SYMMETRY IN IMAGE REGISTRATION AND DEFORMATION MODELING

STEFAN SOMMER AND HENRY O. JACOBS

Abstract. We survey the role of symmetry in diffeomorphic registration of landmarks, curves, surfaces, images and higher-order data. The infinite dimensional problem of finding correspondences between objects can for a range of concrete data types be reduced resulting in compact representations of shape and spatial structure. This reduction is possible because the available data is incomplete in encoding the full deformation model. Using reduction by symmetry, we describe the reduced models in a common theoretical framework that draws on links between the registration problem and geometric mechanics. Symmetry also arises in reduction to the Lie algebra using particle relabeling symmetry allowing the equations of motion to be written purely in terms of Eulerian velocity field. Reduction by symmetry has recently been applied for jet-matching and higher-order discrete approximations of the image matching problem. We outline these constructions and further cases where reduction by symmetry promises new approaches to registration of complex data types.

1. Introduction

Registration, the task of establishing correspondences between multiple instances of objects such as images, landmarks, curves, and surfaces, plays a fundamental role in a range of computer vision applications including shape modeling [You10], motion compensation and optical flow [BBPW04], remote sension [DSS10], and medical imaging [SDP13]. In the subfield of computational anatomy [YAM09], establishing inter-subject correspondences between organs allows the statistical study of organ shape and shape variability. Examples of the fundamental role of registration include quantifying developing Alzheimer's disease by establishing correspondences between brain tissue at different stages of the disease [BRA+06]; measuring the effect of COPD on lung tissue after removing the variability caused by the respiratory process [GJL+10]; and correlating the shape of the hippocampus to schizophrenia after inter-subject registration [JMG97].

In this paper, we survey the role of symmetry in diffeomorphic registration and deformation modeling and link symmetry as seen from the field of geometric mechanics with the image registration problem. We focus on large deformations modeled in subgroups of the group of diffeomorphic mappings on the spatial domain, the approach contained in the Large Deformation Diffeomorphic Metric Mapping (LDDMM, [DGM98, Tro95, CRM02, You10]) framework. Connections with geometric mechanics [HRTY04] have highlighted the role of symmetry and resulted in previously known properties connected with the registration of specific data types being described in a common theoretical framework [Jac13]. We wish to describe these connections in a form that highlights the role of symmetry and points towards future applications of the ideas. It is the aim that the paper will make the role of symmetry in registration and deformation modeling clear to the reader that has no previous familiarity with symmetry in geometric mechanics and symmetry groups in mathematics.

1.1. **Symmetry and Information.** One of the main reasons symmetry is useful in numerics is in it's ability to reduce how much information one must carry. As a toy example, consider the a top spinning in space. Upon choosing some reference configuraiton, the orientation of the top is given by a rotation matrix, i.e. an element $R \in \mathrm{SO}(3)$. If I ask for you to give me the direction of the pointy tip of the top, (which is pointing opposite $\mathbf{k}$ in the reference) it suffices to give me R. However, R is contained in space of dimension 3, while the space of possible directions is the 2-sphere, S^2, which is only of dimension 2. Therefore, providing the full matrix R is excessive in terms of data. It suffices to just provide the vector $R \cdot \mathbf{k} \in S^2$. Note that if $\tilde{R} \cdot \mathbf{k} = \mathbf{k}$, then $R \cdot \mathbf{k} = R \cdot \tilde{R} \cdot \mathbf{k}$. Therefore, given only the direction $\mathbf{k}' = R \cdot \mathbf{k}$, we can only reconstruct R up to an element $\tilde{R}$ which preserves $\mathbf{k}$. The group of element which preserve $\mathbf{k}$ is identifiable with $\mathrm{SO}(2)$. This insight allows us to express the space of directions S^2 as a homogenous space $S^2 \equiv \mathrm{SO}(3)/\mathrm{SO}(2)$. In terms of infomation we can cartoonishly express this by the expression

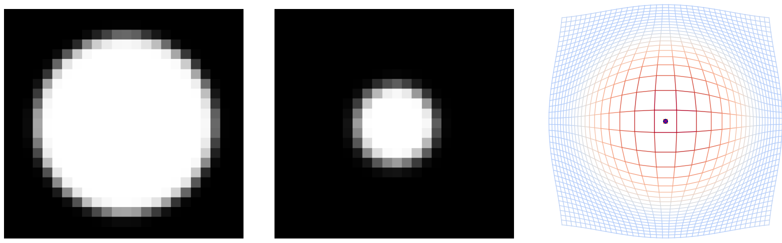
$$\text{“orientation”} = \text{“direction of tip”} + \text{“orientation around the tip”}$$

This example is typically of all group quotients. If X is some universe of objects and G is a group which acts freely upon X, then the orbit space X/G hueristically contains the data of X minus the data which G transforms. Thus

$$\mathrm{data}(X) = \mathrm{data}(X/G) + \mathrm{data}(G).$$

Reduction by symmetry can be implemented when a problem posed on X has G symmetry, and can be rewritten as a problem posed on X/G. The later space containing less data, and is therefore more efficient in terms of memory.

1.2. **Symmetry in Registration.** Registration of objects contained in a spatial domain, e.g. the volume to be imaged by a scanner, can be formulated as the search for a deformation that transforms both domain and objects to establish an inter-object match. The data available



(a) fixed image (b) moving image (c) warp

FIGURE 1. A registration of two discs of different sizes (a,b) with a warp that brings (b) into correspondence with (a) visualized by its effect on an initially regular grid (c). Using symmetry, the dimensionality of the registration problem can be reduced from infinite to finite. In this case, 6 parameters of a 1-jet particle in the center of the moving image encode the entire deformation.

when solving a registration problem generally is incomplete for encoding the deformation of every point of the domain. This is for example the case when images to be matched have areas of constant intensity and no derivative information can guide the registration. Similarly, when 3D shapes are matched based on similarity of their surfaces, the deformation of the interior cannot be derived from the available information. The deformation model is in these cases over-complete, and a range of deformations can provide equally good matches for the data. Here arises *symmetry*: the subspaces of deformations for which the registration problem is symmetric with respect to the available information. When quotienting out symmetry subgroups, a vastly more compact representation is obtained. In the image case, only displacement orthogonal to the level lines of the image is needed; in the shape case, the information left in the quotient is supported on the surface of the shape only.

1.3. **Content and Outline.** We start with background on the registration problem and the large deformation approach from a variational viewpoint. Following this, we describe how reduction by symmetry leads to an Eulerian formulation of the equations of motion when reducing to the Lie algebra. Symmetry of the dissimilarity measure allows additional reductions, and we use isotropy subgroups to reduce the complexity of the registration problem further. Lastly, we survey the effect of symmetry in a range of concrete registration problems and end the paper with concluding remarks.

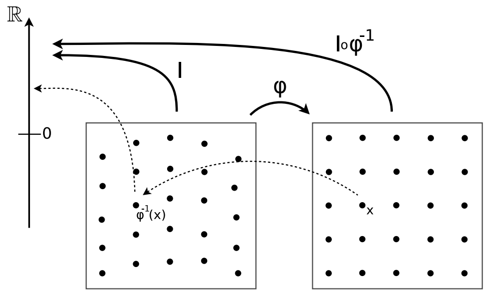


FIGURE 2. A warp $\varphi \in \mathrm{Diff}(M)$ acts on an image $I : M \to \mathbb{R}$ by composition with the inverse warp, $\varphi \cdot I = I \circ \varphi^{-1}$. Given two images $I_0, I_1 : M \to \mathbb{R}$, image registration involves finding a warp φ such that $\varphi \cdot I_0$ is close to I_1 as measured by a dissimilarity measure $F(\varphi \cdot I_0, I_1)$.

2. Registration and Variational Formulation

The registration problem consists in finding correspondences between objects that are typically point sets (landmarks), curves, surfaces, images or more complicated spatially dependent data such as diffusion weighted images (DWI). The problem can be approached by letting M be a spatial domain containing the objects to be registered. M can be a differentiable manifold or, as is often the case in applications, the closure of an open subset of $\mathbb{R}^d$, $d = 2, 3$, e.g. the unit square. A map $\varphi : M \to M$ can deform or warp the domain by mapping each $x \in M$ to $\varphi(x)$.

The deformation encoded in the warp will apply to the objects in M as well as the domain itself. For example, if the objects to be registered consist of points sets $\{x_1, \ldots, x_N\}$, $x_i \in M$, the set will be mapped to $\{\varphi(x_1), \ldots, \varphi(x_N)\}$. For surfaces $S \subset M$, φ similarly results in the warped surface $\varphi(S)$. Because those operations are associative, the mapping φ acts on $\{x_i\}$ or S and we write $\varphi \cdot \{x_i\}$ and $\varphi \cdot S$ for the warped objects. An image is a function $I : M \to \mathbb{R}$, and φ acts on I as well, in this case by composition with its inverse $\varphi \cdot I = I \circ \varphi^{-1}$, see Figure 2. For this φ must be is invertible, and commonly we restrict to the set of invertible and differentiable mappings $\mathrm{Diff}(M)$. For various other types of data objects, the action of a warp on the objects can be defined in a way similar to the case for point sets, surfaces and images. This fact relates a range registration problems to the common case of finding appropriate warps φ that trough the action brings the objects into correspondence. Trough the action, different instances of a shape can be realized by letting warps act on a base instance of the shape, and a class of shape models can therefor be obtained by using deformations to represent shapes [You10].

2.1. **Variations over Warps and Families of Warps.** The search for appropriate warps can be formulated in a variational formulation with an energy

$$E(\varphi) = R(\varphi) + F(\varphi) \tag{1}$$

where F is a dissimilarity measure of the difference between the deformed objects, and R is a regularization term that penalizes unwanted properties of φ such as irregularity. If two objects o_1 and o_2 is to be matched, F can take the form $F(\varphi \cdot o_0, o_1)$ using the action of φ on o_0; for image matching, an often used dissimilarity measure is the L^2-difference or sum of square differences (SSD) that has the form $F(\varphi \cdot I_0, I_1) = \int_M |I_0 \circ \varphi^{-1}(x) - I_1(x)|^2 dx$.

The regularization term can take various forms often modeling physical properties such as elasticity [PSA+05] and penalizing derivatives of φ in order to make it smooth. The free-form-deformation (FFD, [RSH+99]) and related approaches penalize φ directly. For some choices of R, existence and analytical properties of minimizers of (1) have been derived [DLG14], however it is in general difficult to ensure solutions are diffeomorphic by penalizing φ in itself. Instead, flow based approaches model one-parameter families or paths of mappings φ_t, $t \in [0,1]$ where φ_0 is the identity mapping $id \in \mathrm{Diff}(M)$ and the dissimilarity is measured at the endpoint φ_1. The time evolution of φ_t can be described by the differential equation $\frac{d}{dt}\varphi_t(x) = v_t(\varphi(x))$ with the flow field v_t being a vector field on M. The space of such fields is denoted V. In the Large Deformation Diffeomorphic Metric Mapping (LDDMM, [You10]) framework, the regularization is applied to the flow field v_t and integrated over time giving the energy

$$E(\varphi_t) = \int_0^1 \|v_t\|_V^2 dt + F(\varphi_1) \ . \tag{2}$$

If the norm $\|\cdot\|_V$ that measures the irregularity of v_t is sufficiently strong, φ_t will be a diffeomorphism for all t. This approach thus gives a direct way of enforcing properties of the generated warp: Instead of regularizing φ directly, the analysis is lifted to a normed space V that is much easier control. The energy (2) has the same minimizers as the geometric formulation of LDDMM used in the next section.

Direct approaches to solving the optimization problem (2) must handle the fact that the problem of finding a warp is now transfered to finding a time-dependent family of warps implying a huge increase in dimensionality. This problem is therefore vary hard to represent numerically and to optimize. For several data types, it has been shown how optimal paths for (2) have specific properties that reduces the dimensionality of the problem and therefor makes practical solutions feasible. In the next section, we describe the geometric framework and

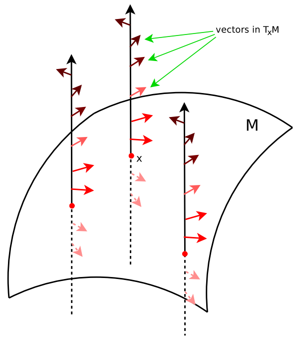


FIGURE 3. The tangent bundle TM of the manifold M consists of pairs (x, v) of points $x \in M$ and tangent vectors $v \in T_xM$. It's a fiber bundle over M with fibers T_xM for each $x \in$M.

the reduction theory that allows the data dependent results to be formulated as specific examples of reduction by symmetry. We survey these examples in the following section.

3. Reduction by symmetry in LDDMM

We here describe a geometric formulation of the registration problem [You10] and how symmetry can be used to reduce the optimization over time-dependent paths of warps to vector fields in the Lie algebra resulting in an Eulerian version of the equations of motion. Secondly, we describe how symmetry of the dissimilarity measure allows further reduction to lower dimensional quotients.

3.1. **Kinematics.** We here introduce a number of notions from differential geometry in a fairly informal manner. For formal definitions we refer to [AM78]. While it is neccessary for the purpose of rigour to learn formal definitions, independent of cartoonish sketches, when learning differential geometry, one can still get quite far with cartoonish sketches. For example, by picturing a manifold, M, as a surface embedded in $\mathbb{R}^3$. This is the approach we will take.

The tangent bundle of M, denoted TM, is the set of pairs (x, v) where $x \in M$ and v is a vector tangential to M at the point x (see Figure 3). A vector-field is a map $u : M \to TM$ such that $u(x) \in TM$ is a vector above x for all $x \in M$. In summary, a vector on M is an "admissible velocity" on M, and TM is the set of all possible admissible velocities.

Given a vector-field u we may consider the initial value problem

$$\begin{cases} x(0) = x_0 \\ \frac{dx}{dt} = u(x(t)) \end{cases}$$

for $t \in [0,1]$. Given an intial condition x_0, the point $x_1 = x(1)$ given by solving this initial value problem is uniquely determined (if it exists). Under reasonable conditions x_1 exists for each x_0, and there is a map $\Phi^u_t : x_0 \in M \mapsto x_1 \in M$ which we call the flow of u. If u is time-dependent we can consider the initial value problem with $\frac{dx}{dt} = u(t, x(t))$. Under certain conditions, this will also yield a flow map, $\Phi^u_{t_0,t_1}$ which is the flow from time $t = t_0$ to $t = t_1$. If u is smooth, the flow map is smooth as well, in particular a diffeophism. We denote the set of diffeomorphisms by $\mathrm{Diff}(M)$.

Conversely, let $\varphi_t \in \mathrm{Diff}(M)$ be a time-dependent diffeomorphism. Thus, for any $x \in M$, we observe that $\varphi_t(x)$ is a curve in M. If this curve is differentiable we may consider its time-derivative, $\frac{d\varphi_t}{dt}(x)$, which is a vector above the point $\varphi_t(x)$. From these observations it imediately follows that $\frac{d\varphi_t}{dt}(\varphi_t^{-1}(x))$ is a vector above x. Therefore the map $u(t) : M \to TM$, given by $u(t) := \left(\frac{d}{dt}\varphi_t\right) \circ \varphi_t^{-1}$ is a vector-field which we call the *Eulerian velocity field of* φ_t.

The Eulerian velocity field contains less data than $\frac{d\varphi_t}{dt}$, and this reduction in data can be viewed from the perspective of symmetry. Given any $\psi \in \mathrm{Diff}(M)$, the curve φ_t can be transformed to the curve $\varphi_t \circ \psi$. We observe that

$$\begin{aligned} u(t) &:= \frac{d\varphi}{dt} \circ \varphi^{-1} = \frac{d\varphi_t}{dt} \circ \psi \circ \psi^{-1} \circ \varphi^{-1} \\ &= \left(\frac{d}{dt}(\varphi_t \circ \psi) \right) \circ (\varphi \circ \psi)^{-1}. \end{aligned}$$

Thus φ_t and $\varphi_t \circ \psi$ both have the same Eulerian velocity fields. In other words, the Eulerian velocity field, $u(t)$, is invariant under particle relablings. Schematically, the following holds

$$\mathrm{data}\left(\frac{d\varphi}{dt} \right) = \mathrm{data}(u(t)) + \mathrm{data}(\varphi_t).$$

Finally, we will denote some linear operators on the space of vector-fields. Let $\Phi \in \mathrm{Diff}(M)$ and let $u \in \mathfrak{X}(M)$. The *push-forward* of u by Φ, denoted $\Phi_* u$, is the vector-field given by

$$[\Phi_* u](x) = D\Phi|_{\Phi^{-1}(x)} \cdot u(\Phi^{-1}(x)).$$

By inspection we see that Φ_* is a linear operator on the vector-space of vector-fields. One can view $\Phi_* u$ as "u in a new coordinate system" because any differential geometric property of u is also inherited by $\Phi_* u$. For example, if $u(x) = 0$ then $[\Phi_* u](y) = 0$ with $y = \Phi(x)$. If S is an invariant under u then $\Phi(S)$ is invariant under $\Phi_* u$.

As Φ_* is linear, we can transpose it. Let $\mathfrak{X}(M)^*$ denote the dual space to the space of vector-fields, i.e. the set of linear maps $\mathfrak{X}(M) \to \mathbb{R}$, and let $m \in \mathfrak{X}(M)^*$. We define $\Phi_* m \in \mathfrak{X}(M)^*$ by the equality

$$\langle \Phi_* m, \Phi_* v \rangle = \langle m, v \rangle$$

for all $v \in \mathfrak{X}(M)$ where $\langle m, v\rangle$ denotes evaluation of m on v.

Finally, we define the Lie-derivative as the linear operator $\pounds_w : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$\pounds_w[u] = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\Phi^w_\epsilon)_* u.$$

As $\pounds_w$ is linear, we can take its transpose. If $m \in \mathfrak{X}(M)^*$, then we can define $\pounds_u[m] \in \mathfrak{X}(M)^*$ by the equation

$$\langle \pounds_u[m], w\rangle + \langle m, \pounds_u[w]\rangle = 0$$

for all $w \in \mathfrak{X}(M)$. This is a satisfying because for a fixed m and w we observe

$$\langle \pounds_u[m], w\rangle + \langle m, \pounds_u[w]\rangle = 0 = \frac{d}{dt}\langle m, w\rangle = \frac{d}{dt}\langle (\Phi^u_t)_* m, (\Phi^u_t)_*\rangle \tag{3}$$

This is nothing but a coordinate free version of the product rule.

3.2. **Reduction to Lie Algebra.** The variational formulation (2) of LDDMM is equivalent to minimizing the energy

$$E = d(id, \varphi) + F(\varphi)$$

where $d : \mathrm{Diff}(M) \times \mathrm{Diff}(M) \to \mathbb{R}$ is a distance metric on $\mathrm{Diff}(M)$, id is the identity diffeomorphism, and $F : \mathrm{Diff}(M) \to \mathbb{R}$ is a function which measures the disparity between the deformed template and the target image.

Example 1. *Given images $I_0, I_1 \in L^2(M)$, we consider the dissimilarity measure*

$$F(\varphi) = \|(I_0 \circ \varphi^{-1}) - I_1\|^2_{L^2(M)}.$$

In this article we will consider the distance metric

$$d(\varphi_0, \varphi_1) = \inf_{\substack{v \in C^0([0,1], \mathfrak{X}(M)) \\ \Phi^v_{0,1} \circ \varphi_0 = \varphi_1}} \left(\int_0^1 \|v(t)\| dt \right),$$

where $\|\cdot\|$ is some norm on $\mathfrak{X}(M)$. If $\|\cdot\|$ is induced by an inner-product, then this distance metric is (formally) a Riemannian distance metric on $\mathrm{Diff}(M)$. Note that the distance metric, d, is written in terms of a norm $\|\cdot\|$, defined on $\mathfrak{X}(M)$. In fact, the norm on $\mathfrak{X}(M)$ induces a Riemannian metric on $\mathrm{Diff}(M)$ given by

$$\|\frac{d\varphi_t}{dt}\|^2 := \|\frac{d\varphi_t}{dt} \circ \varphi_t^{-1}\|^2,$$

and d is the Reimannian distance with respect to this metric. If the norm $\|\cdot\|$ imposes a Hilbert space structure on the vector-fields it can be written in terms of a psuedo-differential operator $P : \mathfrak{X}(M) \to \mathfrak{X}(M)^*$ as $\|u\|^2 = \langle P[u], u\rangle$ [You10].

Given P, minimizers of E must neccessarily satisfy

$$\begin{cases} m(t) = (\Phi^u_{t,1})_* m(1) = P[u(t)] \\ \langle m(1), w\rangle = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(\Phi^w_\epsilon \circ \Phi^u_{0,1}) \quad , \quad \forall w \in \mathfrak{X}(M). \end{cases} \tag{4}$$

This is a vector-calculus statement of Proposition 11.6 of [You10]. That this equation of motion is even well-posed is nontrivial, since P is merely an injective map, and there is no guarantee that it can be inverted to obtain a vector-field u to integrate into a diffeomorphism. Fortunately, safety guards for well-posedness are studied in [TY05]. If the reproducing kernel of P is C^1, then the equations of motion are well-posed for all time.

There is something unsatisfying about using (4) for the purpose of computation. Doing any sort of computation on $\mathrm{Diff}(M)$ is difficult, as it is a nonlinear infinite dimensional space. Moreover, the dissimilarity measure F only comes into play at time $t = 1$ and the distance function is an integral over the vector-space $\mathfrak{X}(M)$. It would be nice if we could rewrite the extremizers purely in terms of the Eulerian velocity field, u and the flow at $t = 1$. In fact this is often the case. Given P, the minimizer of E must neccessarily satisfy the boundary value problem

$$\begin{cases} \partial_t m + \pounds_u[m] = 0, m = P[u] \quad \forall t \in [0,1] \\ \frac{d}{d\epsilon}|_{\epsilon=0} \left[F(\Phi^w_\epsilon \circ \Phi^u_{0,1})\right] + \langle P[u(1)], w\rangle = 0 \quad , \forall w \in \mathfrak{X}(M). \end{cases} \tag{5}$$

This is an alternative formulation of (4), obtained simply by taking a time-derivative. In the language of fluid-dynamics, (5) is an Eulerian version of (4). The advantage of this formulation, is that the bulk of the computation occurs on the vector-space $\mathfrak{X}(M)$, and this observation is the starting point for the algorithm given in [BMTY05].

This reduction of the problem to the space of vector-fields is a first instance of reduction by symmetry. In particular, this corresponds to the fact that the space of vector-fields $\mathfrak{X}(M)$, is identifiable as a quotient space

$$\mathfrak{X}(M) \equiv T\,\mathrm{Diff}(M)/\,\mathrm{Diff}(M).$$

And the map $(\varphi, \frac{d\varphi}{dt}) \in T\,\mathrm{Diff}(M) \mapsto \frac{d\varphi_t}{dt} \circ \varphi^{-1} \in \mathfrak{X}(M)$. is the quotient projection.

3.3. **Isotropy Subgroups.** The reduction to dynamics on $\mathrm{Diff}(M)$ to dynamics on $\mathfrak{X}(M)$ occurs primarily because the distance function is $\mathrm{Diff}(M)$ invariant. However, one can not completely abandon $\mathrm{Diff}(M)$ because the solution requires one to compute the time 1 flow, $\Phi^u_{0,1}$. Fortunately, there is a second reduction which allows us to avoid computing $\Phi^u_{0,1}$ in its entirety. This second reduction corresponds to the invariance properties of the dissimilarity measure F. Let $G_F \subset \mathrm{Diff}(M)$ denote

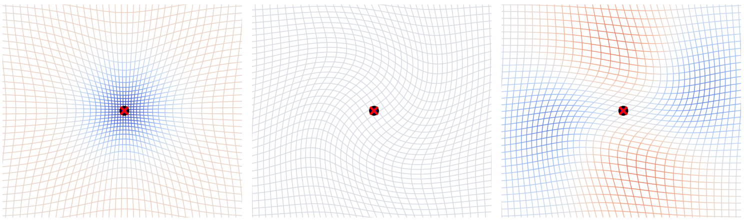


Figure 4. Examples of elements of the isotropy subgroup $\{\psi \in \mathrm{Diff}(M) \mid \psi(x) = x\}$ for a one-point matching problem with dissimilarity measure $F(\varphi) = \|\varphi(x)-y\|^2$ visualized by their effect on an initially square grid. The isotropy subgroup leaves F invariant by not moving x.

the set of diffeomorphisms which leave F invariant, i.e.:

$$G_F := \{\psi \in \mathrm{Diff}(M) \mid F(\varphi \circ \psi) = F(\varphi), \forall \varphi \in \mathrm{Diff}(M)\}.$$

One can readily verify that G_F is a subgroup of $\mathrm{Diff}(M)$, and so we call G_F the *isotropy subgroup of F*.

Having defined G_F we can now consider the homogenous space $Q = \mathrm{Diff}(M)/G_F$, which is the *quotient space* induced by the action of right composition of G_F on $\mathrm{Diff}(M)$. This quotient space is "smaller" in the sense of data. In terms of maps, this can be seen by defining the map $\varphi \in \mathrm{Diff}(M) \mapsto q = [\varphi]_{/G_F} \in Q$, where $[\varphi]_{/G_F}$ denotes the equivalence class of φ. We call this mapping the *quotient projection* because it sends $\mathrm{Diff}(M)$ to Q surjectively. While these notions are theoretically quite complicated, they often manifest more simply in practice.

Example 2. *Let $M \subset \mathbb{R}^n$ be the closure of some open set. Let $x_1, x_2, y_1, y_2 \in M$ with $x_1 \neq x_2$ and consider the dissimilarity measure*

$$F(\varphi) = \|\varphi(x_1) - y_1\|^2 + \|\varphi(x_2) - y_2\|^2.$$

We see that

$$G_F \equiv \{\psi \in \mathrm{Diff}(M) \mid \psi(x_1) = x_1, \psi(x_2) = x_2\},$$

and

$$Q = \mathrm{Diff}(M)/G_F \equiv \{(z_1, z_2) \in M \times M \mid z_1 \neq z_2\} = M \times M - \Delta_{M\times M},$$

where $\Delta_{M\times M}$ denotes the diagonal of $M \times M$. The quotient projection is $\varphi \in \mathrm{Diff}(M) \mapsto (\varphi(x_1), \varphi(x_2)) \in M \times M - \Delta_{M\times M}$. Note that $\mathrm{Diff}(M)$ is infinite dimensional while Q is of dimension $2\dim(M)$. This is massive reduction.

If one is able to understand Q then one can use this insight to reformulate the dissimilarity measure F as function on Q rather than $\mathrm{Diff}(M)$. In particular, there neccessarily exists a unique function $F_Q : Q \to \mathbb{R}$ defined by the property $F_Q([\varphi]_{/G_F}) = F(\varphi)$. Again, this is useful in the sense of data, as is illustrated in the following example.

Example 3. *Consider the dissimilarity measure F of example 2. The function, $F_Q : Q \to \mathbb{R}$ is*

$$F_Q(z_1, z_2) = \|z_1 - y_1\|^2 + \|z_2 - y_2\|^2.$$

Finally, note that $\mathrm{Diff}(M)$ acts upon Q by the left action

$$[\varphi]_{G_F} \in \mathrm{Diff}(M)/G_F \overset{\psi \in \mathrm{Diff}(M)}{\longmapsto} [\psi \circ \varphi]_{/G_F} \in \mathrm{Diff}(M)/G_F.$$

Usually we will simply write $\psi \cdot q$ for the action of $\psi \in \mathrm{Diff}(M)$ on a given $q \in Q$. This means that $\mathfrak{X}(M)$ acts upon Q infinitesimally, as it is the Lie algebra of $\mathrm{Diff}(M)$.

Example 4. *Consider the setup of example 2. Here $Q = M \times M - \Delta_{M\times M}$ and the left action of* $\mathrm{Diff}(M)$ *is given by*

$$\psi \cdot (q_1, q_2) = (\psi(q_1), \psi(q_2))$$

for $\psi \in \mathrm{Diff}(M)$ and $q = (q_1, q_2) \in Q$. The infinitesimal action of $u \in \mathfrak{X}(M)$ on Q is

$$u \cdot (q_1, q_2) = (u(q_1), u(q_2)) \in T_q Q.$$

These constructions allow us to rephrase the initial optimization problem using a reduced curve energy. Minimization of E is equivalent to minimization of

$$E_Q = \int_0^1 \|u\| dt + F_Q\left(q(1)\right)$$

where $q(1)$ is obtained by integrating the ODE, $\frac{dq}{dt} = u \cdot q$ with the intial condition $q(0) = [id]_{/G_F}$ where $id \in \mathrm{Diff}(M)$ is the identity transformation. We see that this curve energy only depends on the Eulerian velocity field and the equivalence class $q(1)$. Minimizers of E_Q must neccessarily satisfy

$$\begin{cases} \partial_t m + \pounds_u[m] \quad , \quad m = P[u] \\ \langle u(1), w\rangle = -DF(q)\cdot(w\cdot q) \quad , \quad \forall w \in \mathfrak{X}(M). \end{cases} \tag{6}$$

Again, the solution only depends on the Eulerian velocity and $q(1)$. For this reason, we see that the G_F symmetry of F provides a second reduction in the data needed to solve our original problem.

3.4. **Orthogonality.** In addition to reducing the amount of data we must keep track of there is an additional consequence to the G_F-symmetry of F. In particular, there is a potentially massive constraint satisfied by the Eulerian velocity u.

To describe this we must introduce an isotropy algebra. Given $q(t) = [\Phi^u_{0,t}]_{/G_F}$ we can define the (time-dependent) isotropy algebra

$$\mathfrak{g}_{q(t)} = \{w \in \mathfrak{X}(M) \mid w \cdot q(t) = 0\}.$$

This is nothing but the Lie-algebra associated to the isotropy group $G_{q(t)} = \{\psi \in \mathrm{Diff}(M) \mid \psi \cdot q(t) = q(t)\}$.

It turns out that the velocity field $u(t)$ which minimizes E (or E_Q) is orthogonal to $\mathfrak{g}_{q(t)}$ with respect to the chosen inner-product. Intuitively this is quite sensible because velocities which do not change $q(t)$ do not alter the data, and simply waste control effort. This intuitive statement is roughly the content of the following proof.

Proposition 1. *Let u satisfy* (5) *or* (6). *Then $m = P[u]$ anihillates $\mathfrak{g}_{q(t)}$.*

Proof. Let u be the solution to (6). We will first prove that $u(1)$ (this is u at time $t = 1$) is orthogonal to $\mathfrak{g}_{z(1)}$. Let $w(1) \in \mathfrak{g}_{z(1)}$. We observe

$$\langle P[u(1)], w(1)\rangle \overset{\text{by (6)}}{=} - \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F_Q(\Phi_\epsilon^{w(1)} z(1)).$$

However, $w(1)$ leaves $z(1)$ fixed, so $\Phi_\epsilon^{w(1)} \cdot q(1) = 0$. Therefore $\langle P[u(1)], w(1)\rangle = 0$. Let $w(t) = [\Phi^u_{t,1}]^* w(1)$ In coordinates this means

$$w^i(t, x) = \partial_j|_{[\Phi^u_{t,1}]^{-1}(x)} [\Phi^u_{t,1}]^i w^j \left(1, [\Phi^u_{t,1}]^{-1}(x)\right)$$

One can directly verify that $w(t) \in \mathfrak{g}_{z(t)}$ for all $t \in [0, 1]$. Denoting $m(t) = P[u(t)]$, as in (6), we find

$$\begin{aligned}\frac{d}{dt}\langle P[u(t)], w(t)\rangle &= \frac{d}{dt}\langle m(t), w(t)\rangle\langle \partial_t m, w\rangle + \langle m, \partial_t w\rangle \\ &= \langle -\pounds_u[m], w\rangle + \langle m, -\pounds_u[w]\rangle = 0.\end{aligned}$$

Where the last equality follows from (3). Thus $\langle P[u(t)], w(t)\rangle$ is constant. We've already verified that at $t = 1$, this inner-product is zero, thus $\langle P[u(t)], w(t)\rangle = 0$ for all time. That $w(1)$ is an arbitrary element of $\mathfrak{g}_{q(1)}$ makes $w(t)$ an arbitrary element of $\mathfrak{g}_{q(t)}$ at each time. Thus $u(t)$ is orthogonal to $\mathfrak{g}_{q(t)}$ for all time. ☐

At this point, we should return to our example to illustrate this idea.

Example 5. *Again consider the setup of example 2. In this case $q(t) = (q_1(t), q_2(t)) \in M \times M - \Delta_{M\times M}$. The space $\mathfrak{g}_{z(t)}$ is the space of vector-fields which vanish at $q_1(t)$ and $q_2(t)$. Therefore, $u(t)$ is orthogonal to $q(t)$ if and only if $m = P[u]$ satisfies*

$$\langle m, v\rangle = p_1 \cdot v(z_1(t)) + p_2 \cdot v(z_2(t))$$

for some covectors p_1, p_2 and for any $v \in \mathfrak{X}(M)$. In other words

$$m = p_1(t) \otimes \delta_{q_1(t)}(\cdot) + p_2 \otimes \delta_{q_2(t)}(\cdot)$$

where $\delta_x(\cdot)$ denotes the Dirac delta functional cetnered at x.

This orthogonality constrain allows one to reduce the evolution equation on $\mathfrak{X}(M)$ to an evolution equation on Q (which might be finite dimensional if G_F is large enough). In particular there is a map

$V : TQ \to \mathfrak{X}(M)$ uniquely defined by the conditions $V(q, \dot{q}) \cdot q = 0$ and $V(q, \dot{q}) \perp \mathfrak{g}_q$ with respect to the chosen inner-product on vector-fields.

Example 6. *Consider the setup of example 2 with $M = \mathbb{R}^n$. Then $Q = \mathbb{R}^n \times \mathbb{R}^n - \Delta_{\mathbb{R}^n \times \mathbb{R}^n}$. Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be the matrix-valued reproducing kernel of P (see* [MG14]*). Then $V : TQ \to \mathfrak{X}(\mathbb{R}^n)$ is given by*

$$V(q, \dot{q})(x) = K(x - q_1) \cdot p_1 + K(x - q_2) \cdot p_2$$

where $p_1, p_2 \in \mathbb{R}^n$ are such that $p_1 + K(q_1 - q_2)p_2 = \dot{q}_1$ and $K(q_2 - q_1)p_1 + p_2 = \dot{q}_2$.

One can immediately observe that V is injective and linear in $\dot{q}$. In other words $V(q, \cdot) : T_qQ \to \mathfrak{X}(M)$ is an injective linear map for fixed $q \in Q$. Because the optimal $u(t)$ is orthogonal to $\mathfrak{g}_{q(t)}$ we may invert $V(q(t), \cdot)$ on $u(t)$. In particular, we may often write the equation of motion on TQ rather than on $\mathfrak{X}(M)$. This is a massive reduction if Q is finite dimensional. In particular, the inner-product structure on $\mathfrak{X}(M)$ induces a Riemannian metric on Q given by

$$g_q(v_1, v_2) = \langle P[V(q, v_1)], V(q, v_2)\rangle.$$

The equations of motion in (5) and (6) map to the geodesic equations on Q.

Proposition 2. *Let u extremize E or E_Q. Then there exists a unique trajectory $q(t) \in Q$ such that $u = V(\frac{dq}{dt})$. Moreover, $q(t)$ is a geodesic with respect to the metric g.*

Proof. Let u minimize E. Thus u satisfies (6). By the previous proposition $u(t)$ is orthogonal to $\mathfrak{g}_{q(t)}$. As $V_{q(t)} : T_{q(t)}Q \to \mathfrak{X}(M)$ is injective on $\mathfrak{g}^{\perp}_{q(t)}$, there exists a unique $\dot{q}(t)$ such that $V(q(t), \dot{q}(t)) = u(t)$. Note that E can be written as

$$E = \int \|V(q(t), \dot{q}(t))\| dt + F(q(1)) = \int g(q, \dot{q}, \dot{q})^{1/2} dt + F(q(1)).$$

Thus, minimizers of E correspond to geodesics in Q with respect to the metric g. □

If we let $H : T^*Q \to \mathbb{R}$ be the Hamiltonian induced by the metric on Q we obtain the most data-efficient form or (5) and (6). Minimizers of E (or E_Q) are:

$$\begin{cases} (q, p)(t) \in T^*Q \text{ satisfies Hamilton's equations} \\ p(1) = -DF_Q(q) \\ q(0) = [e]_{/G_F}. \end{cases} \tag{7}$$

We see that this is a boundary value problem posed entirely on Q. If Q is finite dimensional, this is a massive reduction in terms of data requirements.

Example 7. *Consider the setup of example 2 with $M = \mathbb{R}^n$. The metric on $Q = M \times M - \Delta_{M^2}$ is most easily expressed on the cotangent bundle T^*Q. If K is the matrix valued kernel of P, the metric on T^*Q takes the form*

$$g_q^*(p,p') = \sum_{i,j=1}^{2} p_i^T K(q_i - q_j) p'_j .$$

3.5. **Descending Group Action.** A related approach to defining distances on a space of objects to be registered consists of defining an object space $\mathcal{O}$ upon which $\mathrm{Diff}(M)$ acts transitively[1] with distance

$$d_{\mathcal{O}}(o_1,o_2) = \inf_{\varphi \in \mathrm{Diff}(M)} \{ d(id,\varphi) \,|\, \varphi \cdot o_1 = o_2 \} .$$

Here the distance on $\mathcal{O}$ is defined directly from the distance in the group that acts on the objects, see for example [You10, YAM09]. With this approach, the Riemannian metric descends from $\mathrm{Diff}(M)$ to a Riemannian metric on $\mathcal{O}$ and geodesics on $\mathcal{O}$ lift by horizontality to geodesics on $\mathrm{Diff}(M)$. The quotient spaces Q obtained by reduction by symmetry and their geometric structure corresponds to the object spaces and geometries defined with this approach. Intuitively, reduction by symmetry can be considered a removal of redundant information to obtain compact representations while letting the metric descend to the object space $\mathcal{O}$ constitutes an approach to defining a geometric structure on an already known space of objects. The solutions which result are equivalent to the ones presented in this article because $\mathcal{O} \cong \mathrm{Diff}(M)/G_o$ where $G_o = \{\psi \in \mathrm{Diff}(M) \mid \psi(o) = o\}$ for some fixed reference object $o \in \mathcal{O}$.

4. Examples

We here give a number of concrete examples of how symmetry reduce the infinite dimensional registration problem over $\mathrm{Diff}(M)$ to lower, in some cases finite, dimensional problems. In all examples, the symmetry of the dissimilarity measure with respect to a subgroup of $\mathrm{Diff}(M)$ gives a reduced space by quotienting out the symmetry subgroup.

4.1. **Landmark Matching.** The space Q used in the examples in Section 3 constitutes a special case of the landmark matching problem where sets of landmarks $Q = \{(x_1,\ldots,x_N) |\, x_i \in M,\, x_i \neq x_j \,\forall i \neq j\}$, are placed into spatial correspondence trough the left action $\varphi \cdot (x_1,\ldots,x_N) = (\varphi(x_1),\ldots,\varphi(x_N)\}$ of $\mathrm{Diff}(M)$ by minimizing the dissimilarity measure $F(\varphi) = \sum_{i=1}^N \|\varphi(x_i) - x_i\|^2$. The landmark space Q arises as a quotient of $\mathrm{Diff}(M)$ from the symmetry group G_F as in in Example 2.

[1] This means that for any $o_1, o_2 \in \mathcal{O}$ there exists a $\varphi \in \mathrm{Diff}(M)$ such that $\varphi \cdot o_1 = o_2$

Reduction from $\mathrm{Diff}(M)$ to Q in the landmark case has been used in a series of papers starting with [JM00]. Landmark matching is a special case of jet matching as discussed below. Hamilton's equations (7) take the form

$$\dot{q}_i = \sum_{j=1}^{N} K(q_i - q_j) p_j \quad , \quad \dot{p}_i = -\sum_{j=1}^{N} \left(DK(q_i - q_j) p_j \right)^T p_i$$

on T^*Q where DK denotes the spatial derivative of the reproducing kernel K. Generalizing the situation in Example 5, the momentum field is a finite sum of Dirac measures $\sum_{j=1}^{N} p_j \otimes \delta_{q_j}$ that through the map V gives an Eulerian velocity field as a finite linear combination of the kernel evaluated at q_i: $u(\cdot) = \sum_{j=1}^{N} K(\cdot - q_j) p_j$. Registration of landmarks is often in practice done by optimizing over the initial value of the momentum p in the ODE to minimize E, a strategy called shooting [VMYT04]. Using symmetry, the optimization problem is thus reduced from an infinite dimensional time-dependent problem to an $N \dim(M)$ dimensional optimization problem involving integration of a $2N \dim(M)$ dimensional ODE on T^*Q.

4.2. **Curve and Surface Matching.** The space of smooth non-intersecting closed parametrized curves in $\mathbb{R}^n$ is also known as the space of embeddings, denoted $\mathrm{Emb}(S^1, \mathbb{R}^n)$. The parametrization can be removed by considering the right action of $\mathrm{Diff}(S^1)$ on $\mathrm{Emb}(S^2, \mathbb{R}^n)$ given by

$$c \in \mathrm{Emb}(S^1, \mathbb{R}^n) \overset{\psi \in \mathrm{Diff}(S^1)}{\mapsto} c \circ \psi \in \mathrm{Emb}(S^1, \mathbb{R}^n).$$

Then the quotient space $\mathrm{Gr}(S^1, \mathbb{R}^n) := \mathrm{Emb}(S^1, \mathbb{R}^n) / \mathrm{Diff}(S^1)$ is the space of *unparametrized curves*. The space $\mathrm{Gr}(S^1, \mathbb{R}^n)$ is a special case of a nonlinear Grassmannian [GBV12]. It is not immediately clear if this space is a manifold, although it is certainly an orbifold. In fact the same question can be asked of $\mathrm{Diff}(\mathbb{R}^n)$ and $\mathrm{Emb}(S^1, \mathbb{R}^n)$. A few conditions must be enforced on the space of embeddings and the space of diffeormophisms in order to impose a manifold structure on these spaces, and these conditions along with the metric determine whether or not the quotient $\mathrm{Gr}(S^1, \mathbb{R}^n)$ can inherit a manifold structure. We will not dwell upon these matters here, but instead we refer the reader to the survey article [BBM14].

When the parametrization is not removed, embedded curves and surfaces can be matched with the current dissimilarity measure [VG05, Gla05]. The objects are considered elements of the dual of the space W of differential k-forms on M. In the surface case, the surface S can be evaluated on a 2-form w by

$$S(w) = \int_S w(x)(e_x^1, e_x^2) d\sigma(x) \tag{8}$$

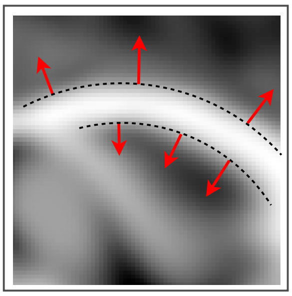


FIGURE 5. In image matching, the gradient of the L^2-difference will be orthogonal to level lines of the image and symmetry implies that the momentum field will be orthogonal to the level lines so that $p(t) = \alpha(t)\nabla\varphi(t).I_0$ for a time-dependent scalar field α.

where (e_x^1, e_x^2) is an orthonormal basis for T_xS and σ the surface element. The dual space W^* is linear an can be equipped with a norm thereby enabling surfaces to be compared with the W^* norm. Note that the evaluation (8) does not depend on the parametrization of S.

The isotropy groups for curves and surfaces generalize the isotropy groups of landmarks by consisting of warps that keeps the objects fixed, i.e.

$$G_F \equiv \{\psi \in \mathrm{Diff}(M) \mid \psi(S) = S\} \ .$$

The momentum field will be supported on the transported curves/surfaces $\varphi(t).S$ for optimal paths for E in $\mathrm{Diff}(M)$.

4.3. **Image Matching.** Images can be registered using either the L^2-difference defined in Example 1 or with other dissimilarity measures such as mutual information or correlation ratio [WIVK96, RMPA98]. The similarity will be invariant to any infinitesimal deformation orthogonal to the gradient of dissimilarity measure. In the L^2 case, this is equivalent to any infinitesimal deformation orthogonal to the level lines of the moving image [MTY06]. The momentum field thus has the form $p(t) = \alpha(t)\nabla\varphi(t).I_0$ for a smooth function $\alpha(t)$ on M and the registration problem can be reduced to a search over the scalar field $\alpha(t)$ instead of vector field $p(t)$.

Minimizers for E follow the PDE [YAM09]

(9)
$$\dot{v} = \int_M K(\cdot - y)\alpha(y)\nabla m(y)dy \quad , \quad \dot{m} = -\nabla m^T v \quad , \quad \dot{\alpha} = -\nabla \cdot (\alpha v)$$

with $m(t)$ representing the deformed image at time t.

4.4. **Jet Matching.** In [SNDP13, Jac13] an extension of the landmark case has been developed where higher-order information is advected with the landmarks. These higher-order particles or *jet-particles* have simultaneously been considered in fluid dynamics [JRD13],

The spaces of jet particles arise as extensions of the reduced landmark space Q by quotienting out smaller isotropy subgroups. Let $G^{(0)}$ be the isotropy subgroup for a single landmark

$$G^{(0)} := \{\psi \in G \mid \psi(q) = q\}$$

Let know k be a positive integer. For any k-differentiable map f from a neighborhood of q, the k-jet of f is denoted $\mathcal{J}_q^{(k)}(f)$. In coordinates, $\mathcal{J}_q^{(k)}(f)$ consists of the coefficients of the kth order Taylor expansions of f about at x. The higher-order isotropy subgroups are then given by

$$G^{(k)} := \{\psi \in G^{(0)} \mid \mathcal{J}_q^{(k)}\psi = \mathcal{J}_q^{(k)}id\} \ .$$

That is, the elements of $G^{(k)}$ fix the Taylor expansion of the deformation φ up to order k. The definition naturally extends to finite number of landmarks, and the quotients $Q^{(k)} = G/G^{(k)}$ can be identified as the sets

$$\begin{aligned}
Q^{(0)} &= \{(q_1, \ldots, q_N) \mid q_i \in M\} \\
Q^{(1)} &= \{((q_i^{(0)}, q_i^{(1)}), \ldots) \mid (q_i^{(0)}, q_i^{(1)}, \ldots) \in M \times \mathrm{GL}(d)\} \\
Q^{(2)} &= \{((q_i^{(0)}, q_i^{(1)}, q_i^{(2)}), \ldots) \mid (q_i^{(0)}, q_i^{(1)}, q_i^{(2)}, \ldots) \in M \times \mathrm{GL}(d) \times S_2^1\}
\end{aligned}$$

with S_2^1 being the space of rank $(1,2)$ tensors. Intuitively, the space $Q^{(0)}$ is the regular landmark space with information about the position of the points; the 1-jet space $Q^{(1)}$ carry for each jet information about the position and the Jacobian matrix of the warp at the jet position; and the 2-jet space $Q^{(2)}$ carry in addition the Hessian matrix of the warp at the jet position. The momentum for $Q^{(0)}$ in coordinates consists of N vectors representing the local displacement of the points. With the 1-jet space $Q^{(0)}$, the momentum in addition contains $d \times d$ matrices that can be interpreted as locally linear deformations at the jet positions [SNDP13]. In combination with the displacement, the 1-jet momenta can thus be regarded locally affine transformations. The momentum fields for $Q^{(2)}$ add symmetric tensors encoding local second order deformation. The local effect effect of the jet particles is sketched in Figure 6.

When the dissimilarity measure F is dependent not just on positions but also on higher-order information around the points, reduction by symmetry implies that optimal solutions for E will be parametrized by k-jets in the same way as $Q^{(0)}$ parametrize optimal paths for E in the landmark case. The higher-order jets can thus be used for landmark matching when the dissimilarity measure is dependent on the local geometry around the landmarks. For example, matching of first order structure such as image gradients lead to 1-order jets, and matching of local curvature leads to 2-order jets.

4.5. **Discrete Image Matching.** The image matching problem can be discretized by evaluating the L^2-difference at a finite number of points. In practice, this alway happens when the integral $\int_M |I_0 \circ \varphi^{-1}(x) - I_1(x)|^2 dx$ is evaluated at finitely many pixels of the image. In [SNDP13, JS14], it is shown how this reduces the image matching PDE (9) to a finite dimensional system on Q when the integral is approximated by pointwise evaluation at a grid Λ_h

$$F^{(0)}(\varphi) \approx \sum_{x \in \Lambda_h} h^d |I_0(\varphi^{-1}(x) - I_1(x)|^2 \tag{10}$$

where $h > 0$ denotes the grid spacing. $F^{(0)}$ approximates F to order $O(h^d)$, $d = \dim(M)$. The reduced space Q encodes the position of the points $\varphi^{-1}(x)$, $x \in \Lambda_h$, and the lifted Eulerian momentum field is a finite sum of point measures $p = \sum_{x \in \Lambda_h} a_x \otimes \delta_{\varphi^{-1}(x)}$. For each grid point, the momentum encodes the local displacement of the point, See Figure 6.

In [JS14], a discretization scheme with higher-order accuracy is in addition introduced with an $O(h^{d+2})$ approximation $F^{(2)}$ of F. The increased accuracy results in the entire energy E being approximated to order $O(h^{d+2})$. The solution space in this cases become the jet-space $Q^{(2)}$. For a given order of approximation, a corresponding reduction in the number of required discretization points is obtained. The reduction in the number of discretization points is countered by the increased information encoded in each 2-jet. The momentum field thus encodes both local displacement, local linear deformation, and second order deformation, see Figure 6. The discrete solutions will converge to solutions of the non-discretized problem as $h \to 0$.

4.6. **DWI/DTI Matching.** Image matching is symmetric with respect to variations parallel to the level lines of the images. With diffusion weighted images (DWI) and the variety of models for the diffusion information (e.g. diffusion tensor imaging DTI CITE, Gaussian Mixture Fields CITE), first or higher-order information can be reintroduced into the matching problem. In essence, by letting the dissimilarity measure depend on the diffusion information, the full $\mathrm{Diff}(M)$ symmetry of the image matching problem is reduced to an isotropy subgroup of $\mathrm{Diff}(M)$.

The exact form of the of DWI matching problem depends on the diffusion model and how $\mathrm{Diff}(M)$ acts on the diffusion image. In [CMWY05], the diffusion is represented by the principal direction of the diffusion tensor, and the data objects to be match are thus vector fields. The action by elements of $\mathrm{Diff}(M)$ is defined by

$$\varphi \cdot I(x) = \frac{\|I \circ \varphi^{-1}\|}{\|D\varphi I \circ \varphi^{-1}\|} D\varphi I \circ \varphi^{-1} \ .$$

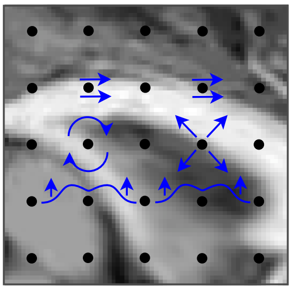


FIGURE 6. With discrete image matching, the image is sampled at a regular grid Λ_h, $h > 0$ and the image matching PDE (9) is reduced to an ODE on a finite dimensional reduced space Q. With the approximation $F^{(0)}$ (10), the momentum field will encode local displacement as indicated by the horizontal arrows (top row). With a first order expansion, the solution space will be jet space $Q^{(1)}$ and locally affine motion is encoded around each grid point (middle row). The $O(h^{d+2})$ approximation $F^{(2)}$ includes second order information and the system reduces to the jet space $Q^{(2)}$ with second order motion encoded at each grid point (lower row).

The action rotates the diffusion vector by the Jacobian of the warp keeping its length fixed. Similar models can be applied to DTI with the Preservation of Principle Direction scheme (PPD, [AGB99, APBG01]) and to GMF based models [CVCM09]. The dependency on the Jacobian matrix implies that a reduced model must carry first order information in a similar fashion to the 1-jet space $Q^{(1)}$, however, any irrotational part of the Jacobian can be removed by symmetry. The full effect of this has yet to be explored.

4.7. **Fluid Mechanics.** Incidentally, the equation of motion

$$\partial_t m + \pounds_u[m] = 0$$
$$u = K * m$$

is an eccentric way of writing Euler's equation for an invicid incompressible fluid if we assume u is initially in the space of divergence free vector-fields and K is a Dirac-delta distribution (which impies $m = u$.) This fact was exploited in [MM13] to create a sequece of regularized models to Euler's equations by considering a sequence of Kernels which converge to a Dirac-delta distribution. Moreover, if one replaces $\mathrm{Diff}(M)$ by the subgroup of volume preserving diffeomorphisms $\mathrm{Diff}_{\mathrm{vol}}(M)$, then (formally) one can produce incompressible particle methods using the same reduction arguments presented here. In fact, jet-particles were independently discoverd in this context as a means of simulating fluids in [JRD13]. It is notable that [JRD13] is a mechanics paper, and the particle methods which were produced were approached from the perspective of reduction by symmetry.

In [CHJM14] one of the Kernel parameters in [MM13] which controls the compressibility of the u was taken to the incompressible limit. This allowed a realization of the particle methods described in [JRD13]. The constructions of [CHJM14] is the same as presented in this survey article, but with $\mathrm{Diff}(M)$ replaced by the group of volume preserving diffeomorphisms of $\mathbb{R}^d$, denoted $\mathrm{Diff}_{\mathrm{vol}}(\mathbb{R}^d)$.

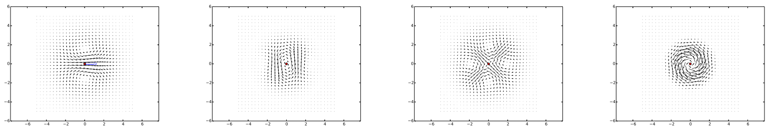


FIGURE 7. Velocity fields induced by first order incompressible jet particles taken from [CHJM14]

5. Discussion and Conclusions

The information available for solving the registration problem is in practice not sufficient for uniquely encoding the deformation between the objects to be registered. Symmetry thus arises in both particle relabeling symmetry that gives the Eulerian formulation of the equations of motion and in symmetry groups for specific dissimilarity measures.

For landmark matching, reduction by symmetry reduces the infinite dimensional registration problem to a finite dimensional problem on the reduced landmark space Q. For matching curves and surfaces, symmetry implies that the momentum stays concentrated at the curve and surfaces allowing a reduction by the isotropy groups of warps that leave the objects fixed. In image matching, symmetry allows reduction by the group of warps that do not change the level sets of the image. Jet particles have smaller symmetry groups and hence larger reduced spaces $Q^{(1)}$ and $Q^{(2)}$ that encode locally affine and second order information.

Reduction by symmetry allow these cases to be handled in one theoretical framework. We have surveyed the mathematical construction behind the reduction approach and its relation to the above mentioned examples. As data complexity rises both in term of resolution an structure, symmetry will continue to be an important tool for removing redundant information and achieving compact data representations.

6. Acknowledgments

HOJ would like to thank Darryl Holm for providing a bridge from geometric mechanics into the wonderful world of image registration algorithms. HOJ is supported by the European Research Council Advanced Grant 267382 FCCA. SS is supported by the Danish Council for Independent Research with the project "Image Based Quantification of Anatomical Change".

References

[AGB99] D. C. Alexander, J. C. Gee, and R. Bajcsy, *Strategies for data reorientation during non-rigid warps of diffusion tensor images*, Medical Image Computing and Computer-Assisted Intervention – MICCAI'99 (Chris Taylor and Alain Colchester, eds.), Lecture Notes in Computer Science, no. 1679, Springer Berlin Heidelberg, January 1999, pp. 463–472 (en).

[AM78] R Abraham and J E Marsden, *Foundations of Mechanics*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978, Second edition, revised and enlarged, with the assistance of Tudor Ratiu and Richard Cushman. Reprinted by AMS Chelsea, 2008.

[APBG01] D. C. Alexander, C. Pierpaoli, P. J. Basser, and J. C. Gee, *Spatial transformations of diffusion tensor magnetic resonance images*, IEEE transactions on medical imaging **20** (2001), no. 11, 1131–1139 (eng).

[BBM14] Martin Bauer, Martins Bruveris, and Peter W. Michor, *Overview of the geometries of shape spaces and diffeomorphism groups*, J Math Imaging Vis **1** (2014), no. 1.

[BBPW04] Thomas Brox, Andrés Bruhn, Nils Papenberg, and Joachim Weickert, *High accuracy optical flow estimation based on a theory for warping*, Computer Vision - ECCV 2004 (Tomás Pajdla and Jiří Matas, eds.), Lecture Notes in Computer Science, no. 3024, Springer Berlin Heidelberg, January 2004, pp. 25–36 (en).

[BMTY05] M. Faisal Beg, Michael I. Miller, Alain Trouvé, and Laurent Younes, *Computing large deformation metric mappings via geodesic flows of diffeomorphisms*, IJCV **61** (2005), no. 2, 139–157.

[BRA+06] Richard G. Boyes, Daniel Rueckert, Paul Aljabar, Jennifer Whitwell, Jonathan M. Schott, Derek L. G. Hill, and Nicholas C. Fox, *Cerebral atrophy measurements using jacobian integration: comparison with the boundary shift integral*, NeuroImage **32** (2006), no. 1, 159–169 (eng).

[CHJM14] C. J. Cotter, D. D. Holm, H. O. Jacobs, and D. M. Meier, *A jetlet hierarchy for ideal fluid dynamics*, arXiv:1402.0086 [math-ph, physics:nlin, physics:physics] (2014).

[CMWY05] Yan Cao, M. I Miller, R. L Winslow, and L. Younes, *Large deformation diffeomorphic metric mapping of vector fields*, IEEE Transactions on Medical Imaging **24** (2005), no. 9, 1216–1230 (English).

[CRM02] GE Christensen, RD Rabbitt, and MI Miller, *Deformable templates using large deformation kinematics*, Image Processing, IEEE Transactions on **5** (2002), no. 10.

[CVCM09] Guang Cheng, Baba C. Vemuri, Paul R. Carney, and Thomas H. Mareci, *Non-rigid registration of high angular resolution diffusion images represented by gaussian mixture fields*, Proceedings of the 12th International Conference on Medical Image Computing and Computer-Assisted Intervention: Part I (Berlin, Heidelberg), MICCAI '09, Springer-Verlag, 2009, pp. 190–197.

[DGM98] Paul Dupuis, Ulf Grenander, and Michael I Miller, *Variational problems on flows of diffeomorphisms for image matching*.

[DLG14] R. Derfoul and C. Le Guyader, *A relaxed problem of registration based on the saint venant–kirchhoff material stored energy for the mapping of mouse brain gene expression data to a neuroanatomical mouse atlas*, SIAM Journal on Imaging Sciences (2014), 2175–2195.

[DSS10] Suma Dawn, Vikas Saxena, and Bhudev Sharma, *Remote sensing image registration techniques: A survey*, Image and Signal Processing (Abderrahim Elmoataz, Olivier Lezoray, Fathallah Nouboud, Driss Mammass, and Jean Meunier, eds.), Lecture Notes in Computer Science, no. 6134, Springer Berlin Heidelberg, January 2010, pp. 103–112 (en).

[GBV12] F Gay-Balmaz and C Vizman, *Dual pairs in fluid dynamics*, Annals of Global Analysis and Geometry **41** (2012), no. 1, 1–24.

[GJL+10] Vladlena Gorbunova, Sander S. A. M. Jacobs, Pechin Lo, Asger Dirksen, Mads Nielsen, Alireza Bab-Hadiashar, and Marleen de Bruijne, *Early detection of emphysema progression*, Medical image computing and computer-assisted intervention: MICCAI ... International Conference on Medical Image Computing and Computer-Assisted Intervention **13** (2010), no. Pt 2, 193–200 (eng).

[Gla05] Joan Glaunès, *Transport par difféomorphismes de points, de mesures et de courants pour la comparaison de formes et l'anatomie numérique*, Ph.D. thesis, Université Paris 13, Villetaneuse, France, 2005.

[HRTY04] Darryl D Holm, J. Tilak Ratnanather, Alain Trouvé, and Laurent Younes, *Soliton dynamics in computational anatomy*, nlin/0411014 (2004), NeuroImage Vol 23, S170-178, 2004.

[Jac13] Henry Jacobs, *Symmetries in LDDMM with higher order momentum distributions*, arXiv:1306.3309 [math] (2013).

[JM00] SC Joshi and MI Miller, *Landmark matching via large deformation diffeomorphisms*, Image Processing, IEEE Transactions on **9** (2000), no. 8, 1357–1370.

[JMG97] Sarang C. Joshi, Michael I. Miller, and Ulf Grenander, *On the geometry and shape of brain sub-manifolds*, International Journal of Pattern Recognition and Artificial Intelligence **11** (1997), no. 08, 1317–1343.

[JRD13] Henry O. Jacobs, Tudor S. Ratiu, and Mathieu Desbrun, *On the coupling between an ideal fluid and immersed particles*, Physica D: Nonlinear Phenomena **265** (2013), 40–56.

[JS14] Henry O. Jacobs and Stefan Sommer, *Higher-order accuracy in diffeomorphic image registration*, in preparation. (2014).

[MG14] M. Micheli and J. A. Glaunès, *Matrix-valued kernels for shape deformation analysis*, Geometry, Imaging, and Computing **1** (2014), no. 1, 57–39.

[MM13] D Mumford and P W Michor, *On Euler's equation and 'EPDiff'*, Journal of Geometric Mechanics **5** (2013), no. 3, 319–344, arXiv:1209.6576 [math.AP].

[MTY06] Michael I. Miller, Alain Trouvé, and Laurent Younes, *Geodesic shooting for computational anatomy*, J. Math. Imaging Vis. **24** (2006), no. 2, 209–228.

[PSA+05] X. Pennec, R. Stefanescu, V. Arsigny, P. Fillard, and N. Ayache, *Riemannian elasticity: A statistical regularization framework for non-linear registration*, MICCAI 2005, 2005, pp. 943–950.

[RMPA98] Alexis Roche, Grégoire Malandain, Xavier Pennec, and Nicholas Ayache, *The correlation ratio as a new similarity measure for multimodal image registration*, Proceedings of the First International Conference on Medical Image Computing and Computer-Assisted Intervention, MICCAI '98, Springer-Verlag, 1998, ACM ID: 709612, pp. 1115–1124.

[RSH+99] D Rueckert, L I Sonoda, C Hayes, D L Hill, M O Leach, and D J Hawkes, *Nonrigid registration using free-form deformations: application to breast MR images*, IEEE Transactions on Medical Imaging **18** (1999), no. 8, 712–721.

[SDP13] A. Sotiras, C. Davatzikos, and N. Paragios, *Deformable medical image registration: A survey*, IEEE Transactions on Medical Imaging **32** (2013), no. 7, 1153–1190.

[SNDP13] Stefan Sommer, Mads Nielsen, Sune Darkner, and Xavier Pennec, *Higher-order momentum distributions and locally affine LDDMM registration*, SIAM Journal on Imaging Sciences **6** (2013), no. 1, 341–367.

[Tro95] Alain Trouvé, *An infinite dimensional group approach for physics based models in patterns recognition*, 1995.

[TY05] A. Trouvé and L. Younes, *Local geometry of deformable templates*, SIAM J. Math. Anal. **37** (2005), no. 1, 17–59.

[VG05] Marc Vaillant and Joan Glaunès, *Surface matching via currents*, Information Processing in Medical Imaging, 2005, pp. 381–392.

[VMYT04] M. Vaillant, M.I. Miller, L. Younes, and A. Trouvé, *Statistics on diffeomorphisms via tangent space representations*, NeuroImage **23** (2004), no. Supplement 1, S161–S169.

[WIVK96] William M Wells, III, Paul Viola, and Ron Kikinis, *Multi-modal volume registration by maximization of mutual information.*

[YAM09] Laurent Younes, Felipe Arrate, and Michael I. Miller, *Evolutions equations in computational anatomy*, NeuroImage **45** (2009), no. 1, Supplement 1, S40–S50.

[You10] Laurent Younes, *Shapes and diffeomorphisms*, Springer, 2010.