ANSWER KEYS

1. (3)	**2.** (40)	**3.** (3)	**4.** (2)	**5.** (4)	**6.** (2)	**7.** (2)	**8.** (1)
9. (4)	**10.** (2)	**11.** (4)	**12.** (2)	**13.** (3)	**14.** (1)	**15.** (2)	**16.** (1)
17. (2)	**18.** (16)	**19.** (3)	**20.** (248)				

1. (3)

Putting, $x=0$ in $y^2+\log_e\left(\cos^2 x\right)=y$ we get $y=0,1$

$2y\cdot y' + \frac{1}{\cos^2 x}\cdot 2\cos x(-\sin x)=y'$

$\Rightarrow 2y\cdot y' - 2\tan x = y' \ldots\ldots(1)$

$y'(0)=0$ for $y=0 \& y=1$.

Differentiating (1), $2y\cdot y'' + 2(y')^2 - 2sec^2x = y''$,

$y''(0)=-2$ for $y=0$

$y''(0)=2$, for $y=1$

$\therefore |y''(0)|=2$

2. (40)

Given:

$\log_e(x+y)=4xy$

When $x=0$, then $y=1$

$\log_e(x+y)=4xy$

$\Rightarrow x+y=e^{4xy}$

Now differentiate w.r.t. x

$1+y'=e^{4xy}(4y+4xy') \quad \ldots$(i)

At $(0,1) \Rightarrow y'(0)+1=4 \Rightarrow y'(0)=3$

Now, again differentiate equation (i), we get

$y''=e^{4xy}\left(4y+4xy'\right)^2+e^{4xy}(4y'+4y'+4xy'')$

At $(0,1)$

$y''(0)=1\left(4\times1+0\right)^2+1\left(4\times3+4\times3+0\right)$

$\Rightarrow y''(0)=16+24=40$

$\Rightarrow y''(0)=40$

3. (3)

Given $h(x)=f(f(x))$

$\Rightarrow h'(x)=f'(f(x)).\, f'(x)$

$\Rightarrow h'(x)=f(f(x)).\, f(x) \quad \ldots(1)$ (as $f'(x)=f(x)$)

Now $f'(x)=f(x)$

$\Rightarrow \frac{f'(x)}{f(x)}=1$

Integrating both sides with respect to x, we get

$\ln|f(x)|=x+c$

$\Rightarrow f(x)=k.\, e^x$

$\Rightarrow f(x)=\frac{2}{e}.\ e^x \quad \ldots(2)\ [\because\ f(1)=2]$

Putting $x=1$ in equation (1), we get

$h'(1)=f(f(1)).\, f(1)=f(2).\, 2=2.\,\frac{2}{e}e^2=4e$ (using equation (2))

4. **(2)**

$e^y + xy = e$

differentiate w. r. t. x

$e^y \frac{dy}{dx} + x\frac{dy}{dx} + y = 0$

$\Rightarrow \frac{dy}{dx}(x+e^y) = -y, \left.\frac{dy}{dx}\right|_{(0,1)} = -\frac{1}{e}$

Again differentiate w. r. t. x

$e^y.\frac{d^2y}{dx^2} + \frac{dy}{dx}.e^y.\frac{dy}{dx} + x.\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$

$\Rightarrow (x+e^y)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2.e^y + 2\frac{dy}{dx} = 0$

Now, at $(0,1)$

$e\frac{d^2y}{dx^2} + \frac{1}{e^2}e + 2\left(-\frac{1}{e}\right) = 0$

$\therefore \frac{d^2y}{dx^2} = \frac{1}{e^2}$

Hence, ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = \left(-\frac{1}{e}, \frac{1}{e^2}\right)$

5. **(4)**

Given

$x = 2\sqrt{2}\cos t\sqrt{\sin 2t}$

Now differentiating w.r.t t both side we get,

$\frac{dx}{dt} = \frac{2\sqrt{2}\cos 3t}{\sqrt{\sin 2t}} \ldots\ldots(1)$

Also given $y(t) = 2\sqrt{2}\sin t\sqrt{\sin 2t}$

Again differentiating w.r.t t both side we get,

$\frac{dy}{dt} = \frac{2\sqrt{2}\sin 3t}{\sqrt{\sin 2t}} \ldots\ldots(2)$

Now dividing equation (2) from (1) we get,

$\frac{dy}{dx} = \tan 3t$

Now finding the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$ we get,

$\frac{dy}{dx} = -1$

Now finding $\frac{d^2y}{dx^2}$ we get,

$\frac{d^2y}{dx^2} = \frac{3}{2\sqrt{2}}\frac{\sec^2 3t\cdot\sqrt{\sin 2t}}{\cos 3t}$

Now finding value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$ we get,

$\frac{d^2y}{dx^2} = -3$

Now putting the value of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in $\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$ we get,

$\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = \frac{1+1}{-3} = -\frac{2}{3}$

6. **(2)**

$\frac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta$

$\frac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta$

$\therefore \frac{dy}{dx} = \frac{\sin 2\theta - \sin\theta}{\cos\theta - \cos 2\theta}$

$= \frac{2\sin\frac{\theta}{2}.\cos\frac{3\theta}{2}}{2\sin\frac{\theta}{2}.\sin\frac{3\theta}{2}} = \cot\frac{3\theta}{2}$

$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = -\frac{3}{2}\text{cosec}^2\frac{3\theta}{2}.\frac{d\theta}{dx}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{3}{2}\text{cosec}^2\frac{3\theta}{2}}{2(\cos\theta - \cos 2\theta)}$

$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{\theta=\pi} = \frac{3}{4(-1-1)} = \frac{3}{8}$

7. **(2)** Consider the equation,

$x \log_e(\log_e x) - x^2 + y^2 = 4$

Differentiate both sides w.r.t. x,

$\log_e(\log_e x) + x \cdot \frac{1}{x \cdot \log_e x} - 2x + 2y\frac{dy}{dx} = 0$

$\log_e(\log_e x) + \frac{1}{\log_e x} - 2x + 2y\frac{dy}{dx} = 0 \quad \ldots(1)$

When $x = e, y = \sqrt{4+e^2}$. Put these values in (1),

$0 + 1 - 2e + 2\sqrt{4+e^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{2e-1}{2\sqrt{4+e^2}}$.

8. **(1)** $2x^y + 3y^x = 20$

$2x^y\left[\frac{y}{x} + (\ln x)y'\right] + 3y'\left[\frac{xy'}{y} + \ln y\right] = 0$

$y' = \frac{-(12\ln 2 + 8)}{12 + 8\ln 2} = -\left(\frac{2+\log_e 8}{3+\log_e 4}\right)$

9. **(4)**

For $\log_e 2\frac{d}{dx}(\log_{\cos x} \operatorname{cosec} x)$

let, $y = \log_{\cos x} \operatorname{cosec} x$

i.e. $y = -\frac{\ln(\sin x)}{\ln(\cos x)}$

$\Rightarrow \frac{dy}{dx} = -\frac{[\cot x \cdot \ln(\cos x) + \tan x \cdot \ln(\sin x)]}{(\ln(\cos x))^2}$

Now $\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = -\frac{\left[\cot\frac{\pi}{4}\cdot\ln\left(\cos\frac{\pi}{4}\right) + \tan\frac{\pi}{4}\cdot\ln\left(\sin\frac{\pi}{4}\right)\right]}{\left(\ln\left(\cos\frac{\pi}{4}\right)\right)^2}$

$= \frac{4}{\ln 2}$

$\Rightarrow \log_e 2 \cdot \frac{4}{\ln 2} = 4$

10. **(2)**

The given equation can be written as

$y = \sin^3\left(\frac{\pi}{3}\cos g(x)\right)$

where,

$g(x) = \frac{\pi}{3\sqrt{2}}\left(-4x^3 + 5x^2 + 1\right)^{3/2}$

$\Rightarrow g'(x) = \frac{\pi}{2\sqrt{2}}\left(-4x^3 + 5x^2 + 1\right)^{1/2}\left(-12x^2 + 10x\right)$

$\Rightarrow g'(1) = \frac{\pi\sqrt{2}}{2\sqrt{2}}(-2) = -\pi$

And,

$g(1) = \frac{2\pi}{3} = \left(\pi - \frac{\pi}{3}\right)$

Now,

$y' = 3\sin^2\left(\frac{\pi}{3}\cos g(x)\right) \times \cos\left(\frac{\pi}{3}\cos g(x)\right) \times \frac{\pi}{3}(-\sin g(x))g'(x)$

$y'(1) = 3\sin^2\left(-\frac{\pi}{6}\right)\cdot\cos\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{3}\left(-\sin\frac{2\pi}{3}\right)g'(1)$

$y'(1) = \frac{3}{4}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\pi}{3}\left(\frac{-\sqrt{3}}{2}\right)(-\pi) = \frac{3\pi^2}{16}$

$y(1) = \sin^3\left(\frac{\pi}{3}\cos\frac{2\pi}{3}\right) = -\frac{1}{8}$

Therefore,

$2y'(1) + 3\pi^2 y(1) = 0$

11. **(4)**

Let $x = \tan\theta$

$y_1 = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$

$x = \sin\phi, \quad y_2 = \tan^{-1}\left(\frac{2\sin\phi\cos\phi}{\cos 2\phi}\right) = \tan^{-1}\left(\tan 2\phi\right) = 2\phi = 2\sin^{-1}x$

$\frac{dy_1}{dy_2} = \frac{dy_1/dx}{dy_2/dx} = \frac{\frac{1}{(1+x^2)}\cdot\frac{1}{2}}{2\cdot\frac{1}{\sqrt{1-x^2}}}$

$= \frac{\sqrt{1-x^2}}{4(1+x^2)} = \frac{\sqrt{1-\frac{1}{4}}}{4\left(1+\frac{1}{4}\right)} = \frac{\sqrt{3}}{10}$

12. (2)

$x=\frac{1}{2}, y=\frac{-1}{4} \Rightarrow xy=\frac{-1}{8}$

$y.\frac{(-2x)}{2\sqrt{1-x^2}}+y'.\sqrt{1-x^2}=0-\left\{1.\sqrt{1-y^2}+\frac{x.(-2y)}{2\sqrt{1-y^2}}y'\right\}$

$-\frac{xy}{\sqrt{1-x^2}}+y'\sqrt{1-x^2}=-\sqrt{1-y^2}+\frac{xy.y'}{\sqrt{1-y^2}}$

$y'\left(\sqrt{1-x^2}-\frac{xy}{\sqrt{1-y^2}}\right)=\frac{xy}{\sqrt{1-x^2}}-\sqrt{1-y^2}$

$y'\left(\frac{\sqrt{3}}{2}+\frac{1}{8.\frac{\sqrt{15}}{4}}\right)=\frac{-1}{8.\sqrt{\frac{3}{4}}}-\frac{\sqrt{15}}{4}$

$y'\left(\frac{\sqrt{45}+1}{2\sqrt{15}}\right)=-\frac{(1+\sqrt{45})}{4\sqrt{3}}$

$y'=-\frac{\sqrt{15}}{2\sqrt{3}}=-\frac{\sqrt{5}}{2}$

13. (3)

$y\left(x\right)=\cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

As we know that, $\sqrt{1-\sin x}=\left|\sin\frac{x}{2}-\cos\frac{x}{2}\right|$ & $\sqrt{1+\sin x}=\left|\sin\frac{x}{2}+\cos\frac{x}{2}\right|$

$y\left(x\right)=\cot^{-1}\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}$

$=\cot^{-1}\frac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}$ for $x\in\left(\frac{\pi}{2},\pi\right)$

$=\cot^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$

$=\cot^{-1}\left(\tan\frac{x}{2}\right)$

$=\cot^{-1}\left(\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)$

$y\left(x\right)=\frac{\pi}{2}-\frac{x}{2}$

$y'\left(x\right)=-\frac{1}{2}$

14. (1)

Put $x=\sin^2\theta, 0<x<1$

$\sin\theta=\sqrt{x}$

$\Rightarrow f(x)=\cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-\sin^2\theta}{\sin^2\theta}}\right)\right)$

$\Rightarrow f(x)=\cos\left(2\tan^{-1}(\sin\theta)\right)$

$\Rightarrow f(x)=\cos\left(2\tan^{-1}\sqrt{x}\right)$

$=\frac{1-\tan^2(\tan^{-1}\sqrt{x})}{1+\tan^2(\tan^{-1}\sqrt{x})}$

$\Rightarrow f(x)=\frac{1-x}{1+x}$

$\Rightarrow f'(x)=\frac{(1+x)(-1)-(1-x)\cdot 1}{(1+x)^2}$

$\Rightarrow f'(x)=\frac{-2}{(1+x)^2}$

Multiply, $(1-x)^2$ on both sides

$\Rightarrow (1-x)^2 f'(x)=\frac{-2(1-x)^2}{(1+x)^2}$

Now, $2(f(x))^2=\frac{2(1-x)^2}{(1+x)^2}$

$\therefore (1-x)^2 f'(x)+2(f(x))^2=0$ option (1) satisfied

15. (2)

Here, $g(f(x))=x$ as $f(x)$ and $g(x)$ are inverse of each other.

Now, $g'(f(x))f'(x)=1$

$\Rightarrow g'(f(x))=\frac{1}{f'(x)} \quad \ldots(i)$

Now $f(x)=63 \Rightarrow x^3+x-5=63$

$\Rightarrow x^3+x-68=0$

So $x=4$ satisfies the above equation

$g'(63)=\frac{1}{f'(4)}$ from (i)

$=\frac{1}{3(4)^2+1}=\frac{1}{49}$

16. (1)

$f(g(x))=x$

$\Rightarrow f'(g(x)).\,g'(x)=1$

Put $x=a$

$\Rightarrow f'(g(a))g'(a)=1$

$\Rightarrow f'(b)\times 5=1$

$\Rightarrow f'(b)=\frac{1}{5}$

17. (2)

Given,

$y=\tan^{-1}\left(\sec x^3-\tan x^3\right)$

$=\tan^{-1}\left(\frac{1-\sin x^3}{\cos x^3}\right)$

$=\tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}-x^3\right)}{\sin\left(\frac{\pi}{2}-x^3\right)}\right)$

$=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x^3}{2}\right)\right)$

Since $\frac{\pi}{4}-\frac{x^3}{2}\in\left(-\frac{\pi}{2},0\right)$ as $\left(\frac{\pi}{2}<x^3<\frac{3\pi}{2}\right)$

So, $y=\left(\frac{\pi}{4}-\frac{x^3}{2}\right)$

Now differentiating we get,

$y'=\frac{-3x^2}{2}, y''=-3x$

Now putting the value of x in term of y'' in $4y=\pi-2x^3$

We get,

$4y=\pi-2x^2\left(\frac{-y''}{3}\right)$

$12y=3\pi+2x^2y''$

$x^2y''-6y+\frac{3\pi}{2}=0$

18. (16)

Given,

(α,α) lies on $C:\left(x^2+y^2-3\right)+\left(x^2-y^2-1\right)^5=0$

So on putting (α,α) we get,

$2\alpha^2-3-1^5=0$

$\Rightarrow\ \alpha=\sqrt{2}$

Now, differentiating the curve C we get,

$2x+2y\cdot y'+5\left(x^2-y^2-1\right)^4(2x-2yy')=0 \quad \cdots(1)$

At $\left(\sqrt{2},\sqrt{2}\right)$

$\sqrt{2}+\sqrt{2}y'+5(-1)^4\left(\sqrt{2}-\sqrt{2}y'\right)=0$

$\Rightarrow y'=\frac{3}{2} \quad \cdots(2)$

Again, Diff. (1) w.r.t. x we get,

$1+(y')^2+yyy''+20\left(x^2-y^2-1\right)^3(x-yy')^2\cdot 2$

$+5\left(x^2-y^2-1\right)^4\left(1-(y')^2-yyy''\right)=0$

At $\left(\sqrt{2},\sqrt{2}\right)$ and $y'=\frac{3}{2}$

We have,

$\left(1+\frac{9}{4}\right)+\sqrt{2}y''-40\left(\sqrt{2}-\sqrt{2}\cdot\frac{3}{2}\right)^2$

$+5(1)\left(1-\frac{9}{4}-\sqrt{2}y''\right)=0$

$\Rightarrow 4\sqrt{2}y''=-23$

$y''=\frac{-23}{4\sqrt{2}}$

So, value of $3y'-y^3y''=\frac{9}{2}+\frac{23}{2}=16$

19. **(3)** Let $\tan^{-1}x = \theta$

$\Rightarrow x = \tan\theta \Rightarrow \sin\theta = \frac{x}{\sqrt{1+x^2}}$

So, $y = \left(\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right)^2 - 1$

$\Rightarrow y = \frac{(x+1)^2}{1+x^2} - 1$

$\Rightarrow y = \frac{2x}{1+x^2} = f(x)$

Now, $\frac{dy}{dx} = \frac{1}{2}\frac{1}{\sqrt{1-f^2}} \times f'(x)$

$= \frac{1}{2}\frac{1}{\sqrt{1-\frac{4x^2}{(1+x^2)^2}}} f(x)$

$= \frac{(1+x^2)}{2(x^2-1)} f'(x)$

$= \frac{1+x^2}{2(x^2-1)} \times 2\frac{1+x^2-2x^2}{(1+x^2)^2}$

$\Rightarrow \frac{dy}{dx} = \frac{1-x^2}{(x^2-1)(1+x^2)}$

$\Rightarrow dy = \frac{1-x^2}{(x^2-1)(1+x^2)} dx$

Integrating both sides with respect to x

$\int dy = \int \frac{1-x^2}{(x^2-1)(1+x^2)} dx$

$\Rightarrow y = -\tan^{-1}x + c$

Given, $y\left(\sqrt{3}\right) = \frac{\pi}{6} \Rightarrow \frac{\pi}{6} = -\frac{\pi}{3} + c \Rightarrow c = \frac{\pi}{2}$

$y = -\tan^{-1}x + \frac{\pi}{2} = \cot^{-1}x$

Now, $y\left(-\sqrt{3}\right) = \cot^{-1}x\left(-\sqrt{3}\right) = \frac{5\pi}{6}$

20. **(248)**

Given, $f(x+y) = 2^x f(y) + 4^y (f(x)$

Put $y = 2$ we get,

$f(x+2) = 2^x \times 3 + 16f(x)$

$f'(x+2) = 16f'(x) + 3 \times 2^x \ln 2$

Now put $x = 2$ we get,

$f'(4) = 16f'(2) + 12\ln 2 \quad \ldots$(i)

Similarly, $f(y+2) = 4f(y) + 3 \times 4^y$

$f'(4) = 4f'(y) + 3 \times 4^y \ln 4$

$f'(4) = 4f'(2) + 96\ln 2 \quad \cdots$(ii)

solving eq. (i) and (ii), we get

$f'(2) = 7\ln 2$

from equation (i), we get

$f'(4) = 124\ln 2$

Now $\Rightarrow 14 \times \frac{f'(4)}{f'(2)}$

$14 \times \frac{124\ln 2}{7\ln 2} = 248$