3.2 3D OBJECT REPRESENTATIONS

3D objects are represented by the following methods,

1. Polygon surfaces } for simple Euclidean objects.
2. Quadric surfaces }
3. Spline surfaces - For curved surfaces
4. Procedural methods - Ex : Fractals, particle systems.
5. Physical based modeling methods.
6. Octree encoding
7. Volume rendering, etc.

3D object representations classified into 2 types. They are,

1. Boundary representations

- ✓ It is also called (B - reps)
- ✓ EX: Polygon facets spline patches

2. Space partitioning representations

- ✓ Ex: Octree representation

✓ Object may also associate with other properties such as mass, volume, so as to determine their response to stress and temperature, etc.

Table 3.1

Boundary representations	Space partition representations
✓ It describes a 3D object as a set of surfaces that separates the object interior from the environment.	✓ It is used to describe interior properties, by partitioning the spatial region containing an object into a set of small, non – overlapping contiguous solids (usually cubes).
Ex: Polygons	Ex: Octree.

3.2.1 Polygon Surfaces

- ✓ The most commonly used boundary representation for a 3D object is a set of polygon surface that enclose the object interior.
- ✓ This method simplifies & speeds up the surface rendering and display of objects.
- ✓ The polygon surface are common in design and solid modeling applications, since their wireframe display can be done quickly to give general indication of surface structure.
- ✓ Then realistic scenes are produced by interpolating shading patterns across polygons surface to illuminate.

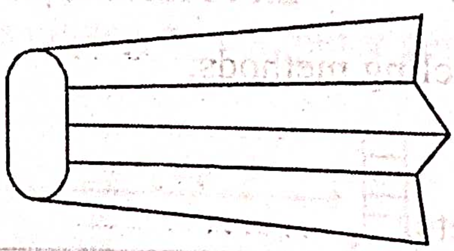


Fig.3.7 A 3D object representation by polygons

P_0 P_0 P_3

Fig.3.15 Curves

3.4 QUADRIC SURFACES

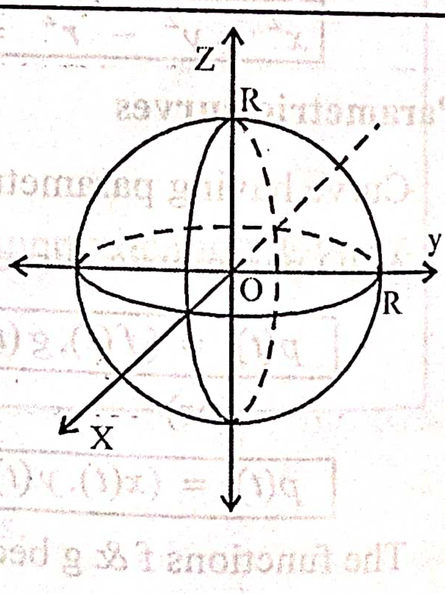
✓ Spheres, cylinders, and cones are part of the family of surfaces called **"Quadric Surface"**.

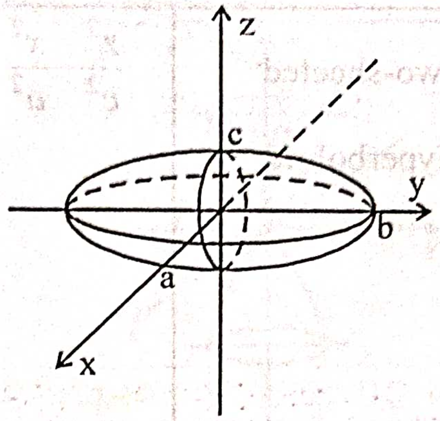
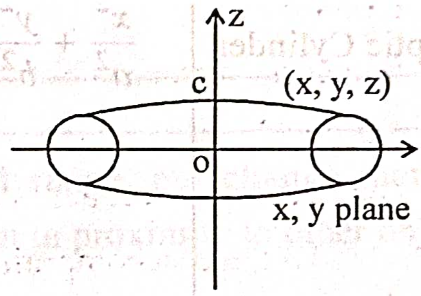
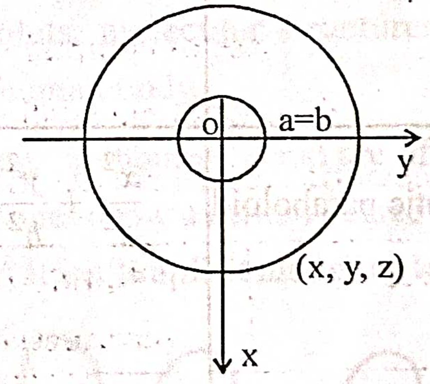
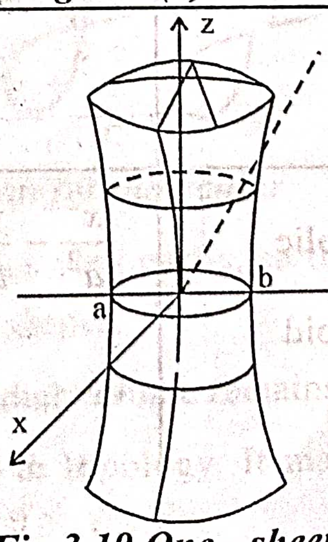
✓ A quadric surface is defined by an equation which is of the second degree (in x, y, or z).

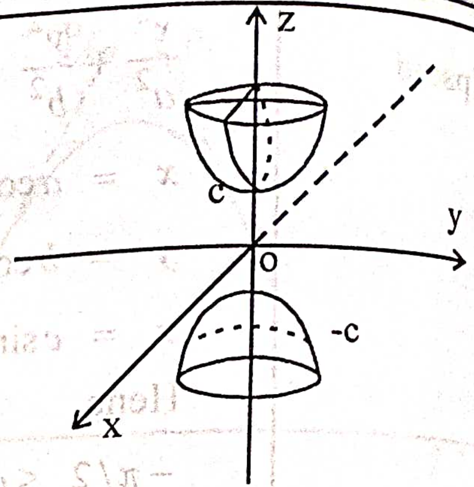
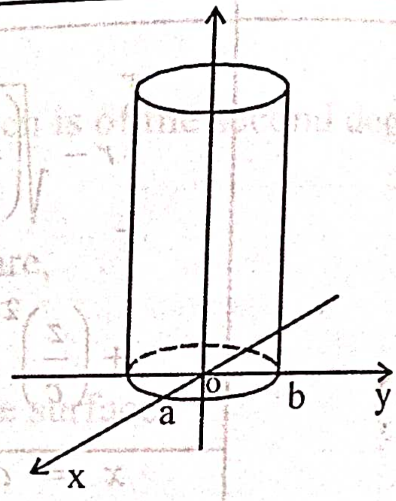
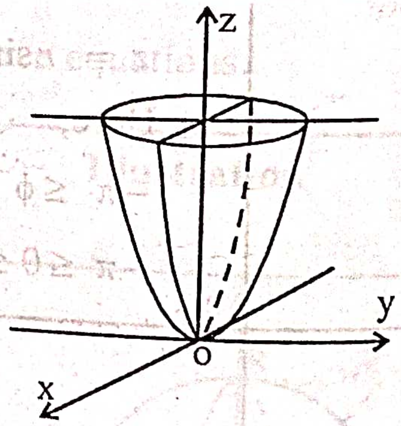
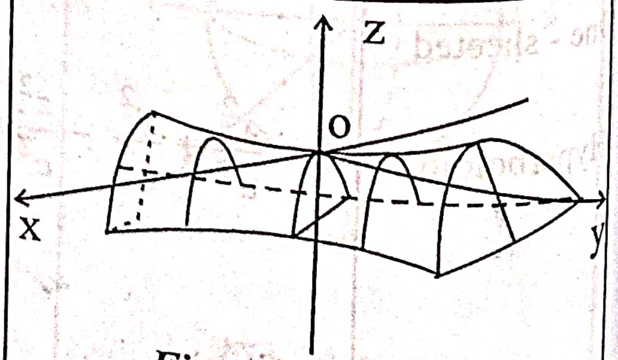
✓ The reason for frequent use of quadric surface are,

1. Ease to computing the surface normal.
2. Ease of testing whether a point is on the surface.
3. Ease of computing z if x & y are give.
4. Ease of calculating intersection of one surface with other.

✓ Quadric surface are described with second degree equations.

Surface Name	Implicit form parametric form	Illustration
Sphere	Implicit from $x^2 + y^2 + z^2 = r^2$ Parametric form: $x = r\cos\phi\cos\theta$ $y = r\cos\phi\sin\theta$ $z = r\sin\phi$ Here, $-\pi/2 \leq \phi \leq \pi/2$ $-\pi \leq \theta \leq \pi$	 *Fig - 3.16 Sphere*

Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $x = a\cos\phi\,\cos\theta$ $y = b\cos\phi\,\sin\theta$ $z = c\sin\phi$ Hene, $-\pi/2 \le \phi \le \pi/2$ $-\pi \le \theta \le \pi$	 *Fig - 3.17 Ellipsoid*
Torus	$\left[r - \sqrt{\left(\frac{x^2}{a}\right) + \left(\frac{y}{b}\right)^2}\right]^2$ $+ \left(\frac{z}{c}\right)^2 = 1$	 *Fig.3.18(a)Torus*
	$x = a(r+\cos\phi)\cos\theta$ $y = b(r+\cos\phi)\sin\theta$ $z = r\sin\phi$ Here, $-\pi \le \phi \le \pi$ $-\pi \le \theta \le \pi$	 *Fig 3.18(b) Torus*
One - sheeted Hyperboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	 *Fig 3.19 One - sheeted*

Two-sheeted Hyperboloid	$\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	 *Fig 3.20 Two-sheeted*
Elliptic Cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	 *Fig.3.21 Elliptic Cylinder*
Elliptic paraboloid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=cz$	 *Fig 3.22 Elliptic paraboloid*
Hyperbolic paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=cz$	 *Fig 3.23 Hyperbolic*

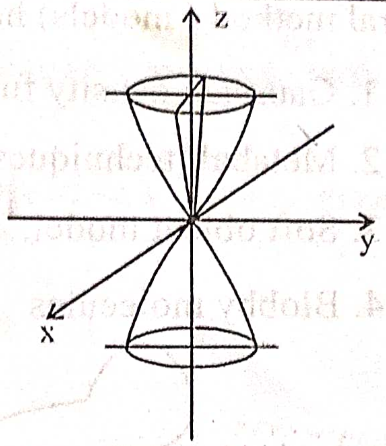
Elliptic cone	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z^2$	 *Fig.3.24 Elliptic cone*

Table 4. b

3.6 SPLINE REPRESENTATIONS

- ✓ A spline is a flexible strip used to produce a smooth curve through a designated set of points.
- ✓ Spline curve refers to any composite curve formed with polynomial sections satisfying specified continuity conditions at the boundary of the pieces.
- ✓ Splines used in graphics applications to design curve & surface shapes, to digitize drawing for computer storage and to specify animation paths for the objects.
- ✓ Typical CAD applications for splines include the design of automobile bodies, aircraft & space craft surfaces and ship hulls.

Control points

✓ A spline curve is specified by giving a set of coordinate positions called "control points".

✓ It indicates the general shape of the curve.

✓ These control points are then fitted with piece wise continuous parametric polynomial functions in one of the two ways.

1. Interpolation splines
2. Approximation Splines

3.6.1 Interpolation Splines

✓ When curve section passes through each. Control point, the curve is said to interpolate the set of control points.

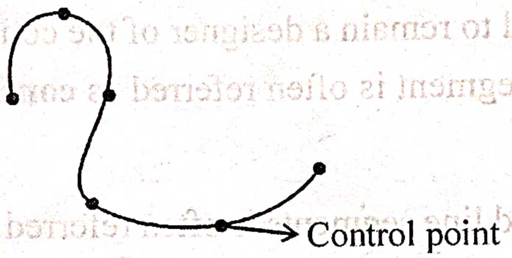


Fig.3.29 Interpolation splines

3.6.2 Approximation Splines

✓ When cuver section follows general control point path without necessarily passing through any control points, the resulting curve is said to approximate the set of control points.

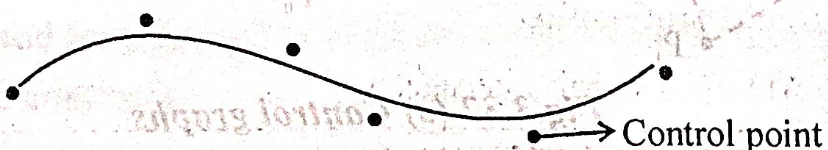


Fig.3.30 Approximation splines

✓ Spline curve can be modified defined and manipulated with operations on the set of control points.

✓ In addition the curve can be translated, rotated or scaled with transformations applied to the control points. CAD package can also insect extra control points to aid a designer in adjusting the curve shapes.

3.6.3 Convex Hull

✓ The convex polygon boundary that encloses aset of control points is called "convex hull".

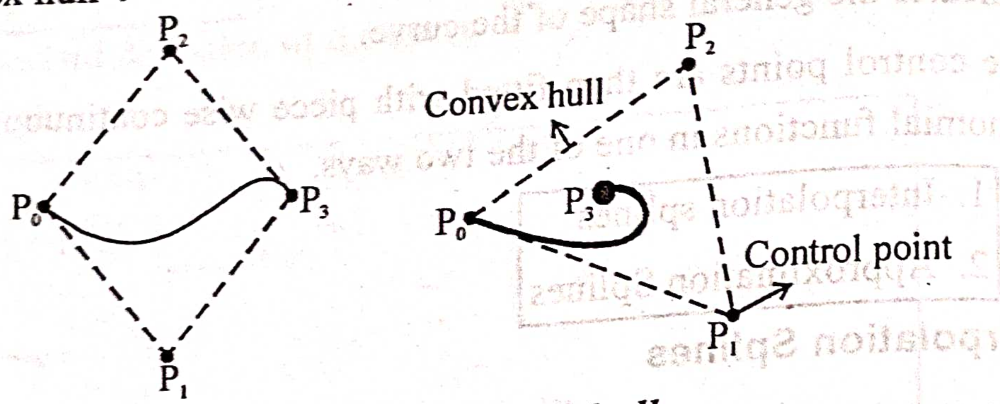


Fig.3. 31 Convex hull

✓ A polyline connecting the sequence of control points for an approximation spline is usually displayed to remain a designer of the control point ordering. This set of connected line segment is often referred as **control graph** of the curve.

3.6.4 Control Graph

✓ The set of connected line segments is often referred to as the "control graph" of the curve. It is also referred as **"control polygon"** (or) **"Characteristic polygon"**.

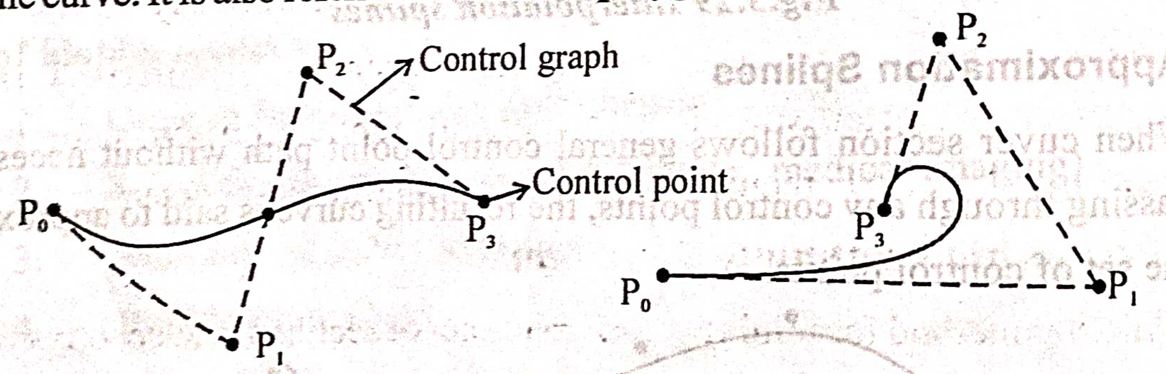


Fig.3.32 (a) Control graphs

3.6.5 Parametric Continuity Condition

✓ For smooth transition from one curve section onto next selection we put various continuity conditions at connection points. It lets parametric coordinate function as,

$$\boxed{\begin{aligned} x &= x(u) \\ y &= y(u) \\ z &= z(u) \qquad u_1 \leq u \leq u_2 \end{aligned}}$$

- Then zero order parametric continuity (C^0) means simply curves meets.

i.e. Last point of first curve section and first points of second curve section are same

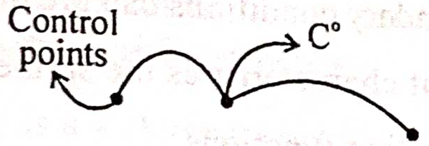


Fig 3.32 (b) Zero order parametric continuity

- Continuity(C^1) means first parametric derivatives are same for both curve section at intersection points.

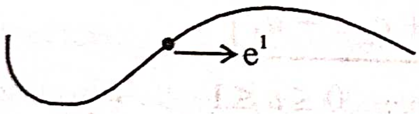


Fig.3.32 (c) First order parametric continuity

- **Second order parametric continuity** means both the first & second derivative of 2 curve section are same at intersection.

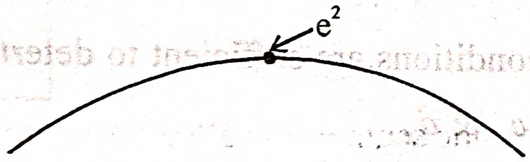


Fig 3.32 (c) Second order parametric continuity

- Higher order parametric continuity can be obtained similarly. First order continuity is often sufficient for general application but some graphics packages like CAD requires second order continuity for accuracy.

3.6.6 Geometric Continuity Condition

- Another method for joining 2 successive curve sections is to specify condition for geometric continuity.
- Zero **order geometry** (g^0) is same as parametric zero order continuity that 2 curve section meets.
- First order geometry (g^1) continuity means that the parametric first derivatives are proportional at the intersection of 2 successive sections but does not necessary. Its magnitude will be equal.
- Second order geometry conditions (g^2) means that the both parametric first & second derivatives are proportional at the intersection of 2 successive sections but does not necessarily magnitude will be equal.

3.9 BEZIER CURVES AND SURFACES

✓ It is developed by French engineer **Pierre Bezier** for the Renault automobile bodies.

✓ It has number of properties and easy to implement, so it is widely available in various CAD & graphics package.

3.9.1 Bezier Curves

✓ Bezier curve section can be fitted to any number of control points.

✓ Number of control points & their relative position gives degree of the Bezier polynomials.

✓ With the interpolation spline Bezier curve can be specified with boundary condition or blending function.

✓ Most convenient method to specify Bezier curve with blending function. Consider we are given n+1 control point position from p_o to p_n where $p_k = (x_k, y_z, z_k)$.

✓ This is blended to gives position vector p(u), which gives path of the approximate Bezier curve is,

$$\boxed{p(u) = \sum_{k=0}^{n} p_k \, \mathrm{BEZ}_{k,n}(u)} \qquad 0 \le u \le 1$$

Here,

$$\mathrm{BEZ}_{k,n}(u) = C(n,k)u^k (1-u)^{n-k}$$

and $C(n,k) = n!/\,k!(n-k)!$

✓ We can also soive Bezier blending function by recursion as follows,

$$\mathrm{BEZ}_{k,n}(u) = (1-u)\mathrm{BEZ}_{k,n-1}(u) + u\,\mathrm{BEZ}_{k-1,n-1}(u) \; [n>k\ge 1]$$

Here,

$$\mathrm{BEZ}_{k,k}(u) = u^k \text{ and}$$

$$\mathrm{BEZ}_{0,k}(u) = (1-u)^k$$

21

✓ Parametric equation from vector equation can be obtain as follows.

$$x(u) = \sum_{k=o}^{n} x_k \text{ BEZ }_{k,n}(u)$$

$$y(u) = \sum_{k=o}^{n} y_k \text{ BEZ }_{k,n}(u)$$

$$z(u) = \sum_{k=o}^{n} z_k \text{ BEZ }_{k,n}(u)$$

✓ Bezier curve is a polynomial of degree one less than the number of control points.

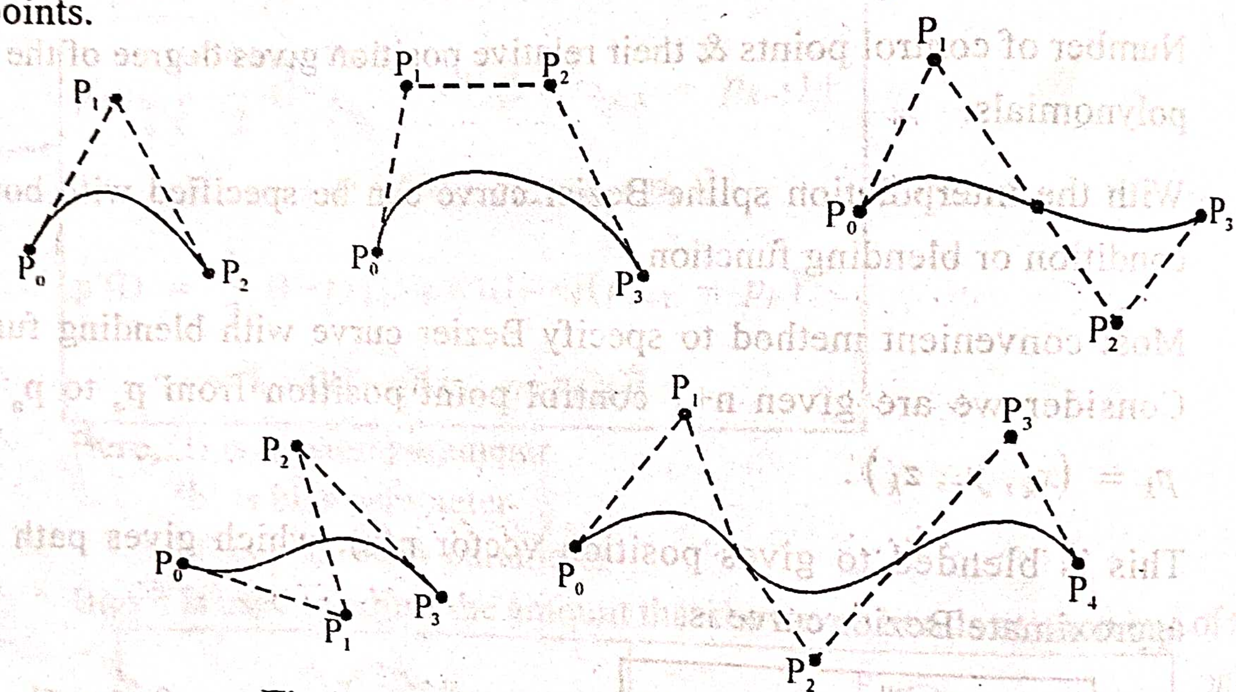


Fig.3.38 Example of 2D Bezier curves

✓ Efficient method for determining coordinate positions along a Bezier curve can be set up using recursive calculation.

For example successive binomial coefficients can be calculated as,

$$C(n,k) = \frac{n-k+1}{k} C(n,k-1)$$

$$n \geq k$$

3.9.2 Properties of Bezier Curves

1. It always passes through first control point. i.e $p(0) = p_0$.

2. It is always passes through last control point.

i.e. $p(1) = p_n$

3. Parametric first order derivatives of a Bezier curve at the endpoints can be obtain from control point coordinates as,

$$p(0) = -np_0 + n p_1$$

$$p(1) = -np_{n-1} + n p_n$$

4. Parametric second order derivative of endpoints are also obtained by control point coordinates as,

$$p(0) = n(n-1)[(p_2 - p_1) - (p_1 - P_0)]$$

$$p(1) = n(n-1)[(p_{n-2} - p_{n-1}) - (p_{n-1} - P_n)]$$

5. Bezier curve always lies within the convex null of the control points.
6. Bezier curve blending function is alwayas positive.
7. Sum of all Bezier blending function is always 1.

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

8. So, any curve position is simple the weighted sum of the control point positions.
9. Bezier curve smoothly follows the control points without erratic oscillations.

3.9.3 Design Technique Using Bezier Curves

✓ For obtaining closed Bezier curve we specify first and last control at same position.

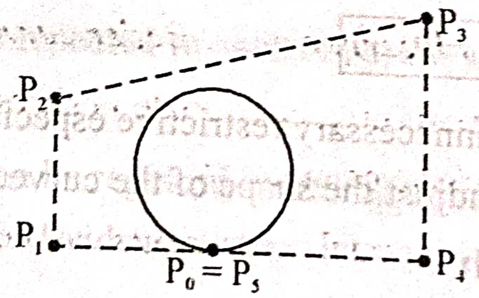


Fig.3.39 A closed Bezier curve

✓ If we specify multiple control points at same position it will get more weight and curve is pull towards that position.

✓ Bezier curve can be fitted for any number of control points but it requires higher order polynomial calculation.

✓ Complicated Bezier curve can be generated by dividing whole curve into several lower order polynomial curves. So we can get better control over the shape of small region.

✓ Since Bezier curve passes through first & last control point it is easy to join two curve sections with zero order parametric continuity (C^0).

✓ For first order continuity we put end point of first curve & start point of second curve at same position and last 2 points of first curve & first 2 points of second curve is collinear. And second control point of second curve is at position,

$$\boxed{p_n + (p_n - p_{n-1})}$$

✓ So that control points are position,

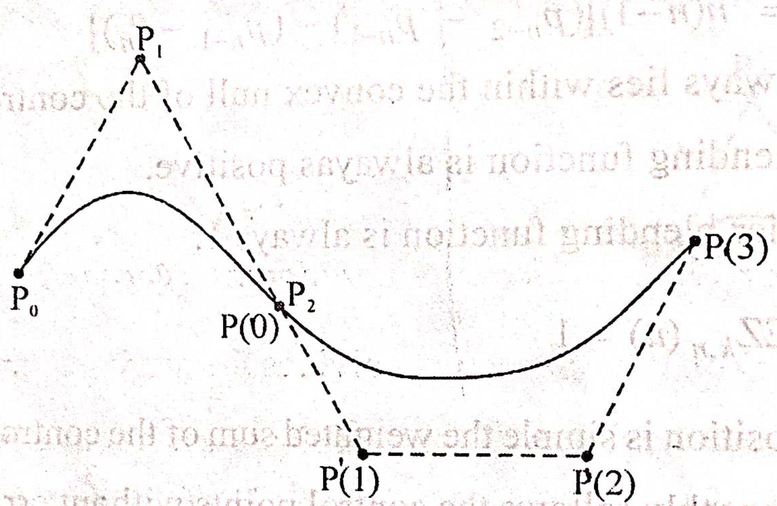


Fig.3.40 Zero and first order continuous curve

✓ Similarly for second order continuity the third control point of second curve in terms of position of the last three control points of fist curve section as,

$$\boxed{p_{n-2} + 4(p_n - p_{n-1})}$$

✓ C^2 continuity can be unnecessary restrictive especially for curve we left only one control point for adjust the shape of the curve.

3.9 4 Midpoint Approach

✓ It is used for construct the Bezier curve.

✓ In this approach, Bezier curve is constructed by midpoints.

✓ In midpoints approach midpoints of lines connecting four control points (A, B, C, D) are determined (AB, BC, CD).

✓ These midpoints are connected by line segments and their midpoints ABC & BCD are determined.

✓ Finally these 2 midpoints are connected by line segments & its midpoint ABCD is determined.

$P_n + (P_n - P_{n-1})$

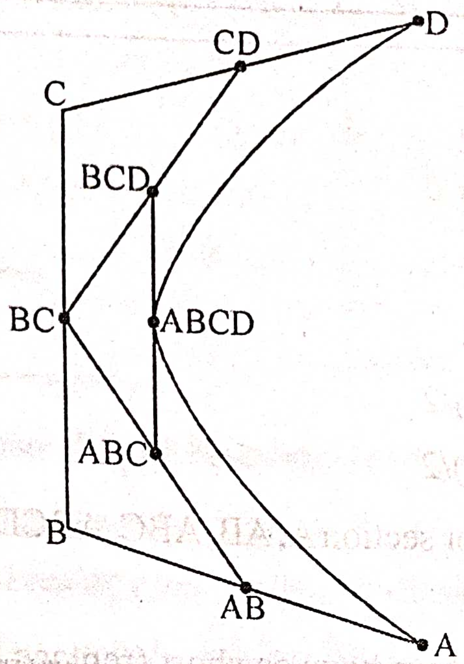


Fig.3.41 Bezier cure – midpoint approach

✓ The point ABCD on the Bezier curve divides the original curve into 2 sections.

✓ It makes the point A, AB, ABC & ABCD are the control points for the first section and the points ABCD, BCD, CD, D are the control points for the second section.

✓ By considering 2 sections separately we can get 2 more sections for each separate section (i.e) the original curve gets divided into 4 different curves.

✓ This process can be repeated to split the curve into smaller section until we have sections.

✓ So short that they can be replaced by straight lines or even until the sections are not bigger than individual pixels.

Algorithm:

1. Get 4 control points say $A(X_A, Y_A)$, $B(X_B, Y_B)$, $C(X_C, Y_C)$ & $D(X_D, Y_D)$.
2. Divide the curve represented by points A, B, C & D in 2 sections.

$X_{AB} = (X_A + X_B)/2$

$Y_{AB} = (Y_A + Y_B)/2$

$X_{BC} = (X_B + X_C)/2$

$Y_{BC} = (Y_B + Y_C)/2$

$X_{CD} = (X_C + X_D)/2$

$Y_{CD} = (Y_C + Y_D)/2$

$X_{ABC} = (X_{AB} + X_{BC})/2$

$Y_{ABC} = (X_{AB} + Y_{BC})/2$

$X_{BCD} = (X_{BC} + X_{CD})/2$

$Y_{BCD} = (Y_{BC} + Y_{CD})/2$

$X_{ABCD} = (X_{ABC} + X_{BCD})/2$

$Y_{ABCD} = (Y_{ABC} + Y_{BCD})/2$

3. Repeat the step (2) for section A, AB, ABC, ABCD & section ABCD, BCD, CD, D.
4. Repeat step (3) until we have so short (replace straight lines).
5. Replace small sections by straight lines,
6. Stop.

3.9.5 Cubic Bezier Curves

✓ Many graphic package provides only cubic spline function, because this given reasonable design flexibility in average calculation.

✓ Cubic Bezier curves are generated using 4 control points.

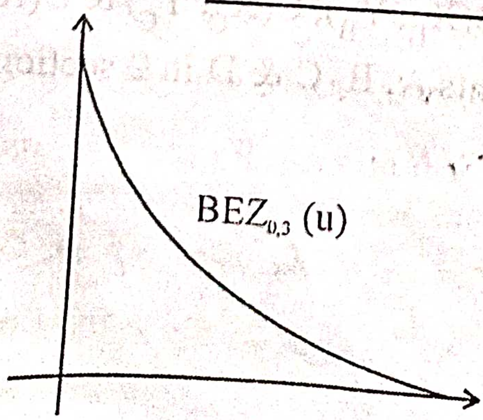
✓ 4 blending function obtained by substituting n=3.

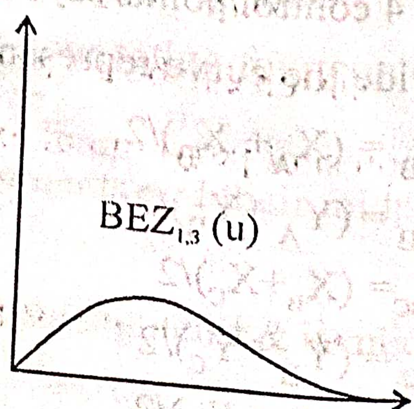
$$BEZ_{0,3}(u) = (1-u)^3$$

$$BEZ_{1,3}(u) = 3u(1-u)^2$$

$$BEZ_{2,3}(u) = 3u^2(1-u)$$

$$BEZ_{3,3}(u) = u^3$$





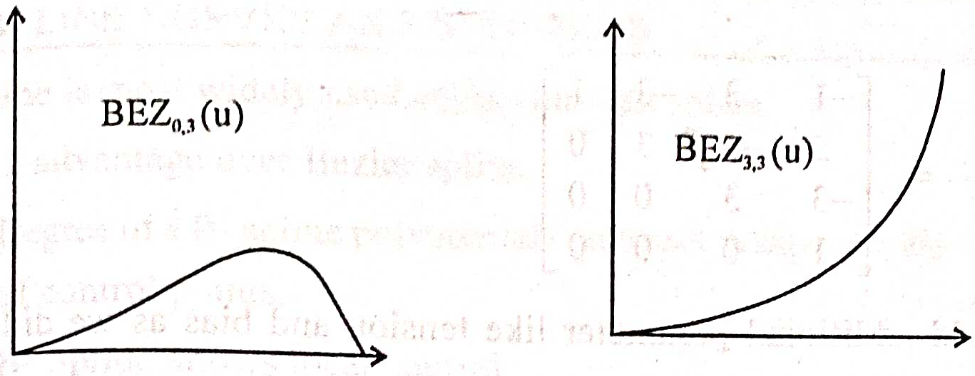


Fig.3.42 Four Bezier blending function for cubic curve

- ✓ The form of blending functions determines how control points affect the shape of the curve for values of parameters u over the range from 0 to 1
- ✓ At $u=0$ $BEZ_{0,3}(u)$ is only nonzero blending function with values 1.
- ✓ At $u=1$ $BEZ_{3,3}(u)$ is only nonzero blending function with values 1.
- ✓ So cubic Bezier always passes through p_0 & p_3
- ✓ Other blending function is affecting the shape of the curve in intermediate values of parameter u.
- ✓ $BEZ_{1,3}(u)$ is maximum at $u=1/3$ and $BEZ_{2,3}(u)$ is maximum at $u=2/3$.
- ✓ Blending function is always nonzero over the entire range of u so, it is not allowed for local control of the curve shape.
- ✓ At end point positions parametric first order derivates are,

$$p'(0) = 3(p_1 - p_0)$$
$$p'(1) = 3(p_3 - p_2)$$

- ✓ And second order parametric derivatives are,

$$p''(0) = 6(p_0 - 2p_1 + p_2)$$
$$p''(1) = 6(p_1 - 2p_2 + p_3)$$

- ✓ This expression can be used construct piecewise curve with c^1 & c^2 continuity.
- ✓ Now represent polynomial expression for blending function in matrix from,

$$p(u) = [u^3\ u^2\ u\ 1].M_{BEZ} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Here,

$$M_{BEZ} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

✓ We can add additional parameter like tension and bias as we did with the interpolating curve.

3.9.6 Bezier Surfaces

✓ Two sets of orthogonal Bezier curves can be used to design an object surface by an input mesh of control points.

✓ By taking Cartesian product of Bezier blending function we obtain parametric vector function as,

$$\boxed{p(u,v) = \sum_{j=0}^{m} \sum_{k=0}^{n} P_{j,k}\, BEZ_{j,m}(v)\, BEZ_{k,n}(u)}$$

✓ $P_{j,k}$ specifying the location of the (m+1) by (n+1) control points.

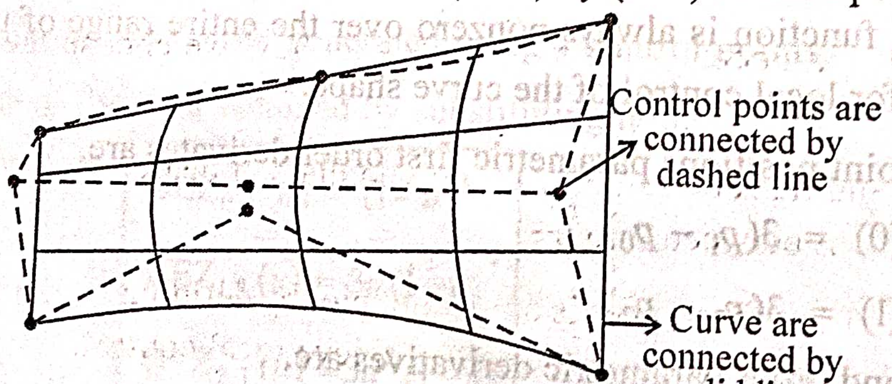


Fig.3.43 Bezier surface

✓ Each curve of constant u is plotted by varying v over interval 0 to 1. And similary we can plot for constant v.

✓ Bezier surfaces have same properties as Bezier curve, so it can be used in interactive design application

✓ For each surface patch we first select mesh of control point xy & then select elevation Z direction.

✓ We can put two or more surfaces & form required surfaces using method similar to curve section joining with continuity c^0, c^1 & c^2 as per need.

3.10 B - SPLINE CURVES AND SURFACES

✓ B- spline is most widely used approximation spline

✓ It has 2 advantage over Bezier spline.

1. Degree of a B- spline polynomial can be set independently of the number of control points.

2. B- Spline allows local control.

Disadvantage

1. B-spline curve is more complex then Bezier curve.

3.10.1 B-Spline Curves

✓ General expression for B - spline curve in terms of blending function is given by,

$$\boxed{p(u) = \sum_{k=0}^{n} p_k B_{k,d}(u)}$$

Here,

$u_{min} \le u \le u_{max}$, $2 \le d \le n+1$

✓ Where P_k is input set of control points.

✓ The range of parameter u is now depends on how we choose the B- spline parameters.

✓ B- Spline blending function $B_{k,d}$ are polynomials of degree d–1, where d can be any values in between 2 to n+1.

✓ We can set d=1 but then curve is only point plot by defining blending function for subintervals of whole range we can achieve local control.

✓ Blending function of B- spline is solved by Cox – deBoor recursion formulas as follows,

$$\boxed{B_{k,1}(u) = \begin{cases} 1 & \text{if } u_k \le u \le u_{k+1} \\ 0 & \text{otherwise} \end{cases}}$$

$$\boxed{B_{k,d}(u) = \frac{u - u_k}{u_{k+d-1} - u_k} B_{k,d-1}(u) + \frac{u_{k+d} - u}{u_{k+d} - u_{k+1}} B_{k+1,d-1}(u)}$$

✓ The selected set of subinterval endpoints u_j is referred to as **knot vector**.

✓ We can set any value as a subinterval end point but it must follow $u_j \leq u_{j+1}$.

✓ Values of u_{min} & u_{max} depends on number of control points, degree d, and knot vector.

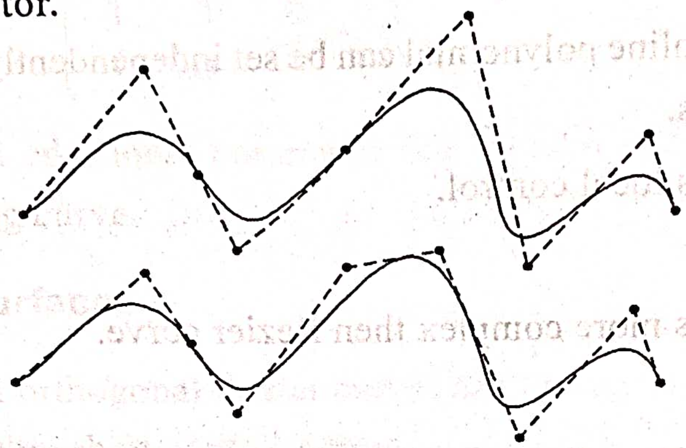


Fig.3.44 Local modification of B- spline curve

✓ B-Spline allows adding or removing control points in the curve without changing the degree of curve.

✓ B spline curve lies within the convex hull of at most d+1 control points. So that B- spline tightly bound to input positions.

✓ For any u in between u_{d-1} to u_{n+1} sum of all blending functions is 1.

$$\boxed{\text{i.e } \sum_{k=0}^{n} B_{k,d}(u) = 1}$$

✓ There are 3 classification of knot vectors.

1. Uniform
2. Open uniform
3. Non - uniform.

3.10.2 Properties of B–Spline Curves

1. It has degree d_{-1} and continuity c^{d-2} over range of u.
2. For n+1 control point we have n+1 blending function.
3. Each blending $B_{k,u}(u)$ is defined over d subintervals of the total range of u, starting at knot value u_k.
4. The range of u is into n+d subintervals by the n+d+1 values specified in the knot vector.

5. With knot values labeled as $\{u_0, u_1, \ldots u_{n+d}\}$ the resulting B - spline curve is defined only in interval from knot values u_{d-1} up to knot values u_{n+1}.
6. Each spline section is influenced by d control points.
7 Any one control point can affect at most d curve section.

3.10.3 Unifrom Periodic B-Spline

✓ When spacing between knot values is constant, the resulting curve is a **"uniform B- spline"**.

Ex:{0.0, 0.1, 0.2, ..., 1.0} or { 0, 1, 2, 3, 4, 5, 6, 7}

✓ Uniform B- spline have periodic blending function. So for given values of n & d all blending function has same shape.

✓ And each successive blending function is simply a shifted version of previous function.

$$\boxed{B_{k,d}(u) = b_{k+1,d}(u+\Delta u) = B_{k+2,d}(u+2\Delta u)}$$

✓ Where, Δu is interval between adjacent knot vectors.

3.10.4 Cubic, Periodic B- Spline

✓ It is commonly used in many graphics packages.

✓ It is particularly useful for generating closed curve.

✓ It any 3 consecutive control points are identical the curve passes through that coordinate position.

✓ Here, for cubic curve d=4 & n=3 knot vector spans d+n+1 = 4+3+1 = 8, so it is {0, 1,2,3,4,5,6,7}

✓ Now boundary conditions for cubic B- spline curve is obtain from,

$$\boxed{p(u) = \sum_{k=0}^{n} p_k B_{k,d}(u)}$$

$$u_{min} \le u \le u_{max}, \; 2 \le d \le n+1$$

That are,

$$\boxed{\begin{aligned} p(0) &= \frac{1}{6}(p_0 + 4p_1 + p_2) \\ p(1) &= \frac{1}{6}(p_1 + 4p_2 + p_3) \\ p'(0) &= \frac{1}{2}(p_2 - p_0) \\ p'(1) &= \frac{1}{2}(p_3 - p_1) \end{aligned}}$$

✓ Matrix formulation for a cubic periodic B splines with 4 control point can then be written as,

$$\boxed{p(u) = [u^3\ u^2\ u\ 1]\cdot M_B \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}}$$

where,

$$M_B = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

✓ We can also madify the B- Spline equation to include a tension parameter t.

✓ The periodic cubic B- Spline with matrix has the following form,

$$M_{Bt} = \frac{1}{6}\begin{bmatrix} -t & 12-9t & 9t-12 & t \\ 3t & 12t-18 & 18-15t & 0 \\ -3t & 0 & 3t & 0 \\ t & 6-2t & t & 0 \end{bmatrix}$$

When, t = 1 and

$$M_{Bt} = M_B$$

✓ We can obtain cubic B – spline blending function for parametric range from to 1 by converting matrix representation in to polynomial form for t=1 we have,

$$B_{0,3}(u) = \frac{1}{6}(1-u)^3$$

$$B_{1,3}(u) = \frac{1}{6}(3u^3 + 6u^2 + 4)$$

$$B_{2,3}(u) = \frac{1}{6}(-3u^3 + 3u^2 + 3u + 1)$$

$$B_{3,3}(u) = \frac{1}{6}u^3$$

3.10.5 Open Uniform B- Spline

- ✓ This class is cross between uniform B- spline & non – uniform B – spline.
- ✓ Sometimes it is treated as a special type of uniform B- spline, and sometimes as non uniform B- spline.
- ✓ For open uniform B–spline (open B- Spline) the knot spacing is uniform except at the ends where k not values are repeated d times.
- ✓ For example {0, 0, 1, 2, 3, 3} for d=2 & n=3, and {0, 0, 0, 0, 1, 2, 2, 2, 2} for d= 4 & n=4.
- ✓ For any values of parameter d & n we can generate an open uniform Knot vector with integer values using the calculations as follows.

$$\boxed{u_j = \begin{cases} 0 & \text{for} \quad 0 \le j, d \\ j-d+1 & \text{for} \quad d \le j \le n \\ n-d+2 & \text{for} \quad j > n \end{cases}}$$

Where,

$$0 \le j \le n+d$$

- ✓ Open uniform B – spline is similar to Bezier spline if we take d= n+1 it will reduce the Bezier spline as all knot values are either 0 or 1.
- ✓ For example cubic open uniform B- spline with d=4 have knot vector is {0, 0, 0, 0, 1, 1, 1, 1}.
- ✓ Open uniform B- Spline curve passes through first & last control points.
- ✓ Also slope at each end is parallel to line joining two adjacent control points at the end.
- ✓ So geometric condition for matching curve sections as for Bezier curves.
- ✓ For closed curve we specify first & last control point at the same position.

3.10.6 Non - Uniform B- Spline

✓ For this class of spline we can specify any values & interval for knot vector.

Ex: {0, 1, 2, 3, 3, 4} & { 0, 0, 1, 2, 2, 3, 4}

✓ It will give more flexible shape of curves.

✓ Each blending function have different shape when plots & different intervals.

✓ By increasing Knot multiplicity we produce variation in curve shape & also introduce discontinuities.

✓ Multiple knot value also reduces continuity by 1 for each repeat of particular value.

✓ We can solve non uniform B- spline using similar method as we used in uniform B- spline.

✓ For set of n+1 control point we set degree d and knot values.

✓ Then using the recurrence relations we can obtain blending function or evaluate curve position directly for display of the curve.

4.10.7 B- Spline Surfaces

✓ B – spline surface formation is also simiar to Bezier splines orthogonal set of curves are used and for connecting 2 surface we can same method which is used in Bezier surfaces.

✓ Vector equation of B- Spline surface is given by Cartesian product of B- spline blending functions.

$$\boxed{p(u,v) = \sum_{k_1=0}^{n1} \sum_{k_2=0}^{n2} p_{k_1,k_2} B_{k_1,d_1}(u) B_{k_2,d_2}(v)}$$

✓ Where, p_{k_1,k_2} specify control point position.

✓ It has same properties of B- spline curve.

Solved problems

Example 1

Construct the Bezier curve of order 3 and with 4 polygon vertices A(1,1), B(2, 3), C(4, 3) & D(6, 4).

P(1,1)

Solution:

✓ Bezier curve of order 3 is

$$p(u) = (1-u)^3 \; p_0 + 3u(1-u)^2 \; p_1 + 3u^2(1-u)p_2 + u^3 \; p_3$$

✓ Where $0 \le u \le 1$ and

p(u) is the point on the curve p_0, p_1, p_2, p_3

$(1-u)^3 p_0$

✓ Let us take $u = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

✓ $$p(0) = p_0 = (1,1)$$

✓ $$p\left(\frac{1}{4}\right) = \left(1-\frac{1}{4}\right)^3 p_0 + \left(3\times\frac{1}{4}\right)\left(1-\frac{1}{4}\right)^2 p_1 + \left(3\times\frac{1}{16}\right)\left(1-\frac{1}{4}\right)p_2 + \left(\frac{1}{4}\right)^3 p_3$$

$$= \left(\frac{3}{4}\right)^3 p_0 + \left(\frac{3}{4}\right)\left(\frac{9}{16}\right)p_1 + \left(\frac{3}{16}\right)\left(\frac{3}{4}\right)p_2 + \frac{1}{64} p_3$$

$$= \frac{27}{64}(1,1) + \frac{27}{64}(2,3) + \frac{9}{64}(4,3) + \frac{1}{64}(6,4)$$

$$= \left(\frac{27}{64}\times1 + \frac{27}{64}\times2 + \frac{9}{64}\times4 + \frac{1}{64}\times6, \frac{27}{64}\times1 + \frac{27}{64}\times3 + \frac{9}{64}\times3 + \frac{1}{64}\times4\right)$$

$$= \left(\frac{123}{64}, \frac{139}{64}\right)$$

$$= (1.9218, 2.1718)$$

✓ $$p\left(\frac{1}{2}\right) = \left(1-\frac{1}{2}\right)^3 p_0 + \left(3\times\frac{1}{2}\right)\left(1-\frac{1}{2}\right)^2 p_1 + \left(3\times\frac{1}{4}\right)\left(1-\frac{1}{2}\right)p_2 + \left(\frac{1}{2}\right)^3 p_3$$

$$= \left(\frac{1}{2}\right)^3 p_0 + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)p_1 + \left(\frac{3}{4}\right)\left(\frac{1}{2}\right)p_2 + \left(\frac{1}{8}\right)p_3$$

$$= \frac{1}{8}(1,1) + \frac{3}{4}(2,3) + \frac{3}{8}(4,3) + \frac{1}{8}(6,4)$$

$$= \left(\frac{1}{8}+\frac{6}{8}+\frac{12}{8}+\frac{6}{8}, \frac{1}{8}+\frac{9}{8}+\frac{9}{8}+\frac{4}{8}\right)$$

$$= \left(\frac{25}{8}, \frac{23}{8}\right)$$

$$= (3.125, 2.875)$$

✓ $$p\left(\frac{3}{4}\right) = \left(1-\frac{3}{4}\right)^3 p_0 + \left(3\times\frac{3}{4}\right)\left(1-\frac{3}{4}\right)^2 p_1 + \left(3\times\frac{9}{16}\right)\left(1-\frac{3}{4}\right)p_2 + \left(\frac{3}{4}\right)^3 p_3$$

$$= \frac{1}{16}(1,1)+\left(\frac{9}{4}\right)\left(\frac{1}{16}\right)(2,3)+\left(\frac{27}{16}\right)\left(\frac{1}{4}\right)(4,3)+\frac{27}{64}(6,4)$$

$$= \left(\frac{1}{64}+\frac{9}{64}\times 2+\frac{27}{64}\times 4+\frac{27}{64}\times 6, \frac{1}{64}+\frac{9}{64}\times 3+\frac{27}{64}\times 3+\frac{27}{64}\times 4\right)$$

$$= \left(\frac{289}{64}, \frac{217}{64}\right)$$

$$= (4.5156, 3.3906)$$

$$p(1) = u^3 p^3 = (6,4)$$

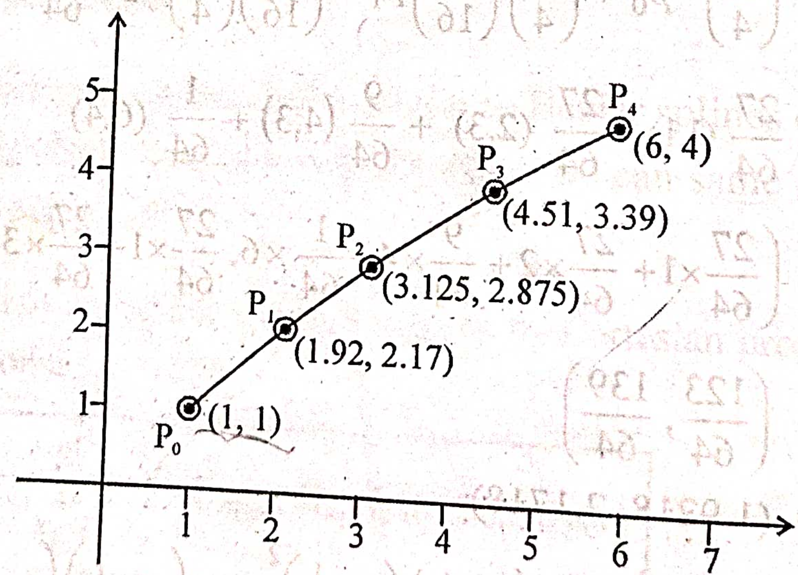


Fig.3.45 Plotted plotted Bezier curve

Example 2

Set up the equation of Bezier curve and roughly trace it for 3 control point (1,1) (2,2) & (3,1).

☺ ***Solution:***

Here number of control points is 3.

n+1 = 3

n = 2

I.e $\boxed{n+1 = 3}$

$\boxed{So,\ n=2}$

✓ Then 3 blending function for Bezier carver using the equation.

$$\boxed{BEZ_{k,n}(u) = C(n,k)\, u^k\, (1-u)^{n-k}}$$

$$\frac{n!}{k!\,(n-k)!}$$

$$BEZ_{0,2}(u) = \frac{2!}{0!\,(2-0)!}\, u^0 (1-u)^{2-0} = (1-u)^2$$

$$BEZ_{1,2}(u) = \frac{2!}{1!\,(2-1)!}\, u^2 (1-u)^{2-1} = 2u(1-u)$$

$$\frac{n}{k(n-k)}$$

$$BEZ_{2,2}(u) = \frac{2!}{2!\,(2-2)!}\, u^2 (1-u)^{2-2} = u^2$$

✓ The $p(u) = (1-u)^2\, p_0 + 2u(1-u)p_1 + u^2 p_2$

$(1-u)^2 p_0 + 2u(1-u)p_1 + u^2 p_2$

✓ Let us take u $=0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1.$

$$p(0) = p_0 = (1,1)$$

✓ $$p\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^2 p_0 + 2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)p_1 + \left(\frac{1}{4}\right)^2 p^2$$

$$= \left(\frac{9}{16}\right)(1,1) + \frac{6}{16}(2,2) + \frac{1}{16}(3,1)$$

$$= \left(\frac{9}{16} + \frac{12}{16} + \frac{3}{16}\right), \left(\frac{9}{16} + \frac{12}{16} + \frac{1}{16}\right)$$

$$= \left(\frac{24}{16}, \frac{22}{16}\right)$$

$$= (1{\cdot}5,\ 1{\cdot}375)$$

$C(n,k)u^k(1-u)^{n-k}$

$C(n,k)u^k(1-u)^{n-k}$

✓ $$p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 p_o + 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) p_1 + \left(\frac{1}{2}\right)^2 p_2$$

$$= \frac{1}{4}(1,1) + \frac{2}{4}(2,2) + \frac{1}{4}(3,1)$$

$$= \left(\frac{1}{4} + \frac{4}{4} + \frac{3}{4},\ \frac{1}{4} + \frac{4}{4} + \frac{1}{4}\right)$$

$$= \left(\frac{8}{4}, \frac{6}{4}\right)$$

$$= (2, 1\cdot5)$$

✓ $$p\left(\frac{3}{4}\right) = \left(\frac{1}{4}\right)^2 p_0 + 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)p_1 + \left(\frac{3}{4}\right)^2 p^2$$

$$= \frac{1}{16}(1,1) + \frac{6}{16}(2,2) + \frac{9}{16}(3,1)$$

$$= \left(\frac{1}{16} + \frac{12}{16} + \frac{27}{16}, \frac{1}{16} + \frac{12}{16} + \frac{9}{16}\right)$$

$$= \left(\frac{40}{16}, \frac{22}{16}\right)$$

$$= (2.5, 1.375)$$

$$p(1) = p_2 = (3, 1)$$

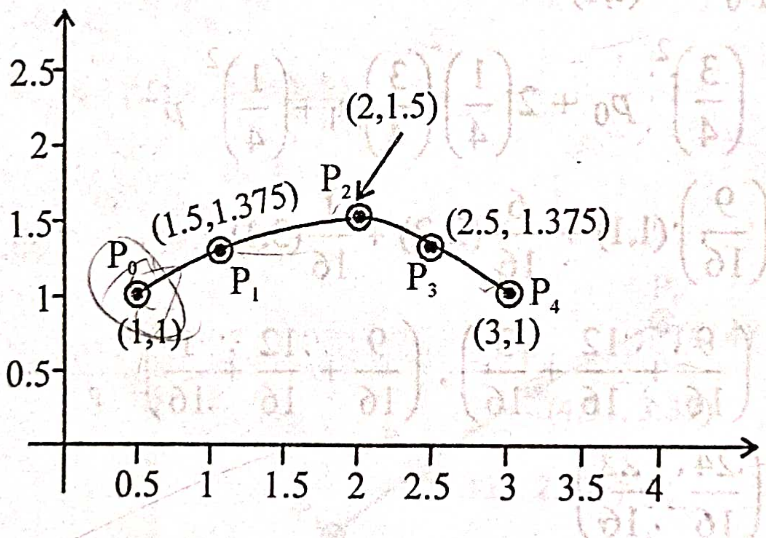


Fig.3.46 Plotted Bezier curve

Example 3

Given control points (10, 100), (50, 100), (70,120),(100,150). Calculate coordinates of any 4 points lying on the corresponding Bezier curve.

☺ ***Solution:***

✓ Here number of control points 4

i.e, $\boxed{n+1=4}$

So, $\boxed{n=3}$

✓ Bezier curve of order 3 is

$$p(u) = (1-u)^3 p_0 + 3u(1-u)^2 p_1 + 3u^2 (1-u) p_2 + u^3 p_3$$

✓ Let us take $u = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

✓ $p(0) = p_0 = (10, 100)$

✓ $$p\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^3 p_0 + 3\left(\frac{1}{4}\right)\left(\frac{9}{16}\right) p_1 + 3\left(\frac{1}{16}\right)\left(\frac{3}{4}\right) p_2 + \left(\frac{1}{4}\right)^3 p_3$$

$$= \frac{27}{64}(10,100) + \frac{27}{64}(50,100) + \frac{9}{64}(70, 120) + \frac{1}{64}(100,150)$$

$$= \left(\frac{270}{64} + \frac{1350}{64} + \frac{630}{64}, \frac{100}{64} + \frac{2700}{64} + \frac{2700}{64} + \frac{1080}{64} + \frac{15}{64}\right)$$

$$= \left(\frac{2350}{64} = \frac{6630}{64}\right)$$

$$= (36.7188, 103.5938)$$

✓ $$p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 p + 3\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) p_1 + 3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) p_2 + \left(\frac{1}{2}\right)^3 p_3$$

$$= \frac{1}{8}(10, 100) + \frac{3}{8}(50,100) + \frac{3}{8}(70,120) + \frac{1}{8}(100, 150)$$

$$= \left(\frac{10}{8} + \frac{150}{8} + \frac{210}{8} + \frac{100}{8}, \frac{100}{8} + \frac{300}{8} + \frac{360}{8} + \frac{150}{8}\right)$$

$$= \left(\frac{470}{8}, \frac{910}{8}\right)$$

$$= (58 \cdot 75, 113 \cdot 75)$$

✓ $$p\left(\frac{3}{4}\right) = \left(\frac{1}{4}\right)^3 p_o + 3\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 p_1 + 3\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right) p_2 + \left(\frac{3}{4}\right)^3 p_3$$

$$= \frac{1}{64}(10,100) + \frac{9}{64}(50,100) + \frac{27}{64}(70, 120) + \frac{27}{64}(100,150)$$

$$= \left(\frac{10}{64} + \frac{450}{64} + \frac{1890}{64} + \frac{2700}{64}, \frac{100}{64} + \frac{900}{64} + \frac{3240}{64} + \frac{4050}{64}\right)$$

$$= \left(\frac{5050}{64}, \frac{8290}{64}\right)$$

$$= (78.9063, 129.5312)$$

✓ $p(1) = p_3 = (100, 150)$

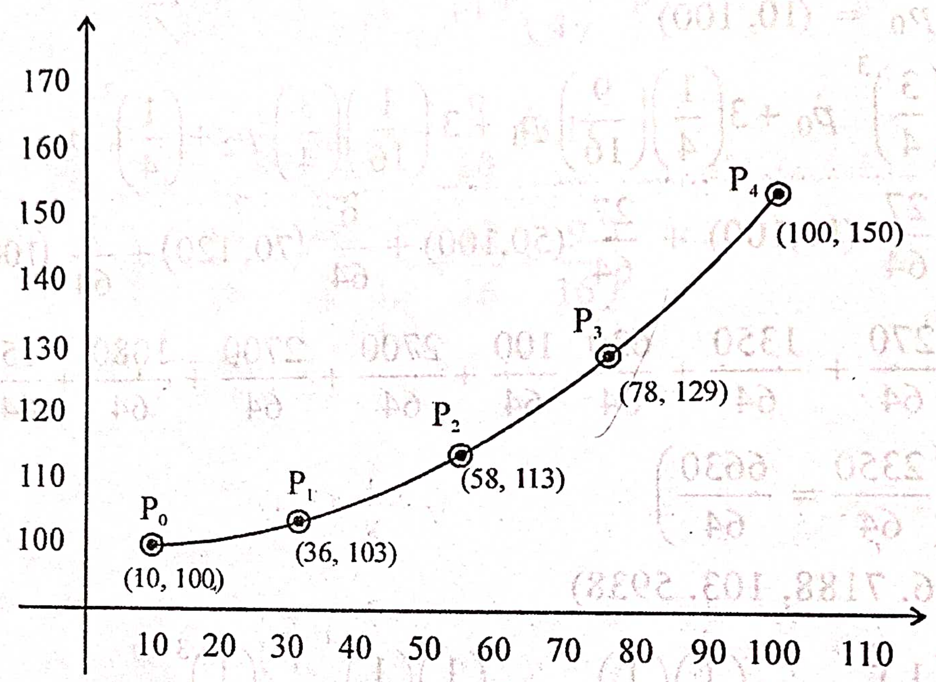


Fig.3.47 Plotted Bezier curve

Example 4

Given the vertices of Bezier polygon as $p_0(1,1)$ $p_1(2,3)$, $P_2(4,3)$ & $p_3(3,1)$ determine five points on Bezier curve.

☺ ***Solution:***

✓ Bezier curve of order 3 is,

$$p(u) = (1-u)^3 \; p_0 + 3u(1-u)^2 + p_1 + 3u^2(1-u)\; p_2 + u^3 p_3$$

✓ Let use take $u = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$$p(0) = p_0 = (1,1)$$

✓ $$p\left(\frac{1}{4}\right) = \left(\frac{3}{4}\right)^3 p_o + 3\left(\frac{1}{4}\right)\left(\frac{9}{16}\right) p_1 + 3\left(\frac{1}{16}\right)\left(\frac{3}{4}\right) p_2 + \left(\frac{1}{64}\right) p_3$$

Bezier polygon

$$= \frac{27}{64}(1, 1) + \frac{27}{64}(2,3) + \frac{9}{64}(4,3) + \frac{1}{64}(3,1)$$

$$= \left(\frac{27}{64} + \frac{54}{64} + \frac{36}{64} + \frac{3}{64}, \frac{27}{64} + \frac{81}{64} + \frac{27}{64} + \frac{1}{64}\right)$$

$$= \left(\frac{120}{64}, \frac{136}{64}\right)$$

$$= (1.875, 2.125)$$

✓ $$p\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 p_0 + 3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) p_1 + 3\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) p_2 + \frac{1}{8} p_3$$

$$= \frac{1}{8}(1, 1) + \frac{3}{8}(2,3) + \frac{3}{8}(4,3) + \frac{1}{8}(3, 1)$$

$$= \left(\frac{1}{8} + \frac{6}{8} + \frac{12}{8} + \frac{3}{8}, \frac{1}{8} + \frac{9}{8} + \frac{9}{8} + \frac{1}{8}\right)$$

$$= \left(\frac{22}{8}, \frac{20}{8}\right)$$

$$= (2.75, 2.5)$$

✓ $$p\left(\frac{3}{4}\right) = \left(\frac{1}{4}\right)^3 p_o + 3\left(\frac{3}{4}\right)\left(\frac{1}{16}\right) p_1 + 3\left(\frac{9}{16}\right)\left(\frac{1}{4}\right) p_2 + \left(\frac{3}{4}\right)^3 p_3$$

$$= \frac{1}{64}(1, 1) + \frac{9}{64}(2,3) + \frac{27}{64}(4,3) + \frac{27}{64}(3,1)$$

$$= \left(\frac{1}{64} + \frac{18}{64} + \frac{108}{64} + \frac{81}{64}, \frac{1}{64} + \frac{27}{64} + \frac{81}{64} + \frac{27}{64}\right)$$

$$= \left(\frac{208}{64}, \frac{136}{64}\right)$$

$$= (3.25, 2.125)$$

✓ $$p(1) = p_3 = (3, 1)$$

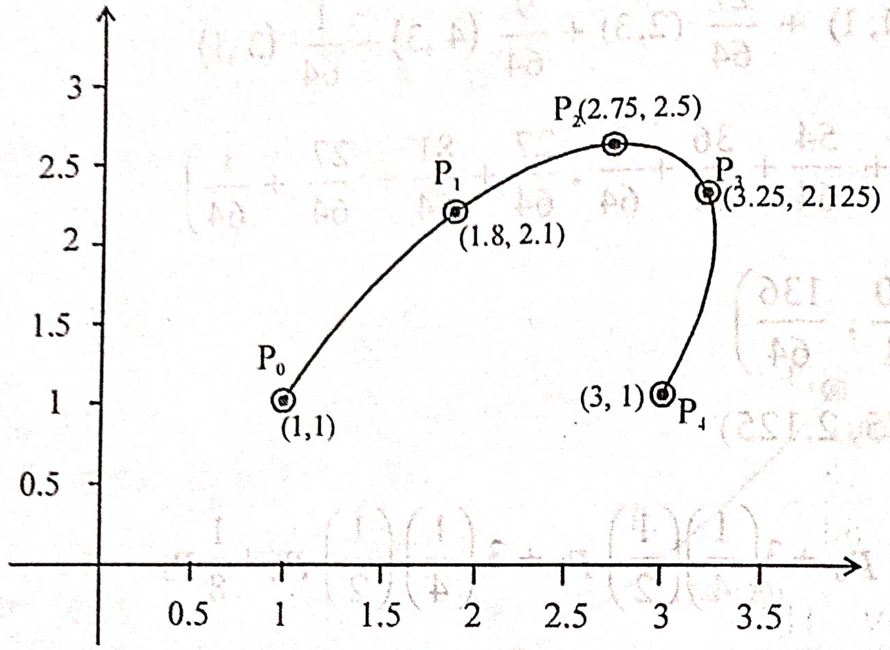


Fig.3.48 Plotted Bezier curve

ample 5

Given a set of four 2D control points(0,1) (2,5), (5,5), (8,0) determine four points on Bezier curve.

Solution:

Bezier curve of order 3 is,

$$p(u) = (1-u)^3\ p_0 + 3u(1-u)^2\ p_1 + 3u^2(1-u)\ p_2 + u^3 p_3$$

Let us take $u = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$$p(0) = p_0(0, 1)$$

$$p\left(\frac{1}{4}\right) = \left(\frac{27}{64}\right)(0,1) + \frac{27}{64}(2, 5) + \frac{9}{64}(5,5) + \frac{1}{64}(8,0)$$

$$= \left(0 + \frac{54}{64} + \frac{45}{64} + \frac{8}{64}, \frac{27}{64} + \frac{135}{64} + \frac{45}{64} + 0\right)$$

$$= \left(\frac{107}{64}, \frac{207}{64}\right)$$

$$= (1.6719, 3.875)$$

$$p\left(\frac{1}{2}\right) = \frac{1}{8}(0,1) + \frac{3}{8}(2,5) + \frac{3}{8}(5,5) + \frac{1}{8}(8,0)$$

$$= \left(0+\frac{6}{8}+\frac{15}{8}+\frac{8}{8}, \frac{1}{8}+\frac{15}{8}+\frac{15}{8}+0\right)$$

$$= \left(\frac{29}{8}, \frac{31}{8}\right)$$

$$= (3\cdot625, 3\cdot875)$$

✓ $$p\left(\frac{3}{4}\right) = \frac{1}{64}(0,1) + \frac{9}{64}(2,5) + \frac{27}{64}(5,5) + \frac{27}{64}(8,0)$$

$$= \left(0+\frac{18}{64}+\frac{135}{64}+\frac{216}{64}, \frac{1}{64}+\frac{45}{64}+\frac{135}{64}+0\right)$$

$$= \left(\frac{369}{64}, \frac{181}{64}\right)$$

$$= (5\cdot7656, 2\cdot8281)$$

✓ $$p(1) = p_3 = (8,0)$$

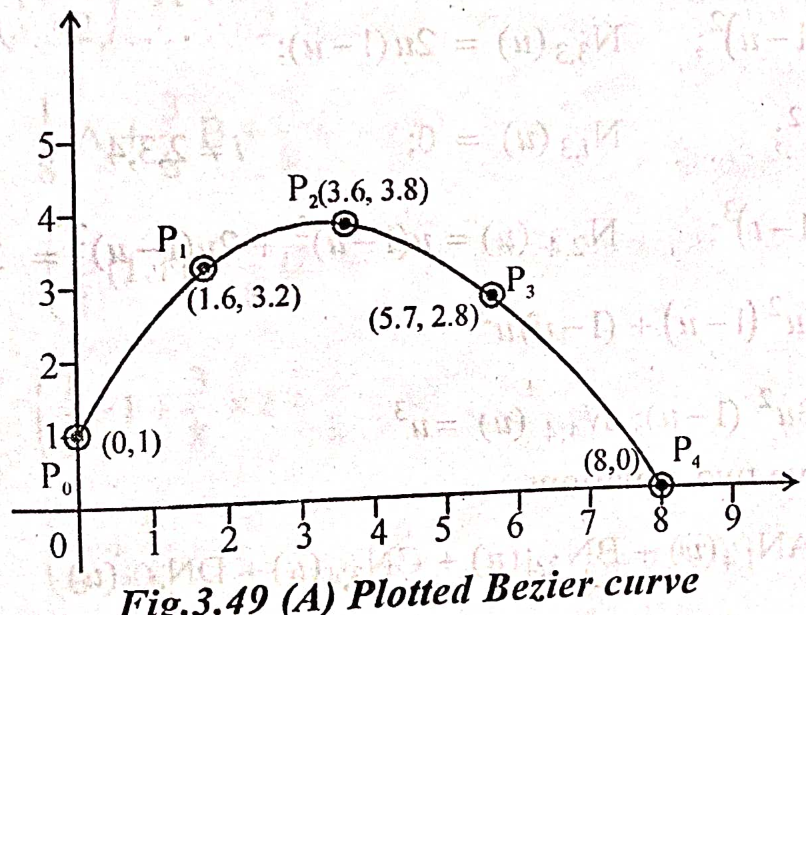


Fig.3.49 (A) Plotted Bezier curve