ANSWER KEYS

1. (1)	**2.** (3)	**3.** (4)	**4.** (3)	**5.** (1)	**6.** (4)	**7.** (3)	**8.** (3)
9. (3)	**10.** (3)						

1. (1)

Since, we know that $\cos^2\theta + \sin^2\theta = 1$

Let $I = \int_{-\frac{\pi}{2}}^{0} \sqrt{\cos x(1-\cos^2 x)}\, dx$

$= \int_{-\pi/2}^{0} \sqrt{\cos x}\, |\sin x| dx$

when $-\frac{\pi}{2} < x < 0$ then $-1 \le \sin x \le 0$

$\therefore \quad I = -\int_{-\pi/2}^{0} \sqrt{\cos x}\, \sin x\, dx$

Put $\cos x = t, \quad -\sin x\, dx = dt,$

$I = \int_0^1 t^{1/2} dt$

$I = \left(\frac{t^{3/2}}{3/2}\right)_0^1 = \frac{2}{3}$

2. (3)

$I = \int_0^1 \left[\sqrt{\frac{1-x}{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}}\right] dx$ (rationalising the denominator)

$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx$

$= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx - \int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$

$\Rightarrow I = [\sin^{-1} x]_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{\sqrt{1-x^2}} dx$

$= \left[\frac{\pi}{2} - 0\right] + \frac{1}{2}\left[2\sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} + [0-1]$

$= \frac{\pi}{2} - 1$

3. (4)

Let $I = \int_0^{\frac{1}{2}} \frac{x\sin^{-1} x}{\sqrt{1-x^2}} dx$

Put, $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

Also, when $x = 0, \theta = 0$

And when, $x = \frac{1}{2}, \theta = \frac{\pi}{6}$

Thus, $I = \int_0^{\frac{\pi}{6}} \frac{\sin\theta \sin^{-1}\left(\sin\theta\right)}{\sqrt{1-\sin^2\theta}} \cos\theta d\theta$

$\Rightarrow I = \int_0^{\frac{\pi}{6}} \theta \sin\theta d\theta$

Integrating the above by parts, we get

$I = [\theta(-\cos\theta)]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} 1.\cos\theta d\theta$

$= [-\theta\cos\theta + \sin\theta]_0^{\frac{\pi}{6}}$

$= \frac{-\pi}{6} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{6-\pi\sqrt{3}}{12}$

4. (3)

$f\left(\frac{1}{x}\right)=\int_{1}^{1/x}\frac{\log t}{1+t}dt$

Put $t=\frac{1}{u}\quad dt=\frac{-1}{u^2}du$

$=\int_{1}^{x}\frac{-\log x}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{u^2}\right)du$

$=\int_{1}^{x}\frac{-\log x}{u(1+u)}du$

$F(x)=f(x)+f\left(\frac{1}{x}\right)=\int_{1}^{x}\frac{\log t}{1+t}+\frac{\log t}{t(1+t)}dt$

$=\int_{1}^{x}\frac{\log t}{t}dt=\left|\frac{(\log t)^2}{2}\right|_1^x$

$=\frac{(\log x)^2}{2}$

$\therefore F(e)=\frac{1}{2}$

5. (1) Consider that, $I=\int_{1}^{x}t\log t\,dt$

$=\left[\log t\cdot\frac{t^2}{2}\right]_1^x-\int_{1}^{x}\frac{1}{t}\cdot\frac{t^2}{2}dt$

$=\frac{x^2}{2}\log x-\frac{1}{2}\left[\frac{t^2}{2}\right]_1^x$

$=\frac{x^2}{2}\log x-\frac{1}{2}\left[\frac{x^2}{2}-\frac{1}{2}\right]$

$\Rightarrow\frac{1}{4}=\frac{x^2}{2}\log x-\frac{1}{4}(x^2-1)$

$\Rightarrow\frac{1}{2}x^2\log x-\frac{1}{4}x^2=0$ (as $x>1$)

$\Rightarrow\quad x^2(2\log x-1)=0$

$\Rightarrow 2\log x-1=0$

$\Rightarrow\quad\log x=\frac{1}{2}$

$\Rightarrow\quad x=e^{\frac{1}{2}}$

$\Rightarrow\quad x=\sqrt{e}$

6. (4)

$P_n=\int_{1}^{e}(\log x)^n dx$

Integrating by parts taking $(\log\ x)^n$ as first function and 1 as second function, we get

$P_n=(\log x)^n x\Big|_1^e-\int_{1}^{e}x.n(\log x)^{n-1}\cdot\frac{1}{x}dx$

$\Rightarrow\quad P_n=e-n\left[(\log x)^{n-1}\cdot x\Big|_1^e-\int_{1}^{e}x\cdot(n-1)(\log x)^{n-2}\cdot\frac{1}{x}dx\right]$

$=e-n\left[(e-0)-(n-1)\int_{1}^{e}(\log x)^{n-2}dx\right]$

$=e-ne+n(n-1)P_{n-2}$

$n=10\Rightarrow P_{10}=e-10e+10(10-1)P_8$

$P_{10}-90P_8=-9e$

7. (3)

Let $I_n=\int_0^\infty x^n e^{-ax}dx$

Using integration by parts,

$\left\{\int u\times vdx=u\int vdx-\int\left(\left(\frac{d}{dx}u\right)\left(\int vdx\right)\right)dx\right\}$

$\Rightarrow I_n=x^n\int e^{-ax}dx-\int_0^\infty\left(\frac{d}{dx}x^n\int e^{-ax}dx\right)dx$

Given: $\int_0^\infty e^{-ax}dx=\frac{1}{a}$

$=\left[x^n\frac{e^{-ax}}{-a}\right]_0^\infty-\int_0^\infty nx^{n-1}\frac{e^{-ax}}{-a}dx$

$=\frac{n}{a}I_{n-1}$

$I_n=\frac{n}{a}I_{n-1}$

Similarly, $I_{n-1}=\frac{n-1}{a}I_{n-2}$,

$I_{n-2}=\frac{n-2}{a}I_{n-3}$ and so on.

$\therefore\ I_n=\frac{n}{a}.\frac{n-1}{a}I_{n-2}$

$=\frac{n(n-1)(n-2)}{a^3}I_{n-3}$

$I_n=\frac{n!}{a^n}\int_0^\infty e^{-ax}dx$

$=\frac{n!}{a_n}I_0=\frac{n!}{a^{n+1}}$

8. (3)

$I_{m,n}=\int_0^1 x^m\left(1-x\right)^n dx$

Take x^m as second & $(1-x)^n$ as first function for Integrating by parts

$I_{m,n}=\left.-x^m\frac{(1-x)}{n+1}^{n+1}\right|_0^1+\int_0^1 mx^{m-1}\frac{(1-x)^{n+1}}{n+1}dx$

$\Rightarrow I_{m,n}=0+\frac{m}{n+1}\int_0^1 x^{m-1}(1-x)^{n+1}dx$

(Repeating above process)

$\Rightarrow I_{m,n}=\frac{m(m-1)}{(n+1)(n+2)}\int_0^1 x^{m-2}\left(1-x\right)^{n+2}dx$

So, $I_{m,n}=\frac{m(m-1)(m-2).....3\cdot2\cdot1}{(n+1)(n+2)....(n+m)}\cdot\int_0^1\left(1-x\right)^{n+m}dx$

$\Rightarrow I_{m,n}=\frac{(m(m-1)...3\cdot2\cdot1)}{(n+1)(n+2)...(n+m)}\left(\frac{n!}{n!}\right)\frac{1}{n+m+1}\left[-\left(1-x\right)^{n+m+1}\right]_0^1$

$I_{m,n}=\frac{m!n!}{(m+n+1)!}$

9. (3)

Let $I=\int_0^{\frac{\pi}{2}}\frac{\cot x}{\cot x+cosecx}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+1}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\frac{\cos x+1-1}{\cos x+1}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{2\cos^2\frac{x}{2}}\right)dx$

$\Rightarrow I=\int_0^{\frac{\pi}{2}}\left(1-\frac{1}{2}\sec^2\frac{x}{2}\right)dx$

$\Rightarrow I=\left(x-\tan\frac{x}{2}\right)_0^{\frac{\pi}{2}}$

$\Rightarrow I=\left(\frac{\pi}{2}-1\right)-(0-0)$

$\Rightarrow I=\frac{\pi}{2}-1=\frac{1}{2}(\pi-2).$

Now using given information $\int_0^{\frac{\pi}{2}}\frac{\cot x}{\cot x+\operatorname{cosec} x}dx=m\left(\pi+n\right)$, clearly $m=\frac{1}{2}$ and $n=-2$.

$\Rightarrow mn=-1$

10. (3)

Given,

$\int_0^x\left(t^2-8t+13\right)dt=x\sin\frac{a}{x}$

$\Rightarrow\left(\frac{t^3}{3}-\frac{8t^2}{2}+13t\right)_0^x=x\sin\frac{a}{x}$

$\Rightarrow\frac{x^3}{3}-4x^2+13x=x\sin\frac{a}{x}$

$\Rightarrow\frac{x^2}{3}-4x+13=\sin\frac{a}{x}$

$\Rightarrow\frac{1}{3}(x-6)^2+1=\sin\frac{a}{x}$

As maximum value of R.H.S. is 1 and minimum value of L.H.S. is 1, so equality will hold only if $x=6$.

Now, putting $x=6\Rightarrow\sin\frac{a}{6}=1$.