ANSWER KEYS

1. (4)	**2.** (25)	**3.** (2)	**4.** (2)	**5.** (17)	**6.** (100)	**7.** (4)	**8.** (1)
9. (1)	**10.** (2)	**11.** (766)	**12.** (204)	**13.** (50)	**14.** (414)	**15.** (2)	**16.** (2)
17. (3)	**18.** (42)	**19.** (4)	**20.** (4)	**21.** (4)	**22.** (17)	**23.** (10)	**24.** (100)
25. (1)	**26.** (5)	**27.** (4)	**28.** (25)	**29.** (1)	**30.** (1)	**31.** (3)	**32.** (414)
33. (16)	**34.** (4)	**35.** (3)					

1. (4)

Given, $A^2 = 3A + \alpha I$

$\therefore A^3 = A^2 . A$

$\Rightarrow A^3 = 3A^2 + \alpha A$

$\Rightarrow A^3 = 3(3A + \alpha I) + \alpha A$

$\Rightarrow A^3 = 9A + \alpha A + 3\alpha I$

Now, $A^4 = A^3 . A$

$A^4 = (9+\alpha)A^2 + 3\alpha A$

$= (9+\alpha)(3A + \alpha I) + 3\alpha A$

$= \mathrm{A}(27 + 6\alpha) + \alpha(9+\alpha) \ldots\ldots.(1)$

Given, $A^4 = 21A + \beta I \ldots\ldots\ldots(2)$

On comparing equation (1) and equation (2),

$\Rightarrow\ 27 + 6\alpha = 21 \Rightarrow \alpha = -1$

$\Rightarrow \beta = \alpha(9+\alpha) = -8$

2. (25)

We have $A = \begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix}$

$A^2 = A \cdot A = \begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix} = \begin{bmatrix} i & 1+i \\ -i+1 & -i \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} i & 1+i \\ 1-i & -i \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ 1 & 1-i \end{bmatrix}$

$A^4 = A^3 \cdot A = \begin{bmatrix} 0 & i \\ 1 & 1-i \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore \quad A^4 = I$

So, $A^5 = A^4 \cdot A = I \cdot A = A$

$A^6 = A^4 \cdot A^2 = I \cdot A^2 = A^2$ and so on

$\because A^1 = A^5 = A^9 = \ldots\ldots = A^{97} = A$

Hence, possible values of n, such that $A^n = A$

$= \{1, 5, 9, \ldots, 97\}$

Clearly, above sequence is in A.P. where

$a = 1,\ d = 4\ \&\ t_n = 97 \Rightarrow a + (n-1)d = 97$

$\Rightarrow\ \ 1 + (n-1)4 = 97\ \ \Rightarrow\ \ n = 25$

$\therefore$ The number of elements in the given set $= 25$.

3. **(2)**

Given:

$$P=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

$$\therefore PP^T=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$\Rightarrow PP^T=I$

Now,

$$P^T\left(PAP^T\right)^{2007}P=P^T\underbrace{\left(PAP^T\right)\left(PAP^T\right)\left(PAP^T\right)\ldots\left(PAP^T\right)}_{2007\text{ times}}P$$

$\Rightarrow P^T\left(PAP^T\right)^{2007}P=A^{2007}$

Now,

$$A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$$

$$A^3=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$$A^{2007}=\begin{bmatrix}1 & 2007\\ 0 & 1\end{bmatrix}$$

So,

$$\Rightarrow P^T\left(PAP^T\right)^{2007}P=\begin{bmatrix}1 & 2007\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow P^TQP=\begin{bmatrix}1 & 2007\\ 0 & 1\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

So,

$a=1,\ b=2007,\ c=0,\ d=1$

$2a+b+3c-4d=2+2007-4=2005$

4. **(2)**

Given $A=\begin{bmatrix}1 & -1\\ 2 & \alpha\end{bmatrix}$ and $B=\begin{bmatrix}\beta & 1\\ 1 & 0\end{bmatrix}, \alpha, \beta \in R,$

So, $A+B=\begin{bmatrix}\beta+1 & 0\\ 3 & \alpha\end{bmatrix}$

Now $(A+B)^2=\begin{bmatrix}\beta+1 & 0\\ 3 & \alpha\end{bmatrix}\begin{bmatrix}\beta+1 & 0\\ 3 & \alpha\end{bmatrix}$

$=\begin{bmatrix}(\beta+1)^2 & 0\\ 3(\beta+1)+3\alpha & \alpha^2\end{bmatrix}$

Also, $A^2=\begin{bmatrix}1 & -1\\ 2 & \alpha\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & \alpha\end{bmatrix}=\begin{bmatrix}-1 & -1-\alpha\\ 2+2\alpha & \alpha^2-2\end{bmatrix}$

Now solving $(A+B)^2=A^2+\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$

$\Rightarrow\begin{bmatrix}(\beta+1)^2 & 0\\ 3(\alpha+\beta+1) & \alpha^2\end{bmatrix}=\begin{bmatrix}1 & -\alpha+1\\ 2\alpha+4 & \alpha^2\end{bmatrix}$

Now on comparing both side we get, $\alpha=1=\alpha_1$

And $B^2=\begin{bmatrix}\beta & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\beta & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}\beta^2+1 & \beta\\ \beta & 1\end{bmatrix}$

Now using $(A+B)^2=B^2$

$\Rightarrow\begin{bmatrix}\beta^2+1 & \beta\\ \beta & 1\end{bmatrix}=\begin{bmatrix}(\beta+1)^2 & 0\\ 3(\beta+1)+3\alpha & \alpha^2\end{bmatrix}$

Again on comparing both side we get, $\beta=0, \alpha=-1=\alpha_2$

So, $|\alpha_1-\alpha_2|=|1-(-1)|=2$

5. (17)

Given, $A=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}$

$\Rightarrow A^2=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\\ 1 & -1 & 0\end{bmatrix}=A$

$\Rightarrow A^n=A$

Now $\forall n \in\{1,2,\ldots,100\}$

Now, $B=A-I=\begin{bmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{bmatrix}$

$B^2=\begin{bmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{bmatrix}=-\begin{bmatrix}1 & -1 & -1\\ 1 & -1 & -1\\ 1 & -1 & -1\end{bmatrix}=-B$

$\Rightarrow B^3=-B^2=B$

$\Rightarrow B^5=B$

$\Rightarrow \mathrm{B}^{99}=\mathrm{B}$

Also, $\omega^{3k}=1$

So, $n=$ common of $\{1,3,5,\ldots,99\}$ and $\{3,6,9,\ldots,99\}=17$

6. (100)

Given,

$A=\begin{bmatrix}-1 & a\\ 0 & b\end{bmatrix}$

$A^2=\begin{bmatrix}-1 & a\\ 0 & b\end{bmatrix}\begin{bmatrix}-1 & a\\ 0 & b\end{bmatrix}=\begin{bmatrix}1 & -a+ab\\ 0 & b^2\end{bmatrix}$

$\therefore T_n=\{A\in S; A^{n(a+1)}=I\}$

$\therefore$ b must be equal to 1

$\therefore$ In this case A^2 will become identity matrix and a can take any value from 1 to 100

$\therefore$ Total number of common element will be 100.

7. (4)

$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 3 & 0 & -1\end{bmatrix}$

$A^2=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}, A^3=\begin{bmatrix}1 & 0 & 0\\ 0 & 8 & 0\\ 3 & 0 & -1\end{bmatrix}$

$A^4=\begin{bmatrix}1 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 1\end{bmatrix}$

Hence,

$A^{20}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2^{20} & 0\\ 0 & 0 & 1\end{bmatrix}, A^{19}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2^{19} & 0\\ 3 & 0 & -1\end{bmatrix}$

So $A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1+\alpha+\beta & 0 & 0\\ 0 & 2^{20}+\alpha.2^{19}+2\beta & 0\\ 3\alpha+3\beta & 0 & 1-\alpha-\beta\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$

Therefore $\alpha+\beta=0$ and $2^{20}+2^{19}\alpha-2\alpha=4$

$\Rightarrow \alpha=\frac{4\left(1-2^{18}\right)}{2\left(2^{18}-1\right)}=-2$

Hence $\beta=2$

So, $(\beta-\alpha)=4$

8. (1)

Given,

Matrix A is symmetric, B is skew symmetric and C is skew symmetric,

So, $A^T = A,\ B^T = -B,\ C^T = -C$

Now let $M = A^{13}B^{26} - B^{26}A^{13}$

Then, $M^T = \left(A^{13}\,B^{26} - B^{26}\,A^{13}\right)^T$

$= \left(A^{13}B^{26}\right)^T - \left(B^{26}\,A^{13}\right)^T$

$= \left(B^T\right)^{26}\left(\,A^T\right)^{13} - \left(A^T\right)^{13}\left(\,B^T\right)^{26}$

$= B^{26}\,A^{13} - A^{13}\,B^{26} = -M$

Hence, M is skew symmetric

Now let, $N = A^{26}C^{13} - C^{13}\,A^{26}$

Then, $N^T = \left(A^{26}C^{13}\right)^T - \left(C^{13}A^{26}\right)^T$

$= -(C)^{13}(A)^{26} + A^{26}C^{13} = N$

Hence, N is symmetric.

So, only $S2$ is true.

9. (1)

Since $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

So $A^2 = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = -4I$

and $A^3 = -4A$

Similarly $A^4 = (-4I)(-4I) = (-4)^2 I$,

$A^5 = (-4)^2 A,\quad A^6 = (-4)^3 I$

Now $M = \sum_{k=1}^{10} A^{2k} = A^2 + A^4 + \ldots . + A^{20}$

$= \left[-4 + (-4)^2 + (-4)^3 + \ldots . + (-4)^{20}\right] I$

$= -k_1 I$

So M is symmetric matrix

$N = \sum_{k=1}^{10} A^{2k-1} = A + A^3 + \ldots\ldots + A^{19}$

$= A\left[1 + (-4) + (-4)^2 + \ldots\ldots + (-4)^9\right]$

$= k_2 A$

So N is skew symmetric

$\Rightarrow N^2$ is symmetric matrix

Hence, MN^2 is non-identity symmetric matrix

10.

(2) $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, a, b, c, d, e, f \in \{0, 1, 2, \ldots 9\}$

Number of matrices $= 10^{\circ}$

11. (766)

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},\ A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$

Diagonal elements of AA^T is $a^2 + b^2 + c^2,\ d^2 + e^2 + f^2,\ g^2 + b^2 + c^2$.

Sum $= a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 9$

$a, b, c, d, e, f, g, h, i \in \{0, 1, 2, 3\}$

	Case	No. of Matrices
(1)	All – $1'$s	$\frac{9!}{9!} = 1$
(2)	One $\rightarrow 3'$s, remaining-$0'$s	$\frac{9!}{1!\times 8!} = 9$
(3)	One-$2'$s Five-$1'$s Three-$0'$s	$\frac{9!}{1!\times 5!\times 3!} = 8 \times 63$
(4)	Two -2’s One-$1'$s Six-0’s	$\frac{9!}{2!\times 6!} = 63 \times 4$

Then, the total no. of ways $= 1 + 9 + 8 \times 63 + 63 \times 4$

$= 766$

12. (204)

Given,

$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ & $a_{11}, a_{12}, a_{21}, a_{22} \in \{0,1,2,3,4\}$

Now given $a_{11}+a_{12}+a_{23}+a_{22}=$ prime number,

So sum can be, $p=3,5,7,11$

Now using multinomial theorem we get,

Sum will be sum of coefficient of $x^3,\ x^5,\ x^7$ & x^{11}in expansion of $\left(x^0+x^1+x^2+x^3+x^4\right)^4$

$\Rightarrow\left(\frac{1-x^5}{1-x}\right)^4=\left(1-x^5\right)^4(1-x)^{-4}$

$=\left({}^4C_{r_1}\left(-x^5\right)^{r_1}\right)\left({}^{4+r_2-1}C_{r_2}x^{r^2}\right)$

$={}^4C_{r1}^{3+r_2}C_{r2}(-1)^{r_1}(x)^{5r_1+r_2}$

Now taking, $5r_1+r_2=3,5,7,11$

when $5r_1+r_2=3 \Rightarrow r_1=0, r_2=3$

when $5r_1+r_2=5 \Rightarrow r_1=0, r_2=5$ or $r_1=1, r_2=0$

when $5r_1+r_2=7 \Rightarrow r_1=1, r_2=2$ or $r_1=0, r_2=7$

when $5r_1+r_2=11 \Rightarrow r_1=0, r_2=11$ or $r_1=1, r_2=6$ or $r_1=2, r_2=1$

So, sum of all coefficient $={}^4C_0\times{}^6C_3+{}^4C_0{}^8C_5-{}^4C_1{}^3C_0+{}^4C_0{}^{10}C_7-{}^4C_1{}^5C_2+{}^4C_0{}^{14}C_{11}-{}^4C_1{}^9C_6+{}^4C_2{}^4C_1$

$=20+56-4+120-40+364-336+24$

$=204$

13. (50)

Given, $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A=A^{-1}$

So, $A^2=A\cdot A^{-1}=I$

$\Rightarrow\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On comparing both side we get,

$\therefore\ \ a^2+bc=1 \quad \cdots(1)$

$ab+bd=0 \quad \cdots(2)$

$ac+cd=0 \quad \cdots(3)$

$bc+d^2=1 \quad \cdots(4)$

Now equation $(1)-(4)$ gives

$a^2-d^2=0$

$\Rightarrow(a+d)=0$ or $a-d=0$

Case-I

$a+d=0 \Rightarrow(a,d)=(-1,1),(0,0),(1,-1)$

Assuming case $(a)\Rightarrow(a,d)=(-1,1)$

Now from equation (1)

$1+bc=1 \Rightarrow bc=0$

When $b=0,\ \ c=12$ possibilities

When $c=0,\ \ b=12$ possibilities

But $(0,0)$ is repeated

$\therefore\ 2\times 12=24$

So, total case will be $24-1$ (repeated) $=23$ pairs.

case $(b) \Rightarrow(a,d)=(1,-1)\Rightarrow bc=0 \rightarrow 23$ pairs

case $(c)\Rightarrow(a,d)=(0,0)\Rightarrow\ bc=1$

$\Rightarrow\ \ (b,c)=(1,1)$ and $(-1,-1)\rightarrow 2$ pairs

Case-II

When $a=d$

from (2) and (3)

$a\neq 0$ then $b=c=0$

$a^2=1$

$\Rightarrow a=\pm 1=d$

$(a,d)=(1,1),(-1,-1)\rightarrow 2$ pairs

$\therefore$ Total $=23+23+2+2$

$=50$ pairs.

14. (414)

Given Matrix $3 \times 3 = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$

Here we can see we have to find nine elements whose sum is 5

Case 1: When five 1's are there & four O's are there

Now by division & distribution Method

We get $\Rightarrow \frac{9!}{5!4!} = 126 \quad \ldots(\mathrm{i})$

Case 2: when six 1's are there & one $\{-1\}$ & two O's are there.

Again by division & distribution Method

we get $\Rightarrow \frac{9!}{6!2!1!} = 252 \quad \ldots(\mathrm{ii})$

Case 3: When seven 1's are there are two $\{-1\}$ are there,

By division & distribution we get

$\Rightarrow \frac{9!}{7!2!} = 36 \quad \ldots(\mathrm{iii})$

Now adding equation (i)+(ii)+(iii)

We get $126 + 252 + 36 = 414$

So total 414 ways will be there.

15. (2)

Sol $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$|A| = 2$

$1(6-1) - 2(2\alpha - 1) + 3(\alpha - 3) = 2$

$5 - 4\alpha + 2 + 3\alpha - 9 = 2$

$-\alpha - 4 = 0$

$\alpha = -4$

$8|Ad(2Adj(2A))|$

$8\left|Adj\left(2 \times 2^2 Adj(A)\right)\right|$

$8\left|Ady\left(2^3 AdA\right)\right|$

$8\left|2^6 Adj(AdjA)\right|$

$2^3\left(2^4\right)^3|Adj(Ad)|$

$2^3 \cdot 2^{18} \mid \mathrm{Al}^4$

$2^{21} \cdot 2^4 = 2^{25} = \left(2^5\right)^5 = (32)^5$

$n = 5$

$\alpha = -4$

16. (2)

$|A| = m - n$

$4m + n = 22$

$17m + 4n = 93$

$m = 5, n = 2$

$|A| = 3$

$\mid 2 \operatorname{adj}(\operatorname{adj} 5\ \mathrm{A})) \mid = 2^5 |5A|^{16}$

$= 2^5, 5^{80}|A|^{16}$

$= 2^5 \cdot 5^{80} \cdot 3^{16}$

$= 3^{11} \cdot 5^{90} \cdot 6^5$

$a + b + c = 96$

17. (3)

Given $|A| = 2$

Now $\left||A|(\operatorname{adj}(\operatorname{adj} A))^3\right| = \left|2(\operatorname{adj}(\operatorname{adj} A))^3\right|$

$= 2^3 \cdot \left|(\operatorname{adj}(\operatorname{adj} A))^3\right| = 2^3\left(|A|^{2^2}\right)^3$

$= 2^3 \cdot 2^{12} = 2^{15}$

18. (42)

Given $A=\begin{bmatrix}\alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{bmatrix}$

$R_3 \rightarrow R_3+R_1$

$A=\begin{bmatrix}\alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma\end{bmatrix}$

$\Rightarrow|A|=|\alpha+\beta+\gamma|\begin{vmatrix}\alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1\end{vmatrix}$

$\Rightarrow|A|=(\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

We know $|\operatorname{adj} A|=|A|^{n-1}$

$\Rightarrow|\operatorname{adj}(\operatorname{adj} A)|=|A|^{(n-1)^2}$

$\Rightarrow|\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(\operatorname{adj} A)))|=|A|^{(n-1)^4}$

Here $|\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(\operatorname{adj} A)))|=|A|^{2^4}=|A|^{16}$

Given $\frac{|(\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(\operatorname{adj} A))))|}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}}=2^{32}\times 3^{16}$

$\Rightarrow \frac{(\alpha+\beta+\gamma)^{16}(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}}=2^{32}\times 3^{16}$

$\therefore\ (\alpha+\beta+\gamma)^{16}=2^{32}\cdot 3^{16}$

$\Rightarrow (\alpha+\beta+\gamma)^{16}=(12)^{16}$

$\Rightarrow \alpha+\beta+\gamma=12$

$\because\ \alpha,\ \beta,\ \gamma \in N$

$(\alpha-1)+(\beta-1)+(\gamma-1)=9$

Possible number all tuples (α, β, γ) will be ${}^{11}C_2=55$

1 case for $\alpha=\beta=\gamma$ and 12 case when any two of these are equal are also included here but $\alpha \neq \beta \neq \gamma$

Hence, number of distinct tuples (α, β, γ)

$=55-13=42$

19. (4)

Given,

$A=\begin{bmatrix}1 & 2 \\ -1 & 4\end{bmatrix} \Rightarrow|A|=6$

So, $A^{-1}=\frac{\operatorname{adj} A}{|A|}=\frac{1}{6}\begin{bmatrix}4 & -2 \\ 1 & 1\end{bmatrix}=\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6}\end{bmatrix}$

Here, $\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6}\end{bmatrix}=\begin{bmatrix}\alpha & 0 \\ 0 & \alpha\end{bmatrix}+\begin{bmatrix}\beta & 2\beta \\ -\beta & 4\beta\end{bmatrix}$

$\left.\begin{array}{l}\alpha+\beta=\frac{2}{3} \\ \beta=-\frac{1}{6}\end{array}\right\} \Rightarrow \alpha=\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$

Hence, $4(\alpha-\beta)=4(1)=4$

20. (4)

Let $C=A^2-B^2 ;|C| \neq 0$

and $A^5=B^5 \quad \ldots(1)$

$A^3 B^2=A^2 B^3 \quad \ldots(2)$

Subtracting equation (2) from (1), we get $A^5-A^3 B^2=B^5-A^2 B^3$

$\Rightarrow A^3\left(A^2-B^2\right)+B^3\left(A^2-B^2\right)=O$

Post multiplying inverse of A^2-B^2 :

$A^3+B^3=O \Rightarrow|A^3+B^3|=0$

21. (4)

Given $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\ldots\begin{bmatrix}1 & n-1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 78\\0 & 1\end{bmatrix}$

And, we have $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1+2\\0 & 1\end{bmatrix}$

Also, $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}$

$=\begin{bmatrix}1 & 6\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1+2+3\\0 & 1\end{bmatrix}$

$\Rightarrow\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\ldots\begin{bmatrix}1 & n-1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 1+2+3+\ldots+n-1\\0 & 1\end{bmatrix}$

$\Rightarrow\begin{bmatrix}1 & 1+2+3+\ldots+n-1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 78\\0 & 1\end{bmatrix}$

Using the sum of first n natural numbers i.e. $1+2+3+\ldots+n=\frac{n(n+1)}{2}$, we get

$\frac{n(n-1)}{2}=78$

$\Rightarrow n^2-n-156=0$

$\Rightarrow(n-13)(n+12)=0$

$\Rightarrow n=13$ or $n=-12$ (reject as n is a natural number)

$\therefore$ We have to find inverse of $\begin{bmatrix}1 & 13\\0 & 1\end{bmatrix}$

Which can be find by using $A^{-1}=\frac{adjA}{|A|}$

Now, $|A|=\begin{vmatrix}1 & 13\\0 & 1\end{vmatrix}=1-0=1$

$A_{11}=1,\ A_{12}=-13,\ A_{21}=0,\ A_{22}=1$

Hence, $adjA=\begin{bmatrix}1 & -13\\0 & 1\end{bmatrix}$

$\therefore A^{-1}=\begin{bmatrix}1 & -13\\0 & 1\end{bmatrix}$.

22. (17) $PQ=kI$

$|P|.|Q|=k^3$

$\Rightarrow\ |P|=2k\neq 0\Rightarrow P$ is an invertible matrix

$\because\ PQ=kI$

$\therefore\ Q=kP^{-1}I$

$\therefore\ Q=\frac{\text{adj}.P}{2}$

$\because\ q_{23}=-\frac{k}{8}$

$\therefore\ \frac{-(3\alpha+4)}{2}=-\frac{k}{8}\Rightarrow k=4$

$\therefore\ |P|=2k\Rightarrow k=10+6\alpha\quad\ldots$(i)

Put value of k in (i).. we get $\alpha=-1$

$\Rightarrow\alpha^2+k^2=17$

23. (10)

Given,

$X=\begin{bmatrix}1\\1\\1\end{bmatrix}$ & $A=\begin{bmatrix}-1&2&3\\0&1&6\\0&0&-1\end{bmatrix}$

Also given, $X^T A^K X=33$

Now putting the value of matrices in $X^T A^K X=33$ we get,

$[1\ \ 1\ \ 1]\begin{bmatrix}-1&2&3\\0&1&6\\0&0&-1\end{bmatrix}^k\begin{bmatrix}1\\1\\1\end{bmatrix}=33$

Now finding $A^2=\begin{bmatrix}-1&2&3\\0&1&6\\0&0&-1\end{bmatrix}\begin{bmatrix}-1&2&3\\0&1&6\\0&0&-1\end{bmatrix}=\begin{bmatrix}1&0&6\\0&1&0\\0&0&1\end{bmatrix}$

And $A^4=\begin{bmatrix}1&0&6\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&6\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&12\\0&1&0\\0&0&1\end{bmatrix}$

Similary $A^8=\begin{bmatrix}1&0&24\\0&1&0\\0&0&1\end{bmatrix}$

And $A^{10}=\begin{bmatrix}1&0&6\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&24\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&30\\0&1&0\\0&0&1\end{bmatrix}$

So, for $K\to$ Even $A^K=\begin{bmatrix}1&0&3K\\0&1&0\\0&0&1\end{bmatrix}$

Now again putting the value in $X^T A^K X=33$ we get,

$\Rightarrow[1\ \ 1\ \ 1]\begin{bmatrix}1&0&3\,K\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=33$

$\Rightarrow[1\ \ 1\ \ 3\,K+1]\begin{bmatrix}1\\1\\1\end{bmatrix}=33$

$\Rightarrow[3\,K+3]=33$

Now assuming 33 as [33]

We get, $3\,K+3=33\ \Rightarrow K=10$

Now, if K is odd $X^T A^K X=33$

We can rewrite above expression as $X^T A A^{K-1} X=33$

$\Rightarrow[1\ \ 1\ \ 1]\begin{bmatrix}-1&2&3\\0&1&6\\0&0&-1\end{bmatrix}\begin{bmatrix}1&0&3k-3\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=33$

$\Rightarrow[-1\ \ 3\ \ 8]\begin{bmatrix}3k-2\\1\\1\end{bmatrix}=[33]$

$\Rightarrow[-3k+13]=[33]$

$\Rightarrow$ k = 20/3 (not possible)

So, $k=10$ is the required answer.

24. (100)

Given, $X=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$, so $X^2=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$

Now, finding $Y=\alpha l+\beta X+\gamma X^2$ & $Z=\alpha^2 I-\alpha\beta X+\left(\beta^2-\alpha\gamma\right)X^2$ by putting the value of X & X^2 we get,

$Y=\begin{bmatrix}\alpha&\beta&\gamma\\0&\alpha&\beta\\0&0&\alpha\end{bmatrix}$ & $Z=\begin{bmatrix}\alpha^2&-\alpha\beta&\beta^2-\alpha\gamma\\0&\alpha^2&-\alpha\beta\\0&0&\alpha^2\end{bmatrix}$

We know that $Y\cdot Y^{-1}=\mathrm{I}$

$\Rightarrow\begin{bmatrix}\alpha&\beta&\gamma\\0&\alpha&\beta\\0&0&\alpha\end{bmatrix}\begin{bmatrix}\frac{1}{5}&\frac{-2}{5}&\frac{1}{5}\\0&\frac{1}{5}&\frac{-2}{5}\\0&0&\frac{1}{5}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$\Rightarrow\begin{bmatrix}\frac{\alpha}{5}&\frac{-2\alpha}{5}+\frac{\beta}{5}&\frac{\alpha}{5}-\frac{2\beta}{5}+\frac{\gamma}{5}\\0&\frac{\alpha}{5}&\frac{-2\alpha}{5}+\frac{\beta}{5}\\0&0&\frac{\alpha}{5}\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

On comparing L.H.S and R.H.S we get,

$\Rightarrow\frac{\alpha}{5}=1\Rightarrow\alpha=5$

$\Rightarrow-\frac{2}{5}\alpha+\frac{\beta}{5}=0\Rightarrow\beta=10$

$\Rightarrow\frac{\alpha}{5}-\frac{2\beta}{5}+\frac{\gamma}{5}=0\Rightarrow\gamma=15$

So, $(\alpha-\beta+\gamma)^2=(5-10+15)^2=100$

25. (1)

Given,

$A=\begin{pmatrix}2&-2\\1&-1\end{pmatrix}$ So, $A^2=\begin{pmatrix}2&-2\\1&-1\end{pmatrix}\begin{pmatrix}2&-2\\1&-1\end{pmatrix}=\begin{pmatrix}2&-2\\1&-1\end{pmatrix}=A$

Also, $B=\begin{pmatrix}-1&2\\-1&2\end{pmatrix}$

$B^2=\begin{pmatrix}-1&2\\-1&2\end{pmatrix}\begin{pmatrix}-1&2\\-1&2\end{pmatrix}=\begin{pmatrix}-1&2\\-1&2\end{pmatrix}=B$

So, $A+B=\begin{pmatrix}2&-2\\1&-1\end{pmatrix}+\begin{pmatrix}-1&2\\-1&2\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I$

Therefore the equation $nA^n+mB^m=I$ is true for $n=1$ & $m=1$ so, only one set is possible.

26. (5)

Given,

$|A|=2$

Now simplifying,

$\left|\operatorname{Adj}\left(2.\operatorname{Adj}\left(2\ A^{-1}\right)\right)\right|$

$=\left|2\cdot\operatorname{Adj}\left(2\ A^{-1}\right)\right|^{n-1}$

$=\left(2^n\left|\operatorname{Adj}\left(2\ A^{-1}\right)\right|\right)^{n-1}$

$=\left(2^n\left|2A^{-1}\right|^{n-1}\right)^{n-1}$

$=2^{n(n-1)}\left(\left(2^n\left|A^{-1}\right|\right)^{(n-1)}\right)^{(n-1)}$

$\therefore\quad\left|A^{-1}\right|=\frac{1}{|A|}=\frac{1}{2}$

$=2^{n(n-1)}\left(\left(2^{(n-1)}\right)^{(n-1)}\right)^{(n-1)}$

$=2^{n(n-1)+(n-1)^3}=2^{84}$

Now comparing both side we get,

$n(n-1)+(n-1)^3=84$

$\Rightarrow(n-1)\left(n+n^2-2n+1\right)=84$

$\Rightarrow(n-1)\left(n^2-n+1\right)=4\times 21$

Now if $n-1=4\Rightarrow n=5$ now checking $n^2-3n+1=25-5+1=21$

Hence, $n=5$

27. **(4)** $|\text{adjadj}(2A)| = |2A|^{(n-1)^2}$

$= |2\text{ A}|^4$

$= (2^3|\text{ A}|)^4$

$= 2^{12}|\text{ A}|^4 \Rightarrow 2^{16}$

$|A| = \frac{1}{5!6!7!}5!6!\begin{vmatrix} 1 & 6 & 42 \\ 1 & 7 & 56 \\ 1 & 8 & 72 \end{vmatrix}$

$R_3 \rightarrow R_3 \rightarrow R_2$

$R_2 \rightarrow R_2 \rightarrow R_1$

$|A| = \begin{vmatrix} 1 & 8 & 42 \\ 0 & 1 & 14 \\ 0 & 1 & 16 \end{vmatrix} = 2$

28. **(25)**

We have $A = \begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix}$

$A^2 = A \cdot A = \begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix} = \begin{bmatrix} i & 1+i \\ -i+1 & -i \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} i & 1+i \\ 1-i & -i \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ 1 & 1-i \end{bmatrix}$

$A^4 = A^3 \cdot A = \begin{bmatrix} 0 & i \\ 1 & 1-i \end{bmatrix}\begin{bmatrix} 1+i & 1 \\ -i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore \quad A^4 = I$

So, $A^5 = A^4 \cdot A = I \cdot A = A$

$A^6 = A^4 \cdot A^2 = I \cdot A^2 = A^2$ and so on

$\because A^1 = A^5 = A^9 = \ldots\ldots = A^{97} = A$

Hence, possible values of n, such that $A^n = A$

$=\{1, 5, 9, \ldots, 97\}$

Clearly, above sequence is in A.P. where

$a = 1,\ d = 4\ \&\ t_n = 97 \Rightarrow a + (n-1)d = 97$

$\Rightarrow \ 1 + (n-1)4 = 97 \ \Rightarrow \ n = 25$

$\therefore$ The number of elements in the given set $= 25$.

29. **(1)**

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^n = \begin{bmatrix} 1 & 0 & 0 \\ n-1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^{2025} - A^{2020} = \begin{bmatrix} 1 & 0 & 0 \\ 2024 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 2019 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^6 - A$

30. **(1)**

$P = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$

$P^2 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$P^3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix}$

$P^4 = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$\vdots$

$\therefore \ P^{50} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} + 49\left(\frac{1}{2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 25 & 1 \end{bmatrix}$

31. **(3)** $A=\begin{bmatrix}0 & 2q & r\\ p & q & -r\\ p & -q & r\end{bmatrix}$

$\therefore \quad A\cdot A^T=\begin{bmatrix}0 & 2q & r\\ p & q & -r\\ p & -q & r\end{bmatrix}\times\begin{bmatrix}0 & p & p\\ 2q & q & -q\\ r & -r & r\end{bmatrix}$

$=\begin{bmatrix}4q^2+r^2 & 2q^2-r^2 & -2q^2+r^2\\ 2q^2-r^2 & p^2+q^2+r^2 & p^2-q^2-r^2\\ -2q^2+r^2 & p^2-q^2-r^2 & p^2+q^2+r^2\end{bmatrix}$

Given, $AA^T=I$

$\therefore \quad 4q^2+r^2=p^2+q^2+r^2=1$

$\Rightarrow \quad p^2-3q^2=0$ and $r^2=1-4q^2$

and $2q^2-r^2=0\Rightarrow r^2=2q^2$

$\therefore \quad p^2=\frac{1}{2}, q^2=\frac{1}{6}$ and $r^2=\frac{1}{3}$

$\therefore \quad |p|=\frac{1}{\sqrt{2}}$.

32. **(414)**

Given Matrix $3\times 3=\begin{bmatrix}- & - & -\\ - & - & -\\ - & - & -\end{bmatrix}$

Here we can see we have to find nine elements whose sum is 5

Case 1: When five 1's are there & four O's are there

Now by division & distribution Method

We get $\Rightarrow \frac{9!}{5!4!}=126 \quad \ldots$(i)

Case 2: when six 1's are there & one $\{-1\}$ & two O's are there.

Again by division & distribution Method

we get $\Rightarrow \frac{9!}{6!2!1!}=252 \quad \ldots$(ii)

Case 3: When seven 1's are there are two $\{-1\}$ are there,

By division & distribution we get

$\Rightarrow \frac{9!}{7!2!}=36 \quad \ldots$(iii)

Now adding equation (i)+(ii)+(iii)

We get $126+252+36=414$

So total 414 ways will be there.

33. **(16)**

.$|A|=ad-bc=15$

where $a,b,c,d\in\{\pm 3,\pm 2,\pm 1,0\}$

Case I : $ad=9$ & $bc=-6$

For ad possible pairs are $(3,3),(-3,-3)$.

For bc possible pairs are $(3,-2),(-3,2),(-2,3),(2,-3)$

So, total number of matrices in case I$=2\times 4=8$

Case II : $ad=6$ & $bc=-9$

Similarly, total number of matrices in case II $=2\times 4=8$

Hence, total number of matrices are 16.

34. (4)

$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 3 & 0 & -1\end{bmatrix}$

$A^2=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}, A^3=\begin{bmatrix}1 & 0 & 0\\ 0 & 8 & 0\\ 3 & 0 & -1\end{bmatrix}$

$A^4=\begin{bmatrix}1 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 1\end{bmatrix}$

Hence,

$A^{20}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2^{20} & 0\\ 0 & 0 & 1\end{bmatrix}, A^{19}=\begin{bmatrix}1 & 0 & 0\\ 0 & 2^{19} & 0\\ 3 & 0 & -1\end{bmatrix}$

So $A^{20}+\alpha A^{19}+\beta A=\begin{bmatrix}1+\alpha+\beta & 0 & 0\\ 0 & 2^{20}+\alpha .2^{19}+2\beta & 0\\ 3\alpha+3\beta & 0 & 1-\alpha-\beta\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$

Therefore $\alpha+\beta=0$ and $2^{20}+2^{19}\alpha-2\alpha=4$

$\Rightarrow \alpha=\frac{4\left(1-2^{18}\right)}{2\left(2^{18}-1\right)}=-2$

Hence $\beta=2$

So, $(\beta-\alpha)=4$

35. (3)

Given $A=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

Now $A^2=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=I$

i.e. $A^{98}=I; A^{49}=A$

$\therefore B_0=A^{98}+2A^{49}=I+2A=\begin{bmatrix}1 & 2 & 0\\ 2 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}$

Given $B_n=\operatorname{adj}(B_{n-1})$

$\therefore B_4=\operatorname{adj}(B_3)=\operatorname{adj}(\operatorname{adj} B_2)=\operatorname{adj}(\operatorname{adj}(\operatorname{adj} B_1))$

$=\operatorname{adj}(\operatorname{adj}(\operatorname{adj}(\operatorname{adj} B_0)))$

$\therefore |B_4|=|B_0|^{2^4}=|B_0|^{16}$

Now $|B_0|=\begin{vmatrix}1 & 2 & 0\\ 2 & 1 & 0\\ 0 & 0 & 3\end{vmatrix}=-9$

Hence, $|B_4|=|B_0|^{16}=9^{16}=3^{32}$