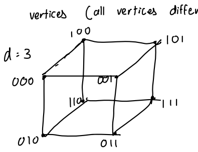
Graphs: Basic Definitions

- nonempty set V of vertices
- finite set E of edges
- function $\varphi: E \to \{\{u,v\} : u,v \in V\}$ associating edge w/ unordered endpoints
- Simple: no loops / multiple edges
- Path Graph: Just a line
- Cycle Graph: Path w/ closing edge
- Complete Graph (K_n): n vertices, all vertices have edge to all other
- Complete Bipartite Graph ($K_{m,n}$): $mn/2$ edges, everything connected to all else.
- Degree of vertex: # of incident edges
  - count self loop twice
  - $\sum$ degrees $= 2|E|$
    - ⇒ # of vertices of odd degree is even
  - A graph is regular if every vertex has the same degree
- Isomorphism: Same graph, diff labels
  - if any graph-theoretical prop. violated, not an isomorphism
- Walk: Any sequence of vertices s.t. adjacent vertices have edge b/w them
  - not as restrictive as path, allow repeats
  - If $v_0 = v_k$ (start, end at same point) this is a closed walk
  - Relation $\sim$ on V is defined as $u \sim v$ if there is u-v walk (this is equivalence relation — sym, reflexive, transitive)
  - → # of classes = # of connected components

Gray Codes

- The cube graph of degree d is 1-skeleton of d-dim cube
  - make 2 copies of graph, connect corresponding vertices (all vertices differing in 1 bit connected)



- Gray Code: HamCycle in cube graph of degree d
  - the reflected gray code of time $b_1 b_2 b_3 \dots b_d$ (time in binary) is defined as: $G(0) = 0$ $G(1) = 1$

$$G(b_1, \dots, b_d) = \begin{cases} 0 \mid G(b_2, \dots, b_d) & \text{if } b_1 = 0 \\ 1 \mid G(\bar b_2, \dots, \bar b_d) & \text{if } b_1 = 1 \end{cases}$$

- $G(b_1, \dots, b_d) = (b_1, b_1 \oplus b_2, \dots, b_{d-1} \oplus b_d)$
- In general: Generating combinatorial objs. is enumerating s.t. adjacent ones differ as little as possible

Flows

- Transportation Network is directed G with $c: E \to \mathbb{R}_{\ge 0}$ (capacity func)
  - source s w/ no incoming edges
  - target t w/ no outgoing edges
  - $c(e) \in \mathbb{Z}$
  - no multiple edges, oriented cycles (optional)

  simplifying assumptions
- Flow: fn $E \to \mathbb{R}$ s.t.:
  - → $0 \le f(e) \le c(e) \quad \forall e \in E$
  - → $\sum_{(*,v)} f(e) = \sum_{(v,*)} f(e) \quad \forall v \in V / \{s,t\}$
- Cut: some set of vertices $\ni s$, $\not\ni t$
  - Capacity of Cut: $\sum_{e\, X \to Y} c(e)$
- Max Flow = Min Cut
- If integral capacities, can always find integral max flow

Trees

- An acyclic graph is called a forest
- A connected forest is a tree
- Lemma: Any two vertices in tree are connected by a unique path
- Lemma: Every tree w/ ≥ 2 vertices has ≥ 1 leaf (deg-1 vertex)
- Lemma: If G is a tree, $|V| - |E| = 1$.
  - induction on # vertices

Euler Characteristic

- 0^{th} Betti #: $b_0(G)$: # of CCs in graph
- 1^{st} Betti #: $b_1(G)$: Max # of edges that can be removed without disconnecting graph (can find efficiently w/ MST)
  - → can think of this as # of independent cycles (not contained in union of 2 other cycles)
- $\chi(G) = |V| - |E| = b_0 - b_1$ for any graph G
  - Eg: if G is a forest $b_1 = 0$ and $b_0 = |V| - |E|$ (# of trees)

Characterization of Trees

Any 3 of these imply the 4^{th}:

(1) G is connected ($b_0 = 1$)
(2) G is acyclic ($b_1 = 0$)
(3) G has n vertices ($|V| = n$)
(4) G has n-1 edges ($|E| = n-1$)

Eulerian Walk

- Walk that goes thru each edge exactly once (7 bridges problem)
- Thm: The following are equiv. for a connected graph G:
  (1) G has closed Eulerian walk
  (2) G is Eulerian
  (3) Every vertex of G has even degree
- Thm: Following are equiv. for connected G:
  (1) G has an Eulerian Walk
  (2) G has either 0/2 vertices of odd degree (not 1 b/c any graph has even # of vertices of odd degree)

Acyclic Orientations

- Acyclic orientation is directed graph gotten by orienting edges of undirected graph s.t. no directed cycle exists.
  - → $ao(G)$:= # of acyclic orientations
  - → Every graph has ≥ 1 AO
  - → Eg: $AO(K_n) = n!$ — take a perm of n, and orient edges from left → right. This acyclic b/c no backwards edge exists.
- In general, at least 1 vertex w/ 0 outgoing edges must exist in any DAG.
- Deletion-Contraction:
  $AO(G) = AO(G-e) + AO(G/e)$
- Thm: $AO(G) = (-1)^{|V|} P_G(-1)$

Ford-Fulkerson:

flow = 0
find s→t path in G_f
set flow along all these edges s.t. saturated
repeat until no s→t path in G_f

- G_f: Residual/Auxillary graph
  - same vertices as G
  - For unsaturated $u \to v \in G$, add $u \to v$ to G_f
  - For non-zero flow $u \to v \in G$, add $v \to u$ to G_f

Planarity

- A graph is planar if it can be drawn in the plane s.t. edges don't cross
- Faces of the graph are connected components of $\mathbb{R}^2 - G$ (for planar only)
  - → always the same # in any planar drawing, but no clear bijection b/w drawings
- Theorem: $|V| - |E| + |F| = b_0 + 1$ for planar G (Note $b_0 = 1$ if connected)
  - can use this to show that some graphs are non-planar
- 1-D skeletons of all polyhedra are planar
- $K_{3,3}$; K_5 are known non-planar
  - show these are subgraphs
- Each edge borders exactly 2 faces, each face must be ≥ 3 sides (if ≥ 3 vertices)
  - So, $3|F| \le 2|E|$ for any planar graph
- Eg: K_5 is non-planar.

Pf: $|V| = 5$, $|E| = 10$. As $|V| - |E| + |F| = 2$, $|F| = 7$. However, $3(7) \not\le 2(10)$ is contradiction

- Eg: $K_{3,3}$ is non-planar.

Notice that every face in $K_{3,3}$ has ≥ 4 sides, So, $4|F| \le 2|E|$. However, $|E| = 9$ and $|V| = 6$, so $|F| = 5$. $4(5) \not\le 2(9)$ is a contradiction.

Ham Cycles, HamPaths

- Ham Cycle: Cycle in a graph that visits each vertex exactly once (except start/end).
  - Ham Path is like HamCycle w/o 1 edge
  - Note: Ham Cycles / Ham Paths UNDIRECTED
- Graph w/ HamCycle is Hamiltonian
- Generally no way to find a HamCycle (NP-Hard), but we can sometimes show that a HamCycle is not possible.
  - Eg: $m \times n$ grid graph is Hamiltonian iff $m \times n$ is even (Pf: bipartite, so 2-colorable. If odd # of vertices, must have 2 adj. of same color)
- The 1-skeleton of any polytope is not nec. Hamiltonian (Herschel's Graph)
  - Polytope is simple if every vertex has degree = 3
  - W. Tutte showed that not all simple 3D polytopes have Hamiltonian skeleton

Grinberg's Theorem

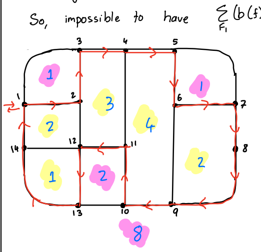
- Let G be a planar graph w/ HamCycle C. F_1: faces inside C; F_2: faces outside C

For $f \in F_1, F_2$, $b(f)$ = # of bounding edges

Then, $\sum_{f \in F_1} (b(f) - 2) = \sum_{f \in F_2} (b(f) - 2) = |V| - 2$

- can use this to prove non-Hamiltonian
- Eg: Grinberg's graph has every inner face a pentagon/octagon, but outer face has 9 edges. $(b(f) - 2)$ is 3/6 for inner, 7 for outer

So, impossible to have $\sum_{F_1} (b(f) - 2) = \sum_{F_2} (b(f) - 2)$.



Vertex Coloring

- (Proper) Vertex Coloring: Assign colors to vertices s.t. adjacent vertices get diff. colors
  - A graph is k-colorable if it can be colored using $\le k$ colors
  - Chromatic # $\chi(G)$ is smallest k s.t. G is k-colorable
  - We assume G is simple (doesn't make a diff if not)
  - Eg: $\chi(K_n) = n$, $\chi(n\text{-cycle}) = 2$ if n even, 3 if n odd
- G is 2-colorable ⟺ G is bipartite
- $\omega(G)$ is size of largest clique in G, $\boxed{\omega(G) \le \chi(G)}$
- $\alpha(G)$ is size of largest independent set in G,

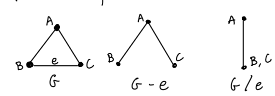
$\boxed{\chi(G) \ge |V| / \alpha(G)}$ • $\boxed{\chi(G) \le \max_{v \in V} \deg(v) + 1}$

Chromatic Polynomials

- Chromatic polynomial of G is function $P_G(k): \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ s.t. $P_G(k)$ = # of vertex colorings w/ $\le k$ colors
  - $\chi(\,)$: smallest k s.t. $P_G(k) \ne 0$.
- Eg: $P_G(k) = \left[k! / n! \right]$ for $G = K_n$
- Eg: $P_G(k) = k(k-1)^{n-1}$ for tree G

Pf: Base Case: $n = 1$, k ways.
Inductive Step: Pick leaf. $k(k-1)^{n-2}$ to cover rest of graph, $(k-1)$ colors for leaf in each scenario

  - Corollary: If G is forest w/ k connected components, $P_G(k) = k^c (k-1)^{n-c}$
- Deletion-Contraction:
  - → $G - e$: Graph with e deleted
  - → G / e: Graph with e contracted (combine u, v)



→ Thm: $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$

- Thm: $P_G(k)$ is monic polynomial in k of deg $|V| = n$
  - → intuitive: Consider $\lim_{k \to \infty} \Pr[\text{Proper Coloring}] = \frac{\#\text{ Proper w/ } k}{\#\text{ colorings w/ } k}$
    - → This converges to 1, and denominator is k^n

- $\hat P_G(a)$: # of ways to color G using exactly a colors
  - Lemma: $\hat P_G(a)$ is not a polynomial as $\hat P_G(a) = 0$ for $a < \chi(G)$, $a > |V|$

    ∞ # of 0s ⇒ not polynomial
- $P_G(k) = \sum_{a=0}^{k} \hat P_G(a) \binom{k}{a} = \sum_{a=\chi(G)}^{|V|} \hat P_G(a) \binom{k}{a}$
- Can recover $\hat P_G(a)$ from $P_G(k)$ w/ diff table

$P_G(k)$: 0 0 2 18 84 260 ...
0 2 16 66 176
2 14 50 110
12 36 60
24 24
0

these are $\hat P_G(a)$

- $G \sqcup H$: Union of G, H (no connections added)
- $G + H$: Join of G, H. Do $G \sqcup H$, and then connect all (g, h), $g \in G$ and $h \in H$.
- $P_{G \sqcup H}(k) = P_G(k) \cdot P_H(k)$
- $\hat P_{G \sqcup H}(a) = \sum_{i+j+h=a} \hat P_G(i+j) \hat P_H(j+k)$
- $P_{G+H}(k) = \sum_a \sum_b \hat P_G(a) \hat P_H(b) \binom{a+b}{a} \binom{k}{a+b}$

  $= \sum_a \sum_b \hat P_G(a) \hat P_H(b) \binom{k}{a,\, b,\, k-a-b}$

Menger's Thm:

- Let G be a directed graph w/ s (no incoming edges), t (no outgoing). Largest # of edge-disjoint $s \to t$ paths is smallest # of edges whose removal disconnects s from t. (consequence of max-flow / min-cut)

Matching: A matching M in a graph is a subset of edges s.t. no two edges in M have a vertex in common. Can think of this as a way to pair up vertices. A matching is called *perfect* if all vertices are "covered" by M.

- **König's Thm**: Max Matching = Min VertexCover

HW 7 Problem 2:

How many HamCycles does K_n have?

- $n!$ permutations of vertices, but cyclical, so $(n-1)!$ Also undirected, so $(n-1)!/2$

HW 7 Problem 3 (CITABLE)

Show if G is a simple planar graph, $|E| \le 3|V| - 6$.

Pf: $|V| - |E| + |F| = b_0 + 1$.

$2|E| \ge 3|F|$

So, $3|F| = 3(|E| + b_0 + 1 - |V|) \le 2|E|$

$$|E| \le 3(|V| - b_0 - 1) \le 3(|V| - 2) \le 3|V| - 6.$$

Show simple planar graph has ≥ 2 vertices of degree ≤ 5.

$$|E| = \tfrac{1}{2} \sum \deg(v) \ge \tfrac{1}{2}(6(|V| - 1)) \ge 3|V| - 3, \text{ Contradiction.}$$

HW 8 #7:

Prove that the signs of the coefficients of the chromatic polynomial alternate.

Pf: Induction on # edges m.

Inductive Step: $G-e$ has $(m-1)$ edges and n vertices. G/e has $\le (m-1)$ edges and $n-1$ vertices.

$P_{G-e} = k^n - c_{n-1}k^{n-1} + c_{n-2}k^{n-2} - \dots (-1)^n c_0$

$P_{G/e} = k^{n-1} - d_{n-2}k^{n-2} + d_{n-3}k^{n-3} - \dots + (-1)^n d_0$

$P_G = P_{G-e} - P_{G/e}$
$= k^n - (c_{n-1}+1)k^{n-1} + (c_{n-2} + d_{n-2})k^{n-2} + \dots$

Pigeonhole Principle

- More than kn items distributed among n bins $\longrightarrow$ at least 1 bin w/ $k+1$
- Usually used to prove existence stmts.
- Eg: Among any 10 pts in unit square, at least 2 points @ distance $< \sqrt{2}/3$.
  - divide into 9 squares, at least one square w/ 2 points.
- Eg: Any simple graph w/ $n \ge 2$ vertices has 2 vertices of same degree.
  - $0, 1, \dots, n-1$ possible degrees. However, cannot be both deg 0 and deg $n-1$ vertex, so $(n-2)$ possible degrees
- Eg: Each cell in 5×41 grid colored red or blue. Prove $\exists$ 3 rows, 3 cols s.t. all intersections are the same color
  - Each column has at least 3 of the same color. $\binom{5}{3}$ ways to choose which 3, 2 colors — so 20 possibilities. One of the patterns must repeat at least thrice as 41 columns

Show that DNE 17 binary strings of length 7 s.t. any 2 differ in at least 3 bits.

- For a binary string w of length 7, consider set consisting of w, strings differing from w in single bit. As $2^7 = 128$ distinct strings and total # elts in these sets $= 8 \cdot 17 = 136$. So, some two sets intersect. Then, those two original binary strings differ in ≤ 2 bits.

HW 9 #4

Let Q_k, $k \ge 2$, denote skeleton of k-dim cube. Prove that Q_k has $\ge 2^{2^{k-2}}$ perfect matchings.

Pf: Induction on k, base case trivial. k dim cube is formed by connecting two $k-1$ dim cubes $\Rightarrow$ any pair of $(k-1)$-perfect matchings works $\Rightarrow 2^{2^{k-3}} \cdot 2^{2^{k-3}} = 2^{2 \cdot 2^{k-3}} = 2^{2^{k-2}}$

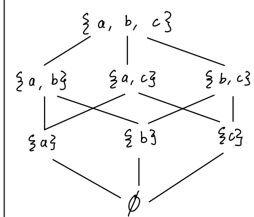
Posets

- A set P with a binary relation $\le$ that is:
  (1) Reflexive; $x \le x \quad \forall x \in P$
  (2) Antisymmetric; $x \le y \wedge y \le x \Rightarrow x = y$
  (3) Transitive; $x \le y \wedge y \le z \Rightarrow x \le z$

This relation is a "partial order".

- Eg: $(\mathbb{R}, \le)$, (Power Set of S, $\subseteq$)

- 2 elements are *comparable* iff $x \le y$ or $y \le x$. If all elements of P are pairwise comparable, $(P, \le)$ is called linearly-ordered
- We can define $<, >, \ge$ in terms of $\le, =, \ne$
- x *covers* y ($x \cdot\!> y$) iff $x > y$ and there is no z s.t. $x > z > y$.
  - multiple elements can cover one elt.
- A finite poset is fully determined by covering relation, which can be depicted in a Hasse diagram.
- Let S be an n-element set. Then, $(2^S, \subseteq)$ is called *Boolean lattice* ($\mathbb{B}_n$).



$\mathbb{B}_3$'s Hasse diagram.

- *Chain*: Any subset of elts. in poset that are pairwise comparable (subset of some path from bottom $\to$ top)
  - $\to$ *maximal* if not contained in any other chain.
  - $\to$ in $\mathbb{B}_n$, $n!$ maximal chains (just what order you add elts. to set
- *Antichain*: A subset of poset s.t. no two elements are comparable
  - $\to$ Eg: Any row in Hasse diagram
- Theorem (Sperner 1928): The largest size of an antichain in $\mathbb{B}_n$ is $\binom{n}{\lfloor n/2 \rfloor}$
  - $\to \binom{n}{\lfloor n/2 \rfloor}$ is middle elt. in Pascal's triangle, size of largest row in Hasse diagram
  - $\to$ Pf: (1) chain, antichain have ≤ 1 elt. in common
    (2) For $0 \le k \le n$, $\binom{n}{k} \le \binom{n}{\lfloor n/2 \rfloor}$ "middle row largest"
    (3) Let $\alpha(x)$ be # of maximal chains containing set x for $x \in \mathbb{B}_n$.
    Then, $\alpha(x) = k!(n-k)! = \dfrac{n!}{\binom{n}{k}} \ge \dfrac{n!}{\binom{n}{\lfloor n/2 \rfloor}}$