ANSWER KEYS

1. (1)	**2.** (2)	**3.** (1)	**4.** (1)	**5.** (64)	**6.** (1)	**7.** (1)	**8.** (7)
9. (4)	**10.** (3)	**11.** (4)	**12.** (4)	**13.** (4)	**14.** (1)	**15.** (3)	**16.** (4)
17. (3)	**18.** (3)	**19.** (1)	**20.** (1)	**21.** (2)	**22.** (1)	**23.** (1)	**24.** (3)
25. (2)	**26.** (28)	**27.** (3)	**28.** (6)	**29.** (3)	**30.** (2)		

1. (1)

Given,

$I=\int\left(\left(\frac{x}{2}\right)^x+\left(\frac{2}{x}\right)^x\right)\log_2 x\,dx$

Now, Let $\left(\frac{x}{2}\right)^x=t$

$\Rightarrow \quad x\log_2\left(\frac{x}{2}\right)=\log_2 t$

$\Rightarrow \quad x\log_e\left(\frac{x}{2}\right)\cdot\log_2 e=\log_e t\cdot\log_2 e$

On differentiating both sides we get :

$\log_e\left(\frac{x}{2}\right)+x\cdot\frac{2}{x}\cdot\frac{1}{2}=\frac{1}{t}\frac{dt}{dx}$

$\Rightarrow \log_e\left(\frac{x}{2}\right)+1=\frac{1}{t}\frac{dt}{dx}$

Then solution is not possible as there is no proper substitution.

Note: This question was bonus in Jee Mains 2023 April session.

2. (2)

We have, $\int\frac{dx}{x^3\left(1+x^6\right)^{\frac{2}{3}}}=xf(x)\left(1+x^6\right)^{\frac{1}{3}}+C$

Now, $I=\int\frac{dx}{x^7\left(\frac{1}{x^6}+1\right)^{\frac{2}{3}}}$

Put, $t=\frac{1}{x^6}+1\Rightarrow dt=-\frac{6}{x^7}dx$

$\therefore I=-\frac{1}{6}\int\frac{dt}{t^{\frac{2}{3}}}=-\frac{1}{2}t^{\frac{1}{3}}+C$, where C is the constant of integration$=-\frac{1}{2}\left(\frac{1}{x^6}+1\right)^{\frac{1}{3}}+C=-\frac{1}{2}\frac{\left(1+x^6\right)^{\frac{1}{3}}}{x^2}+C$

$\therefore f(x)=-\frac{1}{2x^3}$

3. (1)

Let $I=\int x^5e^{-x^2}dx$

Let, $-x^2=t\Rightarrow xdx=-\frac{1}{2}dt$

$\therefore \; I=-\frac{1}{2}\int t^2e^tdt$

By using integrating by parts, i.e. $\int(u\cdot v)dx=u\int vdx-\int\left[\frac{d}{dx}u\int vdx\right]dx+c$, we get

$I=-\frac{1}{2}\left[t^2\int e^tdt-\int\left(2t\int e^tdt\right)dt\right]+c$

$\Rightarrow I=-\frac{1}{2}\left[t^2e^t-2\int te^tdt\right]+c$

Again, applying integrating by parts, we get

$\Rightarrow I=-\frac{1}{2}\left[t^2e^t-2\left(t\int e^tdt-\int 1\cdot e^tdt\right)\right]+c$

$\Rightarrow I=-\frac{1}{2}\left[t^2e^t-2\left(te^t-e^t\right)\right]+c$

$\Rightarrow I=-\frac{1}{2}\left[t^2e^t-2te^t+2e^t\right]+c$

$\Rightarrow I=-\frac{e^t}{2}\left[t^2-2t+2\right]+c$

$\Rightarrow I=-\frac{e^{-x^2}}{2}\left[x^4+2x^2+2\right]+c$

Given, $I=\int x^5e^{-x^2}dx=g(x)e^{-x^2}+c$

$\Rightarrow g(x)=-\frac{1}{2}\left(x^4+2x^2+2\right)$

$\Rightarrow g(-1)=-\frac{1}{2}\left((-1)^4+2(-1)^2+2\right)$

$\Rightarrow g(-1)=-\frac{5}{2}.$

4. (1)

Let

$I=\int \frac{2x}{(x^2+1)(x^2+3)}dx$

Put $x^2=t \Rightarrow 2xdx=dt$

$I=\int \frac{1}{(t+1)(t+3)}dt$

$\Rightarrow I=\frac{1}{2}\int \frac{2}{(t+1)(t+3)}dt$

$\Rightarrow I=\frac{1}{2}\int\left(\frac{1}{t+1}-\frac{1}{t+3}\right)dt$

$\Rightarrow I=\frac{1}{2}[\ln(t+1)-\ln(t+3)]+C$

$\Rightarrow f(x)=\frac{1}{2}\left[\ln\left(x^2+1\right)-\ln\left(x^2+3\right)\right]+C$

Put $x=3$, then

$\frac{1}{2}[\ln 5-\ln 6]=\frac{1}{2}[\ln 10-\ln 12]+C$

$\frac{1}{2}[\ln 5-\ln 6]=\frac{1}{2}[\ln 2+\ln 5-\ln 2-\ln 6]+C$

$\Rightarrow C=0$

So,

$f(x)=\frac{1}{2}\left[\ln\left(x^2+1\right)-\ln\left(x^2+3\right)\right]$

$\Rightarrow f(4)=\frac{1}{2}(\ln 17-\ln 19)$ or $f(4)=\frac{1}{2}(\log_e 17-\log_e 19)$

5. (64)

$\int\sqrt{\frac{x+7}{x}}dx$

Put $x=t^2$

$dx=2tdt$

$\int 2\sqrt{t^2+7}dt=2\int\sqrt{t^2+\sqrt{7}^2}dt$

$I(t)=2\left[\frac{t}{2}\sqrt{t^2+7}+\frac{7}{2}\ln\left|t+\sqrt{t^2+7}\right|\right]+C$

$I(x)=\sqrt{x}\sqrt{x+7}+7\ln\left|\sqrt{x}+\sqrt{x+7}\right|+C$

$I(9)=12+7\ln 7=12+7(\ln(3+4))+C$

$\Rightarrow C=0$

$I(x)=\sqrt{x}\sqrt{x+7}+7\ln(\sqrt{x}+\sqrt{x+7})$

$I(1)=1\sqrt{8}+7\ln(1+\sqrt{8})$

$I(1)=\sqrt{8}+7\ln(1+2\sqrt{2})$

$\alpha=\sqrt{8}$

$\alpha^4=\left(8^{1/2}\right)^4$

$\alpha^4=8^2=64$

6. (1)

Given,

$I(x)=\int \frac{x+1}{x\left(1+xe^x\right)^2}dx$

Now let $1+xe^x=t$

$\Rightarrow e^x(x+1)dx=dt$

So, $I(x)=\int \frac{1}{(t-1)t^2}dt$

$\Rightarrow I(x)=\int \frac{\left(1-t^2\right)+t^2}{(t-1)t^2}dt$

$\Rightarrow I(x)=\int \frac{-(t+1)}{t^2}+\frac{1}{t-1}dt$

$\Rightarrow I(x)=\int -\frac{1}{t}-\frac{1}{t^2}+\frac{1}{t-1}dt$

$\Rightarrow I(x)=-\ln t+\frac{1}{t}+\ln(t-1)+C$

$\Rightarrow I(x)=\ln\left(\frac{xe^x}{xe^x+1}\right)+\frac{1}{xe^x+1}+C$

Also given, $\lim_{x\to\infty} I(x)=0$

$\Rightarrow \lim_{x\to\infty} I(x)=\lim_{x\to\infty}\left[\ln\left(1-\frac{1}{xe^x+1}\right)+\frac{1}{xe^x+1}+C\right]$

$\Rightarrow \lim_{x\to\infty} I(x)=[\ln(1-0)+0+C]$

$\Rightarrow C=0$

Now finding,

$I(1)=\ln\left(\frac{e}{e+1}\right)+\frac{1}{e+1}=1+\frac{1}{e+1}-\ln\left(e+1\right)$

$\Rightarrow I(1)=\frac{e+2}{e+1}-\ln(e+1)$

7. **(1)**

Given,

$\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}dx = g(x)+c$

Put $x = \cos 2\theta$

$dx = -2\sin 2\theta \cdot d\theta$

$= \int \frac{1}{\cos 2\theta}\tan\theta(-4\sin\theta \cdot \cos\theta)d\theta$

$= \int \frac{1}{\cos 2\theta}\left(-4\sin^2\theta\right)d\theta$

$= -2\int \frac{1-\cos 2\theta}{\cos 2\theta}d\theta$

$= -\frac{2}{2}\ln|\sec 2\theta - \tan 2\theta| + 2\theta + c$

$= \ln|\sec 2\theta - \tan 2\theta| + 2\theta + c$

$= \ln\left|\frac{1-\sin 2\theta}{\cos 2\theta}\right| + \cos^{-1}x + c$

$= \underbrace{\ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \cos^{-1}x}_{g(x)} + c$

$\therefore\ g(1) = 0 \Rightarrow c = 0$

So, $g(x) = \ln\left|\frac{1-\sqrt{1-x^2}}{x}\right| + \cos^{-1}x$

$g\left(\frac{1}{2}\right) = \ln|2-\sqrt{3}| + \frac{\pi}{3}$

$g\left(\frac{1}{2}\right) = \ln\left|\frac{\sqrt{3}-1}{\sqrt{3}+1}\right| + \frac{\pi}{3}$

8. **(7)** $I = \int \frac{2e^x+3e^{-x}}{4e^x+7e^{-x}}dx = \int \frac{2e^x+3}{4e^{2x}+7}dx$

Here $2e^{2x}+3 = A\left(8e^{2x}\right) + B\left(4e^{2x}+7\right)$

$\Rightarrow 2e^{2x}+3 = (8A+4B)e^{2x} + 7B$

$\Rightarrow B = \frac{3}{7} \,\&\, A = \frac{1}{28}$

$$I = \int \frac{\frac{1}{28}\left(8e^{2x}\right) + \frac{3}{7}\left(4e^{2x}+7\right)}{4e^{2x}+7}dx$$

$I = \frac{1}{28}\ln\left|4e^{2x}+7\right| + \frac{3}{7}x + C$

$I = \frac{1}{28}\ln\left|e^x\left(4e^x+7e^{-x}\right)\right| + \frac{3}{7}x + C$

$\Rightarrow u = \frac{13}{2}\ \&\ v = \frac{1}{2}$

$\Rightarrow u+v = \frac{13}{2} + \frac{1}{2} = 7$

9. **(4)**

$f(x) = \int \frac{\left(5x^8+7x^6\right)dx}{x^{14}\left(x^{-5}+x^{-7}+2\right)^2}$

Let $x^{-5}+x^{-7}+2 = t$

$\left(-5x^{-6}-7x^{-8}\right)dx = dt$

$\Rightarrow f(x) = \int -\frac{dt}{t^2} = \frac{1}{t} + c$

$f(x) = \frac{x^7}{x^2+1+2x^7}$

$f(1) = \frac{1}{4}$

10. **(3)** $I=\int \frac{\sin \theta \cdot \sin 2 \theta\left(\sin ^6 \theta+\sin ^4 \theta+\sin ^2 \theta\right) \sqrt{2 \sin ^4 \theta+3 \sin ^2 \theta+6}}{1-\cos 2 \theta} d \theta$

$\Rightarrow \quad I=\int \frac{\sin \theta .2 \sin \theta \cos \theta \cdot \sin ^2 \theta\left(\sin ^4 \theta+\sin ^2 \theta+1\right)\left(2 \sin ^4 \theta+3 \sin ^2 \theta+6\right)^{1 / 2}}{2 \sin ^2 \theta} d \theta$

$=\int \sin ^2 \theta \cdot \cos \theta\left(\sin ^4 \theta+\sin ^2 \theta+1\right)\left(2 \sin ^4 \theta+3 \sin ^2 \theta+6\right)^{1 / 2} d \theta$

Let $\sin \theta=t \Rightarrow \cos \theta d \theta=d t$

$\therefore I=\int t^2\left(t^4+t^2+1\right)\left(2 t^4+3 t^2+6\right)^{1 / 2} d t$

$=\int\left(t^5+t^3+t\right) t\left(2 t^4+3 t^2+6\right)^{1 / 2} d t$

$=\int\left(t^5+t^3+t\right)\left(t^2\right)^{1 / 2}\left(2 t^4+3 t^2+6\right)^{1 / 2} d t$

$=\int\left(t^5+t^3+t\right)\left(2 t^6+3 t^4+6 t^2\right)^{1 / 2} d t$

Let $2 t^6+3 t^4+6 t^2=u^2$

$\Rightarrow \quad 12\left(t^5+t^3+t\right) d t=2 u d u$

$\therefore \; I=\int\left(u^2\right)^{1 / 2} \cdot \frac{2 u d u}{12}$

$=\int \frac{u^2}{6} d u=\frac{u^3}{18}+C$

$=\frac{\left(2 t^6+3 t^4+6 t^2\right)^{3 / 2}}{18}+C$

when $t=\sin \theta$

and $t^2=1-\cos ^2 \theta$ will give

$=\frac{1}{18}\left[11-18 \cos ^2 \theta+9 \cos ^4 \theta-2 \cos ^6 \theta\right]^{\frac{3}{2}}+c$

11. **(4)** $\int \frac{\cos x-\sin x}{\sqrt{8-\sin 2 x}} d x$

$=\int \frac{\cos x-\sin x}{\sqrt{9-(\sin x+\cos x)^2}} d x$

Let $\sin x+\cos x=t$

$\int \frac{d t}{\sqrt{9-t^2}}=\sin ^{-1} \frac{t}{3}+c$

$=\sin ^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c$

So $a=1, \; b=3$

12. **(4)**

$I=\int \frac{\cos \theta}{2 \sin ^2 \theta+7 \sin \theta+3} d \theta$

$\sin \theta=t \quad \Rightarrow \quad \cos \theta d \theta=d t$

$=\frac{1}{2} \int \frac{1}{t^2+\frac{7}{2} t+\frac{3}{2}} d t=\frac{1}{2} \int \frac{1}{\left(t+\frac{7}{4}\right)^2-\left(\frac{5}{4}\right)^2} d t=\frac{1}{5} \ln \left|\frac{2 t+1}{t+3}\right|+c=\frac{1}{5} \ln \left|\frac{2 \sin \theta+1}{\sin \theta+3}\right|+c$

so $A=\frac{1}{5}$

$B(\theta)=\frac{5(2 \sin \theta+1)}{\sin \theta+3}$

13. **(4)**

$I=\int \sin ^{-1}\left(\frac{\sqrt{x}}{1+x}\right) d x$

$=\int \underbrace{\tan ^{-1}(\sqrt{x})}_{I} \underbrace{(1)}_{II} d x$

$=x \tan ^{-1} \sqrt{x}-\int \frac{1}{1+x} \cdot \frac{1}{2 \sqrt{x}} . x d x+C$

$=x \tan ^{-1} \sqrt{x}-\frac{1}{2} \int \frac{t .2 t . d t}{1+t^2}+C$ (putting $x=t^2 \Rightarrow d x=2 t d t$)

$=x \tan ^{-1} \sqrt{x}-\int \frac{t^2}{1+t^2} d t+C$

$=x \tan ^{-1} \sqrt{x}-t+\tan ^{-1} t+C$

$=x \tan ^{-1} \sqrt{x}-\sqrt{x}+\tan ^{-1} \sqrt{x}+C$

$=(x+1) \tan ^{-1} \sqrt{x}-\sqrt{x}+C$

Comparing with $A(x) \tan ^{-1}(\sqrt{x})+B(x)+C$, we get

(A(x), B(x))$=(x+1,-\sqrt{x})$

14. **(1)**

$\Rightarrow \int\left(\frac{x-3}{x+4}\right)^{\frac{-6}{7}} \frac{1}{(x+4)^2} d x \ldots$(i)

Let $\frac{x-3}{x+4}=t^7$,

$\frac{7}{(x+4)^2} d x=7 t^6 d t \ldots$ (ii)

From (i) from (ii),

$\int t^{-6} t^6 d t=t+C$

15. (3)

$\int \frac{\sec^2\theta}{\frac{1+\tan^2\theta}{1-\tan^2\theta}+\frac{2\tan\theta}{1-\tan^2\theta}} d\theta$

$= \int \frac{\sec^2\theta\left(1-\tan^2\theta\right)}{(1+tan\theta)^2} d\theta$

$= \int \frac{\sec^2\theta\left(1-\tan\theta\right)}{1+\tan\theta} d\theta$

Let $\tan\theta = t \Rightarrow \sec^2\theta d\theta = dt$

$= \int\left(\frac{1-t}{1+t}\right) dt = \int\left(-1+\frac{2}{1+t}\right) dt$

$= -t + 2\ln(1+t) + C$

$= -\tan\theta + 2\ln(1+\tan\theta) + C$

$\Rightarrow \lambda = -1$ and $f(\theta) = 1+\tan\theta$

16. (4)

Let, $\sin x = t$

$\cos x dx = dt$

$I = \int \frac{dt}{t^3\left(1+t^6\right)^{\frac{2}{3}}} = \int \frac{dt}{t^7\left(1+\frac{1}{t^6}\right)^{\frac{2}{3}}}$

Put $1+\frac{1}{t^6} = r^3$

$\Rightarrow \frac{dt}{t^7} = \frac{-1}{2} r^2 dr$

$I = -\frac{1}{2}\int \frac{r^2 dr}{r^2} = -\frac{1}{2} r + c$

$\Rightarrow I = -\frac{1}{2}\left(\frac{\sin^6 x+1}{\sin^6 x}\right)^{\frac{1}{3}} + c$

$\Rightarrow I = -\frac{1}{2\sin^2 x}\left(1+\sin^6 x\right)^{\frac{1}{3}} + c$

So, $f(x) = -\frac{1}{2}\operatorname{cosec}^2 x$ and $\lambda = 3$

$\lambda f\left(\frac{\pi}{3}\right) = -2$

17. (3)

$I = \int \frac{2x^3-1}{x^4+x} dx$

$\Rightarrow I = \int \frac{2x-\frac{1}{x^2}}{x^2+\frac{1}{x}} dx$

Put, $x^2+\frac{1}{x} = t \Rightarrow \left(2x-\frac{1}{x^2}\right) dx = dt$

$\therefore I = \int \frac{dt}{t} = \log_e(t) + C$

$= \log_e\left(x^2+\frac{1}{x}\right) + C$

$= \log_e\left|\frac{\left(x^3+1\right)}{x}\right| + C$

18. (3)

Given integral can be written as

$I = \int \frac{\sec^2 x}{(\tan x)^{\frac{4}{3}}} dx$

Let $\tan x = t$

$\Rightarrow \sec^2 x dx = dt$

$\Rightarrow I = \int t^{-\frac{4}{3}} dt$

Using $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, we get

$I = \frac{t^{-\frac{1}{3}}}{\left(-\frac{1}{3}\right)} + C$

$\Rightarrow I = -\frac{3}{t^{\frac{1}{3}}} + C$

$\Rightarrow I = -3\tan^{-\frac{1}{3}} x + C.$

19. (1)

We have, $I=\int \frac{\sin \frac{5x}{2}}{\sin \frac{x}{2}} dx$

$=\int \frac{\sin\left(2x+\frac{x}{2}\right)}{\sin \frac{x}{2}} dx$

Using $\sin(A+B)=\sin A\cos B+\cos A\sin B$

$=\int \frac{\sin 2x\cos \frac{x}{2}+\cos 2x\sin \frac{x}{2}}{\sin \frac{x}{2}} dx$

$=\int\left(\frac{(2\ \sin x\ \cos x)\cos \frac{x}{2}}{\sin \frac{x}{2}}+\cos 2x\right)dx$

$=\int\left(\frac{2\left(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)\cos x\ \cos\left(\frac{x}{2}\right)}{\sin \frac{x}{2}}+\cos\ 2x\right)dx$

$=\int\left(4\cos^2\left(\frac{x}{2}\right)\cos x+\cos 2x\right)dx$

Using $\cos 2x=2\cos^2 x-1 \Rightarrow 2\cos^2 x=1+\cos 2x$

$=\int(2(1+\cos x)\cos x+\cos 2x)dx$

$=\int\left(2\cos x+2\cos^2 x+\cos 2x\right)dx$

$=\int(2\cos x+(1+\cos 2x)+\cos 2x)dx$

$=\int(2\cos x+2\cos 2x+1)dx$

$=2\sin x+\sin 2x+x+c$, where c is the constant of integration.

20. (1) $A(x)\left(\sqrt{1-x^2}\right)^m+C=\int \frac{\sqrt{1-x^2}}{x^4}dx$

$=\int \frac{\sqrt{\frac{1}{x^2}-1}}{x^3}dx$

Let $\frac{1}{x^2}-1=u^2$

$\Rightarrow -\frac{2}{x^3}=\frac{2udu}{dx}$

$\frac{dx}{x^3}=-udu$

$A(x)\left(\sqrt{1-x^2}\right)^m+C=\int\left(-u^2\right)du=-\frac{u^3}{3}+C$

$=-\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{\frac{3}{2}}+C$

$=-\frac{1}{3}\cdot\frac{1}{x^3}\cdot\left(1-x^2\right)^{\frac{3}{2}}+C$

$=\frac{-1}{3x^3}\left(\sqrt{1-x^2}\right)^3+C$

Compare both sides,

$\Rightarrow A(x)=-\frac{1}{3x^3}$ and $m=3$

$\Rightarrow \quad (A(x))^3=\frac{-1}{27x^9}$

21. (2) $I(x)=\int \underset{II}{\frac{e^{\sin x}\cdot\sin 2x}{}}\cdot\underset{I}{\frac{\cos x}{}}dx-\int e^{\sin^2 x}\cdot\sin x dx$

$\Rightarrow I(x)=e^{\sin^2 x}-\int(-\sin x)\cdot e^{\sin^2 x}dx-\int e^{\sin^2 x}\cdot\sin x dx$

$\Rightarrow I(x)=e^{\sin^2 x}\cdot\cos x+c$

Put $x=0, c=0$

$\therefore I\left(\frac{\pi}{3}\right)=e^{\frac{3}{4}}\cdot\cos\frac{\pi}{3}=\frac{1}{2}e^{\frac{3}{4}}$

22. (1)

Given,

$I(x)=\int \frac{\sec^2 x-2022}{\sin^{2022} x} dx$

On rearranging we get,

$\Rightarrow I(x)=\int \underbrace{\sec^2 x}_{II} \cdot \underbrace{\sin^{-2022}}_{I} x dx-2022\int \sin^{-2022} x dx$

Now using integration by parts we get,

$\Rightarrow I(x)=\tan x.(\sin x)^{-2022}+\int(2022)\tan x\cdot(\sin x)^{-2023}\cos x dx-2022\int(\sin x)^{-2022}dx$

$\Rightarrow I(x)=\tan x.(\sin x)^{-2022}+\int(2022)\tan x\cdot(\sin x)^{-2022}\frac{\cos x}{\sin x}dx-2022\int(\sin x)^{-2022}dx$

$\Rightarrow I(x)=\tan x.(\sin x)^{-2022}+\int(2022)(\sin x)^{-2022}dx-2022\int(\sin x)^{-2022}dx$

$\Rightarrow I(x)=(\tan x)(\sin x)^{-2022}+C$

Now at $x=\frac{\pi}{4}$

$I\left(\frac{\pi}{4}\right)=2^{1011}\Rightarrow 2^{1011}=1\times\left(\frac{1}{\sqrt{2}}\right)^{-2022}+C\ \Rightarrow C=0$

Hence $I(x)=\frac{\tan x}{(\sin x)^{2022}}$

Now finding the value of $I\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}\left(\frac{1}{2}\right)^{2022}}=\frac{2^{2022}}{\sqrt{3}}$

And $I\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)^{2022}}=\frac{2^{2022}}{(\sqrt{3})^{2021}}=\frac{1}{3^{1010}}I\left(\frac{\pi}{6}\right)$

So, $3^{1010}I\left(\frac{\pi}{3}\right)=I\left(\frac{\pi}{6}\right)$

23. (1)

$\int\frac{(x^2+1)e^x dx}{(x+1)^2}=f(x)e^x+C$

$\int e^x\left(\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right)dx=f(x)e^x+C$

$\int e^x\left(\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right)dx=f(x)e^x+C$

We know that $\int e^x(f(x)+f'(x))=e^x f(x)+c$

Here $f(x)=\frac{x-1}{x+1}$ & $f'(x)=\frac{2}{(x+1)^2}$

So $\int e^x\left(\left(\frac{x-1}{x+1}\right)+\frac{2}{(x+1)^2}\right)dx=f(x)e^x+C$

$\Rightarrow e^x\left(\frac{x-1}{x+1}\right)+C=e^x f(x)+C$

On comparing both sides we get $f(x)=\frac{x-1}{x+1}$

So $f'(x)=\frac{2}{(x+1)^2}$ & $f''(x)=\frac{-4}{(x+1)^3}$

$f'''(x)=\frac{12}{(x+1)^4}$,

Now $f'''(1)=\frac{12}{(1+1)^4}=\frac{12}{16}=\frac{3}{4}$

24. (3)

Given

$\alpha=\frac{I_2}{I_1}=\frac{\int_0^1\left(1-x^{50}\right)^{101}dx}{\int_0^1\left(1-x^{50}\right)^{100}dx}$

Now,

$I_2=\int_0^1\left(1-x^{50}\right)\left(1-x^{50}\right)^{100}dx$

$I_2=I_1-\int_0^1 x.x^{49}\left(1-x^{50}\right)^{100}dx$

$I_2=I_1-\left[\frac{-x\left(1-x^{50}\right)^{101}}{5050}\right]_0^1-\int_0^1\frac{\left(1-x^{50}\right)^{101}}{5050}dx$

$I_2=I_1-\frac{I_2}{5050}$

$\Rightarrow\alpha=\frac{I_2}{I_1}=\frac{5050}{5051}$

25. (2)

$\int e^{\sec x}\left(\sec x\tan x f(x)+\left(\sec x\tan x+\sec^2 x\right)dx=e^{\sec x}f(x)+C\right.$

Differentiating both sides w.r.t 'x' we get

$e^{secx}\left(secx\tan x f(x)+\left(secx\tan x+\sec^2 x\right)\right)=\left(e^{\sec x}\cdot\sec x\tan x\right)f(x)+e^{\sec x}f'(x)$

$\Rightarrow f'(x)=\sec^2 x+\tan x\sec x$

$\Rightarrow f(x)=\int\left(\sec^2 x+\tan x\sec x\right)dx$

$\Rightarrow f(x)=\tan x+\sec x+c,\ c\in R$

Hence, possible choice is $f(x)=\sec x+\tan x+\frac{1}{2}$.

26. (28)

$f(x)=\int \frac{dx}{(3+4x^2)\sqrt{4-3x^2}}$

$x=\frac{1}{t}$

$=\int \frac{\frac{-1}{t^2}dt}{\frac{(3t^2+4)}{t^2}\frac{\sqrt{4t^2-3}}{t}}$

$=\int \frac{-dt\cdot t}{(3t^2+4)\sqrt{4t^2-3}}$: Put $4t^2-3=z^2$

$=-\frac{1}{4}\int \frac{zdz}{\left(3\left(\frac{z^2+3}{4}\right)+4\right)z}$

$=\int \frac{-dz}{3z^2+25}=\frac{1}{3}\int \frac{dz}{z^2+\left(\frac{5}{\sqrt{3}}\right)^2}$

$=-\frac{1}{3}\frac{\sqrt{3}}{5}\tan^{-1}\left(\frac{\sqrt{3}z}{5}\right)+C$

$=-\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{4t^2-3}\right)+C$

$f(x)=-\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\sqrt{\frac{4-3x^2}{x^2}}\right)+C$

$\because f(0)=0 \therefore c=\frac{\pi}{10\sqrt{3}}$

$f(1)=-\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{5}\right)+\frac{\pi}{10\sqrt{3}}$

$f(1)=\frac{1}{5\sqrt{3}}\cot^{-1}\left(\frac{\sqrt{3}}{5}\right)=\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$

$\alpha=5:\beta=\sqrt{3}\therefore \alpha^2+\beta^2=28$

27. (3)

$\int \frac{\sin x}{\sin^3 x+\cos^3 x}dx=\alpha\log_e\left|1+\tan x\right|+\beta\log_e\left|1-\tan x+\tan^2 x\right|+\gamma\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C,\ldots\ldots\ldots(i)$

Let, $I=\int \frac{\sin x}{\sin^3 x+\cos^3 x}dx$

$I=\int \frac{\tan x\sec^2 x}{\tan^3+1}dx$

Put $\tan x=t \Rightarrow \sec^2 xdx=dt$

$=\int \frac{tdt}{t^3+1}=\int \frac{t}{(t+1)(t^2-t+1)}dt$

Now $\frac{t}{(t+1)(t^2-t+1)}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+1}$

$t=A(t^2-t+1)+(Bt+C)(t+1)$

$t=t^2(A+B)-t(A-B-C)+A+C$

Comparing on both sides

$A+B=0,\ -A+B+C=1,\ A+C=0$

Solving these equations

$A=-\frac{1}{3},\ B=\frac{1}{3},\ C=\frac{1}{3}$

Hence $I=\int\left\{\frac{-\frac{1}{3}}{t+1}+\frac{1}{3}\left(\frac{t+1}{t^2-t+1}\right)\right\}dt$

$=-\frac{1}{3}\int\frac{1}{(t+1)}dt+\frac{1}{3}\int\frac{\frac{1}{2}(2t-1)+\frac{3}{2}}{t^2-t+1}dt$

$=-\frac{1}{3}\int\frac{1}{(t+1)}dt+\frac{1}{6}\int\frac{2t-1}{(t^2-t+1)}+\frac{1}{2}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=-\frac{1}{3}\ln|t+1|+\frac{1}{6}\ln|t^2-t+1|+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+C$

$=-\frac{1}{3}\ln|\tan x+1|+\frac{1}{6}\ln|\tan^2 x-\tan x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x-1}{\sqrt{3}}\right)+C$

From equation (i), Comparing from both sides

$\Rightarrow \alpha=-\frac{1}{3},\ \beta=\frac{1}{6}$ and $\gamma=\frac{1}{\sqrt{3}}$

So $18(\alpha+\beta+\gamma^2)$

$=18\left(-\frac{1}{3}+\frac{1}{6}+\frac{1}{3}\right)$

$=3$

28. (6)

$\int \frac{(x^2-1)\,dx}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} + \int \frac{dx}{x^4+3x^2+1}$

$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(\left(x+\frac{1}{x}\right)^2+1\right)\tan^{-1}\left(x+\frac{1}{x}\right)} + \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)\,dx}{x^4+3x^2+1}$

Put $\tan^{-1}\left(x+\frac{1}{x}\right)=t$

$\int \frac{dt}{t} + \frac{1}{2}\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+5} - \frac{1}{2}\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2+1}$

Put $x-\frac{1}{x}=y, x+\frac{1}{x}=z$

$\log_e t + \frac{1}{2}\int \frac{dy}{y^2+5} - \frac{1}{2}\int \frac{dz}{z^2+1}$

$= \log_e \tan^{-1}\left(x+\frac{1}{x}\right) + \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)$

$-\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right)+C$

$\alpha=1, \beta=\frac{1}{2\sqrt{5}}, \gamma=\frac{1}{\sqrt{5}}, \delta=\frac{-1}{2}$

or $\alpha=1, \beta=\frac{-1}{2\sqrt{5}}, \gamma=\frac{-1}{\sqrt{5}}, \delta=\frac{-1}{2}$

$10\left(\alpha+\beta\gamma+\delta\right)=10\left(1+\frac{1}{10}-\frac{1}{2}\right)=6$

29. (3)

$\int \frac{(\sin^n\theta-\sin\theta)^{\frac{1}{n}}\cos\theta}{\sin^{n+1}\theta}d\theta$

Put, $\sin\theta=t \Rightarrow \cos\theta d\theta=dt$

$= \int \frac{(t^n-t)^{\frac{1}{n}}dt}{t^{n+1}}$

$= \int \frac{t\left(1-\frac{1}{t^{n-1}}\right)^{\frac{1}{n}}}{t^{n+1}}dt$

$= \int \frac{\left(1-\frac{1}{t^{n-1}}\right)^{\frac{1}{n}}}{t^n}dt$

Put $1-\frac{1}{t^{n-1}}=z$

$\Rightarrow \frac{(n-1)}{t^n}dt=dz$

$\Rightarrow I=\frac{1}{n-1}\int z^{\frac{1}{n}}dz$

Using $\int x^n dx=\frac{x^{n+1}}{n+1}+c$

$\Rightarrow I=\frac{z^{\frac{1}{n}+1}}{\left(\frac{1}{n}+1\right)(n-1)}+c$

$\Rightarrow I=\frac{n\left(1-t^{1-n}\right)^{\frac{1}{n}+1}}{n^2-1}+c$, where c is the constant of integration.

$\Rightarrow I=\frac{n}{n^2-1}\left(1-\frac{1}{\sin^{n-1}\theta}\right)^{\frac{n+1}{n}}+c.$

30. (2)

$\int \frac{x}{e^x+1}(\cos x-\sin x)dx + \int g(x)\frac{e^x+1-xe^x}{(e^x+1)^2}dx$

$= \frac{x}{e^x+1}(\sin x+\cos x) - \int \frac{e^x+1-xe^x}{(e^x+1)^2}(\sin x+\cos x)dx + \int g(x)\frac{e^x+1-xe^x}{(e^x+1)^2}dx$

By comparison, we get, $g(x)=\sin x+\cos x$

$\Rightarrow g(x)=\sqrt{2}\left[\sin\left(x+\frac{\pi}{4}\right)\right]$

Since $x\in\left(0,\frac{\pi}{4}\right)$ so, $x+\frac{\pi}{4}\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

So $g(x)$ is increasing in $\left(0,\frac{\pi}{4}\right)$

$g'(x)=\cos x-\sin x$

i.e. $g(x)-g'(x)=2\sin x$ is an increasing function in $\left(0,\frac{\pi}{2}\right)$.