boldsymbol{a}'\boldsymbol{y}+\boldsymbol{b}'\boldsymbol{y})/2)]^2$ and substituting Eq. (8.8) into the above expression, we have

$$
\begin{aligned}
& V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})}\int_{-\infty}^{+\infty}\int_{u-\boldsymbol{b}'\boldsymbol{y}}^{u-\boldsymbol{a}'\boldsymbol{y}} g(u)\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w\mathrm{d}u \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\int_{w+\boldsymbol{a}'\boldsymbol{y}}^{w+\boldsymbol{b}'\boldsymbol{y}} g(u)\mathrm{d}u\mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\left[\frac{(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2}{12}+(w-\boldsymbol{\mu}'\boldsymbol{x})^2\right]\mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)(w-\boldsymbol{\mu}'\boldsymbol{x})^2\mathrm{d}w \\
&\quad +\frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\frac{(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2}{12}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w \\
&= \sigma^2(\boldsymbol{x})+\frac{(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2}{12}.
\end{aligned}
$$

The last equality holds since

$$\int_{-\infty}^{+\infty}\exp(-x^2)\mathrm{d}x=\sqrt{\pi},\quad \text{and}\quad \int_{-\infty}^{+\infty}x^2\exp(-x^2)\mathrm{d}x=\sqrt{\pi}/2.$$

The theorem is proved.

Theorem 8.6 (Qin 2015). *Assume that $\eta_j \sim \mathcal{Z}(a_j,b_j,c_j)$ is a zigzag uncertain variable for $j=1,2,\cdots,m$. Then*

$$
\begin{aligned}
V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] &= \sigma^2(\boldsymbol{x}) \\
&+\frac{5(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2+5(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})^2+6(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})}{48}.
\end{aligned}
\tag{8.9}
$$

Proof. It follows from the operational law of uncertain variables that $\boldsymbol{y}'\boldsymbol{\eta}$ is also a zigzag uncertain variable with expected value $(\boldsymbol{a}'\boldsymbol{y}+2\boldsymbol{b}'\boldsymbol{y}+\boldsymbol{c}'\boldsymbol{y})/4$ and the following uncertainty distribution

$$\Upsilon(z)=\frac{z-\boldsymbol{a}'\boldsymbol{y}}{2\boldsymbol{b}'\boldsymbol{y}-2\boldsymbol{a}'\boldsymbol{y}}I_{\{\boldsymbol{a}'\boldsymbol{y}\le z\le \boldsymbol{b}'\boldsymbol{y}\}}+\frac{z+\boldsymbol{c}'\boldsymbol{y}-2\boldsymbol{b}'\boldsymbol{y}}{2\boldsymbol{c}'\boldsymbol{y}-2\boldsymbol{b}'\boldsymbol{y}}I_{\{\boldsymbol{b}'\boldsymbol{y}\le z\le \boldsymbol{c}'\boldsymbol{y}\}}+I_{\{z\ge \boldsymbol{c}'\boldsymbol{y}\}}.$$

It follows from Eqs. (8.3) and (8.6) that

$$
\begin{aligned}
\Phi(u) &= \int_{-\infty}^{+\infty} \frac{u-w-\boldsymbol{a}'\boldsymbol{y}}{2\boldsymbol{b}'\boldsymbol{y}-2\boldsymbol{a}'\boldsymbol{y}} I_{\{\boldsymbol{a}'\boldsymbol{y}\le u-w\le \boldsymbol{b}'\boldsymbol{y}\}}\psi(w)\mathrm{d}w \\
&\quad + \int_{-\infty}^{+\infty} \frac{u-w+\boldsymbol{c}'\boldsymbol{y}-2'\boldsymbol{y}}{2\boldsymbol{c}'\boldsymbol{y}-2\boldsymbol{b}'\boldsymbol{y}} I_{\{\boldsymbol{b}'\boldsymbol{y}\le u-w\le \boldsymbol{c}'\boldsymbol{y}\}}\psi(w)\mathrm{d}w \\
&\quad + \int_{-\infty}^{+\infty} I_{\{u-w\ge \boldsymbol{c}'\boldsymbol{y}\}}\psi(w)\mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{u-\boldsymbol{b}'\boldsymbol{y}}^{u-\boldsymbol{a}'\boldsymbol{y}} \frac{u-w-\boldsymbol{a}'\boldsymbol{y}}{2\boldsymbol{b}'\boldsymbol{y}-2\boldsymbol{a}'\boldsymbol{y}} \exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{u-\boldsymbol{c}'\boldsymbol{y}}^{u-\boldsymbol{b}'\boldsymbol{y}} \frac{u-w+\boldsymbol{c}'\boldsymbol{y}-2\boldsymbol{b}'\boldsymbol{y}}{2\boldsymbol{c}'\boldsymbol{y}-2\boldsymbol{b}'\boldsymbol{y}} \exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{u-\boldsymbol{c}'\boldsymbol{y}} \exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w.
\end{aligned}
$$

Taking the derivative with respect to u on both sides of the above equation, we have

$$
\begin{aligned}
\frac{\mathrm{d}\Phi(u)}{\mathrm{d}u} &= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})(2\boldsymbol{b}'\boldsymbol{y}-2\boldsymbol{a}'\boldsymbol{y})}\int_{u-\boldsymbol{b}'\boldsymbol{y}}^{u-\boldsymbol{a}'\boldsymbol{y}} \exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w \\
&\quad + \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})(2\boldsymbol{c}'\boldsymbol{y}-2\boldsymbol{b}'\boldsymbol{y})}\int_{u-\boldsymbol{c}'\boldsymbol{y}}^{u-\boldsymbol{b}'\boldsymbol{y}} \exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w.
\end{aligned}
$$

Write $g(u)=[u-(\boldsymbol{\mu}'\boldsymbol{x}+(\boldsymbol{a}'\boldsymbol{y}+2\boldsymbol{b}'\boldsymbol{y}+\boldsymbol{c}'\boldsymbol{y})/4]^2$. It follows from Eq. (8.5) that the variance is

$$
\begin{aligned}
& V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] \\
&= \int_{-\infty}^{+\infty}\left[u-\left(\boldsymbol{\mu}'\boldsymbol{x}+\frac{\boldsymbol{a}'\boldsymbol{y}+2\boldsymbol{b}'\boldsymbol{y}+\boldsymbol{c}'\boldsymbol{y}}{4}\right)\right]^2\mathrm{d}\Phi(u) \\
&= \frac{1}{2\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{b}-\boldsymbol{a})'\boldsymbol{y}}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\int_{w+\boldsymbol{a}'\boldsymbol{y}}^{w+\boldsymbol{b}'\boldsymbol{y}} g(u)\mathrm{d}u\mathrm{d}w \\
&\quad + \frac{1}{2\sqrt{2\pi}\sigma(\boldsymbol{x})(\boldsymbol{c}-\boldsymbol{b})'\boldsymbol{y}}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\int_{w+\boldsymbol{b}'\boldsymbol{y}}^{w+\boldsymbol{c}'\boldsymbol{y}} g(u)\mathrm{d}u\mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)(w-\boldsymbol{\mu}'\boldsymbol{x})^2\mathrm{d}w
\end{aligned}
$$

$$+\frac{(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2+(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})^2}{6\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w$$
$$-\frac{(2\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y}-\boldsymbol{c}'\boldsymbol{y})^2}{16\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{+\infty}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w$$
$$=\sigma^2(\boldsymbol{x})+\frac{5(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2+5(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})^2+6(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})}{48}$$

The theorem is proved.

Theorem 8.7 (Qin 2015). *Assume that $\eta_j \sim \mathcal{N}(e_j, \delta_j)$ is a normal uncertain variable for $j = 1, 2, \cdots, n$. Then*

$$V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})]=\sigma^2(\boldsymbol{x})+(\boldsymbol{\delta}'\boldsymbol{y})^2. \tag{8.10}$$

Proof. It follows from the operational law of uncertain variables that $\boldsymbol{y}'\boldsymbol{\eta}$ is also a normal uncertain variable with expected value $\boldsymbol{e}'\boldsymbol{y}$ and the following uncertainty distribution

$$\Upsilon(z)=\left(1+\exp\left(\frac{\pi(\boldsymbol{e}'\boldsymbol{y}-z)}{\sqrt{3}\boldsymbol{\delta}'\boldsymbol{y}}\right)\right)^{-1}.$$

It follows from Eqs. (8.3) and (8.6) that

$$\Phi(u)=\frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})}\int_{-\infty}^{+\infty}\Upsilon(u-w)\cdot\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w.$$

Write $g(u,w)=\exp[\pi(\boldsymbol{e}'\boldsymbol{y}-u+w)/(\sqrt{3}\boldsymbol{\delta}'\boldsymbol{y})]$. Taking the derivative with respect to u on both sides of the above equation, we have

$$\frac{\mathrm{d}\Phi(u)}{\mathrm{d}u}=\frac{\sqrt{\pi}}{\sqrt{6}\sigma(\boldsymbol{x})\boldsymbol{\delta}'\boldsymbol{y}}\int_{-\infty}^{+\infty}\frac{g(u,w)}{(1+g(u,w))^2}\exp\left(-\frac{(w-\boldsymbol{\mu}'\boldsymbol{x})^2}{2\sigma^2(\boldsymbol{x})}\right)\mathrm{d}w.$$

Write $h(w)=\exp(-(w-\boldsymbol{\mu}'\boldsymbol{x})^2/(2\sigma^2(\boldsymbol{x})))$. It follows from Eq. (8.5) that the variance is

$$V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})]$$
$$=\int_{-\infty}^{+\infty}\left[u-\left(\boldsymbol{\mu}'\boldsymbol{x}+\boldsymbol{e}'\boldsymbol{y}\right)\right]^2\mathrm{d}\Phi(u)$$
$$=\frac{\sqrt{\pi}}{\sqrt{6}\sigma(\boldsymbol{x})\boldsymbol{\delta}'\boldsymbol{y}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{g(u,w)h(w)}{(1+g(u,w))^2}\left[u-\left(\boldsymbol{\mu}'\boldsymbol{x}+\boldsymbol{e}'\boldsymbol{y}\right)\right]^2\mathrm{d}w\mathrm{d}u$$

$$= \frac{\sqrt{\pi}}{\sqrt{6}\sigma(\boldsymbol{x})\boldsymbol{\delta}'\boldsymbol{y}} \int_{-\infty}^{+\infty} h(w) \int_{-\infty}^{+\infty} \frac{g(u,w)}{(1+g(u,w))^2} \left[u - \left(\boldsymbol{\mu}'\boldsymbol{x} + \boldsymbol{e}'\boldsymbol{y}\right)\right]^2 \mathrm{d}u\mathrm{d}w.$$

By making the change of variables $u = \boldsymbol{e}'\boldsymbol{y} + w - v\sqrt{3}\boldsymbol{\delta}'\boldsymbol{y}/\pi$, we obtain $g(u,w) = \exp(v)$,

$$\begin{aligned}
&V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{-\infty}^{+\infty} h(w) \int_{-\infty}^{+\infty} \left(\frac{\sqrt{3}\boldsymbol{\delta}'\boldsymbol{y}}{\pi} v + \boldsymbol{\mu}'\boldsymbol{x} - w\right)^2 \frac{e^v}{(1+e^v)^2} \mathrm{d}v\mathrm{d}w \\
&= \frac{1}{\sqrt{2\pi}\sigma(\boldsymbol{x})} \int_{-\infty}^{+\infty} h(w) \left[(\boldsymbol{\delta}'\boldsymbol{y} + (w - \boldsymbol{\mu}'\boldsymbol{x})^2\right] \mathrm{d}w \\
&= \sigma^2(\boldsymbol{x}) + (\boldsymbol{\delta}'\boldsymbol{y})^2.
\end{aligned}$$

The theorem is proved.

8.4 Model Formulation

Mean-variance model aims at finding out the most desirable portfolio only by the first two moments. Expected value of the total return is regarded as the investment return, and the variance is used to measure the investment risk. A typical mean-variance model is to compromise between risk and return. For example, we may choose a portfolio with minimum investment risk on the condition that the acceptable return level is given. Suppose that $\mathbf{1}_m$ and $\mathbf{1}_n$ are respectively the m-dimensional column vector and n-dimensional column vector of which all entries are 1. Following the spirit, Qin (2015) formulated the following uncertain random mean-variance model for hybrid portfolio optimization problem,

$$\begin{cases} \min\ V[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] = V[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] \\ \text{s.t.}\quad E[r(\boldsymbol{x},\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\eta})] = E[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] = \mu_0 \\ \qquad \mathbf{1}_m'\boldsymbol{x} + \mathbf{1}_n'\boldsymbol{y} = 1 \end{cases} \tag{8.11}$$

where μ_0 is the acceptable return level and the second constraint ensures that the net expected return is exactly equal to the return level μ_0.

We may also choose a portfolio with maximum investment return on the condition that the tolerable risk level is given. The detailed formulation is proposed by Qin (2015) as follows,

$$\begin{cases} \max\ E[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] \\ \text{s.t.}\quad V[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] = \lambda_0 \\ \qquad \mathbf{1}_m'\boldsymbol{x} + \mathbf{1}_n'\boldsymbol{y} = 1 \end{cases} \tag{8.12}$$

where λ_0 is the tolerable risk level.

In practice, the investor does not always know how to set μ_0 and ν_0. An alternative is to provide a factor representing the risk aversion by trading off return and risk. Thus, Qin (2015) established the following compromise model,

$$\begin{cases} \min\ \phi V[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] - E[\boldsymbol{x}'\boldsymbol{\xi} + \boldsymbol{y}'\boldsymbol{\eta}] \\ \text{s.t.}\quad \mathbf{1}_m'\boldsymbol{x} + \mathbf{1}_n'\boldsymbol{y} = 1 \end{cases} \tag{8.13}$$

where ϕ is a factor of risk aversion.

Next, we consider the solution procedure of the proposed models. Qin (2015) made the third assumption.

Assumption 3. The covariance matrix Σ is positive definite and the expectation $\boldsymbol{\mu}$ is not a multiple of $\mathbf{1}_m'$.

Assumption 3 guarantees a nondegenerate situation and implies that all the existing securities are indeed risky. The detailed remark can be found in Steinbach (2001). Based on Assumption 3, we write $\rho = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}, \kappa = \mathbf{1}_m'\Sigma^{-1}\mathbf{1}_m', \varsigma = \mathbf{1}_m'\Sigma^{-1}\boldsymbol{\mu}$ that will be used throughout this section.

According to Assumptions 1 and 2, it follows from Theorems 8.5, 8.6, and 8.7 that model (8.13) is equivalent to the following form,

$$\begin{cases} \min\ \phi[\boldsymbol{x}'\Sigma\boldsymbol{x} + \boldsymbol{y}'\boldsymbol{p}\boldsymbol{p}'\boldsymbol{y}] - (\boldsymbol{x}'\boldsymbol{\mu} + \boldsymbol{y}'\boldsymbol{\nu}) \\ \text{s.t.}\quad \mathbf{1}_m'\boldsymbol{x} + \mathbf{1}_n'\boldsymbol{y} = 1. \end{cases} \tag{8.14}$$

Model (8.14) is a convex quadratic programming. We may use the method of Lagrange multipliers to find out its solution.

The Lagrange function of model (8.14) is

$$L(\boldsymbol{x},\boldsymbol{y},\lambda) = \phi(\boldsymbol{x}'\Sigma\boldsymbol{x} + \boldsymbol{y}'\boldsymbol{p}\boldsymbol{p}'\boldsymbol{y}) - (\boldsymbol{x}'\boldsymbol{\mu} + \boldsymbol{y}'\boldsymbol{\nu}) - \lambda[(\mathbf{1}_m'\boldsymbol{x} + \mathbf{1}_n'\boldsymbol{y}) - 1].$$

Further, the first-order necessary conditions are

$$\frac{\partial}{\partial \boldsymbol{x}}L(\boldsymbol{x},\boldsymbol{y},\lambda) = 2\phi\Sigma\boldsymbol{x} - \boldsymbol{\mu} - \lambda\mathbf{1}_m = 0, \tag{8.15}$$

$$\frac{\partial}{\partial \boldsymbol{y}}L(\boldsymbol{x},\boldsymbol{y},\lambda) = 2\phi\boldsymbol{p}\boldsymbol{p}'\boldsymbol{y} - \boldsymbol{\nu} - \lambda\mathbf{1}_n = 0, \tag{8.16}$$

$$\frac{\partial}{\partial \lambda}L(\boldsymbol{x},\boldsymbol{y},\lambda) = (\mathbf{1}_m'\boldsymbol{x} + \mathbf{1}_n'\boldsymbol{y}) - 1 = 0, \tag{8.17}$$

in which Eq. (8.17) is essentially the original constraint of model (8.14). If $(\boldsymbol{x}^*, \boldsymbol{y}^*, \lambda^*)$ is the solution of Eqs. (8.15), (8.16), and (8.17), then it is called a stationary point for the Lagrange function $L(\boldsymbol{x},\boldsymbol{y},\lambda)$. It is known that a stationary point of a convex function is always a global minimum point. Thus, the problem of finding the optimal solution of model (8.14) is reduced to find stationary points of Eqs. (8.15), (8.16), and (8.17).

Theorem 8.8 (Qin 2015). *Let $n = 1$ and $\boldsymbol{p} = (p) > 0$. Then the solution of model (8.13) is*

$$\boldsymbol{x}^* = \frac{1}{2\phi}\Sigma^{-1}\boldsymbol{\mu} + \frac{\lambda^*}{2\phi}\Sigma^{-1}\mathbf{1}_m, \quad y^* = \frac{1}{2\phi p^2}(\nu + \lambda^*) \tag{8.18}$$

where

$$\lambda^* = \frac{(2\phi - \varsigma)p^2 - \nu}{p^2\kappa + 1}.$$

Proof. First, Eqs. (8.15) and (8.16) yield

$$\boldsymbol{x}^* = \frac{1}{2\phi}\Sigma^{-1}\boldsymbol{\mu} + \frac{\lambda^*}{2\phi}\Sigma^{-1}\mathbf{1}_m, \quad y^* = \frac{1}{2\phi p^2}(\nu + \lambda^*).$$

Substituting into the Eq. (8.17) yields

$$\frac{1}{2\phi}\mathbf{1}_m'\Sigma^{-1}\boldsymbol{\mu} + \frac{\lambda^*}{2\phi}\mathbf{1}_m'\Sigma^{-1}\mathbf{1}_m + \frac{1}{2\phi p^2}(\nu + \lambda^*) = 1$$

which implies that λ^* must be equal to be $(2\phi p^2 - \varsigma p^2 - \nu)/(p^2\kappa + 1)$. The theorem is proved.

Example 8.1. We consider a simplified portfolio selection problem with one existing security and one newly listed one. In this case, $m = 1$ and $n = 1$. Write $\Sigma = \sigma_{11} = \sigma^2$, $\boldsymbol{\mu} = \mu$. Then we have

$$\rho = \frac{\mu^2}{\sigma^2}, \quad \kappa = \frac{1}{\sigma^2}, \quad \varsigma = \frac{\mu}{\sigma^2}, \quad \lambda^* = \frac{(2\phi\sigma^2 - \mu)p^2 - \nu\sigma^2}{p^2 + \sigma^2}.$$

The optimal proportions invested these into two securities are

$$x^* = \frac{\mu - \nu + 2\phi p^2}{2\phi(p^2 + \sigma^2)}, \quad y^* = \frac{\nu - \mu + 2\phi\sigma^2}{2\phi(p^2 + \sigma^2)}$$

respectively. In particular, they have the following relationship,

$$\begin{cases} x^* > y^*, \text{ if } \mu + \phi p^2 > \nu + \phi\sigma^2 \\ x^* = y^*, \text{ if } \mu + \phi p^2 = \nu + \phi\sigma^2 \\ x^* < y^*, \text{ if } \mu + \phi p^2 < \nu + \phi\sigma^2. \end{cases}$$

Theorem 8.9 (Qin 2015). *Let $n = 2$ and $\boldsymbol{p} = (p_1, p_2)'$ with $p_1 \neq p_2$. Then the solution of model* (8.13) *is*

$$\boldsymbol{x}^* = \frac{1}{2\phi}\Sigma^{-1}\boldsymbol{\mu} + \frac{\lambda^*}{2\phi}\Sigma^{-1}\mathbf{1}_m \tag{8.19}$$

and

$$y_1^* = \frac{(\nu_1 - \nu_2) - \kappa p_2(p_1\nu_2 - p_2\nu_1)}{2\phi(p_2 - p_1)^2} + \frac{(2\phi - \varsigma)p_2}{2\phi(p_2 - p_1)}$$
$$y_2^* = \frac{(\nu_2 - \nu_1) + \kappa p_1(p_1\nu_2 - p_2\nu_1)}{2\phi(p_2 - p_1)^2} - \frac{(2\phi - \varsigma)p_1}{2\phi(p_2 - p_1)}$$

where

$$\lambda^* = \frac{p_1\nu_2 - p_2\nu_1}{p_2 - p_1}.$$

Proof. When $n = 2$, it follows from Eq. (8.16) that

$$p_1y_1^* + p_2y_2^* = \frac{\nu_1 + \lambda^*}{2\phi p_1}, \quad p_1y_1^* + p_2y_2^* = \frac{\nu_2 + \lambda^*}{2\phi p_2} \tag{8.20}$$

which yields $\lambda^* = (p_1\nu_2 - p_2\nu_1)/(p_2 - p_1)$. Equation (8.15) yields

$$\boldsymbol{x}^* = \frac{1}{2\phi}\Sigma^{-1}\boldsymbol{\mu} + \frac{\lambda^*}{2\phi}\Sigma^{-1}\mathbf{1}_m.$$

Substituting $\boldsymbol{x}$ into the Eq. (8.17) yields

$$y_1^* + y_2^* = \frac{2\phi - (\varsigma + \lambda^*\kappa)}{2\phi}. \tag{8.21}$$

Substituting $\lambda^* = (p_1\nu_2 - p_2\nu_1)/(p_2 - p_1)$ into Eq. (8.21) and considering the simultaneous equations (8.20) and (8.21), y_1^* and y_2^* are uniquely obtained. The theorem is proved.

Remark 8.1. Assume that $p_1 = p_2$, i.e., the variances are same for two newly listed securities. Then investors generally prefer the security with larger expected return. Therefore, the security with smaller expected return is naturally not considered and the problem is reduced to the one in Example 8.1.

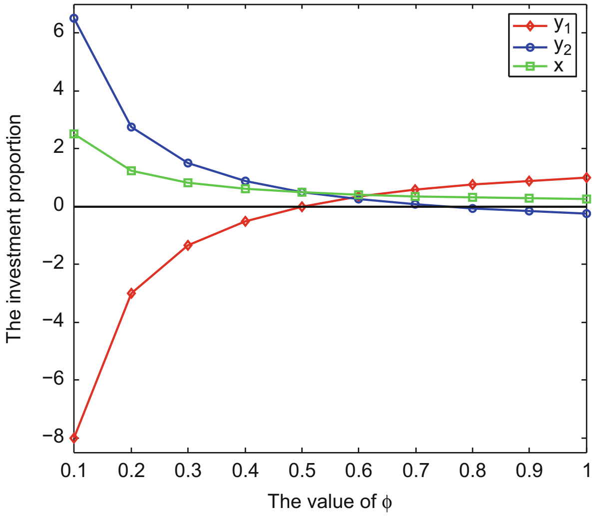

Fig. 8.1 Investment proportions of different securities with different values of ϕ

Example 8.2. Let $m = 1$. That is, we consider a simplified portfolio selection problem with one existing security and two newly listed ones with different risks. According to Theorem 8.9, we have

$$x^* = \frac{p_1(\nu_2-\mu)+p_2(\mu-\nu_1)}{2\phi\sigma^2(p_2-p_1)},$$

$$y_1^* = \frac{(\nu_1-\nu_2)\sigma^2 - p_2(p_1\nu_2-p_2\nu_1)}{2\phi\sigma^2(p_2-p_1)^2} + \frac{(2\phi\sigma^2-\mu)p_2}{2\phi\sigma^2(p_2-p_1)},$$

$$y_2^* = \frac{(\nu_2-\nu_1)\sigma^2 + p_1(p_1\nu_2-p_2\nu_1)}{2\phi\sigma^2(p_2-p_1)^2} - \frac{(2\phi\sigma^2-\mu)p_1}{2\phi\sigma^2(p_2-p_1)}.$$

Let $\nu_1 = 2, p_1 = 1, \nu_2 = 3, p_2 = 2, \mu = 3$ and $\sigma^2 = 4$. Figure 8.1 visually shows the investment proportions of different securities with the changing of the values of the risk aversion factor ϕ.

Suppose that $\mu = \nu_1 = \nu_2$. Then

$$x^* = 0, \quad y_1^* = \frac{p_2}{p_2-p_1}, \quad y_2^* = -\frac{p_1}{p_2-p_1}$$

which implies that the existing security is not selected if it has the same expected return with the newly listed securities regardless of its variance.

Next we consider the case that the number n of newly listed securities is more than 2. First note that the rank of matrix $\boldsymbol{pp}'$ is no more than 1 which implies that the solution of simultaneous equations (8.15), (8.16), and (8.17) is either no solution or infinitely many solutions. In the latter case, we may find the stationary point to further obtain the optimal solution because of the convexity of the proposed models. We also may directly solve the original problem since the proposed models are convex quadratic programmings, which can be solved by available softwares such as Matlab.

8.5 Numerical Examples

In this section, we illustrate the application of uncertain random mean-variance model by a numerical experiment presented in Qin (2015).

Example 8.3. Suppose that there are five newly listed stocks and their monthly return rates are estimated by experts based on the available information. Without loss of generality, suppose that the return rates are denoted by zigzag uncertain variables. The simulated data are shown in Table 8.1 including their expected values and variances.

The existing securities are from Shanghai Stock Exchange (SSE) and the codes of randomly chosen 20 stocks are shown in Table 8.2. The analyzed transaction data of the 20 stocks are from January 29, 2010 to April 1, 2014 and 51 monthly return rates are obtained for each stock. From these data, the sample mean vector $\boldsymbol{\mu}$ and covariance matrix Σ are computed and they are shown in Tables 8.2 and 8.3, respectively.

Next, we consider model (8.11). In many security markets, short selling is not allowed. In this case, we need to add the constraint of nonnegativity and model (8.11) is reformulated as follows,

Table 8.1 Uncertain returns (UR), expected values (E), variances (V) of simulated 5 newly listed stocks

	η_1	η_2	η_3	η_4	η_5
UR	$(-20, 0, 18)$	$(-25, 0, 25)$	$(-30, 2, 20)$	$(-30, 5, 25)$	$(-35, 5, 20)$
E	-0.50	0.00	-1.50	1.25	-1.25
V	120.4	208.3	212.4	256.8	265.1

Table 8.2 The codes and sample means of the randomly chosen 20 stocks from SSE

No.	1	2	3	4	5
Code	600000	600008	600009	600016	600019
Mean	−1.566	0.893	−0.611	1.040	−2.590
No.	6	7	8	9	10
Code	600027	600030	600050	600064	600085
Mean	−0.857	−2.179	−2.984	−1.953	0.804
No.	11	12	13	14	15
Code	600104	600118	600170	600177	600271
Mean	−0.779	0.893	−1.881	−1.780	0.953
No.	16	17	18	19	20
Code	600489	600887	600999	601088	601299
Mean	−6.093	2.581	−2.892	−2.709	0.412

$$\begin{cases} \min \boldsymbol{x}'\Sigma\boldsymbol{x} + \dfrac{5(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})^2 + 5(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})^2 + 6(\boldsymbol{b}'\boldsymbol{y}-\boldsymbol{a}'\boldsymbol{y})(\boldsymbol{c}'\boldsymbol{y}-\boldsymbol{b}'\boldsymbol{y})}{48} \\ \text{s.t.} \quad \boldsymbol{x}'\boldsymbol{\mu} + \boldsymbol{y}'\boldsymbol{v} = \mu_0, \\ \qquad \mathbf{1}_{20}'\boldsymbol{x} + \mathbf{1}_5'\boldsymbol{y} = 1, \\ \qquad \boldsymbol{x}, \boldsymbol{y} \geq 0, \end{cases} \tag{8.22}$$

where $\boldsymbol{\mu}$, Σ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{v}$ can be found in Tables 8.2, 8.3 and 8.1, respectively. Obviously, it is a convex quadratic programming. We employ the function "quadprog" function in Matlab to solve it. The computational results are listed in Table 8.4 which shows the investment proportions of the selected stocks for each given return level μ_0. The last column of Table 8.4 refers to the minimum variance of all possible portfolios for each given return level. Further, the efficient frontier is obtained and depicted in a solid line in Fig. 8.2. In this figure, the vertical axis represents the given return level and the horizontal axis represents the corresponding minimum variance.

Table 8.3 The sample covariance matrix of the 20 stocks

231.2	69.2	94.5	174.4	76.9	94.7	188.5	54.6	135.6	41.0	131.1	95.6	85.1	125.3	45.1	91.1	109.2	141.4	80.0	70.4
69.2	228.7	59.5	55.4	41.9	52.6	66.1	42.8	90.0	63.0	50.4	135.1	78.0	87.1	48.6	14.3	105.1	84.4	25.5	52.0
94.5	59.5	115.1	69.9	55.1	41.5	106.8	44.9	111.6	14.2	76.8	93.5	68.5	88.7	75.7	22.9	64.6	84.9	45.0	57.8
174.4	55.4	69.9	193.2	52.5	79.2	150.0	45.5	97.3	20.6	88.1	85.7	50.2	85.9	26.0	72.9	53.8	126.4	68.4	33.0
76.9	41.9	55.1	52.5	76.3	27.6	78.0	41.5	81.3	21.0	94.2	75.9	83.9	64.1	32.3	28.9	36.8	71.6	64.4	71.4
94.7	52.6	41.5	79.2	27.6	166.8	69.6	10.6	48.9	79.7	59.4	23.8	−14.4	68.7	−19.2	46.6	34.1	54.6	45.5	−14.6
188.5	66.1	106.8	150.0	78.0	69.6	309.0	47.0	207.5	65.0	179.4	121.7	105.5	113.4	135.6	123.4	55.3	185.4	107.6	90.7
54.6	42.8	44.9	45.5	41.5	10.6	47.0	66.3	47.4	−9.8	42.2	50.6	33.2	50.9	41.4	−10.6	27.1	49.8	32.8	74.0
135.6	90.0	111.6	97.3	81.3	48.9	207.5	47.4	250.4	29.6	145.8	166.0	142.1	119.7	145.3	37.9	62.9	156.0	94.1	103.3
41.0	63.0	14.2	20.6	21.0	79.7	65.0	−9.8	29.6	307.5	41.3	50.7	47.3	3.9	16.0	86.8	106.9	106.9	24.6	19.8
131.1	50.4	76.8	88.1	94.2	59.4	179.4	42.2	145.8	41.3	215.2	144.3	119.6	101.2	79.5	59.2	28.0	114.5	110.2	102.1
95.6	135.1	93.5	85.7	75.9	23.8	121.7	50.6	166.0	50.7	144.3	517.4	168.4	156.8	201.6	−30.8	96.9	173.5	77.0	88.4
85.1	78.0	68.5	50.2	83.9	−14.4	105.5	33.2	142.1	47.3	119.6	168.4	316.4	98.7	75.1	42.8	86.0	95.8	90.6	52.9
125.3	87.1	88.7	85.9	64.1	68.7	113.4	50.9	119.7	3.9	101.2	156.8	98.7	152.0	86.3	25.1	89.7	92.4	74.6	56.8
45.1	48.6	75.7	26.0	32.3	−19.2	135.6	41.4	145.3	16.0	79.5	201.6	75.1	86.3	240.9	31.5	52.1	97.4	46.8	74.7
91.1	14.3	22.9	72.9	28.9	46.6	123.4	−10.6	37.9	86.8	59.2	−30.8	42.8	25.1	31.5	235.7	41.5	57.8	46.9	20.2
109.2	105.1	64.6	53.8	36.8	34.1	55.3	27.1	62.9	106.9	28.0	96.9	86.0	89.7	52.1	41.5	273.6	86.4	6.0	13.5
141.4	84.4	84.9	126.4	71.6	54.6	185.4	49.8	156.0	106.9	114.5	173.5	95.8	92.4	97.4	57.8	86.4	237.3	71.2	72.6
80.0	25.5	45.0	68.4	64.4	45.5	107.6	32.8	94.1	24.6	110.2	77.0	90.6	74.6	46.8	46.9	6.0	71.2	100.5	43.6
70.4	52.0	57.8	33.0	71.4	−14.6	90.7	74.0	103.3	19.8	102.1	88.4	52.9	56.8	74.7	20.2	13.5	72.6	43.6	262.9

Table 8.4 The investment proportions of the selected stocks in the optimal portfolio of model (8.22)

	60	60	60	60	60	60	60						
μ_0	0008	0016	0027	0085	0271	0887	1299	η_1	η_2	η_3	η_4	η_5	Variance
0.20	1.97	7.66	6.67	3.34	7.76	7.46	6.47	18.86	12.04	8.45	12.09	7.22	26.04
0.30	2.04	8.84	4.99	3.69	7.88	8.76	6.39	18.37	12.13	7.46	12.89	6.57	27.14
0.40	2.11	10.02	3.30	4.03	8.00	10.05	6.31	17.88	12.21	6.46	13.70	5.91	28.55
0.50	2.19	11.20	1.62	4.38	8.12	11.35	6.23	17.39	12.29	5.47	14.51	5.26	30.27
0.60	2.25	12.35	0	4.70	8.25	12.66	6.16	16.89	12.37	4.46	15.32	4.59	32.28
0.70	1.97	12.89	0	4.56	8.57	14.45	6.25	16.02	12.33	3.06	16.23	3.66	34.65
0.80	1.69	13.43	0	4.42	8.90	16.23	6.34	15.16	12.30	1.66	17.15	2.72	37.40
0.90	1.41	13.97	0	4.28	9.23	18.02	6.43	14.30	12.26	0.26	18.06	1.78	40.54
1.00	0.99	14.50	0	4.01	9.53	20.23	6.44	12.83	12.00	0	19.04	0.42	44.11
1.10	0.37	14.98	0	3.54	9.79	23.03	6.34	10.48	11.40	0	20.08	0	48.23
1.20	0	15.38	0	2.96	10.00	25.96	6.14	7.76	10.65	0	21.15	0	53.02
1.30	0	15.71	0	2.34	10.18	28.81	5.88	4.97	9.88	0	22.22	0	58.51
1.40	0	16.04	0	1.72	10.36	31.67	5.62	2.19	9.11	0	23.29	0	64.70
1.50	0	16.25	0	0.93	10.44	34.81	5.22	0	8.02	0	24.31	0	71.59
1.60	0	15.91	0	0	10.16	38.97	4.19	0	5.60	0	25.17	0	79.47
1.70	0	15.45	0	0	9.82	42.97	2.94	0	2.85	0	25.97	0	88.57
1.80	0	14.98	0	0	9.48	46.97	1.69	0	0.11	0	26.77	0	98.91
1.90	0	12.84	0	0	7.79	52.60	0	0	0	0	26.77	0	110.81
2.00	0	9.70	0	0	4.93	58.98	0	0	0	0	26.39	0	124.90
2.10	0	6.56	0	0	2.07	65.36	0	0	0	0	26.01	0	141.28
2.20	0	2.83	0	0	0	71.82	0	0	0	0	25.35	0	159.97
2.30	0	0	0	0	0	78.89	0	0	0	0	21.11	0	181.72
2.40	0	0	0	0	0	86.40	0	0	0	0	13.60	0	209.00
2.50	0	0	0	0	0	93.91	0	0	0	0	6.09	0	242.26
2.58	0	0	0	0	0	100.00	0	0	0	0	0	0	273.60

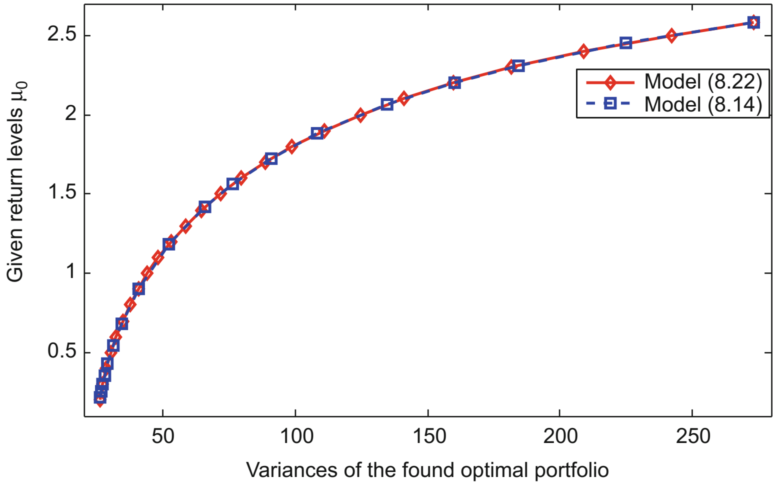

Fig. 8.2 Comparisons of efficient frontiers of models (8.14) and (8.22)

We also employ model (8.14) to construct the desirable portfolio by changing the factor of risk aversion. In fact, we can obtain the same optimal portfolio by adjusting the selections of parameters μ_0 and ϕ. In order to highlight it, we solve model (8.14) by using quadprog function for different values of ϕ. From Fig. 8.2, it is easy to see that the efficient frontier of model (8.14) is the same as that of model (8.22).

Chapter 9
Fuzzy Random Mean-Variance Adjusting Model

9.1 Introduction

Fuzzy portfolio optimization overemphasizes the experts' subjective experiences, which results in that it is difficult to accurately estimate security returns. In fact, historical data and subjective experiences may play equally important roles in making investment decisions. This motivates the researchers to investigate the portfolio optimization problem with mixture of randomness and fuzziness such as Huang (2007a) and Gupta et al. (2013). Fuzzy random variable proposed by Kwakernaak (1978) is a useful tool to integrate these two kinds of information. It has actually been applied to many financial optimization problems such as risk model (Huang et al. 2009), risk assessment (Shen and Zhao 2010), life annuity (de Andres-Sanchez and Puchades 2012; Shapiro 2013) and so forth.

In particular, several authors have applied fuzzy random variables to characterize security returns and studied fuzzy random portfolio selection problems. For example, Hao and Liu (2009) and Li and Xu (2009) both considered security returns as fuzzy random variables and respectively proposed mean-variance models from different perspectives. Yoshida (2009) proposed an estimation model of value-at-risk portfolio under fuzzy random environment. After that, Liu et al. (2012) further developed two types of chance-variance optimization models to provide more decision criteria for investors. Li and Xu (2013) established a multi-objective portfolio selection model by adding the consideration of liquidity to being closer to actual investment problems, and Sadati and Nematian (2013) presented two-level linear programming for the problem by maximizing the degrees of both possibility and necessity that the objective function satisfies the fuzzy goals. It is easy to see that there is an increasing trend in the literature on fuzzy random portfolio optimization.

The above works focus on the single-period portfolio optimization without considering transaction costs. However, transaction costs should obviously not be ignored in practice. Moreover, due to financial market volatility, investors prefer to make periodic adjustments of their holding portfolios during the whole

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investment horizon to achieve the best goal. Actually, many works have been devoted to studying the portfolio adjusting problem. For example, in the stochastic environment, Yu and Lee (2011) developed portfolio rebalancing model by using multiple criteria, and Woodside-Oriakhi et al. (2013) considered Markowitz mean-variance rebalancing portfolio within an investment horizon. In fuzzy environment, Zhang et al. (2010) first solved the portfolio adjusting problem based on credibility theory, and then Zhang et al. (2011) considered the problem with added assets. Different from their works, Gupta et al. (2013) established a multi-objective portfolio rebalancing model for fuzzy portfolio optimization. However, these works either assume random security returns or consider security returns as fuzzy variables.

Different from the above existing portfolio adjusting models, Qin and Xu (2016) prefers establishing a more realistic model with minimum transaction lots in the framework of mean-variance analysis. Minimum transaction lots has actually attracted attentions from several authors such as Huang and Ying (2013) and Woodside-Oriakhi et al. (2013). Specifically, Qin and Xu (2016) took into account not only transaction costs but also minimum transaction lots to formulate two kinds of mean-variance adjusting models following the rules of stock exchange. The first one is called a “theoretical” model by only considering transaction costs, which is in spirit consistent with the existing literature such as Zhang et al. (2010). The other one is called a “practical” model since it simultaneously considers transaction costs and minimum transaction lots. In order to make the proposed models solvable, they focused on two cases of fuzzy random returns and converted the proposed models into a nonlinear programming and a nonlinear integer programming, respectively.

This chapter focuses on portfolio adjusting problem by using fuzzy random variables to describe security returns. The main contents include review of fuzzy random variable, formulation of mean-variance adjusting model, crisp equivalents and numerical examples.

9.2 Fuzzy Random Variable

In this section, we review some basic definitions and results about fuzzy random variable.

Definition 9.1 (Liu and Liu 2003). A fuzzy random variable ξ is a function from a probability space $(\Omega, \mathcal{A}, \Pr)$ to the set of fuzzy variables such that $\mathrm{Cr}\{\xi(\omega) \in B\}$ is a measurable function of ω for any Borel set B of real numbers.

Definition 9.2 (Liu and Liu 2003). Let ξ be a fuzzy random variable ξ defined on a probability space $(\Omega, \mathcal{A}, \Pr)$. Its expected value is defined as

$$E[\xi] = \int_0^{\infty} \Pr\{\omega \in \Omega | E[\xi(\omega)] \geq r\} \mathrm{d}r$$

$$-\int_{-\infty}^{0} \Pr\{\omega \in \Omega | E[\xi(\omega)] \le r\} \mathrm{d}r \tag{9.1}$$

provided that at least one of the two integrals is finite.

Definition 9.3 (Liu and Liu 2003). Let ξ be a fuzzy random variable ξ with a finite expected value. The variance of ξ is defined as

$$V[\xi] = E[(\xi - E[\xi])^2]. \tag{9.2}$$

Example 9.1. Let $(\Omega, \mathcal{A}, \Pr)$ be a probability space with finite sample points $\omega_1, \omega_2, \cdots, \omega_n$, and let $\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n$ be fuzzy variables. Then the function defined by

$$\xi(\omega) = \begin{cases} \tilde{a}_1, \text{ if } \omega = \omega_1 \\ \tilde{a}_2, \text{ if } \omega = \omega_2 \\ \cdots \\ \tilde{a}_n, \text{ if } \omega = \omega_n \end{cases}$$

is just a fuzzy random variable, and we call it a simple fuzzy random variable.

Example 9.2. Suppose that X is a random variable defined on $(\Omega, \mathcal{A}, \Pr)$ with finite expected value. We call ξ a triangular fuzzy random variable, if for each $\omega \in \Omega$, $\xi(\omega) = (X(\omega) - \alpha, X(\omega), X(\omega) + \beta)$ is a triangular fuzzy variable. For each ω, the membership function of fuzzy variable $\xi(\omega)$ is

$$\mu_{\xi(\omega)}(x) = \begin{cases} (x - X(\omega) + \alpha)/\alpha, & \text{if } X(\omega) - \alpha \le x \le X(\omega) \\ (-x + X(\omega) + \beta)/\beta, & \text{if } X(\omega) \le x \le X(\omega) + \beta \\ 0, & \text{otherwise.} \end{cases}$$

Further, it follows from Definition 9.2 that $E[\xi] = E[X] + (\beta - \alpha)/4$. If $\alpha = \beta$, ξ is called a symmetrical triangular fuzzy random variable with expected value $E[\xi] = E[X]$.

Theorem 9.1 (Liu and Liu 2003). *Let ξ and η be two independent fuzzy random variables with finite expected values, and a and b two real numbers. Then we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{9.3}$$

In particular, we have $E[a\xi + b] = aE[\xi] + b$ and $V[a\xi + b] = a^2 V[\xi]$.

9.3 Mean-Variance Adjusting Model

This section first states the problem of finding the optimal investment strategy by adjusting the existing portfolio in the presence of transaction costs and/or minimum transaction lots and then introduces the corresponding mean-variance adjusting models. For sake of convenience, the notations are listed as follows,

- n the total number of risky securities
- A_0 the initial capital which is equal to the value of the existing portfolio
- ξ_i the future return of the ith risky security, which is a fuzzy random variable, $i = 1, 2, \cdots, n$
- x_i^0 the holding proportion of security i before adjusting, $i = 1, 2, \cdots, n$
- x_i the holding proportion of security i after adjusting, $i = 1, 2, \cdots, n$
- x_i^+ the buying quantity (proportion) of security i, $i = 1, 2, \cdots, n$
- x_i^- the selling quantity (proportion) of security i, $i = 1, 2, \cdots, n$
- b_i the unit transaction cost for buying security i, $i = 1, 2, \cdots, n$
- s_i the unit transaction cost for selling security i, $i = 1, 2, \cdots, n$
- p_i the current price of security i in the adjusting time, $i = 1, 2, \cdots, n$
- μ_0 the minimum return level desired by the investor
- ν_0 the maximum risk level beared by the investor

The decision variables of the problem is x_i^+ and x_i^- for $i = 1, 2, \cdots, n$. The existing portfolio is denoted by $(x_1^0, x_2^0, \cdots, x_n^0)$ and, without loss of generality, we have $x_1^0 + x_2^0 + \cdots + x_n^0 = 1$. According to the above notations, the intrinsic relationship

$$x_i = x_i^0 + x_i^+ - x_i^-$$

naturally holds for $i = 1, 2, \cdots, n$. To exclude short selling from our models, we assume that $x_i^+ \geq 0$ and $0 \leq x_i^- \leq x_i^0$. In addition, for an optimal strategy, both x_i^+ and x_i^- are impossible to simultaneously be positive, which means that the constraint $x_i^+ \cdot x_i^- = 0$ holds for $i = 1, 2, \cdots, n$. The adjusting cost of security i is composed of two parts: buying cost $b_i x_i^+$ and selling cost $s_i x_i^-$ which implies that the total adjusting cost is $\sum_{i=1}^n (b_i x_i^+ + s_i x_i^-)$. It is assumed that the whole investment process is self-financing, i.e., we neither add new fund to the portfolio nor take fund out of the portfolio, which implies that

$$\sum_{i=1}^n x_i^0 - \sum_{i=1}^n (b_i x_i^+ + s_i x_i^-) = \sum_{i=1}^n x_i.$$

Substituting $x_i = x_i^0 + x_i^+ - x_i^-$ into the above equation, it is obtained that

$$\sum_{i=1}^n (b_i + 1) x_i^+ + \sum_{i=1}^n (s_i - 1) x_i^- = 0.$$

After adjusting, the net return on the portfolio $(x_1, x_2, \cdots, x_n)$ is

$$\sum_{i=1}^{n} x_i\xi_i - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-)$$

which is still a fuzzy random variable according to the operational law of fuzzy random variables. Thus, the expression $\sum_{i=1}^{n} x_i\xi_i - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-)$ essentially takes various values associated with some chance, which implies that it is meaningless if we directly maximize a fuzzy random variable. Similar to mean-variance analysis, we employ, respectively, expected value and variance of fuzzy random return on portfolio as the investment return and risk (also refer to Hao and Liu 2009 and Li and Xu 2009). It follows from Theorem 9.1 that

$$\begin{aligned} & E\left[\sum_{i=1}^{n} x_i\xi_i - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-)\right] \\ & = E\left[\sum_{i=1}^{n} x_i\xi_i\right] - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-), \end{aligned} \tag{9.4}$$

and

$$V\left[\sum_{i=1}^{n} x_i\xi_i - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-)\right] = V\left[\sum_{i=1}^{n} x_i\xi_i\right]. \tag{9.5}$$

While not considering other constraints, a typical portfolio optimization model is to compromise between risk and return since maximizing investment return and minimizing investment risk are always two conflicting objectives. In general, there are two ways to trade-off risk and return. One is to choose a portfolio with minimum investment risk on the condition that the acceptable return level is given. The other is to choose a portfolio with maximum investment return on the condition that the tolerable risk level is given. According to the first way, a theoretical mean-variance adjusting model is formulated by Qin and Xu (2016) as follows,

$$\text{(Model T)} \quad \begin{cases} \min\ V\left[\sum_{i=1}^{n}(x_i^0 + x_i^+ - x_i^-)\xi_i\right] \\ \text{s.t.}\ \ E\left[\sum_{i=1}^{n}(x_i^0 + x_i^+ - x_i^-)\xi_i\right] - \sum_{i=1}^{n}(b_i x_i^+ + s_i x_i^-) \geq \mu_0 \\ \qquad \sum_{i=1}^{n}(b_i + 1)x_i^+ + \sum_{i=1}^{n}(s_i - 1)x_i^- = 0 \\ \qquad x_i^+ \cdot x_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad x_i^+ \geq 0,\ i = 1, 2, \cdots, n \\ \qquad 0 \leq x_i^- \leq x_i^0,\ i = 1, 2, \cdots, n \end{cases}$$

in which the first constraint ensures that the net expected return is no less than the return level μ_0. According to the second way, Qin and Xu (2016) regarded maximizing the expected value of the net return as the decision objective, and established another kind of theoretical mean-variance adjusting model for fuzzy random portfolio adjusting problem,

$$\begin{cases} \max\ E\left[\sum_{i=1}^{n}(x_i^0+x_i^+-x_i^-)\xi_i\right]-\sum_{i=1}^{n}(b_ix_i^++s_ix_i^-) \\ \text{s.t.}\quad V\left[\sum_{i=1}^{n}(x_i^0+x_i^+-x_i^-)\xi_i\right]\le \nu_0 \\ \qquad \sum_{i=1}^{n}(b_i+1)x_i^+ + \sum_{i=1}^{n}(s_i-1)x_i^- = 0 \\ \qquad x_i^+\cdot x_i^- = 0,\ i=1,2,\cdots,n \\ \qquad x_i^+\ge 0,\ i=1,2,\cdots,n \\ \qquad 0\le x_i^-\le x_i^0,\ i=1,2,\cdots,n. \end{cases} \tag{9.6}$$

Model T is consistent with that proposed in Zhang et al. (2010) when fuzzy random returns degenerate to the case of fuzzy returns (see next section). Although it considers the transaction cost, Model T does not take into account the limits of minimum transaction lots. It is known that the values taken by x_i^+ and x_i^- are subject to rules of stock exchange. For example, in Shanghai Stock Exchange, the minimum unit of buying securities is a round lot which is equal to 100 shares and the minimum unit of selling securities is one share. In order to follow the rule, we introduce new decision variables y_i^+ and y_i^-, respectively, to represent the buying round lots of security i and selling shares of security i for $i=1,2,\cdots,n$. Note that y_i^+ and y_i^- are required to be integers. Furthermore, we have $A_0x_i^+ = 100y_i^+p_i$ and $A_0x_i^- = y_i^-p_i$ for each security i. Substituting $x_i^+ = 100y_i^+p_i/A_0$ and $x_i^- = y_i^-p_i/A_0$ into Model T, the following practical mean-variance adjusting model for fuzzy random portfolio adjusting problem is reformulated according to Qin and Xu (2016),

$$\text{(Model P)}\quad \begin{cases} \min\ V\left[\sum_{i=1}^{n}(A_0x_i^0+100p_iy_i^+-p_iy_i^-)\xi_i\right] \\ \text{s.t.}\quad E\left[\sum_{i=1}^{n}(A_0x_i^0+100p_iy_i^+-p_iy_i^-)\xi_i\right] \\ \qquad\qquad -\sum_{i=1}^{n}(100b_ip_iy_i^++s_ip_iy_i^-)\ge A_0\mu_0 \\ \qquad \sum_{i=1}^{n}100(b_i+1)y_i^+ + \sum_{i=1}^{n}(s_i-1)y_i^- = 0 \\ \qquad y_i^+\cdot y_i^- = 0,\ i=1,2,\cdots,n \\ \qquad p_iy_i^-\le A_0x_i^0,\ i=1,2,\cdots,n \\ \qquad y_i^+,y_i^-\in 0\cup N^+,\ i=1,2,\cdots,n \end{cases}$$

where $0\cup N^+$ is the set of nonnegative integers. Similar to Model (9.6), another type of mean-variance adjusting model can be established by maximizing the expected return as the objective function and controlling the variance as a constraint.

9.4 Equivalent Crisp Models

Model T and Model P are formulated without considering the types of fuzzy random returns. Thus, it will be difficult to be solved since it is impossible to obtain the analytical forms of expected value and variance for general fuzzy random variable.

However, the situation may not be so bad in practice such as in the stochastic environment, security returns are often assumed to be normally distributed (Ziemba et al. 1974) or lognormally distributed (Ohlson and Ziemba 1976). Similar in spirit to stochastic case, it is assumed that all the fuzzy random returns have the same type of chance distributions. In particular, we consider two special cases: simple fuzzy random return and triangular fuzzy random return in this section, and then translate Model T and Model P into the equivalent deterministic models.

9.4.1 Case I: Simple Fuzzy Random Return

Let $(\Omega, \mathcal{P}, \Pr)$ be a discrete probability space with $\Omega = \{\omega_1, \omega_2, \cdots, \omega_m\}$, $\mathcal{P} = 2^{\Omega}$ and $\Pr\{\omega_j\} = \pi_j$ with $\sum_{j=1}^{m} \pi_j = 1$. Suppose that the return on the ith security is described by a function ξ_i defined as,

$$\xi_i(\omega) = \begin{cases} \tilde{a}_1^i, & \text{if } \omega = \omega_1 \\ \tilde{a}_2^i, & \text{if } \omega = \omega_2 \\ \cdots \\ \tilde{a}_m^i, & \text{if } \omega = \omega_m. \end{cases} \tag{9.7}$$

It follows from Definition 9.1 that ξ_i is a fuzzy random variable defined on $(\Omega, \mathcal{P}, \Pr)$. It is called a simple fuzzy random variable since it is regarded as an extension of a simple random variable when $\tilde{a}_1^i, \tilde{a}_2^i, \cdots, \tilde{a}_m^i$ are all real numbers rather than fuzzy variables.

Different sample points represent distinct market scenarios such as pessimistic, optimistic or normative. In a specific scenario ω_j, the value $\tilde{a}_j^i$ taken by ξ_i is a fuzzy variable representing the ambiguous return estimated by experts. If only one scenario appears, i.e., $m = 1$, then ξ_i will only take a fuzzy variable $\tilde{a}_1^i$ for $i = 1, 2, \cdots, n$. That is, $\xi_i \equiv \tilde{a}_1^i$, which implies that the return ξ_i is essentially a fuzzy variable. Then the corresponding portfolio adjusting problem is consistent with that in Zhang et al. (2010). However, the more market scenarios will more describe the natural statuses of decision-making problem which means that a simple fuzzy random return with more than two scenarios will provide more information than a fuzzy return. Under this assumption, we have the following theorem.

Theorem 9.2 (Qin and Xu 2016). *Let security returns $\xi_1, \xi_2, \cdots, \xi_n$ be simple fuzzy random variables defined on a discrete probability space $(\Omega, \mathcal{P}, \Pr)$ with the expression in Eq. (9.7) for $i = 1, 2, \cdots, n$. If for each ω_j, $j = 1, 2, \cdots, m$, $\tilde{a}_j^i \sim (\alpha_j^i - \beta_j^i, \alpha_j^i, \alpha_j^i + \beta_j^i)$ is a triangular fuzzy variable and $\tilde{a}_j^1, \tilde{a}_j^2, \cdots, \tilde{a}_j^n$ are independent in the sense of credibility measure, then Model T can be translated into the following equivalent model,*

$$\begin{cases} \min \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\pi_j\beta_j^i\beta_j^k(x_i^0+x_i^+-x_i^-)(x_k^0+x_k^+-x_k^-) \\ s.t.\ \ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}x_i^+\left(\pi_j\alpha_j^i-b_i\right)-\sum_{i=1}^{n}\sum_{j=1}^{m}x_i^-\left(\pi_j\alpha_j^i+s_i\right) \\ \qquad\qquad \geq \mu_0-\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}x_i^0\pi_j\alpha_j^i \\ \qquad \displaystyle\sum_{i=1}^{n}(b_i+1)x_i^+ +\sum_{i=1}^{n}(s_i-1)x_i^- =0 \\ \qquad x_i^+\cdot x_i^- =0,\ i=1,2,\cdots,n \\ \qquad x_i^+\geq 0,\ i=1,2,\cdots,n \\ \qquad 0\leq x_i^-\leq x_i^0,\ i=1,2,\cdots,n \end{cases} \tag{9.8}$$

which is a nonlinear programming.

Proof. Denote by $r(\boldsymbol{\xi})$ the gross return on portfolio $(x_1,x_2,\cdots,x_n)$ in which $x_i=x_i^0+x_i^+-x_i^-$ for $i=1,2,\cdots,n$. That is, $r(\boldsymbol{\xi})=x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n$. According to the definitions of $\xi_1,\xi_2,\cdots,\xi_n$, we can get

$$r(\boldsymbol{\xi})(\omega)=\begin{cases}\sum_{i=1}^n x_i\tilde{a}_1^i,\ \text{if }\omega=\omega_1\\ \sum_{i=1}^n x_i\tilde{a}_2^i,\ \text{if }\omega=\omega_2\\ \cdots\\ \sum_{i=1}^n x_i\tilde{a}_m^i,\ \text{if }\omega=\omega_m\end{cases}$$

which is also a simple fuzzy random variable. Taking expectation of $\sum_{i=1}^n x_i\tilde{a}_j^i$ for each ω_j, it is obtained that

$$E[r(\boldsymbol{\xi})](\omega)=\begin{cases}E[\sum_{i=1}^n x_i\tilde{a}_1^i],\ \text{if }\omega=\omega_1\\ E[\sum_{i=1}^n x_i\tilde{a}_2^i],\ \text{if }\omega=\omega_2\\ \cdots\\ E[\sum_{i=1}^n x_i\tilde{a}_m^i],\ \text{if }\omega=\omega_m\end{cases}$$

which is a simple random variable with the following expected value and variance

$$E[r(\boldsymbol{\xi})]=\sum_{j=1}^m\pi_jE\left[\sum_{i=1}^n x_i\tilde{a}_j^i\right],\quad V[r(\boldsymbol{\xi})]=\sum_{j=1}^m\pi_jV\left[\sum_{i=1}^n x_i\tilde{a}_j^i\right]$$

respectively. Further, by the assumption of independence, we have

$$E[r(\boldsymbol{\xi})]=\sum_{j=1}^m\pi_j\sum_{i=1}^n x_iE[\tilde{a}_j^i]=\sum_{i=1}^n\sum_{j=1}^m x_i\pi_j\alpha_j^i.$$

For each j, since $\tilde{a}_j^1, \tilde{a}_j^2, \cdots, \tilde{a}_j^n$ are all triangular fuzzy variables, we have

$$\sum_{i=1}^{n} x_i \tilde{a}_j^i \sim \left(\sum_{i=1}^{n} x_i(\alpha_j^i - \beta_j^i),\ \sum_{i=1}^{n} x_i \alpha_j^i,\ \sum_{i=1}^{n} x_i(\alpha_j^i + \beta_j^i) \right),$$

which implies that $V[\sum_{i=1}^{n} x_i \tilde{a}_j^i] = (\sum_{i=1}^{n} x_i \beta_j^i)^2/6$. Further, substituting this expression into the variance $V[r(\boldsymbol{\xi})]$, it is obtained that

$$V[r(\boldsymbol{\xi})] = \frac{1}{6} \sum_{j=1}^{m} \pi_j \left(\sum_{i=1}^{n} x_i \beta_j^i \right)^2.$$

Substituting the above expressions of expected value and variance into Model T, the theorem is proved.

Corollary 9.1. *Under the conditions of Theorem 9.2, and by substituting $x_i^+ = 100 p_i y_i^+ / A_0$ and $x_i^- = p_i y_i^- / A_0$ into Model (9.8), we can convert Model P into the following equivalent deterministic model*

$$\begin{cases} \min \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} \pi_j \beta_j^i \beta_j^k (A_0 x_i^0 + 100 p_i y_i^+ - p_i y_i^-)(A_0 x_k^0 + 100 p_k y_k^+ - p_k y_k^-) \\ \text{s.t. } \ 100 \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} p_i y_i^+ \left(\pi_j \alpha_j^i - b_i\right) - \sum_{i=1}^{n} \sum_{j=1}^{m} p_i y_i^- \left(\pi_j \alpha_j^i + s_i\right) \\ \qquad \qquad \geq A_0 \mu_0 - A_0 \displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} x_i^0 \pi_j \alpha_j^i \\ \qquad 100 \displaystyle\sum_{i=1}^{n} (b_i + 1) y_i^+ + \sum_{i=1}^{n} (s_i - 1) y_i^- = 0 \\ \qquad y_i^+ \cdot y_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad p_i y_i^- \leq A_0 x_i^0,\ i = 1, 2, \cdots, n \\ \qquad y_i^+, y_i^- \in 0 \cup N^+,\ i = 1, 2, \cdots, n \end{cases} \tag{9.9}$$

which is a nonlinear integer programming.

9.4.2 Case II: Triangular Fuzzy Random Return

Another commonly used type is triangular fuzzy random variable which is proposed in Hao and Liu (2009). Let $X_i \sim \mathcal{N}(e_i, \sigma_i^2)$ be a normally distributed random variable defined on the probability sapce $(\Omega, \mathcal{P}, \Pr)$ for $i = 1, 2, \cdots, n$. If X_i is the return on security i, then, for each $\omega \in \Omega$, $X_i(\omega)$ is a real number representing the realization of X_i. Due to lack of information, each realization of X_i is difficult to be accurately obtained at the beginning of investment period. A better alternative is to extend the crisp number to a fuzzy variable. That is to say, for each $\omega \in \Omega$, we assume that $X_i(\omega)$ is a triangular fuzzy variable which is evaluated by experts based on data and/or experiences. In this situation, X_i is a triangular fuzzy random variable, denoted by $\xi_i = (X_i - \rho_i, X_i, X_i + \lambda_i)$ where ρ_i and λ_i are two real numbers. Note that, for each fixed ω, $\xi_i(\omega) = (X_i(\omega) - \rho_i, X_i(\omega), X_i(\omega) + \lambda_i)$ is just a triangular fuzzy variable. If $\rho_i = \lambda_i$, then ξ_i is a symmetrical triangular fuzzy random variable. Hao and Liu (2009) have given the formulas of variances of symmetrical and asymmetrical fuzzy random variables, respectively. Next we only give the following theorem in the symmetrical case since the corresponding results can be easily extended to the cases of asymmetrical triangular fuzzy random returns and trapezoidal fuzzy random returns (Liu et al. 2012).

Theorem 9.3 (Qin and Xu 2016). *Let $\xi_i = (X_i - \rho_i, X_i, X_i + \rho_i)$ be a symmetrical triangular fuzzy random variable where $X_i \sim \mathcal{N}(e_i, \sigma_i^2)$ is a normally distributed random variable defined on the probability space $(\Omega, \mathcal{P}, \Pr)$ with probability distribution Φ_i for $i = 1, 2, \cdots, n$. If $X_1, X_2, \cdots, X_n$ are independent, then Model T can be translated into the following equivalent form,*

$$\begin{cases} \min \dfrac{2\sigma^3 + 3\sigma\rho^2}{3\sqrt{2\pi}\rho} + \dfrac{3\sigma^2 + \rho^2}{2} - \dfrac{3\sigma^2 + \rho^2}{3}\Phi\left(\dfrac{\rho}{\sigma}\right) - \dfrac{2\sigma^3 + \sigma\rho^2}{3\sqrt{2\pi}\rho}\exp\left(-\dfrac{\rho^2}{2\sigma^2}\right) \\ \text{s.t. } \sum_{i=1}^n (e_i - b_i)x_i^+ - \sum_{i=1}^n (e_i + s_i)x_i^- \geq \mu_0 - \sum_{i=1}^n x_i^0 e_i \\ \qquad \sum_{i=1}^n (b_i + 1)x_i^+ + \sum_{i=1}^n (s_i - 1)x_i^- = 0 \\ \qquad \sum_{i=1}^n (x_i^0 + x_i^+ - x_i^-)\rho_i = \rho \\ \qquad \sum_{i=1}^n (x_i^0 + x_i^+ - x_i^-)^2\sigma_i^2 = \sigma^2 \\ \qquad x_i^+ \cdot x_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad \sigma, x_i^+ \geq 0,\ i = 1, 2, \cdots, n \\ \qquad 0 \leq x_i^- \leq x_i^0,\ i = 1, 2, \cdots, n \end{cases} \tag{9.10}$$

which is a nonlinear programming.

Proof. Let x_i be the holding proportion of security i after adjusting for $i = 1, 2, \cdots, n$. It follows from the operational law of fuzzy random variables that for each $\omega \in \Omega$, we have

$$\sum_{i=1}^n x_i\xi_i(\omega) = \left(\sum_{i=1}^n x_i(X_i(\omega) - \rho_i), \sum_{i=1}^n x_iX_i(\omega), \sum_{i=1}^n x_i(X_i(\omega) + \rho_i)\right),$$

where $\sum_{i=1} x_iX_i \sim \mathcal{N}(\sum_{i=1}^n x_ie_i, \sum_{i=1}^n x_i^2\sigma_i^2)$. By the expected value of triangular fuzzy variable, for each $\omega \in \Omega$, we have

$$E\left[\sum_{i=1}^n x_i\xi_i(\omega)\right] = \sum_{i=1}^n x_iX_i(\omega).$$

It follows from Definition 9.2 that

$$E\left[\sum_{i=1}^{n} x_i \xi_i\right] = E\left[\sum_{i=1}^{n} x_i X_i\right] = \sum_{i=1}^{n} x_i e_i.$$

Based on the conditions of this theorem, we can further obtain the expression of the variance $V\left[\sum_{i=1}^{n} x_i \xi_i\right]$ according to Hao and Liu (2009). The theorem is proved.

Corollary 9.2. *Similarly, substituting* $x_i^+ = 100 p_i y_i^+ / A_0$ *and* $x_i^- = p_i y_i^- / A_0$ *into Model (9.10), it follows from the conditions of Theorem 9.3 that Model P is converted into the following equivalent deterministic model*

$$\begin{cases} \min \dfrac{2\sigma^3 + 3\sigma\rho^2}{3\sqrt{2\pi}\rho} + \dfrac{3\sigma^2 + \rho^2}{2} - \dfrac{3\sigma^2 + \rho^2}{3}\Phi\left(\dfrac{\rho}{\sigma}\right) - \dfrac{2\sigma^3 + \sigma\rho^2}{3\sqrt{2\pi}\rho}\exp\left(-\dfrac{\rho^2}{2\sigma^2}\right) \\ s.t.\ \ 100\sum_{i=1}^{n} p_i y_i^+ (e_i - b_i) - \sum_{i=1}^{n} p_i y_i^- (e_i + s_i) \ge A_0\mu_0 - A_0 \sum_{i=1}^{n} x_i^0 e_i \\ \qquad \sum_{i=1}^{n} 100(b_i + 1) y_i^+ + \sum_{i=1}^{n} (s_i - 1) y_i^- = 0 \\ \qquad \sum_{i=1}^{n} [x_i^0 + p_i(100 y_i^+ - y_i^-)/A_0]\rho_i = \rho \\ \qquad \sum_{i=1}^{n} [x_i^0 + p_i(100 y_i^+ - y_i^-)/A_0]^2 \sigma_i^2 = \sigma^2 \\ \qquad y_i^+ \cdot y_i^- = 0,\ i = 1, 2, \cdots, n \\ \qquad p_i y_i^- \le A_0 x_i^0,\ i = 1, 2, \cdots, n \\ \qquad \sigma \ge 0,\ y_i^+, y_i^- \in 0 \cup N^+,\ i = 1, 2, \cdots, n \end{cases} \tag{9.11}$$

which is a nonlinear integer programming.

9.5 Numerical Examples

In this section, we illustrate the application of fuzzy random mean-variance adjusting model by two numerical examples presented in Qin and Xu (2016). The first example assumes that the returns on individual securities are simple fuzzy random variables, and the second one assumes that they are triangular fuzzy random variables.

Example 9.3. Consider a single-period portfolio optimization problem with $n = 10$ securities where their returns $\xi_1, \xi_2, \cdots, \xi_{10}$ are all simple fuzzy random variables. In order to estimate individual security return, the data are collected from Shanghai Stock Exchange (SSE). The data set consists of monthly return rates of 10 securities during the period January 2007–April 2014, yielding 87 observations for each security. As stated before, a simple fuzzy random return may describe the case with multiple scenarios. We suppose that there are three market scenarios $\omega_1 =$ optimistic, $\omega_2 =$ normative and $\omega_3 =$ pessimistic with probabilities $\pi_1 = 0.30, \pi_2 = 0.45, \pi_3 = 0.25$, respectively. In the "optimistic" scenario, the investors prefer to believe future returns on securities at a high level. For example, 45, 70 and 95 percentage points are extracted to construct a triangular fuzzy variable to describe the future return. Following the same logic, 25, 50 and 75 percentage points and 10, 35 and 60 percentage points are chosen to respectively construct the triangular fuzzy returns in the "normative" and "pessimistic" scenarios. The return rates for each security in each scenario are shown in Table 9.1.

The current holding proportion x_i^0 on security i is given as known model inputs. It can be obtained by calculating the product of the holding shares and buying prices. Without loss of generality, we assume that the initial holding share of each security is 10 lots and the buying price is

$$
\begin{aligned}
&(p_1^0, p_2^0, p_3^0, p_4^0, p_5^0, p_6^0, p_7^0, p_8^0, p_9^0, p_{10}^0) \\
&= (4.37, 7.86, 3.94, 13.36, 12.17, 17.82, 17.64, 12.23, 7.90, 14.82),
\end{aligned}
$$

then the current existing portfolio before adjusting is

$$
\begin{aligned}
&(x_1^0, x_2^0, x_3^0, x_4^0, x_5^0, x_6^0, x_7^0, x_8^0, x_9^0, x_{10}^0) \\
&= (0.04, 0.07, 0.03, 0.12, 0.11, 0.16, 0.16, 0.11, 0.07, 0.13).
\end{aligned}
$$

The initial capital A_0 is 112110 which can be correspondingly calculated. The unit buying and selling costs are $b_i = 0.00896$ and $s_i = 0.01896$, respectively, for $i = 1, 2, \cdots, 10$, which are estimated according to the trading rules of SSE. Further, the minimum return level μ_0 is set as 0.015. Then model (9.8) (i.e., Model T) is applied to seek the optimal strategy of adjusting the current portfolio. Taking into account that it is a deterministic programming, we employ the "fmincon" function in Matlab to solve model (9.8). The results are shown in the second to fourth column of Table 9.2.

In order to apply Model P, the transaction prices $p_1, \cdots, p_{10}$ need to be known. Here, we choose the current prices of individual securities as their respective transaction ones as follows,

$$
\begin{aligned}
&(p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}) \\
&= (6.27, 7.87, 3.08, 11.17, 10.04, 17.05, 14.06, 15.93, 6.78, 18.22).
\end{aligned}
$$

Since model (9.9) is a nonlinear integer programming, we employ the function "BNB20" in Matlab to solve it. BNB20 is a branch-and-bound type algorithm proposed by Kuipers to solve integer nonlinear optimization problem. The results are shown in the fifth to ninth column of Table 9.2 where x_i^+ and x_i^- are calculated according to the equations $x_i^+ = 100 p_i y_i^+ / A_0$ and $x_i^- = p_i y_i^- / A_0$, respectively, for $i = 1, 2, \cdots, 10$.

The mean and variance of the optimal portfolio after adjusting are presented in the last two rows of Table 9.2. It is easy to see that these two portfolios provided by Modet T and Model P have the same expected return which is equal to the given

Table 9.1 The triangular fuzzy returns of 10 securities in each scenario in Example 9.3

Scenario	ω_1 = optimistic	ω_2 = normative	ω_3 = pessimistic
$\xi_1(\omega)$	$(0.11-0.24, 0.11, 0.11+0.24)$	$(-0.01-0.11, -0.01, -0.01+0.11)$	$(-0.07-0.12, -0.07, -0.07+0.12)$
$\xi_2(\omega)$	$(0.08-0.16, 0.08, 0.08+0.16)$	$(-0.01-0.10, -0.01, -0.01+0.10)$	$(-0.06-0.11, -0.06, -0.06+0.11)$
$\xi_3(\omega)$	$(0.10-0.20, 0.10, 0.10+0.20)$	$(-0.01-0.09, -0.01, -0.01+0.09)$	$(-0.08-0.12, -0.08, -0.08+0.12)$
$\xi_4(\omega)$	$(0.12-0.26, 0.12, 0.12+0.26)$	$(-0.01-0.15, -0.01, -0.01+0.15)$	$(-0.10-0.17, -0.10, -0.10+0.17)$
$\xi_5(\omega)$	$(0.11-0.19, 0.11, 0.11+0.19)$	$(0.00-0.12, 0.00, 0.00+0.12)$	$(-0.07-0.10, -0.07, -0.07+0.10)$
$\xi_6(\omega)$	$(0.10-0.18, 0.10, 0.10+0.18)$	$(0.01-0.11, 0.01, 0.01+0.11)$	$(-0.07-0.12, -0.07, -0.07+0.12)$
$\xi_7(\omega)$	$(0.15-0.25, 0.15, 0.15+0.25)$	$(0.01-0.14, 0.01, 0.01+0.14)$	$(-0.07-0.14, -0.07, -0.07+0.14)$
$\xi_8(\omega)$	$(0.13-0.25, 0.13, 0.13+0.25)$	$(-0.01-0.14, -0.01, -0.01+0.14)$	$(-0.08-0.12, -0.08, -0.08+0.12)$
$\xi_9(\omega)$	$(0.09-0.19, 0.09, 0.09+0.19)$	$(-0.02-0.11, -0.02, -0.02+0.11)$	$(-0.08-0.09, -0.08, -0.08+0.09)$
$\xi_{10}(\omega)$	$(0.11-0.14, 0.11, 0.11+0.14)$	$(0.01-0.13, 0.01, 0.01+0.13)$	$(-0.07-0.12, -0.07, -0.07+0.12)$

Table 9.2 The optimal adjusting strategies of Model T and Model P in Example 9.3

	Model T			Model P				
Security	x_i^+	x_i^-	x_i	y_i^+	y_i^-	x_i^+	x_i^-	x_i
1	0.000	0.000	0.040	0	32	0.000	0.002	0.038
2	0.000	0.000	0.070	2	0	0.014	0.000	0.084
3	0.000	0.001	0.029	0	5	0.000	0.000[a]	0.030
4	0.000	0.047	0.073	0	616	0.000	0.061	0.059
5	0.001	0.000	0.111	0	12	0.000	0.001	0.109
6	0.010	0.000	0.170	0	0	0.000	0.000	0.160
7	0.004	0.000	0.164	2	0	0.025	0.000	0.185
8	0.000	0.000	0.110	0	23	0.000	0.003	0.107
9	0.000	0.000	0.070	0	912	0.000	0.055	0.015
10	0.032	0.000	0.162	5	0	0.081	0.000	0.211
Sum	0.047	0.048	0.999	9	1600	0.120	0.122	0.998
Mean		0.0150				0.0150		
Variance		0.0038				0.0038		

[a] $x_3^- = p_3 y_3^- / A_0 = 3.08 \times 5/112110 = 0.00014 \approx 0.000$

return level 0.015, and also have the same variance 0.0038. However, the holdings after adjusting are different for these two optimal strategies, which is shown from the comparison the fourth column and the ninth one.

Example 9.4. We continue considering the problem in Example 9.3. The security returns are still estimated based on the same data, however, they are considered as symmetrical triangular fuzzy random variables, denoted by $\xi_i(\omega) = (X_i(\omega) - \rho_i, X_i(\omega), X_i(\omega) + \rho_i)$ for $i = 1, 2, \cdots, 10$.

We first describe how to construct the triangular fuzzy random returns based on the collected 87 observations for each security. A similar method to Li and Xu (2013) is employed to determine the values of three parameters e_i, σ_i^2 and ρ_i for $i = 1, 2, \cdots, 10$. Here e_i and σ_i^2 are estimated by the sample mean and sample variance, respectively. A fuzzy random variable may be regarded as an extension of a random variable by replacing the expectation e_i with a fuzzy variable $(e_i - \rho_i, e_i, e_i + \rho_i)$. The support of the fuzzy variable is $[e_i - \rho_i, e_i + \rho_i]$ which is calculated as a 90 % confidence interval. All the calculations are obtained by using data analysis software Eviews 11.0. The corresponding returns of 10 securities are shown in Table 9.3.

Other input parameters are set as same as Example 9.3. The computational results of model (9.10) (i.e., Model T) and model (9.11) (i.e., Model P) are shown in Table 9.4. The optimal adjusting strategy provided by Model (9.10) corresponds to a portfolio with expected return 0.015 and variance 0.0052. Compared with the results in Example 9.4, the variance becomes larger due to the change of types of fuzzy random returns. In addition, both mean and variance of the return on the optimal portfolio found by model (9.11) are larger than those obtained by model (9.10). Actually the first constraint of model (9.11) is no longer tight. This is because that

Table 9.3 The X_i and ρ_i of each security (%) in Example 9.4

Security no.	1	2	3	4	5	6	7	8	9	10
X_i	2.5	0.6	2.0	0.2	2.5	1.8	3.5	1.8	1.5	0.2
σ_i^2	4.9	2.9	4.8	5.0	5.6	4.1	5.2	5.0	5.2	3.2
ρ_i	3.9	3.0	3.9	4.0	4.2	3.6	4.0	4.0	4.0	3.2

Table 9.4 The optimal adjusting strategies of Model T and Model P in Example 9.4

	Model T			Model P				
Security	x_i^+	x_i^-	x_i	y_i^+	y_i^-	x_i^+	x_i^-	x_i
1	0.049	0.000	0.089	7	0	0.039	0.000	0.079
2	0.014	0.000	0.084	0	0	0.000	0.000	0.070
3	0.048	0.000	0.078	0	1	0.000	0.000[a]	0.030
4	0.000	0.043	0.077	0	14	0.000	0.001	0.119
5	0.000	0.005	0.105	0	17	0.000	0.002	0.108
6	0.000	0.026	0.134	0	16	0.000	0.002	0.158
7	0.000	0.029	0.131	0	254	0.000	0.032	0.128
8	0.000	0.004	0.106	0	3	0.000	0.001	0.109
9	0.000	0.000	0.070	0	0	0.000	0.000	0.070
10	0.000	0.006	0.124	0	14	0.000	0.002	0.128
Sum	0.111	0.113	0.998	7	319	0.039	0.040	0.999
Mean		0.0150				0.0157		
Variance		0.0052				0.0056		

[a] $x_3^- = p_3 y_3^- / A_0 = 3.08 \times 1/112110 = 0.00003 \approx 0.000$

the feasible set of model (9.11) becomes smaller than that of model (9.10) due to the requirements of minimum transaction lots.

Examples 9.3 and 9.4 are presented based on using the same observational data about security returns. In Example 9.3, we handle the security returns as simple fuzzy random variables and employ models (9.8) and (9.9) to construct the optimal adjusting strategy. As a comparison, Example 9.4 regards the security returns as triangular fuzzy random variables and employs models (9.10) and (9.11) to seek the desirable solution. The choice of the model will depend on the given types of fuzzy random returns. Thus, an important step is to choose an appropriate fuzzy random return in applying the proposed models. Actually the computational results show that the introduction of minimum transaction lots has a greater impact to efficient frontier for the model with triangular fuzzy random returns than with simple ones. In addition, the introduction of minimum transaction lots leads to a low computational efficiency since models (9.9) and (9.11) are both nonlinear integer programming models. The numerical studies show that although the function "BNB20" is a feasible solution tool, it depends on the initial guess. It will also require larger computational time for large-scale portfolio adjusting problem.

Chapter 10
Random Fuzzy Mean-Risk Model

10.1 Introduction

In practice, the investors may encounter varieties of uncertainties when constructing the optimal portfolio. For example, the probability distributions of security returns may be partially known, which can be described by stochastic returns with not crisp information. We employ fuzzy variables to characterize these information, which implies that security returns are random variables with fuzzy parameters. Liu (2002) proposed the concept of random fuzzy variable to model the case, which is different from fuzzy random variable. From the mathematical viewpoint, a random fuzzy variable is a measurable function from a credibility space to the set of random variables, however a fuzzy random variable is a measurable function from a probability space to the set of fuzzy variables. Although they are different, both of them may describe the portfolio optimization with mixture of randomness and fuzziness.

This chapter will use random fuzzy variables to describe security returns. Actually several authors have applied the tool to successfully model portfolio optimization such as Hasuike et al. (2009) and Huang (2007b).

Most mean-absolute deviation models are devoted to either stochastic portfolio optimization or fuzzy/uncertain one. As extensions, Qin (2016) proposed random fuzzy mean-absolute deviation model by using random fuzzy variable to describe the stochastic return on individual security with ambiguous information. He first defined the absolute deviation of random fuzzy variable and considered it as a risk measure to seek the optimal portfolio.

For the case with asymmetric returns, Qin et al. (2013) defined semivariance of random fuzzy variable to measure the downside risk. They formulated random fuzzy mean-semivariance model for portfolio optimization with hybrid returns.

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This chapter focuses on random fuzzy mean-risk models for hybrid portfolio optimization. The main contents include concept of random fuzzy variable, absolute deviation and semivariance of random fuzzy variable, formulation of random fuzzy mean-risk model, and numerical examples.

10.2 Random Fuzzy Variable

In this section, we review random fuzzy variable and the definitions of expected value and variance, as well as chance measure of a random fuzzy event.

Definition 10.1 (Liu 2002). A random fuzzy variable ξ is a measurable function from the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of random variables.

It is known that a crisp number may be regarded as a special random variable. Accordingly, the random fuzzy variable ξ becomes a fuzzy variable. If there is only one element θ in Θ with membership degree 1, then the random fuzzy variable ξ is essentially a random variable. Thus, both fuzzy variable and random variable are the special case of random fuzzy variable.

Remark 10.1. Assume that ξ is a random fuzzy variable defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. For each fixed $\theta \in \Theta$, $\xi(\theta)$ is a random variable, and its expected value $E[\xi(\theta)]$ is a crisp number. That is, $E[\xi]$ is a function from $(\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers. Therefore, $E[\xi]$ is a fuzzy variable.

Example 10.1. Let $\eta_1, \eta_2, \cdots, \eta_m$ be random variables, and $\nu_1, \nu_2, \cdots, \nu_m$ real numbers in [0, 1] such that $\max\{\nu_1, \nu_2, \cdots, \nu_m\} = 1$. Then

$$\xi = \begin{cases} \eta_1 & \text{with membership degree } \nu_1 \\ \eta_2 & \text{with membership degree } \nu_2 \\ \cdots\cdots \\ \eta_m & \text{with membership degree } \nu_m \end{cases}$$

is a random fuzzy variable which is called a simple random fuzzy variable.

Example 10.2. A random fuzzy variable ξ is said to be normal if for each $\theta \in \Theta$, $\xi(\theta)$ is a normally distributed random variable with the following probability density function

$$\phi_{\xi(\theta)}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \tilde{\mu}(\theta))^2}{2\sigma^2}\right), \quad x \in \Re$$

where $\tilde{\mu}$ is a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. It is denoted by $\xi \sim \mathcal{N}(\tilde{\mu}, \sigma^2)$, and the fuzziness of ξ is characterized by fuzzy variable $\tilde{\mu}$.

Definition 10.2 (Liu and Liu 2003). Let ξ be a random fuzzy variable defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. Then the expected value of ξ is defined as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\theta \in \Theta | E[\xi(\theta)] \geq r\}\,\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{\theta \in \Theta | E[\xi(\theta)] \leq r\}\,\mathrm{d}r \quad (10.1)$$

provided that at least one of the two integrals is finite.

It follows from Definition 10.2 that $E[\xi] = E[E[\xi(\theta)]]$, i.e., the expected value of ξ is equal to the expected value of fuzzy variable $E[\xi]$. Note that, for each fixed $\theta \in \Theta$, $E[\xi(\theta)]$ represents the expected value of random variable $\xi(\theta)$. In addition, if ξ and η are random fuzzy variables with finite expected values, then for any real numbers a and b, Liu and Liu (2003) has proved that $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ holds for independent case.

Example 10.3. By Definition 10.2, the expected value of a simple random fuzzy variable in Example 10.1 is

$$E[\xi] = \frac{1}{2}\sum_{i=1}^{m} w_i E[\eta_i]$$

and the weights are given by

$$w_i = \max_{1\leq l\leq n}\{\nu_l | E[\eta_l] \leq E[\eta_i]\} - \max_{1\leq l\leq n}\{\nu_l | E[\eta_l] < E[\eta_i]\} + \max_{1\leq l\leq n}\{\nu_l | E[\eta_l] \geq E[\eta_i]\} - \max_{1\leq l\leq n}\{\nu_l | E[\eta_l] > E[\eta_i]\}.$$

The expected value of a normal random fuzzy variable in Example 10.2 is $E[\xi] = E[\tilde{\mu}]$.

Definition 10.3 (Liu and Liu 2003). Let ξ be a random fuzzy variable with finite expected value. Then the variance of ξ is defined as

$$V[\xi] = E[(\xi - E[\xi])^2]. \quad (10.2)$$

Definition 10.4 (Liu 2002). Let ξ be a random fuzzy variable, and B a Borel set of $\Re$. Then the chance measure of random fuzzy event $\xi \in B$ is a function of α from $(0, 1]$ to $[0, 1]$, defined as

$$\mathrm{Ch}\{\xi \in B\}(\alpha) = \sup_{\mathrm{Cr}\{A\}\geq\alpha} \inf_{\theta\in A} \Pr\{\xi(\theta) \in B\}. \quad (10.3)$$

10.3 Absolute Deviation of Random Fuzzy Variable

In this section, we review the definition of absolute deviation of random fuzzy variable and its mathematical properties.

Definition 10.5 (Qin 2016). Let ξ be a random fuzzy variable with finite expected value e. Then the absolute deviation of ξ is defined as

$$A[\xi] = E\left[|\xi - e|\right]. \tag{10.4}$$

Remark 10.2. According to Definition 10.2, the absolute deviation of ξ is equal to

$$\begin{aligned} A[\xi] &= \int_0^{\infty} \mathrm{Cr}\{\theta \in \Theta | E[|\xi(\theta) - e|] \geq r\} \mathrm{d}r \\ &= E\left[E[|\xi(\theta) - e|]\right], \end{aligned} \tag{10.5}$$

where the first symbol E represents the expected value operator of fuzzy variable, and the second one is the expected value operator of random variable.

Example 10.4. Let $\xi \sim \mathcal{N}(\tilde{\mu}, \sigma^2)$ be a normal random fuzzy variable with $\sigma > 0$ and fuzzy variable $\tilde{\mu}$. Write $\hat{\mu} = (E[\tilde{\mu}] - \tilde{\mu})/\sigma$. Then the absolute deviation of ξ is

$$A[\xi] = \sigma E\left[\hat{\mu}(2\Phi(\hat{\mu}) - 1) + \sqrt{\frac{2}{\pi}} \exp\left(-\frac{\hat{\mu}^2}{2}\right)\right] \tag{10.6}$$

where the items in brackets are a function of fuzzy variable $\hat{\mu}$ and $\Phi(\cdot)$ is the standard normal probability distribution function. In general, we cannot obtain the analytical expression of $A[\xi]$, but which can be approximately calculated by using fuzzy simulation.

Example 10.5. If ξ is a simple random fuzzy variable defined in Example 10.1, then its absolute deviation can be obtained similar to the calculation of its expected value by replacing $E[\eta_i]$ with $E[|\eta_i - e|]$ where $e = E[\xi]$. Particularly, if $0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_m \leq 1$, then

$$A[\xi] = \frac{1}{2} \sum_{i=1}^{m} \nu_i y_i$$

where y_i are given by

$$y_i = \max_{j \geq i} E[|\eta_j - e|] - \max_{j > i} E[|\eta_j - e|] + \min_{j \geq i} E[|\eta_j - e|] - \min_{j > i} E[|\eta_j - e|].$$

Theorem 10.1 (Qin 2016). *Let a and b be two real numbers. If ξ is a random fuzzy variable with finite expected value, then $A[a\xi + b] = |a| \cdot A[\xi]$.*

Proof. It follows from Definition 10.5 that

$$\begin{aligned} A[a\xi + b] &= E\left[|a\xi + b - aE[\xi] - b|\right] \\ &= E\left[|a(\xi - E[\xi])|\right] \\ &= |a|E\left[|\xi - E[\xi]|\right] \\ &= |a| \cdot A[\xi]. \end{aligned}$$

The theorem is proved.

Theorem 10.2 (Qin 2016). *Let ξ be a random fuzzy variable defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ with finite expected value. Then we have*

$$A[E[\xi]] \leq A[\xi]. \tag{10.7}$$

Proof. Denote by e the expected value of ξ. For each $\theta \in \Theta$, it follows from the definition of random fuzzy variable that $\xi(\theta)$ is a random variable. Applying the Jensen's Inequality in probability theory, we find

$$|E[\xi(\theta)] - e| \leq E\left[|\xi(\theta) - e|\right], \quad \forall\ \theta \in \Theta.$$

Taking the expected values on both sides with regard to θ, we have

$$A[E[\xi]] = E\left[|E[\xi(\theta)] - e|\right] \leq E\left[E\left[|\xi(\theta) - e|\right]\right] = A[\xi]$$

The theorem is proved.

Theorem 10.2 states that, the absolute deviation of fuzzy variable $E[\xi]$ is no more than the absolute deviation of random fuzzy variable ξ.

Theorem 10.3 (Qin 2016). *Let ξ be a random fuzzy variable with finite expected value e. Then $A[\xi] = 0$ if and only if $\mathrm{Ch}\{\xi = e\}(1) = 1$.*

Proof. Assume that ξ is defined on a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. If $A[\xi] = 0$, then it follows Remark 10.2 that

$$A[\xi] = E[|\xi - e|] = \int_0^{+\infty} \mathrm{Cr}\{\theta \in \Theta | E[|\xi(\theta) - e|] \geq r\}\mathrm{d}r = 0$$

which implies that $\mathrm{Cr}\{\theta \in \Theta | E[|\xi(\theta) - e|] \geq r\} = 0$ for any $r > 0$. By the self-duality of credibility measure, we have $\mathrm{Cr}\{\theta \in \Theta | E[|\xi(\theta) - e|] = 0\} = 1$. That is to say, there exists a set $B^* \subset \Theta$ with $\mathrm{Cr}\{B^*\} = 1$ such that $E[|\xi(\theta) - e|] = 0$ for each $\theta \in B^*$. Note that $|\xi(\theta) - e|$ is a random variable for each θ. For each $\theta \in B^*$, we have

$$E[|\xi(\theta) - e|] = \int_0^{+\infty} \Pr\{|\xi(\theta) - e| \geq u\}\mathrm{d}u = 0.$$

Therefore, for each $\theta \in B^*$, $\Pr\{|\xi(\theta) - e| \geq u\} = 0$ for any $u > 0$, which is equivalent that $\Pr\{|\xi(\theta) - e| = 0\} = 1$ or $\Pr\{\xi(\theta) = e\} = 1$. According to Definition 10.4, we have

$$\begin{aligned} 1 &\geq \mathrm{Ch}\{\xi = e\}(1) \\ &= \sup_{\mathrm{Cr}\{A\}=1} \inf_{\theta \in A} \Pr\{\xi(\theta) = e\} \\ &\geq \inf_{\theta \in B^*} \Pr\{\xi(\theta) = e\} = 1 \end{aligned}$$

which proves the necessity. Conversely, let $\mathrm{Ch}\{\xi = e\}(1) = 1$. It follows from Definition 10.4 that there exists a set B with $\mathrm{Cr}\{B\} = 1$ such that

$$\inf_{\theta \in B} \Pr\{\xi(\theta) = e\} = 1.$$

Therefore, we have $\Pr\{|\xi(\theta) - e| = 0\} = 1$ for each $\theta \in B$, which is equivalent that for each $\theta \in B$, $\Pr\{|\xi(\theta) - e| \geq u\} = 0$ for any $u > 0$. Following the definition of expected value of random variable, for each $\theta \in \Theta$, we have

$$E[|\xi(\theta) - e|] = \int_0^{+\infty} \Pr\{|\xi(\theta) - e| \geq u\}\mathrm{d}u = 0.$$

By the definition of credibility measure, for any given $r > 0$, we have

$$\begin{aligned} 0 &\leq \mathrm{Cr}\{\theta \in \Theta | E[|\xi(\theta) - e| \geq r\} \\ &\leq \mathrm{Cr}\{\theta \in B^c\} \\ &= 1 - \mathrm{Cr}\{B\} = 0, \end{aligned}$$

which means that

$$A[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\theta \in \Theta | E[|\xi(\theta) - e| \geq r\}\mathrm{d}r = 0.$$

The theorem is proved.

10.4 Semivariance of Random Fuzzy Variable

In this section, we review the semivariance of random fuzzy variable and its mathematical properties.

Definition 10.6 (Qin et al. 2013). Let ξ be a random fuzzy variable with finite expected value e. Then its semivariance is defined as

$$S_v[\xi] = E\left[|(\xi - e) \wedge 0|^2\right]. \tag{10.8}$$

The semivariance of a random fuzzy variable ξ is defined as the expected value of random fuzzy variable $|(\xi - e) \wedge 0|^2$. According to Definition 10.2, we have

$$\begin{aligned} S_v[\xi] &= E\left[|(\xi - e) \wedge 0|^2\right] \\ &= \int_0^{+\infty} \mathrm{Cr}\left\{\theta \in \Theta \middle| E\left[|(\xi(\theta) - e) \wedge 0|^2\right] \geq r\right\} \mathrm{d}r \\ &= E\left[E\left[|(\xi(\theta) - e) \wedge 0|^2\right]\right]. \end{aligned} \tag{10.9}$$

Note that $E[|(\xi(\theta) - e) \wedge 0|^2]$ as a function of θ is a fuzzy variable. Thus, $S_v[\xi]$ may also be regarded as the expected value of fuzzy variable $E[|(\xi(\theta) - e) \wedge 0|^2]$.

Theorem 10.4 (Qin et al. 2013). *Let a and b be real numbers with $a > 0$. If ξ is a random fuzzy variable with finite expected value, then we have $S_v[a\xi + b] = a^2 S_v[\xi]$.*

Proof. It follows from Definition 10.6 that

$$\begin{aligned} S_v[a\xi + b] &= E\left[|(a\xi + b - aE[\xi] - b) \wedge 0|^2\right] \\ &= a^2 E\left[|(\xi - E[\xi]) \wedge 0|^2\right] \\ &= a^2 S_v[\xi]. \end{aligned}$$

The theorem is proved.

Theorem 10.5 (Qin et al. 2013). *Let ξ be a random fuzzy variable with finite expected value, and $V[\xi]$ and $S_v[\xi]$ the variance and semivariance of ξ, respectively. Then we have $0 \leq S_v[\xi] \leq V[\xi]$.*

Proof. It follows from Definition 10.6 that the semivariance of ξ is nonnegative. Denote the expected value of ξ by e. For any $r > 0$, we have

$$\left\{\theta \in \Theta \middle| E\left[|\xi(\theta) - e|^2\right] \geq r\right\} \supset \left\{\theta \in \Theta \middle| E\left[|(\xi(\theta) - e) \wedge 0|^2\right] \geq r\right\}$$

which implies that

$$\mathrm{Cr}\left\{\theta \in \Theta \middle| E\left[|\xi(\theta)-e|^2\right] \geq r\right\} \geq \mathrm{Cr}\left\{\theta \in \Theta \middle| E\left[|(\xi(\theta)-e)\wedge 0|^2\right] \geq r\right\}.$$

It follows from the definitions of variance and semivariance that

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathrm{Cr}\left\{\theta \in \Theta \middle| E\left[|\xi(\theta)-e|^2\right] \geq r\right\} \mathrm{d}r \\ &\geq \int_0^{+\infty} \mathrm{Cr}\left\{\theta \in \Theta \middle| E\left[|(\xi(\theta)-e)\wedge 0|^2\right] \geq r\right\} \mathrm{d}r \\ &= S_v[\xi]. \end{aligned}$$

The theorem is proved.

Theorem 10.6 (Qin et al. 2013). *Let ξ be a random fuzzy variable with expected value e. Then $S_v[\xi]=0$ if and only if* $\mathrm{Ch}\{\xi=e\}(1)=1$.

Proof. If $S_v[\xi]=0$, then it follows from Definition 10.6 that

$$\int_0^{+\infty} \mathrm{Cr}\left\{\theta \in \Theta \middle| E\left[|(\xi(\theta)-e)\wedge 0|^2\right] \geq r\right\} \mathrm{d}r = 0$$

which implies that $\mathrm{Cr}\{\theta \in \Theta | E[|(\xi(\theta)-e)\wedge 0|^2] \geq r\} = 0$ for any real number $r > 0$. In other words, $\mathrm{Cr}\{\theta \in \Theta | E[|(\xi(\theta)-e)\wedge 0|^2] = 0\} = 1$. As a consequence, there exists a set A_1^* with $\mathrm{Cr}\{A_1^*\}=1$ such that, for each $\theta \in A_1^*$, $E[|(\xi(\theta)-e)\wedge 0|^2]=0$. That is, the random variable $\xi(\theta)$ is greater than or equal to e almost everywhere. Thus, for each $\theta \in A_1^*$, we have $\xi(\theta)-e=(\xi(\theta)-e)\vee 0+(\xi(\theta)-e)\wedge 0 = (\xi(\theta)-e)\vee 0$ almost everywhere which implies that $E[\xi(\theta)]-e=E[(\xi(\theta)-e)\vee 0]$. Further,

$$\begin{aligned} 0 = E[\xi]-e &= E\left[E[\xi(\theta)]\right]-e \\ &= E\left[E[(\xi(\theta)-e)\vee 0]\right] \\ &= \int_0^{+\infty} \mathrm{Cr}\{\theta \in \Theta | E[(\xi(\theta)-e)\vee 0] \geq r\}\mathrm{d}r \end{aligned}$$

which implies that $\mathrm{Cr}\{\theta \in \Theta | E[(\xi(\theta)-e)\vee 0]=0\}=1$. That is, there exists a set A_2^* with $\mathrm{Cr}\{A_2^*\}=1$ such that $E[(\xi(\theta)-e)\vee 0]=0$ for each $\theta \in A_2^*$. Let $A^*=A_1^*\cap A_2^*$. It follows from self-duality and subadditivity of credibility measure that

$$\begin{aligned} 1 \geq \mathrm{Cr}\{A^*\} &= \mathrm{Cr}\{A_1^* \cap A_2^*\} \\ &= 1-\mathrm{Cr}\{(A_1^*)^c \cup (A_2^*)^c\} \end{aligned}$$

$$\begin{aligned} &\geq 1 - (\mathrm{Cr}\{(A_1^*)^c\} + \mathrm{Cr}\{(A_2^*)^c\}) \\ &= 1 - (1 - \mathrm{Cr}\{A_1^*\} + 1 - \mathrm{Cr}\{A_2^*\}) = 1, \end{aligned}$$

i.e., $\mathrm{Cr}\{A^*\} = 1$. For each $\theta \in A^*$, we have $\xi(\theta) - e = (\xi(\theta) - e) \vee 0 + (\xi(\theta) - e) \wedge 0 = 0$ almost everywhere, i.e., $\Pr\{\xi(\theta) = e\} = 1$. Hence

$$\mathrm{Ch}\{\xi = e\}(1) = \sup_{\mathrm{Cr}\{A\}=1} \inf_{\theta \in A} \Pr\{\xi(\theta) = e\} = 1.$$

Conversely, if $\mathrm{Ch}\{\xi = e\}(1) = 1$, it follows from Theorem 6.14 of Liu (2002) that $V[\xi] = 0$. According to Theorem 10.5, we have $0 \leq S_v[\xi] \leq V[\xi] = 0$. That is, $S_v[\xi] = 0$. The theorem is proved.

Theorem 10.7 (Qin et al. 2013). *Assume that ξ is a random fuzzy variable with finite expected value e. Then $S_v\,[E[\xi]] \leq S_v[\xi]$.*

Proof. Define $f(x) = [(x-e) \wedge 0]^2$. It is easy to prove that $f(x)$ is a convex function. For each θ, $\xi(\theta)$ is a random variable. It follows from the Jensen's inequality that $f(E[\xi(\theta)]) \leq E[f(\xi(\theta))]$, i.e.,

$$|(E[\xi(\theta)] - e) \wedge 0|^2 \leq E\left[|(\xi(\theta) - e) \wedge 0|^2\right].$$

Since both sides are fuzzy variables with regard to θ, taking the expected values on both sides, we obtain

$$\begin{aligned} S_v[E[\xi]] &= E\left[|(E[\xi(\theta)] - e) \wedge 0|^2\right] \\ &\leq E\left[E\left[|(\xi(\theta) - e) \wedge 0|^2\right]\right] = S_v[\xi]. \end{aligned}$$

The theorem is proved.

Theorem 10.7 indicates that the semivariance of fuzzy variable $E[\xi]$ does not exceed the semivariance of random fuzzy variable ξ.

10.5 Random Fuzzy Mean-Risk Models

In this section, we formulate general mean-risk model for random fuzzy portfolio optimization problem. As stated before, random fuzzy variables are applied to characterize security returns in the situation with mixture of randomness and fuzziness. Furthermore, absolute deviation and semivariance are chosen to measure the risk associated with a portfolio of securities, respectively.

Assume that an investor faces an investment decision problem to allocate his/her fund among the candidate set of n risk securities. Denote by ξ_i the return on security

i, which is a random fuzzy variable, and denote by x_i the proportion on security i for $i = 1, 2, \cdots, n$. Then, the total return on a portfolio $(x_1, x_2, \cdots, x_n)$ is $\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$, which is also a random fuzzy variable according to the operational laws. Thus, we cannot maximize the uncertain return directly since it is not a crisp value for each feasible solution. In general, the investment return is described by expected value $E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$. Next we respectively employ absolute deviation and semivariance to quantify the investment risk.

More generally, an investor sets a desired return level in advance, and then seeks a portfolio to minimize the investment risk. Thus, an optimal portfolio should be the one with minimal risk for the given return level. Following the idea, the following random fuzzy mean-risk model is formulated,

$$\begin{cases} \min D[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{s.t.} \quad E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \geq r \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad 0 \leq x_i \leq u_i, \ i = 1, 2, \cdots, n \end{cases} \tag{10.10}$$

where E and D denote the expected value operator and the risk measure of random fuzzy return, respectively. Here r is a given return level provided by the investor and u_i is the upper bound of the proportion on security i. The second constraint means that all the fund are invested to these n securities. The constraint $x_i \geq 0$ implies that short selling is not allowed.

Alternatively, an investor may maximize the investment return on the condition that the risk does not exceed a given risk level. Then an optimal portfolio will be the one with maximal expected return for the given risk level. The corresponding random fuzzy mean-risk model is formulated as follows,

$$\begin{cases} \max E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{s.t.} \quad D[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \leq d \\ \qquad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad 0 \leq x_i \leq u_i, \ i = 1, 2, \cdots, n \end{cases} \tag{10.11}$$

where d is the tolerable risk level predetermined by the investor.

In some situations, the investor does not know how to set the parameters such as the return level and risk level. Then he/she may maximize the investment return and meanwhile minimize the investment risk. The idea can be expressed in a bi-objective random fuzzy mean-risk which is formulated as follows,

$$\begin{cases} \min \ D[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \max \ E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{s.t.} \quad x_1 + x_2 + \cdots + x_n = 1 \\ \qquad 0 \leq x_i \leq u_i, \ i = 1, 2, \cdots, n. \end{cases} \tag{10.12}$$

A solution $(x_1, \cdots, x_n)$ of model (10.12) is called Pareto optimal if there is no other solution $(y_1, \cdots, y_n)$ such that $E[\xi_1 x_1 + \cdots + \xi_n x_n] \le E[\xi_1 y_1 + \cdots + \xi_n y_n]$ and $D[\xi_1 x_1 + \cdots + \xi_n x_n] \ge D[\xi_1 y_1 + \cdots + \xi_n y_n]$. By solving model (10.12), the investor can obtain a series of the Pareto optimal portfolios.

In a normal security market, to maximize investment return and to minimize investment risk are two conflicting objectives. It implies that it is difficult to find all the Pareto optimal solutions. In order to simplify the problem, we may transform model (10.12) into a single-objective programming model, for example, the following compromise model by weighting two objectives,

$$\begin{cases} \min\ wD[\xi_1 x_1 + \cdots + \xi_n x_n] - (1-w)E[\xi_1 x_1 + \cdots + \xi_n x_n] \\ \text{s.t.}\ \ x_1 + x_2 + \cdots + x_n = 1 \\ \qquad 0 \le x_i \le u_i,\ i = 1, 2, \cdots, n \end{cases} \tag{10.13}$$

where $w \in [0, 1]$ is a risk aversion factor of the investor. Here $w = 1$ means that the investor is extremely conservative because he/she only considers the investment risk and $w = 0$ means that the investor is extremely aggressive to purse the investment return and completely ignores the risk.

10.5.1 Random Fuzzy Mean-Absolute Deviation Model

If we need to use absolute deviation instead of variance to measure risk, then models (10.10), (10.11), (10.12), and (10.13) become the corresponding random fuzzy mean-absolute deviation models introduced by Qin (2016).

If $\xi_1, \xi_2, \cdots, \xi_n$ are all fuzzy variables, model (10.10) becomes fuzzy mean-absolute deviation portfolio optimization model (Qin et al. 2011). On the other hand, if $\xi_1, \xi_2, \cdots, \xi_n$ are all random variables, model (10.10) becomes stochastic mean-absolute deviation model proposed by Konno and Yamazaki (1991). It can be seen that models (10.10), (10.11), (10.12), and (10.13) provide mean-absolute deviation criteria for the most general portfolio optimization problem which includes not only randomness but also fuzziness.

Let $(x_1, x_2, \cdots, x_n)$ be a portfolio and $\xi_i \sim \mathcal{N}(\tilde{\mu}_i, \sigma_i^2)$ a normal random fuzzy variables defined on credibility space $(\Theta_i, \mathcal{P}_i, \mathrm{Cr}_i)$ with $\sigma_i > 0$ for $i = 1, 2, \cdots, n$. For given $(\theta_1, \theta_2, \cdots, \theta_n)$ where $\theta_i \in \Theta_i$ for $i = 1, 2, \cdots, n$, assume that random variables $\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n)$ are mutually independent. Then $x_1\xi_1(\theta_1) + \cdots + x_n\xi_n(\theta_n)$ is a normal random variable with expected value $x_1\tilde{\mu}_1(\theta_1) + \cdots + x_n\tilde{\mu}_n(\theta_n)$ and variance $x_1^2\sigma_1^2 + \cdots + x_n^2\sigma_n^2$, for given $(\theta_1, \theta_2, \cdots, \theta_n)$. Let $(\Theta, \mathcal{P}, \mathrm{Cr})$ be the product credibility space of $(\Theta_i, \mathcal{P}_i, \mathrm{Cr}_i)$, $i = 1, 2, \cdots, n$. We define

$$\xi(\theta_1, \cdots, \theta_n) = x_1\xi_1(\theta_1) + \cdots + x_n\xi_n(\theta_n),\ (\theta_1, \cdots, \theta_n) \in \Theta = \Theta_1 \times \cdots \times \Theta_n.$$

It is evident that ξ is a function from $(\Theta, \mathcal{P}, \mathrm{Cr})$ to set of random variables. Also, ξ is a normal random fuzzy variable denoted by $\xi \sim \mathcal{N}(x_1\tilde{\mu}_1(\theta_1) + \cdots + x_n\tilde{\mu}_n(\theta_n), x_1^2\sigma_1^2 + \cdots + x_n^2\sigma_n^2)$. Furthermore, we have $E[x_1\xi_1 + \cdots + x_n\xi_n] = x_1E[\tilde{\mu}_1] + \cdots + x_nE[\tilde{\mu}_n]$ and

$$
\begin{aligned}
&A[x_1\xi_1 + \cdots + x_n\xi_n] \\
&= \sqrt{x_1^2\sigma_1^2 + \cdots + x_n^2\sigma_n^2} \cdot E\left[\hat{\mu}(2\Phi(\hat{\mu}) - 1) + \sqrt{\frac{2}{\pi}}\exp\left(-\frac{\hat{\mu}^2}{2}\right)\right]
\end{aligned}
$$

where

$$
\hat{\mu} = \frac{x_1(E[\tilde{\mu}_1] - \tilde{\mu}_1) + \cdots + x_n(E[\tilde{\mu}_n] - \tilde{\mu}_n)}{\sqrt{x_1^2\sigma_1^2 + \cdots + x_n^2\sigma_n^2}}.
$$

Replacing expected value and absolute deviation with the above expressions, models (10.10) and (10.13) will be simplified and become easier to be solved.

Similarly, if each security return is described by a simple random fuzzy variable, then models (10.10) and (10.13) are also correspondingly simplified.

10.5.2 Random Fuzzy Mean-Semivariance Model

For the case with asymmetrical return, we need to use semivariance, instead of variance or absolute deviation to measure risk. Then models (10.10), (10.11), (10.12), and (10.13) become the corresponding random fuzzy mean-semivariance models introduced by Qin et al. (2013).

If $\xi_1, \xi_2, \cdots, \xi_n$ are all fuzzy variables, model (10.10) becomes fuzzy mean-semivariance portfolio optimization model (Huang 2008a). On the other hand, if $\xi_1, \xi_2, \cdots, \xi_n$ are all random variables, model (10.10) becomes stochastic mean-semivariance model. It can be seen that models (10.10), (10.11), (10.12), and (10.13) provide mean-semivariance deviation criteria for the most general portfolio optimization problem which includes not only randomness but also fuzziness.

It follows that $E[\xi_1x_1 + \xi_2x_2 + \cdots + \xi_nx_n] = E[\xi_1]x_1 + E[\xi_2]x_2 + \cdots + E[\xi_n]x_n$. Thus, model (10.12) may be converted into the following form,

$$
\begin{cases}
\min\ D[\xi_1x_1 + \xi_2x_2 + \cdots + \xi_nx_n] \\
\text{s.t.}\quad E[\xi_1]x_1 + E[\xi_2]x_2 + \ldots + E[\xi_n]x_n \\
\qquad x_1 + x_2 + \cdots + x_n = 1 \\
\qquad 0 \le x_i \le u_i,\ i = 1, 2, \cdots, n.
\end{cases}
\tag{10.14}
$$

Generally speaking, if the expected value of each security return is known, the constraints of models (10.10), (10.12) and (10.13), and the objective of model (10.11) are completely deterministic. However, it is very hard to translate absolute deviation and semivariance into a deterministic expression. Fortunately, we can employ random fuzzy simulation approaches to approximately compute them.

10.6 Random Fuzzy Simulation

As mentioned above, it is difficult to obtain the exact values of absolute deviation and semivariance of random fuzzy return. Thus, we use random fuzzy simulation to approximately calculate them. Random fuzzy simulation was proposed by Liu (2002), Liu and Liu (2003), and has been applied to portfolio optimization (Huang 2008a; Qin et al. 2013) and other fields. The following algorithms come from Qin (2016).

Assume that ξ_i is defined on the credibility space $(\Theta_i, \mathcal{P}_i, \mathrm{Cr}_i)$ for $i = 1, 2, \cdots, n$. We write $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ to denote a portfolio. For each given $\boldsymbol{x}$, we write

$$F_{\boldsymbol{x}}(\xi_1, \cdots, \xi_n) = |\xi_1 x_1 + \cdots + \xi_n x_n - E[\xi_1 x_1 + \cdots + \xi_n x_n]|$$

whose expected value is just absolute deviation defined in Definition 10.5. Note that $E[\xi_1 x_1 + \cdots + \xi_n x_n]$ is a real number and should be calculated in advance. Denote by $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$ the Cartesian product of $\Theta_1, \Theta_2, \cdots, \Theta_n$. For given $(\theta_1, \theta_2, \cdots, \theta_n) \in \Theta$, according to the definition of random fuzzy variable, $F_{\boldsymbol{x}}(\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n))$ is a random variable, thus whose expected value can be estimated by the following random simulation,

Algorithm 10.1 (Random simulation for $E[F_{\boldsymbol{x}}(\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n))]$)

Step 1. Randomly generate a real vector $(u_{1k}, u_{2k}, \cdots, u_{nk})$ based on the joint probability distribution of random vector $(\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n))$ for $k = 1, 2, \cdots, K$, in which K is a sufficiently large integer;

Step 2. Return $\sum_{k=1}^{K} F_{\boldsymbol{x}}(u_{1k}, u_{2k}, \cdots, u_{nk})/K$ as the estimate value of $E[F_{\boldsymbol{x}}(\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n))]$.

Here K is called the cycle of random simulation. The strong law of large numbers may ensure that the estimate by above random simulation will converge to $E[F_{\boldsymbol{x}}(\xi_1(\theta), \xi_2(\theta), \cdots, \xi_n(\theta))]$ as K tends to infinity. For the case that probability distributions are known, $E[F_{\boldsymbol{x}}(\xi_1, \xi_2, \cdots, \xi_n)]$ can be analytically obtained for given $(\theta_1, \theta_2, \cdots, \theta_n)$. In this case, it is not necessary to apply random simulation, which will significantly reduce the calculation time.

Suppose that $\nu_1, \nu_2, \cdots, \nu_n$ are the membership functions associated with $\xi_1, \xi_2, \cdots, \xi_n$, respectively. Let M be a sufficiently large integer to represent the cycle of fuzzy simulation. The process of estimating $E[F_{\boldsymbol{x}}(\xi_1, \xi_2, \cdots, \xi_n)]$ is summarized as follows,

Algorithm 10.2 (Random fuzzy simulation for absolute deviation)

Step 1. For $m = 1$ to M do:

(i) Randomly generate $(\theta_{1m}, \theta_{2m}, \cdots, \theta_{nm})$ from Θ such that $\nu_i(\theta_{im}) \geq \epsilon$ for $i = 1, 2, \cdots, n$ where ϵ is a sufficiently small positive number;

(ii) Calculate $u_m = \nu_1(\theta_{1m}) \wedge \nu_2(\theta_{2m}) \wedge \cdots \wedge \nu_n(\theta_{nm})$;

(iii) Calculate $z_m = E[F_{\boldsymbol{x}}(\xi_1(\theta_{1m}), \xi_2(\theta_{2m}), \cdots, \xi_n(\theta_{nm}))]$ by using random simulation;

Step 2. For $m = 1$ to M calculate

$$\begin{aligned} w_m &= \max_{1\leq l\leq M}\{u_l | z_l \leq z_m\} - \max_{1\leq l\leq M}\{u_l | z_l < z_m\} \\ &\quad + \max_{1\leq l\leq M}\{u_l | z_l \geq z_m\} - \max_{1\leq l\leq M}\{u_l | z_l > z_m\}; \end{aligned}$$

Step 3. Return $0.5\sum_{m=1}^{M} w_m z_m$ as the estimate of $E[F_{\boldsymbol{x}}(\xi_1, \xi_2, \cdots, \xi_n)]$.

This estimation is based on the discretization of fuzzy variable. The value w_m in Step 2 is the weight of the value z_m taken by fuzzy variable $E[F_{\boldsymbol{x}}(\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n))]$. If $F_{\boldsymbol{x}}(\xi_1, \xi_2, \cdots, \xi_n)$ is replaced with $\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$, then repeating the above process, we can obtain the expected value $E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$.

If for each given $\boldsymbol{x}$, we define

$$\begin{aligned} F_{\boldsymbol{x}}(\xi_1, \cdots, \xi_n) &= ((\xi_1 x_1 + \cdots + \xi_n x_n - E[\xi_1 x_1 + \cdots + \xi_n x_n]) \wedge 0)^2 \\ &= ([(\xi_1 - E[\xi_1])x_1 + \cdots + (\xi_n - E[\xi_n])x_n] \wedge 0)^2, \end{aligned}$$

then we can obtain the estimate value of the semivariance $S_v[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$ by applying Algorithms 10.1 and 10.2.

10.7 Numerical Examples

In this section, we illustrate the application of random fuzzy mean-absolute deviation model and mean-semivariance model by numerical examples in Qin (2016) and Qin et al. (2013).

Example 10.6. We present a numerical example with $n = 10$ risk securities, and all the data are shown in Table 10.1 in which each security return is a normal random fuzzy variable. We take $\xi_1 = \mathcal{N}(\tilde{\mu}_1, 0.07)$ as an example for simple explanation, in which $\tilde{\mu}_1 = (0.01, 0.05, 0.07)$ is a triangular fuzzy variable. Denote by $\nu_1(x)$ the membership function of $\tilde{\mu}_1$. Then $\tilde{\mu}_1$ takes the value 0.04, which implies that the return on the first security is a normal random variable $\mathcal{N}(0.04, 0.07)$ with membership degree $\nu_1(0.04) = 0.75$. Similarly, $\tilde{\mu}_1$ takes the value 0.06, which implies that the return is a normal random variable $\mathcal{N}(0.06, 0.07)$ with membership degree $\nu_1(0.06) = 0.5$.

Table 10.1 Random fuzzy returns of ten securities

No.	Security return	
1	$\xi_1 = \mathcal{N}(\tilde{\mu}_1, 0.07)$,	with $\tilde{\mu}_1 = (0.01, 0.05, 0.07)$
2	$\xi_2 = \mathcal{N}(\tilde{\mu}_2, 0.10)$,	with $\tilde{\mu}_2 = (0.02, 0.06, 0.10)$
3	$\xi_3 = \mathcal{N}(\tilde{\mu}_3, 0.06)$,	with $\tilde{\mu}_3 = (0.01, 0.04, 0.07)$
4	$\xi_4 = \mathcal{N}(\tilde{\mu}_4, 0.08)$,	with $\tilde{\mu}_4 = (0.02, 0.05, 0.08)$
5	$\xi_5 = \mathcal{N}(\tilde{\mu}_5, 0.09)$,	with $\tilde{\mu}_5 = (0.03, 0.05, 0.09)$
6	$\xi_6 = \mathcal{N}(\tilde{\mu}_6, 0.11)$,	with $\tilde{\mu}_6 = (0.04, 0.06, 0.10)$
7	$\xi_7 = \mathcal{N}(\tilde{\mu}_7, 0.09)$,	with $\tilde{\mu}_7 = (0.01, 0.06, 0.09)$
8	$\xi_8 = \mathcal{N}(\tilde{\mu}_8, 0.09)$,	with $\tilde{\mu}_8 = (-0.01, 0.05, 0.11)$
9	$\xi_9 = \mathcal{N}(\tilde{\mu}_9, 0.12)$,	with $\tilde{\mu}_9 = (0.03, 0.08, 0.11)$
10	$\xi_{10} = \mathcal{N}(\tilde{\mu}_{10}, 0.13)$,	with $\tilde{\mu}_{10} = (0.04, 0.07, 0.14)$

It follows that expected value of a triangular fuzzy variable $\tilde{\mu} = (a, b, c)$ is $E[\tilde{\mu}] = (a + 2b + c)/4$. In a similar way, we can obtain the value of $E[\tilde{\mu}_i]$ and denote it by μ_i for $i = 1, 2, \cdots, 10$. According to Definition 10.2, we have $E[\xi_i] = E[\tilde{\mu}_i] = \mu_i$ for $i = 1, 2, \cdots, 10$, which implies that we can get the real value of $E[\xi_i]$ in this example. To test the performance of random fuzzy simulation, we employ it to estimate $E[\xi_i]$ and then compare the simulated value with the real one. Next we take $E[\xi_{10}]$ as an example. The computational results are shown in Table 10.2. The last column gives the relative error (RE) for each case, in which relative error is the absolute value of (real value – simulated value)/real value$\times$100 %. Table 10.2 shows that although there are fluctuations, the maximal relative error is 2.50 % when K or M is greater than 3000, and the relative errors are no more than 1.00 % in most cases.

We further test the performance of random fuzzy simulation by comparing the simulated values and real expected returns of other nine securities. Table 10.3 shows the computational results which includes two cases: one is $K = M = 3000$ and the other is $K = M = 5000$. The results indicate that random fuzzy simulation works well since the maximal relative error is 1.15 %. In order to save computational time, we choose $K = M = 3000$ in random fuzzy simulation when simulating the absolute deviation of the return on a portfolio in the following genetic algorithm.

Example 10.7. We next consider applying model (10.10) to construct an optimal portfolio with minimum absolute deviation. The corresponding parameters of genetic algorithm are set as follows: (a) crossover probability $P_c = 0.4$; (b) mutation probability $P_m = 0.3$; (c) the parameter in the rank-based evaluation function $v = 0.05$; (d) the population size *pop_size* $= 100$. The numerical example is performed on a personal computer with 2.40 GHz CPU and 2.0 GB memory. For a given return level $r = 0.06$, a run of the algorithm with 500 generations shows that from the feasible portfolios, an investor should allocate his/her capital according to Table 10.4. The corresponding absolute deviations of the best found portfolio are 0.0827. As a comparison, we also show the optimal allocation in Table 10.4 when $r = 0.07$. It can be seen that the investment proportions on securities 6, 9 and 10 rapidly increase since although they have larger absolute deviations, they also have larger expected values. As a result, more funds are invested in these three securities

Table 10.2 Comparisons of real value and simulated one by random fuzzy simulation of $E[\xi_{10}]$

No.	K	M	Simulated value	Real value	RE (%)
1	1000	500	0.0843	0.08	5.38
2	1000	1000	0.0823	0.08	2.88
3	1000	2000	0.0828	0.08	3.50
4	1000	3000	0.0804	0.08	0.50
5	1000	5000	0.0798	0.08	0.25
6	2000	500	0.0805	0.08	0.63
7	2000	1000	0.0797	0.08	0.38
8	2000	2000	0.0797	0.08	0.38
9	2000	3000	0.0820	0.08	2.50
10	2000	5000	0.0808	0.08	1.00
11	3000	500	0.0806	0.08	0.75
12	3000	1000	0.0806	0.08	0.75
13	3000	2000	0.0812	0.08	1.50
14	3000	3000	0.0801	0.08	0.13
15	3000	5000	0.0796	0.08	0.50
16	5000	500	0.0805	0.08	0.63
17	5000	1000	0.0802	0.08	0.25
18	5000	2000	0.0796	0.08	0.50
19	5000	3000	0.0793	0.08	0.88
20	5000	5000	0.0807	0.08	0.88

Table 10.3 Comparisons of real value and simulated one by random fuzzy simulation for other 9 returns in Example 10.6

	Real value	Simulated[a]	RE (%)[a]	Simulated[b]	RE (%)[b]
$E[\xi_1]$	0.045	0.04506	0.13	0.04533	0.73
$E[\xi_2]$	0.060	0.06010	0.17	0.06055	0.92
$E[\xi_3]$	0.040	0.04002	0.05	0.04041	1.03
$E[\xi_4]$	0.050	0.04994	0.12	0.05033	0.66
$E[\xi_5]$	0.055	0.05524	0.44	0.05531	0.56
$E[\xi_6]$	0.065	0.06540	0.62	0.06530	0.46
$E[\xi_7]$	0.055	0.05563	1.15	0.05547	0.85
$E[\xi_8]$	0.050	0.05049	0.98	0.05026	0.52
$E[\xi_9]$	0.075	0.07505	0.07	0.07540	0.53

[a] Corresponds to the results in the case of $K = 3000$ and $M = 3000$
[b] Corresponds to the results in the case of $K = 5000$ and $M = 5000$

Table 10.4 Investment proportions on 10 securities with different return levels α

r	Investment proportion (%) on 10 securities	AD
0.06	(8.32, 9.24, 4.82, 9.61, 10.24, 10.04, 10.08, 9.17, 15.61, 12.86)	0.0827
0.07	(1.41, 13.48, 0.63, 1.79, 3.87, 17.67, 1.99, 1.45, 28.12, 29.59)	0.1284

to satisfy the return constraint. As r becomes smaller, the proportions on securities $1, 3$ and 4 will become larger since they have relatively smaller absolute deviations.

Example 10.8. We consider another numerical example which is performed on a personal computer with 2.99 GHz CPU and 3.0 GB memory. The parameters in genetic algorithm are set as follows: (a) fuzzy simulation: $M = 3000$ cycles; (b) random simulation: $K = 3000$ cycles; (c) maximum number of generations: $Gen = 500$; (d) crossover probability: $P_c = 0.3$; (e) mutation probability: $P_m = 0.4$; (f) the parameter in the rank-based evaluation function: $v = 0.05$; (g) the population size: $pop\_size = 80$. To compare the results, we use the data from Huang (2007b) which considered nine securities with random fuzzy returns denoted by $\xi_1, \xi_2, \cdots, \xi_9$, listed in Table 10.5.

The computational results are shown in Table 10.6, in which the second column is the real value $E[\xi_i]$, and the third column is its stimulated value. The fourth column shows the relative error of each case. From Table 10.6, we can see that the maximal relative error is 0.67 %, which means that random fuzzy simulation works very well. In addition, we calculate the semivariance of each security return by random fuzzy simulation, which is listed in the fifth column in Table 10.6.

Table 10.5 Random fuzzy returns of nine securities

No.	Security return
1	$\xi_1 = \mathcal{N}(\tilde{a}_1, 0.2)$, with $\tilde{a}_1 = (0.0, 0.5, 0.8)$
2	$\xi_2 = \mathcal{N}(\tilde{a}_2, 0.2)$, with $\tilde{a}_2 = (0.3, 0.4, 0.7)$
3	$\xi_3 = \mathcal{N}(\tilde{a}_3, 0.2)$, with $\tilde{a}_3 = (0.1, 0.6, 0.9)$
4	$\xi_4 = \mathcal{N}(\tilde{a}_4, 0.3)$, with $\tilde{a}_4 = (0.0, 0.5, 1.0)$
5	$\xi_5 = \mathcal{N}(\tilde{a}_5, 0.3)$, with $\tilde{a}_5 = (0.1, 0.6, 0.8)$
6	$\xi_6 = \mathcal{N}(\tilde{a}_6, 0.3)$, with $\tilde{a}_6 = (0.3, 0.5, 0.7)$
7	$\xi_7 = \mathcal{N}(\tilde{a}_7, 0.4)$, with $\tilde{a}_7 = (0.4, 0.6, 1.0)$
8	$\xi_8 = \mathcal{N}(\tilde{a}_8, 0.4)$, with $\tilde{a}_8 = (0.5, 0.7, 0.9)$
9	$\xi_9 = \mathcal{N}(\tilde{a}_9, 0.4)$, with $\tilde{a}_9 = (0.0, 0.9, 1.2)$

Table 10.6 Allocation of capital to nine securities

Security	Mean	Simulated mean	Relative error (%)	Semivariance
1	0.450	0.447	0.67	0.130
2	0.450	0.449	0.22	0.108
3	0.550	0.547	0.55	0.130
4	0.500	0.497	0.60	0.193
5	0.525	0.523	0.38	0.174
6	0.500	0.499	0.20	0.158
7	0.650	0.649	0.15	0.217
8	0.700	0.699	0.14	0.209
9	0.750	0.746	0.53	0.273

Table 10.7 Allocation of money to nine securities (%)

No.	1	2	3	4	5	6	7	8	9
x_i	15.23	18.57	21.44	0.63	3.40	17.57	15.78	5.74	1.65

Table 10.8 Comparisons of the proposed model with the existing model

	Model (10.16)	Model of Huang (2007b)
Mean	0.57	0.57
Semivariance	0.16	0.17
Variance	0.31	0.32

If the minimum return level the investor can bear is 0.5, then we employ the following random fuzzy mean-semivariance model,

$$\begin{cases} \min\ S_v[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_9 x_9] \\ \text{s.t.}\ \ E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_9 x_9] \geq 0.5 \\ \qquad x_1 + x_2 + \cdots + x_9 = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 9 \end{cases} \tag{10.15}$$

which is equivalent to the following form,

$$\begin{cases} \min\ S_v[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_9 x_9] \\ \text{s.t.}\ \ 0.45x_1 + 0.45x_2 + 0.55x_3 + 0.50x_4 + 0.525x_5 \\ \qquad +0.50x_6 + 0.65x_7 + 0.70x_8 + 0.75x_9 \geq 0.5 \\ \qquad x_1 + x_2 + \cdots + x_9 = 1 \\ \qquad x_i \geq 0, \quad i = 1, 2, \cdots, 9. \end{cases} \tag{10.16}$$

Obviously, the feasible set of model (10.16) is deterministic and convex. But the objective function need to be estimated by random fuzzy simulation.

A run of genetic algorithm shows that among the portfolios satisfying the constraints, in order to minimize the semivariance of total return, the investor should allocate his/her asset according to Table 10.7. The minimal semivariance of the best found portfolio is 0.14.

Next we consider to compare our model with the existing models. Here, we choose model (3) from Huang (2007b), which seeks the maximal investment return the investor can obtain at a given chance with a given credibility. Based on the results of Huang (2007b), we may obtain that the expected value of its best found portfolio is 0.57. We regard 0.57 as a benchmark and suppose that it is the minimal return level the investor can accept. And then we employ the hybrid intelligent algorithm to solve Model (10.16), and the computational results are shown in Table 10.8. The best found portfolio is (4.2 %, 23.8 %, 12.5 %, 6.0 %, 7.1 %, 8.7 %, 10.1 %, 20.2 %, 7.4 %). It can be seen from Table 10.8 that our model obtains a better portfolio with smaller risk than Huang's one on the condition that the investment returns are just the same.

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List of Frequently Used Symbols

Pr	probability measure
Cr	credibility measure
$\mathcal{M}$	uncertain measure
Ch	chance measure
$(\Omega, \mathcal{A}, \mathrm{Pr})$	probability space
$(\Theta, \mathcal{P}, \mathrm{Cr})$	credibility space
$(\Gamma, \mathcal{L}, \mathcal{M})$	uncertainty space
E	expected value
V	variance
S_v	semivariance
A	absolute deviation
S_a	semiabsolute deviation
S_k	Skewness
H	entropy
D	cross-entropy
$\tilde{a}$, $\tilde{b}$, $\tilde{c}$	fuzzy variables
μ, ν, μ_i, ν_i	membership functions
$\mathcal{S}(x_i, \mu_i; m)$	simple fuzzy variable
$\mathcal{E}(a-\alpha, a+\alpha)$	equipossible fuzzy variable
$\mathcal{T}(a-\alpha, a, a+\beta)$	triangular fuzzy variable
$\mathcal{TP}(a-\alpha, a, b, b+\beta)$	trapezoidal fuzzy variable
$\mathcal{N}(a, \delta)$	normal fuzzy variable
$\mathcal{EXP}(m)$	exponential fuzzy variable
$\mathcal{L}(a, b)$	linear uncertain variable
$\mathcal{Z}(a, b, c)$	zigzag uncertain variable
$\mathcal{N}(e, \sigma)$	normal uncertain variable

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$\vee$	maximum operator
$\wedge$	minimum operator
$\forall$	universal quantifier
$\emptyset$	the empty set
$\Re$	the set of real numbers
C	chromosome
Eval	evaluation function in genetic algorithm
GA	genetic algorithm
RE	relative error