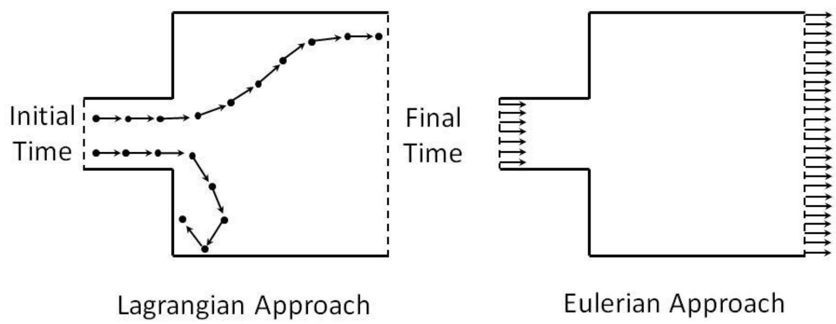
Chapter 3

FLUID KINEMATICS

Fluid kinematics *is defined as that branch of science which deals with motion of particles without considering the forces causing the motion. The velocity at any point in the flow field at any time is studied in this branch of Fluid Mechanics. Once the velocity is known, then the pressure distribution and hence forces acting on the fluid can be determined. In this chapter, the methods of determining velocity and acceleration are discussed.*

Lagrangian and Eulerian Descriptions

The fluid motion can be described by two methods. They are i) Lagrangian Descriptions and ii) Eulerian Descriptions method. In the Lagrangian descriptions method, a single fluid particle is followed during its motion and its velocity, acceleration, density, etc., are described. In case of Eulerian method, the velocity, acceleration, density, etc., are described at a point in flow field. Tracking and recording of individual particle in case of Lagrangian method is very difficult. So the Eulerian method is most commonly used in fluid mechanics.



Types of Fluid Flow:

The fluid flow is classified as:

1) *Internal and External Flow*
2) *Compressible and Incompressible Flow*
3) *Laminar and Turbulent Flow*
4) *Steady and Unsteady Flow*
5) *Uniform and Non uniform Flow*
6) *Rotational and Irrotational Flow*
6) *One, Two, and Three-Dimensional Flows*

1) Internal and External Flow

A fluid flow is classified as being internal or external, depending on whether the fluid is forced to flow in a confined channel or over a surface. The flow of an unbounded fluid over a surface such as a plate, a wire, or a pipe is external flow. The flow in a pipe or duct is internal flow if the fluid is completely bounded by solid surfaces. Water flow in a pipe, for example, is internal flow, and airflow over a ball or over an exposed pipe during a windy day is external flow.

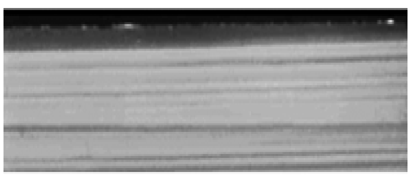
2) Compressible and Incompressible Flow

A flow is classified as being compressible or incompressible, depending on the level of variation of density during flow. Incompressibility is an approximation, and a flow is said to be incompressible if the density remains nearly constant throughout. Therefore, the volume of every *portion of fluid remains unchanged over the course of its motion when the flow (or the fluid) is incompressible. The densities of liquids are essentially constant, and thus the flow of liquids is typically incompressible. Therefore, liquids are usually referred to as incompressible substances.*

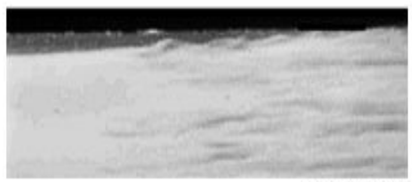
3) Laminar and Turbulent Flow

Some flows are smooth and orderly while others are rather chaotic. The highly ordered fluid motion characterized by smooth layers of fluid is called laminar. The word laminar comes from the movement of adjacent fluid particles together in "laminates." The flow of high-viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluid motion that typically occurs at high velocities and is characterized by velocity fluctuations is called turbulent (Figure). The flow of low-viscosity fluids such as air at high velocities is typically turbulent. The flow regime greatly influences the required power for pumping. A flow that alternates between being laminar and turbulent is called transitional.

Reynolds number, Re, as the key parameter for the determination of the flow regime in pipes.



Laminar



Transitional

Turbulent

4) Steady and Unsteady Flow

Steady flow is defined as that type of flow in which the fluid characteristic like velocity, pressure, density, etc., at a point do not change with change in time. Unsteady flow is that type of flow in which velocity, pressure or density at a point changes with respect to change in time.

5) Uniform and Non uniform Flow

Uniform flow is defined as that type of flow in which the velocity at any given time does not change with respect to distance or space (i.e. length of direction of the flow). Non uniform flow is that type of flow in which the velocity at any given time changes with respect to distance or space.

6) Rotational and Irrotational Flow

Rotational flow is that type of flow in which the fluid particles while flowing along stream lines, also rotates about their own axis. And if the fluid particles while flowing along stream lines do not rotates about their own axis then that type of flow is called Irrotational flow.

7) One, Two, and Three-Dimensional Flows

A flow field is best characterized by the velocity distribution, and thus a flow is said to be one, two, or three-dimensional if the flow velocity varies in one, two, or three primary dimensions, respectively. A typical fluid flow involves a three-dimensional geometry, and the velocity may vary in all three dimensions, rendering the flow three-dimensional [$V\,(x, y, z)$ in rectangular or $V\,(r, \theta, z)$ in cylindrical coordinates]. However, the variation of velocity in certain directions can be small relative to the variation in other directions and can be ignored with negligible error. In such cases, the flow can be modeled conveniently as being one or two dimensional, which is easier to analyze.

Flow Patterns and Flow Visualization

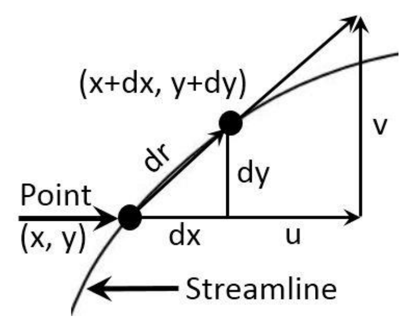
While quantitative study of fluid dynamics requires advanced mathematics, much can be learned from flow visualization, the visual examination of flow field features. Flow visualization is useful not only in physical experiments, but in numerical solutions as well i.e. computational fluid dynamics. In fact, the very first thing an engineer using CFD does after obtaining a numerical solution is simulate some form of flow visualization, so that he or she can see the "whole picture" rather than merely a list of numbers and quantitative data. There are many types of flow patterns that can be visualized, both physically (experimentally) and/or computationally.

Streamlines

A streamlines is a curve that is everywhere tangent to the instantaneous local velocity vector Streamlines are useful as indicators of the instantaneous direction of fluid motion throughout the flow field.

Consider an infinitesimal are length dr = dx i + dy j + dz k along a streamline, dr must be parallel to the local velocity vector V = u i + v j + w k by definition of the streamline. By simple geometric arguments using similar triangles, we know that the components of dr must be proportional to those of V. Hence

Equation of streamline: $$\frac{dr}{V} = \frac{dx}{u} + \frac{dy}{v} + \frac{dz}{w}$$



where dr is magnitude and V is the speed.

If above equation is in two dimensions for simplicity, in two dimensions (x, y), (u, v), the following differential equation is obtained:

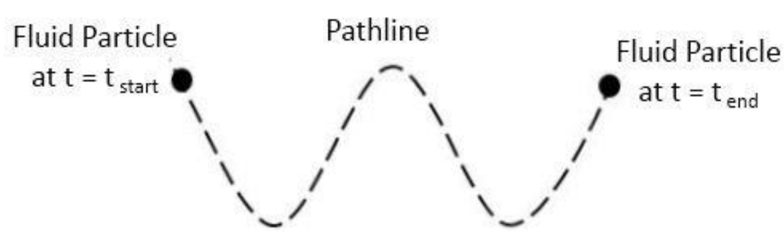
Streamline in the xy – plane: $$\frac{dy}{dx} = \frac{v}{u}$$

Streamtubes

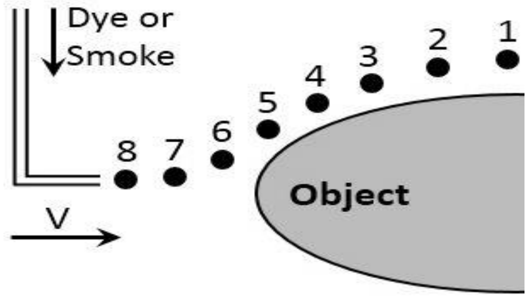
A streamtubes consist of a bundle of streamlines, much like a communication cable consist of bundle of fiber optic cables. Since streamlines are everywhere parallel to the local velocity, fluid cannot cross a streamline by definition. By extension, fluid within a streamline must remain there and cannot cross the boundary of the streamtube.

Pathlines

A pathline is the actual path traveled by an individual fluid particle over some time period. Pathlines are the easiest of the flow patterns to understand. A pathline is a Lagrangian concept in that we simply follow the path of an individual fluid particle as it moves around in the flow field.

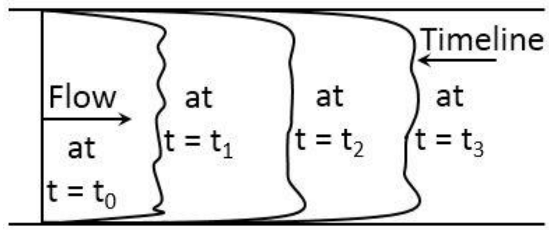


Streaklines



A streakline is the locus of fluid particles that have passes sequentially through a prescribed point in the flow. Streaklines are the most common flow pattern generated in a physical experiment. If you insert a small tube into a flow and introduce a continuous steam of tracer fluid (dye in a water flow or smoke in an airflow), the observed pattern is a streakline.

Timelines



A timeline is a set of adjacent fluid particles that were marked at the same (earlier) instant in time. Timelines are particularly useful in situations where the uniformity of a flow is to be examined. Figure illustrates timelines in a channel flow between two parallel walls. Because of friction at the walls, the fluid velocity there is zero (the no slip condition) and the top and bottom of the timeline are anchored at their starting locations. In regions of the flow away from the walls, the marked fluid particle move at the local fluid velocity, deforming the timeline.

Plots of Fluid Flow Data

Profile Plot

A profile plot indicates how the value of a scalar properties varies along some desired direction in the flow field.

Vector Plot

A vector plot is an array of arrows indicating the magnitude and direction of a vector properties at an instant in time.

Contour Plot

A contour plot shows curves of constant values of a scalar properties (or magnitude of a vector properties) at an instant in time.

Rate of Flow OR Discharge (Q)

It is defined as the quantity of a fluid flowing per second through a section of a pipe or channel. For an incompressible fluid (or liquid) the rate of flow or discharge is expressed as the volume of fluid flowing across the section per second. For compressible fluids, the rate of flow is usually expressed as the weight of fluid flowing across the section. Thus,

1) For liquids the units of Q are m^3/s or liters/sec

2) For gases the units of Q is kgf/sec or Newton/sec

The discharge Q is given as follows,

$$Q = A \times V$$

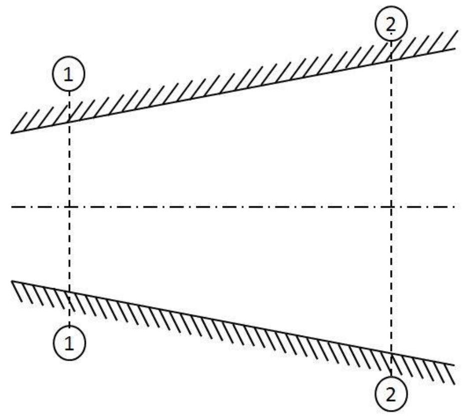
Where, A = cross sectional area of pipe

V = average velocity of fluid across the section

Continuity Equation

The continuity equation is based on the principle of conservation of mass. Thus for a fluid flowing through the pipe at all the cross section, the quantity of fluid per second is constant.

Consider two cross sections of a pipe as shown in figure.



Let, V_1 = *average velocity at cross section 1-1*

ρ_1 = *density at section 1-1*

A_1 = *area of pipe at section 1-1*

And V_2, ρ_2, A_2 are corresponding values at section 2-2

Rate of flow at section 1-1 = $\rho_1 A_1 V_1$

Rate of flow at section 2-2 = $\rho_2 A_2 V_2$

According to law of conservation of mass

Rate of flow at section 1-1 = Rate of flow at section 2-2

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2$$

Above equation is applicable to the compressible as well as incompressible fluids and is called as Continuity equation. If the fluid is incompressible, then $\rho_1 = \rho_2$ and continuity equation reduced to,

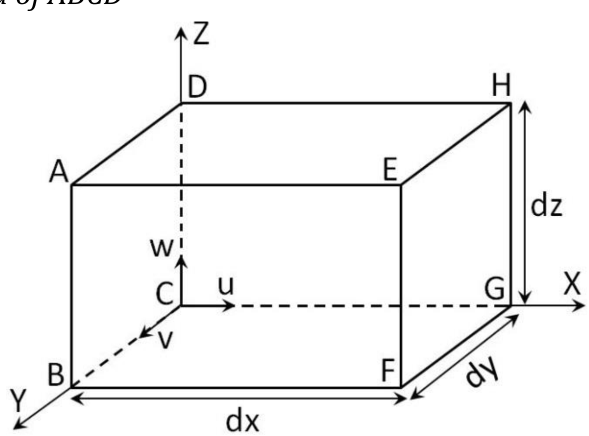
$$A_1 V_1 = A_2 V_2$$

Continuity Equation in Three Dimensions

Consider a fluid element of lengths dx, dy and dz in the direction of x, y and z. Let u, v and w are the inlet velocity components in x, y and z direction respectively. Mass of fluid entering the face ABCD per second,

= ρ × *velocity in x-direction × area of ABCD*

$= \rho \times u \times (dy \times dz)$



Then mass of fluid leaving the face EFGH per second

$= \rho\, u\, (dy.dz) + \frac{\partial}{\partial x} (\rho\, u\, dy\, dz)\, dx$

Gain of mass in x-direction

= Mass through ABCD – Mass through EFGH

$= \rho\, u\, (dy.dz) - \rho\, u\, (dy.dz) - \frac{\partial}{\partial x} (\rho\, u\, dy\, dz)\, dx$

$= -\frac{\partial}{\partial x} (\rho\, u)\, dx\, dy\, dz$

Similarly, the gain of mass in y-direction

$$= -\frac{\partial}{\partial y}(\rho\, v)\, dx\, dy\, dz$$

And in z-direction $= -\frac{\partial}{\partial z}(\rho\, w)\, dx\, dy\, dz$

Net gain of masses = $-\left[\frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w)\right] dx\, dy\, dz$

Since the mass is neither created nor destroyed in the fluid element, the net increase of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the fluid in the element. But mass of fluid in the element is ρ, dx, dy, dz and its rate of increase with time is $\frac{\partial}{\partial t}(\rho\, dx\, dy\, dz)$ *or* $\frac{\partial \rho}{\partial t}(dx\, dy\, dz)$

Equating the two expressions,

$$-\left[\frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w)\right](dx\, dy\, dz) = \frac{\partial \rho}{\partial t}(dx\, dy\, dz)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w) = 0$$

Above equation is the continuity equation in Cartesian coordinates in its most general form. This equation is applicable to,

1) Steady and unsteady flow
2) Uniform and non uniform flow
3) Compressible and incompressible flow

For steady flow, $\frac{\partial \rho}{\partial t} = 0$, *equation becomes,*

$$\frac{\partial}{\partial x}(\rho\, u) + \frac{\partial}{\partial y}(\rho\, v) + \frac{\partial}{\partial z}(\rho\, w) = 0$$

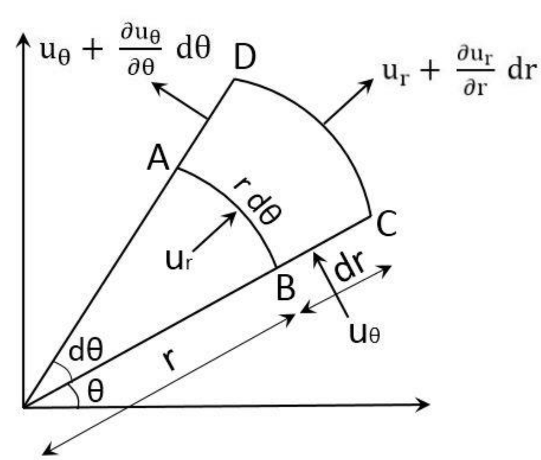
If the fluid is incompressible, then ρ is constant and above equation becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Above equation is three dimensional continuity equation in Cartesian coordinates.

Continuity Equation in Cylindrical Polar Co-ordinates

Consider a two dimensional incompressible flow field. The two dimensional polar co-ordinates are r and θ. Consider a fluid element ABCD between the radii r and r+dr as shown in figure. The angle subtended by the element at the centre is dθ. The components of the velocity V are u_r *in the radial direction and* u_θ *in the tangential direction. The sides of the element are having the length as,*



Side $AB = r.d\theta$, $BC = dr$,
$DC = (r+dr).d\theta$, $AD = dr$

The thickness of the element perpendicular to the plane of the paper is assumed to be unity.

Consider the flow in radial direction
Mass of fluid entering the face AB per unit time

$= \rho \times$ *Velocity in r – direction* $\times$ *Area*

$= \rho \times u_r \times (AB \times 1)$ $\qquad\because$ *Area = AB* $\times$ *thickness* $= (r\, d\theta \times 1)$

$= \rho \times u_r \times (r\, d\theta \times 1)$

$= \rho\, u_r\, r\, d\theta$

Mass of fluid leaving the face CD per unit time

$= \rho \times$ *Velocity* $\times$ *Area*

$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}\, dr\right) \times (CD \times 1)$ $\qquad\because$ *Area = CD* $\times$ *thickness* $= (CD \times 1)$

$= \rho \times \left(u_r + \frac{\partial u_r}{\partial r}\, dr\right) \times (r + dr)\, d\theta$ $\qquad\because$ *CD = (r+dr) dθ*

$= \rho \times \left(u_r\, r + u_r\, dr + r\, \frac{\partial u_r}{\partial r}\, dr + \frac{\partial u_r}{\partial r}\, dr^2\right) \times d\theta$

$= \rho \times \left(u_r\, r + u_r\, dr + r\, \frac{\partial u_r}{\partial r}\, dr\right) \times d\theta$

The term containing dr^2 *is very small and has been neglected*

Gain of mass in r – direction per unit time

= (Mass through AB – Mass through CD) per unit time

$= \rho\, u_r\, r\, d\theta - \rho\left(u_r\, r + u_r\, dr + r\, \frac{\partial u_r}{\partial r}\, dr\right) d\theta$

$= \rho\, u_r\, r\, d\theta - \rho\, u_r\, r\, d\theta - \rho\left(u_r\, dr + r\, \frac{\partial u_r}{\partial r}\, dr\right) d\theta$

$= -\rho\left(u_r\, dr + r\, \frac{\partial u_r}{\partial r}\, dr\right) d\theta$

$= -\rho\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right) r\;\, dr\, d\theta$ $\qquad$ *This is written in this form because* $(r.dr.d\theta.1)$ *is equal to volume of element*

Now consider the flow in θ – direction

Gain of mass in θ – direction per unit time

= (Mass through BC – Mass through AD) per unit time

= (ρ × Velocity through BC × Area) – (ρ × Velocity through AD × Area)

$= \rho\, u_\theta\, dr \times 1 - \rho\left(u_\theta + \frac{\partial u_\theta}{\partial \theta}\, d\theta\right) dr \times 1$

$= -\rho\left(\frac{\partial u_\theta}{\partial \theta}\, d\theta\right) dr \times 1$ $\qquad\because$ *Area = dr × 1*

$= -\rho\, \frac{\partial u_\theta}{\partial \theta}\, \frac{r\, d\theta\, dr}{r}$ $\qquad$ *Multiplying and dividing by r*

Total gain in fluid mass per unit mass

= (Gain of mass in r – direction + Gain of mass in θ – direction) per unit time

$= -\rho\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right) r\;\, dr\, d\theta - \rho\, \frac{\partial u_\theta}{\partial \theta}\, \frac{r\, d\theta\, dr}{r}$ $\qquad$ *… Equation A*

But mass of fluid element

$= \rho \times$ *Volume of fluid element*

$= \rho \times (r\, d\theta \times dr \times 1)$

$= \rho \times r\, d\theta \times dr$

Rate of increase of fluid mass in the element with time

$$= \frac{\partial}{\partial t}(\rho \, r d\theta \, dr) \qquad \text{... Equation B}$$

$$= \frac{\partial \rho}{\partial t}(r \, d\theta \, dr) \qquad \because \; r \, d\theta \, dr \; \times 1 \text{ is the volume of element and is a constant.}$$

Since the mass is neither created nor destroyed in the fluid element, hence net gain of mass per unit time in the fluid element must be equal to the rate of increase of mass of fluid in the element.

Hence equating the two expressions given by equation A and B, we get

$$-\rho \left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right) r \; dr \, d\theta - \rho \, \frac{\partial u_\theta}{\partial \theta} \; \frac{r \, d\theta \, dr}{r} = \frac{\partial \rho}{\partial t}(r \, d\theta \, dr)$$

$$-\rho \left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right) - \rho \, \frac{\partial u_\theta}{\partial \theta} \; \frac{1}{r} = \frac{\partial \rho}{\partial t}$$

$$\frac{\partial \rho}{\partial t} + \rho \left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right) + \rho \, \frac{\partial u_\theta}{\partial \theta} \; \frac{1}{r} = 0$$

Above equation is known as continuity equation in polar co-ordinates for two dimensional flow.

For steady flow, $\frac{\partial \rho}{\partial t} = 0$

Above equation reduced to,

$$\rho \left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r}\right) + \rho \, \frac{\partial u_\theta}{\partial \theta} \; \frac{1}{r} = 0$$

$$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} \; \frac{1}{r} = 0$$

$$u_r + r \, \frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0$$

Or $$\frac{\partial}{\partial r}(r.u_r) + \frac{\partial}{\partial \theta}(u_\theta) = 0 \qquad \because \frac{\partial}{\partial r}(r.u_r) = u_r + r\,\frac{\partial u_r}{\partial r}$$

Above equation represents the continuity equation in polar co-ordinates for two dimensional steady incompressible flow.

Velocity and Acceleration

Let 'V' is the resultant velocity at any point in a fluid flow. Let u, v and w are its component in x, y and z directions. The velocity components are function of space co-ordinates and time.

Mathematically the velocity components are given as

$$u = f_1(x, y, z, t)$$

$$v = f_2(x, y, z, t)$$

$$w = f_3(x, y, z, t)$$

and *Resultant velocity,* $V = ui + vj + wk = \sqrt{u^2 + v^2 + w^2}$

Let a_x a_y and a_z are the total acceleration in x, y and z direction respectively. Then by the chain rule of differentiation, we have

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t}$$

But $\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w,$

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

Similarly, $$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

For steady flow, $\frac{\partial V}{\partial t} = 0$, *where V is resultant velocity.*

Or $$\frac{\partial u}{\partial t} = 0 \qquad \frac{\partial v}{\partial t} = 0 \qquad \frac{\partial w}{\partial t} = 0$$

Hence acceleration in x, y and z directions becomes

$$a_x = \frac{du}{dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{dv}{dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{dw}{dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Acceleration vector

$$A = a_x i + a_y j + a_z k = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Local Acceleration and Convective Acceleration

For example,

$$a_x = \frac{du}{dt} = \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{Convective Term}} + \overbrace{\frac{\partial u}{\partial t}}^{\text{Local Term}}$$

Local acceleration is defined as the rate of increase of velocity with respect to time at a given point in a flow field. In the above equations, $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ *and* $\frac{\partial w}{\partial t}$ *is known as local acceleration.*

Convective acceleration is defined as the rate of change of velocity due to the change of position of fluid particles in a fluid flow. The expression other than $\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}$ *and* $\frac{\partial w}{\partial t}$ *this term is called convective acceleration.*

Velocity Potential Function and Stream Function

Velocity Potential Function

It is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. It is defined by ϕ *(phi). Mathematically, the velocity potential function is defined as* $\phi = f(x, y, z)$ *for steady flow such that*

$$u = -\frac{\partial \phi}{\partial x} \qquad v = -\frac{\partial \phi}{\partial y} \qquad w = -\frac{\partial \phi}{\partial z}$$

where u, v and w are the components of velocity in x, y and z directions respectively.

The velocity components in cylindrical polar co-ordinates in terms of velocity potential function are given by

$$u_r = \frac{\partial \phi}{\partial r} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

where, u_r = *velocity component in radial direction (i.e. in r – direction)*

u_θ = *velocity component in tangential direction (i.e. in* θ *– direction)*

The continuity equation for an incompressible steady flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Substituting the values of u, v and w in terms of velocity potential function, we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right)=0$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

Above equation is called as Laplace equation.

For two dimensional case, equation reduced to, $\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0$

If any value of ϕ that satisfies the Laplace equation, will corresponding to some case of fluid flow.

Properties of the Potential Function. The rotational components are given by

$$\omega_x=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right) \quad \omega_y=\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right) \quad \omega_z=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

Substituting the values of u, v and w in terms of velocity potential function, we get

$$\omega_x=\frac{1}{2}\left[\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial z}\right)-\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial y}\right)\right] \quad =\frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial y\,\partial z}+\frac{\partial^2 \phi}{\partial z\,\partial y}\right]$$

$$\omega_y=\frac{1}{2}\left[\frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial x}\right)-\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial z}\right)\right] \quad =\frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial z\,\partial x}+\frac{\partial^2 \phi}{\partial x\,\partial z}\right]$$

$$\omega_z=\frac{1}{2}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial x}\right)\right] \quad =\frac{1}{2}\left[-\frac{\partial^2 \phi}{\partial x\,\partial y}+\frac{\partial^2 \phi}{\partial y\,\partial x}\right]$$

If ϕ is a continuous function, then

$$\frac{\partial^2 \phi}{\partial x\,\partial y}=\frac{\partial^2 \phi}{\partial y\,\partial x} \qquad \frac{\partial^2 \phi}{\partial z\,\partial x}=\frac{\partial^2 \phi}{\partial x\,\partial z} \qquad \frac{\partial^2 \phi}{\partial y\,\partial z}=\frac{\partial^2 \phi}{\partial z\,\partial y}$$

So $\omega_x=\omega_y=\omega_z=0$

When rotational components are zero, the flow is called irrotational. Hence the properties of the potential function are:

1) If velocity potential (ϕ) exists, the flow should be irrotational

2) If velocity potential (ϕ) satisfies the Laplace equation, it represents the possible steady incompressible irrotational flow.

Stream Function

It is defined as the scalar function of space and time, such that its partial derivative with respect to any direction gives the velocity component at right angles to that direction. It is denoted by ψ (Psi) and defined only for two dimensional flow. Mathematically, for steady flow it is defined as $\psi = f(x, y)$, such that $\frac{\partial \psi}{\partial x}=v$ *and* $\frac{\partial \psi}{\partial y}=-u$

The velocity components in cylindrical polar co-ordinates in terms of stream function are given as

$$u_r=\frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad u_\theta=-\frac{\partial \psi}{\partial r}$$

where, u_r = radial velocity and u_θ = tangential velocity

The continuity equation for two dimensional flow is, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

Substituting the values of u and v in terms of stream function, we get

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=0 \quad or \quad -\frac{\partial^2 \psi}{\partial x\,\partial y}+\frac{\partial^2 \psi}{\partial x\,\partial y}=0$$

Hence existence of ψ means a possible case of fluid flow. The flow may be rotational or irrotational.

The rotational component ω_z is given by, $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Substituting the values of u and v in terms of stream function in above rotational component

$$\omega_z = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)\right] \qquad = \frac{1}{2}\left[\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right]$$

For irrotational flow, $\omega_z = 0$

Hence above equation becomes as $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

Which is Laplace equation for ψ.

The properties of stream function (ψ) are:

1) If stream function (ψ) exists, it is a possible case of fluid flow which may be rotational or irrotational

2) If stream function (ψ) satisfies the Laplace equation, it is a possible case of an irrotational flow.

Equipotential Line

A line along which the velocity potential ϕ is constant, is called equipotential line.

For equipotential line, ϕ = *constant*

$d\phi = 0$

But, $\phi = f(x, y)$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$= -u\,dx - v\,dy \qquad \because \quad \frac{\partial \phi}{\partial x} = -u \qquad \frac{\partial \phi}{\partial y} = -v$

$= -(u\,dx + v\,dy)$

For equipotential line, $d\phi = 0$

Or, $-(u\,dx + v\,dy) = 0$ *Or* $(u\,dx + v\,dy) = 0$

$$\frac{dy}{dx} = -\frac{u}{v} \qquad \text{...Equation (I)}$$

But $\frac{dy}{dx}$ = *Slope of equipotential line*

Line of constant Stream Function

ψ = *Constant*

$d\psi = 0$

But, $d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$

$= +v\,dx - u\,dy \qquad \because \quad \frac{\partial \psi}{\partial x} = v \qquad \frac{\partial \psi}{\partial y} = -u$

For a line of constant stream function

$= d\psi = 0$ *Or* $v\,dx - u\,dy = 0$

$$\frac{dy}{dx} = -\frac{v}{u} \qquad \text{...Equation (II)}$$

But $\frac{dy}{dx}$ = *Slope of stream line*

From equation I and II, it is cleared that the product of the slope of the equipotential line and the slope of the stream line at the point of intersection is equal to -1. Thus the equipotential line are orthogonal to the stream lines at all points of intersection.

Fluid Mechanics

Flow net

A grid obtained by drawing a series of equipotential lines and stream lines is called a flow net. The flow net is an important tool in analyzing two dimensional irrotational flow problems.

Relation between Stream Function and Velocity Potential Function

We have, $\frac{\partial \phi}{\partial x} = -u \qquad \frac{\partial \phi}{\partial y} = -v \qquad and \qquad \frac{\partial \psi}{\partial x} = v \qquad \frac{\partial \psi}{\partial y} = -u$

Thus, we have $u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \qquad and \qquad v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$

Hence

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

And $\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

Vortex Flow

Vortex flow is defined as the flow of a fluid along a curved path or the flow of a rotating mass of fluid is known a "Vortex Flow". The vortex flow is of two types namely:

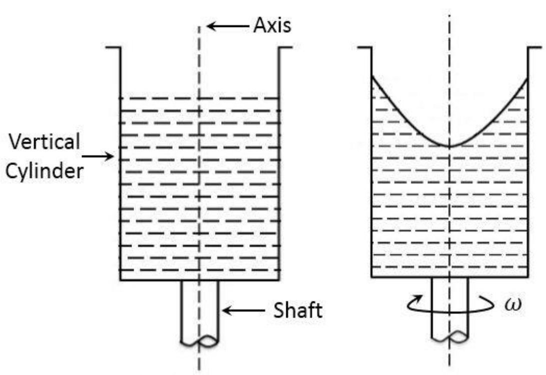
1) Forced vortex flow
2) Free vortex flow

Forced vortex flow

Forced vortex flow is defined as that type of vortex flow, in which some external torque is required to rotate the fluid mass. The fluid mass in this type of flow, rotates at constant angular velocity, ω. The tangential velocity of any fluid particle is given by

v = r ω

where, *r = radius of fluid particle from the axis of rotation*



A) Cylinder is stationary B) Cylinder is rotating

Hence angular velocity ω is given by

$$\omega = \frac{v}{r} = constant$$

Example of forced vortex are:

1) A vertical cylinder containing liquid which is rotated about its central axis with a constant angular velocity ω, as shown in figure.
2) Flow of liquid inside the impeller of a centrifugal pump.
3) Flow of water through the runner of a turbine.

Free Vortex Flow

When no external torque is required to rotate the fluid mass, that type of flow is called free vortex. Thus the liquid in case of free vortex is rotating due to the rotation which is imparted to the fluid previously.

Example of the free vortex flow are;

1) Flow of liquid through a hole provided at the bottom of a container.
2) Flow of liquid around a circular bend in a pipe.
3) A whirlpool in a river.
4) Flow of fluid in a centrifugal pump casing.

The relation between velocity and radius, in free vortex is obtained by putting the value of external torque equal to zero, or the time of change of angular momentum, i.e. moment of momentum must be zero. Consider a fluid particle of mass "m" at a radial distance r from the axis of rotation, having a tangential velocity v. Then

Angular momentum = Mass x Velocity = m v

Moment of momentum = Momentum x r = m v r

Time rate of change of angular momentum = $\frac{\partial}{\partial x}(mrv)$

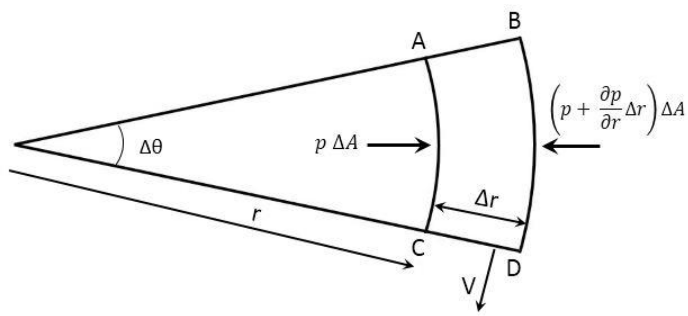
For free vortex $\frac{\partial}{\partial x}(mrv) = 0$

Integrating, we get *m r v = constant*

or $v\,r = \frac{Constant}{m}$ *= Constant*

Equation of Motion for Vortex Flow

Consider a fluid element ABCD (shown in figure) rotating at a uniform velocity in a horizontal plane about an axis perpendicular to the plane of paper and passing through O.
Let,



r = radius of the element from O
Δθ = angle subtended by the element at O
Δr = radial thickness of the element
ΔA = area of cross section of element

The force acting on the element are
1) Pressure force, p. ΔA, on the face AB
2) Centrifugal force (mv^2)/r acting in the direction away from the centre O.

Now, the mass of the element = Mass density × Volume

$$= \rho \times \Delta A \times \Delta r$$

Centrifugal force $= \rho.\Delta A.\Delta r\,\frac{V^2}{r}$

Equating the forces in the radial direction, we get

$$\left(p + \frac{\partial p}{\partial r}\Delta r\right)\Delta A - p.\Delta A = \rho.\Delta A.\Delta r\,\frac{V^2}{r}$$

Or, $$\frac{\partial p}{\partial r}\,\Delta r\,\Delta A = \rho.\Delta A.\Delta r\,\frac{V^2}{r}$$

Cancelling $(\Delta r.\Delta A)$ *from both sides, we get* $\frac{\partial p}{\partial r} = \rho\,\frac{V^2}{r}$ *...Equation (I)*

Equation (I) gives the pressure variation along the radial direction for a forced or free vortex flow in a horizontal plane. The expression $\left(\frac{\partial p}{\partial r}\right)$ *is called pressure gradient in the radial direction. As* $\left(\frac{\partial p}{\partial r}\right)$ *is positive, hence pressure increases with increase with the increase of radius 'r'.*

The pressure variation in the vertical plane is given by the hydrostatic law, i.e.

$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{...Equation (II)}$$

In equation (II), 'z' is measured vertically in the upward direction.

The pressure, p varies with respect to 'r' and 'z' or 'p' is a function of 'r' and 'z' and hence total derivatives of 'p' is

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz$$

Substituting the values of $\left(\frac{\partial p}{\partial r}\right)$ *from equation (I) and* $\left(\frac{\partial p}{\partial z}\right)$ *from equation (II), we get*

$$dp = \rho \frac{V^2}{r} dr - \rho g\, dz \qquad \text{...Equation (III)}$$

Equation (III) gives the variation of pressure of a rotating fluid in any plane.

Equation of Forced Vortex Flow

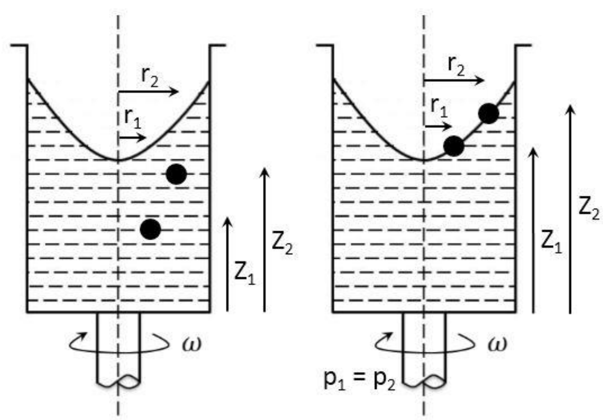
For the forced vortex flow, we have $v = \omega\, r$

where ω *= angular velocity = constant*

Substituting the value of 'v' in equation (III), we get

$$dp = \rho \frac{(\omega\, r)^2}{r} dr - \rho g\, dz$$

Consider two points 1 and 2 in the fluid having forced vortex flow as shown in figure.



Integrating the above equation for points 1 and 2, we get

$$\int_1^2 dp = \int_1^2 \rho\, \omega^2\, r\, dr - \int_1^2 \rho g\, dz$$

Or,
$$(p_2 - p_1) = \left[\rho\, \omega^2\, \frac{r^2}{2}\right]_1^2 - \rho g\, [z]_1^2$$

Or,
$$(p_2 - p_1) = \rho\, \omega^2\, [r_2^2 - r_1^2] - \rho g\, [z_2 - z_1]$$

$$(p_2 - p_1) = \frac{\rho}{2}\, [\omega^2\, r_2^2 - \omega^2\, r_1^2] - \rho g\, [z_2 - z_1]$$

$$(p_2 - p_1) = \frac{\rho}{2}\, [v_2^2 - v_1^2] - \rho g\, [z_2 - z_1] \qquad \because \quad v = \omega\, r$$

If the points 1 and 2 lie on the free surface of the liquid, then $p_1 = p_2$ and hence above equation becomes

$$0 = \frac{\rho}{2}\ [v_2^2 - v_1^2] - \rho g\ [z_2 - z_1]$$

Or,
$$\rho g\ [z_2 - z_1] = \frac{\rho}{2}\ [v_2^2 - v_1^2]$$

$$[z_2 - z_1] = \frac{1}{2\,g}\ [v_2^2 - v_1^2]$$

If the point 1 lies on the axis of rotation, then $v_1 = \omega\ r_1 = \omega\ (0) = 0$. The above equation becomes as

$$[z_2 - z_1] = \frac{1}{2\,g}\ [v_2^2] = \frac{v_2^2}{2\,g}$$

Let $z_2 - z_1 = Z$

then we have $$Z = \frac{v_2^2}{2\,g} = \frac{\omega^2\,r_2^2}{2\,g}$$

Thus Z varies with the square of 'r'. Hence above equation is an equation of parabola. This means the free surface of the liquid is a parabolic.

Closed Cylindrical Vessels

If a cylindrical vessel is closed at the top, which contains some liquid, the shape of parabolic formed due to rotation of the vessel will as shown in figure for different speed of rotations.

Figure (A) shown the initial stage of the cylinder, when it is not rotated. Figure (B) shows the shape of the parabolic formed when the speed of rotation is ω_1. If the speed is increased further say ω_2, the shape of parabolic formed will be as shown in figure (C). In this case the radius of the parabola at the top of the vessel is unknown. Also the height of the parabolic formed corresponding to angular speed ω_2 is unknown. Thus to solve the two unknown, we should have two equations. One equation is

$$Z = \frac{v_2^2}{2\,g} = \frac{\omega^2\,r_2^2}{2\,g}$$

The second equation is obtained from the fact that for closed vessel, volume of air before rotation is equal to the volume of air after rotation.

Volume of air before rotation = Volume of closed vessel – Volume of liquid in vessel

Volume of air before rotation = Volume of parabolic formed = $\frac{\pi\,r^2 \times Z}{2}$.

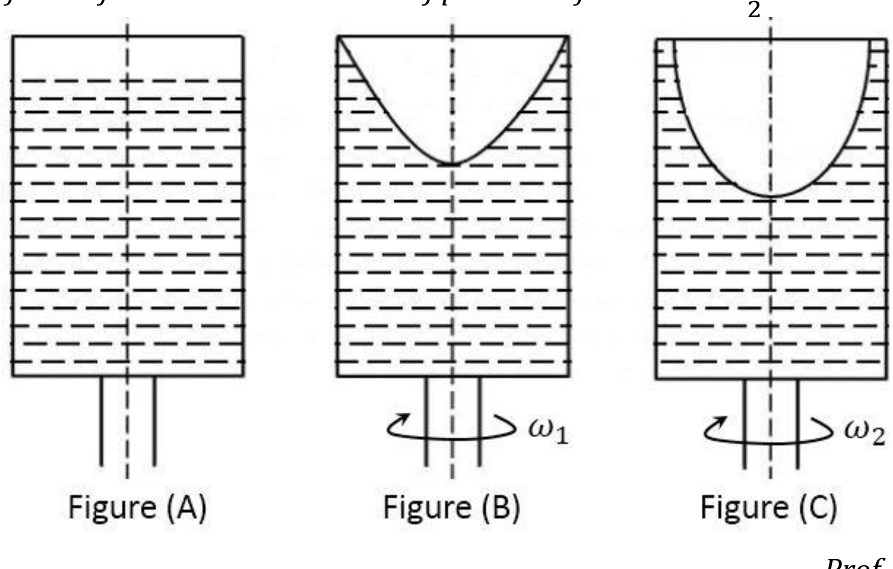


Figure (A) Figure (B) Figure (C)

Ideal Flow (Potential Flow)

Ideal flow is a fluid which is incompressible and inviscid. Incompressible fluid is a fluid for which density remains constant. Inviscid fluid is a fluid for which viscosity is zero. Hence a fluid for which density is constant and viscosity is zero, is known as an ideal fluid.

The shear stress is given by, $\tau = \mu \frac{du}{dy}$*. Hence for ideal fluid the shear stress will be zero as* $\mu = 0$ *for ideal fluid. Also the shear force (which is equal to shear stress multiplied by area) will be zero in case of ideal or potential flow. The ideal fluids will be moving with uniform velocity. All the fluid particles will be moving with the same velocity.*

The concept of ideal fluid simplifies the typical mathematical analysis. Fluids such as water and air have low viscosity. Also when the speed of air is appreciably lower than that of sound in it, the compressibility is so low that air is assumed to be incompressible. Hence under certain conditions, certain real fluids such as water and air may be treated like ideal fluids.

Important cases of Potential flow

The following are the important cases of potential flow

1) Uniform flow   *2) Source flow*
3) Sink flow   *4) Free vortex flow*
5) Superimposed flow

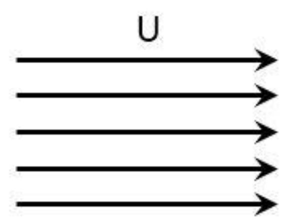
1) Uniform flow

In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity. The uniform flow may be

i) Parallel to x-axis   *ii) Parallel to y-axis*

Uniform Flow Parallel to x – axis

Figure shows the uniform flow parallel to x-axis. In a uniform flow, the velocity remains constant. All the fluid particles are moving with the same velocity.



Let, U *= Velocity which is uniform or constant along x-axis*
u *&* v *= Components of uniform velocity U along x & y axis*

For the uniform flow, parallel to x-axis, the velocity components u and v are given as

$$u = U \quad \text{and} \quad v = 0 \qquad \text{...Equation (I)}$$

But the velocity u in terms of stream function is given by

$$u = \frac{\partial \psi}{\partial y}$$

and in terms of velocity potential the velocity u is given by

$$u = \frac{\partial \phi}{\partial x}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \text{...Equation (II)}$$

Similarly, it can be shown that

$$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad \text{...Equation (II)}$$

But u = U from equation (I) and (II), we have

$$U = \frac{\partial \psi}{\partial y} \quad \text{and also} \quad U = \frac{\partial \phi}{\partial x}$$

First part gives $\partial\psi = U\,dy$ *whereas second part gives* $\partial\phi = U\,dx$

Integrating of these parts gives as

$$\psi = Uy + C_1 \qquad \text{and} \qquad \phi = Ux + C_2$$

where C_1 and C_2 are constant of integration.

Now let us plot the stream lines and potential lines for uniform flow parallel to x-axis.

Plotting of Stream Lines

For stream lines, the equation is

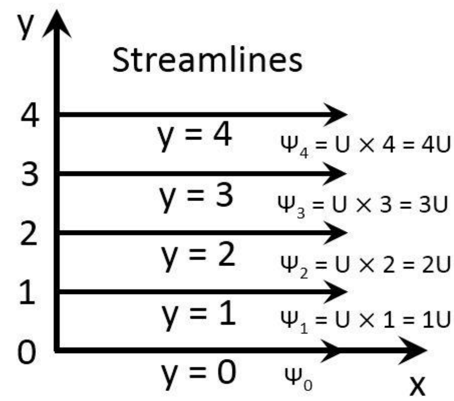
$$\psi = Uy + C_1$$

Let, $\psi = 0$, *where y = 0. Substituting these values in the above equation, we get*

$$0 = U(0) + C_1 \qquad or \qquad C_1 = 0$$

Hence the equation of stream lines becomes as

$$\psi = Uy \qquad \text{...Equation (III)}$$



The stream lines are straight lines parallel to x-axis and at a distance y from the x-axis as shown in figure. In equation (III), 'U.y' represents the volume flow rate (i.e. m³/s) between x-axis and that stream line at a distance y.

Note: The thickness of the fluid stream perpendicular to the plane is assumed to be unity. Then 'y × 1' or 'y' represents the area of flow. And 'U.y' represents the product of velocity and area. Hence 'U.y' represents the volume flow rate.

Plotting of Potential lines:

For potential lines, the equation is

$$\phi = Ux + C_2$$

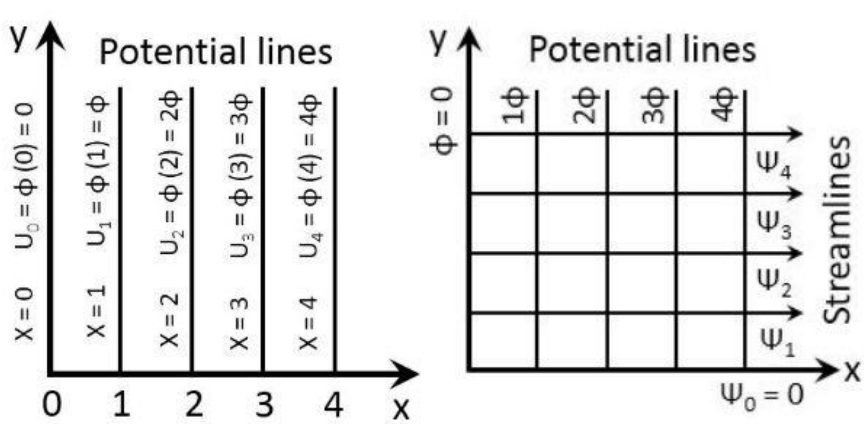
Let, $\phi = 0$, *where* $x = 0$. *Substituting these values in above equation, we get $C_2 = 0$*

Hence equation of potential lines becomes as

$$\phi = Ux$$

The above equation shows that potential lines are straight lines parallel to y-axis and at a distance of 'x' from y-axis as shown in figure.

Next figure shows the plot of stream lines and potential lines for uniform flow parallel to x-axis. The stream lines and potential lines intersect each other at right angles.



Uniform Flow Parallel to y-axis

Figure shows the uniform potential flow parallel to y-axis in which U is the uniform velocity along y-axis.

The velocity components u, v along the x-axis and y-axis are given by

$$u = 0 \quad \text{and} \quad v = U \qquad \text{...Equation (I)}$$

These velocity components in terms of stream function (Ψ) and velocity potential function (ϕ) are given as

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \qquad \text{...Equation (II)}$$

and $$v = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad \text{...Equation (III)}$$

But from equation (I), v = U. substituting v = U in equation (III), we get

$$U = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \qquad \text{Or} \qquad U = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad U = \frac{\partial \phi}{\partial y}$$

First part gives $\partial\psi = -U\,dx$ *whereas second part gives* $\partial\phi = U\,dy$

Integrating of these parts gives as

$$\psi = -Ux + C_1 \qquad \text{and} \qquad \phi = Uy + C_2$$

where C_1 and C_2 are constant of integration. Let us now plot the stream lines and potential lines.

Plotting of Stream lines:

For stream lines, the equation is $\psi = -Ux + C_1$

Let, $\psi = 0$*, where x = 0. Then $C_1 = 0$*

Hence the equation of stream lines becomes as $\psi = -Ux$

The above equation shows that stream lines are straight lines parallel to y-axis and at a distance of 'x' from the y-axis as shown in figure. The –ve sign shows that the stream lines are in the downward direction.

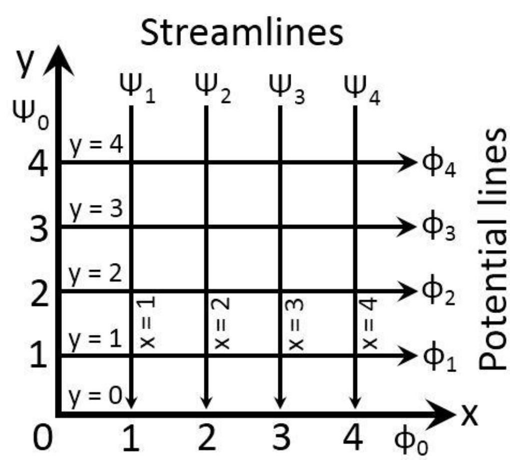
Plotting of Potential lines:

For potential lines, the equation is $\phi = Uy + C_2$

Let, $\phi = 0$*, where y = 0. Then $C_2 = 0$*

Hence the equation of potential lines becomes as $\phi = Uy$

The above equation shows that potential lines are straight lines parallel to x-axis and at a distance of 'y' from the x-axis as shown in figure.



2) Source Flow

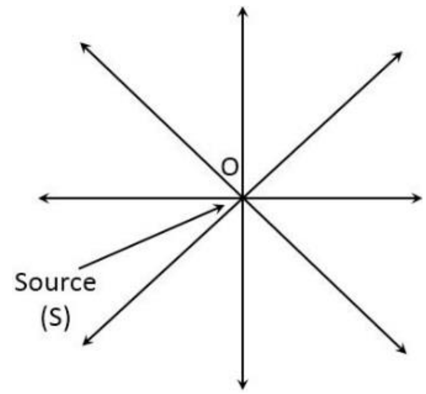
The source flow is the flow coming from a point (source) and moving out radially in all directions of a plane at uniform rate. Figure shows a source flow in which the point O is the source from which the fluid moves radially outward. The strength of a source is defined as the volume flow rate per unit depth. The unit of strength of source is m^2/s. It is represented by 'q'.

Let,

u_r *= radial velocity of flow at a radius 'r' from the source O*

q = volume flow rate per unit depth

r = radius



The radial velocity u_r, at any radius 'r' is given by

$$u_r = \frac{q}{2\,\pi\,r}$$

The above equation shows that with the increase of 'r', the radial velocity decreases. And at a large distance away from the source, the velocity will be approximately equal to zero. The flow is in radial direction, hence the tangential velocity $u_\theta = 0$

Let us now find the equation of stream function and velocity potential function for the source flow. As in this case, $u_\theta = 0$, the equation of stream functions and velocity potential function will be obtained from u_r

Equation of Stream function:

By definition, the radial velocity and tangential velocity components in terms of stream function are given by

$$u_r = \frac{1}{r}\frac{\partial\Psi}{\partial\theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial\Psi}{\partial r}$$

But $$u_r = \frac{q}{2\,\pi\,r}$$

$$\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{q}{2\,\pi\,r}$$

$$d\Psi = r\,\frac{q}{2\,\pi\,r}\,d\theta = \frac{q}{2\,\pi}\,d\theta$$

Integrating the above equation w.r.t θ, we get

$$\Psi = \frac{q}{2\,\pi}\,\theta + C_1$$

Let, $\Psi = 0$*, when* $\theta = 0$*, then* $C_1 = 0$

Hence the equation of stream function becomes as

$$\Psi = \frac{q}{2\,\pi}\,\theta$$

In the above equation, 'q' is constant.

The above equation shows that stream function is a function of θ. For a given value of θ, the stream function Ψ will be constant. And this will be a radial line. The stream lines can be plotted by having different values of θ. Hence θ is taken in radians.

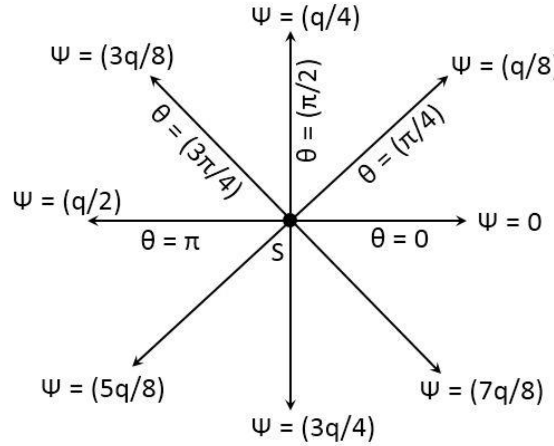
Plotting of stream lines
When θ = 0, Ψ = 0

$\theta = 45^0 = \frac{\pi}{4}$ *radians,* $\Psi = \frac{q}{2\,\pi}\,\frac{\pi}{4} = \frac{q}{8}$ *units*

$\theta = 90^0 = \frac{\pi}{2}$ *radians,* $\Psi = \frac{q}{2\,\pi}\,\frac{\pi}{2} = \frac{q}{4}$ *units*

$\theta = 135^0 = \frac{3\pi}{4}$ *radians,* $\Psi = \frac{q}{2\,\pi}\,\frac{3\,\pi}{4} = \frac{3\,q}{8}$ *units*

The stream lines will be radial lines as shown in figure.



Equation of Potential Function:

By definition, the radial & tangential components in terms of velocity function are given by

$$u_r = \frac{\partial \phi}{\partial r} \qquad \text{and} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

But, we have

$$u_r = \frac{q}{2\,\pi\,r}$$

Equating the two values of u_r, we get

$$\frac{\partial \phi}{\partial r} = \frac{q}{2\,\pi\,r}$$

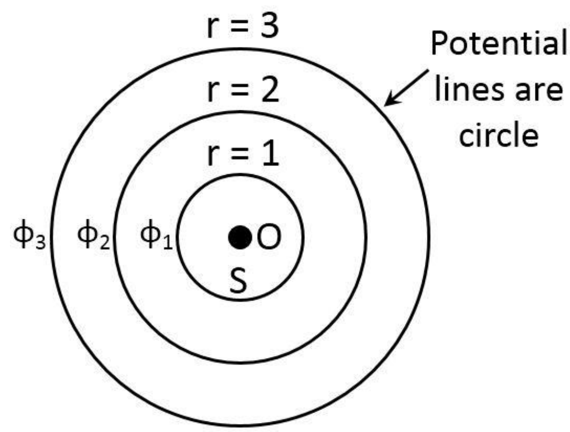
$$\partial \phi = \frac{q}{2\,\pi\,r}\,\partial r$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{q}{2\,\pi\,r}\,dr$$

$$\phi = \frac{q}{2\,\pi}\int \frac{1}{r}\,dr$$

$$\phi = \frac{q}{2\,\pi}\log_e r$$



In the above equation, 'q' is constant.

The above equation shows that the velocity potential function is a function of 'r'. For a given value of 'r', the velocity function ϕ will be constant. Hence it will be a circle with origin at the source. The velocity potential lines will be circles with origin at the source as shown in figure.

Pressure distribution in a plane source flow:

The pressure distribution in a plane source flow can be obtained with the help of Bernoulli's equation. Let us assume that the plane of the flow is horizontal. In that case the datum head will be same for two points of flow.

Let,

p = pressure at point 1 which is at a radius r from the source at point 1
u_r = velocity at point 1
p_0 = pressure at point 2, which is at a large distance away from the source. The velocity wil be zero at point 2.

Applying Bernoulli's equation, we get

$$\frac{p}{\rho\, g} + \frac{u_r^2}{2\, g} = \frac{p_0}{\rho\, g} + 0$$

Or

$$\frac{(p - p_0)}{\rho\, g} = -\frac{u_r^2}{2\, g}$$

$$(p - p_0) = -\frac{\rho\, u_r^2}{2}$$

But, we have

$$u_r = \frac{q}{2\,\pi\, r}$$

Substituting the value of u_r, in the above equation, we get

$$(p - p_0) = -\left(\frac{\rho}{2}\right)\left(\frac{q}{2\,\pi\, r}\right)^2$$

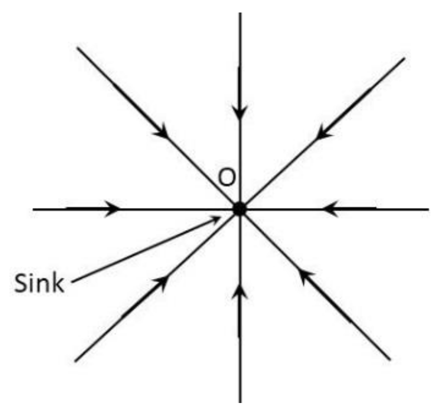
$$(p - p_0) = -\frac{\rho\, q^2}{8\,\pi^2\, r^2}$$

In the above equation, ρ and q are constant.

The above equation shows that the pressure is inversely proportional to the square of the radius from the source.

3) Sink Flow

The sink flow is the flow in which fluid moves radially inwards towards a point where it disappears at a constant rate. This flow is just opposite to the source flow. Figure shows a sink flow in which the fluid moves radially inwards towards point O, where it disappears at a constant rate. The pattern of stream lines and equipotential lines of a sink flow is the same as that of source flow. All the questions derived for a source flow shall hold good for sink flow except that in sink flow equations, 'q' is to be replaced by '- q'.



Free-Vortex Flow

Free vortex is a circulatory flow of a fluid such that its stream lines are concentric circles.

For a free vortex flow

$$u_\theta \times r = \text{constant (say } C)$$

Also, circulation around a stream line of an irrotation vortex is

$$\Gamma = 2\,\pi\, r \times u_\theta = 2\,\pi\, C$$

where u_θ = tangential velocity at any radius r from the center.

$$u_\theta = \frac{\Gamma}{2\,\pi\, r}$$

The circulation Γ is taken positive if the free vortex is anticlockwise.

For a free vortex flow, the velocity components are

$$u_\theta = \frac{\Gamma}{2\,\pi\, r} \quad \text{and} \quad u_r = 0$$

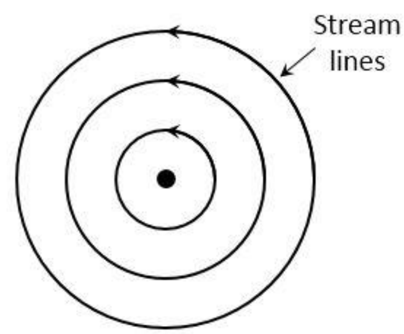
Equation of Stream Function

By definition, the stream function is given by

$$u_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial \Psi}{\partial r}$$

In case of free vortex flow, the radial velocity (u_r) is zero. Hence equation of stream function will be obtained from tangential velocity, u_θ. The value of u_θ is given by

$$u_\theta = \frac{\Gamma}{2\,\pi\,r}$$



Equating the two values of u_θ, we get

$$-\frac{\partial \Psi}{\partial r} = \frac{\Gamma}{2\,\pi\,r}$$

$$d\Psi = -\frac{\Gamma}{2\,\pi\,r}\,dr$$

Integrating the above equation, we get

$$\int d\Psi = \int -\frac{\Gamma}{2\,\pi\,r}\,dr = \left(-\frac{\Gamma}{2\,\pi}\right)\int \frac{1}{r}\,dr$$

Or $$\Psi = \left(-\frac{\Gamma}{2\,\pi}\right)\log_e r$$

The above equation shows that stream function is a function of radius. For a given value of r, the stream function is constant. Hence the stream lines are concentric circles as shown in figure.

Equation of Potential Function

By definition, the potential function is given by

$$u_r = \frac{\partial \phi}{\partial r} \qquad \text{and} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Here $u_r = 0$ & $u_\theta = \frac{\Gamma}{2\,\pi\,r}$. Hence, the equation of potential function will be obtained from u_θ.

Equating the two values of u_θ, we get

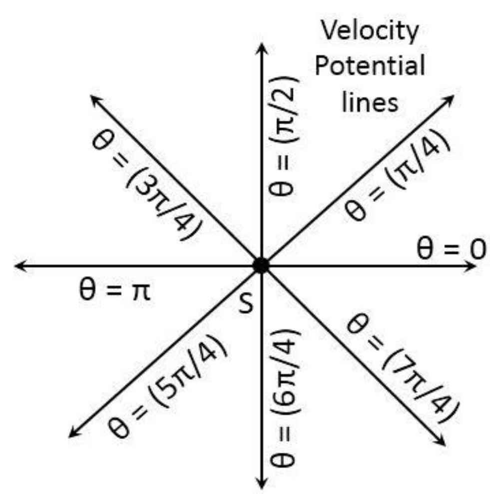
$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\,\pi\,r}$$

$$d\phi = r\,\frac{\Gamma}{2\,\pi\,r}\,d\theta = \frac{\Gamma}{2\,\pi}\,d\theta$$

Integrating the above equation, we get

$$\int d\phi = \int \frac{\Gamma}{2\,\pi}\,d\theta$$

Or $$\phi = \frac{\Gamma}{2\,\pi}\,\theta$$



The above equation shows that velocity potential function is a function of θ. For a given value of θ, potential function is a constant. Hence equipotential lines are radial as shown in figure.

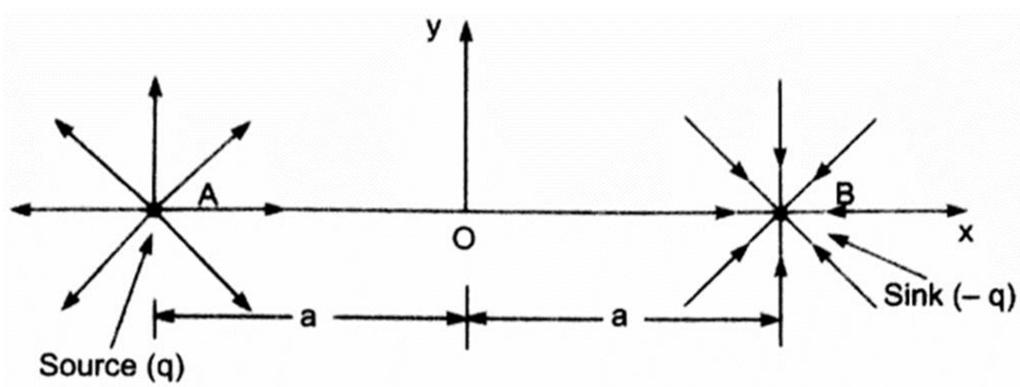
Super Imposed Flow

The flow pattern due to uniform flow, a source flow, a sink flow and a free vortex flow can be super imposed in any linear combination to get a resultant flow which closely resemble the flow around bodies. The resultant flow will still be potential and ideal. The following are the important super imposed flow.

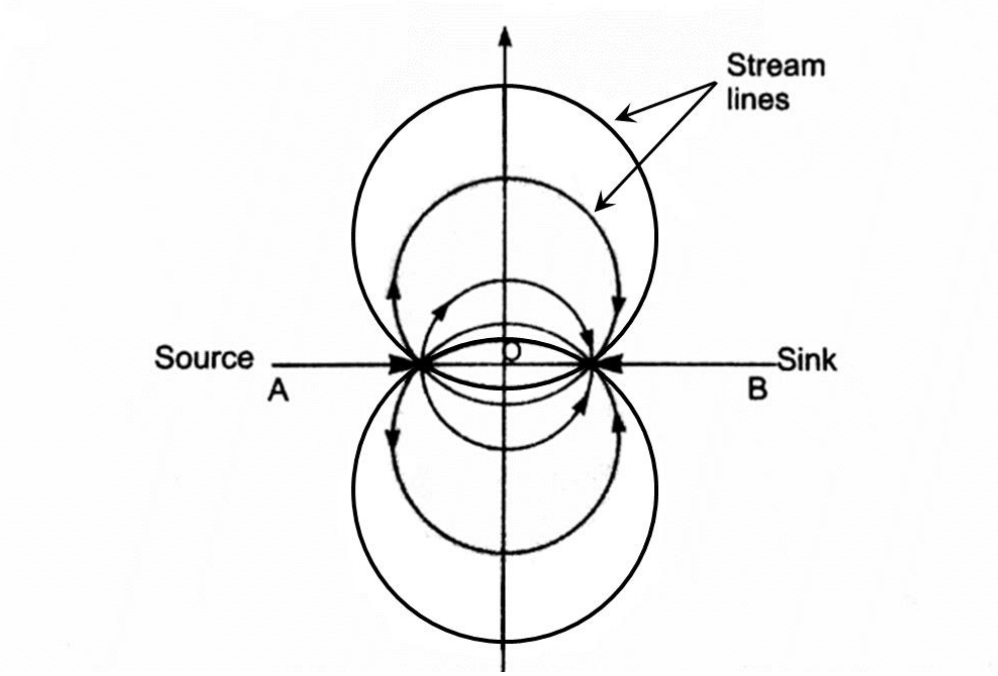
1) Source and sink pair
2) Doublet (special case of source and sink combination
3) A plane source in a uniform flow (flow past a half body)
4) A source and sink pair in a uniform flow
5) A doublet in a uniform flow

1) Source and sink flow

Figure shows a source and a sink of strength 'q' and (-q) placed at A and B respectively at equal distance from the point O on the x-axis. Thus the source and sink are placed symmetrically on the x-axis. The source of strength q is placed at A and sink of strength (-q) is placed at B. The combination of the source and the sink would result in a flownet where stream lines will be circular arcs starting from point A and ending at point B as shown in figure.



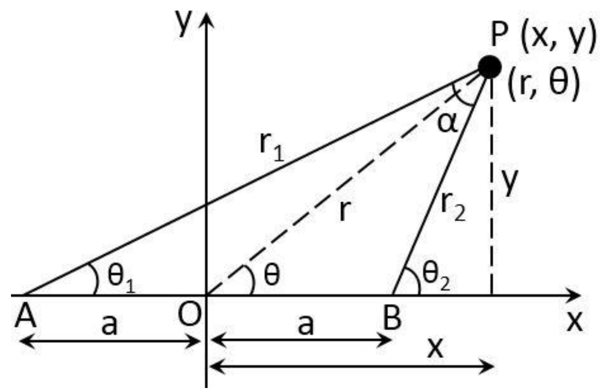
Source and sink pair



Equation of stream function and potential function

Let P be any point in the resultant flownet of source and sink as shown in figure.

Let, *r, θ = cylindrical coordinates of point P with respect to origin O*
x, y = corresponding coordinates of point P
r_1, θ_1 = position of point P with respect to source placed at A
r_2, θ_2 = position of point P with respect to sink placed at B
α = angle subtended at P by the join of source and sink i.e. angle APB



Let us find the equation for the resultant stream function and velocity potential function. The equation for stream function due to source is given as $\Psi_1 = \frac{q}{2\,\pi}\,\theta_1$ *whereas due to sink it is given by* $\Psi_2 = \frac{(-\,q)}{2\,\pi}\,\theta_2$ *. The equation for resultant stream function (Ψ) will be the sum of these two stream function.*

$$\Psi = \Psi_1 + \Psi_2$$

$$\Psi = \frac{q}{2\,\pi}\,\theta_1 + \frac{(-\,q)}{2\,\pi}\,\theta_2 = \frac{(-\,q)}{2\,\pi}\,(\theta_2 - \theta_1)$$

From the figure, $\alpha = \theta_2 - \theta_1$*. In triangle ABP,* $\theta_1 + \alpha + (180^0 - \theta_2) = 180^0$

$$\Psi = \frac{(-\,q)}{2\,\pi}\,\alpha$$

The equation for potential function due to source is given by $\phi_1 = \frac{q}{2\,\pi}\,\log_e r_1$ *and due to sink it is given by* $\phi_2 = \frac{(-\,q)}{2\,\pi}\,\log_e r_2$*. The equation for resultant potential function (ϕ) will be the sum of these two potential function.*

$$\phi = \phi_1 + \phi_2$$

$$\phi = \frac{q}{2\,\pi}\,\log_e r_1 + \frac{(-\,q)}{2\,\pi}\,\log_e r_2 = \frac{q}{2\,\pi}\,[\log_e r_1 - \log_e r_2]$$

$$\phi = \frac{q}{2\,\pi}\,\log_e\left(\frac{r_1}{r_2}\right)$$

To prove that resultant stream lines will be circular arc passing through source and sink

The resultant stream function is given as

$$\Psi = \frac{(-\,q)}{2\,\pi}\,\alpha = \frac{(-\,q)}{2\,\pi}\,(\theta_2 - \theta_1)$$

For a given stream line Ψ = constant. In the above equation the term $\left(\frac{q}{2\,\pi}\right)$ *is also constant. This means that* $(\theta_2 - \theta_1)$ *or angle α will also be constant for various positions of P in the plane.*

To satisfy this, the locus of P must be a circle with AB as chord, having its center on y-axis, as shown in figure.

Consider the equation as

$$\Psi = \frac{q}{2\,\pi}\,(\theta_1 - \theta_2)$$

$$(\theta_1 - \theta_2) = \frac{2\,\pi\,\Psi}{q}$$

Taking tan to both sides, we get

$$\tan(\theta_1 - \theta_2) = \tan\left(\frac{2\,\pi\,\Psi}{q}\right)$$

$$\frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 . \tan\theta_2} = \tan\left(\frac{2\,\pi\,\Psi}{q}\right) \qquad \text{...Equation (I)}$$

But $\tan\theta_1 = \frac{y}{x+a}$ *and* $\tan\theta_2 = \frac{y}{x-a}$

Substituting the above value in equation (I)

$$\frac{\left(\frac{y}{x+a}\right) - \left(\frac{y}{x-a}\right)}{1 + \left(\frac{y}{x+a}\right)\left(\frac{y}{x-a}\right)} = \tan\left(\frac{2\,\pi\,\Psi}{q}\right)$$

On solving

$$\frac{-\,2\,a\,y}{x^2 - a^2 + y^2} = \tan\left(\frac{2\,\pi\,\Psi}{q}\right)$$

$$x^2 - a^2 + y^2 = -2\,a\,y\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right)$$

$$x^2 - a^2 + y^2 + 2\,a\,y\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right) = 0$$

$$x^2 + y^2 + 2\,a\,y\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right) - a^2 = 0$$

Adding and subtracting $a^2\cot^2\left(\frac{2\,\pi\,\Psi}{q}\right)$

$$x^2 + y^2 + 2\,a\,y\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right) + a^2\cot^2\left(\frac{2\,\pi\,\Psi}{q}\right) - a^2\cot^2\left(\frac{2\,\pi\,\Psi}{q}\right) - a^2 = 0$$

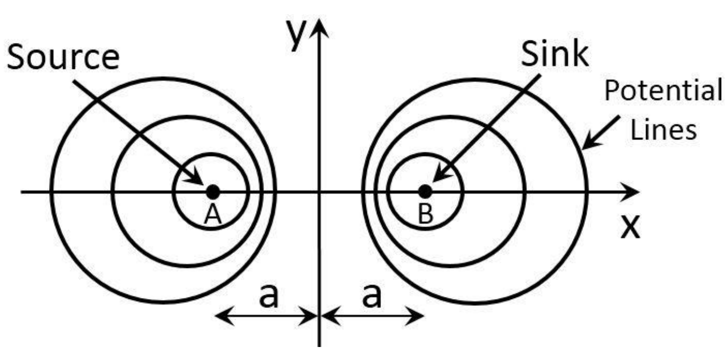
$$x^2 + \left[y + a\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right)\right]^2 = a^2\left[1 + \cot^2\left(\frac{2\,\pi\,\Psi}{q}\right)\right]$$

$$x^2 + \left[y + a\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right)\right]^2 = a^2\,\mathrm{cosec}^2\left(\frac{2\,\pi\,\Psi}{q}\right)$$

Or

$$x^2 + \left[y + a\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right)\right]^2 = \left[a.\,\mathrm{cosec}\left(\frac{2\,\pi\,\Psi}{q}\right)\right]^2$$

The above is the equation of a circle with center on y-axis at a distance of $\left[\pm\,a\;\cot\left(\frac{2\,\pi\,\Psi}{q}\right)\right]$ *from the origin. The radius of the circle will be* $\left[a.\,\mathrm{cosec}\left(\frac{2\,\pi\,\Psi}{q}\right)\right]$.



Similarly, it can be shown that the potential lines for the source sink pair will be eccentric non-intersecting circles with their centers on the x-axis is as shown in figure.

2) Doublet

It is a special case of a source and sink pair (both of them are of equal strength) when the two approach each other in such a way that the distance 2a between them approaches zero and the product 2a. q remains constant. This product 2a. q is known as doublet strength and is denoted by μ.

Doublet strength, $\mu = 2a.\, q$

Let q and (-q) may be the strength of the source and the sink respectively as shown in figure. Let 2a be the distance between them and P be any point in the combined field of source and sink.

Let θ is the angle made by P at A whereas (θ + δθ) is the angle at B.

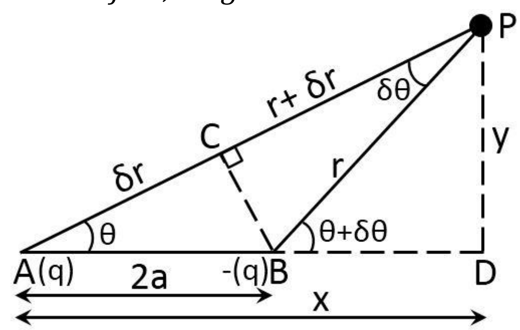
Now the stream function at P,

$$\Psi = \frac{q}{2\,\pi}\theta - \frac{q}{2\,\pi}(\theta - \delta\theta) = -\frac{q}{2\,\pi}\,\delta\theta \qquad \text{...Equation (I)}$$

From B, draw BC ⊥ on AP. Let AC = δr, CP = r and AP = r + δr. Also angle BPC = δθ. The angle δθ is very small. The distance BC can be taken equal to r × δθ. In triangle ABC, angle BCA = 90^0 *and hence distance BC is also equal to 2a. sin θ. Equating the two values of BC, we get*

$$r \times \delta\theta = 2a.\sin\theta$$

$$\delta\theta = \frac{2a.\sin\theta}{r}$$



Substituting the value of δθ in equation (I), we get

$$\Psi = -\frac{q}{2\,\pi} \times \frac{2a.\sin\theta}{r}$$

As μ = 2a. q

$$\Psi = -\frac{\mu}{2\,\pi} \times \frac{\sin\theta}{r} \qquad \text{...Equation (II)}$$

In figure, when 2a → 0, the angle δθ subtended by point P with A and B becomes very small. Also δr → 0 and AP becomes equal to r. Then

$$\sin\theta = \frac{PD}{AP} = \frac{y}{r}$$

Also $AP^2 = AD^2 + PD^2$ *or* $r^2 = x^2 + y^2$

Substituting the value of $\sin\theta$ *in equation (II), we get*

$$\Psi = -\frac{\mu}{2\,\pi} \times \frac{y}{r} \times \frac{1}{r} = -\frac{\mu\, y}{2\,\pi\, r^2} = -\frac{\mu\, y}{2\,\pi\,(x^2 + y^2)}$$

Or

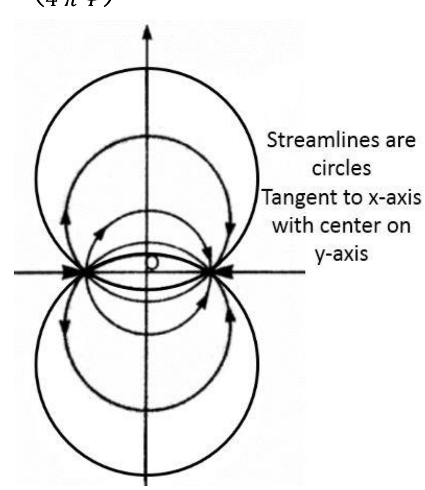
$$(x^2 + y^2) + \frac{\mu\, y}{2\,\pi\,\Psi} = 0$$

The above equation can be written as, adding and subtracting $\left(\frac{\mu}{4\,\pi\,\Psi}\right)^2$

$$x^2 + y^2 + 2\,y\,\frac{\mu}{4\,\pi\,\Psi} + \left(\frac{\mu}{4\,\pi\,\Psi}\right)^2 - \left(\frac{\mu}{4\,\pi\,\Psi}\right)^2 = 0$$

Or

$$x^2 + \left(y + \frac{\mu}{4\,\pi\,\Psi}\right)^2 = \left(\frac{\mu}{4\,\pi\,\Psi}\right)^2$$



The above equation is the equation of circle with center $\left(0, \frac{\mu}{4\,\pi\,\Psi}\right)$ *and radius* $\left(\frac{\mu}{4\,\pi\,\Psi}\right)$*. The center of the circle lies on y-axis at a distance of* $\left(\frac{\mu}{4\,\pi\,\Psi}\right)$ *from x-axis. As the radius of the circle is also equal to* $\left(\frac{\mu}{4\,\pi\,\Psi}\right)$*, hence the circle will be tangent to the x-axis. Hence stream lines of the doublet will be the family of circles tangent to the x-axis as shown in figure.*