CHAPTER 12

Distance Formula (2 points):

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Sketching 3D graphs:

- fix one variable (input value) to get cross section

Dot Product (2 vectors → scalar)

$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$

$\vec{v} \cdot \vec{v} = |\vec{v}|^2$

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutative)

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive)

$(r\vec{v}) \cdot \vec{w} = r(\vec{v} \cdot \vec{w})$ (associative)

$\vec{v} \cdot \vec{0} = 0.$

If $\vec{v} \cdot \vec{w} = 0$, then $\vec{v}$ is perpendicular to $\vec{w}$. (Converse holds too).

$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$

Proj on $\vec{w}$ of $\vec{v}$ ($\text{Proj}_{\vec{w}} \vec{v}$):

$$\boxed{\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}}$$

Line through (x_0, y_0, z_0) with direction vector (a,b,c):

$\vec{\ell}(t) = \langle at + x_0,\ bt + y_0,\ ct + z_0 \rangle$

Can be parametrized in infinitely many ways.

Need:

- 2 points on line
- Point + Direction Vector

PLANE:

- 3 non-collinear pts
- Point + normal vector

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

with $\vec{n} = \langle a, b, c \rangle$

$P = (x_0, y_0, z_0)$

When given 3 points, cross 2 vecs to find normal.

Intersection of two planes:

cross two normals to get direction vector of line; solve algebraically for point

Quadric surface: given by degree 2 equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ELLIPSE

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ HYPERBOLOID OF ONE SHEET

$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{z^2} = 1$ HYPERBOLOID OF TWO SHEETS

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ CONE

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ PARABOLOID

$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ HYPERBOLIC PARABOLOID

Cross Product (2 vec → vec)

$$\vec{v} \times \vec{w} = \left\langle \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\rangle$$

- $\vec{v} \times \vec{v} = \vec{0}$
- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

Distributive, Scalar assoc.

- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin \theta$
- $\vec{v} \times \vec{w}$ perpendicular to $\vec{v}, \vec{w}$.
  - RH rule with $\vec{v} = x$, $\vec{w} = y$, $\vec{v} \times \vec{w} = z$.
- Area of triangle PQR: $\frac{1}{2} |\vec{PQ} \times \vec{PR}|$...

CHAPTER 13

Vector function:
function who's output is a vector; creates space curve
line is special vector function
$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

continuous if component functions continuous
$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$\int_a^b \vec{r}(t) = \langle \int_a^b f(t), \int_a^b g(t), \int_a^b h(t) \rangle$

$\int_a^b \vec{r}'(t) = \vec{r}(b) - \vec{r}(a)$

$r'(t)$ is TANGENT VECTOR to $r(t)$

ARC LENGTH ℓ between $r(a)$, $r(b)$ =

$$\int_a^b |r'(t)|\, dt$$

$|\ |$: MAGNITUDE

Parametrization by arc length:

$$a = \int_0^a |r'(t)|\, dt$$

every curve can be uniquely parametrized by arc length.

- How to find:
  Set s to be equal to $\int_0^t |r'(u)|\, du$.
  Find s in terms of t
  Find t in terms of s
  Substitute s for t in $r(t)$

$\vec{r}(t)$ represents position of moving object @ time t. ($r'(t)$ = velocity
$r''(t)$ = acceleration)

COLLIDE: Set components equal w/ single var
INTERSECT: Set components equal w/ diff vars

MIN DISTANCE (POINT TO POINT):
Use distance formula to find dist. as function of t.
Minimize (can square to minimize).

MIN DISTANCE (TRAJECTORY):
= min distance between two lines
$\vec{d}$ is perpendicular to both dir. vectors
$\vec{\ell}_1 \times \vec{\ell}_2 = \vec{n}$.
Then find plane w/ normal $\vec{n}$ containing $\vec{\ell}_1$. Find distance to random $\vec{\ell}_2$ point using point-to-plane

on formula sheet

CHAPTER 14

$f(x,y)$: func of two variables.

Often $z = f(x,y)$.

Level Curve: $f(x,y) = k$ (cross-section in xy).

Contour Map = collection of level curves.

Get LEVEL SURFACES for $f(x,y,z)$.

Drawing level curves: plug in, figure it out. (≥ 5)

Chain Rule:

$f(x,y)$; $x(t)$, $y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

If z implicitly $f(x,y)$ w/ $f(x,y,z) = C$, then:

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \qquad \frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$

PARTIALS:

Keep all other vars constant, move one var.

$f_{xy} = f_{yx}$ if f_{xy}, f_{yx} continuous.

Estimates: do two-sided, then average

Find signs from contour map:

- first order: just see if level curves increase
- second order: level curves getting closer together or further apart?

f_x −ve, closer : −ve

f_x −ve, further : +ve

f_x +ve, closer : +ve

f_x +ve, further : −ve

f_{xy} test: see f_x at diff y-values.

TANGENT PLANE:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\vec{n} = \langle f_x, f_y, -1 \rangle$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$ (estimate small changes).

Directional Derivative + Gradient:

(1) $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$ (normal to tangent plane)

(2) $D_{\vec{u}}$: Directional Derivative of $f(x,y,z)$ in $\vec{u}$ direction

$$D_{\vec{u}} f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$$

(3) $\nabla f(a,b,c)$: fastest increase

$|\nabla f(a,b,c)|$: max directional derivative

(4) If $x(t)$, $y(t)$, $z(t)$:

$$\frac{df}{dt} = \nabla f \cdot r'(t)$$

Sections 14.6 – 15.4 / materials in previous sections could be indirectly used

14.6 14.7 14.7 14.8 15.1 15.2 15.2 15.3 15.4

Chapter 14

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

$$D_{\vec{u}} f(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$$

is the directional derivative along unit vector $\vec{u} = \langle u_1, u_2, u_3 \rangle$.

- ∇f is vector ; $D_{\vec{u}} f$ is a scalar
- $D_{\vec{i}} f(x,y,z) = f_x$; $D_{\vec{j}} f(x,y,z) = f_y$; $D_{\vec{k}} f(x,y,z) = f_z$
- $D_{\vec{u}} f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$
- $\vec{\nabla f}(a,b,c)$ is direction of fastest increase @ (a,b,c)
- $-\vec{\nabla f}(a,b,c)$ is direction of fastest decrease @ (a,b,c)
- $|\nabla f(a,b,c)|$ = max directional derivative @ (a,b,c)
- $-|\nabla f(a,b,c)|$ = min directional derivative @ (a,b,c)
- $\nabla f(a,b,c)$ is normal vector of tangent plane to level curve $f(x,y,z) = k$ @ (a,b,c)

- $f(x,y)$ has critical pt at (a,b) if:

  $\nabla f(a,b) = \vec{0}$ or undefined

- $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$

  $= \nabla f \cdot r'(t)$

$$H = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

(If f_{xx}, f_{yy} have diff signs, then it is saddle).

(1) If $H > 0$ and $f_{xx} > 0$, (a,b) is a local min

(2) If $H > 0$ and $f_{xx} < 0$, (a,b) is a local max

(3) If $H < 0$, (a,b) is a saddle point

(4) If $H = 0$, the test is inconclusive.

Saddle: 2 intersecting level curves of same value

Max/Min: Concentric circles

Lagrange Multipliers

- $f(x,y,z)$ with constraint $g(x,y,z) = 0$

Solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$; $g(x,y,z) = 0$

Evaluate $f(x,y,z)$ @ all solutions

Compare values from step 2

Types of domains:

- Open: None of boundary included
- Closed: Contains all points on boundary
- Bounded: If domain finite (contained in some large enough circle)

$D = \mathbb{R}^2 \rightarrow$ closed AND open, not bounded

$D = \{(x,y) \mid x,y \geq 0\}$ closed, not bounded

- Extreme Values
  - $f(x,y)$ @ critical points
  - Find min/max on boundary
  - Compare values from S1/S2

hyperbolas on level curves $\rightarrow$ saddle

Can use to find extreme value of functions on boundary of domain for global extrema

To show a func has no max/min, fix variables and show single-variable limit as $x\to\infty$ is ∞ / $-\infty$.

CHAPTER 15

$$\iint_R f(x,y)\,dA = \lim_{m,n\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\,\Delta A$$

(Divide rectangular domain into $m \times n$ squares)

(Multiply sample point in each subregion w/ area of subregion)

Midpoint Rule: Use midpoint of each subregion as sample points.

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Type I Domain: dy dx

Type II Domain: dx dy

- Bounds for outer integral must be const.
- Cannot randomly flip — must draw out to convert

(1) $\iint_D f+g\,dA = \iint_D f\,dA + \iint_D g\,dA$

(2) $c\iint_D f(x,y)\,dA = c\iint_D f(x,y)\,dA$

$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$$

bounds in r, θ — JACOBIAN

$x = r\cos\theta$

$y = r\sin\theta \qquad r^2 = x^2 + y^2$

For lamina w/ density $\rho(x,y)$ over domain D

- Mass: $\iint_D \rho(x,y)\,dA = m$
- COM $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA$$

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA$$

pay attention to symmetry

(3) If $D = D_1 + D_2$:

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

(4) $\iint_D 1\,dA = \text{Area}(D)$

(5) Average Value of $f(x,y)$ on D $\frac{1}{\text{Area}(D)}\iint_D f(x,y)\,dA$

(6) $\iint_D f(x,y)\,dA$ = signed volume of solid between $z = f(x,y)$ and D on xy-plane.

If $f(x,y)$ is odd in x and is symmetric about the y-axis, then $\iint_D f(x,y)\,dA = 0$.

Similarly, if $f(x,y)$ is odd in y and symmetric about x-axis, then $\iint_D f(x,y)\,dA = 0$.

If $f(x,y)$ even, and domain symmetric, can be split up & multiplied.

Final Notes: 15.5 - 16.9

Chapter 15

Area of graph $z=f(x,y)$ over domain D:

$$= \iint_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dA$$

Volume (E) $= \iiint_E 1\,dV$

Mass $m = \iiint_E \rho(x,y,z)\,dV$

$\bar{x} = \frac{1}{m}\iiint_E x\rho(x,y,z)\,dV$

$\bar{y} = \frac{1}{m}\iiint_E y\rho(x,y,z)\,dV$

$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV$

Setting up triple integral:

(1) Projection Method

- Choose the innermost variable
- Find bounds for outer double integral
- Find the bounds for innermost integral.

→ have to sketch in 3D but simpler algebra

→ only good for reasonable solids (easy to sketch)

(2) Cross - Section Method

- Choose the outermost variable
- Find bounds for outermost integral
- Find the bounds for inner double integral.

→ no 3D sketch, but harder algebra

Switching order of vars: can use algebra + sketch

Cylindrical:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$r^2 = x^2+y^2$

Jacobian: r

Spherical:

$x = \rho\sin(\varphi)\cos(\theta)$

$y = \rho\sin(\varphi)\sin(\theta)$

$z = \rho\cos(\varphi)$

$x^2+y^2+z^2 = \rho^2$

Jacobian: $\underbrace{\rho^2\sin(\varphi)}$

$r = \rho\sin(\varphi)$

$0 \le \varphi \le \pi$

θ is angle on xy plane (proj.)

ρ is dist. from origin

φ is angle from +ve z.

Chapter 16

VECTOR FIELDS: $\vec{F} = \nabla f$

Takes $(x, y, [z])$ input, returns $(x, y, [z])$ output

$\vec{F}(x,y,z) = \langle p(x,y,z), q(x,y,z), r(x,y,z) \rangle$

Differentiable / continuous if COMPONENT FUNCTIONS diff/cont.

C is parametrized by $\vec{r}(t)$ w/ $a \le t \le b$ (curve)

$$\left.\begin{aligned} \int_C f\, ds &= \int_a^b f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt \\ \int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \end{aligned}\right\} \text{Line Integrals}$$

$$\text{Length} = \int_C 1\, ds = \int_a^b |r'(t)|\, dt$$

SIGN DEPENDS ON ORIENTATION

Some formulas for mass using scalar function $\rho(x,y,z)$

Work over curve: $\int_C \vec{F}\, d\vec{r}$ — $\vec{F}$: force field

A vector field $\vec{F}$ is CONSERVATIVE if $\vec{F} = \nabla f$ for some potential function $f(x,y,z)$.

If $\vec{F}$ is conservative $(\vec{F} = \nabla f)$,

$$\int_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

Note: This means that $\int_C \vec{F} \cdot d\vec{r} = 0$ if C is a closed loop and $\vec{F}$ is conservative.

Iff $\vec{F}(P(x,y), Q(x,y))$ conservative,

$$P_y = Q_x.$$

$\vec{F}(P, Q, R)$ conservative iff

$$P_y = Q_x \; ; \; Q_z = R_y \; ; \; P_z = R_x$$

* BEING CONSERVATIVE CAN DEPEND ON DOMAIN *

GREEN'S THEOREM:

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA$$

on CLOSED, SIMPLE, POSITIVELY ORIENTED domain.

Positive Orientation: CW for inner boundary, CCW for outer boundary

(D on left side if walking in dir.)

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$$

$$\text{div}(\text{curl}(\vec{F})) = 0 \quad \forall \vec{F}$$

$\text{curl}(\vec{F}) = \vec{0} \Leftrightarrow \vec{F}$ is conservative

$$\iint_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA = \iint_D \text{curl}(\vec{F}) \cdot \hat{k} \, dA$$

Surface in $\mathbb{R}^3$ parametrized by u, v:

$$\vec{r}(u,v) = \langle x(u,v),\ y(u,v),\ z(u,v) \rangle$$

$$\vec{r}_u = \langle \partial x/\partial u,\ \partial y/\partial u,\ \partial z/\partial u \rangle$$

$$\vec{r}_v = \langle \partial x/\partial v,\ \partial y/\partial v,\ \partial z/\partial v \rangle$$

Normal vector to tangent plane: $\vec{r}_u \times \vec{r}_v$

GRID CURVE: set one of u, v to constant

study as single-var parametrized curves

SURFACE INTEGRALS

$$\iint_S f\ dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\ dA$$

$$\iint_S \vec{F}\ d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\ dA$$

FLUX! (orientation matters) — direction is given by $\vec{r}_u \times \vec{r}_v$ (normals)

$$\text{Area}(S) = \iint_S 1\ dS = \iint_D |\vec{r}_u \times \vec{r}_v|\ dA$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\ dS$$

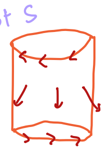
UNIT normal of S

Some mass, COM formulas

STOKES:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

simple, closed, positively oriented boundary of S



Positive orientation: Align head with normal vectors from surface and walk along boundary; surface should be to the left.

Stokes: Let E be a SOLID whose boundary ∂E is oriented outward (away from solid)

For a differentiable $\vec{F}$ on E, $\displaystyle\iint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E \text{div}(\vec{F})\ dV$

Flux = Trip Int of Divergence

important for smiley-face type 3D problem (inner oriented in)

Vortex Field (2D):

$$\vec{V}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle ; \ (x,y) \neq (0,0)$$

$\int_C \vec{V}\, d\vec{r}$, C is circle centered @ $(0,0)$: $\boxed{2\pi}$ (normal derivation)

$\vec{V}$ has potential function $\arctan(y/x)$, $x \neq 0$.

$\int_C \vec{V} \cdot d\vec{r}$, C is closed simple curve oriented CCW:

0 if C does not enclose $(0,0)$

2π if C encloses $(0,0)$ — trick of circle

Inverse Square Field (3D):

$$\vec{F}(x,y,z) = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$\iint_S \vec{F} \cdot d\vec{S}$, S is a sphere centered @ $(0,0)$ oriented outward: $\boxed{4\pi}$

(compute using $\iint_S \vec{F} \cdot \vec{n}\, dS$ formula)

$\operatorname{div}(\vec{F}) = 0$

$\iint_S \vec{F} \cdot dS$, S does not enclose $(0,0)$: 0

$\iint_S \vec{F}\, dS$, S does enclose $(0,0)$: 4π

Note The vortex field $\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ has the following properties:

(1) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

(2) For any circle C centered at $(0,0)$ with ccw orientation, $\int_C \vec{V}\cdot d\vec{r} = 2\pi$

(3) For any closed curve not enclosing $(0,0)$,
$\int_C \vec{V}\cdot d\vec{r} = 0$.

(4) For any closed curve enclosing $(0,0)$ with ccw orientation, $\int_C \vec{V}\cdot d\vec{r} = 2\pi$.

For this semester, you may not directly quote any of these facts on the exam.

* Explanation of (1):

$$P = -\frac{y}{x^2+y^2}, \quad Q = \frac{x}{x^2+y^2}$$

$$\frac{\partial P}{\partial y} = -\frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = \frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

* Explanation of (2) :

R : radius of C

$\rightsquigarrow$ C is parametrized by

$$\vec{r}(t) = (R\cos t, R\sin t) \text{ on } 0 \le t \le 2\pi.$$

$$\int_C \vec{V}\cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}(t))\cdot \vec{r}'(t)\,dt.$$

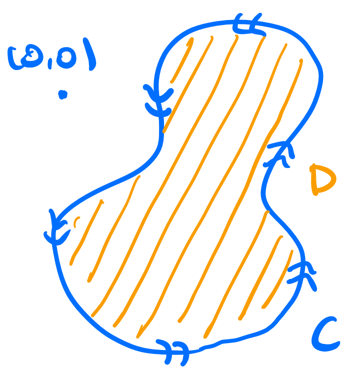
$$\vec{V}(\vec{r}(t)) = \left(-\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2}\right) = \left(-\frac{\sin t}{R}, \frac{\cos t}{R}\right)$$

$$\vec{r}'(t) = (-R\sin t, R\cos t)$$

$$\vec{V}(\vec{r}(t))\cdot\vec{r}'(t) = \sin^2 t + \cos^2 t = 1$$

$$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \int_0^{2\pi} 1\,dt = 2\pi.$$

* Explanation of (3) :



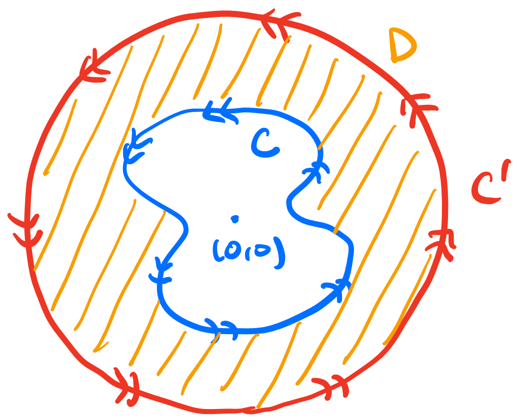
D : region enclosed by C

$\Rightarrow \partial D = C$ is positively oriented

$\vec{V}$ is defined on D.

$$\int_C \vec{V}\cdot d\vec{r} \underset{\substack{\uparrow \\ C=\partial D}}{=} \int_{\partial D}\vec{V}\cdot d\vec{r} \underset{\substack{\uparrow \\ \text{Green's} \\ \text{thm}}}{=} \iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\substack{\| \\ 0 \text{ by (1)}}}\,dA = 0$$

* Explanation of (4) :



You cannot take D to be the region enclosed by C because $\vec{v}$ is not defined at (0,0)

C' : a circle centered at (0,0) which encloses C with CCW orientation

D : region between C and C'.

$\Rightarrow \partial D = -C \cup C'$ ($\because$ C is negatively oriented)

$\vec{v}$ is defined on D.

$$\int_{\partial D} \vec{v} \cdot d\vec{r} = -\int_C \vec{v} \cdot d\vec{r} + \int_{C'} \vec{v} \cdot d\vec{r}$$

Green's thm $\longrightarrow$ ||

$$\iint_D \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{\overset{\shortparallel}{0} \text{ by (1)}} dA = 0$$

$$\Rightarrow \int_C \vec{v} \cdot d\vec{r} = \int_{C'} \vec{v} \cdot d\vec{r} \underset{\uparrow\,(2)}{=} 2\pi$$

Note The inverse square field

$$\vec{F} = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$$

has the following properties:

(1) $\operatorname{div}(\vec{F}) = 0$

(2) For any sphere S centered at $(0,0,0)$ with outward orientation, $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$.

(3) For a closed surface S which does not enclose $(0,0,0)$, $\iint_S \vec{F} \cdot d\vec{S} = 0$

(4) For a closed surface S which encloses $(0,0,0)$ with with outward orientation, $\iint_S \vec{F} \cdot d\vec{S} = 4\pi$.

For this semester, you may not directly quote any of these facts on the exam.

* Explanation of (1):

$P = x(x^2+y^2+z^2)^{-3/2}$, $Q = y(x^2+y^2+z^2)^{-3/2}$, $R = z(x^2+y^2+z^2)^{-3/2}$

$$\frac{\partial P}{\partial x} = 1 \cdot (x^2+y^2+z^2)^{-3/2} + x \cdot \left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2} \cdot 2x$$

$$= (x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2-3x^2)$$

Similarly, $\dfrac{\partial Q}{\partial y} = (x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2-3y^2)$

$$\frac{\partial R}{\partial z} = (x^2+y^2+z^2)^{-5/2}(x^2+y^2+z^2-3z^2)$$

$$\Rightarrow \operatorname{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

$\ast$ Explanation of (2):

$S: x^2+y^2+z^2=R^2$

See the notes after F16 #6

$\Rightarrow$ The unit normal vector on S is $\vec{n}=\left(\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right)$

$$\vec{F}=\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\vec{F}\cdot\vec{n}=\frac{x^2}{R(x^2+y^2+z^2)^{3/2}}+\frac{y^2}{R(x^2+y^2+z^2)^{3/2}}+\frac{z^2}{R(x^2+y^2+z^2)^{3/2}}$$

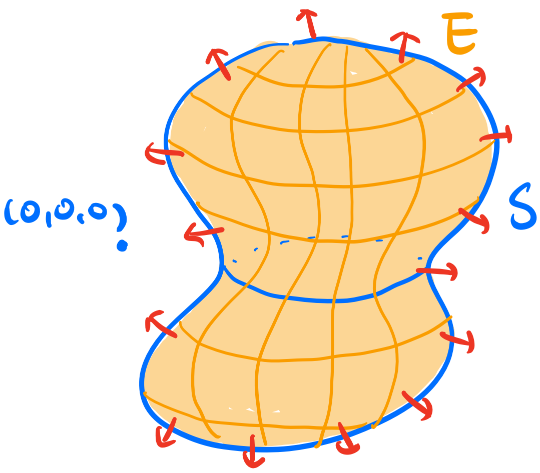
$$=\frac{x^2+y^2+z^2}{R(x^2+y^2+z^2)^{3/2}}=\frac{R^2}{R\cdot R^3}=\frac{1}{R^2}$$

$x^2+y^2+z^2=R^2$ on S

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S}=\iint_S \vec{F}\cdot\vec{n}\,dS=\iint_S \frac{1}{R^2}\,dS=\frac{1}{R^2}\,\text{Area}(S)$$

$$=\frac{1}{R^2}\cdot 4\pi R^2=\boxed{4\pi}$$

$\ast$ Explanation of (3):



E: solid enclosed by S

$\Rightarrow \partial E=S$ is oriented outward

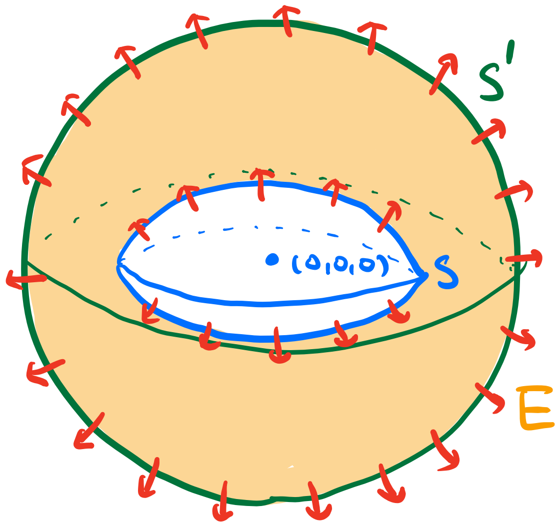
$\vec{F}$ is defined on E.

$$\iint_S \vec{F}\cdot d\vec{S}=\iint_{\partial E}\vec{F}\cdot d\vec{S}=\iiint_E \underbrace{\text{div}(\vec{F})}_{=0}\,dV=0$$

divergence thm.

0 by (1)

* Explanation of (4) :



* We can't take E to be the solid enclosed by S because $\vec{F}$ is not defined at (0,0,0).

S' : a sphere centered at (0,0,0) which encloses S with outward orientation.

E : the solid bounded by S and S'.

$\Rightarrow \partial E = -S \cup S'$ ($\because$ S is oriented inward for E)

$\vec{F}$ is defined on E.

$$\iint_{\partial E} \vec{F}\cdot d\vec{S} = -\iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S}$$

divergence thm $\longrightarrow$ $\|$

$$\iiint_E \underbrace{\operatorname{div}(\vec{F})}_{\substack{\| \\ 0 \text{ by } (1)}} dV = 0$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{S'} \vec{F}\cdot d\vec{S} \underset{\substack{\uparrow \\ (2)}}{=} 4\pi$$

A COMPREHENSIVE GUIDE ON LINE/SURFACE INTEGRALS IN MATH 215

Line integrals:

(I) If you integrate a scalar function f over a curve C, you should use the definition

$$\int_C f\ ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\ dt.$$

(II) If you integrate a vector field $\vec{F}$ over a curve C, there are several possibilities.

(1) If the vector field is conservative (i.e., has a potential function f), then you should always use the Fundamental theorem.

(2) If the curve C is closed, then there are several subcases.

(a) If $\vec{F}$ is two dimensional and defined everywhere inside C, then you should use Green's theorem.

(b) If $\vec{F}$ is two dimensional but undefined at some points inside C with C not being a circle centered at the origin, then the best way is probably to use Green's theorem by choosing a large circle enclosing C.

(c) If $\vec{F}$ is three dimensional and $\mathrm{curl}(\vec{F})$ is easy to compute (or is mentioned on the problem), then you should try to use Stokes' theorem.

(3) If none of the above applies, then you should use the definition

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)dt.$$

Surface integrals:

(1) If you integrate a scalar function f over a surface S, there are two possibilities:

(a) If S is not a sphere, then you should use the definition

$$\iint_S f\ dS = \iint_D f(\vec{r}(u,v))|\vec{r}_u\times\vec{r}_v|\ dA.$$

(b) If S is a sphere $x^2+y^2+z^2 = R^2$, then you can use the unit normal vector $\vec{n} = (x/R, y/R, z/R)$ and find a vector field $\vec{F}$ with $\vec{F}\cdot\vec{n} = f$ to convert it to a vector field integral $\iint_S \vec{F}\cdot d\vec{S}$ and use the divergence theorem.

(2) If you integrate a vector field $\vec{F}$ over a surface S, there are several possibilities.

(1) If you integrate the curl of a vector field, then you should use Stokes' theorem.

(2) If the surface S is closed, there are several subcases.

(a) If $\vec{F}$ is defined everywhere inside S, then you should use the divergence theorem.

(b) If $\vec{F}$ is undefined at some points inside S with S not being a sphere centered at the origin, then the best way is probably to use the divergence theorem by choosing a large sphere enclosing S.

(c) If $\vec{F}$ is undefined at some points inside S with S being a sphere $x^2 + y^2 + z^2 = R^2$, it's usually best to compute $\iint \vec{F} \cdot \vec{n}\, dS$ with $\vec{n} = (x/R, y/R, z/R)$.

(3) If the surface S is flat and parallel to one of the xy, yz, or zx planes, then it's often best to compute $\iint \vec{F} \cdot \vec{n}\, dS$.

(4) If S is almost closed (meaning you can make it closed by closing the top or the base, etc), then you should think about using the divergence theorem after adding an appropriate surface to make it closed.

(5) If none of the above applies, then you should use the definition

$$\iint_S \vec{F} \cdot\, d\vec{S} = \iint_D \vec{F}(\vec{r}(u, v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$