ANSWER KEYS

1. (2)	**2.** (1)	**3.** (06.00)	**4.** (2)	**5.** (4)	**6.** (32)	**7.** (1)	**8.** (2)
9. (63)	**10.** (4)	**11.** (3)	**12.** (575)	**13.** (14)	**14.** (1)	**15.** (3)	**16.** (4)
17. (2)	**18.** (512)	**19.** (1)	**20.** (21)	**21.** (2)	**22.** (8)	**23.** (4)	**24.** (2)
25. (385)	**26.** (3)	**27.** (1)	**28.** (2)	**29.** (5)	**30.** (34)		

1. (2)

Given,

$I=\int_0^{\frac{\pi}{2}} \frac{dx}{3+2\sin x+\cos x}$

$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{dx}{3+2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$

$\Rightarrow I=\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}\cdot dx}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}$

Put $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=dt$, so

$\Rightarrow I=\int_0^1 \frac{dt}{(t+1)^2+1}$

$\Rightarrow I=\tan^{-1}(t+1)\Big|_0^1$

$\Rightarrow I=\tan^{-1}2-\frac{\pi}{4}$

2. (1)

Let $I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^{\frac{2}{3}}x\cdot \operatorname{cosec}^{\frac{4}{3}}x\,dx$

$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{\cos^{\frac{2}{3}}x\cdot\sin^{\frac{4}{3}}x}\right)dx$

$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{\cos^{\frac{4}{3}}x}{\cos^{\frac{2}{3}+\frac{4}{3}}x\cdot\sin^{\frac{4}{3}}x}\right)dx$

$\Rightarrow I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{\sec^2 x}{\tan^{\frac{4}{3}}x}\right)dx$

Let $\tan x=t$, $\sec^2 x dx=dt$ and at $x=\frac{\pi}{6}$, $t=\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}$ and at $x=\frac{\pi}{3}$, $t=\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$

$\Rightarrow I=\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\frac{dt}{t^{\frac{4}{3}}}$

Using $\int x^n dx=\frac{x^{n+1}}{n+1}$

$\Rightarrow I=\left[\frac{t^{-\frac{4}{3}+1}}{-\frac{4}{3}+1}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$\Rightarrow I=\left[-3t^{-\frac{1}{3}}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$\Rightarrow I=-3\left\{\left(\sqrt{3}\right)^{-\frac{1}{3}}-\left(\frac{1}{\sqrt{3}}\right)^{-\frac{1}{3}}\right\}$

$\Rightarrow I=-3\left\{\left(3^{\frac{1}{2}}\right)^{-\frac{1}{3}}-\left(\frac{1}{3^{\frac{1}{2}}}\right)^{-\frac{1}{3}}\right\}$

$=-3\left(\frac{1}{3^{\frac{1}{6}}}-3^{\frac{1}{6}}\right)$

$=\frac{-3}{3^{\frac{1}{6}}}+3\times 3^{\frac{1}{6}}$

$=3^{\frac{7}{6}}-3^{\frac{5}{6}}$.

3. (06.00)

Given that $\int_0^{2.4}[x^2]dx=\alpha+\beta\sqrt{2}+\gamma\sqrt{3}+\delta\sqrt{5}$, then $\alpha+\beta+\gamma+\delta$

We know that for $1\le x<2$, $[x]=1 \Rightarrow \int_0^{2.4}[x^2]dx=\int_0^1 0dx+\int_1^{\sqrt{2}}1dx+\int_{\sqrt{2}}^{\sqrt{3}}2dx+\int_{\sqrt{3}}^{2}3dx+\int_2^{\sqrt{5}}4dx+\int_{\sqrt{5}}^{2.4}5dx$

$=0+\left(\sqrt{2}-1\right)+2\left(\sqrt{3}-\sqrt{2}\right)+3\left(2-\sqrt{3}\right)+4\left(\sqrt{5}-2\right)+5\left(2.4-\sqrt{5}\right)$

$=9-\sqrt{2}-\sqrt{3}-\sqrt{5}$

This is in the form of $\alpha+\beta\sqrt{2}+\gamma\sqrt{3}+\delta\sqrt{5}$

$\alpha=9,\ \beta=-1,\ \gamma=-1,\ \delta=-1$

$\Rightarrow \alpha+\beta+\gamma+\delta=9-1-1-1=6$

Hence this is the required answer.

4. **(2)**

Given $I=\int\limits_0^\pi \frac{\sin x \cdot e^{\cos x}}{\left(1+\cos^2 x\right)\left(e^{\cos x}+e^{-\cos x}\right)} d x \quad \ldots$(i)

Using property of integral $\int\limits_a^b f(a+b-x)=\int\limits_a^b f(x)$

we get $I=\int\limits_0^\pi \frac{\sin x \cdot e^{-\cos x}}{\left(1+\cos^2 x\right)\left(e^{-\cos x}+e^{\cos x}\right)} d x \quad \ldots$(ii)

Adding equation (i) & (ii) we get

$2 I=\int\limits_0^\pi \frac{\sin x}{\left(1+\cos^2 x\right)} \cdot \frac{\left(e^{\cos x}+e^{-\cos x}\right)}{e^{\cos x}+e^{-\cos x}} d x$

$2 I=\int\limits_0^\pi \frac{\sin x}{1+\cos^2 x} d x \quad \Rightarrow 2 I=2 \int\limits_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} d x$

Let $\cos x=t \Rightarrow-\sin x d x=d t$

$I=-\int\limits_1^0 \frac{d t}{1+t^2} \Rightarrow I=-\left[\tan^{-1} t\right]_1^0=-\left[0-\frac{\pi}{4}\right]=\frac{\pi}{4}$

5. **(4)**

Let $I=\int_0^{20 \pi}(|\sin x|+|\cos x|)^2 d x$

We know that the period of $|\sin x|+|\cos x|$ is $\frac{\pi}{2}$

i.e. $I=40 \int_0^{\frac{\pi}{2}}(\sin x+\cos x)^2 d x$

$I=40 \int_0^{\frac{\pi}{2}}(1+\sin 2 x) d x$

$=40\left(x-\frac{\cos 2 x}{2}\right)_0^{\frac{\pi}{2}}=40\left(\frac{\pi}{2}-\frac{\cos \pi}{2}+\frac{\cos 0}{2}\right)$

$I=20[\pi+2]$

6. **(32)**

$\alpha(\mathrm{m}, \mathrm{n})=\int_0^2 \mathrm{t}^{\mathrm{m}}(1+3 \mathrm{t})^{\mathrm{n}} \mathrm{dt}$

If $11 \alpha(10,6)+18 \alpha(11,5)=\mathrm{p}(14)^6$ then P

$=11 \int_0^2 \underset{\mathrm{II}}{\mathrm{t}^{10}} \underset{\mathrm{I}}{(1+3 \mathrm{t})^6}+10 \int_0^2 \mathrm{t}^{11}(1+3 \mathrm{t})^5 \mathrm{dt}$

$=11\left[(1+3 \mathrm{t})^6 \cdot \frac{\mathrm{t}^{11}}{11}-\int 6(1+3 \mathrm{t})^5 \cdot 3 \frac{\mathrm{t}^{11}}{11}\right]_0^2+18 \int_0^2 \mathrm{t}^{11}(1+3 \mathrm{t})^5 \mathrm{dt}$

$=\left(\mathrm{t}^{11}(1+3 \mathrm{t})^6\right)_0^2$

$=2^{11}(7)^6$

$=2^5(14)^6$

$=32(14)^6$

7. **(1)** $I=\int_{-\ln 2}^{\ln 2} e^x\left(\ln \left(e^x+\sqrt{1+e^{2 x}}\right)\right) d x$

Put $\mathrm{e}^{\mathrm{x}}=\mathrm{t} \Rightarrow \mathrm{e}^{\mathrm{x}} \mathrm{dx}=\mathrm{dt}$

$I=\int_{1/2}^2 \ln \left(t+\sqrt{1+t^2}\right) d t$

Applying integration by parts.

$$=\left[\mathrm{t} \ln \left(\mathrm{t}+\sqrt{1+\mathrm{t}^2}\right)\right]_{\frac{1}{2}}^2-\int_{1/2}^2 \frac{t}{t+\sqrt{1+t^2}}\left(1+\frac{2 \mathrm{t}}{2 \sqrt{1+t^2}}\right) d t$$

$$=2 \ln (2+\sqrt{5})-\frac{1}{2} \ln \left(\frac{1+\sqrt{5}}{2}\right)-\int_{1/2}^2 \frac{t}{\sqrt{1+t^2}} d t$$

$$=2 \ln (2+\sqrt{5})-\frac{1}{2} \ln \left(\frac{1+\sqrt{5}}{2}\right)-\frac{\sqrt{5}}{2}$$

$$=\ln \left(\frac{(2+\sqrt{5})^2}{\left(\frac{\sqrt{5}+1}{2}\right)^{\frac{1}{2}}}\right)-\frac{\sqrt{5}}{2}$$

8. **(2)**

Given,

$g(x)=\int_x^{\frac{\pi}{4}}(f'(t)\sec t+\tan t\sec t f(t))dt$

$\Rightarrow g(x)=\int_x^{\frac{\pi}{4}} d(f(t)\cdot\sec t)\Rightarrow g(x)=[f(t)\sec t]_x^{\frac{\pi}{4}}$

$g(x)=f\left(\frac{\pi}{4}\right)\sec\frac{\pi}{4}-f(x)\cdot\sec x$

$g(x)=2-f(x)\sec x=2-\left(\frac{f(x)}{\cos x}\right)$

Now taking limit both side, we get

$\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-} g(x)=2-\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{f(x)}{\cos x}\right)$

Using L'Hospital Rule

$=2-\lim\limits_{x\to\left(\frac{\pi}{2}\right)^-}\frac{f'(x)}{(-\sin x)}$

$=2+\frac{f'\left(\frac{\pi}{2}\right)}{\sin\frac{\pi}{2}}=2+\frac{1}{1}=3$

9. **(63)**

Let

$I=\int(x^{20}+x^{13}+x^6)(2x^{21}+3x^{14}+6x^7)^{\frac{1}{7}}dx$

Put

$2x^{21}+3x^{14}+6x^7=t$

$\Rightarrow 42(x^{20}+x^{13}+x^6)dx=dt$

$\therefore I=\frac{1}{42}\int_0^{11}t^{\frac{1}{7}}dt$

$\Rightarrow I=\frac{1}{42}\left[\frac{7t^{\frac{8}{7}}}{8}\right]_0^{11}$

$\Rightarrow I=\frac{1}{48}(11)^{\frac{8}{7}}$

So,

$l=48,\ m=8, n=7$

$\therefore l+m+n=63$

10. **(4)**

Given,

$I=16\int_1^2\frac{dx}{x^3(x^2+2)^2}$

$\Rightarrow I=16\int_1^2\frac{dx}{x^3x^4\left(1+\frac{2}{x^2}\right)^2}$

Now let $1+\frac{2}{x^2}=t\Rightarrow\frac{-4}{x^3}dx=dt$

Then, $I=-4\int_3^{\frac{3}{2}}\frac{dt}{\left(\frac{2}{t-1}\right)^2t^2}$

$\Rightarrow I=-4\int_3^{\frac{3}{2}}\left(\frac{t-1}{2}\right)^2\frac{dt}{t^2}$

$\Rightarrow I=-\frac{4}{4}\int_3^{\frac{3}{2}}\left(1-\frac{2}{t}+\frac{1}{t^2}\right)\text{dt}$

$\Rightarrow I=-1\left[\text{t}-2\ln|t|-\frac{1}{t}\right]_3^{\frac{3}{2}}$

$\Rightarrow I=-1\left[\left(\frac{3}{2}-2\ln\frac{3}{2}-\frac{2}{3}\right)-\left(3-2\ln 3-\frac{1}{3}\right)\right]$

$\Rightarrow I=-1\left[2\ln 2-\frac{11}{6}\right]$

$\Rightarrow I=\frac{11}{6}-\ln 4$

11. **(3)**

Let $\text{y}=(\text{ex})^{\text{x}}$

$\ln\text{y}=x[1+\ln\text{x}]$

$\frac{1}{\text{y}}\frac{\text{dy}}{\text{dx}}=(2+\ln\text{x})$

$\Rightarrow\text{dy}=(\text{ex})^{\text{x}}(2+\ln\text{x})\ \text{dx}$

$\int\limits_1^2\text{e}^{\text{x}}.\text{x}^{\text{x}}(2+\log_{\text{e}}\text{x})\text{dx}$

$=(\text{y})_1^2$

$=((\text{ex})^{\text{x}})_1^2$

$=4\text{e}^2-\text{e}$

12. (575) $\int_{-0.15}^{0.15}|100x^2-1|\,dx=2\int_0^{0.15}|100x^2-1|\,dx$

Now $100x^2-1=0\Rightarrow x^2=\frac{1}{100}\Rightarrow x=0.1$

$I=2\left[\int_0^{0.1}(1-100x^2)\,dx+\int_{0.1}^{0.15}(100x^2-1)\,dx\right]$

$I=2\left[x-\frac{100}{3}x^3\right]_0^{0.1}+2\left[\frac{100x^3}{3}-x\right]_{0.1}^{0.15}$

$=2\left[0.1-\frac{0.1}{3}\right]+2\left[\frac{0.3375}{3}-0.15-\frac{0.1}{3}+0.1\right]$

$=2\left[0.2-\frac{0.2}{3}+0.1125-0.15\right]$

$=2\left[\frac{5}{100}-\frac{2}{30}+\frac{1125}{10000}\right]=2\left(\frac{1500-2000+3375}{30000}\right)$

$=\frac{575}{3000}\Rightarrow \text{k}=575$

13. (14)

Let,

$I=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}(8[\operatorname{cosec}x]-5[\cot x])dx$

$\Rightarrow I=8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\operatorname{cosec}x]dx-5\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\cot x]dx$

$\Rightarrow I=8I_1-5I_2$, where $I_1=8\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\operatorname{cosec}x]dx$ & $I_2=5\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\cot x]dx$

Now solving,

$I_1=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\operatorname{cosec}x]dx$

$\Rightarrow I_1=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}1dx=\frac{2\pi}{3}$

As when $x\in\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$, $\operatorname{cosec}x\in\left[1,2\right)$,

So, $[\operatorname{cosec}x]=1$

Now solving,

$I_2=\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}[\cot\ x]dx$

$\Rightarrow I_2=\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}1dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}0dx+\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}(-1)dx+\int_{\frac{3\pi}{4}}^{\frac{5\pi}{6}}(-2)dx$

$\Rightarrow I_2=\left(\frac{\pi}{4}-\frac{\pi}{6}\right)-\left(\frac{3\pi}{4}-\frac{\pi}{2}\right)-2\left(\frac{5\pi}{6}-\frac{3\pi}{4}\right)$

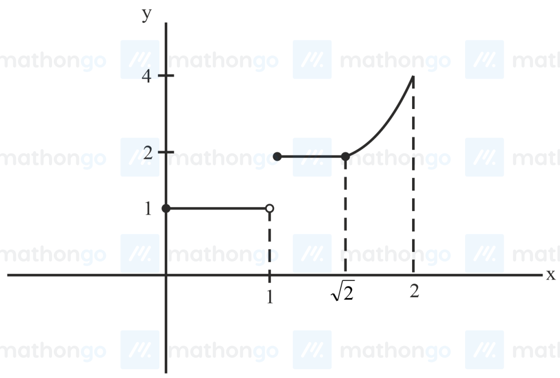
$\Rightarrow I_2=-\frac{\pi}{3}$

Required value will be,

$\frac{2}{\pi}I=\frac{2}{\pi}\left[8\times\frac{2\pi}{3}+5\times\frac{\pi}{3}\right]=14$

14. (1)

Plotting the diagram of the given function $\text{f}(\text{x})=\max\{\text{x}^2,1+[x]\}$ we get,



Now when $x\in[0,1)\Rightarrow f(x)=1$

When $x\in\left[1,\sqrt{2}\right)\Rightarrow f(x)=2$

And when $x\in\left[\sqrt{2},2\right]\Rightarrow f(x)=x^2$

So, from above diagram area can be calculated as,

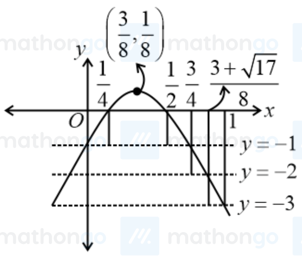
$A=\int_0^1 1.\,dx+\int_1^{\sqrt{2}}2dx+\int_{\sqrt{2}}^2 x^2dx$

$\Rightarrow A=1+2\sqrt{2}-2+\frac{8}{3}-\frac{2\sqrt{2}}{3}$

$\Rightarrow A=\frac{5}{3}+\frac{4\sqrt{2}}{3}$

15. (3)

The graph of $y=-8x^2+6x-1$ will be as shown



Now $\int_0^1\left[-8x^2+6x-1\right]dx = \int_0^{\frac{1}{4}}(-1)dx+\int_{\frac{1}{4}}^{\frac{1}{2}}(0)dx+\int_{\frac{1}{2}}^{\frac{3}{4}}(-1)dx$

$+\int_{\frac{3}{4}}^{\frac{3+\sqrt{17}}{8}}(-2)dx+\int_{\frac{3+\sqrt{17}}{8}}^{1}(-3)dx$

$=-[x]_0^{\frac{1}{4}}+0-[x]_{\frac{1}{2}}^{\frac{3}{4}}+-2[x]_{\frac{3}{4}}^{\frac{3+\sqrt{17}}{8}}-3[x]_{\frac{3+\sqrt{17}}{8}}^{1}$

$=-\left(\frac{1}{4}-0\right)-\left(\frac{3}{4}-\frac{1}{2}\right)-2\left(\frac{3+\sqrt{17}}{8}-\frac{3}{4}\right)-3\left(1-\frac{3+\sqrt{17}}{8}\right)$

$=-\frac{1}{4}-\frac{1}{4}+\frac{-6-2\sqrt{17}}{8}+\frac{3}{2}-3+\frac{9+3\sqrt{17}}{8}$

$=\frac{\sqrt{17}-13}{8}$

16. (4)

Given,

$f(x)=\frac{\left|x^3+x\right|}{\left(e^{x|x|}+1\right)}dx$

and $\int_{-2}^{2}f(x)dx=\int_0^2(f(x)+f(-x))dx$

$=\int_0^2\left(\frac{\left|x^3+x\right|}{\left(e^{x|x|}+1\right)}+\frac{\left|-x^3-x\right|}{\left(e^{-x|-x|}+1\right)}\right)dx$

$=\int_0^2\left(\frac{\left|x^3+x\right|}{\left(e^{x|x|}+1\right)}+\frac{\left|x^3+x\right|}{\left(e^{-x|x|}+1\right)}\right)dx$

$=\int_0^2\left(\frac{x^3+x}{\left(e^{x^2}+1\right)}+\frac{x^3+x}{\left(e^{-x^2}+1\right)}\right)dx$

$I=\int_0^2\left(\frac{x^3+x}{1+e^{x^2}}+\frac{e^{x^2}\left(x^3+x\right)}{1+e^{x^2}}\right)dx$

$=\int_0^2\left(x^3+x\right)dx$

$=\left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^2$

$=4+2=6$

17. (2)

$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{\left(1+e^{x\cos x}\right)\left(\sin^4x+\cos^4x\right)}\quad\ldots(1)$

Using $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$I=\int_{-\pi/4}^{\pi/4}\frac{dx}{\left(1+e^{-x\cos x}\right)\left(\sin^4x+\cos^4x\right)}\quad\ldots(2)$

Add (1) and (2)

$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{\sin^4x+\cos^4x}$

$\Rightarrow 2I=2\int_0^{\frac{\pi}{4}}\frac{dx}{\sin^4x+\cos^4x}$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\left(1+\tan^2x\right)\sec^2x}{\tan^4x+1}dx$

$\Rightarrow I=\int_0^{\frac{\pi}{4}}\frac{\left(1+\frac{1}{\tan^2x}\right)\sec^2x}{\left(\tan x-\frac{1}{\tan x}\right)^2+2}dx$

Put $\tan x-\frac{1}{\tan x}=t$

$\Rightarrow\left(1+\frac{1}{\tan^2x}\right)\sec^2x\,dx=dt$

So, $I=\int_{-\infty}^{0}\frac{dt}{t^2+2}=\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right]_{-\infty}^{0}$

$\Rightarrow I=0-\frac{1}{\sqrt{2}}\left(-\frac{\pi}{2}\right)=\frac{\pi}{2\sqrt{2}}$

18. (512)

$I=\int_{-4}^{4} f(x^2)dx=2\int_0^4 f(x^2)dx$ {Even function}

$=2\int_0^4 (4x^3-g(4-x))dx$

$=2\left(\frac{4x^4}{4}\Big|_0^4-\int_0^4 g(4-x)dx\right)=512-2I_1$

$I_1=\int_0^4 g(4-x)dx=\int_0^2 g(4-x)dx+\int_2^4 g(4-x)dx$

Let $I_2=\int_2^4 g(4-x)dx$

If $4-x=t$ then

$I_2=-\int_2^0 g(t)dt=\int_0^2 g(t)dt=\int_0^2 g(x)dx$

So, $I_1=\int_0^2 g(4-x)dx+\int_0^2 g(x)dx=0$

Hence, $I=512$

19. (1)

Let $I=\int_0^{10}\frac{[\sin 2\pi x]}{e^{x-[x]}}dx=\int_0^{10}\frac{[\sin 2\pi x]}{e^{\{x\}}}dx$, where $x-[x]=\{x\}$ is the fractional part function.

We know that the $\{x\}$ is a periodic function with period 1 and also that $\sin 2\pi x$ is a periodic function with period $\frac{2\pi}{2\pi}=1$.

Thus, the function $f(x)=\frac{[\sin 2\pi x]}{e^{\{x\}}}$ is periodic with period 1.

If $f(x)$ is a periodic function with period T then $\int_0^{nT} f(x)dx=n\int_0^T f(x)dx,\ n\in N$

Therefore $I=10\int_0^1\frac{[\sin 2\pi x]}{e^{\{x\}}}dx$

$\Rightarrow I=10\int_0^1\frac{[\sin 2\pi x]}{e^x}dx$

Now, for $0\le x\le 1\ \Rightarrow -1\le \sin 2\pi x\le 1$, hence $[\sin 2x]$ can take values 0 and -1

$\Rightarrow I=10\left(\int_0^{1/2}\frac{[\sin 2\pi x]}{e^x}dx+\int_{1/2}^1\frac{[\sin 2\pi x]}{e^x}dx\right)$

$\Rightarrow I=10\left(0+\int_{1/2}^1\frac{(-1)}{e^x}dx\right)$

$\Rightarrow I=-10\int_{1/2}^1 e^{-x}dx$

$\Rightarrow I=10[e^{-x}]_{1/2}^1$

$\Rightarrow I=10(e^{-1}-e^{-1/2})$

Given, $I=\alpha e^{-1}+\beta e^{-\frac{1}{2}}+\gamma$, thus $\alpha=10,\ \beta=-10,\ \gamma=0.$

And, $\alpha+\beta+\gamma=10-10=0.$

20. (21)

$\int_0^n\{x\}dx=n\int_0^1 xdx=n\left(\frac{x^2}{2}\right)_0^1=\frac{n}{2}$

and $\int_0^n[x]dx=\int_0^n(x-\{x\})dx=\left(\frac{x^2}{2}\right)_0^n-\int_0^n\{x\}dx=\frac{n^2}{2}-\frac{n}{2}$

Now, $\frac{n}{2},\frac{n^2-n}{2}$ and $10(n^2-n)$ are in Geometric progression

So, $\left(\frac{n^2-n}{2}\right)^2=\frac{n}{2}\cdot 10(n^2-n)$

$\Rightarrow \frac{n^2(n-1)^2}{4}=5.n^2\left(n-1\right)$

$\Rightarrow n-1=20\Rightarrow n=21\ (\because n\neq 1)$

21. (2)

Given, $\int_{-3}^{101}\left([\sin \pi x]+e^{[\cos 2\pi x]}\right)dx$

Now let $I_1 = \int_{-3}^{101}[\sin \pi x]dx$ and $I_2 = \int_{-3}^{101} e^{[\cos 2\pi x]}dx$

Now checking periodicity of $[\sin \pi x]$ we get,

$0 < x < 1 \Rightarrow [\sin \pi x] = 0$

$1 < x < 2 \Rightarrow [\sin \pi x] = -1$

$[\sin \pi x] \rightarrow$ periodic with period 2

So, $I_1 = 52\int_0^2 [\sin \pi x]dx$

$= 52\left[\int_0^1 0 + \int_1^2 -1dx\right] = -52$

Now checking periodicity of $[\cos 2\pi x]$ we get,

$0 < x < \frac{1}{4} \Rightarrow [\cos 2\pi x] = 0$

$\frac{1}{4} < x < \frac{1}{2} \Rightarrow [\cos 2\pi x] = -1$

$\frac{1}{2} < x < \frac{3}{4} \Rightarrow [\cos 2\pi x] = -1$

$\frac{3}{4} < x < 1 \Rightarrow [\cos 2\pi x] = 0$

So, $[\cos 2\pi x]$ has periodicity of 1

So, $I_2 = 104\left(\int_0^{\frac{1}{4}} e^0 dx + \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-1}dx + \int_{\frac{3}{4}}^{1} e^0 dx\right)$

$\Rightarrow I_2 = 104\left(\frac{1}{4} + \frac{1}{e} \times \frac{1}{2} + \frac{1}{4}\right)$

$\Rightarrow I_2 = 52 + \frac{52}{e}$

So, $I_1 + I_2 = \frac{52}{e}$ or $\int_{-3}^{101}\left([\sin \pi x]+e^{[\cos 2\pi x]}\right)dx = \frac{52}{e}$

22. (8)

Given,

$f(x)+\int_0^x (x-t)f'(t)dt = \left(e^{2x}+e^{-2x}\right)\cos 2x + \frac{2x}{a} \quad \cdots$(i)

Putting $x = 0$ both side we get, $f(0) = 2 \quad \cdots$(ii)

On differentiating equation (i) w.r.t. x we get:

$f'(x)+\int_0^x f'(t)dt + xf'(x) - xf'(x) = 2\left(e^{2x}-e^{-2x}\right)\cos 2x - 2\left(e^{2x}+e^{-2x}\right)\sin 2x + \frac{2}{a}$

$\Rightarrow f'(x)+f(x)-f(0) = 2\left(e^{2x}-e^{-2x}\right)\cos 2x - 2\left(e^{2x}+e^{-2x}\right)\sin 2x + \frac{2}{a}$

Replace x by 0 we get:

$\Rightarrow 4 = \frac{2}{a} \Rightarrow a = \frac{1}{2}$

Now putting the value of a in$(2a+1)^5 \cdot a^2$

We get, $2^5 \cdot \frac{1}{2^2} = 2^3 = 8$

23. (4)

Given the function $f(x) = x + \int_0^{\frac{\pi}{2}} \sin x \cdot \cos y f(y)dy$

$f(x) = x + k\sin x \quad \ldots(i)$

$f(y) = y + k\sin y \quad \ldots(ii)$

$k = \int_0^{\frac{\pi}{2}} \cos y f(y)dy$

From equations (i) & (ii)

$f(x) = x + \int_0^{\frac{\pi}{2}} \sin x \cos y(y + k\sin y)dy$

$f(x) = x + \sin x\int_0^{\frac{\pi}{2}} y\cos ydy + \frac{k}{2}\sin x\int_0^{\frac{\pi}{2}} \sin 2ydy$

$f(x) = x + \sin x\left[y\int_0^{\frac{\pi}{2}} \cos ydy - \int_0^{\frac{\pi}{2}}\left\{\frac{d}{dy}y\int_0^{\frac{\pi}{2}} \cos ydy\right\}dy\right] + \frac{k}{2}\sin x\int_0^{\frac{\pi}{2}} \sin 2ydy$

$f(x) = x + \sin x\left[\frac{\pi}{2} - 1\right] + \frac{k}{2}\sin x \times (1)$

$f(x) = x + \frac{(\pi-2)\sin x}{2} + \frac{k\sin x}{2} \quad \ldots(iii)$

from (i) and (iii)

$k = \frac{\pi}{2} - 1 + \frac{k}{2}$

$k = \pi - 2$

Hence, $f(x) = x + (\pi-2)\sin x$

24. (2)

$I=\int_0^1 \frac{dx}{7^{\left[\frac{1}{x}\right]}}=\int_0^1\left(\frac{1}{7}\right)^{\left[\frac{1}{x}\right]} dx$

Let $\frac{1}{7}=k \Rightarrow I=\int_0^1 k^{\left[\frac{1}{x}\right]} dx$

$I=\int_0^1 k^{\left[\frac{1}{x}\right]} dx$

$\Rightarrow I=\int_{\frac{1}{2}}^1 k^{\left[\frac{1}{x}\right]} dx+\int_{\frac{1}{3}}^{\frac{1}{2}} k^{\left[\frac{1}{x}\right]} dx+\int_{\frac{1}{4}}^{\frac{1}{3}} k^{\left[\frac{1}{x}\right]} dx+\cdots$

$\Rightarrow \quad I=k\left(1-\frac{1}{2}\right)+k^2\left(\frac{1}{2}-\frac{1}{3}\right)+k^3\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$

$\Rightarrow I=\left(k+\frac{k^2}{2}+\frac{k^3}{3}+\cdots\right)-\frac{1}{k}\left(\frac{k^2}{2}+\frac{k^3}{3}+\cdots\right)$

$\Rightarrow I=-\ln(1-k)-\frac{1}{k}[-\ln(1-k)-k]$

$\Rightarrow \quad I=-\ln\frac{6}{7}-7\left[-\ln\frac{6}{7}-\frac{1}{7}\right]\left(\because k=\frac{1}{7}\right)$

$\therefore I=1+6\ln\frac{6}{7}$

25. (385)

Given, $f(x)=\min\{[x-1],[x-2],\ldots,[x-10]\}$

$\Rightarrow f(x)=[x]-10$

Now finding $\int_0^{10} f(x)\cdot dx=-10-9-8-\ldots\ldots-1$

$=-\frac{10\cdot 11}{2}=-55$

Finding $\int_0^{10}\left(f(x)^2\right)dx=10^2+9^2+8^2+\ldots+1^2$

$=\frac{10\cdot 11\cdot 21}{6}=385$

And $\int_0^{10}|f(x)|=10+9+8+\ldots.+1$

$=\frac{10\cdot 11}{2}=55$

So, $\int_0^{10} f(x)dx+\int_0^{10}(f(x))^2dx+\int_0^{10}|f(x)|dx$

$=-55+385+55=385$

26. (3)

$I=\int_0^{10}[x]\cdot e^{[x]-x+1}$

$I=\int_0^1 0dx+\int_1^2 1\cdot e^{2-x}+\int_2^3 2\cdot e^{3-x}+\ldots\ldots+\int_9^{10} 9\cdot e^{10-x}dx$

$\Rightarrow I=\sum_{n=0}^{9}\int_n^{n+1} n\cdot e^{n+1-x}dx$

$=-\sum_{n=0}^{9} n\left(e^{n+1-x}\right)_n^{n+1}$

$=-\sum_{n=0}^{9} n\cdot\left(e^0-e^1\right)$

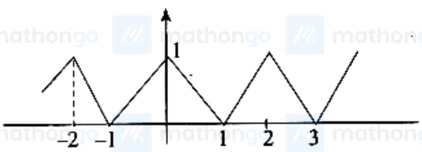
$=(e-1)\sum_{n=0}^{9} n$

$=(e-1)\cdot\frac{9\cdot 10}{2}$

$=45(e-1)$

27. (1)

$f(x)=\begin{cases} x-1, & 1\le x<2 \\ 1-x, & 0\le x<1 \end{cases}$



$f(x)$ is periodic with period 2, and is even.

$\therefore I=\int_{-10}^{10} f(x)\cos\pi x\,dx$

$=2\int_0^{10} f(x)\cos\pi x\,dx$

$=2\times 5\int_0^2 f(x)\cos\pi x\,dx$

$=10\left[\int_0^1(1-x)\cos\pi x\,dx+\int_1^2(x-1)\cos\pi x\,dx\right]=10(I_1+I_2)$

$I_2=\int_1^2(x-1)\cos\pi x\,dx$

Put $(x-1)=t$

$I_2=-\int_0^1 t\cos\pi t dt$

$I_1=\int_0^1(1-x)\cos\pi x\,dx=-\int_0^1 x\cos(\pi x)\,dx$

$\therefore I=10\left[-2\int_0^1 x\cos\pi x\,dx\right]=-20\left[x\frac{\sin\pi x}{\pi}+\frac{\cos\pi x}{\pi^2}\right]_0^1$

$=-20\left[-\frac{1}{\pi^2}-\frac{1}{\pi^2}\right]=\frac{40}{\pi^2}$

$\therefore \frac{\pi^2}{10}I=4$

28. (2)

Let

$I=\int_0^\alpha \frac{t^{50}}{1-t}dt$

$\Rightarrow I=-\int_0^\alpha\left(\frac{1-t^{50}-1}{1-t}\right)dt$

$\Rightarrow I=-\int_0^\alpha\left(\frac{1-t^{50}}{1-t}\right)dt+\int_0^\alpha\left(\frac{1}{1-t}\right)dt$

$\Rightarrow I=-\int_0^\alpha(1+t+t^2+\ldots+t^{49})dt-[\ln(1-t)]_0^\alpha$

$\Rightarrow I=-\left[t+\frac{t^2}{2}+\frac{t^3}{3}+\ldots+\frac{t^{50}}{50}\right]_0^\alpha-\ln(1-\alpha)$

$\Rightarrow I=-\left[\alpha+\frac{\alpha^2}{2}+\frac{\alpha^3}{3}+\ldots+\frac{\alpha^{50}}{50}\right]-\ln(1-\alpha)$

$\Rightarrow I=-(\beta+P_{50}(\alpha))$

29. (5)

Given $a_n=\int_{-1}^n\left(1+\frac{x}{2}+\frac{x^2}{3}+\ldots.+\frac{x^{n-1}}{n}\right)dx$

$=\left[x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\ldots\ldots+\frac{x^n}{n^2}\right]_{-1}^n$

i.e. $a_n=\frac{n+1}{1^2}+\frac{n^2-1}{2^2}+\frac{n^3+1}{3^2}+\frac{n^4-1}{4^2}+\ldots+\frac{n^n+(-1)^{n+1}}{n^2}$

Here $a_1=2,\ a_2=\frac{2+1}{1}+\frac{2^2-1}{2}=\frac{9}{2}$,

$a_3=4+2+\frac{28}{9}=\frac{100}{9}$,

$a_4=5+\frac{15}{4}+\frac{65}{9}+\frac{255}{16}>31$.

$\because\ a_n\in(2,30)$, so the required set is $\{2,3\}$.

$\therefore$ sum of all elements $=5$.

30. (34)

Here $y=\frac{9-x^2}{5-x}=5+x+\frac{16}{x-5}$

$\frac{dy}{dx}=1-\frac{16}{(x-5)^2}$

When $\frac{dy}{dx}=0\Rightarrow 1-\frac{16}{(x-5)^2}=0$

So critical point is $x=1$ in $[0,2]$

Now we get $y(0)=\frac{9}{5}, y(1)=2, y(2)=\frac{5}{3}$

So $\alpha=2$ and $\beta=\frac{5}{3}$

$I=\int_{-1}^3\max\left(\frac{9-x^2}{5-x},x\right)$

$I=\int_{-1}^{\frac{9}{5}}\left(5+x+\frac{16}{x-5}\right)dx+\int_{\frac{9}{5}}^3 xdx$

$=\left[5x+\frac{x^2}{2}+16\ln(x-5)\right]_{-1}^{\frac{9}{5}}+\left[\frac{x^2}{2}\right]_{\frac{9}{5}}^3$

$I=18+16\ln\left(\frac{8}{15}\right)$

$\alpha_1=18$ and $\alpha_2=16$

$\Rightarrow \alpha_1+\alpha_2=34$