ANSWER KEYS

1. (3)	**2.** (3)	**3.** (3)	**4.** (1)	**5.** (3)	**6.** (2)	**7.** (1)	**8.** (3)
9. (3)	**10.** (1)	**11.** (8)	**12.** (1)	**13.** (5)	**14.** (3)	**15.** (1)	**16.** (1)
17. (3)	**18.** (3)	**19.** (2)	**20.** (1)				

1. (3)

Since $\lim\limits_{x \to 7} \frac{18-[1-x]}{[x-3a]}$ exists $(a \in \mathbb{I})$

$\therefore$ L.H.L $=$ R.H.L

$\Rightarrow \lim\limits_{h \to 0} \frac{18-(-6)}{[7-h-3a]} = \lim\limits_{h \to 0} \frac{18-(-7)}{[7+h-3a]}$

$\Rightarrow \lim\limits_{h \to 0} \frac{24}{(7-3a)+[-h]} = \lim\limits_{h \to 0} \frac{25}{(7-3a)+[h]}$

$\Rightarrow \frac{24}{6-3a} = \frac{25}{7-3a} \ \Rightarrow a = -6$

2. (3)

Given,

$f(x) = [1+x] + \frac{\alpha^{2[x]+\{x\}}+[x]-1}{2[x]+\{x\}}$

$\lim\limits_{x \to 0^-} f(x) = \alpha - \frac{4}{3}$

$\lim\limits_{x \to 0^-} [1+x] + \frac{\alpha^{[x]+[x]+\{x\}}+[x]-1}{2[x]+\{x\}} = \alpha - \frac{4}{3}$

$\Rightarrow 0 + \frac{\alpha^{-1}-2}{-1} = \alpha - \frac{4}{3}$

$\Rightarrow 2 - \frac{1}{\alpha} = \alpha - \frac{4}{3}$

$\Rightarrow \alpha + \frac{1}{\alpha} = \frac{10}{3}$

On solving above equation we get,

$\Rightarrow \alpha = 3; \alpha \in I$

3. (3)

Given, $\lim\limits_{n \to \infty} \sqrt{n^2 - n - 1} + \alpha n + \beta = 0$

Now rewriting the expression we get,

$\lim\limits_{n \to \infty} (n^2 - n - 1)^{\frac{1}{2}} + \alpha n = -\beta$

$\Rightarrow \lim\limits_{n \to \infty} n\left(1 - \frac{1}{n} - \frac{1}{n^2}\right)^{\frac{1}{2}} + \alpha n = -\beta$

Now on using binomial approximation $(1+x)^n = 1 + nx$ we get,

$\lim\limits_{n \to \infty} n\left(1 - \frac{1}{2n} - \frac{1}{2n^2}\right) + \alpha n = -\beta$

$\Rightarrow \lim\limits_{n \to \infty} n - \frac{1}{2} - \frac{1}{2n} + \alpha n = -\beta$

$\Rightarrow \lim\limits_{n \to \infty} n(1+\alpha) - \frac{1}{2} - \frac{1}{2n} = -\beta$

Now for limit to exist $(1+\alpha) = 0 \Rightarrow \alpha = -1$

Now on putting the value of α we get,

$\lim\limits_{n \to \infty} n(1+\alpha) - \frac{1}{2} - \frac{1}{2n} = -\beta$

$\Rightarrow \lim\limits_{n \to \infty} -\frac{1}{2} - \frac{1}{2n} = -\beta$

$\Rightarrow -\frac{1}{2} - 0 = -\beta$

$\Rightarrow \beta = \frac{1}{2}$

So, $8(\alpha + \beta) = 8\left(-1 + \frac{1}{2}\right) = 8 \times \frac{-1}{2} = -4$

4. (1)

Let $f(x) = (x-1)\left(x^4 + ax^3 + bx^2 + cx + d\right)$

$f(x) + f'(x) + f''(x) = x^5 + (a+4)x^4 + (b+3a+16)x^3 +$
$(9a + 2b + c - 12)x^2 + (d + c + 4b - 6a)x + (c - 2b)$

$\because \ f(x) + f'(x) + f''(x) = x^5 + 64$

$\Rightarrow \ a + 4 = 0, \ b + 3a + 16 = 0, \ 9a + 2b + c - 12 = 0, \ d + c + 4b - 6a = 0$

$\Rightarrow \ a = -4, \ b = -4, \ c = 56, \ d = -64$

So $\lim\limits_{x \to 1} \frac{f(x)}{x-1} = \lim\limits_{x \to 1} \left(x^4 - 4x^3 - 4x^2 + 56x - 64\right)$

$= 1 - 4 - 4 + 56 - 64 = -15$

5. (3)

The given limit $\lim\limits_{n\to\infty}\frac{1+2-3+4+5-6+\ldots+(3n-2)+(3n-1)-3n}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$ can be expressed as, $\lim\limits_{n\to\infty}\frac{\sum\limits_{r=1}^{n}(3r-2)+(3r-1)-3r}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$

$=\lim\limits_{n\to\infty}\frac{\sum\limits_{r=1}^{n}3(r-1)}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}$

$=\lim\limits_{n\to\infty}\frac{0+3+6+9+\ldots n\ terms}{\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}}\ldots\ldots(1)$

We know the sum of the nth term of A.P. is given by

$S_n=\frac{n}{2}[2a+(n-1)d]$

Using above formula in equation(1) we get,

$=\lim\limits_{n\to\infty}\frac{\frac{3n(n-1)}{2}}{\left(\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}\right)}$

$=\lim\limits_{n\to\infty}\frac{3n(n-1)}{2\left(\sqrt{2n^4+4n+3}-\sqrt{n^4+5n+4}\right)}$

$=\lim\limits_{n\to\infty}\frac{3\left(1-\frac{1}{n}\right)}{2\left(\sqrt{2+\frac{4}{n^3}+\frac{3}{n^4}}-\sqrt{1+\frac{5}{n^3}+\frac{4}{n^4}}\right)}$

$=\frac{3(1-0)}{2\left(\sqrt{2+0+0}-\sqrt{1+0+0}\right)}$

$=\frac{3}{2\left(\sqrt{2}-1\right)}=\frac{3}{2}\left(\sqrt{2}+1\right)$

6. (2)

Consider $\lim\limits_{x\to0}\frac{\cos(\sin x)-\cos x}{x^4}\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim\limits_{x\to0}\frac{2\sin\left(\frac{\sin x+x}{2}\right)\cdot\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$

$=\lim\limits_{x\to0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{\sin x+x}{2}\right)\times\left(\frac{x-\sin x}{2}\right)\times\frac{1}{x^4}$

$\lim\limits_{x\to0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{x^2-\sin^2x}{4x^4}\right)$

$\lim\limits_{x\to0}2\times\left(\frac{x^2-\sin^2x}{4x^4}\right)\quad\left(\frac{0}{0}\text{ form}\right)\quad\left(\because\lim\limits_{t\to0}\frac{\sin t}{t}=1\right)$

Applying L'Hospital's Rule,

$\lim\limits_{x\to0}2\times\left(\frac{2x-2\sin x\cos x}{4\cdot4x^3}\right)=\lim\limits_{x\to0}\left(\frac{2x-\sin2x}{8x^3}\right)\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim\limits_{x\to0}\left(\frac{2-2\cos2x}{24x^2}\right)\quad\left(\frac{0}{0}\text{ form}\right)$

$\lim\limits_{x\to0}\left(\frac{4\sin2x}{48x}\right)=\frac{1}{6}\lim\limits_{x\to0}\left(\frac{\sin2x}{2x}\right)=\frac{1}{6}$

7. (1)

$\lim\limits_{x\to\frac{\pi}{4}}\frac{8\sqrt{2}-(\cos x+\sin x)^7}{\sqrt{2}-\sqrt{2}\sin2x}\quad\left(\frac{0}{0}\text{form}\right)$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{-7(\cos x+\sin x)^6(-\sin x+\cos x)}{-2\sqrt{2}\cos2x}$ (using L'Hospital Rule)

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{56(\cos x-\sin x)}{2\sqrt{2}\cos2x}\quad\left(\frac{0}{0}\text{ form}\right)$

$=\lim\limits_{x\to\frac{\pi}{4}}\frac{-56(\sin x+\cos x)}{-4\sqrt{2}\sin2x}$ (using L'Hospital Rule)

$=\frac{56\sqrt{2}}{4\sqrt{2}}=14$

8. (3)

$\lim\limits_{x\to0}\frac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin x-x+1}}\quad\left(\frac{0}{0}\text{form}\right)$

By using L'Hospital Rule,

$=\lim\limits_{x\to0}\frac{1+2\cos x}{\frac{1(2x+2\cos x)}{2\sqrt{x^2+2\sin x+1}}-\frac{1(\sin2x-1)}{2\sqrt{\sin^2x-x+1}}}$

$=\frac{1+2}{1+\frac{1}{2}}=\frac{3}{\frac{3}{2}}=2.$

9.

(3) $L=\lim\limits_{h\to 0}\left(\dfrac{\sqrt{3}\left(\frac{1}{2}\cos h+\frac{\sqrt{3}}{2}\sin h\right)-\left(\frac{\sqrt{3}}{2}\cos h-\frac{\sin h}{2}\right)}{\left(\sqrt{3}h\right)\left(\sqrt{3}\right)}\right)$

$L=\lim\limits_{h\to 0}\frac{2\sin h}{3h}$

$\Rightarrow L=\frac{2}{3}$

10. (1)

Given,

Function $f(x)=\begin{cases}\frac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$, $g(x)=\begin{cases}\frac{\sin(x+1)}{(x+1)}, & x\neq -1\\ 1, & x=-1\end{cases}$ and $h(x)=2[x]-f(x)$

To find $\lim\limits_{x\to 1}g(h(x-1))$

Taking L. H. L, we take $x=1-h$

$\lim\limits_{h\to 0^-}g\left(h\left(1-h-1\right)\right)$

$\Rightarrow \lim\limits_{h\to 0^-}g\left(2[-h]-\frac{-h}{|-h|}\right)$

$\Rightarrow \lim\limits_{h\to 0^-}g\left(2\left(-1\right)+1\right)$

$\Rightarrow \lim\limits_{h\to 0^-}g(-1)=1$

Now taking R. H. L, we take $x=1+h$

$\lim\limits_{h\to 0^+}g\left(h\left(1+h-1\right)\right)$

$=\lim\limits_{h\to 0^+}g(2[h]-f(h))$

$=\lim\limits_{h\to 0^+}g\left(0-1\right)$

$\Rightarrow \lim\limits_{h\to 0^+}1=1$

11. (8)

$\lim\limits_{x\to 0}\left\{\frac{1}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)\right\}$

$=\lim\limits_{x\to 0}\dfrac{\left(1-\cos\frac{x^2}{2}\right)\left(1-\cos\frac{x^2}{4}\right)}{x^8}$

$=\lim\limits_{x\to 0}\dfrac{\left(1-\cos\frac{x^2}{2}\right)}{x^4}\times\dfrac{\left(1-\cos\frac{x^2}{4}\right)}{x^4}$

$=\lim\limits_{x\to 0}\dfrac{\left(1-\cos\frac{x^2}{2}\right)}{4\times\left(\frac{x^4}{4}\right)}\times\dfrac{\left(1-\cos\frac{x^2}{4}\right)}{16\times\left(\frac{x^4}{16}\right)}$

Using $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

$=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{2}\times\frac{1}{16}=\frac{1}{256}=2^{-8}$

$\Rightarrow 2^{-8}=2^{-k}\quad\Rightarrow k=8$

12. (1)

$\lim\limits_{x\to 1^+}\dfrac{(1-|x|+\sin|1-x|)\sin\left([1-x]\frac{\pi}{2}\right)}{|1-x|[1-x]}$

$=\lim\limits_{h\to 0}\dfrac{(1-|(1+h)|+\sin|1-(1+h)|)\sin\left(\frac{\pi}{2}[1-(1+h)]\right)}{|1-(1+h)|[1-(1+h)]}$

$=\lim\limits_{h\to 0}\dfrac{(1-h-1+\sin|(1-h-1)|)\sin\left(\frac{\pi}{2}[1-1-h]\right)}{|(1-1-h)|[1-1-h]}$

$=\lim\limits_{h\to 0}\dfrac{(-h+\sin h)\sin\left(\frac{\pi}{2}[-h]\right)}{|-h|[-h]}$

$=\lim\limits_{h\to 0}\dfrac{(-h+\sin h)\sin\left(-\frac{\pi}{2}\right)}{h(-1)}=\lim\limits_{h\to 0}\dfrac{(-h+\sin h)(-1)}{-h}$

$=1-1=0.$

13. (5)

$\lim\limits_{x\to 0}\frac{ax-\left(e^{4x}-1\right)}{ax\left(e^{4x}-1\right)}\ \left(\frac{0}{0}\right)$

$=\lim\limits_{x\to 0}\frac{ax-\left(e^{4x}-1\right)}{ax\cdot 4x}$ Use $\lim\limits_{x\to 0}\frac{e^{4x}-1}{4x}=1$

Apply L'Hospital Rule

$=\lim\limits_{x\to 0}\frac{a-4e^{4x}}{8ax}$ ($\frac{a-4}{0}$ form)

limit exists only when $a-4=0\Rightarrow a=4$

$=\lim\limits_{x\to 0}\frac{4-4e^{4x}}{32x}$

$=\lim\limits_{x\to 0}\frac{1-e^{4x}}{8x}\ \left(\frac{0}{0}\right)$

$=\lim\limits_{x\to 0}\frac{-e^{4x}\cdot 4}{8}=-\frac{1}{2}\Rightarrow b=-\frac{1}{2}$

$a-2b=4-2\left(-\frac{1}{2}\right)$

$=5$

14. (3)

We have,

$\lim\limits_{x\to 0}\left(2-\cos x\sqrt{\cos 2x}\right)^{\left(\frac{x+2}{x^2}\right)}\rightarrow 1^{\infty}$

Hence,

$=e^{\lim\limits_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\right)\times(x+2)}$

Now,

$\lim\limits_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos 2\mathrm{x}}}{x^2}\right)\rightarrow\left(\frac{0}{0}\right)$

Using L'Hospital Rule

$\lim\limits_{x\to 0}\left\{\frac{\sin x\sqrt{\cos 2x}-\cos x\times\frac{1}{2\sqrt{\cos 2x}}\times(-2\sin 2x)}{2x}\right\}$

$=\lim\limits_{x\to 0}\left\{\frac{\sin x\cos 2x+\sin 2x\cdot\cos x}{2x\sqrt{\cos 2x}}\right\}$

$=\lim\limits_{x\to 0}\left\{\frac{\sin 3x}{2x\sqrt{\cos 2x}}\right\}$

$=\frac{3}{2}\lim\limits_{x\to 0}\left\{\frac{\sin 3x}{3x}\right\}\times\lim\limits_{x\to 0}\left\{\frac{1}{\sqrt{\cos 2x}}\right\}$

$=\frac{3}{2}\lim\limits_{x\to 0}\left\{\frac{\sin 3x}{3x}\right\}\times 1$

$=\frac{3}{2}$

So,

$e^{\lim\limits_{x\to 0}\left(\frac{1-\cos x\sqrt{\cos 2x}}{x^2}\right)(x+2)}$

$=e^{\frac{3}{2}\times 2}=e^3$

$\Rightarrow\boxed{a=3}$

15. (1)

This limit is 1^{∞} form

$\therefore\lim\limits_{x\to 0}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{1+f(3+x)-f(3)}{1+f(2-x)-f(2)}-1\right)}$

$=e^{\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{f(3+x)-f(3)-f(2-x)+f(2)}{1+f(2-x)-f(2)}\right)}\left(\frac{0}{0}\text{ form}\right)$

$=e^{\lim\limits_{x\to 0}\frac{f'(3+x)+f'(2-x)}{x(-f'(2-x))+(1+f(2-x)-f(2))}}$

$=e^0=1.$

16. (1)

Given, $\lim\limits_{x\to 0}\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)^{\frac{100}{x}}$

Now by putting the value of limit we can see it is in form 1^{∞}

Now we know that,

$\lim\limits_{x\to 0}(f(x))^{g(x)}$ when is in form of 1^{∞}, we can write this as $e^{\lim\limits_{x\to 0}(f(x)-1)g(x)}$

Now using the above rule we get,

$= e^{\lim\limits_{x\to 0}\left[\left(\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right)-1\right]\times\frac{100}{x}}$

$= e^{\lim\limits_{x\to 0}\left[\frac{100}{x}\left[\frac{(x+2\cos x)^3+2(x+2\cos x)^2+3\sin(x+2\cos x)-\left((x+2)^3+2(x+2)^2+3\sin(x+2)\right)}{(x+2)^3+2(x+2)^2+3\sin(x+2)}\right]\right]}$

$= e^{\lim\limits_{x\to 0}\frac{100}{x}\left[\left(\frac{(x+2\cos x)^3-(x+2)^3+2(x+2\cos x)^2-2(x+2)^2+3\sin(x+2\cos x)-3\sin(x+2)}{8+8+3\sin 2}\right)\right]}$

Using L-hospital method we get,

$= e^{\frac{100}{16+3\sin 2}\lim\limits_{x\to 0}\frac{3(x+2\cos x)^2\times(1+2\sin x)-3(x+2)^2+4(x+2\cos x)\times(1-2\sin x)-4(x+2)+3\cos(x+2\cos x)\times(1-2\sin x)-3\cos(x+2)}{1}}$

$= e^{\frac{100}{16+3\sin 2}\left(\frac{12-3(4)+8\times 1-8+3\cos 2-3\cos 2}{1}\right)}$

$= e^{\frac{100}{16+3\sin 2}\times 0}$

$= e^{0} = 1$

17. (3)

Given,

$\lim\limits_{x\to 0}\frac{\alpha e^{x}+\beta e^{-x}+\gamma\sin x}{x\sin^2 x} = \frac{2}{3}$

$\Rightarrow \lim\limits_{x\to 0}\frac{\alpha e^{x}+\beta e^{-x}+\gamma\sin x}{x^3\frac{\sin^2 x}{x^2}} = \frac{2}{3}$

$\Rightarrow \lim\limits_{x\to 0}\frac{\alpha e^{x}+\beta e^{-x}+\gamma\sin x}{x^3} = \frac{2}{3}$

Now using expansion of given function in terms of x we get,

$= \lim\limits_{x\to 0}\frac{\alpha\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)+\beta\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)+\gamma\left(x-\frac{x^3}{3!}+\ldots\right)}{x^3}$

Now for limit to exist constant terms should be zero

$\Rightarrow a+\beta = 0\ldots.(1)$

Also coefficient of x should be zero

$\Rightarrow \alpha-\beta+\gamma = 0\ldots\ldots.(2)$

Also coefficient of x^2 should be zero

$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2} = 0\ldots\ldots.(3)$

So limit becomes, $\lim\limits_{x\to 0}\frac{x^3\left(\frac{\alpha}{3!}-\frac{\beta}{3!}-\frac{\gamma}{3!}\right)+x^4\left(\frac{\alpha}{3!}-\frac{\beta}{3!}-\frac{\gamma}{3!}\right)}{x^3} = \frac{2}{3}$

Now putting the value of limit we get,

$\frac{\alpha}{6}-\frac{\beta}{6}-\frac{\gamma}{6} = \frac{2}{3}\ldots\ldots.(4)$

Now on solving equation 1, 2, 3 & 4 we get,

$\Rightarrow \alpha = 1, \beta = -1, \gamma = -2$

Now putting these in all option we get, $\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2+3 = 0$

18. (3)

We have,

$$\lim_{x\to0}\left[\frac{\alpha xe^{x}-\beta\log_e(1+x)+\gamma x^2e^{-x}}{x\sin^2 x}\right]=10$$

$$\Rightarrow \lim_{x\to0}\left[\frac{\alpha x\left(1+x+\frac{x^2}{2!}+..\right)-\beta\left(x-\frac{x^2}{2}+\frac{x^3}{3}-...\right)+\gamma x^2\left(1-x+\frac{x^2}{2!}-..\right)}{x^3\left(\frac{\sin^2 x}{x^2}\right)}\right]=10$$

$$\Rightarrow \lim_{x\to0}\left[\frac{\alpha x\left(1+x+\frac{x^2}{2}\right)-\beta\left(x-\frac{x^2}{2}+\frac{x^3}{3}\right)+\gamma x^2(1-x)}{x^3}\right]=10$$

Neglecting higher powers.

$$\lim_{x\to0}\frac{x(\alpha-\beta)+x^2\left(\alpha+\frac{\beta}{2}+\gamma\right)+x^3\left(\frac{\alpha}{2}-\frac{\beta}{3}-\gamma\right)}{x^3}=10$$

For limit to exist

$\alpha-\beta=0$

$\alpha+\frac{\beta}{2}+\gamma=0$

$\frac{\alpha}{2}-\frac{\beta}{3}-\gamma=10\ \ldots$(i)

$\beta=\alpha, \gamma=-\frac{3\alpha}{2}$

Putting in (i), we get

$\frac{\alpha}{2}-\frac{\alpha}{3}+\frac{3\alpha}{2}=10$

$\Rightarrow \frac{\alpha}{6}+\frac{3\alpha}{2}=10$

$\Rightarrow \frac{\alpha+9\alpha}{6}=10$

$\Rightarrow \alpha=6$

Hence,

$\alpha=6, \beta=6, \gamma=-9$

$\Rightarrow \alpha+\beta+\gamma=3$

19. (2)

$$\left(2^{\frac{1}{2}}-2^{\frac{1}{3}}\right)^n<\left(2^{\frac{1}{2}}-2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}}-2^{\frac{1}{5}}\right)\left(2^{\frac{1}{2}}-2^{\frac{1}{7}}\right)$$

$$---\left(2^{\frac{1}{2}}-2^{\frac{1}{2n+1}}\right)<\left(2^{\frac{1}{2}}-2^{\frac{1}{2n+1}}\right)^n$$

$$\left(2^{\frac{1}{2}}-2^{\frac{1}{3}}\right)^n<L<\left(2^{\frac{1}{2}}-2^{\frac{1}{2n+1}}\right)^n$$

$$\lim_{n\to\infty}\left(2^{\frac{1}{2}}-2^{\frac{1}{3}}\right)^n=0 \text{ and } \lim_{n\to\infty}\left(2^{\frac{1}{2}}-2^{\frac{1}{2n+1}}\right)^n=0$$

$$\Rightarrow \lim_{n\to\infty}L=0$$

20. (1)

We know that $r\leq[r]<r+1$

$2r\leq[2r]<2r+1$

$3r\leq[3r]<3r+1$

$\vdots\ \vdots\ \vdots$

$nr\leq[nr]<nr+1$

On adding all the above inequalities, we get

$r+2r+\ldots+nr\leq[r]+[2r]+\ldots+[nr]<r+2r+\ldots nr+n$

Using $1+2+3+\ldots+n=\frac{n(n+1)}{2}$, $n\in N$,

$\frac{n(n+1)}{2}\cdot r\leq[r]+[2r]+\ldots+[nr]<\frac{n(n+1)}{2}\cdot r+n$

$$\Rightarrow \frac{\frac{n(n+1)}{2}\cdot r}{n^2}\leq\frac{[r]+[2r]+\ldots+[nr]}{n^2}<\frac{\frac{n(n+1)}{2}\cdot r+n}{n^2}$$

Now, $\lim_{n\to\infty}\frac{n(n+1)\cdot r}{2\cdot n^2}=\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)\cdot r}{2\cdot n^2}=\frac{r}{2}$

and $\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}\cdot r+n}{n^2}=\lim_{n\to\infty}\frac{n^2\left\{\left(1+\frac{1}{n}\right)\cdot r+\frac{1}{n}\right\}}{2\cdot n^2}=\frac{r}{2}$

So, by Sandwich Theorem, we can conclude that

$$\lim_{n\to\infty}\frac{[r]+[2r]+\ldots+[nr]}{n^2}=\frac{r}{2}.$$