Relation & Lattices

Hasse Diagram:

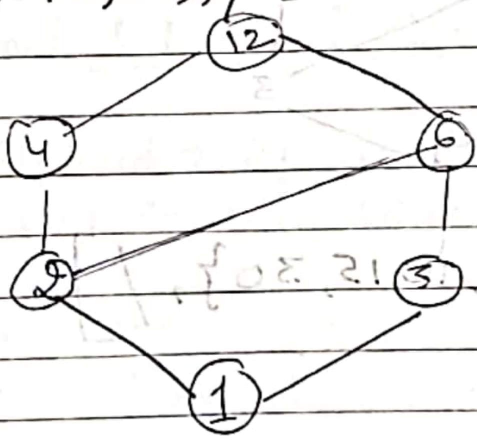
Q1) [{1,2,3,4,5}, ≤]

[A, ≤] → This representation shows a set along with a relationship and is known as a poset

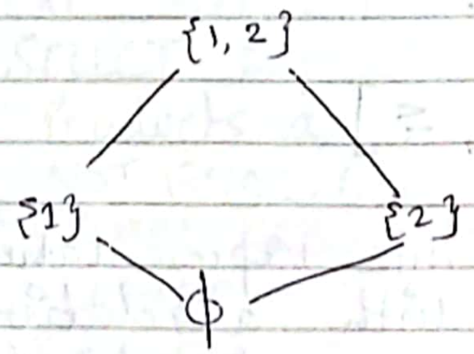
This ρ means, if $(a \leq b)$ then $(a,b) \in R$
In hasse diagram small elements are below and large on top.

(self loop & transitive edges are there but no need to represent them again & again)

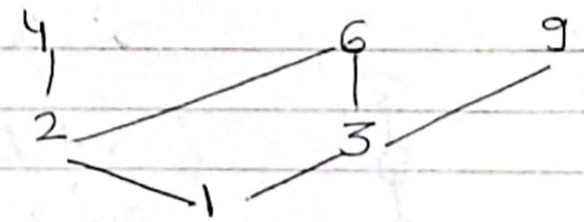
Q2) [{1,2,3,4,6,12}, /]



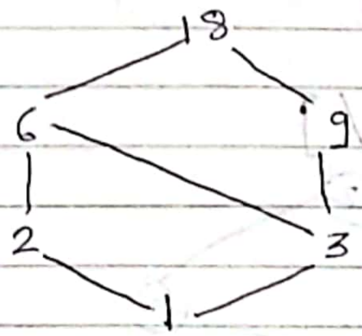
Q3) $[\{\phi, \{1\}, \{2\}, \{1,2\}\}, \subseteq]$



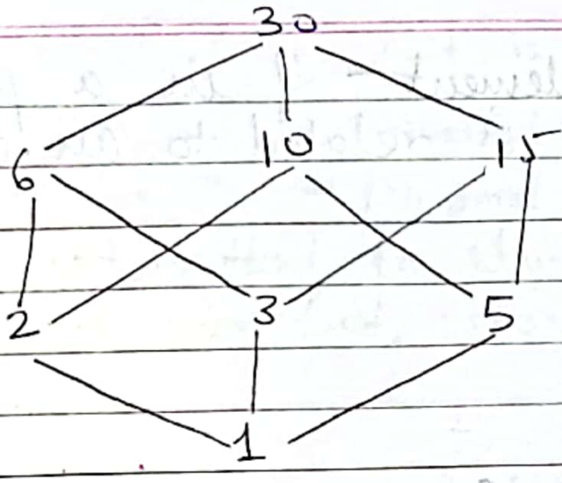
Q4) $[\{1,2,3,4,6,9\}, /]$



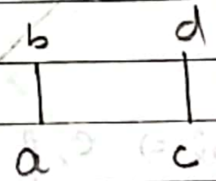
Q5) $[\{1, 2, 3, 6, 9, 18\}, /]$



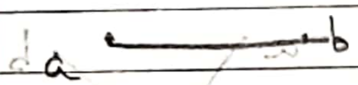
Q6) $[\{1, 2, 3, 5, 6, 10, 15, 30\}, /]$



✓ a b c → Its a valid hasse diagram

✓ 

→ Also valid hasse diagram



→ ☒ Not Valid

- Valid hasse diagram can never have a horizontal edge.

Maximal & Minimal Elements:

Maximal Element → If in a poset, an element is not related to any other element

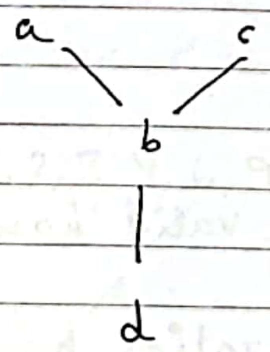
In hasse diagram, element at the bottom relates to all the elements coming at the top (those which get linked)

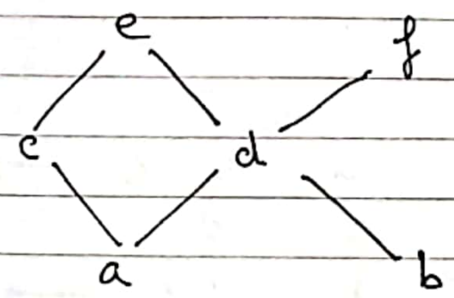
- So elements on the top are maximal elements as they aren't related to any other element.

Minimal element – If in a poset, no element is related to an element.

→ The elements at bottom, the relate to all element, but no element relates to them

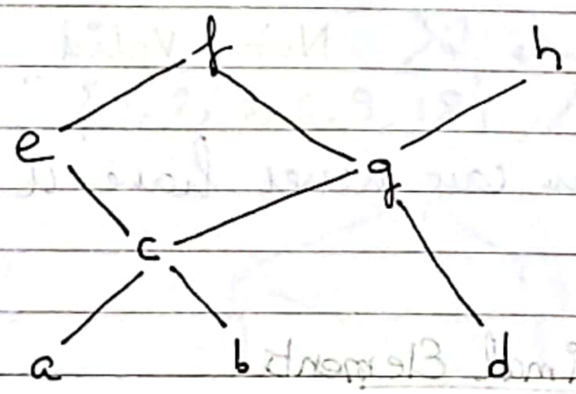
Eg

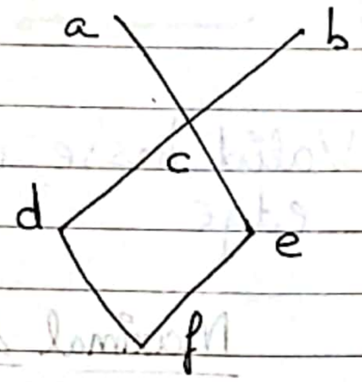




Maximal ⇒ c, a
Minimal ⇒ d

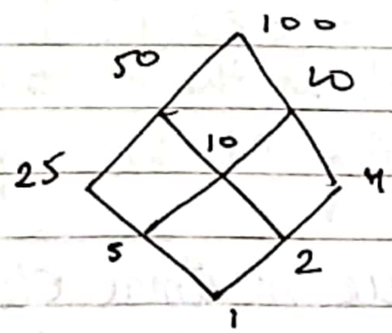
Maximal ⇒ e, f
Minimal ⇒ a, b



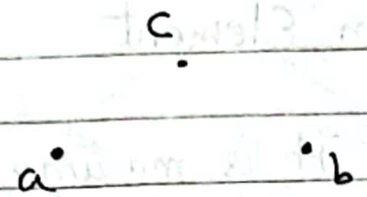


Maximal ⇒ f, h
Minimal ⇒ a, b, d

Maximal ⇒ a, b
Minimal ⇒ f



Maximal ⇒ 100
minimal ⇒ 1

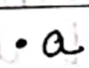


(It is valid hasse diagram)

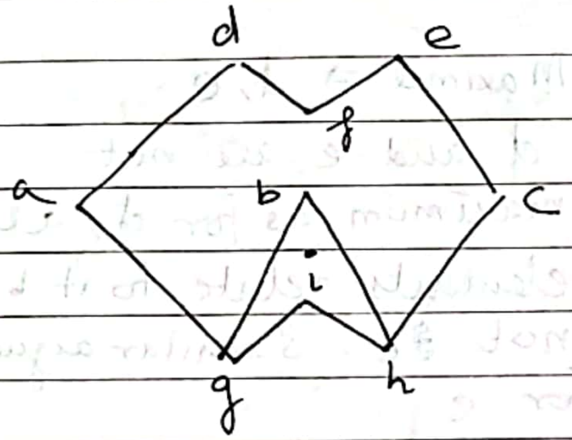
Maximal ⇒ a, b, c

Minimal ⇒ a, b, c

⇒ {(a,a), (b,b) (c,c)}

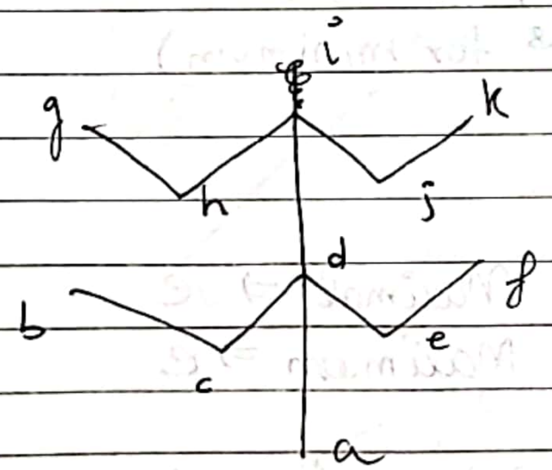


Maximal & Minimal ⇒ a



Maximal ⇒ d, e, b, i

Minimal ⇒ f, g, h



Maximal ⇒ i, g, k, b, f

Minimal ⇒ h, j, c, e, a

- Every hasse diagram will have atleast one maximal and minimal element.

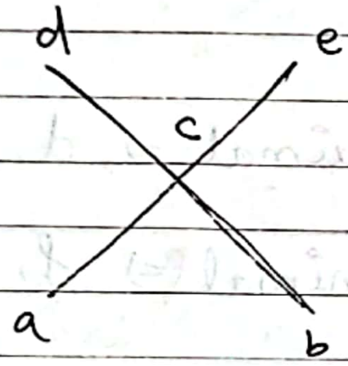
• There can be only 1 maximum & 1 minimum element

Maximum & Minimum Element

Maximum element – If it is maximal & every element is related to it.
(All elements of the system)

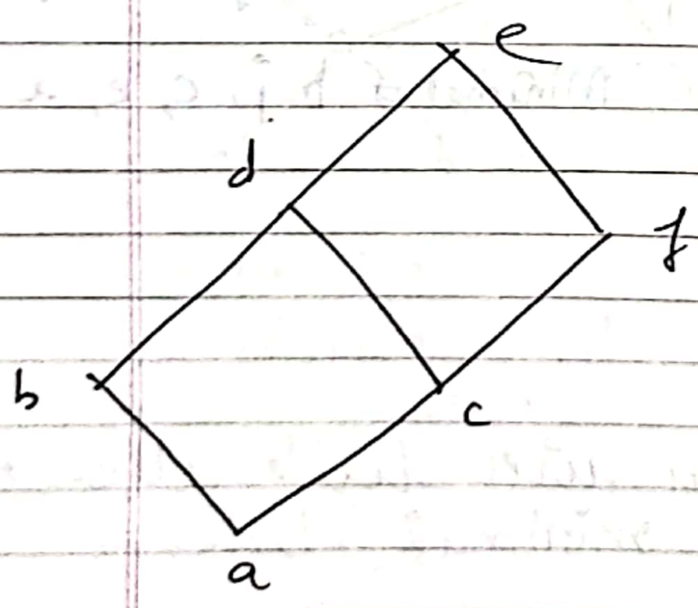
Minimum element – If it is minimal and it is related to every element in poset

Eg



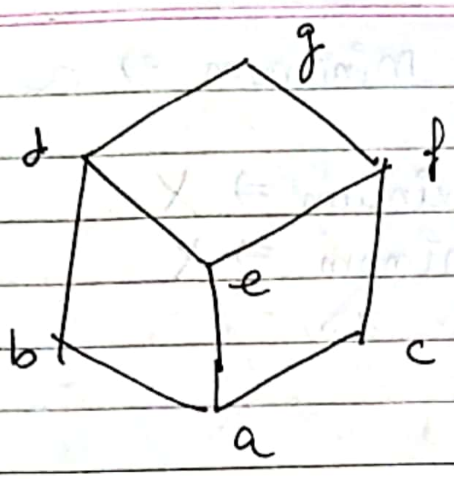
Maximal ⇒ d, e
⇒ d and e are not maximum as for d, all elements relate to it but not e. Similar arguement for e.

Minimal ⇒ a, b
(Same arguements for minimum)



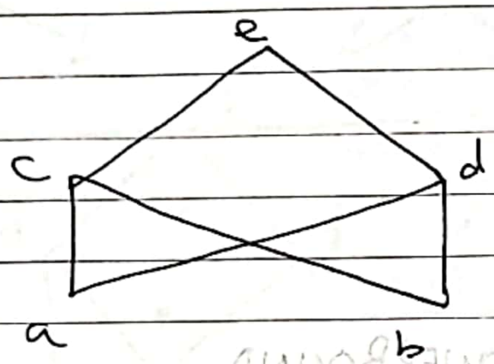
Maximal ⇒ e
Maximum ⇒ e

Minimal ⇒ a
Minimum ⇒ a



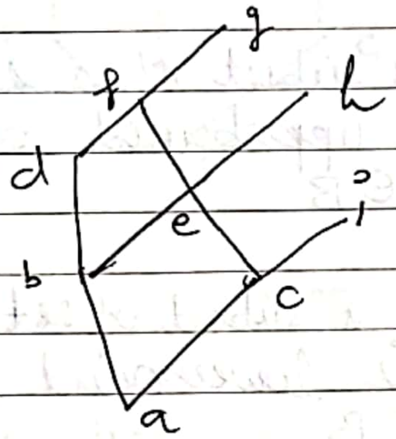
Maximal ⇒ g
Maximum ⇒ g.

Minimal ⇒ a
Minimum ⇒ a



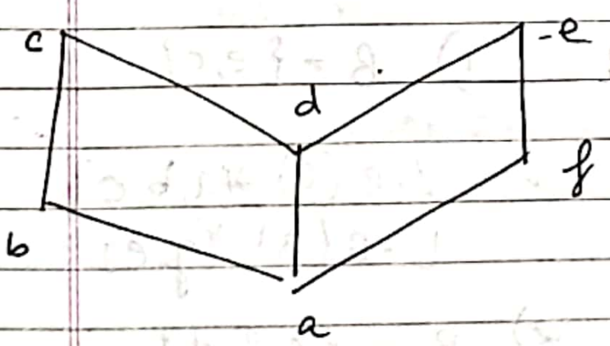
Maximal ⇒ e
Maximum ⇒ e

Minimal ⇒ a, b
Minimum ⇒ X



Maximal ⇒ g, h, i
Maximum ⇒ X

Minimal ⇒ a
Minimum ⇒ a

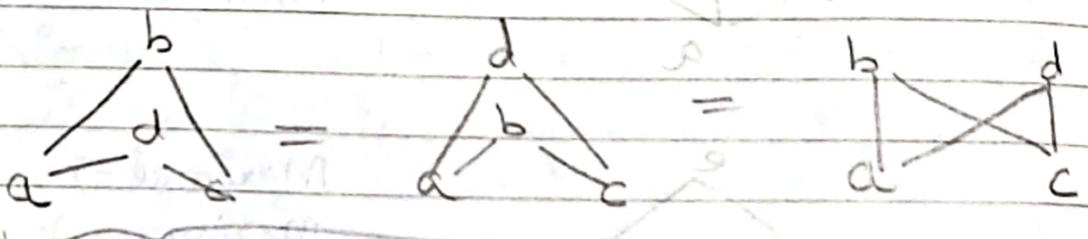


Maximal ⇒ c, d, e
Maximum ⇒ X

Minimal ⇒ a
Minimum ⇒ a

• a   Maximum, Minimum ⇒ a

• a • b • c   Maximum ⇒ X
Minimum ⇒ X



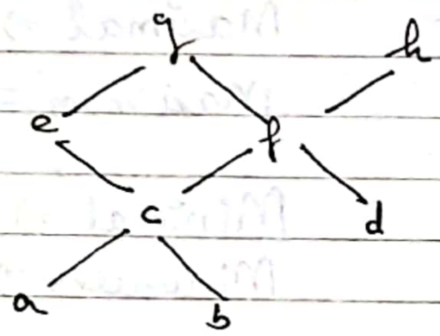
No maximum
No minimum

UPPERBOUND & LOWERBOUND

Upperbound → Let B be a subset of a set A. An element $x \in A$ is in upperbound of B if $(y, x) \in$ Poset $\forall\, y \in B$

LowerBound → Let B be a subset of set A, an element $x \in A$ is in lowerbound of B if $(x, y) \in$ Poset $\forall\, y \in B$

Eg



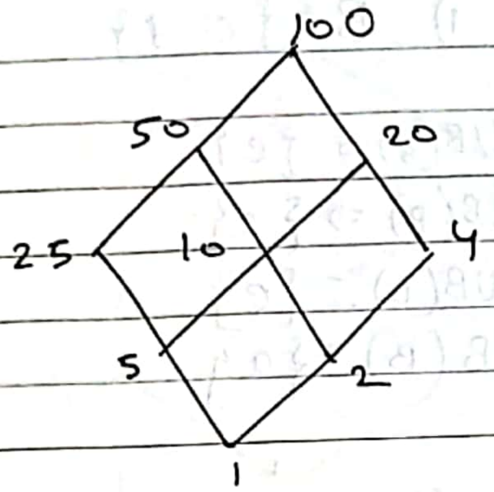
1) $B = \{e, c\}$

$L.B(B) \Rightarrow \{a, b, c\}$
$U.B(B) \Rightarrow \{g, e\}$

2) $B = \{c, f, d\}$

$L.B(B) \Rightarrow X \Rightarrow \phi$
$U.B(B) \Rightarrow \{g, h, f\}$

Scanned with CamScanner



1) $B = \{5, 10\}$

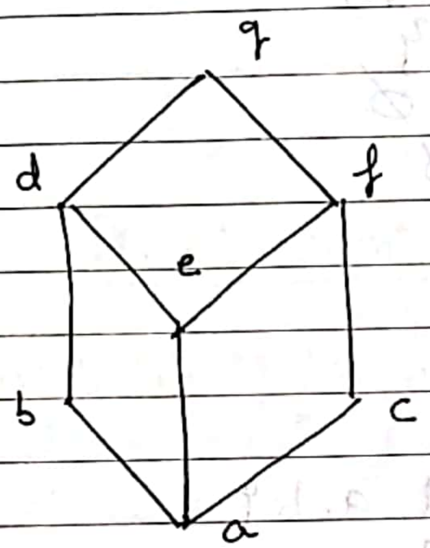
$L\cdot B(B) \Rightarrow \{1, \cancel{2}, 5, \cancel{10}\}$

$U\cdot B(B) \Rightarrow \{10, 20, 50, 100\}$

2) $B = \{5, 10, 2, 4\}$

$L\cdot B(B) \Rightarrow \{1\}$

$U\cdot B(B) \Rightarrow \{20, 100\}$



1) $B = \{d, g\}$

$L\cdot B(B) \Rightarrow \{a, b, e, d\}$

$U\cdot B(B) \Rightarrow \{g\}$

2) $B = \{e, f\}$

$L\cdot B(B) \Rightarrow \{a, e\}$

$U\cdot B(B) \Rightarrow \{g, f\}$

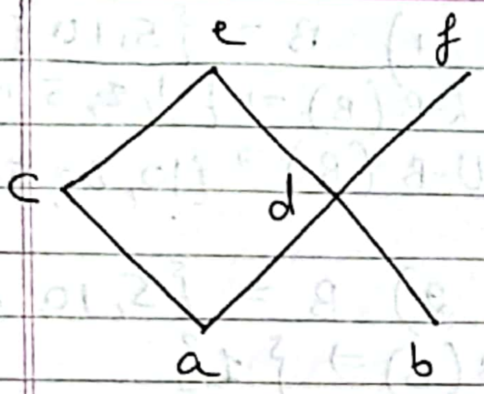
LEAST UPPER BOUND & GREATEST LOWER BOUND

Least Upper Bound – (LUB)(Supremum)(Join)($\vee$)

$\Rightarrow$ Least (minimum) element in Upper Bound

Greatest Lower Bound – (GLB)(Infimum)(Meet)($\wedge$)

$\Rightarrow$ Greatest (maximum) element in lower bound.



1) $B = \{c, d\}$

$UB(B) \Rightarrow \{e\}$

$LB(B) \Rightarrow \{a\}$

$LUB(B) = \{e\}$

$GLB(B) = \{a\}$

2) $B = \{a, b\}$

$UB(B) \Rightarrow \{d, e, f\}$

$LUB(B) \Rightarrow \{d\}$

$L\cdot B(B) \Rightarrow \emptyset$

$GLB(B) \Rightarrow \emptyset$

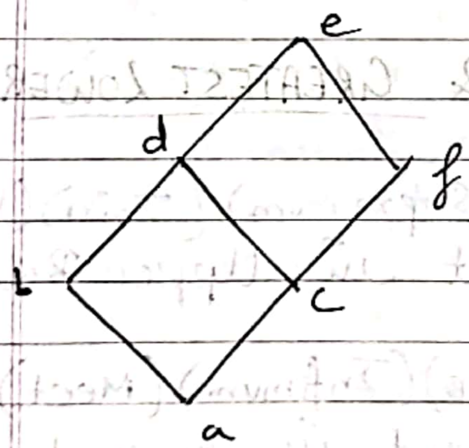
3) $B = \{e, f\}$

$U\cdot B(B) \Rightarrow \emptyset$

$LUB(B) \Rightarrow \emptyset$

$L\cdot B(B) \Rightarrow \{d, a, b\}$

$GLB(B) \Rightarrow \{d\}$



1) $B = \{a, c, f\}$

$UB(B) \Rightarrow \{e, f\}$

$LUB(B) \Rightarrow \{f\}$

$L\cdot B(B) \Rightarrow \{a\}$

$GLB(B) \Rightarrow \{a\}$

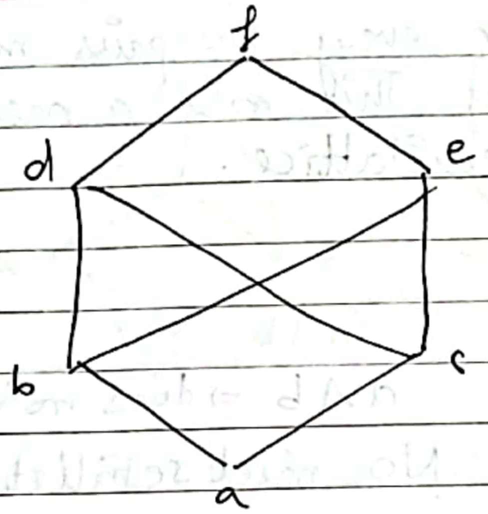
2) $B = \{d, c\}$

$UB(B) \Rightarrow \{e, d\}$

$LUB(B) \Rightarrow \{d\}$

$LB(B) \Rightarrow \{a, c\}$

$GLB(B) \Rightarrow \{c\}$

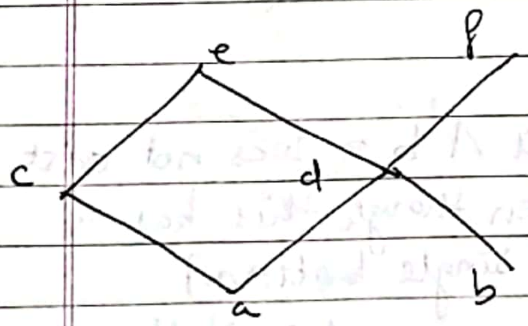


$B \Rightarrow \{d, e\}$
$UB(B) \Rightarrow \{f\}$
$LUB(B) \Rightarrow \{f\}$
$LB(B) \Rightarrow \{b, a, c\}$
$GLB(B) \Rightarrow \phi$

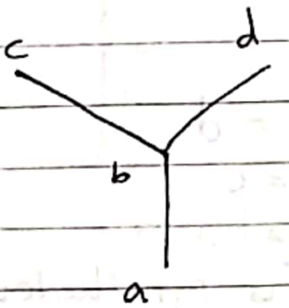
$B \Rightarrow \{b, c\}$
$UB(B) = \{d, f, e\}$
$LUB(B) \Rightarrow \phi$
$LB(B) \Rightarrow \{a\}$
$GLB(B) \Rightarrow \{a\}$

Meet Semilattic:

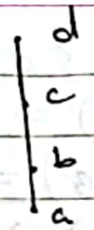
Meet Semilattice - In a poset if GLB / Meet / Infimum / $\wedge$ exist for any pair of elements, then poset is called meet semilattice.



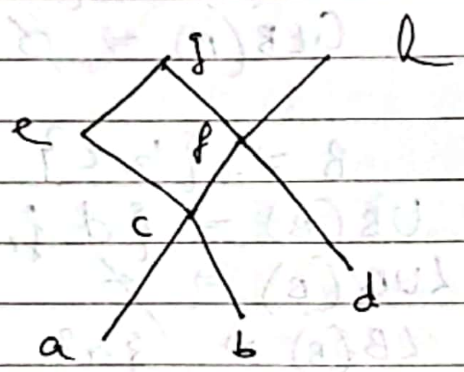
$e \wedge f = d$ (e, f intersect at d first)
$a \wedge b =$ does not exist
So for one pair condition false, this structure can't be meet semilattic.



$a \wedge b = a$
$c \wedge d = b$
(And for all pairs meet exists)
This is a meet semilattice



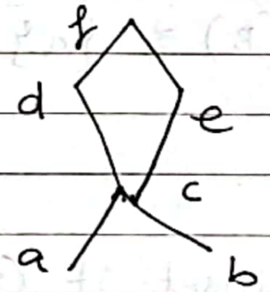
For every two pairs meet exists. This is a meet semilattice.



$(a \wedge b = ?)$

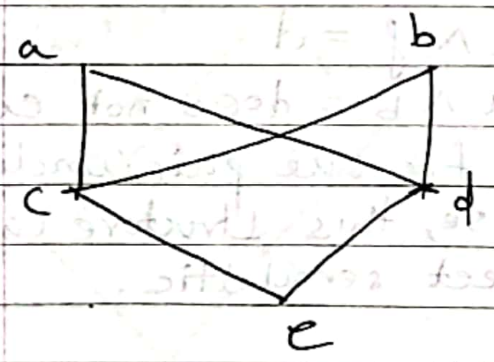
$a \wedge b \Rightarrow$ does not exist

No meet semillatice



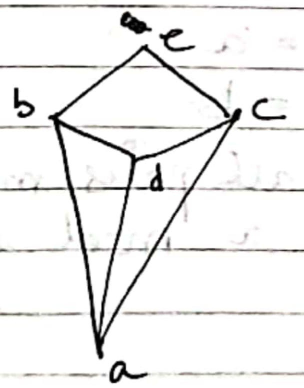
No a meet semilattice

★ For a meet semilattice we must have a single bottom.



$a \wedge b =$ does not exist
(even though this has a single bottom)

No a meet semilattice.



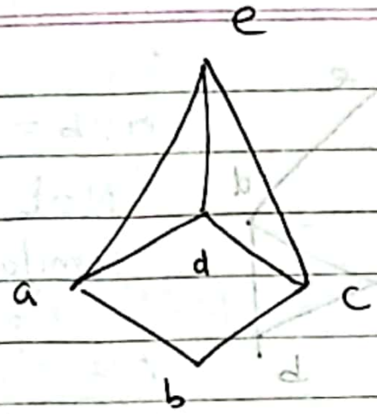
$b \wedge c = d$

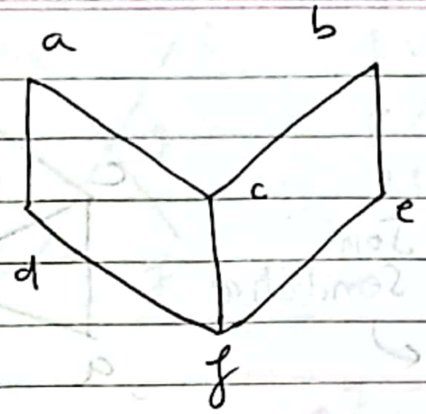
$e \wedge c = c$

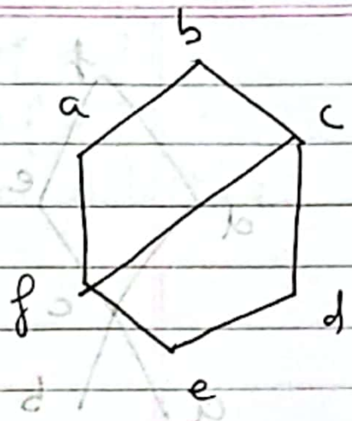
$e \wedge b = b$

For all pairs on checking meet will exists

∴ Meet Semilattice



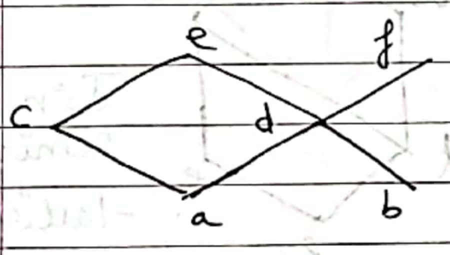




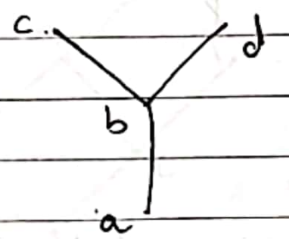
All 3 of them are meet semi lattices.

Join Semilattice

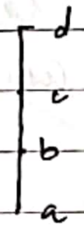
Join semilattice → In a poset if LUB/ JOIN/ Supremum / V exists for any$^{\text{all}}$ pair of elements, then poset is called Join-semilattice.



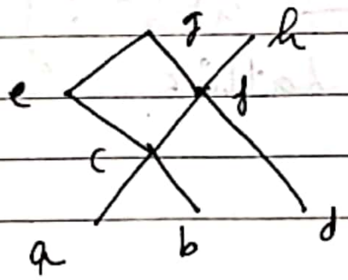
$a \vee b = d$
$a \vee e = e$
$e \vee f \Rightarrow$ does not exist
Not join semilattice



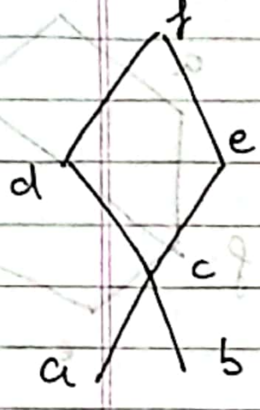
$a \vee b = b$
$c \vee d = ?$
Not join semi lattice



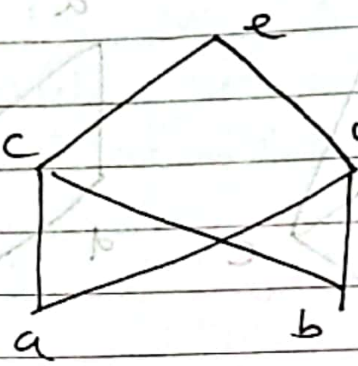
This join semilattice



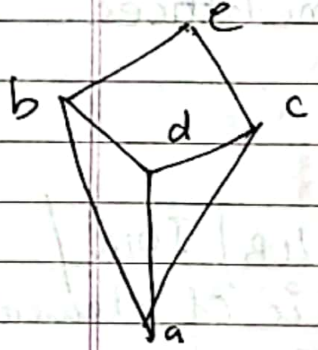
$a \vee d = f$, $e \vee b = e$, $a \vee h = h$
$g \vee h = ?$
Not join semilattice

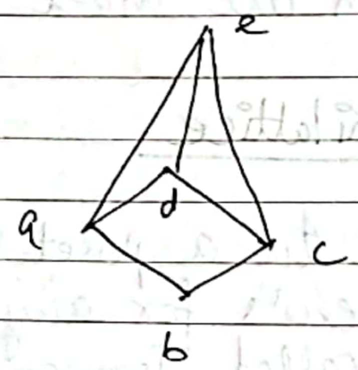


Join Semilattice

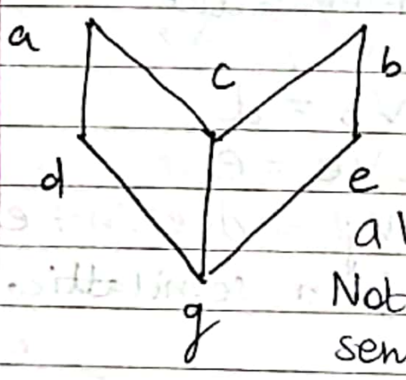


$a \vee b = ?$
Not join Semilattice
(Single top still not join semi-lattice)

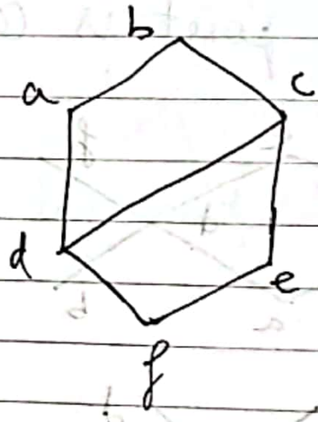




Both are join semi-lattices



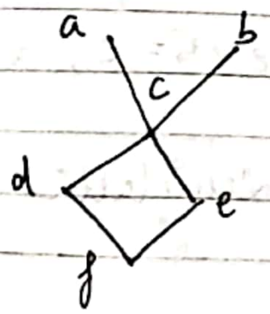
$a \vee b = ?$
Not join semi lattice



Join semi-lattice

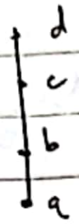
LATTICE

Lattice → A poset is called a lattice if it is both Meet Semi-Lattice and Join Semi lattice [A, ∧, ∨]

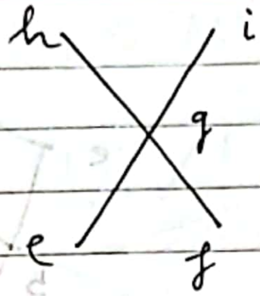


~~$a \wedge b = ?$~~ $a \vee b = ?$
Not a lattice

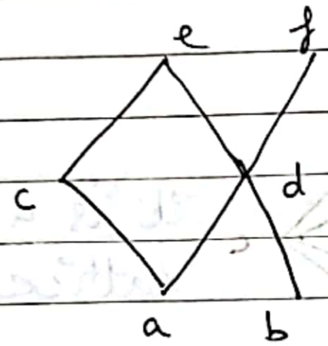
$\wedge \rightarrow$ meet $\quad \vee \rightarrow$ join



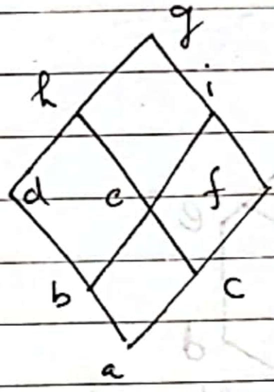
It is a lattice



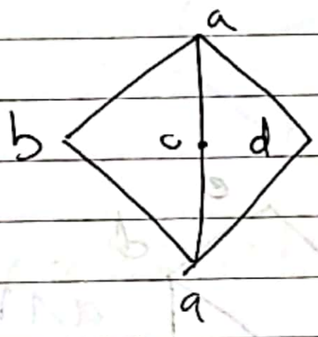
$h \vee i = ?$
$e \wedge f = ?$
Not a lattice



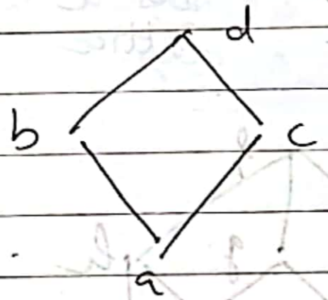
$e \vee f = ?$
$a \wedge b = ?$
Not a lattice

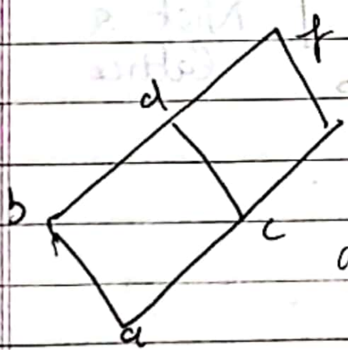


It is a lattice

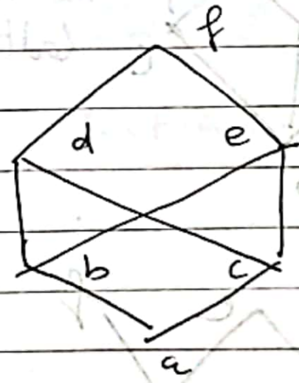


Both are lattices.

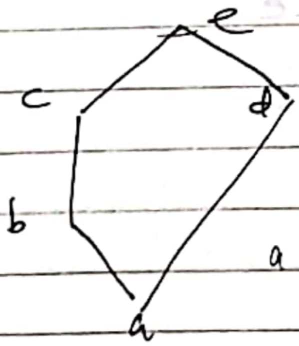




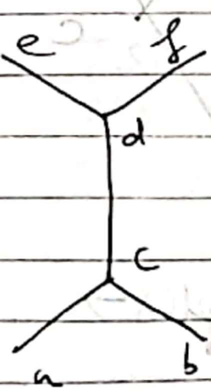
It is a lattice



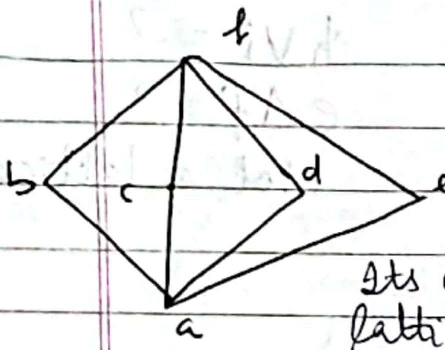
$d \wedge e = ?$
$b \vee c = ?$
Not a lattice



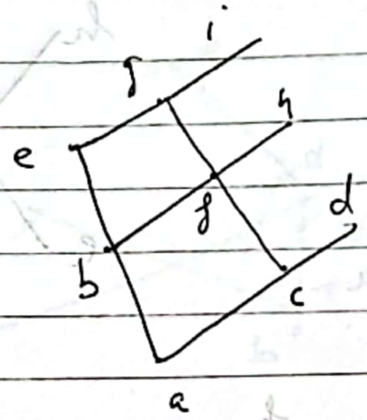
It is a lattice



$e \vee f = ?$
$a \wedge b = ?$
Not a lattice

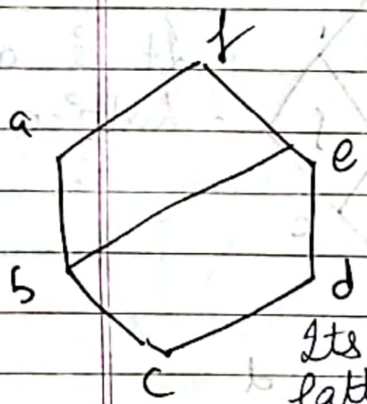


Its a lattice

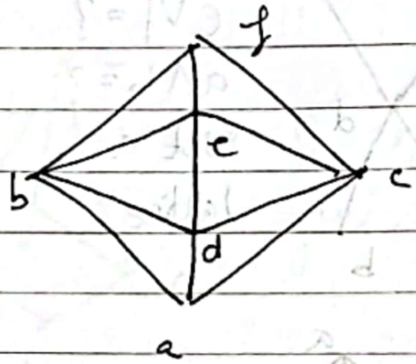


Not a join semi-L

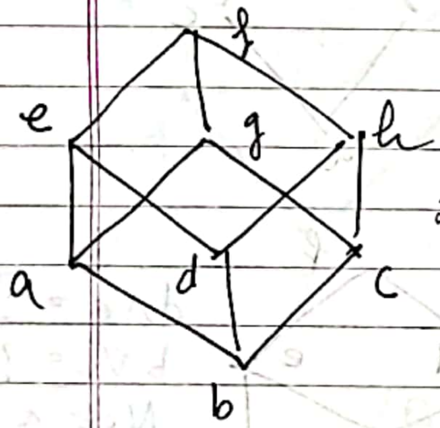
∴ Not a lattice



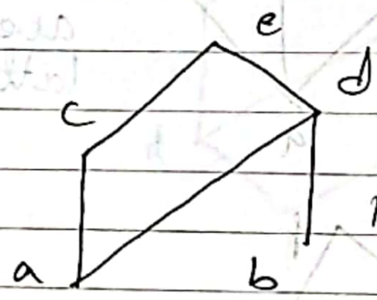
Its a lattice



It is a lattice

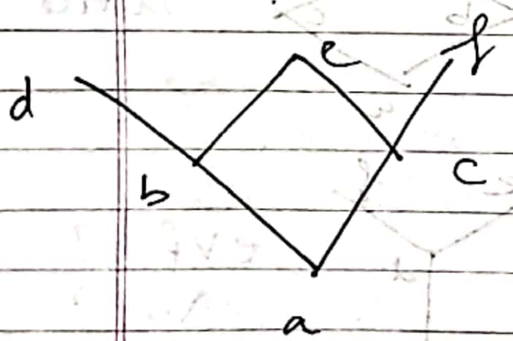


It is a lattice



$a \wedge b = ?$

Not a lattice



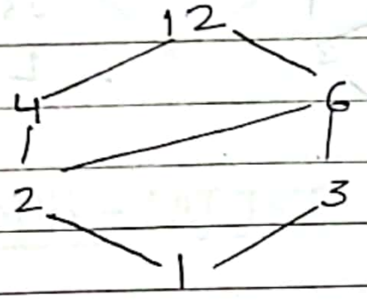
$d \vee e = ?$

Not a lattice

Examples →

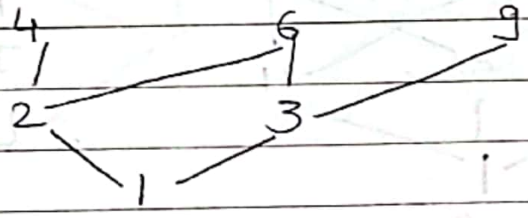
1) $[D_{12}, /]$ $D_{12} \Rightarrow$ All factors of 12

$[\{1, 2, 3, 4, 6, 12\}, /]$



It is a Lattice

2) $\{1, 2, 3, 4, 6, 9\}, /$

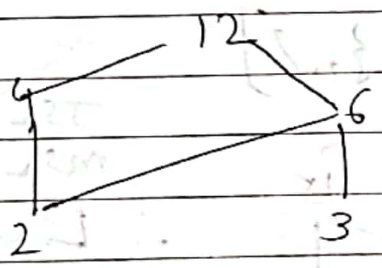


(Not a single top)

Meet S.L ✓

Not a Lattice

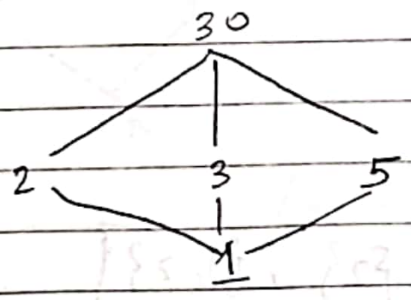
3) $\{2, 3, 4, 6, 12\}, /$



Join S.L ✓

Not a lattice

4) $\{1, 2, 3, 5, 30\}, /$

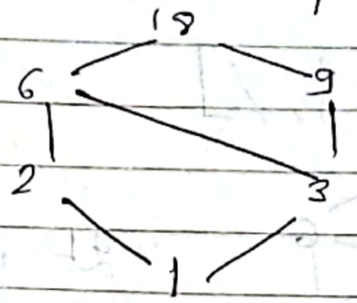


JSL ✓

MSL ✓

Lattice ✓

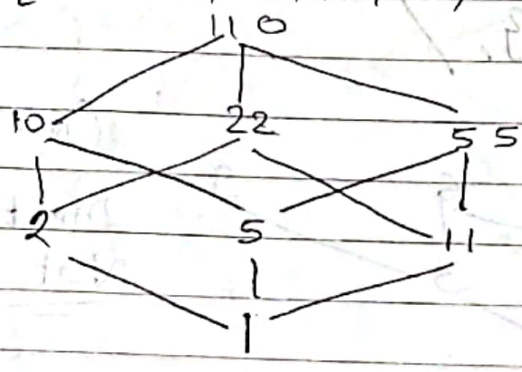
5) $\{1, 2, 3, 6, 9, 18\}, /$



MSL ✓
JSL ✓
$\therefore$ L ✓

6) $[D_{110}, /]$

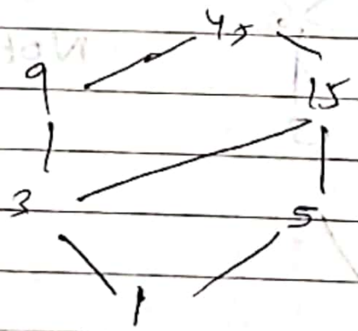
$[\{1, 2, 5, 10, 11, 22, 55, 110\}, /]$



MSL ✓
JSL ✓
$\therefore$ L ✓

7) $[D_{45}, /]$

$[\{1, 3, 5, 9, 15, 45\}, /]$



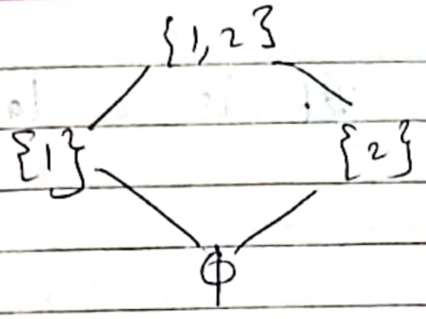
JSL ✓
MSL ✓
$\therefore$ L ✓

8) $A = \{1, 2\}$

$[P(A), \subseteq]$

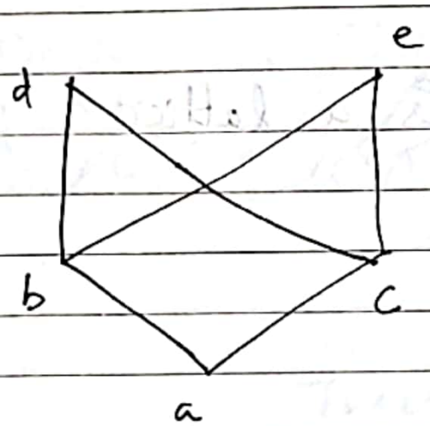
$P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Scanned with CamScanner

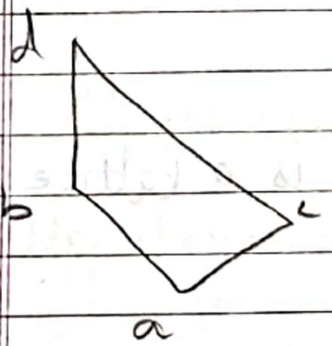


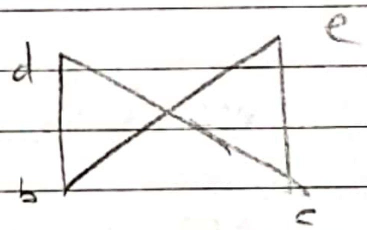
Lattice ✓

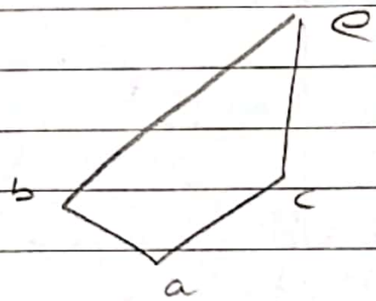
SUB - LATTICE

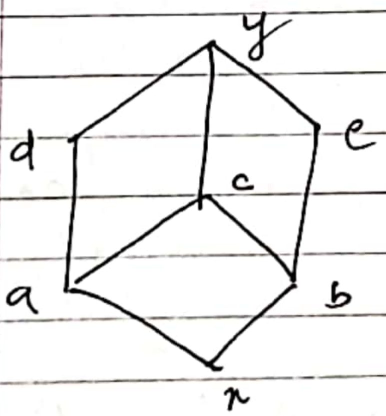


a) It is a Lattice $\boxed{X}$
b) Subset {a,b,c,d} is L ✓
c) Subset {b,c,d,e} is L $\boxed{X}$
d) Subset {a,b,c,e} is L ✓





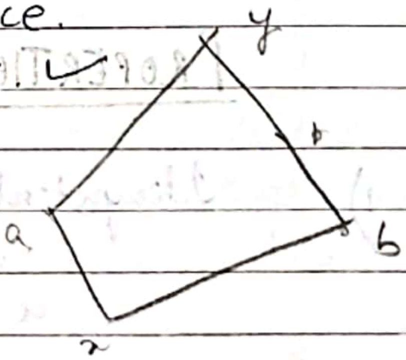




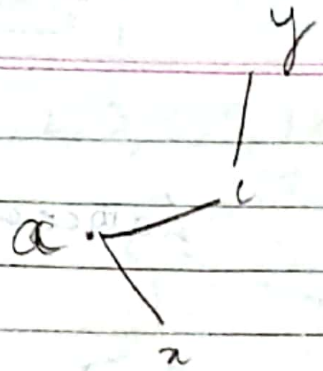
It is a Lattice.

$L_1 = \{x, a, y, b\}$ ✓

It is a Lattice

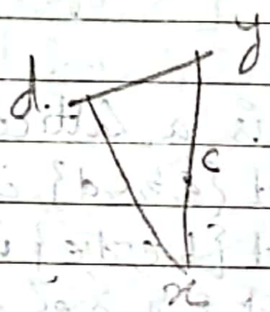


$L_2 = \{x, a, c, y\}$



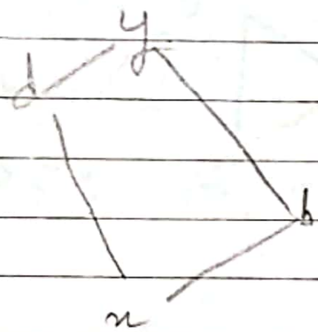
It is a lattice

$L_3 = \{x, c, d, y\}$



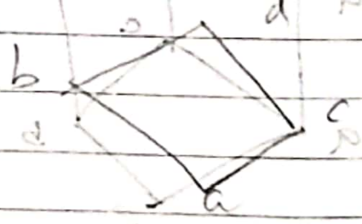
It is a lattice

$L_4 = \{x, b, d, y\}$



It is a lattice

PROPERTIES OF LATTICES



1) Idempotent law

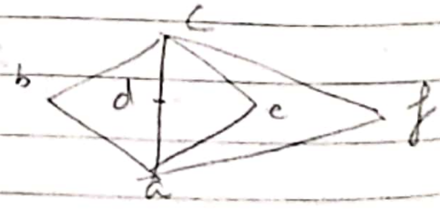
$a \vee a = a$

$a \wedge a = a$

2) Associative law

$(a \vee b) \vee c = a \vee (b \vee c)$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$



3) Commutative law

$a \vee b = b \vee a$

$a \wedge b = b \wedge a$

~~4) Distributive law~~ Does not hold in lattice

$\cancel{a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)}$

$\cancel{a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)}$

~~4) De morgan's Law:~~

$\cancel{(a \vee c)^c = a^c \wedge c^c}$

Due to this:

→ De morgan

→ Identity

→ Complement

→ Involution

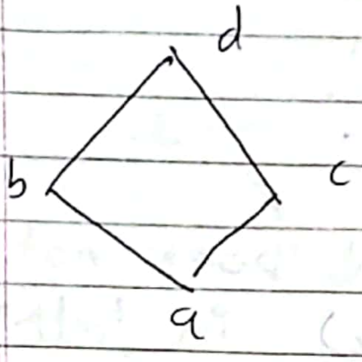
These laws fail to work

Upper Bound & Lower Bound in Lattice:

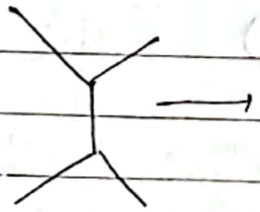
(I) Upperbound in Lattice (Maximum): In a lattice L if there exists an element I, such that $\forall a \in L$ $(a R I)$, then I is called upperbound of the lattice.

(0) LowerBound in Lattice (Minimum): In a lattice 'L' if there exists an element '0' such that $\forall a \in L$ $(0 R a)$, then 0 is called lower Bound of the lattice.

If a lattice contains both lower bound & upper bound then it is said to be bounded



Maximum $\rightarrow \{d\}$ $\}$
Minimum $\rightarrow \{a\}$



No U.B or L.B

→ A set can be infinite so a lattice can also be infinite.

→ For infinite lattices upperbound, lower bound don't exist.

⇒ Finite Lattices are bounded.

Prop:—

1) $a \vee I = I$
2) $a \wedge I = a$
3) $a \vee O = a$
4) $a \wedge O = O$
5) $a \vee a^c = I$
6) $a \wedge a^c = O$

COMPLEMENT OF ELEMENT IN LATTICE

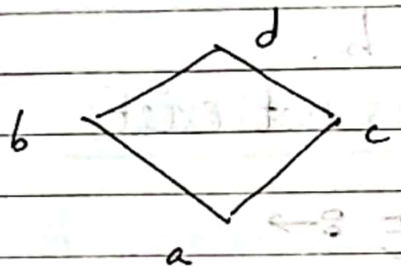
Complement of element in lattice :—
In a bounded lattice L, for any element $a \in L$, if there exist an element $b \in L$, such that $a \vee b = I$, $a \wedge b = O$, that b

Is called complement of a, we can say a & b are complements of each other.

$I^c = O$

$O^c = I$

1)



$a \vee d = d$

$a \wedge d = a$

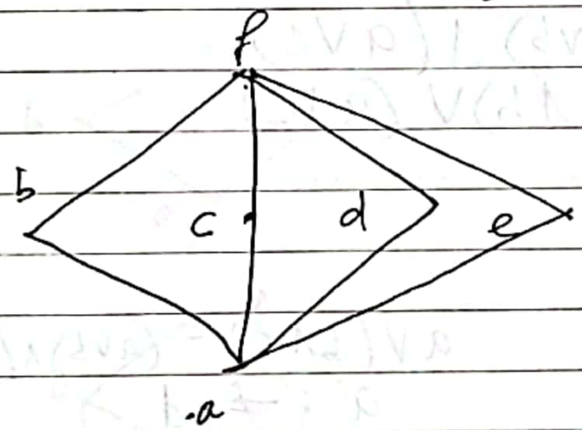
$\therefore a^c = d, \; d^c = a$

$b \vee c = d$

$b \wedge c = a$

$\therefore b^c = c, \; c^c = b$

2)



$a^c = f, \; f^c = a$

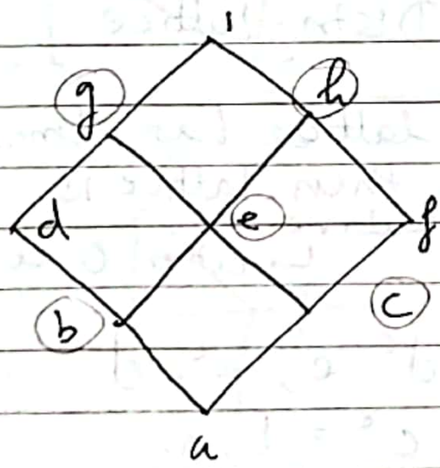
$b^c = c, d, e$

$c^c = b, d, e$

$d^c = b, c, e$

$e^c = b, c, d$

3)



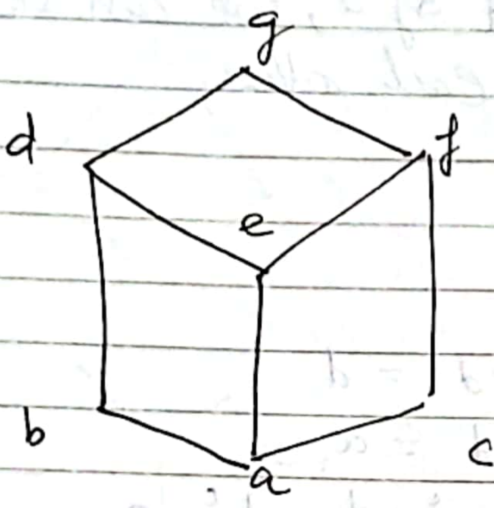
$d \vee f = i, \; d \wedge f = a$

$d^c = f, \; f^c = d$

$a^c = i, \; i^c = a$

$e, g, h, b, c \rightarrow$ Do not have complement

There can be elements which either have no complement or have more than 1 complement.



$a^c = g, \; g^c = a$

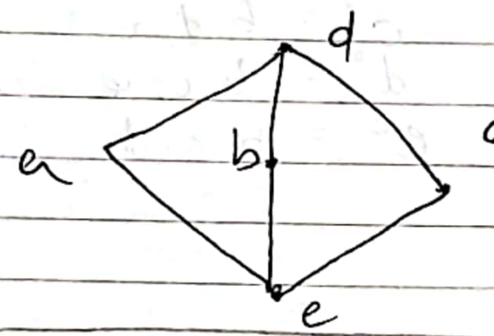
$b^c = c, f$
$c^c = b, d$
$d^c = c$
$f^c = b$
e^c does not exist

DISTRIBUTIVE LATTICE :→

A lattice is said to be distributive if $\forall\, a, b, c \in L$

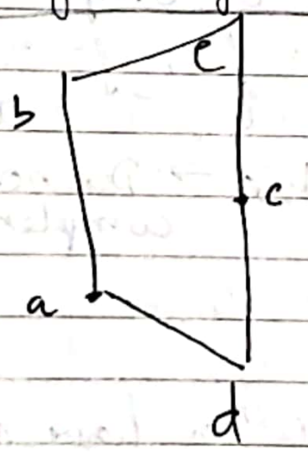
a) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
b) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$a^c = b, c$
$b^c = c, a$
$c^c = a, b$

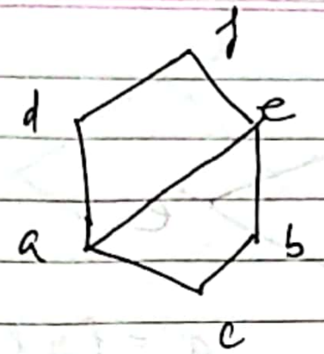


$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
$a \neq d$
Not Distri. Lattice

- If every element in lattice has atmost 1 element then lattice is distributive.
  ↳ either 0 or 1 comple



$d^c = e, \; e^c = d$
$c^c = b, a$
Not distributive

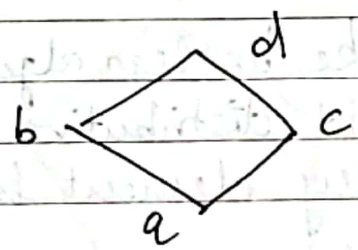


$c^c = f, \; f^c = c$
$d^c = b, \; b^c = d$
$a^c, e^c \rightarrow$ Not exist

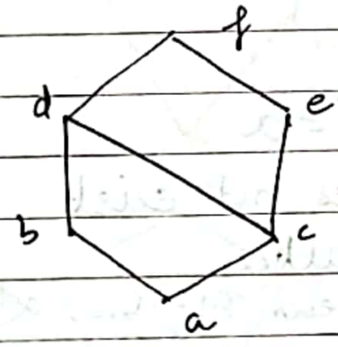
$\therefore$ Lattice is distributive

COMPLEMENTED LATTICE

A lattice L is said to be complemented if every element $\forall a \in L$ must have atleast 1 complement

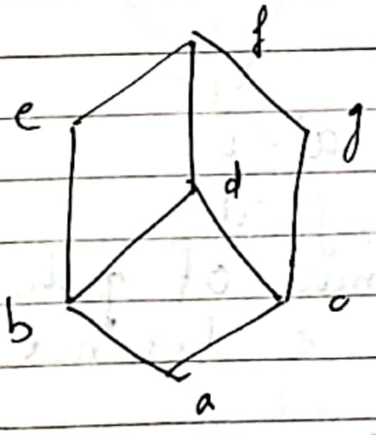


$d^c = a, \; a^c = d$
$b^c = c, \; c^c = b$
Complemented Latice ✓

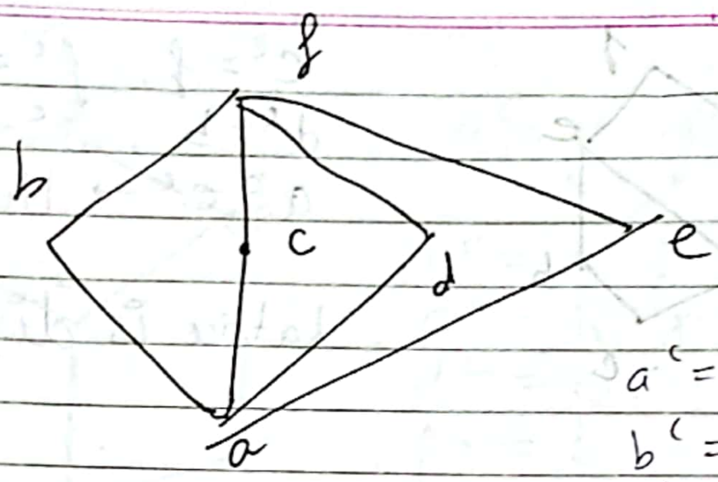


$d^c = ? \quad c^c = ?$

$\therefore$ Not Complemented Lattice ✗



$b^c = g$
$c^c = e$
$a^c = f, \; f^c = a$
$e^c = g, c$
$g^c = e, b$
$d^c \rightarrow$ Not exist
Not Complemented Lattice ✗

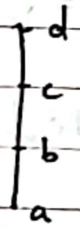


It is a Complemented lattice

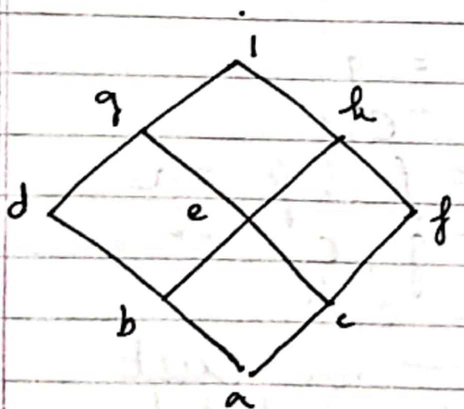
$a^c = f$, $f^c = a$
$b^c = c, d, e$
$c^c = b, d, e$
$d^c = b, c, e$
$e^c = b, c, d$

BOOLEAN ALGEBRA

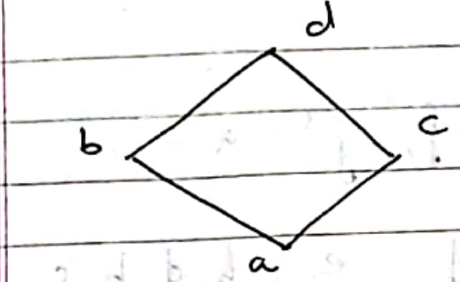
A lattice 'L' is said to be boolean algebra, if it is complemented and distributive. This is possible when every element has exactly one complement.



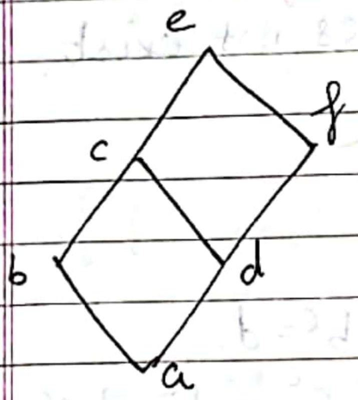
$a^c = d$, $d^c = a$
c^c and b^c does not exist
Distributive lattice
Not follows Boolean Algebra ✗



$i^c = a$, $a^c = i$
$d^c = f$, $f^c = d$
Complement of g, h, b, c, e does not exist.
B.A ✗

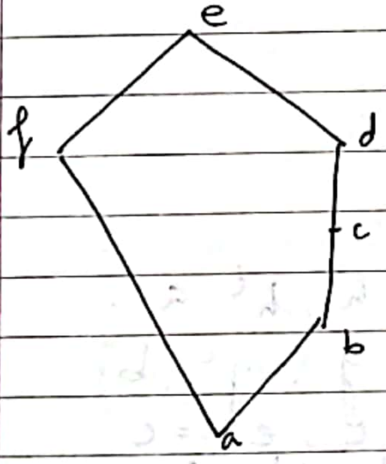


$a^c = d$, $d^c = a$
$b^c = c$, $c^c = b$
BA ✓

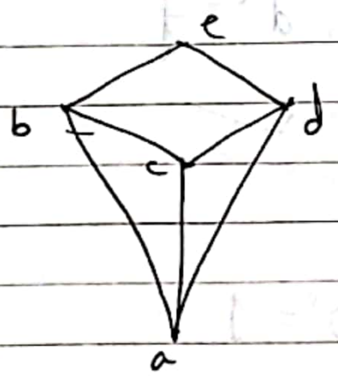


$a^c = e$, $e^c = a$
$b^c = f$, $f^c = b$
$c^c, d^c \to$ does not exist

BA ×

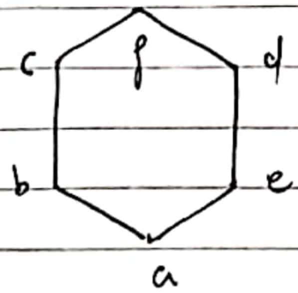


$a^c = e$, $e^c = a$
$f^c = d, c, b$
(Its Complemented Lattice)
BA ×

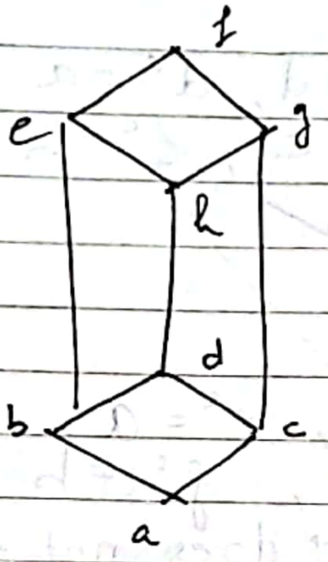


$a^c = e$, $e^c = a$

$b, c, d \to$ No complement
BA ×



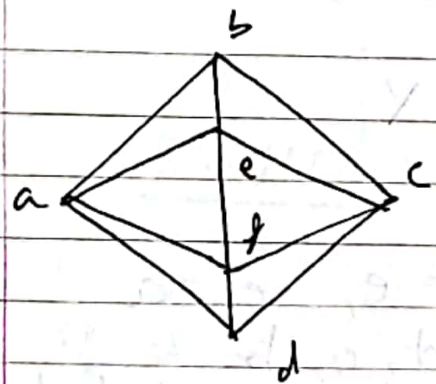
$f^c = a$, $a^c = f$
$b^c = d, e$; $c^c = d, e$
$e^c = b, c$ $d^c = c, b$
BA ×



$a^c = f$, $f^c = a$

Comp of e, g, h, d, b, c → Does not exist

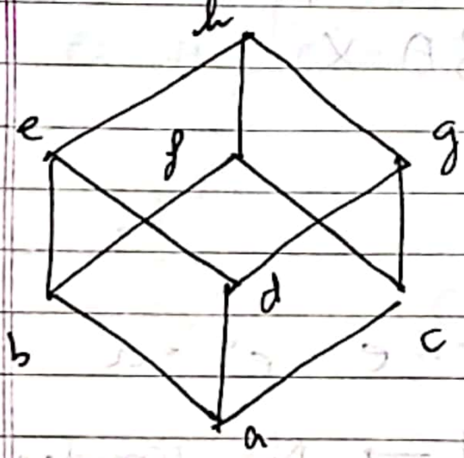
BA ✗



$d^c = b$, $b^c = d$

e^c, c^c, a^c, f^c → ✗

BA ✗

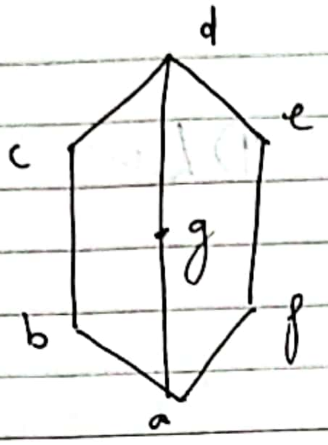


$a^c = h$, $h^c = a$
$b^c = g$, $g^c = b$
$c^c = e$, $e^c = c$
$f^c = d$, $d^c = f$

BA ✓

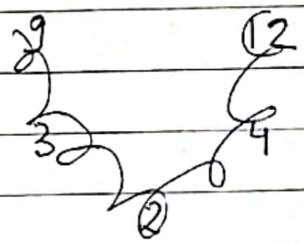
$1^c = 0$, $0^c = 1$

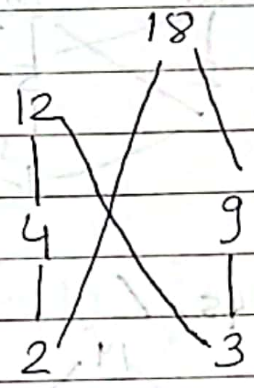
BA ✓



$a^c = d,\ d^c = a$
$b^c = g, f, e$
$g^c = c, e, b, f$
(CL)
BA ✗

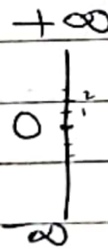
① $[\{2, 3, 4, 9, 12, 18\}, /]$





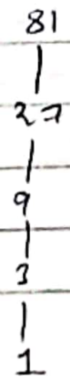
Not a Lattice

② $[R, \leq]$ (Real No.s)



It is an unbounded lattice
Concept of UBound & LowerB not applicable because its infinite
Its Distributive lattice
Not BA ✗

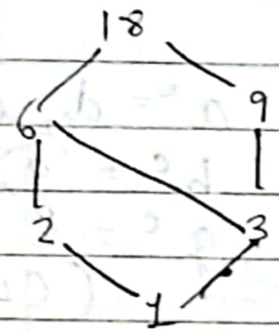
③ $D_{81}, /$



Distri Lattice ✓
BA ✗

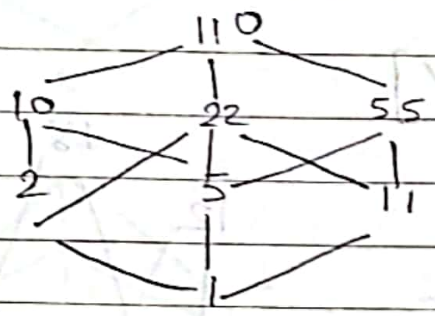
[Dn, /] ⟹ if n is perfect sq then for sure its BA not BA. Or if in the fact of n there is perfect sq. the BA ✗

(4) D_{18}, /



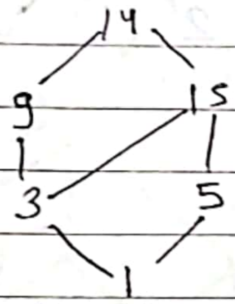
DL ✓

(5) D_{110}, /



Perfect BA ✓

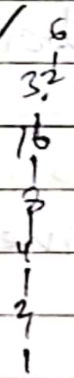
(6) D_{45}, /



Comp of 3 & 15 not exist

BA ✗

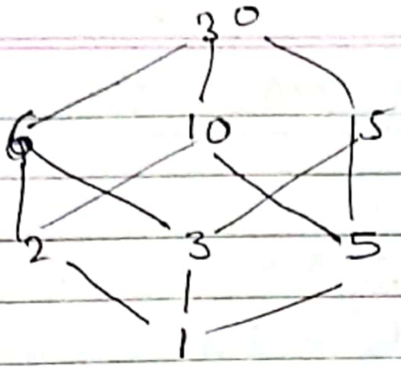
(7) D_{64}, /



D.L ✓

BA ✗

(8) D_{30}, /



BA ✓

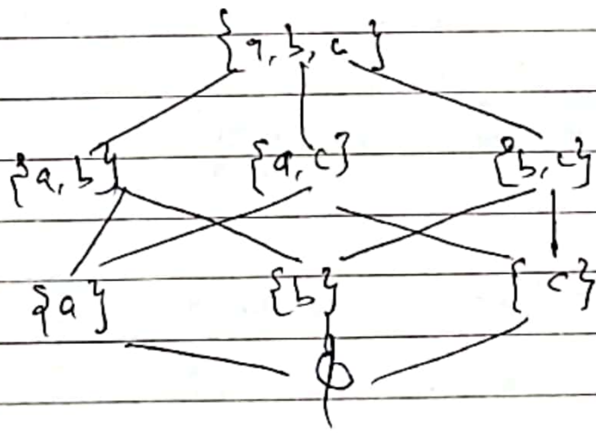
(9) $[D_{10}, /]$ $\underbrace{1, 2, 5, 10}$

NO perfect sq in the factors

BA ✓

(10) $[P(A), \subseteq]$ $A = \{a, b, c\}$

$P(A) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$



BA ✓

Here Complement of an element will be same as its comp in Set theory.

Eg. For $\{a\}$, Complement is $\{b, c\}$

For $\{b\} \longrightarrow \{a, c\}$

$\phi \longrightarrow \{a, b, c\}$

Meet is the Intersection of sets.