Chapter 1

- Set of equations of m variables has solution set in $\mathbb{R}^m$
  - some dimension of hyperplane — $\overset{0}{\text{point}}$, $\overset{1}{\text{line}}$, $\overset{2}{\text{plane}}$, $\overset{3}{\text{3D}}$ - hyperplane, ...
  - dim = # of free variables
- Reduced Row Echelon Form
  - First non-zero entry is a 1 in any row
  - If column has non-zero entry, all other things in column = 0
  - If row is non-zero (has leading 1), each row above has leading 1 further to left.
- All elementary row ops correspond to some operation on I_n → swap rows, scale rows, add a multiple of a row to another
- Linear System
  - Consistent: At least one solution (1 or ∞)
  - Inconsistent: 0 solutions
- Rank(A) → # of leading 1s in rref(A)
- n linear equations with m variables → $n \times m$ matrix
  - → If $n < m$ (less rows than columns, less equations than variables)
    - → 0 OR ∞ solutions
- $$A\vec{x} = \begin{bmatrix} | & & | \\ V_1 & \cdots & V_m \\ | & & | \end{bmatrix}\begin{bmatrix} x_1 \\ | \\ x_m \end{bmatrix} = x_1\vec{V}_1 + \cdots + x_m\vec{V}_m \quad , \quad V_i \in \mathbb{R}^n$$ ☆☆

Chapter 2

- Linear Transformation $T: X \to Y$
  - $T(x_1 + x_2) = T(x_1) + T(x_2)$ for all $x_1, x_2 \in X$
  - $T(cx) = cT(x)$ for all $x \in X$, $c \in \mathbb{R}$
  - If $X, Y = \mathbb{R}^?$, $T(\vec{x}) = A\vec{x}$ (for coordinate spaces)
- If $T: \mathbb{R}^m \to \mathbb{R}^n$, $A_T \in \mathbb{R}^{n \times m}$

$$A = \begin{bmatrix} | & & | \\ T(e_1) & \cdots & T(e_m) \\ | & & | \end{bmatrix}$$

- Given a line L thru origin with unit direction vector $\vec{w}$:

  $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ (w.r.t. L)

  - $T(\vec{x}) = \vec{x}^{\parallel}$ — orthogonal projection of $\vec{x}$ onto L
    - → so $\vec{x}^{\parallel} = k\vec{w}$; so $\vec{x} - k\vec{w} = \vec{x}^{\perp}$,

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0$$
$$\vec{x} \cdot \vec{w} - k\vec{w} \cdot \vec{w} = 0$$
$$k = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \Rightarrow \vec{x} = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}$$

. If $\vec{w}$ is unit vector,

$$\vec{x} = (\vec{x} \cdot \vec{w})\vec{w}$$

- All matrices of lin transforms:
  - Scaling by k:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

  - Rotation through θ

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  - Orthogonal projection onto line L, with $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$$\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

  - Horizontal shear (x-coords change)

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+ky \\ y \end{bmatrix}$$

  - Vertical shear (y-coords change)

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx+y \end{bmatrix}$$

  - Reflection about a line

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} 2u_1^2 - 1 & 2u_1 u_2 \\ 2u_1 u_2 & 2u_2^2 - 1 \end{bmatrix}$$

- $$\underbrace{B}_{n \times p} A = B \begin{bmatrix} | & & | \\ \underbrace{v_1}_{v_i \in \mathbb{R}^p} & \cdots & v_m \\ | & & | \end{bmatrix} = \underbrace{\begin{bmatrix} | & & | \\ B\vec{v_1} & \cdots & B\vec{v_m} \\ | & & | \end{bmatrix}}_{n \times m}$$

  B is $n \times p$, A is $p \times m$

  - not commutative; same as applying transformation A, then B
  - matrix mult. is associative

- $T: X \to Y$ is invertible iff injective, surjective

  $\exists\ T^{-1}$ s.t. $T^{-1}(T(x)) = x$, $T(T^{-1}(y)) = y$. T^{-1} is unique

- Matrix is invertible iff $\text{rref}(A) = I_n$.
  - $AA^{-1} = I_n$
  - $A^{-1}A = I_n$

- For $A \in \mathbb{R}^{n \times n}$,

  $A\vec{x} = \vec{b}$ has 1 soln. iff A invertible – else, 0 or ∞ solns.

- If $A, B \in \mathbb{R}^{n \times n}$ and $AB = I_n$,
  - $BA = I_n$, $B = A^{-1}$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Chapter 3

- Image: All values a function takes in its target space
  - → For $f: X \to Y$, if $im(f) = Y$, f is surjective
  - → For T_A, $im(T_A) = span(A)$
    - → linear combinations of columns – why? b/c $A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots$
  - → image is SUBSPACE of Y – includes $\vec{0}$, closed under addition, scalar mult.
    - CODOMAIN
- Kernel
  - → All zeros of transformation ($A\vec{x} = \vec{0}$) for T_A
  - → Subspace of DOMAIN of transformation — includes $\vec{0}$, closed under addition, scalar mult.
    - → How? solve linear system
    - → $T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \vec{x}$; $rref(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

      $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t$ → $ker(T) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} t \right\}$

- $\dim(im(T)) + \dim(ker(T)) = m$, where $T: \mathbb{R}^m \to \mathbb{R}^n$ (size of domain)

  $rank(T) + nullity(T) = m$ Rank – Nullity

  # of pivot cols + _ _ = dim(input space)

  # of vecs in basis

- A is invertible iff ($A \in \mathbb{R}^{n \times n}$):
  - $A\vec{x} = \vec{b}$ has unique sol ($A^{-1} b$) for all b
  - $rref(A) = I_n$
  - $rank(A) = n$
  - $im(A) = \mathbb{R}^n$
  - $ker(A) = \{\vec{0}\}$ → implies lin. indep. column vectors
  - column vectors of A are basis of $\mathbb{R}^n$
  - column vectors of A span $\mathbb{R}^n$
  - column vectors of A are linearly independent
- Subspace S of vector space V:
  - → S contains $\vec{0}$
  - → S closed under addition, scalar multiplication
  - → Note: Image, Kernel subspaces
- $\mathbb{R}^n$ has infinite subspaces
  - → $(n+1)$ categories (dimensions)
- Consider vectors $\vec{v}_1, \dots, \vec{v}_m$:
  - redundant if some v_i linear combination of prev. vectors
  - linearly independent if no redundant vectors
  - Alternatively: $\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$ is a trivial linear relation

  linear dependence: opposite of this
- Basis of V
  - set of linearly independent vectors B that span V
  - basis of $im(A)$ are all column vectors, omitting redundant vec
  - Unique lin. comb. $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = v$ for all $v \in V$

- Characterizations of Linear Independence
  - For vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$:
    - i) $\vec{v}_1, \dots, \vec{v}_m$ are linearly independent
    - ii) None of $\vec{v}_1, \dots, \vec{v}_m$ are redundant
    - iii) None of vectors are lin. comb. of any other vectors
    - iv) $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{0}$ is a trivial relation
    - v) $\ker\left(\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}\right) = \{\vec{0}\}$
    - vi) $\text{rank}\left(\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}\right) = m$ $(\text{im}(A)) = \mathbb{R}$

- Basis is:
  1) Minimal Spanning Set
  2) Maximal Linearly Independent Set
  3) Equal in size to dimension of its vector space

- Basis of image: run rref, pick pivot cols from ORIGINAL

- RANK - NULLITY THM:

  $\dim(\ker(A)) + \dim(\text{im}(A)) = m$ for $A \in \mathbb{R}^{n\times m}$

  $\text{nullity}(A) + \text{rank}(A) = m$

- Coordinates in subspace of $\mathbb{R}^n$:
  - Consider basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of $V \subseteq \mathbb{R}^n$
  - For all $\vec{x} \in V$, $\vec{x} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ with unique $c_1, \dots, c_m$

    $\rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} = \mathcal{B}$-coordinates of $\vec{x}$, denoted $[\vec{x}]_{\mathcal{B}}$

    $\rightarrow \vec{x} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} [\vec{x}]_{\mathcal{B}}$; $S = \underbrace{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix}}_{\text{BASIS VECTORS}}$, $S \in \mathbb{R}^{n\times m}$

    $\vec{x} = S[\vec{x}]_{\mathcal{B}}$

    $S^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$

$S: \mathcal{B} \rightarrow \mathcal{E}$

Note: $\rightarrow$ S is coordinate-transform $S_{\mathcal{B}\rightarrow\mathcal{E}}$ matrix from $\mathcal{B}$ to standard

write basis of $\mathcal{B}$ in $\mathcal{E}$

$\rightarrow S^{-1}$ is transform from standard to $\mathcal{B}$

write $\mathcal{E}$ in $\mathcal{B}$

- Linearity of Coordinates:
  - $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$
  - $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}}$

- To find $\mathcal{B}$-coordinates of vector $\vec{x}$, solve $S[\vec{x}]_{\mathcal{B}} = \vec{x}$.

- To find original coordinates of vector $[\vec{x}]_{\mathcal{B}}$, $\vec{x} = S[\vec{x}]_{\mathcal{B}}$

- Matrix of a linear transformation

$$\begin{tikzcd}
\vec{x} \arrow[r, "T"] \arrow[d, "\mathfrak{B} \to S^{-1}"'] & T(\vec{x}) \arrow[d, "\mathfrak{B} = S^{-1}"] \\
{[\vec{x}]_{\mathfrak{B}}} \arrow[r, "B"'] & {[T(x)]_{\mathfrak{B}}}
\end{tikzcd}$$

↑ NOT 1

$B = S^{-1} \circ T \circ S$

$= S^{-1} T S.$

$S = \begin{bmatrix} | & & | \\ b_1 & \cdots & b_m \\ | & & | \end{bmatrix}$

$B = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathfrak{B}} & \cdots & [T(v_m)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}$, where $v_1, \dots, v_m$ form basis $\mathfrak{B}$

$TS = SB$

$B = S^{-1} A S$

$A = SBS^{-1}$

T is just special case where $v_1, \dots, v_m = e_1, \dots, e_m$

- A is similar to B if $\exists\, S$ (invertible S) s.t. $AS = SB$

  same linear transformation, diff bases

- $\mathfrak{B}$ - matrix is DIAGONAL iff $T(\vec{v}_j)$ is parallel to $\vec{v}_j$ for all $\vec{v}_j$

CHAPTER 4

- VECTOR SPACES

  1. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
  2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
  3. $\vec{u} + \vec{0} = \vec{u}$
  4. For all $\vec{u} \in V$, $\exists\, \vec{v} \in V$ such that $\vec{u} + \vec{v} = \vec{0}$, with $\vec{v} = -\vec{u}$
  5. $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$, $k \in \mathbb{R}$
  6. $(c + k)\vec{u} = c\vec{u} + k\vec{u}$, $c, k \in \mathbb{R}$
  7. $c(k\vec{u}) = (ck)\vec{u}$, $c, k \in \mathbb{R}$
  8. $1\vec{u} = \vec{u}$

  Examples: Polynomials, $F(\mathbb{R}, \mathbb{R})$, $\mathbb{C}$

- Subset W of vector space V is subspace if
  - → W contains $\vec{0}$
  - → W closed under addition, scalar mult.

- Some defns. of span, linear ind., basis, coords

- $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ is $\mathfrak{B}$-coordinate vector of v, $[v]_{\mathfrak{B}}$

  $L_{\mathfrak{B}}$ is $\mathfrak{B}$-coordinate transformation (to $\mathbb{R}^n$) $L: V \to \mathbb{R}^n$

  $\equiv$

  Note $L_{\mathfrak{B}}(\vec{v}) = [v]_{\mathfrak{B}}$

  $S^{-1}: \mathcal{E} \to \mathfrak{B}$

  $L_{\mathfrak{B}}^{-1}(\vec{e}_i) = v_i$,

  when $\mathfrak{B} = \{v_1, \dots, v_i, \dots, v_m\}$

- Finding basis of linear space V
  - Write down typical element of V in terms of arbitrary consts.
  - Use consts. to express arbitrary element as lin. comb. of other elements of V
- Some defn. of linear, image, kernel
- Isomorphism:
  - invertible (bijective) linear transformation
  - Vec space V isomorphic to W if $\exists$ isomorphism T from V to W
  - $L_\mathcal{B}$ is an isomorphism
    - → All vector spaces w/ dim n are isomorphic

- Properties of Transformations
  - → Linear $T: V \to W$ is isomorphism iff $\underline{\ker(T) = \{\vec{0}\}}$ and $\underline{\operatorname{im}(T) = W}$
  - → V is isomorphic to W iff $\underline{\dim(V) = \dim(W)}$
  - → Suppose linear $T: V \to W$ has $\ker(T) = \{\vec{0}\}$. If $\dim(V) = \dim(W)$, T is isomorphism
  - → Suppose linear $T: V \to W$ has $\operatorname{im}(T) = W$. If $\dim(V) = \dim(W)$, T is isomorphism

$$
\begin{tikzcd}
V \arrow[rr, "L_1"] \arrow[d, "L_2^{-1} \circ L_1"'] & & \mathbb{R}^n \\
W \arrow[urr, "L_2"'] & &
\end{tikzcd}
$$

- MATRIX OF A LINEAR TRANSFORM

$$
\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "L_\mathcal{B}"] & T(V) \arrow[d, "L_\mathcal{B}"] \\
{[V]_\mathcal{B}} \arrow[r, "B"] & {[T(V)]_\mathcal{B}}
\end{tikzcd}
\qquad [T(V)]_\mathcal{B} = B([V]_\mathcal{B})
$$

$$
B = \begin{bmatrix} | & & | \\ [T(v_1)]_\mathcal{B} & \cdots & [T(v_m)]_\mathcal{B} \\ | & & | \end{bmatrix} \quad ; \quad (v_1, \dots, v_m) = \mathcal{B}
$$

Columns are $\mathcal{B}$-coordinate vectors of transforms of basis elements $(v_1, \dots, v_m)$ of $\mathcal{B}$

Same as S

- CHANGE - OF - BASIS MATRIX
  - Consider 2 bases $\mathcal{U}$ and $\mathcal{B}$ of vector space V, $\dim(V) = n$

$$
\begin{tikzcd}
& \mathbb{R}^n \\
V \arrow[ur, "L_\mathcal{U}"] \arrow[dr, "L_\mathcal{B}"'] & \\
& \mathbb{R}^n \arrow[uu, "S"']
\end{tikzcd}
$$

$S = L_\mathcal{U} L_\mathcal{B}^{-1}$

$S(\vec{x}) = L_\mathcal{U}(L_\mathcal{B}(\vec{x}))$ for all $x \in \mathbb{R}^n$

$S_{\mathcal{B} \to \mathcal{U}} : \mathbb{R}^n \to \mathbb{R}^n$

$$
\begin{tikzcd}
& {[\vec{v}]_\mathcal{U}} \\
\vec{v} \arrow[ur, "L_\mathcal{U}"] \arrow[dr, "L_\mathcal{B}"'] & \\
& {[\vec{v}]_\mathcal{B}} \arrow[uu, "S"']
\end{tikzcd}
$$

$[\vec{v}_\mathcal{U}] = S([\vec{v}_\mathcal{B}])$

If $\mathcal{B} = (b_1, \dots, b_i, \dots, b_n)$:

$[b_i]_\mathcal{U} = S[b_i]_\mathcal{B} = S\vec{e}_i$

So, $S_{\mathcal{B} \to \mathcal{U}} = \begin{bmatrix} | & & | \\ [b_1]_\mathcal{U} & \cdots & [b_n]_\mathcal{U} \\ | & & | \end{bmatrix}$

- EXAMPLE: $V :=$ Subspace of C^∞ spanned by e^x, e^{-x}

  $\mathcal{U} = (e^x, e^{-x})$

  $\mathcal{B} = (e^x + e^{-x}, e^x - e^{-x})$

  Find $S_{\mathcal{B} \to \mu}$

$$\begin{tikzcd}
 & {[v]_\mu} \\
v \arrow[ur, "L_\mu"] \arrow[dr, "L_\mathcal{B}"'] & \\
 & {[v]_\mathcal{B}} \arrow[uu, "S"']
\end{tikzcd}$$

$$S_{\mathcal{B} \to \mu} = S\,[v]_{\mathcal{B}} = [v]_\mu$$

$$S = \begin{bmatrix} | & & | \\ [b_1]_\mu & \cdots & [b_m]_\mu \\ | & & | \end{bmatrix}$$

$$\begin{array}{cc|c} e^x + e^{-x} & e^x - e^{-x} & \\ \hline \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \begin{matrix} e^x \\ e^{-x} \end{matrix} \end{array}$$

- $\mathcal{U} = (a_1, \ldots, a_m)$ and $\mathcal{B} = (b_1, \ldots, b_m)$

$$\begin{bmatrix} | & & | \\ b_1 & \cdots & b_m \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix} S_{\mathcal{B} \to \mu}$$

NOTE: opposite of what you think

PROOF METHODS

- For all : choose arbitrary element and show statement holds (using properties true for all elements)
- And: show statements true independently
- Exists: Show example
- Implies ($A \to B$): Suppose A, conclude B

  Suppose not-B, conclude not-A
- Iff ($A \Leftrightarrow B$): $A \to B$, $B \to A$
- (P OR Q):

  ① Assume not P, show Q

  OR

  ② Assume not Q, show P
- (P XOR Q)

  ① Assume P, show not Q

  OR

  ② Assume Q, show not P

CHAPTER 5

- For $\vec{v}, \vec{w} \in \mathbb{R}^n$:
  - $\vec{v}, \vec{w}$ are orthogonal iff $\vec{v} \cdot \vec{w} = 0$
  - $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$
  - $\dfrac{\vec{v}}{\|\vec{v}\|}$ is a unit vector (length = 1)

- $\vec{x} \in \mathbb{R}^n$ is orthogonal to subspace $V \subseteq \mathbb{R}^n$ iff $\vec{x} \cdot \vec{v} = 0$ for all $\vec{v} \in V$.
  - equivalently, if $\vec{x} \cdot \vec{b} = 0$ for all $\vec{b} \in \mathcal{B}$, a basis of V

- Orthonormal vectors:
  - $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ are orthonormal iff

    $\vec{u}_i \cdot \vec{u}_j = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ – set of orthogonal unit vectors
  - Set of n orthormal vectors in V w/ $\dim(V) = n$ form Orthonormal Basis (ONB) of V

- Orthogonal Projection (onto subspace V)
  - $\vec{x} = x^{\parallel} + x^{\perp}$
  - $\text{proj}_V\, x = x^{\parallel}$ — this is a linear transform $\mathbb{R}^n \to \mathbb{R}^n$
  - If V has ONB $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$:

    $\text{proj}_V \vec{x} = (u_1 \cdot x)\, u_1 + (u_2 \cdot x)\, u_2 + \dots + (u_m \cdot x)\, u_m$
  - If $\mathcal{U} = u_1, \dots, u_n$ is ONB spanning V, $\dim(V) = n$:

    $[\vec{v}]_{\mathcal{U}} = \begin{bmatrix} u_1 \cdot v \\ u_2 \cdot v \\ \vdots \\ u_n \cdot v \end{bmatrix}$

- $\|\text{proj}_V \vec{x}\| \leq \|\vec{x}\|$; = iff $\vec{x} \in V$
- CAUCHY-SCHWARZ:

  $|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$
- $\theta = \arccos\left(\dfrac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|\|\vec{y}\|}\right)$

- Orthogonal complement:
  - $V^{\perp} = \{\vec{x} \in \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0 \text{ for all } v \in V\}$
  - $V^{\perp} = \ker(\text{proj}_V)$
  - $V^{\perp}$ is subspace of $\mathbb{R}^n$
  - $V \cap V^{\perp} = \{\vec{0}\}$
  - $\dim(V) + \dim(V^{\perp}) = n$
  - $(V^{\perp})^{\perp} = V$

- Random note: $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$

- GRAM-SCHMIDT
  - In general, any basis / lin. indep. set of vectors can be made into ORTHONORMAL set
  - Process:

    >> start with $\mathcal{B} = v_1, \dots v_n$

    >> for $i = 1 \to n$:

    >> Take set $u_1, \dots, u_{i-1}$ of orthonormal vectors

    $v_i^{\perp} = v_i - v_i^{\parallel} = v_i - (u_1(u_1 \cdot v_i) + u_2(u_2 \cdot v_i) + \dots + u_{i-1}(u_{i-1} \cdot v_i))$

    $u_i = v_i^{\perp} / \|v_i^{\perp}\|$

    >> Then, $u_1, \dots, u_n$ is ONB

- QR Factorization:

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & v_m \\ | & & | \end{bmatrix}}_{m} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} R$$

$R_{\mathcal{B}\to\mathcal{U}}$

```tikzcd
{[\vec{x}]_{\mathcal{U}}} \arrow[rd, "Q"] & \\
 & \vec{x} \\
{[\vec{x}]_{\mathcal{B}}} \arrow[uu, "R"] \arrow[ru, "M"'] &
```

Gram-Schmidt is just change of basis

$$R = \begin{bmatrix} \|v_1\| & u_1 \cdot v_2 & u_1 \cdot v_3 & \cdots & u_1 \cdot v_n \\ 0 & \|v_2^{\perp}\| & u_2 \cdot v_3 & & u_2 \cdot v_n \\ 0 & 0 & \|v_3^{\perp}\| & & \vdots \\ \vdots & 0 & 0 & & \vdots \\ \vdots & \vdots & 0 & & \vdots \\ \vdots & \vdots & \vdots & & u_{n-1} \cdot v_n \\ 0 & 0 & 0 & \cdots & \|v_n^{\perp}\| \end{bmatrix}$$

$\perp$ indicates $\perp$ to all previous vectors

- Orthogonal Transformations
  - A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is ORTHOGONAL iff $\|T(\vec{x})\| = \|\vec{x}\|$ $\forall x \in \mathbb{R}^n$
  - If $T(\vec{x}) = A\vec{x}$ is orthogonal, A is orthogonal matrix
  - $\vec{x} \cdot \vec{y} = T(\vec{x}) \cdot T(\vec{y})$ for all $x, y \in \mathbb{R}^n$ under orth. T
  - T is orthogonal iff $T(e_1), \ldots, T(e_n)$ is ONB of $\mathbb{R}^n$

    $\iff$ columns of A form ONB

- For orthogonal $A, B \in \mathbb{R}^{n \times n}$:
  - AB is orthogonal
  - A^{-1} is orthogonal
  - $A^T = A^{-1}$; so, $A^T A = A A^T = I_n$ } ALTERNATE DEFN. OF ORTH. MAT.

- PROPERTIES OF ORTHOGONAL MATRICES
  - Consider an $n \times n$ matrix A. Then:

    A is orthogonal iff

    $\iff T(\vec{x}) = A\vec{x}$ preserves length

    $\iff$ Columns of A form ONB of $\mathbb{R}^n$

    $\iff A^T A = I_n$

    $\iff A^{-1} = A^T$

    $\iff A$ preserves dot product

- PROPERTIES OF TRANSPOSE:
  - $A^T + B^T = (A+B)^T$
  - $(kA)^T = kA^T$
  - $(AB)^T = B^T A^T$
  - $\mathrm{rank}(A^T) = \mathrm{rank}(A)$
  - $(A^T)^{-1} = (A^{-1})^T$

- Matrix of an orthogonal projection

  $u_1 (u_1 \cdot x) = u_1 u_1^T x$

  So, $\text{proj}_V \vec{x} = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{bmatrix} \begin{bmatrix} — & u_1^T & — \\ & \vdots & \\ — & u_m^T & — \end{bmatrix} \vec{x}$

  $= \underbrace{Q}_{n \times m} \underbrace{Q^T}_{m \times n} \underbrace{\vec{x}}_{n \times 1} \to n \times 1$

  NOTE: $Q^{-1} = Q^T$ iff Q is $n \times n$, which makes sense b/c then $V = \mathbb{R}^n$. $u_1, \ldots, u_m$ are ONB for V in $\mathbb{R}^n$; these are rectangle matrices

- $(\text{im } A)^\perp = \ker(A^T)$

  $\Leftrightarrow (\text{im } A^T)^\perp = \ker(A)$

  $\Leftrightarrow (\text{im } A) = \ker(A^T)^\perp$

  $\Leftrightarrow (\text{im } A^T) = \ker(A)^\perp$

- From this:
  - $\ker(A) = \ker(A^T A)$ — pf: consider $\vec{x} \in \ker(A^T A)$.
    $A^T A \vec{x} = 0$; $A\vec{x} \in \text{im}(A)$, $A\vec{x} \in \ker(A^T)$.
    $\text{im}(A) = \ker(A^T)^\perp$; $A\vec{x} = \vec{0}$.
  - If $\ker(A) = \{\vec{0}\}$ for ANY $n \times m$ A, $A^T A$ is invertible
    - pf: $\ker(A^T A) = \ker(A) = \{\vec{0}\}$

LEAST SQUARES

- Orthogonal Projections: $\text{proj}_V \vec{x}$ is vector in V closest to $\vec{x}$:

  $\|\vec{x} - \text{proj}_V \vec{x}\| < \|\vec{x} - \vec{v}\|$ for all $\vec{v} \in V$ s.t. $\vec{v} \neq \text{proj}_V \vec{x}$

- LSS:

  For $A \in \mathbb{R}^{n \times m}$, consider $A\vec{x} = \vec{b}$.

  $\vec{x}^* \in \mathbb{R}^m$ is LSS iff $\|b - A\vec{x}^*\| \leq \|\vec{b} - A\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^m$ (closest vector to $\vec{b}$ in $\text{im}(A)$)

- If $\vec{x}^*$ is LSS of $A\vec{x} = \vec{b}$:

  $\|b - Ax^*\| \leq \|b - Ax\| \quad \forall \ x \in \mathbb{R}^m$

  $\Leftrightarrow Ax^* = \text{proj}_V \vec{b}$, $V = \text{im}(A)$

  $\Leftrightarrow b - Ax^* \in V^\perp$; $V^\perp = \text{im}(A)^\perp = \ker(A^T)$

  $\Leftrightarrow A^T(b - Ax^*) = \vec{0}$

  $\Leftrightarrow A^T A \vec{x}^* = A^T \vec{b}$ ← normal equation

- If $\ker(A) = \{\vec{0}\}$,

  then normal equation has unique LSS $x^* = (A^T A)^{-1} A^T \vec{b}$

- Matrix of an orthogonal projection:

V has basis $\mathcal{B} = v_1, v_2, \ldots, v_m$

$$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{bmatrix} \; ; \quad \text{proj}_V \vec{x} = (A(A^T A)^{-1} A^T)\vec{x}$$

- Inner products

Maps pair of vectors to a scalar

→ $\langle f, g \rangle = \langle g, f \rangle$ symmetric

→ $\langle f+h, g \rangle = \langle f, g \rangle + \langle h, g \rangle$

→ $\langle cf, g \rangle = c \langle f, g \rangle$

bilinear

→ $\langle f, f \rangle > 0 \; \forall$ nonzero $f \in V$ positive definite

everything from dot products applies

Vector space w/ inner product = INNER PRODUCT SPACE

CHAPTER 6

- Pattern in $n \times n$ matrix: Choose one entry in each row, each column ($n!$ total patterns)

$$\begin{bmatrix} \cdot & \cdot & \odot & \cdot \\ \cdot & \odot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \odot \\ \odot & \cdot & \cdot & \cdot \end{bmatrix}$$

prod_P = product of entries in pattern P

inversion = pair of entries in pattern s.t. top one is RIGHT of bottom one

$\text{sign}_P = (-1)^{\#\text{ of inversions}}$

- $\det(A) = \sum_P (\text{sign } P)(\text{prod } P)$

- $\det(\text{Triangular } A) = \text{prod (diagonal)}$

- PROPERTIES OF DETERMINANTS:
  - $\det(A) = \det(A^T)$
  - Linear in rows, columns (BUT NOT LINEAR ITSELF ~~to det~~)

$$\det\left(\begin{bmatrix} | & & & & | \\ \vec{v}_1 & \cdots & \vec{x}+\vec{y} & \cdots & \vec{v}_n \\ | & & & & | \end{bmatrix}\right) = \det\left(\begin{bmatrix} \vec{v}_1 & \cdots & \vec{x} & \cdots & \vec{v}_n \end{bmatrix}\right) + \det\left(\begin{bmatrix} \vec{v}_1 & \cdots & \vec{y} & \cdots & \vec{v}_n \end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix} | & \\ k\vec{x} & \cdots \\ | & \end{bmatrix}\right) = k \det\left(\begin{bmatrix} | & \\ \vec{x} & \cdots \\ | & \end{bmatrix}\right)$$

- Elementary Row Ops:
  - → Scaling: scale det by same factor
  - → Swap: $(-1) * \det$
  - → Add multiple of one row to another: NO CHANGE in det

  same for elem column ops

- $A \in \mathbb{R}^{n \times n}$: A invertible iff $\det(A) \neq 0$
- $\det(AB) = \det(A)\det(B)$
- $\det(A^m) = \det(A)^m$
- $\det(A^{-1}) = 1/\det(A)$
- If A is similar to B ($A = S^{-1}BS$), $\det(A) = \det(B)$
- $\det(T: \mathbb{R}^n \to \mathbb{R}^n) = \det([T]_{\mathcal{B}})$ for any basis $\mathcal{B}$

- Laplace Expansion

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

- Determinant of orthogonal matrix is 1 or -1
  - Orthogonal matrix w/ det = 1 → rotation matrix
  - — — — — det = -1 → rotation + reflection matrix
- $|\det A|$ is volume of parallelpiped spanned by $v_1, v_2, \dots, v_n$, $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$
  - $|\det A| = \|v_1\| \, \|v_2^{\perp}\| \cdots \|v_n^{\perp}\|$
- In general for non-square A:

  m-volume of n-parallelpiped $= \sqrt{\det(A^T A)}$

- $|\det A|$ is expansion factor of $T(\vec{x}) = A\vec{x}$

CHAPTER 7

- Stuff is easier to do for diagonal matrices
- $T(\vec{x}) = A\vec{x}$ is DIAGONALIZABLE if $[T]_{\mathcal{B}}$ is diagonal for some basis $\mathcal{B}$
  - A is similar to some diagonal matrix D
  - $A = PDP^{-1}$
- Eigenvectors & Eigenvalues:

  → $A\vec{v} = \lambda\vec{v}$; $\vec{v}$ is eigenvector and λ an eigenvalue

  → Then, $P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$, $D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

  → So A is diagonalizable iff ∃ eigenbasis of $\mathbb{R}^n$ for A / T_A (enough independent eigenvectors)

- 0 is an eigenvalue iff $\ker(A) \neq \{\vec{0}\}$
- # of eigenvalues:

  At most n

  If n is odd, at least 1 real λ.

  Finding EVs : Use characteristic eq.

- Eigenspaces:

$$E_\lambda = \ker(A - \lambda I_n) = \{\vec{v} \text{ in } \mathbb{R}^n : A\vec{v} = \lambda\vec{v}\}$$

- $\text{gemu}(\lambda) = \text{nullity}(A - \lambda I_n) = n - \text{rank}(A - \lambda I_n)$
- Concatenating eigenbases ⇒ lin. indep. vectors
- If A is similar to B:
  - A, B have same characteristic polynomial
  - rank(A) = rank(B) ; nullity(A) = nullity(B)
  - A, B have same eigenvalues w/ same almus, gemus

    → not necessarily same eigenvectors

  - det(A) = det(B) ; tr(A) = tr(B)
- $\text{gemu}(\lambda) \leq \text{almu}(\lambda)$
- Complex things: SAME STUFF

Chapter 8

- A matrix A is orthogonally diagonalizable iff A is symmetric
  $A = SDS^{-1}$, with S being orthogonal

  } Spectral Theorem

Worksheet Notes (1 - 14)

- Matrix - Vector Products

$$A\vec{x} = \sum_{i=1}^{n} x_i \vec{a}_i \quad , \quad A \in \mathbb{R}^{m \times n}$$

- Matrix - Matrix Products

$$AB = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_n \\ | & | & & | \end{bmatrix}.$$

So, every column is new linear comb. of columns of A.

- Linear Functions
  - recall that $f(x) = x + 1$ is not linear
  - $f(\vec{0}) = \vec{0}$ for every linear function f

$$\rightarrow f(\vec{0}) = f(0\vec{0}) = 0 f(\vec{0}) = \vec{0}$$

- pf: Linear Transformations preserve linear combinations

$$f\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = f(c_1\vec{v}_1 + \cdots + c_n v_n)$$

$$= f(c_1\vec{v}_1) + \cdots + f(c_n v_n)$$

$$= c_1 f(\vec{v}_1) + \cdots + c_n f(\vec{v}_n)$$

$$= \sum_{i=1}^{n} c_i f(\vec{v}_i)$$

- pf: For every linear function $f: \mathbb{R}^n \to \mathbb{R}^m$, there is a matrix A s.t. $f(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

  For all $\vec{x} \in \mathbb{R}^n$,

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$f(\vec{x}) = f\left(\sum_{i=1}^{n} x_i \vec{e}_i\right) = \sum_{i=1}^{n} x_i f(\vec{e}_i) = \begin{bmatrix} | & & | \\ f(\vec{e}_1) & \cdots & f(\vec{e}_n) \\ | & & | \end{bmatrix} \vec{x}.$$

Thus, $A = \begin{bmatrix} | & & | \\ f(\vec{e}_1) & \cdots & f(\vec{e}_n) \\ | & & | \end{bmatrix}$

- $C^\infty([0,1])$ is set of all smooth (forever-differentiable) from $[0,1] \to \mathbb{R}$

  → this is a vector space

- Is $F: \mathcal{P}_2 \to \mathcal{P}_2$ defined by $F(p)(x) = x + p(x)$ linear?

  → $F(p+q) = x + (p+q)(x)$

  → $F(p) + F(q) = x + p(x) + x + q(x) = 2x + (p+q)(x)$.

  NO! Pay attention to linearizing over right argument

- $$\begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$ is reflection matrix

- Can only cancel matrix from both sides of eq. if known invertible

- pf: If A is invertible, there is a unique matrix B s.t. $AB = BA = I_n$.

  Suppose $AB = BA = I_n$ and $AC = CA = I_n$.

  $B = I_n B = (CA)B = C(AB) = CI_n = C$

- Suppose $T_A: \mathbb{R}^n \to \mathbb{R}^m$, with $A \in \mathbb{R}^{m \times n}$

  $m < n$ → T_A is not injective.

  Note if $m < n$, rref(A) will have non-pivot columns → free variables → multiple solutions for $A\vec{x} = \vec{0}$

  $m > n$ → T_A is not surjective

- $\text{Span}(v_1, \dots, v_n) = \text{im}\left(\begin{bmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{bmatrix}\right)$

- pf: For linear $T: V \to W$, T is injective iff $\ker(T) = \{\vec{0}\}$.

  i) Suppose T is injective.

  $T(\vec{0}) = \vec{0}$, so $\vec{0} \in \ker(T)$.

  Suppose $x \in V$ s.t. $T(x) = \vec{0}$. Then, $x = \vec{0}$ by defn. of injective.

  ii) Suppose $\ker(T) = \{\vec{0}\}$

  Suppose $\vec{x}, \vec{y} \in V$, $T(x) = T(y)$.

  Then, $T(x) - T(y) = T(x-y) = 0$.

  So, $x - y = 0$ → $x = y$. So, T is injective.

- To find ordered bases of image, kernel of a transform: - this is correct as columns are in $\mathbb{R}^m$ for $m \times n$ matrix, which maps $\mathbb{R}^n \to \mathbb{R}^m$.

  → image: Take rref, choose pivot columns

  → kernel: Solve equations in terms of free variables — these coefficients for $t_1, \dots$ form basis of kernel

- To find a transform with a certain solution set:

  → $S = \begin{cases} x_1 + 2x_3 + 3x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases}$

  → $\text{im}(T) = S$ → $T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 0 \\ 1 & 0 \end{bmatrix}$; $\ker(T) = S = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

- Set of all even polynomials, odd polynomials both subspaces of all $\mathcal{P}$.

- $X \cup \{v\}$ is linearly indep or $v \in \text{Span}(X)$

  Suppose $X \cup \{v\}$ is lin. dependent. Then,

  $c_0 v + c_1 x_1 + \cdots c_n x_n = \vec{0}$ is a non-trivial liner relation. Note $c_0 \neq 0$ as then $c_1 x_1 + \cdots + c_n x_n$ would be a non-trivial linear relation, implying X is not lin. indep.

  So, $v = -\frac{1}{c_0}(c_1 x_1 + \cdots + c_n x_n)$, meaning $v \in \text{Span}(X)$

- B is maximal linearly independent subset of V
- B is minimal spanning subset of V.

- $\mathcal{E} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$; $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$

  $A = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$

  $\rightarrow [I_2]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\rightarrow [A]_{\mathcal{E}} = \begin{bmatrix} 3 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

  $\rightarrow [I_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\rightarrow [A]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

- V given by $x_1 + x_2 - 2x_3 = 0$

  a) Find basis $\mathcal{B}$ of V s.t. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

  $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

  arbitrary   based on first

- Suppose $T: V \to \mathbb{R}^n$ is an isomorphism.

  Pick a basis for V based on T.

  $$\begin{tikzcd} V \arrow[d, shift right, "T"', "L_{\mathcal{B}}"'] \\ \mathbb{R}^n \arrow[u, shift right, "T^{-1}"'] \end{tikzcd}$$

  $L_{\mathcal{B}}^{-1}$ columns are basis vectors

  Why? T^{-1} of e_i is T^{-1} of standard basis vector, converting back to normal coords.

Worksheet Notes (15 - 27)

- If $\mathcal{B} = (b_1, \dots, b_n)$ is a basis of V and $T: V \to W$ is an isomorphism, $T(b_1), T(b_2), \dots, T(b_n)$ is a basis of W

- Equivalent Statements for linear transfor $T: \mathbb{R}^n \to \mathbb{R}^n$, standard matrix A:
  - → T preserves length
  - → T preserves distance
  - → T is an orthogonal transformation (preserves dot product)
  - → T maps any ONB to an ONB
  - → T maps standard basis of $\mathbb{R}^n$ to an ONB
  - → Columns of A form ONB of $\mathbb{R}^n$
  - → $AA^T = I_n$
  - → $A^TA = I_n$
  - → rows of A form ONB of $\mathbb{R}^n$

- All orthogonal transforms in $\mathbb{R}^2$ are given by matrix

rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ reflection $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

- Show, for any $m \times n$ matrix A, $\ker(A^T) = \operatorname{im}(A)^\perp$

pf: $x \in \ker(A^T) \iff A^Tx = \vec{0}$

$\iff A\vec{e}_j \cdot x = 0$ for all $1 \le j \le n$

$\iff x \cdot y = 0$ for all $y \in \operatorname{im}(A)$

$\iff x \in \operatorname{im}(A)^\perp$

→ $\ker(A^T) = \operatorname{im}(A)^\perp$ → $\ker(A) = \operatorname{im}(A^T)^\perp$

→ $\ker(A^T)^\perp = \operatorname{im}(A)$ → $\ker(A)^\perp = \operatorname{im}(A^T)$

- Show $\ker(A^TA) = \ker(A)$:

($\Rightarrow$) pf: Suppose $x \in \ker(A)$. Then, $A\vec{x} = \vec{0}$. So, $A^TA\vec{x} = A^T(A\vec{x}) = A^T(\vec{0}) = \vec{0}$, so $x \in \ker(A^TA)$.

($\Leftarrow$) pf: Suppose $x \in \ker(A^TA)$. Then, $A^TA\vec{x} = \vec{0}$.

So, $A\vec{x} \in \ker(A^T)$. Furthermore, $A\vec{x} \in \operatorname{im}(A) = \ker(A^T)^\perp$.

So, $A\vec{x} \in \ker(A^T) \cap \ker(A^T)^\perp = \{\vec{0}\}$. So, $A\vec{x} = \vec{0} \Rightarrow \vec{x} \in \ker(A)$.

- Projection matrix:
  - → $P = A(A^TA)^{-1}A^T$
  - → $P^2 = P = P^T$

$A^TA_{(i,j)} = a_i \cdot a_j$

$AA^T_{(i,j)} =$

- Example Inner Products:
  - V: P_2

    $\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$
  - V: $\mathbb{R}^{m \times n}$

    $\langle A, B \rangle = tr(A^TB)$

- Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the direct image of the unit circle under linear transform $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ax \\ by \end{bmatrix}$, find formula for area of ellipse.

  So, $[T]_e = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. So, Area$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right) = \det(T)\left[\pi(1^2)\right]$

  $= \pi ab$

- T preserves volume iff $|\det T| = 1$

- Given eigenvalue λ and corresponding eigenvector $\vec{v}$:

  T^n has eigenvalue λ^n w/ eigenvector $\vec{v}$

- Show if eigenvectors v_1, v_2 correspond to distinct eigenvalues λ_1, λ_2, v_1 and v_2 are lin independent.

  Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$.

  $\lambda_1(c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0})$

  $c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}$

  $T(c_1\vec{v}_1) + T(c_2\vec{v}_2) = T(\vec{0})$

  $c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0}$

  $(\lambda_1 - \lambda_2)c_2\vec{v}_2 = \vec{0}$; so $c_2 = 0$. Similarly, $c_1 = 0$.

- Extend this proof to r eigenvectors.

  By INDUCTION:

  Base Case: $r = 1$ is trivial

  Inductive Step:

  Suppose all lists $v_1, \ldots, v_{r-1}$ are lin. indep

  Suppose $\sum_{i=1}^{r} c_i v_i = \vec{0}$

  λ_r: $\sum_{i=1}^{r} \lambda_r c_i v_i = \vec{0}$

  T: $\sum_{i=1}^{r} \lambda_i c_i v_i = \vec{0}$

  $\sum_{i=1}^{r-1} c_i(\lambda_i - \lambda_r)\vec{v}_i = \vec{0}$, so $c_i = 0$ for $1 \le i \le r-1$

  Then, $c_r = 0$ by original eqn.

- A is diagonalizable iff $\sum \text{gemu}(\lambda) = n$

-

		INVERTIBLE Y	INVERTIBLE N
DIAGONALIZABLE	Y	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
	N	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

- $\overline{z + w} = \bar{z} + \bar{w}$

  $\overline{zw} = (\bar{z})(\bar{w})$

  $\bar{z} = z$ iff $z \in \mathbb{R}$

- Show for any symmetric $A \in \mathbb{R}$, A has n real eigenvalus (counting multiplicities).
  - >> By fundamental theorem of algebra, A has n complex eigenvalus as characteristic equation is of degree n.
  - >> Let λ be a complex eigenvalue of A w/ corresponding complex eigenvector $\bar{z}$. We will show λ is real.
  - >> As $Az = \lambda z$, $A\bar{z} = \bar{\lambda}\bar{z}$

$$\begin{aligned}
\bar{z}^T A^T &= \bar{\lambda}\, \bar{z}^T \\
\bar{z}^T A^T z &= \bar{\lambda}\, \bar{z}^T z \\
\bar{z}^T A z &= \bar{\lambda}(\bar{z} \cdot z) \\
\bar{z}^T (\lambda z) &= \bar{\lambda}(\bar{z} \cdot z) \\
\lambda(\bar{z}^T \cdot z) &= \bar{\lambda}(\bar{z} \cdot z) \\
\lambda &= \bar{\lambda} \quad \Rightarrow \lambda \in \mathbb{R}.
\end{aligned}$$

- Prove backward direction of spectral thm (orthogonally diagonalizable $\Rightarrow$ symmetric)

  Suppose $A \in \mathbb{R}^{n \times n}$ is orthogonally diagonalizable

  Then, $A = QDQ^{-1}$ for some orthogonal matrix Q, diagonal matrix D.

$$\begin{aligned}
\text{So, } A^T &= (QDQ^{-1})^T \\
&= Q^{-1^T} D^T Q^T \\
&= Q^{T^T} D^T Q^{-1} && \text{as } Q^T = Q^{-1} \text{ as } Q \text{ is orthogonal} \\
&= QDQ^{-1} && \text{as } D^T = D \text{ as } D \text{ is diagonal} \\
&= A.
\end{aligned}$$

  As $A^T = A$, A is symmetric.

- Show forward direction of spectral theorem (symmetric $\Rightarrow$ orthogonally diagonalizable)

  <u>BASE CASE</u>: Consider symmetric $A \in \mathbb{R}^{1 \times 1}$.

  Then, let $A = [a]$, $a \in \mathbb{R}$.

  $[a] = [1][a][1]^{-1}$, so A is orthogonally diagonalizable as $[1]$ is orthogonal matrix.

  <u>INDUCTIVE STEP</u>: Suppose for all symmetric $A \in \mathbb{R}^{n \times n}$, A is orthogonally diagonalizable.

  Let $\mathcal{U} = \{\hat{u}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$, a basis of $\mathbb{R}^{n+1}$.

  Let $Q = \begin{bmatrix} | & | & & | \\ \vec{u} & \vec{u}_1 & \cdots & u_n \\ | & | & & | \end{bmatrix}$. Then $Q^T = Q^{-1} = S_{\mathcal{E} \to \mathcal{U}}$

$$Q^T A Q \vec{e}_1 = Q^T A \hat{u} = Q^T \lambda \hat{u} = \lambda S_{\mathcal{E}\to\mathcal{U}}[\vec{u}]_{\mathcal{E}} = \lambda [u]_{\mathcal{U}} = \lambda \vec{e}_1$$

$$\text{So} \quad Q^T A Q = \left[\begin{array}{c|ccc} \lambda & 0 & 0 \cdots & 0 \\ \hline 0 & & & \\ 0 & & B & \\ \vdots & & & \\ 0 & & & \end{array}\right]$$

We know by I.H. that $B = RDR^T$, D is diagonal.

So, $$Q^T A Q = \left[\begin{array}{c|c} \lambda & 0 \\ \hline 0 & RDR^T \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & R^T \end{bmatrix}$$

$$Q^T A Q = P D' P^{-1}$$

$$P^T Q^T A Q = D' P^T$$

$$P^T Q^T A Q P = D'$$

$(QP)^{-1} A (QP) = D'$, so A is orth. diag.

Handouts Notes

Complex Numbers: