ANSWER KEYS

1. (3)	**2.** (2)	**3.** (3)	**4.** (3)	**5.** (3)	**6.** (2)	**7.** (4)	**8.** (3)
9. (1)	**10.** (3)						

1. (3)

We know, $\{x\} = x - [x]$

$\therefore \quad I = \int_0^{1000} e^{x-[x]} dx$

$\Rightarrow I = \int_0^{1000 \times 1} e^{\{x\}} dx$

Since, $\{x\}$ is a periodic function with a period 1.

$= 1000 \int_0^1 e^{\{x\}} dx \quad \left[\int_0^{nT} f(x) dx = n \int_0^T f(x) dx\right]$

$= 1000 \int_0^1 e^x dx$

$= 1000[e^x]_0^1 = 1000\left[e^1 - e^0\right] = 1000[e-1]$

$= 1000(e-1)$.

2. (2) Let, $I = \int\limits_{-\pi}^{199\pi} \sqrt{\left(\frac{1-\cos 2x}{2}\right)} dx$

$= \int\limits_{-\pi}^{199\pi} |\sin x| dx$

$=(199-(-1)) \int\limits_0^{\pi} |\sin x| dx$

($\because |\sin x|$ is periodic with period π and $\int_{mT}^{nT} f(x) dx = (n-m) \int_0^T f(x) dx$ if T is the period of the function $f(x)$).

$= 200 \int\limits_0^{\pi} \sin x dx$

$= 200 |-\cos x|_0^{\pi}$

$= 200(1-(-1)) = 400$.

3. (3)

Given,

$P = \int_0^{3\pi} f\left(\cos^2 x\right) dx \quad \ldots$(i)

$Q = \int_0^{\pi} f\left(\cos^2 x\right) dx \quad \ldots$(ii)

Now, we know that

$\int_0^{nT} g(x) dx = n \int_0^T g(x) dx$ where, $n \in I$ and T is the period of $g(x)$.

Now, from Eq. (i), we have

$P = 3 \int_0^{\pi} f\left(\cos^2 x\right) dx = 3Q$

$\therefore P - 3Q = 0$

4. (3)

Let $I = \int_0^{100\pi+\alpha} |\sin\ x|\ dx$

$= \int_0^{100\pi} |\sin\ x|\ dx + \int_{100\pi}^{100\pi+\alpha} |\sin\ x|\ dx$

$= 100 \int_0^{\pi} \sin\ x\ dx + \int_0^{\alpha} \sin\ x\ dx$

$\because$ Period of $|\sin\ x|$ is π & $0 < \alpha < \pi$

$= 100\ (-\cos x)_0^{\pi} + (-\cos x)_0^{\alpha}$

$= 100\ (1+1) + (-\cos\ \alpha + 1) = 201 - \cos\ \alpha$

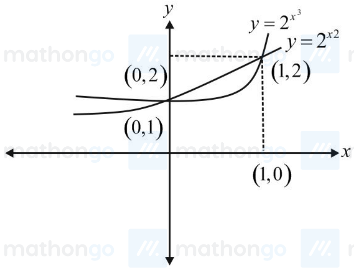
$\therefore\ k = 201$

5. (3)

Given that,

$I_1=\int_0^1 2^{x^2}dx,\ I_2=\int_0^1 2^{x^3}dx,\ I_3=\int_1^2 2^{x^2}dx$

and $I_4=\int_1^2 2^{x^3}dx$



From the graph it is clear that

$\because\ 2^{x^3}<2^{x^2},\ 0<x<1$ and $2^{x^3}>2^{x^2},\ x>1$

$\therefore\ I_4>I_3$ and $I_2<I_1$

6. (2)

$g(x)=\int_0^x f(t)dt$

Substitute $x=2$, then

$g(2)=\int_0^2 f(t)dt$

$\Rightarrow g(2)=\int_0^1 f(t)dt+\int_1^2 f(t)dt$

Now

$\frac{1}{2}\le f(t)dt\le 1$ for $t\in[0,\ 1]$

$\Rightarrow \frac{1}{2}\le \int_0^1 f(t)dt\le 1\ \ldots$(i)

Again

$0\le f(t)dt\le \frac{1}{2}$ for $t\in[1,\ 2]$

$\Rightarrow 0\le \int_1^2 f(t)dt\le \frac{1}{2}\quad \ldots$(ii)

Adding (i) and (ii),

$\frac{1}{2}\le g(2)\le \frac{3}{2}$

7. (4)

$$\underset{x\to 0}{Lt}\left\{\frac{\int_0^{x^2}\sec^2 t\,dt}{x\sin x}\right\}\quad\left(\frac{0}{0}\text{ form}\right)$$

Apply L'Hospital's Rule, we get

$$=\underset{x\to 0}{Lt}\frac{(\sec^2 x^2)\,2x}{x\cos x+\sin x}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\underset{x\to 0}{Lt}\frac{2\sec^2 x^2}{\cos x+\frac{\sin x}{x}}=\frac{2}{1+1}=1$$

8. (3)

$$\int_{\sin x}^{1} t^2 f(t)\,dt=1-\sin x$$

Applying Leibnitz Rule,

$-\sin^2 x\, f(\sin x)\cos x=-\cos x$

$\Rightarrow f(\sin x)=\frac{1}{\sin^2 x}$

$\Rightarrow f(t)=\frac{1}{t^2}$

$$f\left(\frac{1}{\sqrt{3}}\right)=\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2}=3$$

9. (1)

Given,

$f(x) = \sin x + \int_0^x f'(t)\left(2\sin t - \sin^2 t\right)dt$

$f'(x) = \cos x + f'(x)\left(2\sin x - \sin^2 x\right)$

$f'(x)\left(1 - 2\sin x + \sin^2 x\right) = \cos x$

$f'(x) = \frac{\cos x}{\sin^2 x - 2\sin x + 1}$

$f(x) = \int \frac{\cos x}{(\sin x - 1)^2}dx$

$= -\frac{1}{\sin x - 1} + c = \frac{1}{1-\sin x} + c$

when $x=0$

$\therefore\ f(0) = 1 + c = 0 \Rightarrow c = -1$

$\therefore\ f(x) = \frac{1}{1-\sin x} - 1$

$= \frac{1-1+\sin x}{1-\sin x}$

$= \frac{\sin x}{1-\sin x}$

10. (3)

Here, $f(x+1) - f(x)$

$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^2(n+1)\theta - \sin^2 n\theta}{\sin^2\theta}$

$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\theta \cdot \sin\theta}{\sin^2\theta}d\theta$

$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)\theta}{\sin^2\theta}d\theta$

$= \frac{1}{\pi}\left(\int_0^{\frac{\pi}{2}} \frac{\sin 2n\theta \cdot \cos\theta}{\sin\theta}d\theta + \int_0^{\frac{\pi}{2}} \cos 2n\theta d\theta\right)$

Now using, $\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots + \cos(2n-1)\theta$

$= \frac{\sin\frac{n(2\theta)}{2}}{\sin\left(\frac{2\theta}{2}\right)}\cos(\theta + (n-1)\theta)$

$\cos\theta + \cos 3\theta + \cos 5\theta + \ldots\ldots\ldots + \cos(2n-1)\theta = \frac{1}{2}\left(\frac{\sin 2n\theta}{\sin\theta}\right)$

$\Rightarrow f(n+1) - f(n) = \frac{1}{\pi}$

$\left[2\int_0^{\frac{\pi}{2}} \cos\theta + \cos 3\theta + \ldots. + \cos(2n-1)\theta\cos\theta d\theta + 0\right]$

$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\{(2\cos\theta\cos\theta) + (2\cos 3\theta\cos\theta) + \ldots(2\cos(2n-1)\cdot\cos\theta)\}d\theta \quad = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1 \cdot d\theta$

$= \frac{1}{\pi}[\theta]_0^{\frac{\pi}{2}}$

$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0\right]$

$= \frac{\pi}{2} \times \frac{1}{\pi} = \frac{1}{2}$

$\Rightarrow \int_0^{\frac{\pi}{2}} \cos 2n\theta d\theta = 0$

$f(n+1) - f(n) = \frac{1}{2}$

If $n = 1$, $f(2) - f(1)$

$= \frac{1}{2}$ as $\left(f(1) = \frac{1}{2}\right)$

$\Rightarrow f(2) = \frac{2}{2}$

If $n = 2$, $f(3) - f(2) = \frac{1}{2}$

$n = 2$, $f(3) - f(2) = \frac{1}{2}$ and so on

$f(n) = \frac{n}{2}$

Hence, $\frac{f(15) + f(3)}{f(15) - f(9)}$

$= \frac{\frac{15}{2} + \frac{3}{2}}{\frac{15}{2} - \frac{9}{2}} = 3$