7/1/2/21

UNIT-3: ALGORITHMS, INDUCTION AND RECURSION

Mathematical Induction:-

→ Mathematical induction have two part, first part they show that the statement holds for +ve integer.

→ Second part. Assume that statement is true for large positive integer. then they show that the statement is true for next-large +ve integer.

→ Mathematic induction is key goal of learning Discrete Mathematics. A ~~cod~~ rule is given for finding subsequent values from the known values.

(2) Prove that $1^2+3^2+5^2+\dots+(2x+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

where n non negative integer

Sol: case (i) :- Basic step

for $n=0$

LHS $(2n+1)^2 = 1^2 = 1$

RHS $= \frac{(n+1)(2n+1)(3n+3)}{3}$

$= (1)(1)(1)/3 = 1$

for $n=0$

LHS = RHS

The given statement is true for $n=0$

i.e $P(0)$ is true

for $n=1$

LHS $1^2+3^2 = 10$ $\left(\because n=1 \atop (2n+1)^2 = 3^2 \right)$

RHS $\frac{(n+1)(2n+1)(2n+3)}{3} = \frac{(1+1)(2(1)+1)+(2(1)+3)}{3}$

$= 10$

for $n=1$ LHS = RHS

The given statement is true for $n=1$

i.e, $P(1)$ is true

case-(ii):- The statement is true for $n=k$

i.e, Assume $P(k)$ is true

$$\Rightarrow 1^2+2^2+3^2+\dots+(2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3} \quad \text{--- (1)}$$

Consider,

$$1^2+2^2+3^2+\dots(2k+1)^2+(2(k+1)+1)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$$

$$= (2k+3)\left[\frac{(k+1)(2k+1)}{3} + (2k+3)\right]$$

$$= (2k+3)\left[\frac{(k+1)(2k+1)+3(2k+3)}{3}\right]$$

$$= (2k+3)\left[\frac{2k^2+3k+1+6k+9}{3}\right]$$

$$= (2k+3)\left[\frac{2k^2+9k+10}{3}\right]$$

$= (2k+3)\left[\frac{(k+2)(2k+5)}{3}\right]$

$= \frac{((k+1)+1)[2(k+1)+1][2(k+1)+1]}{3}$

for $n=k+1$

L.H.S - RHS

The given statement is true for $n=k+1$

i.e $P(k+1)$ is true

Hence, By mathematical Induction. the given statement is true for every non-negative integer 'n'.

(Pb) Prove that 133 is divisor for the integer $11^{n+1}+12^{2n-1}$ where n is any integer.

Sol: case (i):- Basic step

for $n=1$

$$11^{n+1}+12^{2n-1} = 11^{1+1}+12^{2(1)-1}$$

$$= 11^2+12$$

$$= 121+12$$

$$= 133$$

133 divides $11^{n+1}+12^{2n-1}$ for $n=1$

P(1) is true

from $n=2$

$$11^{n+1}+12^{2n-1} = 11^{2+1}+12^{(2(2)-1)}$$

$$= 11^3+12^3$$

$$= 3059$$

$133 | 3059$ & $\frac{3059}{133} = 23$

133 is divisior of $11^{n+1}+12^{2n-1}$ for $n=2$

$P(2)$ is true

case (ii):- Inductive step

Assume that $P(k)$ is true

i.e 133 is divisor of $11^{k+1}+12^{2k-1}$

Consider

$$11^{(k+1)+1} + 12^{2(k+1)-1}$$

$$\Rightarrow 11^{(k+2)} + 12^{2k+2-1}$$

$$\Rightarrow 11^{k+1} \cdot 11 + 12^{2k+1} \cdot 12^2$$

$$\Rightarrow 11^{k+1} \cdot 11 + 12^{2k-1}(144)$$

$$\Rightarrow 11^{k+1} \cdot 11 + 12^{2k-1}(133+11)$$

$$\Rightarrow 11^{k+1} \cdot 11 + (11(12^{2k-1})) + (133 \cdot (12^{2k-1}))$$

$$\Rightarrow 11(11^{k+1}+12^{2k-1}) + 133 \cdot (12^{2k-1})$$

is divisible by 133 [$\because P(k)$ is true

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$\therefore P(k+1)$ is true

Hence the given statement is true for every +ve integer "n".

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(16) Prove that $1+1/4+1/9+ \cdots +1/n^2 < 2-1/n$ where n is a positive integer and $n \geq 2$.

Sol: Given that, n is a positive integer

case (i): $\Rightarrow$ for $n=2$

$\Rightarrow$ LHS

$\Rightarrow \frac{1}{1^2}+\frac{1}{2^2} = 1+\frac{1}{4} = 5/4 = 1.25$

$\Rightarrow$ RHS

$\Rightarrow 2-\frac{1}{2} = \frac{4-1}{2} = 3/2 = 1.5$

LHS < R.H.S

for n=2 the given statement is true

i.e, P(2) is true

for n=3

LHS: $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2} = \frac{1}{1}+\frac{1}{4}+\frac{1}{9} = \frac{49}{36} = 1.3611$

R.H.S: $2-\frac{1}{3} = \frac{6-1}{3} = 5/3 = 1.666$

L.H.S < R.H.S

For $n=3$ the given statement is true

i.e $P(3)$ is true

Case (ii): Assume that $P(k)$ is true

$$\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\frac{1}{k^2}<2-1/k \quad -①$$

Consider LHS $P(k+1)$

$$\Rightarrow 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\frac{1}{k^2}\times\frac{1}{(k+1)^2} \quad -②$$

$$\Rightarrow <2-\frac{1}{k}+\frac{1}{(k+1)^2} \quad (\because \text{from } ①)$$

$$\Rightarrow <2-\left[\frac{1}{k}-\frac{1}{(k+1)^2}\right]$$

$$<2-\left[\frac{(k+1)^2-k}{k(k+1)^2}\right]$$

$$<2-\left[\frac{(k^2+2k+1)-k}{k(k+1)^2}\right]$$

$$<2-\left[\frac{(k^2+k+1)}{k(k+1)^2}\right]$$

$$<2-\left[\frac{k^2}{k(k+1)^2}+\frac{(k+1)}{k(k+1)^2}\right]$$

$$< 2 - \left[\frac{k}{(k+1)^2} + \frac{1}{k(k+1)}\right]$$

$$< 2 - \left[\frac{k}{(k+1)^2} - \frac{1}{(k+1)^2} + \frac{1}{k(k+1)}\right]$$

$$< 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} - \frac{1}{k(k+1)}$$

$$< 2 - \frac{1}{k+1} + \frac{k-(k+1)}{k(k+1)^2}$$

$$< 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2}$$

$$< 2 - \frac{1}{k+1}$$

L.H.S < R.H.S

P(k+1) is true

Hence given statement is true for $n \geq 2$.

(P6) Prove that $1+2+3+ \cdots n = \frac{n(n+1)}{2}$ where n is a positive integer.

Sol: Given that

$$1+2+3+ \cdots n = \frac{n(n+1)}{2}$$

⇒ case(i):-

for $n=1$

L.H.S

$t = 1$

R·H·S

$$\frac{1(1+1)}{2} = 1$$

L·H·S = R·H·S for $n=1$ the given statement is true i.e $P(1)$ is true

for $n=2$

for

L·H·S

$1+2=3$

R·H·S

$$= \frac{\cancel{2}(2+1)}{\cancel{2}} = 3$$

L·H·S = R·H·S

for $n=2$ the given statement is true i.e $P(2)$ is true

Case (ii):- ~~P(k+~~ Assume that $P(k)$ is true

$$1+2+3+ \cdots k = \frac{k(k+1)}{2} \quad \text{--- } (1)$$

consider $P(k+1)$ is

$$\Rightarrow 1+2+3+ \cdots k+(k+1)$$

$$\frac{k(k+1)}{2} + k+1 \quad [\because \text{from } (1)]$$

$$\frac{k(k+1)+2(k+1)}{2}$$

$\Rightarrow \dfrac{k^2+k+2k+2}{2}$

$\Rightarrow \dfrac{k^2+3k+2}{2}$

$\Rightarrow \dfrac{k^2+2k+k+2}{2}$

$\Rightarrow \dfrac{k(k+1)+1(k+2)}{2}$

$\Rightarrow \dfrac{(k+2)(k+1)}{2}$

$= \dfrac{(k+1)((k+1)+1)}{2}$

$\Rightarrow$ R.H.S for $n=k+1$

$\therefore P(k+1)$ is true

By mathematical induction

$1+2+3+ \cdots n = \dfrac{n(n+1)}{2}$

for all $n \in Z^+$

(Pb) Prove that $1+2+2^2+2^3+ \cdots +2^n = 2^{n+1}-1$ for non-integer n.

Sol. Given that

$1+2+2^2+2^3+ \cdots +2^n = 2^{n+1}-1$

for $n=0$

L.H.S

$2^0=1$

R.H.S

$2^{0+1}=1$

$\Rightarrow 2^1-1$

$2-1=1$

L.H.S = R.H.S

for $n=0$ the given statement is true

i.e $P(0)$ is true

for $n=1$

$\Rightarrow$ L.H.S

$\Rightarrow 2^0+2^1$

$= 1+2 = 3$

R.H.S

$\Rightarrow 2^{1+1}-1 \Rightarrow 2^2-1 \Rightarrow 4-1=3$

L.H.S = R.H.S

for $n=1$ the given statement is true

i.e $P(1)$ is true

$\Rightarrow$ case (ii):-

Assume that $P(k)$ is true

$\Rightarrow 1+2+2^2+2^3+ \cdots 2^k = 2^{k+1}-1$ — (1)

$\Rightarrow$ consider

L.H.S = P(k+1)

$\Rightarrow 1+2+2^2+2^3+ \cdots 2^k + 2^{(k+1)}$

$\Rightarrow 2^{k+1}-1+2^{k+1}$ [$\because$ from ①]

$\Rightarrow 2 \cdot 2^{(k+1)}-1$

$\Rightarrow 2^{k+1+1}-1$

$\Rightarrow 2^{k+2}-1$

$\Rightarrow$ R.H.S

$\therefore$ P(k+1) is true

By mathemitcal induction

$1+2+2^2+2^3+ \cdots 2^n = 2^{n+1}-1$

for all non-negative integer

* Sum of Geometric progression

(Pb) the sum of finite number of terms of geometric progression $\sum_{j=0}^{n} ar^j$ $(= a + ar + ar^2 + ar^3 + \dots + ar^n) = \frac{a(r^{n+1}-1)}{r-1}$ where n is non-negative integer and also $r \neq 1$.

Sol: wh n is non - negative integer

$r \neq 1$.

given that

case (i):

for $n=0$

$L.H.S = 0$

$R.H.S \quad \frac{a(r^{0+1}-1)}{(r-1)} = \frac{a\cancel{(r-1)}}{\cancel{(r-1)}} = a$

$L.H.S = R.H.S$

statement is true for $n=0$

$\therefore$ ie $P(0)$ is true

for $n=1$

$L.H.S \quad a + ar$

$= a(1+r)$

R.H.S $\frac{a(r^{1+1}-1)}{r-1}$

$$= \frac{a(r^2-1^2)}{r-1} = \frac{a\cancel{(r-1)}(r+1)}{\cancel{r-1}} = a(r+1)$$

L.H.S = R.H.S

Statement is true for n=1

i.e, P(1) is true

case (ii):

~~introd~~ Assume that P(k) is true

$$\sum_{j=0}^{k} ar^j = \frac{a(r^{k+1}-1)}{r-1}$$

$$\Rightarrow a+ar+ar^2+ar^3+ \dots +ar^k = \frac{a(r^{k+r}-1)}{r-1} \quad \text{—} (1)$$

consider

L.H.S for n=k+1

$$a+ar+ar^2+ \dots + ar^k + ar^{k+1}$$

$$\Rightarrow \frac{a(r^{k+1}-1)}{r-1} + ar^{k+1} \quad [\because \text{from } (1)]$$

$$\Rightarrow a\left[\frac{r^{k+1}-1+r^{k+1}(r-1)}{(r-1)}\right]$$

$$\Rightarrow a\left[\frac{\cancel{r^{k+1}}+r^{k+2}-1-\cancel{r^{k+1}}}{r-1}\right]$$

$\Rightarrow \dfrac{a(r^{(k+1)+1}-1)}{r-1} \Rightarrow$ RHS

$\Rightarrow p(k+1)$ is true

By mathematical induction

$\sum_{j=0}^{4} ar^j = (a + ar + ar^2 \dots ar^n) = \dfrac{a(r^{n+1}-1)}{r-1}$

for non-negative integer $n, r \neq 1$.

(16) Prove that 21 divides $4^{n+1} + 5^{2n-1}$ where n is positive integer.

Sol: Given that

n is positive integer

for $n=1$

$4^{1+1} + 5^{2-1}$

$4^2 + 5^1$

$16 + 5 = 21$ 21 divides $4^{n+1} + 5^{2n-1}$ for $n=1$

$P(1)$ is true.

for $n=2$

$\Rightarrow 4^{2+1} + 5^{2(2)-1}$

$\Rightarrow 4^3 + 5^3$

$\Rightarrow 64 + 125 \Rightarrow 189$

$21/189$ & $= \frac{189}{21} = 9$

~~21 divides~~ 0^{0}

21 is divisior of $4^{n+1} + 5^{2n-1}$ for $n=2$

P(2) is true

case (ii):

Assume that P(k) is true

$\therefore 4^{k+1} + 5^{2k-1}$

consider

P(k+1)

$\Rightarrow 4^{(k+1)+1} + 5^{2(k+1)-1}$

$\Rightarrow 4^{k+2} + 5^{2k+1}$

$\Rightarrow 4^{k+1} \cdot 4 + 5^{2k-1} \cdot 5^{2}$

$\Rightarrow 4^{k+1} \cdot 4 + 5^{2k-1} (21+4)$

$\Rightarrow 4^{k+1} \cdot 4 + 4 \cdot 5^{2k-1} + 21 \cdot 5^{2k-1}$

$\Rightarrow 4(4^{k-1} + 5^{2k-1}) + 21 \cdot 5^{2k-1}$

$\Rightarrow$ is divisible by 21

$\therefore$ P(k+1) is true

the statement is true for n=k+1

by induction 21 divides $4^{n+1} + 5^{2n-1}$ for

$n \in Z^{+}$

**

(Pb) Prove that if $h > -1$, then $1+nh \leq (1+h)^n$ for all non-negative integer n

(or)

Prove that the Bernoullis inequivalent by use the mathematical induction

Sol: Given that

for $n=0$

L.H.S

$\Rightarrow 1+0 = 1$

$\Rightarrow$ R.H.S

$(1+1)^0 = 1$

L.H.S = R.H.S

$\Rightarrow$ for $n=0$ the given statement is true

$\Rightarrow$ $P(0)$ is true

for $n=1$

L.H.S

$\Rightarrow 1+h$

$\Rightarrow$ R.H.S

$= (1+h)^1$

$= (1+h)$

$\Rightarrow$ L.H.S $\leq$ R.H.S

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for $n=1$ the given statement is true

$P(1)$ is true

Inductive part:-

Assume $P(k)$ is true

$\Rightarrow 1+kh \leq (1+h)^k$

$(1+h)^k \geq (1+kh)$

Consider

R.H.S for $n=k+1$

$\Rightarrow (1+h)^k \cdot (1+h)$

$\Rightarrow (kh+1)(1+h)$

$\Rightarrow \geq 1+kh+h+kh^2$

$\geq 1+(k+1)h+kh^2$

$\geq 1+(k+1)h$

L.H.S $P(k+1)$ is true

By the mathematical induction

i.e $(P(k+1))$ is true

$\therefore$ For a non-negative n

$1+nh \leq (1+h)^n$ is true

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H.W

* ① $1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

H.W

* ② $1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

Sol: Given that

$$1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\Rightarrow$ for $n=0$

$L.H.S = n^2 = 0$

$R.H.S = \frac{0(0+1)(0+1)}{6} = 0/6 = 0$

$L.H.S = R.H.S$

for $(n=0)$ the given statement is true

$P(0)$ is true

$\Rightarrow$ for $n=1$

L.H.S

$n = 0^2+1^2 = 1+0 = 1$

R.H.S

$\frac{1(1+1)(2+1)}{6} = \frac{1\cancel{(2)(3)}}{\cancel{6}}$

$\Rightarrow$ L.H.S = R.H.S

for $(n=1)$ the given statement is true

$P(1)$ is true

Case ii:-

$\Rightarrow$ Assume that $P(k)$ is true

$$\Rightarrow 1^2+2^2+3^2+\cdots(k)^2 = \frac{k(k+1)(2k+1)}{6}$$

Consider that

L.H.S $P(k+1)$ is

$$\Rightarrow 1^2+2^2+3^2+\cdots(k)^2+(k+1)^2$$

$$\Rightarrow \frac{k(k+1)(2k+1)}{6}+(k+1)^2$$

$$\Rightarrow \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$\Rightarrow \frac{(k^2+k)(2k+1)+6(k+1)^2}{6}$$

$$\Rightarrow \frac{2k^3+k^2+2k^2+k+6[k^2+2k+1]}{6}$$

$$\Rightarrow \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$$

$$\Rightarrow \frac{2k^3+9k^2+13k+6}{6}$$

$$\Rightarrow (k+1)\frac{(2k^2+7k+6)}{6}$$

$$\Rightarrow \frac{(k+1)(k+2)+(2k+3)}{6}$$

$$\Rightarrow \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

R.H.S $\Rightarrow P(k+1)$ is true

For By the mathematical induction

$$1^2+2^2+3^2+\cdots n^2 = \frac{n(n+1)(2n+1)}{6} \text{ is true}$$

for all $\mathbb{Z}$

Sol: $1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

→ Given that

$$1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

⇒ for $n=0$

$L.H.S = 0$

$R.H.S = \frac{0(0+1)}{4} = 0$

$L.H.S = R.H.S$

for $n=0$ the given statement is true

$P(0)$ is true

⇒ $1+8 = 9 \;\Big|\; \frac{4(3)^2}{4} = 9$

for $n=1$

⇒ L.H.S

$= n^2 = 1$

⇒ RHS

$⇒ \frac{1(1+1)^2}{4} = \frac{1(2)^2}{4} = 4/4 = 1$

for n=1 the given statement is true

p(1) is true

Case (ii):

Assume that p(k) is true

$$1^3+2^3+3^3+4^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}$$

consider that

LHS is p(k+1)

$$\Rightarrow 1^3+2^3+3^3+4^3+\cdots+k^3+(k+1)^3$$

$$\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$\Rightarrow \frac{k^2(k+1)^2+4(k+1)^2}{4}$$

$$\Rightarrow \frac{(k+1)^2[k^2+4(k+1)]}{4}$$

$$\Rightarrow \frac{(k+1)^2[k^2+4k+4]}{4}$$

$$\Rightarrow \frac{(k+1)^2((k+2)^2)}{4}$$

$$\Rightarrow \frac{(k+1)^2((k+1)+1))^2}{4}$$

$= R.H.S$

p(k+1) is true

By the mathematical induction

$1^3+2^3+3^3+4^3+ \cdots n^3 = \frac{n^2(n+1)^2}{4}$ is true

for all $n \in \mathbb{Z}$.

17/12/21 Demorgan Law for union and intersection

$(\overline{A \cup B}) = \bar{A} \cap \bar{B}$

$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

(Pb) To proof that the generalization of the Demorgn law $\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \bar{A_j}$ where $a_1, a_2, a_3 \cdots$ is a subset of universal set [U],

$n \geq 2$ $\quad n \geq 2$ & $n \in \mathbb{Z}^+$

Sol: Basic step:- for $n=2$

$L.H.S = \overline{\bigcap_{j=1}^{2} A_j} = \overline{(A_1 \cap A_2)} = \bar{A_1} \cup \bar{A_2}$

(∵ By Demorgan law)

$R.H.S = \bigcup_{j=1}^{2} \bar{A_j} = \bar{A_1} \cup \bar{A_2}$

$L.H.S = R.H.S$

$P(2)$ is true.

for $n=3$

L.H.S. $= \overline{\bigcap_{j=1}^{3} A_j} = [\overline{(A_1 \cap A_2) \cap A_3}] = \overline{(A_1 \cap A_2)} \cup \bar{A}_3$

$= (\bar{A}_1 \cup \bar{A}_2) \cup \bar{A}_3 = (\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3)$

R.H.S $= \bigcup_{j=1}^{3} \bar{A}_j = \bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3$

L.H.S = R.H.S

P(3) is true

Inductive step:-

Assume P(k) is true

i.e, $\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \bar{A}_j \rightarrow ①$

Consider L.H.S for $n=k+1$

$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$

$= \overline{\bigcap_{j=1}^{k} A_j} \cup \bar{A}_{k+1}$ ($\because$ By Demorgan Law)

$= \bigcup_{j=1}^{k} \bar{A}_j \cup \bar{A}_{k+1}$ ($\because$ from ①)

$= \bigcup_{j=1}^{k+1} \bar{A}_j$

= R.H.S for $n=k+1$

LHS = RHS for $n=k+1$

P(k+1) is true

By Induction Demogan law proved

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j} \quad \text{for } n \geq 2, n \in Z^+$$

H.W

Pb) To proof the generalization of the Demorgan law $\overline{\bigcup_{j=1}^{n} A_j} = \bigcap_{j=1}^{n} \overline{A_j}$ where $a_1, a_2, a_3 \ldots$ is a subset of universal set $[\mu]$, $n \geq 2$.

ol:

(Pb) Prove that, If $A_1, A_2, A_3 \ldots A_n$ and B are Subsets of μ [Universal set]. Then

$(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots \cup (A_n \cap B)$

Sol. for $n \in Z^+$

Basic part:-

for $n = 1$

L.H.S $(A_1) \cap B = A_1 \cap B$

R.H.S $(A_1 \cap B)$

L.H.S = R.H.S

$p(1)$ is true

for $n=2$

L.H.S $(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$

$$\boxed{\begin{aligned}(A \cup B) \cap C &= (A \cap B) \cup (B \cap C) \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C)\end{aligned}}$$

R.H.S $= (A_1 \cap B) \cup (A_2 \cap B)$

L.H.S = R.H.S

$P(2)$ is true

Inductive part:-

Assume $P(k)$ is true

i.e, $(A_1 \cup A_2 \cup \cdots \cup A_k) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cdots \cup (A_k \cap B)$ — (1)

Consider L.H.S for $n = k+1$

$$[(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}] \cap B$$

$$= [(A_1 \cup A_2 \cup \cdots \cup A_k) \cap B] \cup [A_{k+1} \cap B]$$

$$= (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_k \cap B) \cup [A_{k+1} \cap B]$$

($\because$ from (1))

= R.H.S for $n = k+1$

$P(k+1)$ is True

By Induction

$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A \cap B)$ for every $n \in Z^+$ is true

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17/12/21

STRONG INDUCTION AND WELL ORDERING

INDUCTION:

(Pb) Prove that every amount of postage 5c or more can be founded by using 3c and 4c stamps

[Hint: Let $P(n)$ = postage can be made using 3c and 4c stamps]

Sol: NOTE:-

* strong induction:

To prove that $P(n)$ is true for all $n \in Z^+$ $\forall n \in Z^+$

Basic part; verify that $P(1)$ and $P(2)$ are true

Inductive step: Prove that if $[P(1) \wedge P(2) \wedge \cdots P(k)]$ is true then $P(k+1)$ is true

(or)

$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all $n \in Z^+$

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5C

Sol: let $p(n)$ = postage can be made by using 3c and 4c stamps

Basic part:- $P(6)$ is true because $6c = 3c + 3c$

Similarly $7c = 3c + 4c \Rightarrow P(7)$ is true

Similarly $8c = 4c + 4c \Rightarrow P(8)$ is true

Inductive step:- Assume that $P(j)$ is true for all j ranges $6 \leq j \leq k$

→ To prove $P(k+1)$ is true by adding the stamps 3c or 4c

→ we need to consider $P(k-2)$ is true i.e $(k-2)c$ can be made by using the postal stamps 3c and 4c.

and also

$$(k-2)c + 3c = (k+1)c$$

$\therefore (k+1)c$ can be made by the summassion of postage stamps of 3c and 4c

i.e $P(k+1)$ is true,

→ By strong induction the given statement is true for $n \in Z^+$ and $n \geq 6$.

(Pb) Suppose every amount of postage 18c are more can be found using 4c and 7c stamps.

Sol: let $P(n)$ = postage can be made by using 4c and 7c stamps.

Basic part:- $P(18)$ is true because

$$18c = 7c + 7c + 4c$$

$19c = 7c + 4c + 4c + 4c$ $\Rightarrow P(19)$ is true

$20c = 4c + 4c + 4c + 4c + 4c$ $\Rightarrow P(20)$ is true

$21c = 7c + 7c + 7c$ $\Rightarrow P(21)$ is true

Inductive part:- Assume that $P(j)$ is true for values of j ranges from $18 \le j \le k$. To prove $P(k+1)$ is true by adding stamps 4c and 7c we need to consider $P(k-3)$ is true

i.e

$(k-3)c$ can be made using postal stamps - 4c and 7c

and also

$$(k-3)c + 4c = (k+1)c$$

$(k+1)c$ can be made by using the

summation of postage stamps 4c and 7c
i.e $P(k+1)$ is true

By strong induction the given statement is true for $n \in Z^+$ $n \geq 18$.

Example: we can reach the first rung of the ladder for every integer k. if we can reach the first k rung then we can reach the $(k+1)^{th}$ rung we can show this example by using strong induction principle that for +ve integer n then $P(n)$ holds the strong induction principle can also called as second Mathematical induction.

(Pb) Suppose every amount of postage 12 c or more can be formed using 4-cent and 5-cent stamps

Sol: let $P(n)$ = postage can be made using 4c and 5c stamps

Basic step:- $P(12)$ is true

because
$12c = 4c + 4c + 4c$
$13c = 4c + 4c + 5c \Rightarrow P(13)$ is true
$14c = 4c + 5c + 5c \Rightarrow P(14)$ is true
$15c = 5c + 5c + 5c \Rightarrow P(15)$ is true

Inductive step:- Assume that $P(j)$ is true for values of j ranges from $12 \leq j \leq k$. To prove $P(k+1)$ is true by adding stamps 4c and 5c we need to consider $P(k-3)$ is true

i.e,
$(k-3)c$ can be made using postal stamps 4c and 5c

and also,
$(k-3)c + 4c = (k+1).c$
$(k+1)c$ can be made by stummation

20/12/2 (Pb)

of postage stamps 4c and 5c

i.e P(k+1) is true

By strong ~~in duction~~ induction, the given statement is true for $n \in Z^+$.

(b) Prove that the Fundamental Theorem of arithmetic by using mathematical induction.

(or)

Proof:- Show that the ~~n is~~ if the n is greater than 1 then n is product of prime numbers.

Proof: (~~Basi~~) n is an integer and $n > 1$.

Basic part:

for $n = 2$

$2 = 1 \times 2$ (product of prime numbers

$\therefore P(2)$ is true

for $n = 3$

$3 = 1 \times 3$ (product of prime numbers

$\therefore P(3)$ is true

for $n = 4$

$4 = 2 \times 2$ (product of prime numbers

$\therefore P(4)$ is true

Inductive part: Assume that $P(j)$ is true for j ranges $2 \leq j \leq k$

Consider a number $n = k+1$

case-1:

→ If $(k+1)$ is a prime number than $(k+1)$ can be expressed as $1 \times (k+1)$ is product of primes

case-2:

→ If $(k+1)$ is a composite number, then $(k+1)$ can be expressed product of its factors then ($\exists\, a, b$, $2 \leq a < b < k$ ∋ $(k+1) = a \times b$)

$\exists\, a, b,\ 2 \leq a < b < k$ ∋ $(k+1) = a \times b$

→ a, b are in between (~~betwar~~ [illegible]) our assumption range that is a, b are satisfying fundamental theorem of arthematic so a can be as product of prime number & so can b

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then product of a and b is also product of prime numbers

Therefore P(k+1) is true

thus, Mathematical Induction $n \geq 1$ then is product of prime numbers. so the fundamental theorem of arithmetic is True.

(Pb) use the Bell ordering property to prove the division algorithm,

"which states that if a is an integer and d is a positive integer, then the unique integers q and r with $0 \leq r < d$ such that $a = dq + r$"

Soll- Note:- "well ordering property":-

Every non-empty of non-negative integer has a least element

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Let s be the set of non-negative integers of the form $a-dq$, where q is an integer. This set is non-empty because $-dq$ can be made as large as desired.

By the well ordered property, every non-empty, non-negative integer set has a least element i.e, s has a least element

Let $r = a - dq_0$ be the least element of set s

there r is non-negative integer

In this case $r<d$ is true

If not $r \geq d$.

then these would be an smaller non-negative element in s. the element namely

$a-d(q_0+1)$ $[=a-dq_0-d]$

which contridicts our assumption $a-dq_0$ is a least element

$\because (a-dq_0-d < a-dq_0)$

$\therefore r < d$

hence $0 \leq r < d$

uniqueness of q & r:-

Assume that

$a = dq_1 + r$ and $a = dq_2 + r$

$\Rightarrow dq_1 + r = dq_2 + r$

$dq_1 - dq_2 = r - r$

$d(q_1 - q_2) = 0$

$q_1 - q_2 = 0 \Rightarrow q_1 = q_2$

$\therefore q$ is unique

(b) Suppose we can reach the first and second runges of an infinite ladder, and we know that if we can reach a rung then, we can reach two rungs higher. can we prove that we can reach every rung using the principle of mathematical induction? can we prove that we can reach every rung using the principle of strong induction.

Sol: The proof by mathematical induction

i, Basic part:- The basic step of this proof holds, it is simply verifies that we can reach the first rung

ii, Inductive ~~part~~ step:: The inductive hypothises is the statement that we can reach kth step (rung) of the ladder.

we need to show that if we assume the inductive hypothesis for a positive integer k then we can reach the (k+1)th rung of the ladder.

However there is no way to complete this inductive step because we dont know from the given information that we can reach the (k+1)th rung from the kth rung.

After all only we know that if we can reach a rung we can reach the rung of two higher.

Proof by strong induction

i) Basic step:- The basic step of this proof holds, it simply verifies that we can reach the first rung

ii) Inductive step:- Assume that $[P(1) \wedge P(2) \wedge P(3) \cdots \wedge P(k)]$ is true. i.e, we reached k rungs in the ladder step by step up to kth rung. then by the inductive hypothesis we can reach $(k+1)^{th}$ rung of the ladder because already we know that we can reach the second rung if we complete the inductive hypothesis step by step. we can reach $(k+1)^{th}$ rung from the $(k-1)th$ rung. By strong induction if we reach the first two rungs of the infinite ladder and for every integer we can reach the $(k+1)^{th}$ rung

∴ we can reach all the rungs of the ladder.

27/12/21

(pb)

Recursive Induction:

* A sequence recursively by specifying how talls of the sequence are found from previous terms, we can use induaction to prove result about the sequence.

* To prove results about recursively defines structure at we use a method called induction. In this Induction every step is depends upon the previous step.

Basic step:- Specifiy the value of the function at zero (at initial stages).

Inductive step:- Give a rule for finding its value at an integer from its values and at smaller integer. It is called recursive (or) Inductive step,

Sol:

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(17) Suppose that f is defined recursively by $f(0)=3$, $f(n+1)=2f(n)+3$. Then find $f(1)$, $f(2)$, $f(3)$ and $f(4)$.

Sol: Given that $f(0)=3$

Given recursive Relation $f(n+1)=2f(n)+3$

$$f(1)=f(0+1)=2f(0)+3$$
$$=2(3)+3$$
$$f(1)=9$$

$$f(2)=f(1+1)=2f(1)+3$$
$$=2(9)+3$$
$$=18+3$$
$$f(2)=21$$

$$f(3)=f(2+1)=2f(2)+3$$
$$=2(21)+3$$
$$=42+3$$
$$f(3)=45$$

$$f(4)=f(3+1)=2f(3)+3$$
$$=2(45)+3$$
$$=90+3$$
$$f(4)=93$$

(Pb) Give an inductive definition ~~after~~ of the factorial function $f(n) = n$ factorial ($n!$)

Sol: Basic step:-

for $n=0$

$F(0) = 0!$

$F(0) = 1$

for $n=1$

$F(1) = 1! = 1 \times 0!$

$= 1 \times F(0)$

for $n=2$

$F(2) = 2! = 2 \times 1!$

$= 2 \times F(1)$

Inductive step:- Assume that $F(k+1) = (k+1)$ $F(k)$

(or)

$F(k) = k\,F(k-1)$

By using above Basic values

Suppose

$F(5) = 5 \cdot F(5-1)$

$= 5 \cdot F(4)$

$= 5 \cdot 4 \cdot F(4-1)$

$= 5 \cdot 4 \cdot F(3)$

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$= 5 \cdot 4 \cdot 3F(3-1)$
$= 5 \cdot 4 \cdot 3F(2)$
$= 5 \cdot 4 \cdot 3 \cdot 2F(2-1)$
$= 5 \cdot 4 \cdot 3 \cdot 2F(1)$
$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1F(0)$
$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \quad (\because F(0) = 1)$
$= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$F(5) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$F(5) = 5!$
$\therefore F(k) = kF(k-1)$

Hence $F(n) = nF(n-1)$ is the Definition (Recursive relation) of factorial notation.

NOTE:- The fibonacci numbers ~~f(a)~~ f_0, f_1, f_2 and soon are defined by the equation

$f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$

where $f_0 = 0$ and $f_1 = 1$.

(pb) Find fibonacci Numbers f_2, f_3, f_4, f_5 and f_6.

Sol. By the defintion of fibonacci series we have $f_0 = 0$, $f_1 = 1$ and also

$f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$

~~No~~ $f_2 = f_{2-1} + f_{2-2}$

$\Rightarrow f_2 = f_1 + f_0 = 1 + 0 = 1$

Now

$f_3 = f_{3-1} + f_{3-2}$

$= f_2 + f_1$

$f_3 = 1 + 1 = 2$

$f_4 = f_{4-1} + f_{4-2}$

$= f_3 + f_2$

$= 2 + 1 = 3$.

$f_5 = f_{5-1} + f_{5-2}$

$= f_4 + f_3$

$= 3 + 2 = 5$

$f_6 = f_{6-1} + f_{6-2}$

$= f_5 + f_4$

$= 5 + 3 = 8$

1-1-22

Recursive Algorithm:

An algorithm that proceeds by reducing a problem to the same problem with smaller input.

Pb) Give a recursive algorithm for computing n! where n is ~~a~~ non-negative integer

Sol: ~~Basic step~~:- Given that n is non-negative integer and define $0! = 1$.

Basic step:- If $n=0$ (It is already defined)
$0! = 1$

If $n=1$
$\Rightarrow n! = 1!$

If n is a positive integer. n! is defined as
$n.(n-1)!$

$1! = 1.0!$
$1! = 1\times 1 = 1$

Inductive step:- The recursive Algorithm for a positive integer n to compute n! is defined in above step.

$$n! = n.(n-1)!$$

If n! is non-negative integer, for n=0 then n! (i.e 0!) is undefined according to the ~~recursive~~ given recursive algorithm.

Therefore, for all +ve integer n

$n! = n.(n-1)!$ and $0! = 1$ is the required recursive algorithm for the given data.

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Recurssicive algorithm for computi

n! :-

Procedure Factorial (n: non negative integer)

If n=0 then Factorial(n) := 1

else factorial (n) := n . Factorial(n-1)

(6) Given recursive algorithm for computing G.C.D of two non-negative integers a and b with $a<b$

Sol Basic step:- This G.C.D ~~of~~ $\cancel{(a,b)}$ $(0,b) = b$

Recursive step or Inductive step:-

Define the $g.c.d(a,b) = g.c.d(b \bmod a, a)$

for all ~~non negative~~ Positive integers a, b and $a<b$

and $g(0,b) = b$

for $a=5$, $b=8$, since $5<8$

$$\begin{aligned}
g.c.d(5,8) &= g.c.d(8 \bmod 5, 5) \\
&= g.c.d(3,5) \\
&= g.c.d(5 \bmod 3, 3) \\
&= g.c.d(2,3) \\
&= g.c.d(3 \bmod 2, 2) \\
&= g.c.d(1,2) \\
&= g.c.d(2 \bmod 1, 1) \\
&= g.c.d(1,1) \\
&= g.c.d(1 \bmod 1, 1) \\
&= g.c.d(0,1)
\end{aligned}$$

$g.c.d(5,8) = 1$

Recursive algorithm for computing $g.c.d(a,b)$:-

Procedure gcd ~~g.c.d~~ (a, b: non negative integers with $a<b$)

If $a=0$ then $gcd(a,b) := b$

else $gcd(a,b) := gcd(b \bmod a, a)$

(Pb) Given recursive algorithm for computing a^n where a is non-zero real number and n is a non-negative integer.

Sol: Given that a is non-zero real number n is non-negative integer and previosuly defined $a^0 = 1$

Basic step:- If $n=0$
$a^0 = 1$

If $n=1$
$\Rightarrow a^1 = a^1 . a^0$
$= a . 1$

If $n=2$
$\Rightarrow a^2 = a^1 . a^1$
$= a . a$

Recursive step (or) Recursive Algorithm (or)
Inductive Step:- Recursive Algorithm to compute a^n is defined as follows

$a^n = a . a^{n-1}$

(or)

$a^{n+1} = a . a^n$ (By using the basic step)

The above recursive algorithm only for $n>0$. for $n=0$ $a^0 = 1$ already defined

Recurssicive algorithm for computing a^n:-

Procedure power (a: nonzero real number, n: nonnegative integer)

If $n=0$ then power $(a,n):=1$

else power $(a,n):=a \cdot$ power $(a,n-1)$

* Recurssicive algorithm for Fibonacci numbers:-

Procedure fibonacci (n: nonnegative integer)

If $n=0$ then fibonacci $(n):=0$

else if $n=1$ then fibonacci $(n):=1$

else fibonacci $(n):=$ fibonacci $(n-1)$ + fibonacci $(n-2)$.