Question 3:

$E[W_s W_t] = \min(s,t)$ for $s,t \geq 0$

Assume, $s \leq t$

$W_t = W_s + (W_t - W_s)$

Since W_s and $W_t - W_s$ are independent and $E[W_t - W_s] = 0$

$\Rightarrow$ $E[W_s W_t] = E[W_s(W_s + (W_t - W_s))] = E[W_s^2] + E[W_s(W_t - W_s)]$

$= s + 0 = s$

$\Rightarrow$ $E[W_s W_t] = \min(s,t)$

Question 4:

$W_t - W_s \sim N(0, t-s)$

By definition of Brownian motion,

- $W_t - W_s \sim N(0, t-s)$ because Brownian increments are normally distributed with $\mu = 0$, $\sigma^2 = t - s$
- if $[s_1, t_1]$ and $[s_2, t_2]$ are non-overlapping intervals, the increments $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independents

Question 5:

To show: $E[W_t / F_s] = W_s$ for, $0 \leq s \leq t$, conclude that Brownian motion is martingale.

$W_t = W_s + (W_t - W_s)$

Then $E[W_t / F_s] = E[W_s + (W_t - W_s) | F_s] = W_s + E[W_t - W_s | F_s]$

Since, $W_t - W_s$ is independent of F_s and has mean $= 0$: $E[W_t - W_s | F_s] = 0 \Rightarrow E[W_t | F_s] = W_s$

$\Rightarrow$ $\{W_t\}$ is martingale.