## ANSWER KEYS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **1.** (6) | **2.** (2) | **3.** (3) | **4.** (4) | **5.** (1) | **6.** (2) | **7.** (1) | **8.** (2) |
| **9.** (2) | **10.** (1) | **11.** (3) | **12.** (2) | **13.** (6) | **14.** (1680) | **15.** (2) | **16.** (3) |
| **17.** (2) | **18.** (13) | **19.** (2) | **20.** (0) | **21.** (40) | **22.** (24) | **23.** (4) | **24.** (4) |
| **25.** (2) | **26.** (2) | **27.** (6) | **28.** (48) | **29.** (26) | **30.** (3) | | |

**1. (6)**

$$\frac{(2i)^n}{(1-i)^{n-2}}$$
$$=\frac{(2i)^n}{\left(-2i\right)^{\frac{n-2}{2}}}$$
$$=\frac{\left(2i\right)^{\frac{n+2}{2}}}{\left(-1\right)^{\frac{n-2}{2}}}$$
$$=\frac{(2)^{\frac{n+2}{2}} i^{\frac{n+2}{2}}}{\left(-1\right)^{\frac{n-2}{2}}}$$

Clearly $n$ must be even $n = 2, 4$ rejected.

So the least positive integer possible is $n = 6$

**2. (2)**

Let,

$z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$\because$ Re$(z_1 z_2)$= 0

$x_1 x_2 - y_1 y_2 = 0 \ldots\ldots(i)$

$\because$ Re$(z_1 + z_2)$= 0

$x_1 + x_2 = 0 \ldots\ldots(ii)$

From equations $(i)$ and $(ii)$ we get

${x_1}^2 + y_1 y_2 = 0$

$\Rightarrow y_1 y_2 = -{x_1}^2$

Therefore $Im(z_1)$ and $Im(z_2)$ are of opposite sign

**3.** **(3)** Let $z_1=\left(\frac{z-\bar{z}+z\bar{z}}{2-3z+5z}\right)$

Let $z=3+iy$

$\bar{z}=3-iy$

$$z_1=\frac{2iy+\left(9+y^2\right)}{2-3(3+iy)+5(3-iy)}$$

$$=\frac{9+y^2+i(2y)}{8-8iy}$$

$$=\frac{\left(9+y^2\right)+i(2y)}{8(1-iy)}$$

$$\operatorname{Re}(z_1)=\frac{\left(9+y^2\right)-2y^2}{8\left(1+y^2\right)}$$

$$=\frac{9-y^2}{8\left(1+y^2\right)}$$

$$=\frac{1}{8}\left[\frac{10-\left(1+y^2\right)}{\left(1+y^2\right)}\right]$$

$$=\frac{1}{8}\left[\frac{10}{1+y^2}-1\right]$$

$1+y^2\in[1,\infty]$

$$\frac{1}{1+y^2}\in(0,1]$$

$$\frac{10}{1+y^2}\in(0,10)$$

$$\frac{10}{1+y^2}-1\in(-1,9]$$

$$\operatorname{Re}(z_1)\in\left(\frac{-1}{8},\frac{9}{8}\right]$$

$$\alpha=\frac{-1}{8},\beta=\frac{9}{8}$$

$$24(\beta-\alpha)=24\left(\frac{9}{8}+\frac{1}{8}\right)=30$$

**4.** **(4)**

$u=\frac{2(x+iy)+i}{(x+iy)-ki}=\frac{2x+(2y+1)i}{x+(y-k)i}\times\frac{x-(y-k)i}{x-(y-k)i}$

Real part of $u=Re(u)=\frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2}$

Imaginary part of $u=\operatorname{Im}(u)\frac{x(2y+1)-2x(y-k)}{x^2+(y-k)^2}$

Now $\operatorname{Re}(u)+\operatorname{Im}(u)=1$

$\frac{2x^2+(2y+1)(y-k)+x(2y+1)-2x(y-k)}{x^2+(y-k)^2}=1$

for $y$-axis put $x=0$

$\Rightarrow\frac{(2y+1)(y-k)}{(y-k)^2}=1$

$\Rightarrow(y-k)(y+1+k)=0$

$y=k,\ -(1+k)$

Now point $P(0,\ k),\ Q(0,\ -(1+k))$

$PQ=|2K+1|=5$

$2k+1=\pm5$

$2k=4,-6$

$k=2,-3$

Hence, $k=2\quad(k>0)$.

**5.** **(1)**

The given system of equations has more than one solution, then it must have infinitely many solutions.

So, $\frac{4i}{8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)}=\frac{1+i}{\alpha+i\beta}$ $\left(\because \begin{aligned} a &= \alpha - i\beta \\ \Rightarrow \bar{a} &= \alpha + i\beta \end{aligned}\right)$

$\Rightarrow \frac{4i}{8\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)}=\frac{1+i}{\alpha+i\beta}$

$\Rightarrow \alpha i-\beta=-1-i+\sqrt{3}i-\sqrt{3}$

$\Rightarrow -\beta+\alpha i=\left(-1-\sqrt{3}\right)+\left(-1+\sqrt{3}\right)i$

$\Rightarrow \alpha=-1+\sqrt{3}\ \&\ -\beta=-1-\sqrt{3}$

$\therefore \frac{\alpha}{\beta}=\frac{-1+\sqrt{3}}{1+\sqrt{3}}=\frac{-1+\sqrt{3}}{1+\sqrt{3}}\times\frac{1-\sqrt{3}}{1-\sqrt{3}}$

$=\frac{-\left(1-\sqrt{3}\right)^2}{1-3}=\frac{1+3-2\sqrt{3}}{2}=2-\sqrt{3}$

**6.** **(2)**

$\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right)\leq 2$

$\Rightarrow \frac{|z|+11}{(|z|-1)^2}\geq\frac{1}{2}, \{\because \log_a(b)\leq c\Rightarrow b\geq a^c\ if\ 0<a<1\}$

$\Rightarrow 2|z|+22\geq(|z|-1)^2, \left\{\because |z|\neq 1\ \&\ (|z|-1)^2>0\right\}$

$\Rightarrow 2|z|+22\geq|z|^2+1-2|z|$

$\Rightarrow |z|^2-4|z|-21\leq 0$

$\Rightarrow(|z|-7)(|z|+3)\leq 0$

$\Rightarrow|z|\leq 7$

$\therefore$ Largest value of $|z|$ is 7

**7.** **(1)**

Let $\alpha=\frac{\sqrt{3}}{2}+\frac{i}{2}$, then $z=\alpha^5+(\bar{\alpha})^5$

We know $\bar{z}^n=\bar{z}^n$ ,

then, $z=2Re\left(\alpha^5\right)$

Hence, $I(z)=0$.

**8.** **(2)** Let $Z=x+iy, x\in R, y\in R$

$x-iy=i\left(x^2-y^2+(2xy)i+x\right)$

$x=-2xy$

$-y=-y^2+x^2+x$

$\Rightarrow x=0, y=-\frac{1}{2}(\text{ from }(1))$

If $x\neq 0$, then $y=0,1$

If $y=-\frac{1}{2}$, then $x=\frac{1}{2},-\frac{3}{2}$

$Z=0+i0, 0+i, \frac{1}{2}-\frac{i}{2}, -\frac{3}{2}-\frac{i}{2}$

**9.** **(2)**

Given,

$z=\alpha+i\beta\ \&\ |z+2|=z+4(1+i)$

Now putting the value of $z=\alpha+i\beta$ in $|z+2|=z+4(1+i)$ we get,

$|z+2|=z+4(1+i)$

$\Rightarrow \sqrt{(\alpha+2)^2+\beta^2}=(\alpha+4)+i(\beta+4)$

Now on comparing real and imaginary part, we get

$\sqrt{(\alpha+2)^2+\beta^2}=\alpha+4\ldots(i)$ and $\beta+4=0\Rightarrow\beta=-4\ \ldots(ii)$

Now solving,

$\sqrt{(\alpha+2)^2+16}=\alpha+4$

$\Rightarrow \alpha^2+4\alpha+20=\alpha^2+8\alpha+16$

$\Rightarrow \alpha=1$

So, $\alpha+\beta=-3,\ \alpha\beta=-4$

We know that quadratic equation is given by,

$x^2-$(sum of roots)$x+$(product of roots)$=0$

So, equation with roots $-3$ and $-4$ will be,

$x^2+7x+12=0$

**10. (1)**

Given, the minimum value $v_0$ of $v = |z|^2 + |z-3|^2 + |z-6i|^2$, $z \in \mathbb{C}$ is attained at $z = z_0$,

Let $z = x + iy$

$v = x^2 + y^2 + (x-3)^2 + y^2 + x^2 + (y-6)^2$

$= (3x^2 - 6x + 9) + (3y^2 - 12y + 36)$

$= 3(x^2 + y^2 - 2x - 4y + 15)$

$= 3\left[(x-1)^2 + (y-2)^2 + 10\right]$

$V_{\text{min}}$ at $z = 1 + 2i = z_0$ and $v_0 = 30$

So, $\left|2z_0^2 - \bar{z}_0^3 + 3\right|^2 + v_0^2$

$= \left|2(1+2i)^2 - (1-2i)^3 + 3\right|^2 + 900$

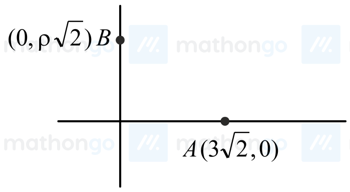
$= |-6 + 8i - (1 + 8i - 6i - 12) + 3|^2 + 900$

$= |8 + 6i|^2 + 900 = 1000$

**11. (3)**

Note that $|z - 3\sqrt{2}|$ is the distance between the point $z$ and $\left(3\sqrt{2}, 0\right)$ on argand plane.

Note that $|z - p\sqrt{2}i|$ is the distance between the point $z$ and $\left(0, p\sqrt{2}\right)$ on argand plane.




$|z - 3\sqrt{2}| + |z - p\sqrt{2}i|$ is sum of distance of $z$ from $A\left(3\sqrt{2}, 0\right)$ and $B\left(0, p\sqrt{2}\right)$

For minimising the sum, $z$ should lie on $AB$ and $AB$ should be equal to $5\sqrt{2}$

Now, $(AB)^2 = 18 + 2p^2$

$\Rightarrow 18 + 2p^2 = \left(5\sqrt{2}\right)^2$

$\Rightarrow p^2 = 16$

i.e. $p = \pm 4$

**12. (2)**

Given $z^2+3\bar{z}=0$

Put $z=x+iy$, then we know that $\bar{z}=x-iy$, hence, we have

$(x+iy)^2+3(x-iy)=0$

$\Rightarrow\ x^2+i^2y^2+2ixy+3x-3iy=0$

We also, know that $i^2=-1$, hence, we get

$\Rightarrow\ x^2-y^2+2ixy+3x-3iy=0$

$\Rightarrow\ (x^2-y^2+3x)+i(2xy-3y)=0+i0$

On comparing the real and imaginary parts, we get

$x^2-y^2+3x=0\quad \ldots(1)$

And $2xy-3y=0\quad \ldots(2)$

$\Rightarrow\ y(2x-3)=0$

$\Rightarrow x=\frac{3}{2},\ y=0$

Put $x=\frac{3}{2}$ in equation (1), we get $\frac{9}{4}-y^2+\frac{9}{2}=0$

$\Rightarrow\ y^2=\frac{27}{4}$

$\Rightarrow\ y=\pm\frac{3\sqrt{3}}{2}$.

$\Rightarrow\ (x,\ y)=\left(\frac{3}{2},\ \frac{3\sqrt{3}}{2}\right),\ \left(\frac{3}{2},\ \frac{-3\sqrt{3}}{2}\right)$.

Now, put $y=0$, in the equation (1), we get $x^2+3x=0$

$\Rightarrow x=0,\ -3$.

$\therefore\ (x,\ y)=(0,\ 0),\ (-3,\ 0)$.

$\therefore$ No of solutions $=n=4$

Now, $\sum_{k=0}^{\infty}\left(\frac{1}{n^k}\right)=\sum_{k=0}^{\infty}\left(\frac{1}{4^k}\right)$

$=\frac{1}{1}+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots\ldots$

The above progression is a geometric progression, with first term $a=1$ and common ratio $r=\frac{1}{4}$ and the sum of infinite terms of the geometric progression is $\frac{a}{1-r}$

Thus, $\sum_{k=0}^{\infty}\left(\frac{1}{n^k}\right)=\frac{1}{\left(1-\frac{1}{4}\right)}$

$=\frac{1}{\left(\frac{3}{4}\right)}=\frac{4}{3}$.

**13. (6)**

Given,

$z^2=\bar{z}\cdot 2^{1-|z|}\quad \cdots(1)$

On taking modulus both side we get,

$\Rightarrow |z|^2=|\bar{z}|\cdot 2^{1-|z|}$

$\Rightarrow |z|=2^{1-|z|},\quad \because\ b\neq 0\Rightarrow |z|\neq 0$

So comparing both side we get $|z|=1\quad \cdots(2)$

Now putting $z=a+ib$ then $\sqrt{a^2+b^2}=1\quad \cdots(3)$

Now again from equation (1), equation (2), equation (3) we get:

$a^2-b^2+i2ab=(a-ib)2^0=a-ib$

Now on comaparing imagenary and real part we get,

$\therefore\ a^2-b^2=a$ and $2ab=-b$

Now solving we get, $a=-\frac{1}{2}$ and $b=\pm\frac{\sqrt{3}}{2}$

So, $z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ or $=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Now solving $z^n=(z+1)^n\Rightarrow\left(\frac{z+1}{z}\right)^n=1$

$\Rightarrow\left(1+\frac{1}{z}\right)^n=1$

$\Rightarrow\left(\frac{1+\sqrt{3}i}{2}\right)^n=1$

$\Rightarrow\left(-\omega^2\right)^n=1$, then minimum value of $n$ is 6

**14. (1680)** $\left(\frac{z^2+8iz-15}{z^2-3iz-2}\right) \in R$

$\Rightarrow 1+\frac{(11iz-13)}{(z^2-3iz-2)} \in R$

Put $z=\alpha-\frac{13}{11}i$

$\Rightarrow \left(z^2-3iz-2\right)$ is imaginary

Put $z=x+iy$

$\Rightarrow \left(x^2-y^2+2xyi-3ix+3y-2\right) \in$ Imag inary

$\Rightarrow \operatorname{Re}\left(x^2-y^2+3y-2+(2xy-3x)i\right)=0$

$\Rightarrow x^2-y^2+3y-2=0$

$x^2=y^2-3y+2$

$x^2=(y-1)(y-2) \therefore z=\alpha-\frac{13}{11}i$

Put $x=\alpha, y=\frac{-13}{11}$

$$\alpha^2=\left(\frac{-13}{11}-1\right)\left(\frac{-13}{11}-2\right)$$

$$\alpha^2=\frac{(24\times 35)}{121}$$

$242\alpha^2=48\times 35=1680$

**15. (2)**

Given:

$\left(1-\sqrt{3}i\right)^{200}=2^{199}\left(p+iq\right)$

$\Rightarrow \left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{200}=\frac{1}{2}(p+iq)$

$\Rightarrow \left(\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right)^{200}=\frac{1}{2}(p+iq)$

$\Rightarrow \left(\cos\left(\frac{200\pi}{3}\right)-i\sin\left(\frac{200\pi}{3}\right)\right)=\frac{1}{2}(p+iq)$

$\Rightarrow \left(\cos\left(201\pi-\frac{\pi}{3}\right)-i\sin\left(201\pi-\frac{\pi}{3}\right)\right)=\frac{1}{2}(p+iq)$

$\Rightarrow \left(-\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right)=\frac{1}{2}(p+iq)$

$\Rightarrow 2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=p+iq$

$\Rightarrow -1-i\sqrt{3}=p+iq$

So,

$p=-1,\ q=-\sqrt{3}$

$\alpha=p+q+q^2=2-\sqrt{3}$

$\beta=p-q+q^2=2+\sqrt{3}$

So,

$\alpha+\beta=4$

$\alpha\cdot\beta=1$

Required equation is

$x^2-4x+1=0$

**16. (3)**

Given,

$\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$

Now let $z=\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}$,

So, $\bar{z}=\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}=\frac{1}{z}$

So, $\left(\frac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$

$=\left(\frac{1+z}{1+\bar{z}}\right)^3$

$=\left(\frac{1+z}{1+\frac{1}{z}}\right)^3$

$=z^3\left(\frac{1+z}{1+z}\right)^3$

$=z^3$

$=\left(\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}\right)^3$

$=i^3\left(\cos\frac{2\pi}{9}-i\sin\frac{2\pi}{9}\right)^3$

$=-i\left(\cos\left(3\times\frac{2\pi}{9}\right)-i\sin\left(3\times\frac{2\pi}{9}\right)\right)$

$=-i\left(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}\right)$

$=-i\left(\frac{-1}{2}-i\frac{\sqrt{3}}{2}\right)$

$=-\frac{1}{2}\left(\sqrt{3}-i\right)$

**17. (2)** As

$|z\omega|=1$

If $|z|=r$, then $|\omega|=\frac{1}{r}$

Let $\arg(z)=\theta$

$\therefore\ \arg(\omega)=\left(\theta-\frac{3\pi}{2}\right)$

So,

$z=re^{i\theta}$

$\Rightarrow\bar{z}=re^{i(-\theta)}$

$\Rightarrow\omega=\frac{1}{r}e^{i\left(\theta-\frac{3\pi}{2}\right)}$

Now, consider

$\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}=\frac{1-2e^{i\left(-\frac{3\pi}{2}\right)}}{1+3e^{i\left(-\frac{3\pi}{2}\right)}}=\left(\frac{1-2i}{1+3i}\right)$

$=\frac{(1-2i)(1-3i)}{(1+3i)(1-3i)}=\frac{1-5i+6i^2}{10}=\frac{-5+5i}{10}=-\frac{1}{2}(1+i)$

Then,

principal $\arg\left(\frac{1-2\bar{z}\omega}{1+3\bar{z}\omega}\right)$

$=$ principal $\arg\left(-\frac{1}{2}(1+i)\right)$

$=-\left(\pi-\frac{\pi}{4}\right)=\frac{-3\pi}{4}$

**18. (13)**

Given:

$z=\frac{1-i\sqrt{3}}{2}$

$\Rightarrow z=-\left(\frac{-1+i\sqrt{3}}{2}\right)$

$\Rightarrow z=-\omega$

where, $\omega$ is the cube root of unity. So,

$1+\omega+\omega^2=0$

$\omega^3=1$

Now, let

$A=\left(z+\frac{1}{z}\right)^3+\left(z^2+\frac{1}{z^2}\right)^3+\left(z^3+\frac{1}{z^3}\right)^3+\ldots+\left(z^{21}+\frac{1}{z^{21}}\right)^3$

$\Rightarrow A=\left(-\omega-\frac{1}{\omega}\right)^3+\left(\omega^2+\frac{1}{\omega^2}\right)^3+\left(-\omega^3-\frac{1}{\omega^3}\right)^3+\ldots.+\left(-\omega^{21}-\frac{1}{\omega^{21}}\right)^3$

$\Rightarrow A=-\left(\frac{\omega^2+1}{\omega}\right)^3+\left(\frac{\omega^4+1}{\omega^2}\right)^3+\left(-1-\frac{1}{1}\right)^3+\ldots.+\left(-\omega^{21}-\frac{1}{\omega^{21}}\right)^3$

$\Rightarrow A=-\left(\frac{-\omega}{\omega}\right)^3+\left(\frac{-\omega^2}{\omega^2}\right)^3+(-1-1)^3+\ldots.+\left(-\omega^{21}-\frac{1}{\omega^{21}}\right)^3$

$\Rightarrow A=1+(-1)+(-1-1)^3+1+(-1)+(-1-1)^3\ldots.+(-1-1)^3$

$\Rightarrow A=3[1+(-1)]+(-1-1)^3$

$\Rightarrow A=-8$

Then,

$21+A=21-8$

$=13$

**19. (2)**

Given, $z^2+z+1=0\Rightarrow z=w,w^2$

Now, $\left|\sum_{n=1}^{15}\left(z^n+(-1)\frac{1}{z^n}\right)^2\right|=\left|\sum_{n=1}^{15}\left(z^{2n}+\frac{1}{z^{2n}}+2(-1)^n\right)\right|$

$=\left|\sum_{n=1}^{15}w^{2n}+\frac{1}{w^{2n}}+2(-1)^n\right|$

$=\left|\frac{w^2\left(1-w^{30}\right)}{1-w^2}+\frac{\frac{1}{w^2}\left(1-\frac{1}{w^{30}}\right)}{1-\frac{1}{w^2}}+2(-1)\right|$

$=\left|\frac{w^2(1-1)}{1-w^2}+\frac{\frac{1}{w^2}(1-1)}{1-\frac{1}{w^2}}-2\right|$

$=|0+0-2|=2$

**20. (0)**

$P(x)=f(x^3)+xg(x^3)$

$P(1)=f(1)+g(1)\quad\ldots(1)$

Now $P(x)$ is divisible by $x^2+x+1$

$\Rightarrow P(x)=Q(x)(x^2+x+1)$

$P(\omega)=0=P(\omega^2)$ where $\omega,\omega^2$ are non-real cube roots of units

$P(x)=f(x^3)+xg(x^3)$

$P(\omega)=f(\omega^3)+\omega g(\omega^3)=0$

$f(1)+\omega g(1)=0\quad\ldots(2)$

$P(\omega^2)=f(\omega^6)+\omega^2 g(\omega^6)=0$

$f(1)+\omega^2 g(1)=0\quad\ldots(3)$

Now, $(2)+(3)$

$\Rightarrow 2f(1)+(\omega+\omega^2)g(1)=0$

$2f(1)=g(1)\quad\ldots(4)$

Now, $(2)-(3)$

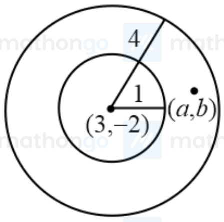
$\Rightarrow(\omega-\omega^2)g(1)=0$

$g(1)=0=f(1)$ from (4)

from (1), $P(1)=f(1)+g(1)=0$

**21. (40)**

We know that $|z-(3-2i)|=r$ represents a circle with centre $(3,-2)$ and radius $r$.




$1<|z-3+2i|<4$

$1<(a-3)^2+(b+2)^2<16$

The ordered pairs of $(a,b)$ satisfying the above inequality are $(0,\pm2),(\pm2,0),(\pm1,\pm2),(\pm2,\pm1)$

$(\pm2,\pm3),(3\pm,\pm2),(\pm1,\pm1),(2\pm,\pm2)$

$(\pm3,0),(0,\pm3),(\pm3\pm1),(\pm1,\pm3)$

i.e. total 40 elements are present in the given set

**22. (24)**

$\omega=z\bar{z}+k_1z+k_2iz+\lambda(1+i)$

$\operatorname{Re}(w)=x^2+y^2+k_1x-k_2y+\lambda=0$

$\text{Centre } \equiv\left(\frac{-k_1}{2},\frac{k_2}{2}\right)\equiv(1,2)$

$\Rightarrow k_1=-2,k_2=4$

radius $=1\Rightarrow\lambda=4$

$\operatorname{Im}=k_1y+k_2x+\lambda=0$

$\therefore 2x-y+2=0$

$d=\frac{2}{\sqrt{5}}$

$\frac{1^2}{4}=1-\frac{4}{5}=\frac{1}{5}$

$\therefore 301^2=24$

**23. (4)**

Given:

$\left|\frac{z-2}{z-3}\right|=2$

Let $z=x+iy$, then we have

$\left|\frac{x-2+iy}{x-3+iy}\right|=2$

$\Rightarrow\sqrt{(x-2)^2+y^2}=2\sqrt{(x-3)^2+y^2}$

$\Rightarrow x^2+y^2-4x+4=4x^2+4y^2-24x+36$

$\Rightarrow 3x^2+3y^2-20x+32=0$

$\Rightarrow x^2+y^2-\frac{20}{3}x+\frac{32}{3}=0$

So, $(\alpha,\beta)\equiv\left(\frac{10}{3},0\right)$

And,

$\gamma=\sqrt{\frac{100}{9}-\frac{32}{3}}=\sqrt{\frac{4}{9}}=\frac{2}{3}$

Therefore,

$3(\alpha+\beta+\gamma)=3\left(\frac{10}{3}+\frac{2}{3}\right)=12$

**24. (4)**

Here $S_n:|z-(3-2i)|=\frac{n}{4}$ represents a circle with center $C_1(3,-2)$ and radius $\frac{n}{4}$

and $T_n:|z-(2-3i)|=\frac{1}{n}$ represents a circle with center $C_2(2,-3)$ and radius $\frac{1}{n}$

For $S_n\cap T_n=\phi$, both circles do not intersect each other.

When $C_1C_2>\frac{n}{4}+\frac{1}{n}$

i.e. $\sqrt{2}>\frac{n}{4}+\frac{1}{n}$

then possible values of $n=1,2,3,4$

When $C_1C_2<\left|\frac{n}{4}-\frac{1}{n}\right|$

$\Rightarrow\sqrt{2}<\left|\frac{n^2-4}{4n}\right|$

then $n$ has infinite solutions for $n\in N$

Hence, there are total four values possible.

**25.** **(2)**

$\bar{z}=iz^2 \Rightarrow (x-iy)=i(x+iy)^2$

$\Rightarrow \ x-iy=(x^2-y^2)i-2xy$

i.e., $x=-2xy$ and $-y=x^2-y^2$

$\Rightarrow \ x=0, y=-\frac{1}{2}$

When $x=0$; $y=0,1$

When $y=-\frac{1}{2}$; $x=\pm\frac{\sqrt{3}}{2}$

$(0,0)$ will be rejected as vertices would be non-real roots.

So, the vertices will be $(0,1)$, $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2},-\frac{-1}{2}\right)$

Hence, area of $\Delta=\frac{1}{2}\times\sqrt{3}\times\frac{3}{2}=\frac{3\sqrt{3}}{4}$

**26.** **(2)**

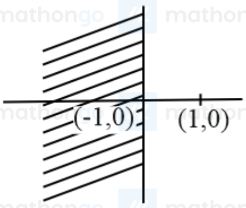
Set $A$ is $\left\{z\in C:\left|\frac{z+1}{z-1}\right|<1\right\}$

$\Rightarrow\left|\frac{\bar{z}+1}{\bar{z}-1}\right|<1$

$\Rightarrow|\bar{z}+1|<|\bar{z}-1|$

$\Rightarrow (x+1)^2+y^2<(x-1)^2+y^2$

$\Rightarrow \ x<0$



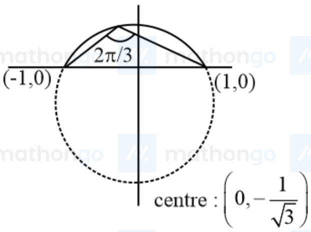

Set $B$ is $\left\{z\in C:\arg\left(\frac{z-1}{z+1}\right)=\frac{2\pi}{3}\right\}$

$\Rightarrow \arg\left(\frac{z-1}{z+1}\right)=\frac{2\pi}{3}$

$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right)-\tan^{-1}\left(\frac{y}{x+1}\right)=\frac{2\pi}{3}$

$\Rightarrow x^2+y^2+\frac{2y}{\sqrt{3}}-1=0$

$\Rightarrow$ Centre $\left(0,-\frac{1}{\sqrt{3}}\right)$




Hence, $A\cap B$ will represent a portion of a circle centred at $\left(0,-\frac{1}{\sqrt{3}}\right)$ that lies in the second quadrant only

**27.** **(6)**

$z^2+az+12=0$

$z_1+z_2=-a$ and $z_1z_2=12$

If $0,\ z_1,\ z_2$ are vertices of equilateral triangles

$z_2=z_1e^{\frac{i\pi}{3}}$

$z_2=z_1\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$2z_2-z_1=\sqrt{3}iz_1$

squaring both sides

$\Rightarrow 4z_2^2+z_1^2-4z_1z_2=-3z_1^2$

$\Rightarrow (z_1+z_2)^2=3z_1z_2$

$\Rightarrow a^2=3\times12$

$\Rightarrow|a|=6$

**28. (48)**

Let $z=x+iy$

Given $|z+5|\leq 4$

$(x+5)^2+y^2\leq 16 \;\ldots(1)$

$z(1+i)+\bar{z}(1-i)\geq -10$

$(z+\bar{z})+i(z-\bar{z})\geq -10$

$x-y+5\geq 0 \quad \ldots(2)$

$|z+1|^2=|z-(-1)|^2$

Let $P(-1,0)$

$\boxed{|z+1|^2_{\text{Max.}}=PB^2}$ (where $B$ is in $3^{\text{rd}}$ quadrant)

for point of intersection

$$\left.\begin{array}{r}(x+5)^2+y^2=16\\ x-y+5=0\end{array}\right\}y=\pm 2\sqrt{2}$$

$A\left(2\sqrt{2}-5,2\sqrt{2}\right)$ $B\left(-2\sqrt{2}-5,-2\sqrt{2}\right)$

$PB^2=\left(+2\sqrt{2}+4\right)^2+\left(2\sqrt{2}\right)^2$

$|z+1|^2=8+16+16\sqrt{2}+8$

$\alpha+\beta\sqrt{2}=32+16\sqrt{2}$

$\alpha=32,\beta=16\Rightarrow\alpha+\beta=48$

**29. (26)**

Given $|z-2|\leq 1$

$(x-2)^2+y^2\leq 1 \quad \cdots(i)$ represents interior region of a circle with centre $(2,0)$ and radius 1

Now $z(1+i)+\bar{z}(1-i)\leq 2$

Puting $z=x+iy$, we get

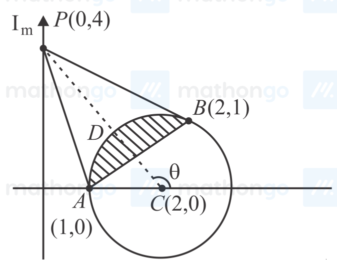
$(x+iy)(1+i)+(x-iy)(1-i)\leq 2$

$\Rightarrow x-y+i(x+y)+x-y-i(x+y)\leq 2$

$\therefore\; x-y\leq 1 \quad \ldots(i)$

Let $A(1,0)$ and $B(2,1)$ be the points on the line $x-y=1$ and the circle $(x-2)^2+y^2=1$

$|z-4i|$ represents the distance from $P(0,4)$ to any point in the region $S$




Here $PA=\sqrt{17},\; PB=\sqrt{13}$

So $A(1,0)$ is the point representing $z_2$

Let $D(2+\cos\theta,0+\sin\theta)$ be the point representing $z_1$

$\therefore\; m_{CP}=\tan\theta=-2$

$\cos\theta=-\frac{1}{\sqrt{5}},\sin\theta=\frac{2}{\sqrt{5}}$

$\therefore\; D\equiv\left(2-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right)$

$\Rightarrow z_1=\left(2-\frac{1}{\sqrt{5}}\right)+\frac{2i}{\sqrt{5}}$

So $|z_1|=\frac{25-4\sqrt{5}}{5}$ & $z_2=1$

$\therefore\; |z_2|^2=1$

$\therefore\; 5\left(|z_1|^2+|z_2|^2\right)=30-4\sqrt{5}$

$\therefore\alpha=30$

$\beta=-4$

Hence $\alpha+\beta=26$

**30.** **(3)**

We know that the complex number $z$ satisfying $|z - z_0| = r$ represents a circle with centre $z_0$ and radius $r$ units.

Hence, for $S_1 = |z - 1| \leq \sqrt{2},\ z$ lies on and inside the circle of radius $\sqrt{2}$ units and centre $(1,\ 0)$.

For $S_2$, let $z = x + iy$

Now, $(1 - i)(z) = (1 - i)(x + i\,y)$

$\Rightarrow (1 - i)(z) = x + iy - ix - i^2 y$

$\Rightarrow (1 - i)(z) = x + iy - ix + y$

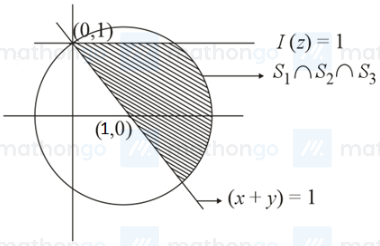
$\Rightarrow Re((1 - i)z) = x + y$

$\Rightarrow x + y \geq 1.$

And, for $S_3$, again let $z = x + iy$,

$\Rightarrow y \leq 1.$

Plotting all the inequalities on the graph, we get




Now, the common part is shown in the shaded part, hence, we get infinite points in the shaded region.

$\Rightarrow S_1 \cap S_2 \cap S_3$ has infinitely many elements.