Common Formulas

- Alphabet: some finite set of symbols
- A word ∈ Alphabet: some sequence of letters from A. This is a function mapping position to symbol @ position
- Subsets: 2^n for n-elt set
- Words of length k: n^k for n-len alphabet
- Falling Factorial: $(n)_k = n(n-1)\dots(n-k+1)$
  - $(n)_k$ words w/o repetition over len-n alphabet $\binom{n}{k}k! = (n)_k$
- Multiset: Each elt. has some "count". # perms $= \binom{n}{n_1, n_2, \dots, n_k} = \frac{n!}{n_1!\, n_2! \dots n_k!}$
- Ordered Set Partitions: Bijection to multisets, this is $\binom{n}{n_1, n_2, \dots, n_k}$ for k parts of size $n_1, \dots, n_k$
- Lattice Paths: $\binom{m+n}{m}$ paths from $(0,0)$ to (m,n)
- Pascal's Recurrence: $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
- Weak Compositions: n w/ k parts.
  # of sols to $a_1 + \dots + a_k = n$.
  $\binom{n+k-1}{k-1}$ (for $a_i \geq 0$).
  For $a_i \geq 1$, $\binom{n-1}{k-1}$

If $f: A \to B$, $\#f^{-1}(b) = d \Rightarrow \#A = d\#B$

Newton's Binomial Thm:

For $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

Eg: $\binom{-n}{k} = \binom{n+k-1}{k}(-1)^k$, $n > 0$

So, we define $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$

Thm: For non-formal power series, $|x| < 1$: the above holds for subbing in #s.

Thm: Generating func for $a_k = \binom{n+k-1}{k}$ is $(1-x)^{-n} = \sum_{k \geq 0} \binom{-n}{k}(-x)^k$.

Note: This is WEAK COMPOSITIONS of n into k

Stirling Numbers of the First Kind:

- $c(n,k) = \#\{w \in S_n \mid c(w) = k\}$ (cycles)
- $\sum_{w \in S_n} x^{c(w)} = \prod_{i=1}^{n}(x+i-1) = \sum_{k \geq 1} c(n,k)x^k$

  $c(n,n) = 1$, $c(n,1) = (n-1)!$
- $(n,k) = C(n-1,k-1) + (n-1)C(n-1,k)$

$\sum C(n,k)x^k = x(x+1)\dots(x+n-1)$

$\sum C(n,k)x^k = (x+n-1)[x(x+1)\dots(x+n-2)]$

$\sum C(n,k)x^k = x\sum_k C(n-1,k) + (n-1)\sum_k C(n-1,k)$

$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k)$

Signed Version: $s(n,k) = (-1)^{n-k}c(n,k)$

$$\sum_k s(n,k)x^k = (x)_n = x!/(x-n)!$$
$$= (-1)^n \sum_k (-1)^k c(n,k)x^k$$
$$= (-1)^n \sum_k c(n,k)(-x)^k$$
$$= (-1)^n(-x)(-x+1)\dots(-x+n-1)$$
$$= (x)(x-1)(x-2)\dots(x-n+1) = (x)_n$$

Note: $(x)_n$ is NOT $S(n,k)$ — that just represents the generating function

Binomial / Multinomial Thm:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

$$(x_1 + \dots + x_j)^n = \sum_{\sum n_i = n}\binom{n}{n_1, \dots, n_j}x_1^{n_1}x_2^{n_2}\dots x_j^{n_j}$$

Prop: $2^n = \sum_{k=0}^{n}\binom{n}{k}$

Combinatorial Pf: Power Set ⟺ Subsets of size k

Binomial Thm Pf: $x=1$, $y=1$.

Prop: $0 = \sum_{k=0}^{n}(-1)^k\binom{n}{k}$

Combinatorial: Show a bijection btwn even-elt subsets and odd-elt subsets of [n]. Any even subset maps to odd one by removing {1} if it contains it, and adding {1} if it doesn't.

Algebraic: Binomial Thm; $x = -1$, $y = 1$

Prop: $\sum_{k=0}^{\infty}\binom{n}{k}\binom{m}{\ell-k} = \binom{n+m}{\ell}$

Combinatorial: Choose ℓ people for a committee from n men and m women.

Algebraic: $(x+y)^{n+m} = (x+y)^n(x+y)^m$

$$\sum_{\ell=0}^{n+m}\binom{n+m}{\ell}x^\ell y^{n+m-\ell} = \sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}\sum_{j=0}^{m}\binom{m}{j}x^j y^{m-j}$$
$$= \sum_{i=0}^{n}\sum_{j=0}^{m}\binom{n}{i}\binom{m}{j}x^{i+j}y^{m+n-i-j}$$

$i + j = \ell$;

$$\binom{m+n}{\ell}x^\ell y^{n+m-\ell} = \sum_{i=0}^{n}\binom{n}{i}\binom{m}{\ell-i}x^\ell y^{m+n-\ell}$$

Prop: $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$

Combinatorial: $\binom{2n}{n}$ is # of lattice paths from $(0,0)$ to (n,n). Partition this set by where each path crosses the diagonal from $(0,n) \to (n,0)$.

Algebraic: Let $n = m = \ell$ in prev. prop.

Prop: $\sum_k k\binom{n}{k} = n2^{n-1}$

Combinatorial: Choose a committee from n ppl of any size k and then choose a chairperson ⟺ choose chairperson first, then remaining members

Algebraic: $(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$

$$n(x+1)^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}x^{k-1}$$
$$n2^{n-1} = \sum_{k=0}^{n}k\binom{n}{k}$$

Stirling #s of Second Kind (2^{nd})

$S(n,k)$ = # unordered partitions of [n] into k non-empty blocks.

- $S(n,k) = S(n-1,k-1) + kS(n-1,k)$
  - Recurse on n^{th} number — n in own block, or in one of k possible blocks.

  $S(n,1) = 1$, $S(n,n) = 1$
- Thm: $x^n = \sum_{k=1}^{n}S(n,k)(x)_k$

  Note: kinda the opposite of $s(n,k)$.
- x^n is # of ways to color [n] in x colors
- Partition [n] into k blocks s.t. blocks ↔ colors. → $(x)_k$ to assign x colors to k blocks.

Prop: $\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \sum_{j=k}^{n}\binom{j}{k} = \binom{n+1}{k+1}$

RHS is coeff of x^k in $\sum_{j=k}^{n}(1+x)^j$.

$(1+x)^k + (1+x)^{k+1} + \dots + (1+x)^n$. This is a geometric series w/ $n-k+1$ terms and ratio $(x+1)$.

$(1+x)^k\frac{(1+x)^{n-k+1}-1}{x}$, so coeff of x^k is $\binom{n+1}{k+1}$.

Sum of Squares:

$$\sum_{j=1}^{n}j^2 = \sum_{j=1}^{n}\sum_{k=1}^{2}S(n,k)(j)_k = \sum_{j=0}^{n}\binom{j}{1} + 2\binom{j}{2}$$
$$= \sum_{j=0}^{n}\binom{j}{1} + 2\sum_{j=0}^{n}\binom{j}{2} = \binom{n+1}{2} + 2\binom{n+1}{3}$$
$$= n(n+1)(2n+1)/6$$

Generating Functions

- For sequence $(a_0, a_1, \dots)$ of integers, $f(x) = \sum_{n \geq 0}a_n x^n$ is the generating func.
- Don't think of x as numbers — interested in coeff.
- $\sum_i a_i x^i \sum_j b_j x^j = \sum_k \sum_{\ell=0}^{k}a_\ell b_{k-\ell}x^k$
- $1 + x + x^2 + \dots = \frac{1}{1-x}$ IMPORTANT

Constructing Gen Funcs

- Want to count some set A
- Define α, a weight fn on A that maps A to ℤ
- This defines formal power series $\sum_{a \in A}x^{\alpha(a)}$
- this is equal to $\sum_{n=0}^{\infty}a_n x^n$

  a_n: # of things in A that map to n with α
  - So, α is typically obvious b/c we want some specific a_n
- Multiplication principle: $A \longleftrightarrow B_1 \times B_2 \times \dots$

Then, if you have $\alpha: A \to \mathbb{Z}$, $\beta_i: B_i \to \mathbb{Z}$ s.t. for $a \longmapsto (b_1, b_2, \dots)$ you have $\alpha(a) = \beta_1(b_1) + \beta_2(b_2) + \dots$

Then,

$$\sum_{a \in A}x^{\alpha(a)} = \prod_i \underbrace{\left(\sum_{b_j \in B_i}x^{\beta_i(b_j)}\right)}_{\text{gen func of each } B_i}$$

Eg: How many poker hands (5) from double deck of cards?

A = {set of all hands from double-deck}

$\alpha: A \to \mathbb{Z}$ is # cards in hand

So, coeff of x^n is # of n-card hands; note $n = n_1 + n_2 + \dots + n_{52}$ (# of each card).

So, we define B_i: identity, where each $B_i = \{0, 1, 2\}$. So, gen func for A is $(1 + x + x^2)^{52}$.

Counting Inversions:

For a perm $w \in S_n$, # of pairs (i,j) s.t. $i < j$ and $w_i > w_j$ is # of inversions. $inv(w)$ = # inversions in w

$$\sum_{w \in S_n}x^{inv(w)} = (1+x)(1+x+x^2)\dots(1+x+\dots+x^{n-1})$$
$$= \prod_{k=0}^{n}\frac{1-x^k}{1-x} \quad \text{as } 1-x^k = (1-x)(1+x+\dots+x^{k-1})$$

Pf: The code of a perm is a tuple $(c_1, \dots, c_n)$, where c_k = # inversions of (k, ?)

So, $inv(w) = \sum c_i$, $c_k \in \{0, 1, \dots, k-1\}$

There exists a bijection b/w perms and code.

So, $\sum_{w \in S_n}x^{inv(w)} = (x^0)(x^0+x^1)\dots(x^0+x^1+\dots+x^{n-1})$

Counting Cycles:

Can think of a permutation as a bijection from [n] to [n].

$$\begin{matrix}1 & 2 & 3 & 4 & 5\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ 3 & 5 & 1 & 4 & 2\end{matrix} \qquad 1 \rightleftarrows 3 \qquad 4 \circlearrowleft \qquad 2 \rightleftarrows 5$$

Let $c(w)$ be # of cycles in w.

Then, $\sum_{w \in S_n}x^{c(w)} = x(x+1)\dots(x+n-1)$.

At each step from $j = n$ to 1, b_j = the "parent" of element j in cycle (then remove j).

So, $b_1 = 0$ always. Furthermore, # of 0's is # of cycles, and $b_i \in \{0, \dots, i-1\}$.

Then, $\beta_i = 0$ if $b_i \neq 0$; 1 if $b_i = 0$.

Note $c(w) = \beta_1(b_1) + \beta_2(b_2) + \dots + \beta(b_n)$.

So, weight-gen func for B_i: $x^1 + (i-1)x^0$.

So, $x^{c(w)} = \prod_{i=1}^{n}(x+i-1)$.

Solving Linear Recurrences:

Given $h_n + a_1 h_{n-1} + \dots + a_k h_{n-k} = 0$,

$q^k + a_1 q^{k-1} + \dots + a_k = 0$

is called the characteristic equation.

Let $q_1, \dots, q_r$ be the roots w/ multiplicities $m_1, \dots, m_r$. Then,

$h_n = C_1(n)q_1^n + \dots + C_r(n)q_r^n$

is the solution, where $C_i(n)$ is a polynomial of degree $< m_i$ ($\leq m_i - 1$).

Then, you can solve for these coefficients using the initial conditions.

Eg: Solve $h_n = 3h_{n-2} + 2h_{n-3}$. $h_0 = h_1 = 0$, $h_2 = 1$.

$x^3 - 3x + 2 = 0$.

$(x+1)^2(x-2) = 0 \to x = -1\ (2),\ 2\ (1)$

$h_n = (C_0 n + C_1)(-1)^n + C_2 2^n$

$\to C_1 + C_2 = 0$, $2C_0 + C_1 + 4C_2 = 1$

$\to 2C_2 - C_0 - C_1 = 0$ (Init. Cases)

$C_0 = \frac{1}{3}$, $C_1 = -\frac{1}{9}$, $C_2 = \frac{1}{9}$

So, $h_n = [(3n-1)(-1)^n + 2^n]/9$

Difference Tables:

- $\Delta h(n) = h(n+1) - h(n)$.
- $\Delta^k h(n) = \Delta(\Delta^{k-1}h)(n)$
- Let $H(x) = h(0) + h(1)x + h(2)x^2 + \dots$
- $\Delta h = h(0) + (h(1) - h(0))x^1 + \dots$
  $= (1-x)H(x)$.
- Furthermore, coeff. of x^{n+k} in $(1-x)^k H(x)$ is $\Delta^k h(n) = (1-x)^k H(x)$
  $= \sum_{i=0}^{k}(-1)^i\binom{k}{i}x^{n+k-i}$
- A sequence h is given by a polynomial of degree $\leq k-1$ iff k^{th} diffs go to 0.
- $h(n) = \sum_{k=0}^{n}\Delta^k h(0)$

Shift-by-operator

Pf: $h(n) = E^n h(0)$; $(Eh(n) = h(n+1))$

$= (I + \Delta)^n h(0) \quad \Delta h(n) = h(n+1) - h(n)$

$= \sum_{k=0}^{n}\binom{n}{k}\Delta^k h(0)$.

So, can reconstruct polynomial w/ leftmost diffs:

$\boxed{1}$	−2	−5	4	37	106
$\boxed{-3}$	−3	9	33	69	
$\boxed{0}$	12	24	36		
$\boxed{12}$	12	12			

$$h(n) = \binom{n}{0} - 3\binom{n}{1} + 12\binom{n}{3}$$
$$= 1 - 3n + 2n(n-1)(n-2)$$
$$= 2n^3 - 6n^2 + n + 1.$$

Lookup Tables:

Catalan Numbers: 1, 1, 2, 5, 14, 42, 132...

→ BS: |||---, ||-|--, ||--|-, |-||--, |-|-|- (n = 3)

Pascal's Triangle: $\binom{n}{k}$

n = 1: 1
n = 2: 1 1
1 2 1 (k = 2)
1 3 3 1 (k = 3)
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Stirling #s (1st)

n = 1: 1 (k = 1)
n = 2: 1 1
n = 3: 2 3 1
6 11 6 1
24 50 35 10 1
120 274 225 85 15 1
...

Right Parent Contribs $(n-1) \times$

Stirling #s (2^{nd})

n = 1: 1 (k = 1)
n = 2: 1 1 (k = 2)
1 3 1 (k = 3)
1 7 6 1 (k = 4)
1 15 25 10 1
1 31 85 65 15 1

Right parent contribs k (value)

Catalan Numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n+1}$$

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

Thm: # of circular arrangements of $(n+1)$ 1's and $(n-1)$ -1's is $C_n = \frac{1}{2n+1}\binom{2n+1}{n+1}$.

This is just counting circ perms after proving that all cyclic shifts of $(n+1)$ 1s and n -1's are distinct (consider for contradiction that this doesn't hold; then, suppose shift is by k. Then, concatenate k copies of perm and count # 1s in 2 ways).

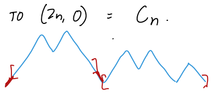
Ballot Sequences: A sequence of length $2n$ with n 1's and n -1's s.t. each initial segment has at least as many 1's as -1's.

Thm: # of ballot seq of len n $= \frac{1}{n+1}\binom{2n}{n}$

Dyck Paths: From $(0,0)$ to $(2n, 0)$ — contained in $y \ge 0$; steps ↗, ↘.

Dyck Paths to $(2n, 0)$ $= C_n$.

Recurrence:



Split on 1st time it hits x-axis (second doesn't work).

Triangulations of an $(n+2)$-gon:

of ways to triangulate $(n+2)$-gon $= C_n$

- Consider a defining edge e.
  - in exactly 1 triangle. When you remove this triangle, left w/ k-gon, $(n+1-k)$-gon
  - So, $C_n = \sum_i C_i C_{n-i-1}$.

of Plane Trees on $(n+1)$ Nodes:

Plane Tree: Has distinguished vertex v (root). P has sequence of smaller $(P_1, P_2, \dots, P_m)$ s.t. $\#P - 1 = \sum_{i=1}^{m} \#P_i$

Claim: C_n plane trees w/ $n+1$ nodes.

Pf: Note that a plane tree w/ $n+1$ nodes has n edges. Then, we can do a DFS of tree, marking a 1 when we go down an edge and a -1 when we come back up. This is a ballot seq!

HW Problems:

1.4: How many ways to choose (A, B) of subsets of $[n]$ such that $A \cap B \neq \emptyset$? $\boxed{4^n - 3^n}$

Insight: 4^n total ordered pairs w/ no restriction. 3^n pairs w/ $A \cap B = \emptyset$, as each elt has 3 choices.

1.6: How many sets of 3 numbers $\in [20]$ s.t. no consecutive #s in a set?

Soln: Bijection to words over $\{0,1\}$ of length 20 w/ exactly 3 non-consec 1s. Once 0's are placed, $\binom{18}{3}$ ways to place 1s. (This is strong comps of 19 into 3).

2.2: Prove $\sum_{i=0}^{k} \binom{n}{i}(-1)^i = \binom{n-1}{k}(-1)^k$ for $0 \le k \le n$

$$(1-x)^{n-1} = (1-x)^n \cdot (1-x)^{-1}$$

$$\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k x^k = \left(\sum_{k=0}^{n}\binom{n}{k}(-1)^k x^k\right)(1 + x + x^2 + \dots)$$

$$\binom{n-1}{k}(-1)^k = \sum_{i=0}^{k}\binom{n}{i}(-1)^i$$ (coeff of x^k on both sides)

2.5 By integrating the Binomial Thm, evaluate:

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \dots + (-1)^n \frac{1}{n+1}\binom{n}{n}$$

$$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$\frac{1}{n+1}(x+1)^{n+1} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}x^{k+1} + C \Big|_{x=-1}^{0}$$

$$\frac{1}{n+1} = 0 - \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}(-1)^{k+1}$$

$$\frac{1}{n+1} = \sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}$$ = original sum

KEY: DON'T FORGET CONSTANT OF INTEGRATION

Inclusion-Exclusion Principle:

$$|S - (A_1 \cup A_2)| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$$\left|S - \bigcup_{i\in[n]} A_i\right| = \sum_{j=0}^{n}(-1)^j \sum_{I \in \binom{[n]}{j}} \left|\bigcap_{i \in I} A_i\right|$$

↳ $|S|$ - (1-way ∩) + (2-way ∩) - (3-way ∩) + ...

Eg: Count the number of surjective functions from $X = [n]$ to $Y = [k]$.

$S = \{\text{functions from } X \to Y\}$ ($\#S = k^n$)

$A_i = \{\text{fns } f: X \to Y \text{ s.t. } i \text{ is not in } Im(f)\}$, $1 \le i \le k$

Note $\#A_i = (k-1)^n$.

Also, $\#\{j\text{-wise intersection}\} = (k-j)^n$.

So, this is: $\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n$.

Corollary: (Stirling Numbers)

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}(-1)^j(k-j)^n$$

As $S(n,k)$ is # of unordered parts into k nonempty blocks, can think of it like a surjective function ($1/k!$ for unordered).

Euler's Totient Function:

$\phi(n) = \#\{k \in [n] : k \text{ relatively prime to } n\}$

$\phi(n) = n\prod_{j=1}^{r}(1 - 1/p_j)$

if n has prime factors $p_1, p_2, \dots, p_r$

Count Weak Comps of 7 w/ 7 parts s.t. no part = 2.

S = All weak comps $\to \binom{n+k-1}{k-1}$

$A_i = \{x \in S \mid x_i = 2\}$, $1 \le i \le 7$

$\binom{10}{5}$ weak comps w/ $x_i = 2$

$\binom{7}{4}$ weak comps w/ $x_i, x_j = 2$, etc.

Derangement: $w \in S_n$, $\forall i \in [n]$, $w_i \neq i$

S = All perms

$A_i = w \in S$ s.t. $w_i = i$

So, # derangements $= n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$

Note that $d_n/n! \to 1/e$ as $n \to \infty$

Count Perms w/ Ascents / Descents

S: All perms w/ just ascent conditions satisfied

A_i: i^{th} descent condition violated

Idea: Conditions going one way easy to count:

$w_1 < w_2 < w_3$, $w_5 < w_6 < w_7$ $\to \binom{7}{3}\binom{4}{3}$

Problem 8. For $n \ge 0$, prove that

$$\sum_{k=1}^{n} c(n,k)x^{n-k} = \prod_{k=1}^{n-1}(1+kx),$$

HW 3.8

where $c(n,k)$ is the signless Stirling number of the first kind.

Solution. We know that

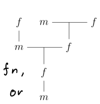
$$\sum_{k=1}^{n} c(n,k)x^k = \prod_{k=0}^{n-1}(x+k).$$

Letting $x \mapsto x^{-1}$ and multiplying everything by x^n, we get

$$\sum_{k=1}^{n} c(n,k)x^{n-k} = x^n\left(\sum_{k=1}^{n} c(n,k)x^{-k}\right) = x^n\prod_{k=0}^{n-1}(x^{-1}+k) = \prod_{k=0}^{n-1}(1+kx).$$

(6) Find the recurrence relation for the number a_n of bees in the nth previous generation of a male bee, if a male bee is born asexually from a single female and a female bee has the normal male and female parents. The ancestral chart below shows that $a_1 = 1, a_2 = 2, a_3 = 3$.

HW 4.6



KEY: Partition recurrence into two recurrences!

Note that $a_n = m_n + f_n$, as each bee is male or female. Then, $m_n = f_{n-1}$ (as each female in the next gen must have a male parent), and $f_n = m_{n-1} + f_{n-1} = a_{n-1}$ as both males and females in next gen require a female parent. So, $a_n = f_{n-1} + a_{n-1} = a_{n-2} + a_{n-1}$.

(7) Prove that the number of permutations $w_1 \cdots w_n \in S_n$ satisfying the condition

(∗) there are no indices $i < j < k$ for which $w_k < w_i < w_j$ 5.7

is a Catalan number. For example, when $n = 3$, all permutations in S_3 satisfy (∗) except 231.

We will partition these on w_1. Let $w_1 = m$. Then, as $w_k < w_1 < w_j$ for all $j < k$, we must have $(m+1 \dots n)$ appearing before $(1 \dots m-1)$. So, we can partition the remainder of the permutation into two parts, each of which individually satisfy this condition. So, summing over all choices for w_1, we have

$$T(n) = \sum_{k=1}^{n} C_{k-1}C_{n-k} = \sum_{i=0}^{n-1} C_i C_{n-i-1}.$$

This is precisely the recurrence for the Catalan numbers.

Partitions:

A partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ such that $\lambda_1 \ge \lambda_2 \ge \dots \ge 0$ (implies $\lambda_i = 0$ for all $i \ge i'$) and $\sum \lambda_i = n$.

- Can think of this as an unordered weak comp.

Thm: $$\sum_{n\ge0} p(n)x^n = \prod_{k\ge1}(1 + x^k + x^{2k} + \dots) = \prod_{k\ge1}\frac{1}{1-x^k}$$

Pf: Let A be set of all partitions. $\alpha(a) = n$ for $a \in A$.

Then, $a = (b_1, b_2, \dots)$, where b_i = contribution from blocks of size i.

So, $B_1 = \{0, 1, 2, 3 \dots\}$, $B_2 = \{0, 2, 4, \dots\}$, etc.

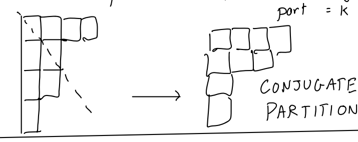
So, $\sum_{B_i} x^{B_i(b_i)} = 1 + x^i + x^{2i} + \dots$

Thm: Gen func w/ each part $\le k$ is $\frac{1}{(1-x)(1-x^2)\dots(1-x^k)}$ (sets B from $1 \to k$)

Thm: Gen func w/ all distinct parts: $(1+x)(1+x^2)(1+x^3)\dots$

Thm: # of partitions s.t. each part odd equals # of partitions into distinct parts.

Thm: # of parts $= k$ ⟺ len of longest part $= k$



6.3 Let $F(n,k)$ be # of partitions of $[n]$ into k blocks such that each block contains at least 2 elements. Express F in terms of $S(n,k)$ — Stirling #s of 2^{nd} kind

S := All partitions into k nonempty $(S(n,k))$

A_i := $x \in S$ s.t. i is in 1-elt block

Note $\#A_i = S(n-1, k-1)$.

$\#(j\text{-wise} \cap \text{ of } A_i\text{s}) = S(n-j, k-j)$.

So, $F(n,k) = \sum_{j=0}^{k}\binom{n}{k}S(n-j, k-j)(-1)^j$

5.6 Show that the number of sequences $1 \le a_1 \le a_2 \le \dots \le a_n$ such that $a_i \le i$, $1 \le i \le n$ is C_n.

Bijection to ballot seq: Let $a_i' = a_i + i - 1$. Then, a' describes positions of 1 in B.S.

Prove correctness of f (every sequence actually produces a B.S.) and f^{-1} (every B.S. can be transformed to seq).

Idea: each term describes how many "off" from earliest position each 1 is