ANSWER KEYS

1. (1)	**2.** (66)	**3.** (4)	**4.** (4)	**5.** (4)	**6.** (60)	**7.** (30)	**8.** (3)
9. (3)	**10.** (1)	**11.** (3)	**12.** (2)	**13.** (1)	**14.** (4)	**15.** (90)	**16.** (1)
17. (2)	**18.** (0.8)	**19.** (1)	**20.** (2)	**21.** (1)	**22.** (1)	**23.** (288)	**24.** (1)
25. (11)	**26.** (1)	**27.** (1)	**28.** (1)	**29.** (29)	**30.** (36)		

1. (1)

Let the four points be $A:\ (3,-4,2)$, $B:\ (1,2,-1)$, $C:\ (-2,-1,3)$ and $D:\ (5,-2\alpha,\ 4)$

$A,\ B,\ C,\ D$ are coplanar points, then

$\Rightarrow \begin{vmatrix} 1-3 & 2+4 & -1-2 \\ -2-3 & -1+4 & 3-2 \\ 5-3 & -2\alpha+4 & 4-2 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} -2 & 6 & -3 \\ -5 & 3 & 1 \\ 2 & -2\alpha+4 & 2 \end{vmatrix} = 0$

$\Rightarrow -3\alpha + 146 = 0$

$\Rightarrow \alpha = \frac{73}{17}$

2. (66)

$|\vec{a}| = \sqrt{11}, \quad |\vec{c}| = \sqrt{22}$

$|\vec{a}| = |\vec{b} \times \vec{c}| = |\vec{b}||\vec{c}| \sin\theta$

$\sqrt{11} = \sqrt{50}\sqrt{22} \sin\theta$

$\Rightarrow \sin\theta = \dfrac{1}{10}$

$|\vec{b} + \vec{c}|^2 = |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c}$

$= |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\theta$

$= 50 + 22 + 2 \times \sqrt{50} \times \sqrt{22} \times \dfrac{\sqrt{99}}{10}$

$= 72 + 66$

$|72 - |\vec{b} + \vec{c}|^2| = 66$

3. (4) $|\vec{a}| = 2, |\vec{b}| = 3$

$|(\vec{a} + 2\vec{b}) \times (2\vec{a} - 3\vec{b})|^2$

$| -3\vec{a} \times \vec{b} + 4\vec{b} \times \vec{a}|^2$

$| -3\vec{a} \times \vec{b} - 4\vec{a} \times \vec{b}|^2$

$| -7\vec{a} \times \vec{b}|^2$

$\left(-7|\vec{a}| \times |\vec{b}| \sin\left(\frac{\pi}{4}\right)\right)^2$

$49 \times 4 \times 9 \times \dfrac{1}{2} = 882$

4. (4)

Given,

Points $A(2,3,9),\ B(5,2,1),\ C(1,\lambda,8)$ & $D(\lambda,2,3)$ are coplanar,

Now we know that condition of coplanarity is $\left[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}\right] = 0$

$\Rightarrow \begin{vmatrix} 3 & -1 & -8 \\ -1 & \lambda-3 & -1 \\ \lambda-2 & -1 & -6 \end{vmatrix} = 0$

$\Rightarrow 3((-6)(\lambda-3)-1)+1(6+\lambda-2)-8(1-(\lambda-3)(\lambda-2)) = 0$

$\Rightarrow 3(-6\lambda+17)+4+\lambda-8+8(\lambda-3)(\lambda-2) = 0$

$\Rightarrow -18\lambda + 51 + 4 + \lambda - 8 + 8\lambda^2 - 40\lambda + 48 = 0$

$\Rightarrow 8\lambda^2 - 57\lambda + 95 = 0$

So product of roots will be $\frac{95}{8}$

5. (4)

Given, $\vec{a} = \alpha\hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \alpha\hat{k}, \alpha > 0$. If the projection of $\vec{a} \times \vec{b}$ on the vector $-\hat{i} + 2\hat{j} - 2\hat{k}$ is 30, then α is equal to

Now $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \alpha & 1 & -1 \\ 2 & 1 & -\alpha \end{vmatrix} = (1-\alpha)\hat{i} + (\alpha^2 - 2)\hat{j} + (\alpha - 2)\hat{k}$

Projection of $\vec{a} \times \vec{b}$ on $-\hat{i} + 2\hat{j} - 2\hat{k}$

$= \left| \frac{\left(\vec{a} \times \vec{b}\right) \cdot \left(-\hat{i} + 2\hat{j} - 2\hat{k}\right)}{\sqrt{1^2 + 2^2 + 2^2}} \right|$

$= \frac{\left((1-\alpha)\hat{i} + (\alpha^2 - 2)\hat{j} + (\alpha - 2)\hat{k}\right) \cdot \left(-\hat{i} + 2\hat{j} - 2\hat{k}\right)}{3}$

$= \frac{-1 + \alpha + 2\alpha^2 - 4 - 2\alpha + 4}{3}$

$= \frac{2\alpha^2 - \alpha - 1}{3}$

Now given length of projection is 30

So $\frac{2\alpha^2 - \alpha - 1}{3} = 30$

$\Rightarrow 2\alpha^2 - \alpha - 91 = 0$

$\Rightarrow \alpha = 7, -\frac{13}{2}$

6. (60)

We have, $\left(\vec{a} + 3\vec{b}\right) \perp \left(7\vec{a} - 5\vec{b}\right)$

Therefore, $\left(\vec{a} + 3\vec{b}\right) \cdot \left(7\vec{a} - 5\vec{b}\right) = 0$

$\Rightarrow 7\left|\vec{a}\right|^2 - 15\left|\vec{b}\right|^2 + 16\vec{a} \cdot \vec{b} = 0 \quad \ldots(1)$

and $\left(\vec{a} - 4\vec{b}\right) \cdot \left(7\vec{a} - 2\vec{b}\right) = 0$

$\Rightarrow 7\left|\vec{a}\right|^2 + 8\left|\vec{b}\right|^2 - 30\vec{a} \cdot \vec{b} = 0 \quad \ldots(2)$

From (1) and (2), we get

$2\left|\vec{a}\right|\cos\theta = \left|\vec{b}\right|$

$\therefore \cos\theta = \frac{\left|\vec{b}\right|}{2\left|\vec{a}\right|}$

$\Rightarrow \theta = 60^\circ$.

7. (30)

$\vec{b} \cdot \vec{c} = 10 \Rightarrow \left|\vec{b}\right|\left|\vec{c}\right|\cos\left(\frac{\pi}{3}\right) = 10 \Rightarrow 5.\left|\vec{c}\right|.\frac{1}{2} = 10$

$\Rightarrow \left|\vec{c}\right| = 4$

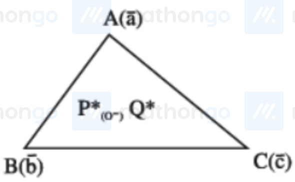
Also, $\vec{a}.\left(\vec{b} \times \vec{c}\right) = 0$

$\left|\vec{a} \times \left(\vec{b} \times \vec{c}\right)\right| = \left|\vec{a}\right|\left|\vec{b} \times \vec{c}\right|\sin\left(\frac{\pi}{2}\right)$

$\Rightarrow \sqrt{3} \times \left|\vec{b}\right|\left|\vec{c}\right|\sin\frac{\pi}{3} \times 1$

$\Rightarrow \sqrt{3} \times 5 \times 4 \times \frac{\sqrt{3}}{2} = 30$

8. (3)



$\overline{PA} + \overline{PB} + \overline{PC} = \bar{a} + \bar{b} + \bar{c}$

$\overline{PG} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$

$\Rightarrow \bar{a} + \bar{b} + \bar{c} = 3\overline{PG} = \overline{PQ}$

Ans. (4)

9. (3) $\vec{a} = \lambda(\vec{b} \times \vec{c})$

$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -2 \\ -1 & 4 & 3 \end{vmatrix} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\vec{d} = \lambda(2\hat{i} - \hat{j} + 2\hat{k})$

$\vec{a} \cdot \vec{d} = 18$

$\lambda = 2$

So $\vec{d} = 2(2\hat{i} - \hat{j} + 2\hat{k})$

$\vec{d} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & 4 \\ 2 & 3 & 4 \end{vmatrix} = -20\hat{i} - 8\hat{j} + 16\hat{k}$

$|\vec{d} \times \vec{a}|^2 = 720$

10. (1)

Given data as below:

$\vec{a} = 2\hat{i} - 7\hat{j} + 5\hat{k}$

$\vec{b} = \hat{i} + \hat{k}$

$\vec{c} = \hat{i} + 2\hat{j} - 3\hat{k}$

Given, $\vec{r} \times \vec{a} = \vec{c} \times \vec{a}$

$\Rightarrow \left(\vec{r} - \vec{c}\right) \times \vec{a} = 0$

It means $\left(\vec{r} - \vec{c}\right)$ is parallel to $\vec{a}$,

$\therefore \vec{r} = \vec{c} + \lambda\vec{a} \ldots\ldots \left(1\right)$

Given, $\vec{r} \cdot \vec{b} = 0$

From equation (1),

$\left(\vec{c} + \lambda\vec{a}\right) \cdot \vec{b} = 0$

$\Rightarrow \left(\vec{c} \cdot \vec{b}\right) + \lambda\left(\vec{b} \cdot \vec{a}\right) = 0$

$\Rightarrow -2 + \lambda\left(7\right) = 0 \Rightarrow \lambda = \frac{2}{7}$

On putting values in equation (1) we get,

$\therefore \vec{r} = \vec{c} + \frac{2}{7}\vec{a} = \frac{1}{7}\left(11\hat{i} - 11\hat{k}\right)$

$\left|\vec{r}\right| = \frac{11\sqrt{2}}{7}$

11. (3)

Given,

$\left|\vec{a}\right|=\sqrt{31},\ 4\left|\vec{b}\right|=\left|\vec{c}\right|=2,\ 2\left(\vec{a}\times\vec{b}\right)=3\left(\vec{c}\times\vec{a}\right)$ and angle between $\vec{b}$ & $\vec{c}$ is given as $\frac{2\pi}{3}$

Now solving, $3\left(\vec{c}\times\vec{a}\right)+2\left(\vec{b}\times\vec{a}\right)=0$

$\Rightarrow\left(3\vec{c}\times2\vec{b}\right)\times\vec{a}=0$

Means $\left(3\vec{c}\times2\vec{b}\right)$ & $\vec{a}$ are parallel vector,

So, let $3\vec{c}\times2\vec{b}=\lambda\vec{a}$

Now squaring both sides we get,

$9\left|\vec{c}\right|^2+4\left|\vec{b}\right|^2+12\left(\vec{b}\cdot\vec{c}\right)=\lambda^2\left|\vec{a}\right|^2$

$\Rightarrow 36+1+12\times\frac{1}{2}\times2\left(\cos\left(2\frac{\pi}{3}\right)\right)=\lambda^2(31)$

$\Rightarrow\lambda^2=1$

$\Rightarrow\lambda=\pm1$

Now putting the value of λ in $\left(3\vec{c}\times2\vec{b}\right)=\lambda\vec{a}$ we get,

$3\vec{c}+2\vec{b}=\pm\vec{a}\qquad\ldots(1)$

Now taking dot product with $\vec{b}$ in above equation we get,

$3\left(\vec{b}\cdot\vec{c}\right)+2\left(\vec{b}\cdot\vec{b}\right)=\pm\vec{a}\cdot\vec{b}$

$\Rightarrow\vec{a}\cdot\vec{b}=\pm\left(-\frac{3}{2}+\frac{1}{2}\right)=\pm(-1)$

$\Rightarrow\left(\vec{a}\cdot\vec{b}\right)^2=1$

Again taking $3\left(\vec{c}\times\vec{a}\right)=2\left(\vec{a}\times\vec{b}\right)$ and sqauring both side,

$\Rightarrow\left(\vec{c}\times\vec{a}\right)^2=\frac{4}{9}\left(\vec{a}\times\vec{b}\right)^2$

$\Rightarrow\left(\vec{c}\times\vec{a}\right)^2=\frac{4}{9}\left[\left|\vec{a}\right|^2\left|\vec{b}\right|^2-\left(\vec{a}\cdot\vec{b}\right)^2\right]$

$\Rightarrow\left(\vec{c}\times\vec{a}\right)^2=\frac{4}{9}\left[\frac{31}{4}-(1)\right]$

$\Rightarrow\left(\vec{c}\times\vec{a}\right)^2=\frac{4}{9}\times\frac{27}{4}=3$

Hence, the value of $\left(\frac{\vec{a}\times\vec{c}}{\vec{a}\cdot\vec{b}}\right)^2=\frac{3}{1}=3.$

12. (2)

Let the position vector of P is $\vec{O}$, Q is $\vec{q}$ and R is $\vec{r}$.

Now, position vector of A is

$\frac{2\vec{q}+\vec{r}}{3}$

Position vector of B is $\frac{2\vec{r}}{3}$.

Position vector of C is $\frac{\vec{q}}{3}$.

$\overrightarrow{AB}=\frac{\vec{r}-2\vec{q}}{3}$

$\overrightarrow{AC}=\frac{-\vec{r}-\vec{q}}{3}$

So,

$\overrightarrow{AB}\times\overrightarrow{AC}=\frac{1}{3}\left[\left(\vec{r}-2\vec{q}\right)\times\left(-\vec{r}-\vec{q}\right)\right]$

$\Rightarrow\overrightarrow{AB}\times\overrightarrow{AC}=\frac{1}{3}\left[-\left(\vec{r}\times\vec{q}\right)+2\left(\vec{q}\times\vec{r}\right)\right]$

$\Rightarrow\overrightarrow{AB}\times\overrightarrow{AC}=\left(\vec{q}\times\vec{r}\right)$

Area of ΔPQR is $=\frac{1}{2}\left|\vec{q}\times\vec{r}\right|$

Area of ΔABC is $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

Area of $\Delta ABC=\frac{1}{2}\left|\vec{q}\times\vec{r}\right|$

Therefore,

$\frac{\text{Area }(\Delta PQR)}{\text{Area}(\Delta ABC)}=1$

13. **(1)** Given $\vec{a}=\alpha\hat{i}+3\hat{j}-\hat{k}$, $\vec{b}=3\hat{i}-\beta\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+2\hat{j}-2\hat{k}$ where $\alpha,\beta\in R$ be three vectors.

Projection of $\vec{a}$ on $\vec{c}$ is $\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}=\frac{10}{3}$ (given)

$\Rightarrow\frac{\alpha+6+2}{\sqrt{1+4+4}}=\frac{10}{3}\Rightarrow\alpha=2$

and $\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\3&-\beta&4\\1&2&-2\end{vmatrix}=-6\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow 2\beta-8=-6\Rightarrow\beta=1$

Hence $\alpha+\beta=3$

14. **(4)**

$\vec{c}=2\sqrt{2}\left(\vec{a}\times\vec{b}\right)-2\vec{b}$

So, $\vec{b}\cdot\vec{c}=2\sqrt{2}\vec{b}\cdot\left(\vec{a}\times\vec{b}\right)-2\vec{b}\cdot\vec{b}=-2\left|\vec{b}\right|^2\quad\ldots(i)$

Since $\left|\vec{a}\cdot\vec{b}\right|=\left|\vec{a}\times\vec{b}\right|$, so angle between $\vec{a}$ & $\vec{b}$ is $\frac{\pi}{4}$

Now, area of triangle is $2\sqrt{2}$

i.e. $\frac{1}{2}\left|\vec{a}\times\vec{b}\right|=\frac{1}{2}\left|\vec{a}\right|\left|\vec{b}\right|\sin\frac{\pi}{4}=2\sqrt{2}\Rightarrow\left|\vec{b}\right|=8$

From (i), $\vec{b}\cdot\vec{c}=-128$

Now, $\left|\vec{c}\right|^2=\left|2\sqrt{2}\left(\vec{a}\times\vec{b}\right)-2\vec{b}\right|^2$

$=8\left|\vec{a}\times\vec{b}\right|^2+4\left|\vec{b}\right|^2-8\sqrt{2}\left(\vec{a}\times\vec{b}\right)\cdot\vec{b}=8\times32+4\times64=512$

Hence, angle between $\vec{b}$ & $\vec{c}$ will be $\cos^{-1}\left(\frac{\vec{b}\cdot\vec{c}}{\left|\vec{b}\right|\left|\vec{c}\right|}\right)=\cos^{-1}\left(\frac{-128}{8\times\sqrt{512}}\right)=\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)=\frac{3\pi}{4}$

15. **(90)**

$\vec{a}\cdot\vec{b}=0$

$1+15+\alpha\beta=0$

$\alpha\beta=-16\quad\ldots(1)$

Also,

$\left|\vec{b}\times\vec{c}\right|^2=75$

$(10+\beta^2)\cdot 14-(5-3\beta)^2=75$

$5\beta^2+30\beta+40=0$

$\beta=-4,\ -2$

$\alpha=4,8$

$\left|\vec{a}\right|^2_{max}=(26+\alpha^2)_{max}=90$

16. **(1)**

$\vec{r}\times\vec{a}-\vec{r}\times\vec{b}=0$

$\Rightarrow\vec{r}\times\left(\vec{a}-\vec{b}\right)=0$

$\Rightarrow\vec{r}=\lambda\left(\vec{a}-\vec{b}\right)$

$\Rightarrow\vec{r}=\lambda\left(-5\hat{i}-4\hat{j}+10\hat{k}\right)$

Also $\vec{r}\cdot\left(\hat{i}+2\hat{j}+\hat{k}\right)=-3$

$\Rightarrow\lambda(-5-8+10)=-3$

$\lambda=1$

Now $\vec{r}=-5\hat{i}-4\hat{j}+10\hat{k}$

$=\vec{r}\cdot\left(2\hat{i}-3\hat{j}+\hat{k}\right)$

$=-10+12+10=12$

17. (2)

$\vec{r}\times\vec{a}=\vec{b}\times\vec{r}\Rightarrow\vec{r}\times\left(\vec{a}+\vec{b}\right)=0$

$\vec{r}=\vec{\lambda}\left(\vec{a}+\vec{b}\right)\Rightarrow\vec{r}=\vec{\lambda}\left(\hat{i}+2\hat{j}-3\hat{k}+2\hat{i}-3\hat{j}+5\hat{k}\right)$

$\vec{r}=\vec{\lambda}\left(3\hat{i}-\hat{j}+2\hat{k}\right)\ \ldots(1)$

$\vec{r}\cdot\left(\alpha\hat{i}+2\hat{j}+\hat{k}\right)=3$

Put $\vec{r}$ from (1)

$\alpha\lambda=1\ \ldots(2)$

$\vec{r}\cdot\left(2\hat{i}+5\hat{j}-\alpha\hat{k}\right)=-1$

Put $\vec{r}$ from (1)

$2\lambda\alpha-\lambda=1\ \ldots(3)$

Solve (2) and (3)

$\alpha=1,\lambda=1$

$\Rightarrow\vec{r}=3\hat{i}-\hat{j}+2\hat{k}$

$\left|\vec{r}\right|^2=14\ \&\ \alpha=1$

$\alpha+\left|\vec{r}\right|^2=15$

18. (0.8)

Position vector of P is $\overrightarrow{OP}=\frac{\vec{a}+\lambda\vec{b}}{\lambda+1}\ \because\ \overrightarrow{OB}\cdot\overrightarrow{OP}-3\left|\overrightarrow{OA}\times\overrightarrow{OP}\right|^2=6$

$\Rightarrow\ \vec{b}\cdot\left(\frac{\vec{a}+\lambda\vec{b}}{\lambda+1}\right)-3\left|\vec{a}\times\left(\frac{\vec{a}+\lambda\vec{b}}{\lambda+1}\right)\right|^2=6$

$\Rightarrow\ \frac{\vec{a}\cdot\vec{b}+\lambda\left|\vec{b}\right|^2}{\lambda+1}-\frac{3\lambda^2}{(\lambda+1)^2}\left|\vec{a}\times\vec{b}\right|^2=6$

$\Rightarrow\ \frac{6+\lambda.14}{\lambda+1}-\frac{3\lambda^2}{(\lambda+1)^2}\cdot6=6$

$\Rightarrow\ \frac{18\lambda^2}{(\lambda+1)^2}+6=6+\frac{8\lambda}{\lambda+1}$

$\Rightarrow\ 18\left(\frac{\lambda}{\lambda+1}\right)^2-\frac{8\lambda}{\lambda+1}=0\ \left(\frac{\lambda}{\lambda+1}\neq0\right)$

$\Rightarrow\ 10\lambda=8\Rightarrow\ \lambda=0.8$

19. (1)

If the vectors are co-planar,

$\begin{vmatrix}a+b+2 & a+2b+c & -b-c\\ b+1 & 2b & -b\\ b+2 & 2b & 1-b\end{vmatrix}=0$

Now $R_1\leftrightarrow R_1-R_2,\ R_3\leftrightarrow R_3-R_2$

So $\begin{vmatrix}a+1 & a+c & -c\\ b+1 & 2b & -b\\ 1 & 0 & 1\end{vmatrix}=0$

$\Rightarrow(a+1)2b-(a+c)(2b+1)-c(-2b)=0$

$\Rightarrow2ab+2b-2ab-a-2bc-c+2bc=0$

$\Rightarrow2b-a-c=0\Rightarrow2b=a+c$

20. (2)

Given,

$\vec{a},\vec{b}$ and $\vec{c}$ be three non-zero non-coplanar vectors,

Also the position vectors of four points A, B, C and D be $\vec{a}-\vec{b}+\vec{c},\ \lambda\vec{a}-3\vec{b}+4\vec{c},-\vec{a}+2\vec{b}-3\vec{c}$ and $2\vec{a}-4\vec{b}+6\vec{c}$ respectively,

So, $\overrightarrow{AB}=\left(\lambda-1\right)\vec{a}-2\vec{b}+3\vec{c}$

$\overrightarrow{AC}=-2\vec{a}+3\vec{b}-4\vec{c}$

$\overrightarrow{AD}=\vec{a}-3\vec{b}+5\vec{c}$

Now using the condition of coplanar we get,

$\begin{vmatrix}\lambda-1 & -2 & 3\\ -2 & 3 & -4\\ 1 & -3 & 5\end{vmatrix}=0$

$\Rightarrow(\lambda-1)(15-12)+2(-10+4)+3(6-3)=0$

$\Rightarrow(\lambda-1)=1\Rightarrow\lambda=2$

21. (1)

Let us assume that vectors are collinear so, $\overrightarrow{AB} \parallel \overrightarrow{AC}$ if $\frac{1}{2} = \frac{\alpha-4}{-6} = \frac{1}{2} \Rightarrow \alpha = 1$

Now, for $\vec{a}$, $\vec{b}$, $\vec{c}$ to be non-collinear smallest positive integer will be $\alpha = 2$

So, mid-point of $BC = M\left(\frac{5}{2}, 0, \frac{9}{2}\right)$

Now, $AM = \sqrt{\frac{9}{4} + 16 + \frac{9}{4}} = \frac{\sqrt{82}}{2}$

22. (1)

$(\vec{a} + \vec{b} + \vec{c}) \times \vec{c} = 0$

$(\vec{a} + \vec{b}) \times \vec{c} = 0$

$\vec{c} = \alpha(\vec{a} + \vec{b}) = \alpha(\lambda + 3)\hat{i} + \alpha\hat{k}$

$\vec{b} \cdot \vec{c} = -20 \Rightarrow 3\alpha(\lambda + 3) + 2\alpha = -20$

$\vec{a}.\vec{c} = -17 \Rightarrow \alpha\lambda(\lambda + 3) - \alpha = -17$

$\Rightarrow \alpha(3\lambda + 9 + 2) = -20$

$\alpha\left(\lambda^2 + 3\lambda - 1\right) = -17$

$17(3\lambda + 11) = 20\left(\lambda^2 + 3\lambda - 1\right)$

$20\lambda^2 + 9\lambda - 207 = 0$

$\lambda = 3 \quad (\lambda \in Z)$

$\Rightarrow \alpha = -1$

$\Rightarrow \vec{c} = -(6\hat{i} + \hat{k})$

$\vec{V} = \vec{c} \times (3\hat{i} + \hat{j} + \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & 0 & -1 \\ 3 & 1 & 1 \end{vmatrix} = \hat{i} + 3\hat{j} - 6\hat{k}$$

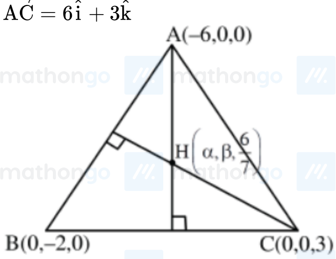
$|\vec{V}|^2 = (-1)^2 + 3^2 + 6^2 = 46$

23. (288)

$\mathrm{A}(-6,0,0) \quad \mathrm{B}(0,-2,0)\mathrm{C} = (0,0,3)$

$\overrightarrow{\mathrm{AB}} = 6\hat{\mathrm{i}} - 2\hat{\mathrm{j}}, \quad \overrightarrow{\mathrm{BC}} = 2\hat{\mathrm{j}} + 3\hat{\mathrm{k}}$

$\overrightarrow{\mathrm{AC}} = 6\hat{\mathrm{i}} + 3\hat{\mathrm{k}}$



$\overrightarrow{\mathrm{AH}} \cdot \overrightarrow{\mathrm{BC}} = 0$

$\left(\alpha + 6, \beta, \frac{6}{7}\right) \cdot (0, 2, 3) = 0$

$\beta = \frac{-9}{7}$

$\overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{AB}} = 0$

$\left(\alpha, \beta, \frac{-15}{7}\right) \cdot (6, -2, 0) = 0$

$6\alpha - 2\beta = 0$

$\alpha = \frac{-3}{7}$

$98(\alpha + \beta)^2 = (98)\frac{(144)}{49} = 288$

24. (1)

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=(2\hat{i}+4\hat{j}-2\hat{k})-(-2\hat{i}+\hat{j}-3\hat{k})$

$=4\hat{i}+3\hat{j}+\hat{k}$

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-2\hat{i}+\hat{j}+2\hat{k}$

$\overrightarrow{AB}\times\overrightarrow{AC}=5\hat{i}-10\hat{j}+10\hat{k}$

$\overrightarrow{OP}=-\hat{i}-2\hat{j}+3\hat{k}$

Projection

$$=\frac{(\overrightarrow{OP})\cdot(\overrightarrow{AB}\times\overrightarrow{AC})}{|\overrightarrow{AB}\times\overrightarrow{AC}|}=3$$

25. (11)

Given,

$\vec{a}=6\hat{i}+9\hat{j}+12\hat{k}$, $\vec{b}=\alpha\hat{i}+11\hat{j}-2\hat{k}$ and $\vec{c}$ be vectors such that $\vec{a}\times\vec{c}=\vec{a}\times\vec{b}$,

$\vec{a}\times\vec{c}-\vec{a}\times\vec{b}=0$

$\Rightarrow\vec{a}\times\left(\vec{c}-\vec{b}\right)=0$

So, $\vec{a}$ & $\left(\vec{c}-\vec{b}\right)$ are parallel vectors,

Hence, $\lambda\vec{a}=\vec{c}-\vec{b}$

$\Rightarrow\vec{c}=\vec{b}+\lambda\vec{a}$

$\Rightarrow\vec{a}\cdot\vec{c}=\vec{a}\cdot\vec{b}+\lambda\left|\vec{a}\right|^2$

$\Rightarrow -12=(6\alpha+75)+\lambda(261)$

$\Rightarrow 2\alpha+87\lambda=-29\ldots.(i)$

Now again using $\vec{c}=\vec{b}+\lambda\vec{a}$ we get,

$\vec{c}=\hat{i}(\alpha+6\lambda)+\hat{j}\left(11+9\lambda\right)+\hat{k}\left(-2+12\lambda\right)$

Also given $\vec{c}\cdot\left(\hat{i}-2\hat{j}+\hat{k}\right)=5$

$\Rightarrow(\alpha+6\lambda)-2(11+9\lambda)+(-2+12\lambda)=5$

$\Rightarrow\alpha=29$

So, $2\alpha+87\lambda=-29$

$\Rightarrow\lambda=-1$

Hence, $\vec{c}=23\hat{i}+2\hat{j}-14\hat{k}$

So, the value of $\vec{c}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=23+2-14=11$

26. (1)

Given,

$\vec{r} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$

$\Rightarrow \vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$

$\Rightarrow \left(\vec{r} - \vec{c}\right) \times \vec{b} = 0$

$\Rightarrow \left(\vec{r} - \vec{c}\right) \| \vec{b}$

Therefore, $\vec{r} - \vec{c} = \lambda \vec{b}$

$\Rightarrow \vec{r} = \vec{c} + \lambda \vec{b}$

Also,

$\vec{r} \cdot \vec{a} = 0$ (given)

$\Rightarrow \left(\vec{c} + \lambda \vec{b}\right) \cdot \vec{a} = 0$

$\Rightarrow \vec{c} \cdot \vec{a} + \lambda \vec{b} \cdot \vec{a} = 0$

$\Rightarrow \lambda = \frac{-\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}}$

Now,

$\vec{r} \cdot \vec{c} = \left(\vec{c} + \lambda \vec{b}\right) \cdot \vec{c}$

$= \left(\vec{c} - \frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}} \vec{b}\right) \cdot \vec{c}$

$= \left|\vec{c}\right|^2 - \left(\frac{\vec{c} \cdot \vec{a}}{\vec{b} \cdot \vec{a}}\right)\left(\vec{b} \cdot \vec{c}\right)$

$= 74 - \left(\frac{15}{3}\right) \times 8$

$= 74 - 40 = 34$

27. (1)

Given:

$\vec{v} \times \vec{w} = \vec{u} + \lambda \vec{v} \quad \ldots(1)$

Taking dot product with $\vec{w}$, we get

$\vec{w} \cdot \left(\vec{v} \times \vec{w}\right) = \vec{w} \cdot \vec{u} + \lambda\left(\vec{w} \cdot \vec{v}\right)$

$\Rightarrow \vec{u} \cdot \vec{w} + \lambda\left(\vec{w} \cdot \vec{v}\right) = 0$

$\Rightarrow \vec{u} \cdot \vec{w} = -2\lambda \quad \ldots(2)$

Taking dot product with $\vec{v}$ in (1), we get

$\vec{v} \cdot \left(\vec{v} \times \vec{w}\right) = \vec{v} \cdot \vec{u} + \lambda\left(\vec{v} \cdot \vec{v}\right)$

$\Rightarrow (2 - 1 + 2) + \lambda(6 + 1 + 1) = 0$

$\Rightarrow \lambda = -\frac{1}{2}$

So, by (2), we get

$\vec{u} \cdot \vec{w} = -2\lambda = 1$

28. (1)

Given,

$\vec{a}=\hat{i}-\hat{j}+2\hat{k}$, $\vec{a}\times\vec{b}=2\hat{i}-\hat{k}$ and $\vec{a}\cdot\vec{b}=3$

Now using the formula $\left|\vec{a}\times\vec{b}\right|^2+\left|\vec{a}\cdot\vec{b}\right|^2=\left(\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\sin^2\theta+\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2\cos^2\theta\right)$

$\Rightarrow\left|\vec{a}\times\vec{b}\right|^2+\left|\vec{a}\cdot\vec{b}\right|^2=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2(\sin^2\theta+\cos^2\theta)=\left|\vec{a}\right|^2\cdot\left|\vec{b}\right|^2$

We get,

$\Rightarrow\left|2\hat{i}-\hat{k}\right|^2+|3|^2=\left|\hat{i}-\hat{j}+2\hat{k}\right|^2\left|\vec{b}\right|^2$

$\Rightarrow\left|\vec{b}\right|^2=\frac{5+9}{6}=\frac{14}{6}=\frac{7}{3}$

Now finding $\left|\vec{a}-\vec{b}\right|=\sqrt{\left|\vec{a}\right|^2+\left|\vec{b}\right|^2-2\left|\vec{a}\right|\left|\vec{b}\right|}=\sqrt{6+\frac{7}{3}-6}=\sqrt{\frac{7}{3}}$

Now projection of $\vec{b}$ on $\vec{a}-\vec{b}=\frac{\vec{b}\cdot\left(\vec{a}-\vec{b}\right)}{\left|\vec{a}-\vec{b}\right|}$

$=\frac{\vec{b}\cdot\vec{a}-\left|\vec{b}\right|^2}{\left|\vec{a}-\vec{b}\right|}=\frac{3-\frac{7}{3}}{\sqrt{\frac{7}{3}}}$

$=\frac{2}{\sqrt{21}}$

29. (29)

Given, $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=\hat{i}+\hat{j}+\hat{k}$

And $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{0}$,

Let $\vec{c}=x\hat{i}+y\hat{j}+z\hat{k}$

So, $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{0}\Rightarrow\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=\vec{0}$

$\Rightarrow\vec{a}\times\vec{b}=-\vec{a}\times\vec{c}\Rightarrow\vec{a}\times\vec{b}=\vec{c}\times\vec{a}$

$\Rightarrow\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&-2&3\\1&1&1\end{vmatrix}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\1&-2&3\end{vmatrix}$

$\Rightarrow-5\hat{i}+2\hat{j}+3\hat{k}=\hat{i}(3y+2z)+\hat{j}(z-3x)+\hat{k}(-2x-y)$

On comparing we get,

$3y+2z=-5$ .....(i)

$z-3x=2$ ....(ii)

$-2x-y=3$ ....(iii)

On solving equation (i), (ii) and (iii) we get,

$x=\frac{-1}{6},\ y=\frac{-8}{3},\ z=\frac{3}{2}$

So, $\vec{c}=\frac{-1}{6}\hat{i}+\frac{-8}{3}\hat{j}+\frac{3}{2}\hat{k}$

So, $\vec{c}\cdot\vec{a}=\left(\frac{-1}{6}\hat{i}+\frac{-8}{3}\hat{j}+\frac{3}{2}\hat{k}\right)\cdot\left(\hat{i}-2\hat{j}+3\hat{k}\right)=\frac{29}{3}$

So, $3\vec{c}\cdot\vec{a}=29$

Note: This question appeared in JEE Main 2022, 29th june shift 2. The question was incorrect, so it is modified to make it correct.

30. (36)

Taking cross product of $\vec{c}$ both side we get,

$\vec{a} \times \vec{b} = 4\vec{c} \Rightarrow \left(\vec{a} \times \vec{b}\right) \times \vec{c} = 0$

$\Rightarrow \left(\vec{a} \cdot \vec{c}\right). \vec{b} - \left(\vec{b} \cdot \vec{c}\right). \vec{a} = 0$

$\Rightarrow \left(\vec{a} \cdot \vec{c}\right). \vec{b} = \left(\vec{b} \cdot \vec{c}\right). \vec{a}$

Since vectors are non-coplanar so,

$\Rightarrow \left(\vec{a} \cdot \vec{c}\right) = \left(\vec{b} \cdot \vec{c}\right) = 0 \ldots\ldots(1)$

Similarly for $\vec{b} \times \vec{c} = 9\vec{a} \Rightarrow \vec{a} \cdot \vec{b} = 0 = \vec{a} \cdot \vec{c} \ldots\ldots(2)$

So, from equation (1) & (2) we can say that $\vec{a}, \vec{b}, \vec{c}$ are mutually $\perp$ set of vectors.

Now taking modulus both side of $\vec{a} \times \vec{b} = 4\vec{c},\ \vec{b} \times \vec{c} = 9\vec{a}$ & $\vec{c} \times \vec{a} = \alpha\vec{b}$ we get,

$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right| = 4\left|\vec{c}\right|, \left|\vec{b}\right|\left|\vec{c}\right| = 9\left|\vec{a}\right|$ & $\left|\vec{c}\right|\left|\vec{a}\right| = \alpha\left|\vec{b}\right|$

$\Rightarrow \frac{\left|\vec{a}\right|}{\left|\vec{c}\right|} = \frac{4}{9}\frac{\left|\vec{c}\right|}{\left|\vec{a}\right|}$

$\Rightarrow \frac{\left|\vec{c}\right|}{\left|\vec{a}\right|} = \frac{3}{2}$

$\therefore$ If $|a| = \lambda, |c| = \frac{3\lambda}{2}$ & $|b| = 6$

Now $\left|\vec{a}\right| + \left|\vec{b}\right| + \left|\vec{c}\right| = 36$

$\Rightarrow \lambda + \frac{3}{2}\lambda + 6 = 36$

$\Rightarrow \lambda = 12$

Now given $\left|\vec{c}\right|\left|\vec{a}\right| = \alpha\left|\vec{b}\right|$

$\Rightarrow \alpha = \frac{\left|\vec{c}\right|\left|\vec{a}\right|}{\left|\vec{b}\right|} = \frac{3\times12}{2} \times \frac{12}{6}$

$\Rightarrow \alpha = 36$

Note this question was bonus in jee main 27th july 2022 shift 2, so we have some modification to the originial question.