CHAPTER 12

Distance Formula (2 points):

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Sketching 3D graphs:

- fix one variable (input value) to get cross section

Dot Product (2 vectors → scalar)

$\vec{v} \cdot \vec{w} = v_x w_x + v_y w_y + v_z w_z$

$\vec{v} \cdot \vec{v} = |\vec{v}|^2$

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (commutative)

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributive)

$(r\vec{v}) \cdot \vec{w} = r(\vec{v} \cdot \vec{w})$ (associative)

$\vec{v} \cdot \vec{0} = 0.$

If $\vec{v} \cdot \vec{w} = 0$, then $\vec{v}$ is perpendicular to $\vec{w}$. (Converse holds too).

$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos \theta$

Proj on $\vec{w}$ of $\vec{v}$ ($\mathrm{Proj}_{\vec{w}} \vec{v}$):

$$\boxed{\frac{\vec{v} \cdot \vec{w}}{|\vec{w}|^2} \vec{w}}$$

Line through (x_0, y_0, z_0) with direction vector (a, b, c):

$\vec{\ell}(t) = \langle at + x_0,\ bt + y_0,\ ct + z_0 \rangle$

Can be parametrized in infinitely many ways.

Need:

- 2 points on line
- Point + Direction Vector

Plane:

- 3 non-collinear pts
- Point + normal vector

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

with $\vec{n} = \langle a, b, c \rangle$

$P = (x_0, y_0, z_0)$

When given 3 points, cross 2 vecs to find normal.

Intersection of two planes:

cross two normals to get direction vector of line; solve algebraically for point

Quadric surface: given by degree 2 equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ELLIPSE

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ HYPERBOLOID OF ONE SHEET

$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{z^2} = 1$ HYPERBOLOID OF TWO SHEETS

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ CONE

$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ PARABOLOID

$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ HYPERBOLIC PARABOLOID

Cross Product (2 vec → vec)

$$\vec{v} \times \vec{w} = \left\langle \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \right\rangle$$

- $\vec{v} \times \vec{v} = \vec{0}$
- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

Distributive, Scalar assoc.

- $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin \theta$
- $\vec{v} \times \vec{w}$ perpendicular to $\vec{v}, \vec{w}$.
  - RH rule with $\vec{v} = x$, $\vec{w} = y$, $\vec{v} \times \vec{w} = z$.
- Area of triangle PQR: $\frac{1}{2} |\vec{PQ} \times \vec{PR}|$ ...

CHAPTER 13

Vector function:

function who's output is a vector; creates space curve

line is special vector function

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

continuous if component functions continuous

$\vec{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

$\int_a^b \vec{r}(t) = \langle \int_a^b f(t), \int_a^b g(t), \int_a^b h(t) \rangle$

$\int_a^b \vec{r}'(t) = \vec{r}(b) - \vec{r}(a)$

$r'(t)$ is TANGENT VECTOR to $r(t)$

ARC LENGTH ℓ between $r(a)$, $r(b)$ =

$$\int_a^b |r'(t)| \, dt$$

$|\ |$: MAGNITUDE

Parametrization by arc length:

$$a = \int_0^a |r'(t)| \, dt$$

every curve can be uniquely parametrized by arc length.

- How to find:
  - Set s to be equal to $\int_0^t |r'(u)| \, du$.
  - Find s in terms of t
  - Find t in terms of s
  - Substitute s for t in $r(t)$

$\vec{r}(t)$ represents position of moving object @ time t. ($r'(t)$ = velocity, $r''(t)$ = acceleration)

COLLIDE: Set components equal w/ single var

INTERSECT: Set components equal w/ diff vars

MIN DISTANCE (POINT TO POINT):

Use distance formula to find dist. as function of t.

Minimize (can square to minimize).

MIN DISTANCE (TRAJECTORY):

= min distance between two lines

$\vec{d}$ is perpendicular to both dir. vectors

$\vec{\ell}_1 \times \vec{\ell}_2 = \vec{n}$.

Then find plane w/ normal $\vec{n}$ containing $\vec{\ell}_1$. Find distance to random $\vec{\ell}_2$ point using point-to-plane

CHAPTER 14

$f(x,y)$: func of two variables.

Often $z = f(x,y)$.

Level Curve: $f(x,y) = k$ (cross-section in xy).

Contour Map = collection of level curves.

Get LEVEL SURFACES for $f(x,y,z)$.

Drawing level curves: plug in, figure it out. (z 5)

Chain Rule:

$f(x,y)$; $x(t), y(t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

If z implicitly $f(x,y)$ w/ $f(x,y,z) = C$, then:

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} \qquad \frac{\partial z}{\partial y} = \frac{-f_y}{f_z}$$

PARTIALS:

Keep all other vars constant, move one var.

$f_{xy} = f_{yx}$ if f_{xy}, f_{yx} continuous.

Estimates: do two-sided, then average

Find signs from contour map:

- first order: just see if level curves increase
- second order: level curves getting closer together or further apart?

  f_x -ve, closer : -ve

  f_x -ve, further : +ve

  f_x +ve, closer : +ve

  f_x +ve, further : -ve

  f_{xy} test: see f_x at diff y-values.

TANGENT PLANE:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\vec{n} = \langle f_x, f_y, -1 \rangle$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$ (estimate small changes).

Directional Derivative + Gradient:

(1) $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$ (normal to tangent plane)

(2) $D_{\vec{u}}$: Directional Derivative of $f(x,y,z)$ in $\vec{u}$ direction

$$D_{\vec{u}} f(a,b,c) = \nabla f(a,b,c) \cdot \vec{u}$$

(3) $\nabla f(a,b,c)$: fastest increase

$|\nabla f(a,b,c)|$: max directional derivative

(4) If $x(t), y(t), z(t)$:

$$\frac{df}{dt} = \nabla f \cdot r'(t)$$