ANSWER KEYS

1. (3)	**2.** (66)	**3.** (1)	**4.** (1)	**5.** (3)	**6.** (1)	**7.** (1)	**8.** (1)
9. (16)	**10.** (1)	**11.** (2)	**12.** (2)	**13.** (2)	**14.** (98)	**15.** (9)	**16.** (2)
17. (1)	**18.** (2)	**19.** (272)	**20.** (2)	**21.** (25)	**22.** (3)	**23.** (3)	**24.** (2)
25. (25)	**26.** (13)	**27.** (2)	**28.** (2)	**29.** (9)	**30.** (1)		

1. (3)

Given $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=15$

For equation $3x^2+\lambda x-1=0$

$\alpha+\beta=\frac{-\lambda}{3}, \alpha\beta=\frac{-1}{3} \Rightarrow \alpha^2\beta^2=\frac{1}{9}$

then $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{\lambda^2}{9}-2\left(\frac{-1}{3}\right)=\frac{\lambda^2}{9}+\frac{2}{3}$

$\Rightarrow \alpha^2+\beta^2=\frac{\lambda^2+6}{9}$

Now $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=15 \Rightarrow \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=15$

$\Rightarrow \frac{\lambda^2+6}{9}\times\frac{1}{\alpha^2\beta^2}=15 \Rightarrow \frac{\lambda^2+6}{9}\times\frac{1}{\frac{1}{9}}=15$

$\Rightarrow \lambda^2+6=15 \Rightarrow \lambda=\pm 3$

Now $6\left(\alpha^3+\beta^3\right)^2=6\left((\alpha+\beta)\left(\alpha^2-\alpha\beta+\beta^2\right)\right)^2$

$=6\times 1\times\left(\frac{15}{9}+\frac{1}{3}\right)^2$

$=6\times 1\times(2)^2=6\times 4=24$

2. (66)

We have,

$\frac{2}{x-1}-\frac{1}{x-2}=\frac{2}{k}$

$\Rightarrow \frac{2x-4-x+1}{(x-1)(x-2)}=\frac{2}{k}$

$\Rightarrow \frac{x-3}{x^2-3x+2}=\frac{2}{k}$

$\Rightarrow kx-3k=2x^2-6x+4$

$\Rightarrow 2x^2-(6+k)x+3k+4=0$

For no real roots $D<0$

$\Rightarrow (6+k)^2-8(3k+4)<0$

$\Rightarrow k^2+12k+36-24k-32<0$

$\Rightarrow (k-6)^2-32<0$

$\Rightarrow |k-6|<\sqrt{32}$

$\Rightarrow 6-\sqrt{32}<k<6+\sqrt{32}$

Integral value of $k=1,2,3,4, 5,6,7,8,9,10,11$

Sum$=\frac{11\times 12}{2}=66$

3. (1)

$\because \alpha, \beta \in \mathrm{R} \Rightarrow$ other root is $1+2i$

$\alpha=-($ sum of roots $)=-(1-2\mathrm{i}+1+2\mathrm{i})=-2$

$\beta=$product of roots $=(1-2i)(1+2i)=5$

$\therefore \alpha-\beta=-7$

option (1)

4. (1)

Let $x = 3 + \frac{1}{4+\frac{1}{3+\frac{1}{4+\frac{1}{3+\ldots\infty}}}}$

So, $x = 3 + \frac{1}{4+\frac{1}{x}} = 3 + \frac{1}{\frac{4x+1}{x}}$

$\Rightarrow (x-3) = \frac{x}{(4x+1)}$

$\Rightarrow (4x+1)(x-3) = x$

$\Rightarrow 4x^2 - 12x + x - 3 = x$

$\Rightarrow 4x^2 - 12x - 3 = 0$

$x = \frac{12 \pm \sqrt{(12)^2 + 12\times 4}}{2\times 4} = \frac{12\pm\sqrt{12(16)}}{8}$

$= \frac{12\pm 4\times 2\sqrt{3}}{8} = \frac{3\pm 2\sqrt{3}}{2}$

$x = \frac{3}{2} \pm \sqrt{3} = 1.5 \pm \sqrt{3}$.

But only positive value is accepted

So, $x = 1.5 + \sqrt{3}$

5. (3)

In given equation $x^2 + px + q = 0,\ p,\ q \in Q$.

And, we know that the irrational roots of a quadratic equation exist in conjugate pair, if the coefficients are rational.

Thus, if one root of the equation $x^2 + px + q = 0$ is $2 - \sqrt{3}$, then, the other root will be $2 + \sqrt{3}$.

We know that the sum and product of the roots of a quadratic equation $ax^2 + bx + c = 0$ are respectively $-\frac{b}{a}$ and $\frac{c}{a}$.

Therefore, the sum of roots $2 + \sqrt{3} + 2 - \sqrt{3} = -p$

$\Rightarrow p = -4$

And, the product of roots $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = q$

$\Rightarrow q = 2^2 - \left(\sqrt{3}\right)^2 = 1$

Thus, we have $p^2 - 4q - 12 = (-4)^2 - 4\times 1 - 12$

$= 16 - 16 = 0$.

Thus, the answer is, $p^2 - 4q - 12 = 0$.

6. (1)

Given, $x^4 + x^2 + 1 = 0$

Using the formula $a^4 + a^2 + 1 = \left(a^2 + a + 1\right)\left(a^2 - a + 1\right)$

We get, $\left(x^2 + x + 1\right)\left(x^2 - x + 1\right) = 0$

$\Rightarrow x = \pm\omega,\ \pm\omega^2$ where $\omega = 1^{1/3}$ and imaginary.

So $\alpha^{1011} + \alpha^{2022} - \alpha^{3033} = 1 + 1 - 1 = 1$

7. (1)

$x^2+\sqrt{6}x+3=0$

$$x=\frac{-\sqrt{6}\pm\sqrt{6-12}}{2}=\frac{-\sqrt{6}\pm\sqrt{6}i}{2}=\frac{\sqrt{3}\sqrt{2}(-1\pm i)}{\sqrt{2}\sqrt{2}}$$

$$=\sqrt{3}\left(\frac{-1}{2}\pm\frac{i}{\sqrt{2}}\right)$$

$$=\sqrt{3}\left(\cos\frac{3\pi}{4}\pm\left(\sin\frac{3\pi}{4}\right)i\right)$$

$$=\sqrt{3}e^{\pm i\left(\frac{3\pi}{4}\right)}$$

$\alpha=\sqrt{3}e^{i\left(\frac{3\pi}{4}\right)}$ and $\beta=\sqrt{3}e^{i\left(\frac{-3\pi}{4}\right)}$

Now, $(\alpha)^{23}+(\beta)^{23}=(\sqrt{3})^{23}\left[e^{i\left(\frac{69\pi}{4}\right)}+e^{-i\left(\frac{69\pi}{4}\right)}\right]$

$$=(\sqrt{3})^{23}\left[2\cos\left(\frac{69\pi}{4}\right)\right]$$

Similarly $\rightarrow(\alpha)^{14}+(\beta)^{14}=(\sqrt{3})^{14}\left[2\cos\left(\frac{42\pi}{4}\right)\right]$

$\rightarrow(\alpha)^{15}+(\beta)^{15}=(\sqrt{3})^{15}\left[2\cos\left(\frac{45\pi}{4}\right)\right]$

$\rightarrow(\alpha)^{10}+(\beta)^{10}=(\sqrt{3})^{10}\left[2\cos\left(\frac{30\pi}{4}\right)\right]$

Now, $\dfrac{\alpha^{23}+\beta^{23}+\alpha^{14}+\beta^{14}}{\alpha^{15}+\beta^{15}+\alpha^{10}+\beta^{10}}$

$$=\frac{(\sqrt{3})^{23}\left[2\cos\left(16\pi+\frac{5\pi}{4}\right)\right]+(\sqrt{3})^{14}\left[2\cos\left(10\pi+\frac{\pi}{2}\right)\right]}{(\sqrt{3})^{15}\left[2\cos\left(10\pi+\frac{5\pi}{4}\right)\right]+(\sqrt{3})^{10}\left[2\cos\left(16\pi+\frac{3\pi}{2}\right)\right]}$$

$$=\frac{(\sqrt{3})^{23}2\cos\left(\frac{5\pi}{4}\right)}{(\sqrt{3})^{15}2\cos\left(\frac{5\pi}{4}\right)}=(\sqrt{3})^{8}=81$$

8. (1)

For given quadratic $375x^2-25x-2=0$, its roots are,

$\alpha,\beta=\frac{25\pm\sqrt{25^2+2\times4\times375}}{2\times375}\Rightarrow|\alpha|<1,|\beta|<1$

Also, $\alpha+\beta=\frac{25}{375}$, $\alpha\beta=\frac{-2}{375}$

Now $\lim\limits_{n\to\infty}\sum_{r=1}^{n}\alpha^r+\lim\limits_{n\to\infty}\sum_{r=1}^{n}\beta^r$

$$=\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\left[\begin{array}{c}a+ar+ar^2+\ldots..\text{ infinite terms}\\=\frac{a}{1-r}\text{ if }|r|<1\end{array}\right]$$

$$=\frac{(\alpha+\beta)-2\alpha\beta}{1-(\alpha+\beta)+\alpha\beta}=\frac{\frac{25}{375}+\frac{4}{375}}{1-\frac{25}{375}-\frac{2}{375}}=\frac{29}{348}=\frac{1}{12}$$

9. **(16)** Given,

$P_n = \alpha^{\mathrm{n}} - \beta^{\mathrm{n}}$ and α & β are roots of $x^2 - x - 4 = 0$

Now replacing n with $n-1$ in $P_n = \alpha^{\mathrm{n}} - \beta^{\mathrm{n}}$ we get,

$P_{n-1} = \left(\alpha^{n-1} - \beta^{n-1}\right)$

Now subtracting $P_n - P_{n-1}$ we get,

$P_n - P_{n-1} = (\alpha^n - \beta^n) - \left(\alpha^{n-1} - \beta^{n-1}\right)$

$\Rightarrow P_n - P_{n-1} = \alpha^{n-2}\left(\alpha^2 - \alpha\right) - \beta^{n-2}\left(\beta^2 - \beta\right)$

Now using the equation $\alpha^2 - \alpha - 4 = 0$ & $\beta^2 - \beta - 4 = 0$ we get,

$P_n - P_{n-1} = 4\left(\alpha^{\mathrm{n}-2} - \beta^{\mathrm{n}-2}\right)$

$P_n - P_{n-1} = 4P_{n-2}$

which implies that $P_{15} - P_{14} = 4P_{13}$ and $P_{16} - P_{15} = 4P_{14}$

Now putting the value in given expression

$$\frac{P_{15}P_{16} - P_{14}P_{16} - P_{15}^2 + P_{14}P_{15}}{P_{13}P_{14}}$$

$$= \frac{P_{16}\left(P_{15} - P_{14}\right) - P_{15}\left(P_{15} - P_{14}\right)}{P_{13}P_{14}}$$

$$= \frac{\left(P_{15} - P_{14}\right)\left(P_{16} - P_{15}\right)}{P_{13}P_{14}} = \frac{\left(4P_{13}\right)\left(4P_{14}\right)}{P_{13}P_{14}} = 16$$

10. **(1)**

Given, $x^2 + 5\sqrt{2}x + 10 = 0$

and $P_n = \alpha^n - \beta^n$

Now $\frac{P_{17}P_{20} + 5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19} + 5\sqrt{2}P_{18}^2} = \frac{P_{17}\left(P_{20} + 5\sqrt{2}P_{19}\right)}{P_{18}\left(P_{19} + 5\sqrt{2}P_{18}\right)}$

$$= \frac{P_{17}\left(\alpha^{20} - \beta^{20} + 5\sqrt{2}\left(\alpha^{19} - \beta^{19}\right)\right)}{P_{18}\left(\alpha^{19} - \beta^{19} + 5\sqrt{2}\left(\alpha^{18} - \beta^{18}\right)\right)}$$

$$= \frac{P_{17}\left(\alpha^{19}\left(\alpha + 5\sqrt{2}\right) - \beta^{19}\left(\beta + 5\sqrt{2}\right)\right)}{P_{18}\left(\alpha^{18}\left(\alpha + 5\sqrt{2}\right) - \beta^{18}\left(\beta + 5\sqrt{2}\right)\right)}$$

Since $\alpha + 5\sqrt{2} = -10/\alpha \quad \ldots(1)$

& $\beta + 5\sqrt{2} = -10/\beta \quad \ldots(2)$

Now put these values in above expression

$$\frac{P_{17}\left(\alpha^{19}\left(\alpha + 5\sqrt{2}\right) - \beta^{19}\left(\beta + 5\sqrt{2}\right)\right)}{P_{18}\left(\alpha^{18}\left(\alpha + 5\sqrt{2}\right) - \beta^{18}\left(\beta + 5\sqrt{2}\right)\right)} = \frac{-10P_{17}P_{18}}{-10P_{18}P_{17}} = 1$$

11. **(2)** Given equation is-

$\left|\left(x^2\right) - 8x + 15\right| - 2x + 7 = 0$

$|(x-5)(x-3)| - 2x + 7 = 0$

Now, as per wavy-curve method and rules of modulus-

For x ≤ 3 or $x \geq 5$

$x^2 - 8x + 15 - 2x + 7 = 0$

$x = 5 + \sqrt{3}$

For $3 < x < 5$,

$x^2 - 8x + 15 + 2x - 7 = 0$

$x = 4$

Hence sum $= 9 + \sqrt{3}$

12. (2)

Case-I

$x \leq 5$

$(x+1)^2 - (x-5) = \frac{27}{4}$

$(x+1)^2 - (x+1) - \frac{3}{4} = 0$

$x+1 = \frac{3}{2}, -\frac{1}{2}$

$x = \frac{1}{2}, -\frac{3}{2}$

Case-II

$x > 5$

$(x+1) + (x-5) = \frac{27}{4}$

$(x+1)^2 + (x+1) - \frac{51}{4} = 0$

$x = \frac{-1 \pm \sqrt{52}}{2}$ (rejected as $x > 5$)

So, the equation have two real root.

13. (2)

Given α, β be the roots of the equation $x^2 - \sqrt{2}x + \sqrt{6} = 0$

So sum of roots will be $\alpha + \beta = \sqrt{2}$ and product of roots will be $\alpha\beta = \sqrt{6}$

And also given $\frac{1}{\alpha^2} + 1$ and $\frac{1}{\beta^2} + 1$ are roots of $x^2 + ax + b = 0$

So sum of roots will be $-a = \frac{1}{\alpha^2} + 1 + \frac{1}{\beta^2} + 1$

$\Rightarrow a = \frac{-1}{\alpha^2} - \frac{1}{\beta^2} - 2 \ldots..(1)$

And similarly product of roots will be,

$b = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + 1 + \frac{1}{\alpha^2\beta^2} \ldots.(2)$

Now adding equation (1) & (2) we get,

$a + b = \frac{1}{(\alpha\beta)^2} - 1 = \frac{1}{6} - 1 = -\frac{5}{6}$ $\{$as $\alpha\beta = \sqrt{6}\}$

Now putting the value of $a+b$ in $x^2 - (a+b-2)x + (a+b+2) = 0$

$\Rightarrow x^2 - \left(-\frac{5}{6} - 2\right)x + \left(2 - \frac{5}{6}\right) = 0$

$\Rightarrow 6x^2 + 17x + 7 = 0$

$\Rightarrow x = -\frac{7}{3}, x = -\frac{1}{2}$ are the roots, both roots are real and negative.

14. (98) Given α & β are roots of $x^2 - 4\lambda x + 5 = 0$

So, $\alpha + \beta = 4\lambda$ and $\alpha\beta = 5$

And α & γ are roots of $x^2 - (3\sqrt{2} + 2\sqrt{3})x + (7 + 3\lambda\sqrt{3}) = 0$

So, $\alpha + \gamma = 3\sqrt{2} + 2\sqrt{3}$ and $\alpha\gamma = 7 + 3\lambda\sqrt{3}$

Given, if $\beta + \gamma = 3\sqrt{2}$

So, from equation (i) and (ii) we get, $\alpha = 2\lambda + \sqrt{3}$ and

$\beta = 2\lambda - \sqrt{3}$, Now by product of roots we get, $4\lambda^2 - 3 = 5 \Rightarrow \lambda = \sqrt{2}$

$\therefore (\alpha + 2\beta + \gamma)^2 = (\alpha + \beta + \beta + \gamma)^2 = (4\lambda + 3\sqrt{2})^2$

$= (4\sqrt{2} + 3\sqrt{2})^2 = (7\sqrt{2})^2 = 98$

15. (9)

$x^2 - 12x + [x] + 31 = 0$

$\{x\} = x^2 - 11x + 31$

$0 \le x^2 - 11x + 31 < 1$

$x^2 - 11x + 30 < 0$

$x \in (5, 6)$

so $[x] = 5$

$x^2 - 12x + 5 + 31 = 0$

$x^2 - 12x + 36 = 0$

$x = 6$ but $x \in (5, 6)$

so $x \in \phi$

$m = 0$

Now

$x^2 - 5|x + 2| - 4 = 0$

$x \ge -2$

$x^2 - 5x - 14 = 0$

$(x - 7)(x + 2) = 0$

$x = 7, -2$

$x < -2$

$x^2 + 5x + 6 = 0$

$(x + 3)(x + 2) = 0$

$x = -3, -2$

$x = \{7, -2, -3\}$

$n = 3$

$m^2 + mn + n^2 = n^2 = 9$

16. (2)

The Given equation is $x^2 + 9y^2 - 4x + 3 = 0$

$\Rightarrow 9y^2 + 0y + (x^2 - 4x + 3) = 0$

Make quadratic of y, we have $D \ge 0$ As it gives real values

$\Rightarrow 0 - 4 \times 9 \times (x^2 - 4x + 3) \ge 0$

$\Rightarrow x^2 - 3x - x + 3 \le 0$

$\Rightarrow (x - 3)(x - 1) \le 0$

$x \in [1, 3]$

Now making quadratic in x equation is $x^2 - 4x + 3 + 9y^2 = 0$

$D \ge 0$

$16 - 4 \times (3 + 9y^2) \ge 0$

$\Rightarrow 4 - 3 - 9y^2 \ge 0$

$\Rightarrow 9y^2 \le 1$

$\Rightarrow y \in \left[\frac{-1}{3}, \frac{1}{3}\right]$

17. (1)

Let α, β be the roots of the equation

$x^2+(3-a)x+1-2a=0$

Then, sum of roots $\alpha+\beta=a-3$

And product of roots $\alpha\beta=1-2a$

We know that $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$\therefore \alpha^2+\beta^2=(a-3)^2-2(1-2a)$

$=a^2-2a+7$

$=(a-1)^2+6$

$\therefore$ Minimum value of $\alpha^2+\beta^2=6$ at $a=1$.

18. (2)

We have,

$x^2+2(a+4)x-5a+64>0$

If $A>0$ and $D=B^2-4AC<0$, then $Ax^2+Bx+C>0\ \forall x \in R$.

Hence,

$D<0$

$\Rightarrow\ 4(\mathrm{a}+4)^2-4(-5\mathrm{a}+64)<0$

$\Rightarrow\ a^2+16+8a+5a-64<0$

$\Rightarrow\ a^2+13a-48<0$

$\Rightarrow\ (a+16)(a-3)<0$

$\Rightarrow a \in(-16,3)$

$\therefore$ Possible values for a are $\{-15,\ -14,\ \ldots\ldots, 2\}$ containing 18 integers.

But given range of a is $[-5,30]$, hence a would take values $\{-5,\ -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2\}$, containing 8 integers.

And, $[-5,\ 30]$ has 36 numbers.

$\therefore$ Required probability

$=\frac{8}{36}$

$=\frac{2}{9}$

19. (272)

Given, $\left(p^2+q^2\right)x^2-2q(p+r)x+q^2+r^2=0$

On simplifying we get, $(px-q)^2+(qx-r)^2=0$

$\Rightarrow px-q=0$ & $qx-r=0$

$\Rightarrow x=\frac{q}{p}=\frac{r}{q}$

$\Rightarrow\ x=\frac{q}{p}=\frac{r}{q}=4$ [because roots of equation $x^2-2x-8=0$ are 4 or -2]

As p, q, r are positive, so x must be 4.

Now, $q=4p$ and $r=4q=16p$

So, $\frac{q^2+r^2}{p^2}=\frac{(4p)^2+(4\times 4p)^2}{p^2}=16+256=272$.

20. (2)

Given $ax^2 - 2bx + 15 = 0 \quad \ldots(\text{i})$

Has repeated roots So $D = 0$

$4b^2 - 4 \times 15 \times a = 0$

$\Rightarrow b^2 = 15a \quad \ldots(\text{ii})$

Also given

$$x^2 - 2bx + 21 = 0 \begin{cases} \alpha \\ \beta \end{cases} \text{Root}$$

Now α will satisfy both quadratic

$ax^2 - 2bx + 15 = 0$ & $x^2 - 2bx + 21 = 0$

Putting the value we get

$a\alpha^2 - 2b\alpha + 15 = 0$

$\alpha^2 - 2b\alpha + 21 = 0$

$-\quad +\quad -$

$-----------$

$(a-1)\alpha^2 = 6$

$\alpha^2 = \frac{6}{a-1}$

Now in equation (1) product of Root $\alpha^2 = \frac{15}{a}$

So $\frac{15}{a} = \frac{6}{a-1} \Rightarrow 2a = 5a - 5 \Rightarrow a = \frac{5}{3}$

Now $b^2 = 15a \Rightarrow b^2 = 15 \times \frac{5}{3} \Rightarrow b^2 = 25$

So $b = \pm 5$

Now in quadratic $x^2 - 2bx + 21 = 0$

Putting the value of b we get

$x^2 - 10x + 21 = 0 \Rightarrow (x-7)(x-3) = 0$

So $x = 3$ or 7.

Or

$x^2 + 10x + 21 = 0 \Rightarrow x = -3 \text{ or } x = -7$

So $\alpha = \pm 3$ & $\beta = \pm 7$

So $\alpha^2 + \beta^2 = 3^2 + 7^2 = 9 + 49 = 58$

21. (25)

Let $f(x)=(x-\alpha)(x-\beta)$

It is given that $f(0)=p \Rightarrow \alpha\beta=p$

and $f(1)=\frac{1}{3} \Rightarrow (1-\alpha)(1-\beta)=\frac{1}{3}$

Now let us assume that α is the common root of $f(x)=0$ and $fofofof\ (x)=0$

$fofofof\ (x)=0$

$\Rightarrow fofofof\ (\alpha)=0$

$\Rightarrow fofof\ (0)=0$

$\Rightarrow fof(p)=0$

So, $f(p)$ is either α or β.

Now assuming $(p-\alpha)(p-\beta)=\alpha$

$\Rightarrow(\alpha\beta-\alpha)(\alpha\beta-\beta)=\alpha \Rightarrow(\beta-1)(\alpha-1)\beta=1$

$\Rightarrow \frac{\beta}{3}=1\left(\text{as } (1-\alpha)(1-\beta)=\frac{1}{3}\right)$

So, $\beta=3$

Now finding α by putting the value of β in $(1-\alpha)(1-\beta)=\frac{1}{3}$,

$\Rightarrow(1-\alpha)(1-3)=\frac{1}{3}$

$\Rightarrow \alpha=\frac{7}{6}$

So, $f(x)=\left(x-\frac{7}{6}\right)(x-3)$

So, $f(-3)=\left(-3-\frac{7}{6}\right)(-3-3)=25$

22. (3) As there is exactly one root between 0 and 1,

$f(0)\cdot f(1)\leq 0$

$\Rightarrow 2\left(\lambda^2+1-4\lambda+2\right)\leq 0 \Rightarrow 2\left(\lambda^2-4\lambda+3\right)\leq 0$

$\Rightarrow(\lambda-1)(\lambda-3)\leq 0$

$\Rightarrow \lambda \in[1,\ 3]$

But at $\lambda=1$, both roots are 1.

So, $\lambda \neq 1, \lambda \in (1,3]$

23. (3)

$x^3-2x^2+2x-1=0$

$x=1$ satisfying the equation

$\therefore\ x-1$ is factor of x^3-2x^2+2x-1

$=(x-1)\left(x^2-x+1\right)=0$

$x=1, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$

$x=1,-\omega^2,-\omega$

sum of 162^{th} power of roots

$=(1)^{162}+\left(-\omega^2\right)^{162}+(-\omega)^{162}$

$=1+(\omega)^{324}+(\omega)^{162}$

$=1+1+1=3$

24. (2)

Given:

$S=\left\{x:\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}+\left(\sqrt{3}-\sqrt{2}\right)^{x^2-4}=10\right\}$

Now,

$\sqrt{3}-\sqrt{2}=\frac{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}{\sqrt{3}+\sqrt{2}}=\frac{1}{\sqrt{3}+\sqrt{2}}$

So,

$\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}+\left(\sqrt{3}-\sqrt{2}\right)^{x^2-4}=10$

$\Rightarrow\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}+\left(\frac{1}{\sqrt{3}+\sqrt{2}}\right)^{x^2-4}=10$

Put $\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}=u$, then

$u+\frac{1}{u}=10$

$\Rightarrow u^2-10u+1=0$

$\Rightarrow u=\frac{10\pm\sqrt{100-4}}{2}$

$\Rightarrow u=5\pm\sqrt{24}$

$\Rightarrow u=5\pm2\sqrt{3}\sqrt{2}$

$\Rightarrow u=\left(\sqrt{3}\pm\sqrt{2}\right)^2$

$\Rightarrow\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}=\left(\sqrt{3}\pm\sqrt{2}\right)^2$

So, $\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}=\left(\sqrt{3}+\sqrt{2}\right)^2$ and $\left(\sqrt{3}+\sqrt{2}\right)^{x^2-4}=\left(\frac{1}{\sqrt{3}+\sqrt{2}}\right)^2$

Therefore,

$x^2-4=2$ & $x^2-4=-2$

$\Rightarrow x=\pm\sqrt{6}$ & $x=\pm\sqrt{2}$

So, $S=\left\{\sqrt{6},\ \sqrt{2},\ -\sqrt{6},\ -\sqrt{2}\right\}$

Hence, $n(S)=4$

25. (25)

Given,

$\log_2\left(9^{2\alpha-4}+13\right)-\log_2\left(3^{2\alpha-4}\cdot\frac{5}{2}+1\right)=2$

Now let $3^{2\alpha-4}=t$, so the equation becomes,

$\log_2\left(t^2+13\right)-\log_2\left(\frac{5t}{2}+1\right)=2$

$\Rightarrow \log_2\frac{\left(t^2+13\right)}{\left(\frac{5t}{2}+1\right)}=2$

$\Rightarrow \frac{\left(t^2+13\right)}{\left(\frac{5t}{2}+1\right)}=2^2$

$\Rightarrow t^2+13=10t+4$

$\Rightarrow t^2-10t+9=0$

$\Rightarrow t=1$ or 9

So,

$3^{2\alpha-4}=1$ or 9

$\Rightarrow 3^{2\alpha-4}=3^0$ or 3^2

$\Rightarrow 2\alpha-4=0$ or 2

$\Rightarrow \alpha=2,\ 3$

Now,

$\mathrm{x}^2-2\left(\sum_{\alpha\in s}\alpha\right)^2\mathrm{x}+\sum_{\alpha\in s}(\alpha+1)^2\beta=0$

$\Rightarrow x^2-2\left((2+3)^2x\right)+\left(3^2+4^2\right)\beta=0$

$\Rightarrow x^2-50x+25\beta=0$

Now for real roots

$D\geq 0$

$\Rightarrow 50^2-4\times 25\beta\geq 0$

$\Rightarrow 50-2\beta\geq 0$

$\Rightarrow \beta\leq 25$

So, maximum value of β is 25.

26. (13) We have, $a+b+c=1, ab+bc+ca=2$ and $abc=3$

Now, $a^2+b^2+c^2=(a+b+c)^2-2\Sigma ab=-3$

$\Rightarrow (ab+bc+ca)^2=\Sigma(ab)^2+2abc\Sigma \mathrm{a}$

$\Rightarrow \Sigma(ab)^2=-2$

So, $a^4+b^4+c^4=\left(a^2+b^2+c^2\right)^2-2\Sigma(ab)^2$

$=9-2(-2)$

$=13.$

27. (2) Let $e^{2x} = t$

$\Rightarrow t^4 - t^3 - 3t^2 - t + 1 = 0$

$\Rightarrow t^2 + \frac{1}{t^2} - \left(t + \frac{1}{t}\right) - 3 = 0$

$\Rightarrow \left(t + \frac{1}{t}\right)^2 - \left(t + \frac{1}{t}\right) - 5 = 0$

Putting $(t + (1/t)) = a$, the equation becomes-

$a^2 - a - 5 = 0$

$a = \frac{1 \pm \sqrt{21}}{2}$ but $\frac{1-\sqrt{21}}{2}$ is rejected as we know value of (a) starts from 2

So, $a = (t + (1/t)) = \frac{1+\sqrt{21}}{2}$

Substituting the value of $t = e^{(2x)}$

$e^{(2x)} + \left(\frac{1}{e^{(2x)}}\right) = \frac{1+\sqrt{21}}{2}$

So, two real solutions and this, graph of the function given cuts x-axis 2 times

28. (2)

Given,

$e^{4x} + 8e^{3x} + 13e^{2x} - 8e^x + 1 = 0,\ x \in R$

Now let $e^x = t$ we get,

$t^4 + 8t^3 + 13t^2 - 8t + 1 = 0$

Now divide complete equation by t^2

$\Rightarrow t^2 + \frac{1}{t^2} + 8\left(t - \frac{1}{t}\right) + 13 = 0$

$\Rightarrow \left(t - \frac{1}{t}\right)^2 + 8\left(t - \frac{1}{t}\right) + 15 = 0$

Now let $t - \frac{1}{t} = z$ we get,

$\Rightarrow z^2 + 8z + 15 = 0$

$\Rightarrow z = -3, -5$

So, $t - \frac{1}{t} = -3$ or $t - \frac{1}{t} = -5$

$\Rightarrow t^2 + 3t - 1 = 0$ or $t^2 + 5t - 1 = 0$

$\Rightarrow t = \frac{-3 \pm \sqrt{13}}{2}, \frac{-5 \pm \sqrt{29}}{2}$

So, $e^x = \frac{-3+\sqrt{13}}{2}, \frac{-5+\sqrt{29}}{2} = \alpha, \beta$ (rejecting negative values as exponential is positive function)

And both $\frac{-3+\sqrt{13}}{2}$ and $\frac{-5+\sqrt{29}}{2} \in (0, 1)$

So, $x = \ln(\alpha),\ \ln(\beta)$ are both negative,

Hence, there are two solution and both are negative.

29. (9)

Given:

$x^7+3x^5-13x^3-15x=0$

$\Rightarrow x\left(x^6+3x^4-13x^2-15\right)=0$

So, $x=0$ is one of the root.

Now,

$\left(x^6+3x^4-13x^2-15\right)=0$

Put $x^2=t$, then we have

$t^3+3t^2-13t-15=0$

$\Rightarrow(t-3)\left(t^2+6t+5\right)=0$

$\Rightarrow(t-3)(t+1)(t+5)=0$

So, $t=3, t=-1, t=-5$

Now we are getting

$x^2=3, x^2=-1, x^2=-5$

$\Rightarrow x=\pm\sqrt{3},\ \pm i, \pm\sqrt{5}i$

From the given condition $|\alpha_1|\geq|\alpha_2|\geq \ldots\ldots \geq|\alpha_7|$

We can clearly say that $|\alpha_7|=0$ and

and $|\alpha_6|=\sqrt{5}=|\alpha_5|$

and $|\alpha_4|=\sqrt{3}=|\alpha_3|$ and

$|\alpha_2|=1=|\alpha_1|$

So we can have,

$\alpha_1=\sqrt{5}i,\ \alpha_2=-\sqrt{5}i,\ \alpha_3=\sqrt{3}$

$\alpha_4=-\sqrt{3},\ \alpha_5=i,\ \alpha_6=-i$

$\alpha_7=0$

Hence,

$\alpha_1\alpha_2-\alpha_3\alpha_4+\alpha_5\alpha_6$

$=5-(-3)+1=9$

30. (1)

Consider the equation $x^2+ax+b=0$

It has two roots (not necessarily real α and β)

Either $\alpha=\beta$ or $\alpha\neq\beta$

Case (1) If $\alpha=\beta$, then it is repeated root. Given than α^2-2 is also root

So, $\alpha=\alpha^2-2$

$\Rightarrow\ (\alpha+1)(\alpha-2)=0$

$\Rightarrow\ \alpha=-1$ or $\alpha=2$

When $\alpha=-1$ then $(a,b)=(2,1)$

$\alpha=2$ then $(a,b)=(-4,4)$

Case (2) If $\alpha\neq\beta$, then four possibilities are there

(I) $\alpha=\alpha^2-2$ and $\beta=\beta^2-2$

Here $(\alpha,\beta)=(2,-1)$ or $(-1,2)$

Hence $(a,b)=(-(\alpha+\beta),\alpha\beta)$

$=(-1,-2)$

(II) $\alpha=\beta^2-2$ and $\beta=\alpha^2-2$

Then $\alpha-\beta=\beta^2-\alpha^2=(\beta-\alpha)(\beta+\alpha)$

Since $\alpha\neq\beta$ we get $\alpha+\beta=\beta^2+\alpha^2-4$

$\alpha+\beta=(\alpha+\beta)^2-2\alpha\beta-4$

Thus $-1=1-2\alpha\beta-4$ which implies

$\alpha\beta=-1$. Therefore $(a,b)=(-(\alpha+\beta),\alpha\beta)$

$=(1,-1)$

(III) $\alpha=\alpha^2-2=\beta^2-2$ and $\alpha\neq\beta$

$\Rightarrow\ \alpha=-\beta$

Thus $\alpha=2,\beta=-2$

$\alpha=-1,\beta=1$

Therefore $(a,b)=(0,-4)$ and $(0,-1)$

(IV) $\beta=\alpha^2-2=\beta^2-2$ and $\alpha\neq\beta$ is same as (III)

Therefore we get 6 pairs of (a,b)

Which are $(2,1)$,$(-4,4)$,$(-1,-2)$,$(1,-1)$,$(0,-4)$,$(0,-1)$