Set Identities:

- $A \cap U = A$, $A \cup \emptyset = A$ — Identity Laws
- $A \cup U = U$, $A \cap \emptyset = \emptyset$ — Domination Laws
- $A \cup A = A$, $A \cap A = A$ — Idempotent Laws
- $\overline{(\overline{A})} = A$ — Complementation Law
- $A \cup B = B \cup A$, $A \cap B = B \cap A$ — Commutative Laws
- $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ — Associative Law
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ — Distributive Law
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$, $\overline{A \cup B} = \bar{A} \cap \bar{B}$ — DeMorgan's Laws
- $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$ — Absorption Laws
- $A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$ — Complement Laws
- $A - B = A \cap \bar{B}$ — Set Minus

To prove $A = B$:
Show $A \subseteq B$, $B \subseteq A$.
Pick arbitrary element in A, show in B.

Set: Unordered, no duplicates

Defined by MEMBERSHIP ($\in$)
$S = \{x \mid x \equiv 1 \pmod 2 \ \wedge\ \text{Prime}(x)\}$

$\mathbb{N}$: Naturals, $0 \to \infty$ — U: Union
$\mathbb{Z}$: Integers, $-\infty \to \infty$ — $\cap$: Intersection
$\mathbb{Q}$: Rationals — $\bar{S}$: Complement
$\mathbb{R}$: Reals — $S-T$: Minus

$S \subseteq T$: S subset of T — $S \oplus T$: $S \cup T - S \cap T$ (XOR / Symmetric Difference)
$S \subset T$: $S \subseteq T \wedge S \neq T$
$S \cap T = \emptyset$: S, T disjoint

Power Set: Set of all subsets of S (Size 2^n)

Cardinality: # of elements (recall no repeats)

PIE: $|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$
[ALTERNATE SIGNS]

Cartesian Product: $A \times B$ (ORDERED)
Set of all tuples (a,b); $a \in A$, $b \in B$.
$|A| = x$, $|B| = y \to |A \times B| = xy$

Functions: Map from Set D to Set C

- EVERY elem in D has ONE elem in C
- Range: Mapped part of Codomain
- ONTO (SURJECTIVE): Every elem in C mapped to
  $\forall b \in B, \exists a \in A\ [f(a) = b]$; $f: A \to B$
- ONE-TO-ONE (INJECTIVE):
  $\forall a, b \in A\ [f(a) = f(b) \to a = b]$
  Can have unmapped elements in CoDomain

DOMAIN, CODOMAIN MATTER!!

- BIJECTIVE: BOTH (One-to-one correspondence) ← IMPORTANT
  - Implies inverse exists

One-to-One Proof: $f(x): \mathbb{R} \to \mathbb{R}$
$f(x) = 2x + 4$.

One-to-One: $\forall a,b \in A\ [f(a) = f(b) \to a = b]$.
Let a, b be arbitrary domain elements such that $f(a) = f(b)$. Then, $2a + 4 = 2b + 4$
$2a = 2b$
$a = b$.

So, $(f(b) = f(a)) \to (b = a)$. As a, b arbitrary, $\forall a,b \in A\ [f(b) = f(a) \to b = a]$. So, $f(x)$ is one-to-one by definition.

Onto Proof: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 4$.

Onto: $\forall b \in B, \exists a \in A\ [f(a) = b]$
Let b be an arbitrary codomain element.
Define $a = \frac{b-4}{2}$. Then, $f(a) = 2a + 4$
Note a is Domain element. $= 2\left(\frac{b-4}{2}\right) + 4$
[Do side work to arrive @ this defn. of a] $= b - 4 + 4$
$= b$.

So, $\exists a \in \mathbb{R}\ [f(a) = b]$. As b was arbitrary,
$\forall b \in B, \exists a \in A\ [f(a) = b]$, so f is onto.

Proof of NOT One-to-One:
$\neg(\forall a, b \in A\ [f(b) = f(a) \to b = a])$
$\equiv \exists a, b \in A\ [(f(b) = f(a)) \wedge (b \neq a)]$
So, find 2 inputs $(a, b \in A)$ such that $f(a) = f(b)$ but $a \neq b$.

Proof of NOT Onto:
$\neg(\forall b \in B\ \exists a \in A\ [f(a) = b])$
$\exists b \in B\ \neg \exists a \in A\ f(a) = b$
$\exists b \in B\ \forall a \in A,\ f(a) \neq b$

So, find a $b \in B$ such that no a exists for $f(a) = b$.

$h = g \circ f = g(f(x))$; if $f: A \to B$ and $g: B \to C$, $g \circ f: A \to C$
* Output of f must be subset of domain of g
$(f_1 + f_2)(x) = f_1(x) + f_2(x)$ — $f_1, f_2: S \to \mathbb{R}$
$(f_1 f_2)(x) = f_1(x)\, f_2(x)$,

If $f: A \to B$ one-to-one, $|A| \leq |B|$
If $f: A \to B$ onto, $|A| \geq |B|$
If $f: A \to B$ bijection, $|A| = |B|$

Assume for ∞ sets; prove for finite

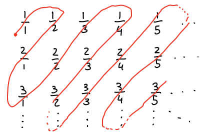
Schroeder-Bernstein Thm:
$(|A| \leq |B|) \wedge (|B| \leq |A|) \to |A| = |B|$.

So, to demonstrate $|A| = |B|$:
Prove: $[f_1: A \to B$ is 1-1 OR $g_1: B \to A$ is onto$]$ $|A| \leq |B|$
AND
$[f_1: A \to B$ is onto OR $g_1: B \to A$ is 1-1$]$ $|B| \leq |A|$

∞ Set Cardinality

Set S is countable iff $|S| \leq |\mathbb{Z}^+|$ — $\mathbb{Z}, \mathbb{N}, \mathbb{Z}^+ \times \mathbb{Z}^+, \mathbb{Q}$ — cardinality: $\aleph_0$ (aleph null)
- one-to-one $f: S \to \mathbb{Z}^+$
- onto $f: \mathbb{Z}^+ \to S$
- all finite sets countable

can list elements in some order



(onto, not one-to-one)
Similar for $\mathbb{Z}^+ \times \mathbb{Z}^+$

$f: \mathbb{Z} \to \mathbb{Z}^+$
$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ -2x & x < 0 \end{cases}$

Diagonalization Proof ($\mathbb{R}$ uncountable)
$[0,1] \subseteq \mathbb{R}$, so $|[0,1]| \leq |\mathbb{R}|$.
For contradiction, assume $[0,1]$ countable. Then $\exists$ onto $f: \mathbb{Z}^+ \to [0,1]$.
List $[0,1]$ out in a table in arbitrary order.

$z \in \mathbb{Z}^+$	$x \in [0,1]$	D_1	D_2	D_3	D_4
1	0	0	0	0	...
2	0.12	1	2	0	...
3	1/2	5	0	0	...
4	$1/\pi$	3	1	8	...

Then, shift diagonal numbers up 1 (9 → 0).
So, every row disagrees w/ diagonal on one digit; but diagonal num $\in \mathbb{R}$.

$\mathbb{R}$, Power set of ∞ S, Set f of $\mathbb{N} \to \{0,1\}$ all uncountable ($\aleph_1$)

Relations (xRy, $R(x,y)$, x related to y)

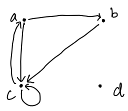
Binary relation between D, C: $R \subseteq D \times C$.
R is a SET of tuples (ordered).
Properties on relations on $S \times S$: SET OPS APPLY

- Reflexive: $\forall x,\ xRx$
- Symmetric: $\forall x,y,\ xRy \leftrightarrow yRx$
- Antisymmetric: $\forall x,y,\ (xRy \wedge yRx) \to x = y$
- Transitive: $\forall x,y,z,\ (xRy \wedge yRz) \to xRz$
- Asymmetric: $\forall x,y\ (xRy \to \neg yRx)$
- Irreflexive: $\forall x,\ \neg xRx$

→ Asymmetric = (Antisymmetric ∧ Irreflexive)
→ Symmetric ∧ Antisymmetric: Only self-loops
→ Sym ∧ Asym ∧ Antisym: $\emptyset$

xRy representations:

x\y	a	b	c	d
a	0	1	1	0
b	0	0	1	0
c	1	0	1	0
d	0	0	0	0



$R = \{(a,b), (a,c), (b,c), (c,c), (c,a)\}$

- Reflexive: main diagonal all 1s
- Symmetric: Symmetry across main diagonal
- Antisymmetric: No 2 1s across from main diagonal (except diagonal itself)
- Irreflexive: main diagonal all 0s
- Graphs:
  - Reflexive: Self-loop @ each vertex
  - Symmetric: Directed edges in pairs

- Composition: $R \subseteq A \times B$, $S \subseteq B \times C$
  $a(S \circ R)c$ iff $\exists b \in B\ (aRb \wedge bRc)$
  On bipartite graph, is there path from a to c
- R^2: $R \circ R$; R^3: $R \circ R^2$: $R \circ R \circ R$, etc.
  - R^n is all nodes you can reach w/ EXACTLY n moves
  - R^* (transitive closure) $= R \cup R^2 \cup R^3 \cup \ldots$ (reachability region)
  - If $R^2 \subseteq R$, R is transitive

Closure (min relation satisfying prop):
- Reflexive, Transitive, Symmetric
- Iteratively find violation → fix violation
- Can only ADD to relation
- Reflexive: $MAT_R \cup$ Identity Mat
- Symm: $MAT_R \cup MAT_R^T$

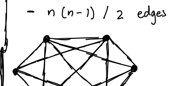
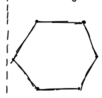
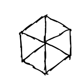
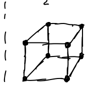
Graphs:
- $G = (V, E)$
  Tuple of set of Vertices & set of edges (tuples)
- Differentiating Graphs
  - Directed Edges?
  - Multiple Edges b/w nodes?
  - Loops?
- NOTE: "ALLOWED TO" have loops and actually having loops dft. Differentiate on ALLOWED behavior.

- In-Degree = $\deg^-(v)$ — incoming
- Out-Degree = $\deg^+(v)$ — outgoing
- $2|E| = \sum_{v \in V} \deg(v)$
- $|E| = \sum_{v \in V} \deg^+(v) = \sum \deg^-(v)$

- Simple Graphs: Undirected, no multiple edges, no loops
- $e = \{u, v\}$ in Undirected; — Degree: # of edge ends on a vertex (loops counted 2x)
- $e = (u, v)$ in Directed.
- Subgraph of $G = (V, E)$: $H = (V', E')$, $V' \subseteq V \wedge E' \subseteq E$
- $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$
  Union $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

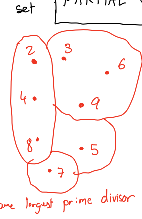
Types of Graphs

- K_n (Complete) – all nodes connected
  - n nodes
  - $n(n-1)/2$ edges
- C_n (Cycle)
  - n nodes
  - n edges
- W_n (Wheel)
  - $(n+1)$ nodes
  - $2n$ edges
- Q_n (Hypercube)
  - 2^n nodes
  - $\frac{n2^n}{2} = n2^{n-1}$ edges

- Bipartite

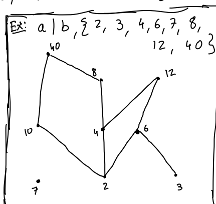
Equivalence Relation: – set to some set
- Reflexive, Symmetric, Transitive (iff)
- $x = y$; $p \equiv q$; $\frac{a}{b} = \frac{c}{d}$; etc.
- Leads to EQUIVALENCE CLASSES
- EQ CLASS $[a]_R = \{b \in S : aRb\}$
- eq classes partition graph
  - disjoint classes, form whole graph
- ANY partition of S is equivalence class



Same largest prime divisor

Partial Orders: Ranks elements (some could be incomparable)
- Reflexive
- Antisymmetric
- Transitive
- Eg: $a \mid b$ over $\{1,2,3,4,5,6\}$ (a divides b) → $(\{1,2,3,4\}, \leq)$
- Tuple (Set, operator) — poset
- Total Order iff poset ∧ all elem comparable
- Hasse Diagrams
  - Height indicates order
  - Implied edges (transitive) omitted
  - No self-loops
  - For total order, straight line

Ex: $a \mid b$, $\{2, 3, 4, 6, 7, 8, 12, 40\}$



- Minimal: No elem smaller (some incomparable)
- Maximal: No elem bigger ("...")
- Minimum: Unique – comparable to AND less than everything else.
- Maximum: Unique – comparable to AND greater than everything else.
- For subset A of S, upper bound for A is $x \in S$ s.t. $\forall a \in A, a \leq x$. x COULD BE IN A
- Similar for lower bound
  - Not necessarily just 2 elems
- LUB: Least Upper Bound
- GLB: Greatest Lower Bound

Strict Poset
- Irreflexive instead of reflexive
- ($>$, not $\geq$)

- Minimal, Maximal exist if set is finite
- Minimum, Maximum not nec. $\exists$. (doesn't have to be total order for them to exist tho)

Topological Sort
- Convert partial order into total order
- Starting from bottom, output any minimal element in S
- $\prec$: ordering symbol for orders

Bipartite
- If G can have vertices partitioned into disjoint V_1, V_2 such that every edge connects vertex in V_1, vertex in V_2.
(V_1, V_2): Bipartition
- Simple graph bipartite iff it is two-colorable (no adj. vertices have same color).
- 2-colorable iff all cycles have even length

Induction: To prove $P(n)$ true for all $n \in \mathbb{N}$:
- Prove $P(0)$
- Prove $P(n) \to P(n+1)$ $\forall n \in \mathbb{N}$
- ALWAYS WORK IN ONE DIRECTION FROM LHS = RHS
- Induction works when set WELL-ORDERED
- Set S well-ordered iff for all non-empty $A \subseteq S$, A has a LEAST ELEMENT according to some order $\prec$
- For ex, reals not well-ordered

Strong Induction
- To prove $P(n)$:
- Prove Base Case(s): $P(1)$, $P(2)$...
- Prove Inductive Step: $(P(1) \wedge P(2) \wedge \ldots P(k-1)) \to P(k)$
- Can use ALL previous results

Induction Template:
Inductive Step: Assume $P(k)$: [. .] for some $k \geq n_0$.
Want to show $P(k+1)$: [. . .].
[Show $P(k+1)$ using assumption for $P(k)$. Remember only work from one side]
Base Case: $P(n_0)$: [. . .]
[Prove $P(n_0)$ holds]
By mathematical induction, $P(n)$ is true for all $n \geq n_0$

Strong Induction Template:
Inductive Step: Assume $P(j)$: [. . .] true for $n_0 \leq j < k$
Want to show $P(k)$: [. . .].
[Show $P(k)$ using assumption for $P(j)$, $n_0 \leq j < k$. Remember only work from one side]
Base Case: $P(n_0)$ . . . (and others)
[Prove $P(n_0)$, etc. hold]
By strong induction, $P(n)$ is true for all $n \geq n_0$

Exam Reminders:
- Define $P(k)$ properly
- $[a]$: equivalence class of a
- READ THE QUESTION
- Asymmetric vs. Antisymm, $\in$ vs. $\subseteq$ — CAREFUL
- DEFINE EVERYTHING
- JUSTIFY STEPS when solving onto / one-to-one / induction
- Conclusions important

$f: \mathbb{Z} \to \mathbb{Z}^+$

a) one-to-one, not onto: $f(x) = 3x+1$ for $x \geq 0$,
$f(x) = -3x+2$ for $x < 0$

b) onto, not one-to-one: $f(x) = |x| + 1$

c) bijective: $f(x) = -2x$ for $x < 0$

d) neither: $f(x) = x^2 + 1$

$f: \mathbb{N} \to \mathbb{N}$

a) One-to-one but not onto: $f(x) = x + 1$

b) Onto but not one-to-one: $f(x) = \begin{cases} 0 & \text{if } x \text{ is odd} \\ n/2 & \text{if } x \text{ is even} \end{cases}$

RELATION PROOF: Prove that if relation R over non-empty A is symmetric, transitive, and irreflexive, then R is empty.

Seeking a contradiction, suppose R is non-empty. So R has some element (a,b), $a,b \in A$. Since R is symmetric, $(b,a) \in R$. Since R is transitive, and $aRb \land bRa$, aRa. However, R is also irreflexive, so $\forall a \in A$, $a \not R a$. So, we have a contradiction as both aRa and $a \not R a$ cannot be true.
So, we disprove our initial assumption that R is non-empty.

Which properties do these satisfy?

Relation	Domain	Ref.	Sym.	Antisym	Trans.
<	**R**	✗	✗	✓	✓
⊆	sets	✓	✗	✓	✓
=	**Z**	✓	✓	✓	✓
"has a non-empty intersection with"	sets	✗, only b/c of ∅	✓	✗	✗
"is a sister of"	people	✗	✗	✗	✗
"is a sibling of"	people	✗	✓	✗	✗
"is a descendant of"	people	✗	✗	✓	✓
"is divisible by"	**Z**	✓	✗	✗	✓

$\boxed{aRb \quad bRa \quad aRa}$

not a sibling/sister to yourself

depends on defn. of divide w/ 0

premise never true

−1, 1

Set Proof: Prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Prove both subsets of each other.

First, lets prove that $A \times (B \cup C) \subseteq (A \times B) \cap (A \times C)$.

Let (x,y) be an arbitrary element of $A \times (B \cup C)$.

Then $x \in A$, $y \in (B \cap C)$ by definition of the Cartesian product.

As $y \in (B \cap C)$, by the definition of intersection, $[(y \in B) \land (y \in C)]$.

Thus as $x \in A$ and $y \in B$, $[(x,y) \in (A \times B)]$ by the definition of Cartesian Produ
Likewise, as $x \in A$ and $y \in C$, $[(x,y) \in (A \times C)]$ by the definition of Cartesian Product.

As $[((x,y) \in (A \times B)) \land ((x,y) \in (A \times C))]$, by the definition of intersection, $(x,y) \in (A \times B) \cap (A \times C)$. Thus, we have proven that any arbitrary element (x,y) of $A \times (B \cup C)$ is also in $((A \times B) \cap (A \times C))$.

Thus, $A \times (B \cup C) \subseteq (A \times B) \cap (A \times C)$.

Second, lets prove that $(A \times B) \cap (A \times C) \subseteq A \times (B \cup C)$.

Let (x,y) be an arbitrary element of $(A \times B) \cap (A \times C)$. By the definiton of intersection, $(x,y) \in (A \times B) \land (x,y) \in (A \times C)$.

By the definition of the Cartesian Product, we can conclude that $x \in A$ and $y \in B$ and $y \in C$. As $y \in B$ and $y \in C$, by the definition of intersection, $y \in B \cap C)$. Thus, as $x \in A$ and $y \in (B \cap C)$, by definition of the Cartesian product, $(x,y) \in [A \times (B \cap C)]$.

Thus we have proven any arbitrary element of $(A \times B) \cap (A \times C)$ is also in $A \times (B \cap C)$.
Thus, $[(A \times B) \cap (A \times C)] \subseteq [A \times (B \cap C)]$.

As both sets are subsets of each other, they must be equal.

Suppose R_1, R_2 equivalence relations on set S. Then,

a) $R_1 \cup R_2$ is NOT nec. an equivalence relation (not nec. transitive)

b) $R_1 \cap R_2$ IS an equivalence relation

c) $R_1 \oplus R_2$ is NOT an equivalence relation (esp. not reflexive)

- K_n is bipartite only for $n = 2$ (not enough vertices w/ $n = 1$)
- C_n is defined for $n \geq 3$. C_n is bipartite iff n is even – otherwise not
- W_n is bipartite only for $n = 1$ $(W_1 = K_2)$
- Q_n is bipartite $\forall n \geq 1$ – division on even # of ones vs. odd # of ones (when vertices represented as bitstrings)
  Each vertex connects to all vertices w/ exactly one bit flipped

ONTO / ONE-TO-ONE

a) $f: \mathbb{R} \to \mathbb{Z}$ with $f(x) = \lfloor x \rfloor$

- First, lets prove that $f(x)$ is $\boxed{\text{NOT one-to-one}}$. For a function f to be one to one w/ domain $\mathbb{R}$, $\forall a,b \in \mathbb{R}\ (f(a) = f(b) \to a = b\}$. Negating this statement,
$[\neg \forall a,b \in \mathbb{R}\ ((f(a) = f(b)) \to (a = b))] \equiv [\exists a,b \in \mathbb{R}\ ((f(a) = f(b)) \land (a \neq b))]$.
Consider $a = 0.1$ and $b = 0.2$, noting that $a,b \in \mathbb{R}$ (domain).
Then $f(a) = \lfloor a \rfloor = \lfloor 0.1 \rfloor = 0$, and $f(b) = \lfloor b \rfloor = \lfloor 0.2 \rfloor = 0$.
Thus $f(a) = f(b)$, but $a \neq b$; therefore f is not one-to-one.
- Now, let's prove that $\boxed{f \text{ is onto.}}$ For a function f to be onto w/ domain $\mathbb{R}$ and codomain $\mathbb{Z}$, $[\forall b \in \mathbb{Z}, \exists a \in \mathbb{R}\ (f(a) = b)]$. Let b be an arbitrary codomain element. Define $a = b + 0.1$, noting that $a \in \mathbb{R}$ and $b < a < b+1$.
Then $f(a) = \lfloor b + 0.1 \rfloor = b$ (as $b < a < b+1$). Thus, as $f(a) = b$ for an arbitrary codomain element b, f is onto.

b) $f: \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 5 - 3x$.

- First, let's prove that $f(x)$ is one-to-one. For a function f to be one-to-one w/ domain $\mathbb{Z}$ $\forall a,b \in \mathbb{Z}, [(f(a) = f(b)) \to (a = b)]$. For arbitrary a, b, assume $f(a) = f(b)$
$f(a) = f(b)$
$5 - 3a = 5 - 3b$
$-3a = -3b$
$a = b$.
Thus, as $f(a) = f(b)$ implies $a = b$ for arbitrary a, b, we can conclude that $f(x)$ is one-to-one.
- Second, let's prove that $f(x)$ is not onto. For a function f to be onto w/ domain $\mathbb{Z}$ and codomain $\mathbb{Z}$, $[\forall b \in \mathbb{Z}\ \exists a \in \mathbb{Z}\ [f(a) = b]]$.
Negating this statement, $\neg \forall b \in \mathbb{Z}, \exists a \in \mathbb{Z}\ [f(a) = b] \equiv \exists b \in \mathbb{Z}\ \forall a \in \mathbb{Z}\ [f(a) \neq b]$.
Consider $b = 0$. Then for $f(x)$ to be onto, $f(a) = 0$ for some a in domain $\mathbb{Z}$.
$f(a) = 0$
$5 - 3a = 0$
$3a = 5$
$a = 5/3$.
So for $f(a) = b$, $a = 5/3$; but $5/3$ is not in the domain $\mathbb{Z}$.
Thus there is no a-value in the domain such that $f(a) = 0$, thus $f(x)$ is not onto.

- Prove $2^n > n^2$ if $n > 5$.

Let $P(n)$: $2^n > n^2$

BASE CASE: LHS: $2^n = 2^5 = 32$
RHS: $n^2 = 5^2 = 25$
As LHS > RHS, $P(5)$ holds true

IS: Assume $P(k)$ is true: $2^k > k^2$.
We want to show $P(k+1)$ is true: $2^{k+1} > (k+1)^2$

$$
\begin{aligned}
2^{k+1} &= 2(2^k) \\
&> 2k^2 && \text{(By the IH)} \\
&= k^2 + k^2 \\
&> k^2 + 4k && \text{(As } k > 4) \\
&> k^2 + 2k + 1 && \text{(As } 2k > 1 \text{ as } k > 4) \\
&= (k+1)^2
\end{aligned}
$$

So, $P(k+1)$ holds true.

Thus, as $P(k) \to P(k+1)$ and $P(5)$ holds true, we have proved $P(n)$ is true for $n > 4$ by mathematical induction.

Use **strong induction** to show that every positive integer $n \geq 2$ can be written as a product of one or more primes. *Note*: the primes do not necessarily need to be distinct.

Hint: Consider the case where n is prime and the case where n is composite separately.

Solution: Inductive Step: Assume that every positive integer $2, 3, 4, \ldots, k$ can be written as the product of one or more primes. In other words, for all $2 \leq j \leq k$, j can be written as the product of one or more primes. We want to prove that $k+1$ can also be written as the product of one or more primes.

If $k+1$ is prime, we can write $k+1$ as the product of one or more primes (itself).
If $k+1$ is composite, then there exist at least two (not necessarily distinct) factors j_1, j_2 of $k+1$ such that $j_1 \cdot j_2 = k+1$ and neither j_1 nor j_2 is equal to 1 (by definition of composite). Since $2 \leq j_1 \leq k$ and $2 \leq j_2 \leq k$, j_1 and j_2 can both be written as the product of one or more primes by the inductive hypothesis. We can combine these products to write $k+1$ as the product of one or more primes.

Base Case: let $n = 2$, 2 is a prime number, so 2 can be written as the product of one or more primes (itself).

By strong induction, we have shown that every positive integer $n \geq 2$ can be written as the product of one or more primes.

Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Solution: We claim that it takes exactly $n-1$ breaks to separate a bar (or any connected piece of a bar obtained by horizontal or vertical breaks) into n pieces. We use strong induction.

Inductive Hypothesis Assume the strong inductive hypothesis, that the statement is true for breaking into k or fewer pieces.

Inductive Step Consider the task of obtaining $k+1$ pieces. We must show that it takes exactly k breaks. The process must start with a break, leaving two smaller pieces. We can view the rest of the process as breaking one of these pieces into $i+1$ pieces and breaking the other piece into $k-i$ pieces, for some i between 0 and $k-1$, inclusive. By the inductive hypothesis it will take exactly i breaks to handle the first piece and $k-i-1$ breaks to handle the second piece. Therefore the total number of breaks will be $1 + i + (k-i-1) = k$, as desired.
Base Case If $n = 1$, this is trivially true (one piece, no breaks).