Divergence and Curl

Divergence: Divergence of a vector point function $\overline{A} = \hat{i}A_1 + jA_2 + kA_3$, denoted by $\nabla \cdot \overline{A}$, is defined as

$$\boxed{\nabla \cdot \overline{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}}$$

Note that the Divergence is defined for vector point functions only and divergence of a vector is scalar.

Solenoidal Vector: Vector $\overline{A} = \hat{i}A_1 + jA_2 + kA_3$ is Solenoidal if $\boxed{\nabla \cdot \overline{A} = 0}$

Curl: Curl of a vector point function $\overline{A} = \hat{i}A_1 + jA_2 + kA_3$, denoted by $\nabla \times \overline{A}$, is defined as

$$\nabla \times \overline{A} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \hat{i}\left[\frac{\partial}{\partial y}A_3 - \frac{\partial}{\partial z}A_2\right] - j\left[\frac{\partial}{\partial x}A_3 - \frac{\partial}{\partial z}A_1\right] + k\left[\frac{\partial}{\partial x}A_2 - \frac{\partial}{\partial y}A_1\right]$$

Note that the Curl is defined for vector point functions only and curl of a vector is vector.

Conservative or Irrotational Vector: If the curl of a vector field vanishes, then the vector field is known as irrotational or conservative. That is, $\nabla \times \overline{F} = \overline{0} \Rightarrow \overline{F}$ is a conservative or irrotational vector.

<u>Note</u>: If vector field $\overline{F}$ is conservative, then there exist scalar potential ϕ such that $\overline{F} = \nabla\phi$

Examples:

1. Find $\nabla\cdot\bar{A}$ at $(1,-1,1)$ where $\bar{A}=3xyz^2\hat{i}+2xy^3\hat{j}-x^2yz\hat{k}$.

2. Find $\nabla\cdot\bar{r}$

3. Prove that $\bar{F}=\dfrac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is solenoidal.

4. Determine constant a so that $\bar{F}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal

5. Prove that $\nabla\cdot\left(\phi\bar{A}\right)=\phi\left(\nabla\cdot\bar{A}\right)+\left(\nabla\phi\right)\cdot\bar{A}$, where $\phi\left(x,y,z\right)$ is a scalar point function and $\bar{A}$ is a vector point function.

6. Prove that $\nabla\cdot\hat{r}=\dfrac{2}{r}$

7. Prove that $\dfrac{\bar{r}}{r^3}$ is Solenoidal

8. Prove that $\nabla\left[\nabla\cdot\left(r^n\bar{r}\right)\right]=n\left(n+3\right)r^{n-2}\bar{r}$

9. If $\bar{f}=3xy^2\hat{i}+5xy\hat{j}+xyz^3\hat{k}$, find $\nabla\times\bar{f}$ at $\left(1,2,3\right)$.

10. Prove that $\nabla\times\bar{r}=\bar{0}$

11. Prove that $\bar{f}=\dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$ is irrotational.

12. Prove that $\bar{f}=(z+\sin y)\hat{i}+(x\cos y-z)\hat{j}+(x-y)\hat{k}$ is irrotational.

13. Find constants *a*, *b* and *c* if $\bar{f}=\left(axy+bz^3\right)\hat{i}+\left(3x^2-cz\right)\hat{j}+\left(3xz^2-y\right)\hat{k}$ is irrotational.

14. Prove that $\bar{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$ is solenoidal and determine constants *a*, *b*, and *c* if $\bar{F}$ is irrotational.

Example 6B.2.3: Prove that $\overline{F}=\dfrac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is solenoidal.

Solution: $\overline{F}=\hat{i}\dfrac{x}{x^2+y^2}+j\dfrac{y}{x^2+y^2}+k0$

$\overline{F}$ is solenoidal if $\nabla\cdot\overline{F}=0$

$$\nabla\cdot\overline{F}=\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)+\frac{\partial}{\partial z}(0)$$

$$\therefore\nabla\cdot\overline{F}=\left[\frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2}\right]+\left[\frac{(x^2+y^2)(1)-(y)(2y)}{(x^2+y^2)^2}\right]$$

$$=\left[\frac{(x^2+y^2-2x^2)}{(x^2+y^2)^2}\right]+\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right]$$

$$=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0$$

$\therefore\boxed{\nabla\cdot\overline{F}=0}$

$\overline{F}$ is solenoidal.

Example 6B.2.4: Determine constant a so that $\overline{F}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal.

Solution: Given that $\overline{F}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$ is solenoidal

$\therefore\nabla\cdot\overline{F}=0$

$$\therefore\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)=0$$

$\therefore 1+1+a=0$

$\therefore\boxed{a=-2}$

Example 6B.2.9: If $\bar{f} = 3xy^2\hat{i} + 5xy\hat{j} + xyz^3\hat{k}$, find $\nabla \times \bar{f}$ at $(1,2,3)$.

Solution: By definition $\nabla \times \bar{f} = \begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3xy^2 & 5xy & xyz^3 \end{vmatrix}$

$$\therefore \nabla \times \bar{f} = \hat{i}\left[\frac{\partial}{\partial y}\left(xyz^3\right) - \frac{\partial}{\partial z}(5xy)\right] - j\left[\frac{\partial}{\partial x}\left(xyz^3\right) - \frac{\partial}{\partial z}\left(3xy^2\right)\right] + k\left[\frac{\partial}{\partial x}(5xy) - \frac{\partial}{\partial y}\left(3xy^2\right)\right]$$

$$= \hat{i}\left[xz^3 - 0\right] - j\left[yz^3 - 0\right] + k\left[5y - 6xy\right]$$

$$\therefore \nabla \times \bar{f} = xz^3\hat{i} - yz^3\, j + \left[5y - 6xy\right]k$$

At $(1,2,3)$, $\nabla \times \bar{f} = (1)(2)^3\hat{i} - (2)(3)^3\, j + \left[5(2) - 6(1)(2)\right]k$

$$\therefore \boxed{\nabla \times \bar{f} = 8\hat{i} - 54\, j - 2\hat{k}}$$

Example 6B.2.11: Prove that $\bar{f} = \frac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$ is irrotational.

Solution: $\bar{f}$ is irrotational, if $\nabla \times \bar{f} = \bar{0}$

$$\bar{f} = \left(\frac{-y}{x^2+y^2}\right)\hat{i} + \left(\frac{x}{x^2+y^2}\right)\hat{j} + 0k$$

$$\therefore \nabla \times \bar{f} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$\therefore \nabla \times \bar{f} = \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)\right] - j\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right)\right] + k\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]$$

$$= \hat{i}[0-0] - j[0-0] + k\left[\frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1)-(y)(2y)}{(x^2+y^2)^2}\right]$$

$$= 0\hat{i} + 0\hat{j} + \hat{k}\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right]$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \bar{0}$$

Thus, $\nabla \times \bar{f} = \bar{0}$

$\therefore$ $\bar{f}$ is irrotational.

Example 6B.2.12: Prove that $\bar{f}=(z+\sin y)\hat{i}+(x\cos y-z)\hat{j}+(x-y)\hat{k}$ is irrotational.

Solution: $\bar{f}$ is irrotational, if $\nabla\times\bar{f}=\bar{0}$

$\bar{f}=(z+\sin y)\hat{i}+(x\cos y-z)\hat{j}+(x-y)\hat{k}$

$$\therefore \nabla\times\bar{f}=\begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (z+\sin y) & (x\cos y-z) & (x-y) \end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]-j\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(z+\sin y)\right]+k\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(z+\sin y)\right]$$

$$=\hat{i}[-1-(-1)]-j[1-1]+k[\cos y-\cos y]$$

$$=0\hat{i}+0j+0k=\bar{0}$$

Thus, $\nabla\times\bar{f}=\bar{0}$

$\therefore$ $\bar{f}$ is irrotational.

Example 6B.2.13: Find constants a, b and c if

$\bar{f}=\left(axy+bz^3\right)\hat{i}+\left(3x^2-cz\right)\hat{j}+\left(3xz^2-y\right)\hat{k}$ is irrotational.

Solution: $\bar{f}$ is irrotational. $\therefore\ \nabla\times\bar{f}=\bar{0}$

$$\bar{f}=\left(axy+bz^3\right)\hat{i}+\left(3x^2-cz\right)\hat{j}+\left(3xz^2-y\right)\hat{k}$$

$$\therefore \nabla\times\bar{f}=\begin{vmatrix}\hat{i} & j & k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \left(axy+bz^3\right) & \left(3x^2-cz\right) & \left(3xz^2-y\right)\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}\left(3xz^2-y\right)-\frac{\partial}{\partial z}\left(3x^2-cz\right)\right]-j\left[\frac{\partial}{\partial x}\left(3xz^2-y\right)-\frac{\partial}{\partial z}\left(axy+bz^3\right)\right]+k\left[\frac{\partial}{\partial x}\left(3x^2-cz\right)-\frac{\partial}{\partial y}\left(axy+bz^3\right)\right]$$

$$=\hat{i}[-1-(-c)]-j\left[3z^2-3bz^2\right]+k[6x-ax]$$

$$=[c-1]\hat{i}-3(b-1)z^2\,j+(6-a)\,x\widehat{k}$$

$$\therefore \nabla\times\bar{f}=\bar{0}\ \Rightarrow\ [c-1]\hat{i}-3(b-1)z^2\,\widehat{j}+(6-a)\,x\widehat{k}=0\hat{i}+0\widehat{j}+0\widehat{k}$$

$$\therefore c-1=0,\ \ b-1=0,\ \ 6-a=0$$

$$\therefore \boxed{a=6,\ b=1,\ c=1}$$

Example 6B.2.14: Prove that $\bar{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$ is solenoidal and determine constants a, b, and c if $\bar{F}$ is irrotational.

Solution: $\bar{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$

$\bar{F}$ is solenoidal if $\nabla\cdot\bar{F}=0$

$$\bar{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$$

$$\nabla\cdot\bar{F}=\frac{\partial}{\partial x}(x+2y+az)+\frac{\partial}{\partial y}(bx-3y-z)+\frac{\partial}{\partial z}(4x+cy+2z)$$

$$\therefore \nabla\cdot\bar{F}=1-3+2=3-3=0$$

$$\therefore \boxed{\nabla\cdot\bar{F}=0}$$

$\bar{F}$ is solenoidal.

$\bar{f}$ is irrotational, *if* $\nabla\times\bar{f}=\bar{0}$

$$\bar{F}=(x+2y+az)\hat{i}+(bx-3y-z)\hat{j}+(4x+cy+2z)\hat{k}$$

$$\therefore \nabla\times\bar{f}=\begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right]-\hat{j}\left[\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]$$

$$\therefore \nabla\times\bar{f}=\hat{i}[c+1]-\hat{j}[4-a]+\hat{k}[b-2]$$

$$\therefore \nabla\times\bar{f}=\bar{0} \Rightarrow \hat{i}[c+1]-\hat{j}[4-a]+\hat{k}[b-2]=0\hat{i}+0\hat{j}+0\hat{k}$$

$$\therefore c+1=0,\quad 4-a=0,\quad b-2=0$$

$$\therefore \boxed{a=4,\ b=2,\ c=-1}$$

Line Integral

Line Integral along curve C is defined as $\int_C \overline{F}\cdot d\overline{r}$

Work done by the force $\overline{F}$ in moving a particle along the cuve C is defined as $\int_C \overline{F}\cdot d\overline{r}$

Note: *Using equation of curve, eliminate variables and get line integral in terms of only one variable and integrate using usual rules of integration.*

Solved Examples

1. Evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F} = 3xy\hat{i} - y^2 j$ and C is the arc of the curve $2x^2 = y$ from $(0,0)$ to $(1,2)$.

2. Evaluate $\int_C \overline{F}\cdot d\overline{r}$, where $\overline{F} = x^2\hat{i} + 2xy\,j$ and C is the arc of the curve $x^2 = y$ from (0,0) to (1,1).

3. Evaluate $\int_C \overline{F}\cdot d\overline{R}$, where $\overline{F} = y^2\hat{i} + 2xy\,j$ and C is the straight line from $(0,0)$ to $(1,2)$

4. Evaluate $\int_C \overline{F}\cdot d\overline{r}$ where $\overline{F} = xy\hat{i} + yz\,j + xzk$ and C is the arc of the curve $x = t,\ y = t^2,\ z = t^3$ from $t = -1$ to $t = 1$.

5. Find the work done in moving a particle in the force field $\overline{F} = 3x^2\hat{i} + (2zx - y)\,j + z\hat{k}$ along
   (*i*) The straight line from (0,0,0) to (2,1,3)
   (*ii*) The curve defined by $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$

6. Find the work done in moving a particle once around the circle $x^2 + y^2 = 9,\ z = 0$ in the force field $\overline{F} = (2x - y + z)\hat{i} + (x + y - z^2)\,j + (3x - 2y + 4z)k$

7. Find the work done when a force $\bar{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from origin to $(4,2)$ along a parabola $y^2 = x$.

8. If a force $\vec{F} = 2x^2 y\hat{i} + 3xy\,j$ displaces a practicle in the XY plane along the curve $y = 4x^2$ from (0,0) to (1,4), find the work done .

9. Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where $\bar{F} = yz\hat{i} + (zx+1)j + xyk$ and C is the straight line joining the points $A(1,0,0)$ to $B(2,1,4)$.

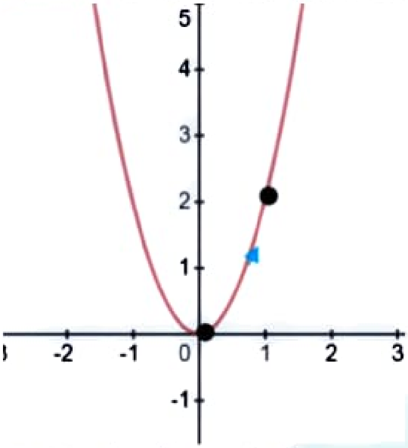
10. Find the work done in moving a particle in a force field $\bar{F} = 3xy\hat{i} - 5z\,j + 10x\hat{k}$ along the curve $C:\ x = t^2 + 1,\ y = 2t^2,\ z = t^3$ from $t = 0$ to $t = 2$.

11. Prove that $\bar{F} = (6xy^2 - 2z^3)\hat{i} + (6x^2 y + 2yz)\hat{j} + (y^2 - 6z^2 x)k$ is irrotational. Find scalar potential of $\bar{F}$. Hence find the work done in moving the particle from $(1,0,2)$ to $(0,1,1)$.

12. Prove that $\bar{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)k$ is irrotational. Find scalar potential of $\bar{F}$. Hence find the work done in moving the particle from $(1,0,1)$ to $(2,1,3)$.

Example 6B.3.1: Evaluate $\int_C \overline{F} \cdot d\overline{r}$ where $\overline{F} = 3xy\hat{i} - y^2 j$ and C is the arc of the curve $2x^2 = y$ from $(0,0)$ to $(1,2)$

Solution: Path of integration is $C: y = 2x^2$ from $(0,0)$ to $(1,2)$



$$\overline{F} \cdot \overline{dr} = \left(3xy\hat{i} - y^2 j\right) \cdot \left(\hat{i}dx + jdy + kdz\right) \qquad \left\{\because \overline{r} = \hat{i}x + jy + kz\right\}$$

$$= 3xydx - y^2 dy$$

Along $C: y = 2x^2, \quad dy = 4xdx$

$$\therefore \overline{F} \cdot \overline{dr} = 3x\left(2x^2\right)dx - \left(2x^2\right)^2 (4xdx)$$

$$= \left(6x^3 - 16x^5\right)dx$$

$$\therefore \int_C \overline{F} \cdot d\overline{r} = \int_0^1 \left(6x^3 - 16x^5\right)dx \qquad \{\because \text{ From } (0,0) \text{ to } (1,2),\ x \text{ varies from } 0 \text{ to } 1\}$$

$$= \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right]_0^1$$

$$= \left[\frac{3}{2} - \frac{8}{3}\right] - [0 - 0] = -\frac{7}{6}$$

$$\therefore \boxed{\int_C \overline{F} \cdot d\overline{r} = -\frac{7}{6}}$$

Example 6B.3.5: Find the work done in moving a particle in the force field $\overline{F}=3x^2\hat{i}+(2zx-y)j+z\hat{k}$ along

(i) The straight line from (0,0,0) to (2,1,3)

(ii) The curve defined by $x^2=4y, 3x^3=8z$ from $x=0$ to $x=2$

Solution: (i) The equation of straight line joining (0,0,0) to (2,1,3) is

$$\frac{x-0}{2-0}=\frac{y-0}{1-0}=\frac{z-0}{3-0}$$

or $\frac{x}{2}=\frac{y}{1}=\frac{z}{3}=t$

$\therefore$ Along $C: x=2t,\ y=t,\ z=3t$

$$\overline{F}\cdot\overline{dr}=\left(3x^2\hat{i}+(2zx-y)j+z\hat{k}\right)\cdot\left(\hat{i}dx+jdy+kdz\right) \quad \left\{\because \overline{r}=\hat{i}x+jy+kz\right\}$$

$$=3x^2dx+(2zx-y)dy+zdz$$

Along $C: x=2t,\ y=t,\ z=3t$

$\therefore dx=2dt,\ dy=dt,\ dz=3dt$

$\therefore \overline{F}\cdot\overline{dr}=3(2t)^2(2dt)+(2(3t)(2t)-t)dt+(3t)(3dt)$

$$=\left(24t^2+12t^2-t+9t\right)dt$$

$\therefore \overline{F}\cdot\overline{dr}=\left(36t^2+8t\right)dt$

$\therefore$ Work Done $=\int_C \overline{F}\cdot d\overline{r}=\int_0^1\left(36t^2+8t\right)dt \quad \{\because$ At $(0,0,0),\ t=0$ and at $(2,1,3),\ t=1\}$

$$=\left[36\frac{t^3}{3}+8\frac{t^2}{2}\right]_0^1$$

$$=[12+4]-[0+0]=16$$

$\therefore$ $\boxed{\text{Work Done}=16\text{ units}}$

(ii) Along the curve C: $x^2=4y, 3x^3=8z$ from $x=0$ to $x=2$,

$y=\frac{x^2}{4}$ and $z=\frac{3x^3}{8}$

$\therefore dy=\frac{x}{2}dx,\ dz=\frac{9}{8}x^2dx$

$\overline{F}\cdot\overline{dr}=3x^2dx+(2zx-y)dy+zdz$

$$=3x^2dx+\left(2\left(\frac{3x^3}{8}\right)x-\frac{x^2}{4}\right)\left(\frac{x}{2}dx\right)+\left(\frac{3x^3}{8}\right)\left(\frac{9}{8}x^2dx\right)$$

$$\therefore \text{Work Done } = \int_C \overline{F}\cdot d\overline{r} = \int_0^1 \left(36t^2 + 8t\right)dt \quad \left\{\because \text{At } (0,0,0),\ t=0 \text{ and at } (2,1,3),\ t=1\right\}$$

$$= \left[36\frac{t^3}{3} + 8\frac{t^2}{2}\right]_0^1$$

$$= [12+4] - [0+0] = 16$$

$$\therefore \boxed{\text{Work Done} = 16 \text{ units}}$$

(*ii*) Along the curve C: $x^2 = 4y, 3x^3 = 8z$ from $x = 0$ to $x = 2$,

$$y = \frac{x^2}{4} \text{ and } z = \frac{3x^3}{8}$$

$$\therefore dy = \frac{x}{2}dx,\quad dz = \frac{9}{8}x^2 dx$$

$$\overline{F}\cdot\overline{dr} = 3x^2dx + (2zx - y)dy + zdz$$

$$= 3x^2dx + \left(2\left(\frac{3x^3}{8}\right)x - \frac{x^2}{4}\right)\left(\frac{x}{2}dx\right) + \left(\frac{3x^3}{8}\right)\left(\frac{9}{8}x^2dx\right)$$

Example 6B.3.10: Find the work done in moving a particle in the force field $\overline{F} = 3xy\hat{i} - 5z\,j + 10x\hat{k}$ along the curve $C:\ x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t=0$ to $t=2$.

Solution: Work done by the force $\overline{F}$ in moving a particle along the cuve C is defined as $\int_C \overline{F}\cdot d\overline{r}$

$\overline{F} = 3xy\hat{i} - 5z\,j + 10x\hat{k}$ along the curve $C:\ x = t^2+1, y = 2t^2, z = t^3$ from $t=0$ to $t=2$.

$$\overline{F} = 3xy\hat{i} - 5z\,j + 10x\hat{k}$$

$$\overline{F}\cdot\overline{dr} = \left[3xy\hat{i} - 5z\,j + 10x\hat{k}\right]\cdot\left(\hat{i}dx + jdy + kdz\right) \qquad \left\{\because \overline{r} = \hat{i}x + jy + kz\right\}$$

$$= 3xydx - 5zdy + 10xdz$$

Along $C:\ x = t^2+1, y = 2t^2, z = t^3$ from $t=0$ to $t=2$

$$\therefore dx = 2tdt,\quad dy = 4tdt,\quad dz = 3t^2dt$$

$$\therefore \overline{F}\cdot\overline{dr} = 3\left(t^2+1\right)\left(2t^2\right)2tdt - 5\left(t^3\right)4tdt + 10\left(t^2+1\right)3t^2dt$$

$$= \left(12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2\right)dt$$

$$\therefore \overline{F}\cdot\overline{dr} = \left(12t^5 + 10t^4 + 12t^3 + 30t^2\right)dt$$

$$\therefore \int_C \overline{F}\cdot d\overline{r} = \int_0^2 \left(12t^5 + 10t^4 + 12t^3 + 30t^2\right)dt$$

$$= \left[\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right]_0^2$$

$$= 2(2^6) + 2(2^5) + 3(2)^4 + 10(2)^3$$

$$= 128 + 64 + 48 + 80 = 320$$

$$\therefore \boxed{\text{Work Done} = \int_C \overline{F}\cdot d\overline{r} = 320 \text{ units}}$$

Example 6B.3.11: Prove that $\bar{F}=\left(6xy^2-2z^3\right)\hat{i}+\left(6x^2y+2yz\right)\hat{j}+\left(y^2-6z^2x\right)k$ is irrotational. Find scalar potential of $\bar{F}$. Hence find the work done in moving the particle from $(1,0,2)$ to $(0,1,1)$.

Solution: $\bar{F}$ is irrotational, if $\nabla\times\bar{F}=\bar{0}$

$$\bar{F}=\left(6xy^2-2z^3\right)\hat{i}+\left(6x^2y+2yz\right)\hat{j}+\left(y^2-6z^2x\right)k$$

$$\therefore \nabla\times\bar{F}=\begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(6xy^2-2z^3\right) & \left(6x^2y+2yz\right) & \left(y^2-6z^2x\right) \end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}\left(y^2-6z^2x\right)-\frac{\partial}{\partial z}\left(6x^2y+2yz\right)\right]-j\left[\frac{\partial}{\partial x}\left(y^2-6z^2x\right)-\frac{\partial}{\partial z}\left(6xy^2-2z^3\right)\right]$$

$$+k\left[\frac{\partial}{\partial x}\left(6x^2y+2yz\right)-\frac{\partial}{\partial y}\left(6xy^2-2z^3\right)\right]$$

$$=\hat{i}\left[2y-2y\right]-j\left[6z^2-6z^2\right]+\hat{k}\left[12xy-12xy\right]$$

$$=0\hat{i}+0\hat{j}+0\hat{k}=\bar{0}$$

Thus, $\nabla\times\bar{F}=\bar{0}$

$\therefore$ $\bar{F}$ is irrotational.

$\therefore$ There exist scalar potential $\phi(x,y,z)$ such that $\bar{F}=\nabla\phi$

$$\left(6xy^2-2z^3\right)\hat{i}+\left(6x^2y+2yz\right)\hat{j}+\left(y^2-6z^2x\right)\hat{k}=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x}=6xy^2-2z^3 \Rightarrow \phi(x,y,z)=\int\left(6xy^2-2z^3\right)dx = 3x^2y^2-2z^3x+h_1(y,z) \quad ----(1)$$

$$\frac{\partial\phi}{\partial y}=6x^2y+2yz \Rightarrow \phi(x,y,z)=\int\left(6x^2y+2yz\right)dx = 3x^2y^2+y^2z+h_2(x,z) \quad ----(2)$$

$$\frac{\partial\phi}{\partial z}=y^2-6z^2x \Rightarrow \phi(x,y,z)=\int\left(y^2-6z^2x\right)dx = y^2z-2z^3x+h_3(x,y) \quad ----(3)$$

From (1), (2) and (3)

$$\phi(x,y,z) = 3x^2y^2-2z^3x+h_1(y,z) = 3x^2y^2+y^2z+h_2(x,z) = y^2z-2z^3x+h_3(x,y)$$

$$\therefore h_1(y,z)=y^2z,\ \ h_2(x,z)=-2z^3x,\ \ h_3(x,y)=3x^2y^2$$

$$\therefore \boxed{\phi(x,y,z) = 3x^2y^2-2z^3x+y^2z}$$

$$\therefore \frac{\partial \phi}{\partial x} = 6xy^2 - 2z^3 \Rightarrow \phi(x,y,z) = \int \left(6xy^2 - 2z^3\right) dx = 3x^2y^2 - 2z^3x + h_1(y,z) \quad ----(1)$$

$$\frac{\partial \phi}{\partial y} = 6x^2y + 2yz \Rightarrow \phi(x,y,z) = \int \left(6x^2y + 2yz\right) dx = 3x^2y^2 + y^2z + h_2(x,z) \quad ----(2)$$

$$\frac{\partial \phi}{\partial z} = y^2 - 6z^2x \Rightarrow \phi(x,y,z) = \int \left(y^2 - 6z^2x\right) dx = y^2z - 2z^3x + h_3(x,y) \quad ----(3)$$

From (1), (2) and (3)

$$\phi(x,y,z) = 3x^2y^2 - 2z^3x + h_1(y,z) = 3x^2y^2 + y^2z + h_2(x,z) = y^2z - 2z^3x + h_3(x,y)$$

$$\therefore h_1(y,z) = y^2z, \quad h_2(x,z) = -2z^3x, \quad h_3(x,y) = 3x^2y^2$$

$$\therefore \boxed{\phi(x,y,z) = 3x^2y^2 - 2z^3x + y^2z}$$

Example 6B.3.12: Prove that $\bar{F}=\left(y^2-z^2+3yz-2x\right)\hat{i}+\left(3xz+2xy\right)\hat{j}+\left(3xy-2xz+2z\right)k$ is irrotational. Find scalar potential of $\bar{F}$. Hence find the work done in moving the particle from $(1,0,1)$ to $(2,1,3)$.

Solution: $\bar{F}$ is irrotational, if $\nabla\times\bar{F}=\bar{0}$

$$\bar{F}=\left(y^2-z^2+3yz-2x\right)\hat{i}+\left(3xz+2xy\right)\hat{j}+\left(3xy-2xz+2z\right)k$$

$$\therefore \nabla\times\bar{F}=\begin{vmatrix} \hat{i} & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(y^2-z^2+3yz-2x\right) & \left(3xz+2xy\right) & \left(3xy-2xz+2z\right) \end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(3xy-2xz+2z)-\frac{\partial}{\partial z}(3xz+2xy)\right]-j\left[\frac{\partial}{\partial x}(3xy-2xz+2z)-\frac{\partial}{\partial z}\left(y^2-z^2+3yz-2x\right)\right]$$

$$+k\left[\frac{\partial}{\partial x}(3xz+2xy)-\frac{\partial}{\partial y}\left(y^2-z^2+3yz-2x\right)\right]$$

$$=\hat{i}[3x-3x]-\hat{j}[3y-2z-(-2z+3y)]+\hat{k}[3z+2y-(2y+3z)]$$

$$=0\hat{i}+0\hat{j}+0\hat{k}=\bar{0}$$

Thus, $\nabla\times\bar{F}=\bar{0}$

$\therefore$ $\bar{F}$ is irrotational.

$\therefore$ There exist scalar potential $\phi(x,y,z)$ such that $\bar{F}=\nabla\phi$

$$\left(y^2-z^2+3yz-2x\right)\hat{i}+\left(3xz+2xy\right)\hat{j}+\left(3xy-2xz+2z\right)\hat{k}=\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial\phi}{\partial x}=y^2-z^2+3yz-2x \Rightarrow \phi(x,y,z)=\int\left(y^2-z^2+3yz-2x\right)dx$$

$$=y^2x-z^2x+3yzx-x^2+h_1(y,z) \quad ----(1)$$

$$\frac{\partial\phi}{\partial y}=3xz+2xy \Rightarrow \phi(x,y,z)=\int(3xz+2xy)dx = 3xzy+xy^2+h_2(x,z) \quad ----(2)$$

$$\frac{\partial\phi}{\partial z}=3xy-2xz+2z \Rightarrow \phi(x,y,z)=\int(3xy-2xz+2z)\,dx = 3xyz-xz^2+2z^2+h_3(x,y) \quad --(3)$$

From (1), (2) and (3)

$$\phi(x,y,z)= y^2x-z^2x+3yzx-x^2+h_1(y,z) = 3xzy+xy^2+h_2(x,z) = 3xyz-xz^2+2z^2+h_3(x,y)$$

$$\therefore h_1(y,z)=2z^2,\ h_2(x,z)=-z^2x-x^2+2z^2,\ h_3(x,y)=y^2x-x^2$$

$$\therefore \boxed{\phi(x,y,z) = y^2x-z^2x+3yzx-x^2+2z^2}$$

Work done by the force $\overline{F}$ in moving a particle along the cuve C is $\int_C \overline{F}\cdot\overline{dr}$

$$\overline{F}\cdot\overline{dr}=\nabla\phi\cdot\overline{dr}=\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}k\right)\cdot\left(\hat{i}dx+\hat{j}dy+kdz\right)$$

$$=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=d\phi$$

$\therefore \overline{F}\cdot\overline{dr}=d\phi$

Work done in moving the particle from $(1,0,1)$ to $(2,1,3)$ is

$$\int_{(1,0,2)}^{(0,1,1)}\overline{F}\cdot\overline{dr}=\int_{(1,0,2)}^{(0,1,1)}d\phi=[\phi]_{(1,0,1)}^{(2,1,3)}$$

$$=\left[y^2x-z^2x+3yzx-x^2+2z^2\right]_{(1,0,1)}^{(2,1,3)}$$

$$=[2-18+18-4+18]-[0-1+0-1+2]$$

$$=16$$

$\therefore$ $\boxed{\text{Work done}=16\text{ units}}$