Chapter 2

FLUID STATIC

Fluid statics *deals with problems associated with fluids at rest. The fluid can be either gaseous or liquid. Fluid statics is generally referred to as hydrostatics when the fluid is a liquid and as aerostatics when the fluid is a gas. In fluid statics, there is no relative motion between adjacent fluid layers, and thus there are no shear (tangential) stresses in the fluid trying to deform it. The only stress we deal with in fluid statics is the normal stress, which is the pressure, and the variation of pressure is due only to the weight of the fluid. Therefore, the topic of fluid statics has significance only in gravity fields, and the force relations developed naturally involve the gravitational acceleration 'g'. The force exerted on a surface by a fluid at rest is normal to the surface at the point of contact since there is no relative motion between the fluid and the solid surface, and thus no shear forces can act parallel to the surface.*

Fluid statics is used to determine the forces acting on floating or submerged bodies and the forces developed by devices like hydraulic presses and car jacks. The design of many engineering systems such as water dams and liquid storage tanks requires the determination of the forces acting on the surfaces using fluid statics. The complete description of the resultant hydrostatic force acting on a submerged surface requires the determination of the magnitude, the direction, and the line of action of the force.

Fluid Pressure at a Point

Consider a small area 'dA' in large mass of fluid. If the fluid is stationary, then the force exerted by the surrounding fluid on the area dA will always be perpendicular to the surface dA. Let 'dF' is the force acting on the area dA in the normal direction. Then ratio of dF/dA is known as the intensity of pressure or simply pressure and this ratio is represented by P.

Hence mathematically the pressure at a point in a fluid at rest is,

$$P = \frac{dF}{dA}$$

If the force (F) is uniformly distributed over the area (A), then pressure at any point is given by

$$P = \frac{F}{A} = \frac{Force}{Area}$$

Force or pressure force, $F = P \times A$

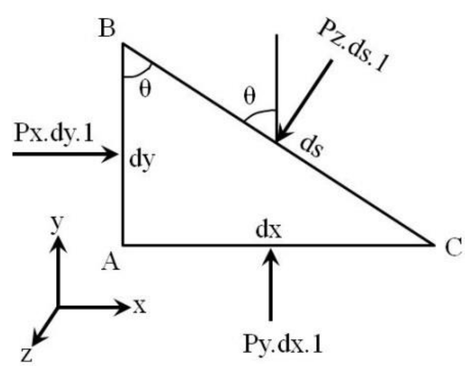
The units of pressure are:

In MKS units, kgf/m^2 *and* kgf/cm^2

In SI units, $Newton/m^2$ *and* N/m^2*, is known as Pascal (Pa)*

$1\ bar = 100kPa = 10^5\ N/m^2$

Pascal's Law



It states that the pressure or intensity of pressure at a point in a static fluid is equal in all direction.

Proof:

The fluid element is of very small dimensions i.e. dx.dy.dz

Consider an arbitrary fluid element of wedge shape in a fluid mass at rest as shown in Figure.

Let the width of the element perpendicular to the plane of paper is unity, and P_X, P_Y and P_z are the pressure intensity or pressure acting on the face AB, AC and BC respectively, angle ABC = θ

The forces acting on the element are:

1. *Pressure forces normal to the surface*
2. *Weight of element in the vertical direction*

The forces on the faces are:

Force on the face AB $= P_X \times$ *area of face AB*

$= P_X \times dy \times 1$

Similarly, Force on the face AC $= P_Y \times dx \times 1$

And Force on the face BC $= P_Z \times ds \times 1$

Weight of element = (*Mass of element*) $\times g$

= (*Volume* $\times \rho$) $\times g$ $= \frac{AB \times AC}{2} \times 1 \times \rho \times g$

Where, ρ = *density of fluid*

Resolving the forces in x-direction, we have

$$P_X \times dy \times 1 - P. (ds \times 1). \sin (90 - \theta) = 0$$

$$P_X \times dy \times 1 - P_Z. (ds \times 1). \cos (\theta) = 0$$

From the Figure, $ds. \cos (\theta) = AB = dy$

$$P_X \times dy \times 1 - P_Z. (dy \times 1) = 0$$

$$P_X = P_Z \quad .....(1)$$

Similarly, resolving the forces in y-direction, we get

$$P_Y \times dx \times 1 - P_Z. \times (ds \times 1). \cos (90 - \theta) - \frac{dx \times dy}{2} \times 1 \times \rho \times g = 0$$

$$P_Y \times dx - P_Z. \times (ds. \sin \theta) - \frac{dx \times dy}{2} \times \rho \times g = 0$$

But (ds. Sin θ) = dx, and also the element is very small and hence weight if negligible.

$$P_Y \times dx - P_Z. \times dx = 0$$

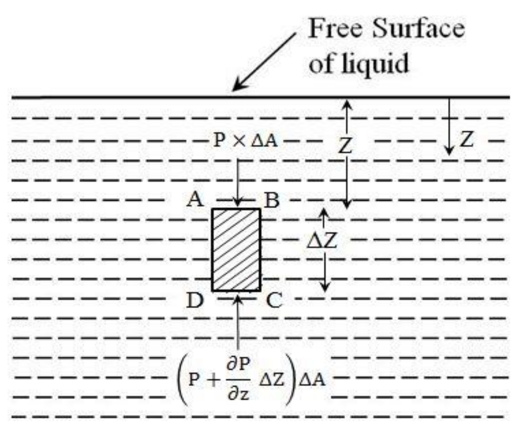
$$P_Y = P_Z \quad .....(2)$$

From equation (1) and (2), we have

$$P_X = P_Y = P_Z \quad .....(3)$$

The above equation shows that the pressure at any point in x, y and z direction is equal.

Pressure variation in a fluid at rest



The pressure at any point in a fluid at rest is obtained by the hydrostatic law which states that the rate of increase of pressure in a vertically downward direction must be equal to the specific weight of the fluid at that point.

This is proved as follows:

Consider a small fluid element as shown in Figure.

Let, ΔA = *cross sectional area of the element*

ΔZ = *height of fluid element*

P = pressure on face AB
Z = distance of fluid element from free surface.

The forces acting on the fluid element are,

1. *Pressure force on AB =* $P \times \Delta A$ *and acting perpendicular to face AB in the downward direction*
2. *Pressure force on CD =* $\left(P + \frac{\partial P}{\partial z}\,\Delta Z\right) \times \Delta A$*, acting perpendicular to face CD vertically upward direction*
3. *Weight of fluid element = density* $\times g \times$ *Volume =* $\rho \times g \times (\Delta A \times \Delta Z)$
4. *Pressure force on surfaces BC and AD are equal and opposite.*

For equilibrium of fluid element, we have

$$P \times \Delta A - \left(P + \frac{\partial P}{\partial z}\,\Delta Z\right) \times \Delta A + \rho \times g \times (\Delta A \times \Delta Z) = 0$$

$$P \times \Delta A - P \times \Delta A - \left(\frac{\partial P}{\partial z}\,\Delta Z\right) \times \Delta A + \rho \times g \times (\Delta A \times \Delta Z) = 0$$

$$-\left(\frac{\partial P}{\partial z}\,\Delta Z\right)\Delta A + \rho \times g \times (\Delta A \times \Delta Z) = 0$$

$$\left(\frac{\partial P}{\partial z}\,\Delta Z\right)\Delta A = \rho \times g \times (\Delta A \times \Delta Z)$$

$$\frac{\partial P}{\partial z} = \rho \times g = w \qquad \because \ (w = \rho \times g) \qquad ......(4)$$

Where w = weight density of fluid

An equation (4) state that rate of increase of pressure in a vertical direction is equal to weight density of the fluid at that point. This is Hydrostatic Law.

By integrating above equation (4) for liquids, we get

$$\int dP = \int \rho.g.dZ$$

$$P = \rho.g.Z \qquad ......(5)$$

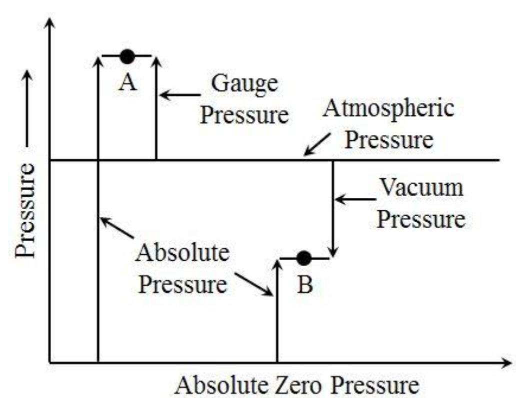
Where P is the pressure above atmosphere pressure and Z is the height of the point from free surfaces.

From equation (5), we have $Z = \frac{P}{\rho g}$

Here Z is called Pressure Head.

Absolute, Gauge, Atmospheric and Vacuum Pressures

The pressure on a fluid is measured in two different systems. In one system, it is measured above the absolute zero or complete vacuum and it is called the Absolute pressure and in other system, pressure is measured above the atmosphere pressure and it is called as Gauge pressure.



1. Absolute Pressure

It is defined as the pressure which is measured with reference to absolute vacuum pressure.

2. Gauge Pressure

It is defined as the pressure which is measured with the help of a pressure measuring instrument, in which the atmospheric pressure is taken as datum. The atmospheric pressure on the scale is marked as zero.

3. Vacuum Pressure

It is defined as the pressure below the atmospheric pressure.

The relationship between the absolute pressure, gauge pressure and vacuum pressure are shown in Figure.

Mathematically,

1) Absolute pressure = Atmospheric pressure + Gauge pressure

2) Vacuum pressure = Atmospheric pressure - Absolute pressure

Total Pressure and Centre of Pressure

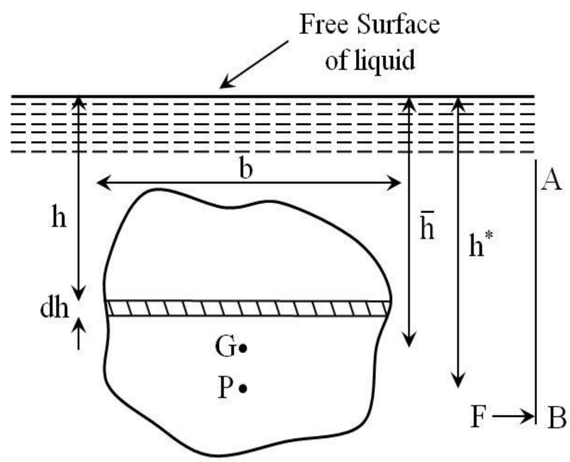
Total pressure is defined as the force exerted by a static fluid on a surface either plane or curved when the fluid comes in contact with the surfaces. This force always acts normal to the surface.

Centre of pressure is defined as the point of application of the total pressure on the surface. There are four cases of submerged surfaces on which the total pressure force and center of pressure is to be determined. The submerged surfaces may be:

1. Vertical plane surface	*2. Horizontal plane surface*
3. Inclined plane surface	*4. Curved surface*

1. Vertical plane surface

Consider a plane vertical surface of an arbitrary shape immersed in a liquid as shown in Figure.



Let,

A = total area of the surface

$\bar{h}$ = distance of CG of the area from free surface of liquid

G = Centre of gravity of the plane surface

P = Centre of pressure

h^* = distance of center of pressure from free surface of liquid

a) Total pressure (F)

The total pressure on the surface may be determined by dividing the entire surface into a number of small strips. The force on small strip is then calculated and the total pressure force on the whole area is calculated by integrating the force on small strip.

Consider a strip of thickness 'dh' and width 'b' at a depth of 'h' from free surface of liquid as shown in Figure.

Pressure intensity on the strip,

$$P = \rho g h$$

Area of strip, $dA = b \times dh$

Total pressure force on the strip,

$$dF = P \times Area$$
$$= \rho g h \times b \times dh$$

Total pressure force on the whole surface,

$$F = \int dF = \int \rho g h \times b \times dh = \rho g \int b \times h \times dh$$

But

$$\int b \times h \times dh = \int h \times dA$$

= *Moment of surface area about the free surface of liquid*

= *Area of surface* × *Distance of CG from free surface*

$= A \times \bar{h}$

$$F = \rho\, g\, A\, \bar{h} \qquad \text{....(6)}$$

b) Centre of pressure (h^*)

Centre of pressure is calculated by using the "Principle of Moment", which states that the moment of the resultant force about an axis is equal to the sum of moments of the components about the same axis.

The resultant force F is acting at P, at a distance h^* from free surface of the liquid as shown in Figure. Hence moment of the force F about free surface of liquid = $F \times h^*$ ....(7)

Moment of force dF, acting on a strip about free surface of liquid

$= dF \times h$

$= \rho\, g\, h \times b \times dh \times h \qquad \because (dF = \rho g h \times b \times dh)$

Sum of moments of all such forces about free surface of liquid

$= \int \rho\, g\, h \times b \times dh \times h$

$= \rho\, g \int b \times h \times h\, dh$

$= \rho\, g \int b \times h^2 \times dh$

$= \rho\, g \int h^2 \times dA \qquad \because (b \times dh = dA)$

But

$$\int h^2 \times dA = \int b \times h^2 \times dh$$

= *Moment of Inertia of the surface about free surface of liquid*

$= I_o$

Sum of moments about free surface

$$= \rho g\, I_o \qquad \text{....(8)}$$

From equation (7) and (8)

$$F \times h^* = \rho g\, I_o$$

$$\rho\, g\, A\, \bar{h} \times h^* = \rho\, g\, I_o \qquad \because (F = \rho\, g\, A\, \bar{h})$$

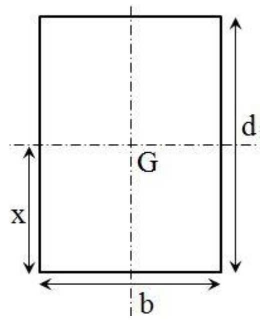
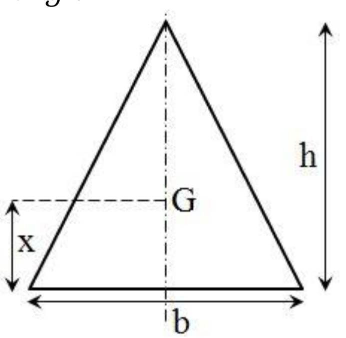
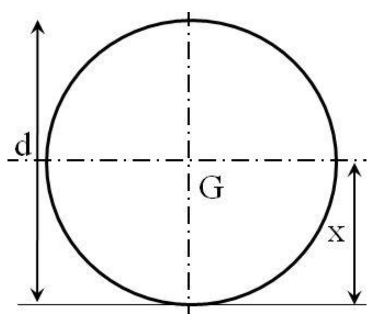
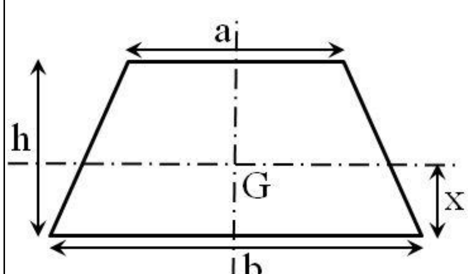
$$h^* = \frac{I_O}{A.\ \bar{h}}$$

By the theorem of parallel axis, we have $I_o = I_G + A \times \overline{h^2}$

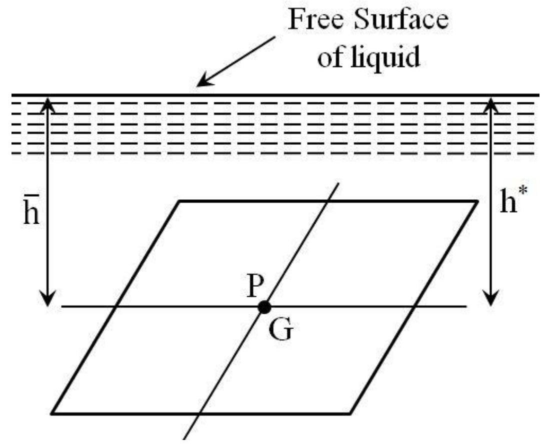
Where I_G = Moment of Inertia of area about an axis passing through the CG of the area and parallel to the free surface of liquid

$$h^* = \frac{I_G + A\,\overline{h^2}}{A.\bar{h}} = \frac{I_G}{A.\bar{h}} + \bar{h} \qquad \text{....(9)}$$

The moments of inertia and other geometric properties of some important plane surface

Plane Surface	*CG from the base*	*Area*	*Moment of inertia about and axis passing through CG and parallel to base (I_G)*	*Moment of inertia about base (I_o)*
1. Rectangle  	$x = \frac{d}{2}$	bd	$\frac{bd^3}{12}$	$\frac{bd^3}{3}$
2. Triangle  	$x = \frac{h}{3}$	$\frac{bh}{2}$	$\frac{bh^3}{36}$	$\frac{bh^3}{12}$
3. Circle  	$x = \frac{d}{2}$	$\frac{\pi d^2}{4}$	$\frac{\pi d^4}{64}$	----
4. Trapezium  	$x = \left(\frac{2a+b}{a+b}\right)\frac{h}{3}$	$\left(\frac{a+b}{2}\right)h$	$\left(\frac{a^2+4ab+b^2}{36(a+b)}\right)h^3$	----

2. Horizontal plane surface



Consider a plane horizontal surface immersed in a static fluid. As every point of the surface is at the same depth from the free surface of the liquid, the pressure intensity will be equal on the entire surface and equal to, P = ρgh, where 'h' is depth of surface.

Let,

A = total area of surface

Then,

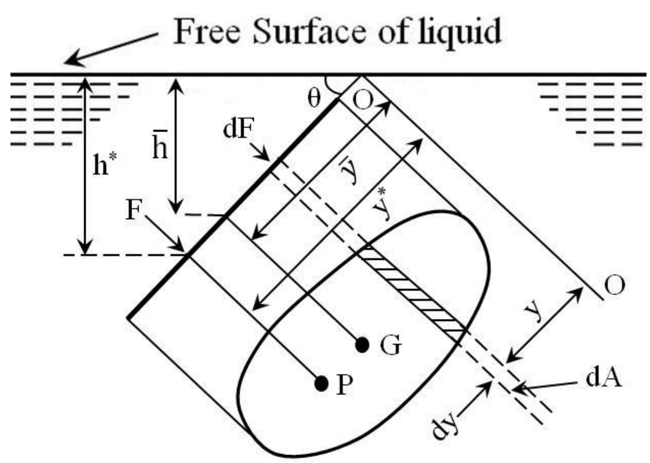
Total force F, on the surface

$$F = P \times area$$
$$= \rho\ g\ h \times A$$
$$= \rho\ g\ A\ \bar{h}$$

Where, $\bar{h}$ = depth of CG from free surface of liquid = h

h^* = depth of center of pressure from free surface = h

3. Inclines plane surface



Consider s plane surface of arbitrary shape immersed in a liquid in such a way that the plane of the surface makes an angle θ with the free surface of the liquid as shown in Figure.

Let,

A = total area of inclined surface

$\bar{h}$ = depth of CG of inclined area from free surface of liquid

h^* = distance of center of pressure from free surface of liquid

θ = angle made by the plane of the surface with free liquid surface

a) Total pressure (F)

Let the plane of the surface, if produced meet the free liquid surface at O. Then O-O is the axis perpendicular to the plane of the surface.

Let

$\bar{y}$ = distance of the CG of the inclined surface from O-O

y^* = distance of the centre of pressure from O-O

Consider a small strip of area 'dA' at a depth of 'h' from free surface and at a distance 'y' from the axis O-O as shown in Figure.

Pressure intensity on the strip, $P = \rho\ g\ h$

Pressure force 'dF' on the strip, $dF = P \times area = \rho\ g\ h \times dA$

Total pressure force on whole area, $F = \int dF = \int \rho\ g\ h \times dA$

But from diagram, $$\sin\theta = \frac{h}{y} = \frac{\bar{h}}{\bar{y}} = \frac{h^*}{y^*}$$

$$h = y \sin\theta$$

$$F = \int \rho\ g\ y \sin\theta \times dA = \rho\ g\ \sin\theta \int y\ dA$$

But $\int y\ dA = A\ \bar{y}$

Where $\bar{y}$ *= distance of CG from axis O - O*

$$F = \rho\ g\ A\ \bar{y} \sin\theta = \rho\ g\ A\ \bar{h} \qquad \because (\bar{h} = \bar{y} \sin\theta)$$

b) Centre of pressure (h^*)

Pressure force on the strip, $dF = \rho\ g\ h \times dA$

$$= \rho\ g\ (y \sin\theta) \times dA \qquad \because (h = y \sin\theta)$$

Moment of the force dF, about axis O - O

$$= dF \times y \quad = \rho\ g\ (y \sin\theta) \times dA \times y$$

Sum of moments of all such forces about O - O

$$= \rho\ g\ \sin\theta \int y^2\ dA$$

But,

$\int y^2\ dA$ *= Moment of Inertia of the surface about O - O =* I_O

Sum of moments of all forces about O - O = $\rho\ g\ \sin\theta\ I_O$ ....(10)

Moment of the total force F, about O - O is also given by

$$= F \times y^* \qquad ....(11)$$

Where y^* *= distance of centre of pressure from O - O*

Equating equation (10) and (11)

$$F \times y^* = \rho\ g\ \sin\theta\ I_O$$

$$y^* = \frac{\rho\ g\ \sin\theta\ I_0}{F} \qquad ....(12)$$

Now $h^* = y^* \sin\theta$

Therefore $F = \rho\ g\ A\ \bar{h}$

And I_O *by theorem of parallel axis =* $I_G + A\ \overline{y^2}$

Substituting these values in equation (12), we get

$$\frac{h^*}{\sin\theta} = \frac{\rho\ g\ \sin\theta}{\rho\ g\ A\ \bar{h}} \left[I_G + A\ \overline{y^2}\right]$$

$$h^* = \frac{\sin^2\theta}{A\ \bar{h}} \left[I_G + A\ \overline{y^2}\right]$$

But $\bar{y} = \dfrac{\bar{h}}{\sin\theta}$

$$h^* = \frac{\sin^2\theta}{A\ \bar{h}} \left[I_G + A \times \left(\frac{\bar{h}}{\sin\theta}\right)^2\right]$$

$$h^* = \frac{I_G\ \sin^2\theta}{A\ \bar{h}} + \bar{h} \qquad ....(13)$$

If $\theta = 90^0$*, equation (13) becomes same as equation (9) which is applicable to vertically plane submerged surfaces.*

4. Curved surface

Consider a curved surface AB, submerged in a static fluid as shown in Figure. Let 'dA' is the area of a small strip at a depth of 'h' from water surface.

Then pressure intensity on the area 'dA' is = $\rho\ g\ h$

And Pressure force, $dF = P \times Area = \rho\ g\ h \times dA$ ....(14)

This force dF acts normal to the surface.

Hence total pressure force on the curved surface should be

$$F = \int \rho\ g\ h\ dA \quad ....(15)$$

But here as the direction of the forces on the small area are not in the same direction, but varies from point to point. Here integrating of equation (14) for curved surface is impossible. The problem can be solved by restoring the force 'dF' in two components dF_x and dF_y in the x and y direction respectively. The total force in the x and y direction, i.e. F_x and F_y are obtained by integrating dF_x and dF_y.

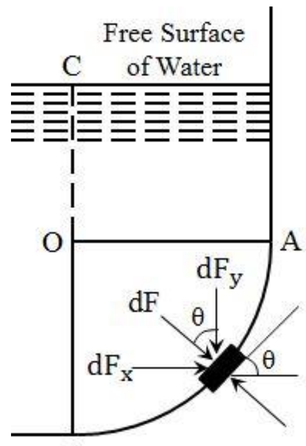


Figure (A)

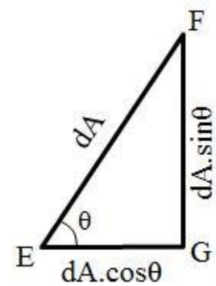


Figure (B)

The total force on curved surface is

$$F = \sqrt{\left(F_x^2 + F_y^2\right)}$$

And inclination of resultant with horizontal is

$$\tan \emptyset = \frac{F_y}{F_x}$$

Resolving the force dF given by equation (14) in x and y direction

$$dF_x = dF \sin\theta = \rho\ g\ h\ dA \sin\theta \qquad \because\ (dF = \rho\ g\ h\ dA)$$

And

$$dF_y = dF \cos\theta = \rho\ g\ h\ dA \cos\theta$$

Total forces in the x and y direction are:

$$F_x = \int dF_x = \int \rho\ g\ h\ dA \sin\theta = \rho\ g \int h\ dA \sin\theta \quad .....(16)$$

$$F_y = \int dF_y = \int \rho\ g\ h\ dA \cos\theta = \rho\ g \int h\ dA \cos\theta \quad ....(17)$$

Figure (B) shows the enlarged area dA, from this Figure i.e. ΔEFG

$EF = dA$

$FG = dA \sin\theta$ *and* $EG = dA \cos\theta$

Thus in equation (16), $dA \sin\theta$ = FG = vertical projection of the area dA and hence the expression $\rho\ g \int h\ dA \sin\theta$ represents the total pressure force on the projected area of the curved surface on the vertical plane. Thus

F_x = total pressure force on the projected area of the curved surface on vertical plane.

Also $dA \cos\theta$ = EG = horizontal projection of dA and hence '$h\ dA \cos\theta$' is the volume of the liquid contained in the elementary area dA upto free surface of the liquid. Thus $\int h\ dA \cos\theta$ is the total volume contained between the curved surface extended upto free surfaces.

Hence $\rho\ g \int h\ dA \cos\theta$ is the total weight supported by the curved surface.

Thus $F_y = \rho\ g \int h\ dA \cos\theta$ = weigth of liquid supported by the curved surface upto free surface of liquid

Buoyancy

When a body is immersed in a fluid, an upward force is exerted by the fluid on the body. This upward force is equal to the weight of the fluid displaced by the body and is called as the force of buoyancy or simply buoyancy.

Centre of Buoyancy

It is defined as the point, through which the force of buoyancy is supposed to act. As the force of buoyancy is a vertical force and is equal to the weight of the fluid displaced by the body, the centre of buoyancy will be the centre of gravity of the fluid displaced.

Meta Centre

It is defined as the point about which a body starts oscillating when the body is tilted by a small angle. The Meta centre may be also defined as the point at which the line of action of the force of buoyancy will meet the normal axis of the body when the body is given a small angular displacement.

Consider a body floating in a liquid as shown in the Figure (B). Let the body is in equilibrium and G is the centre of gravity and B is the centre of buoyancy. For equilibrium, both the points lie on the normal axis, which is vertical

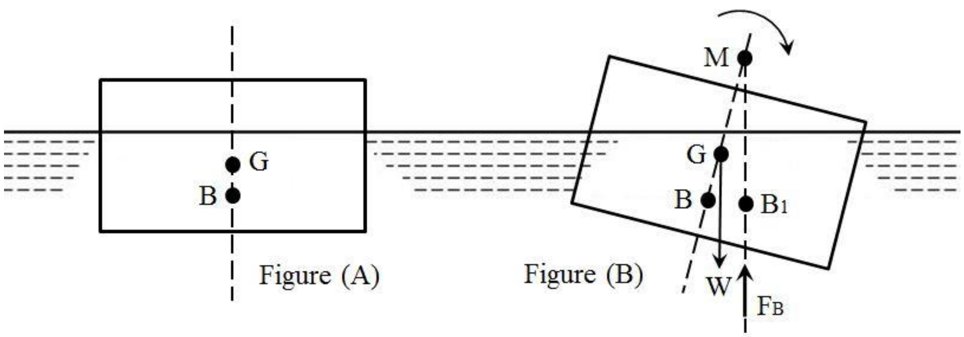


Figure (A) Figure (B)

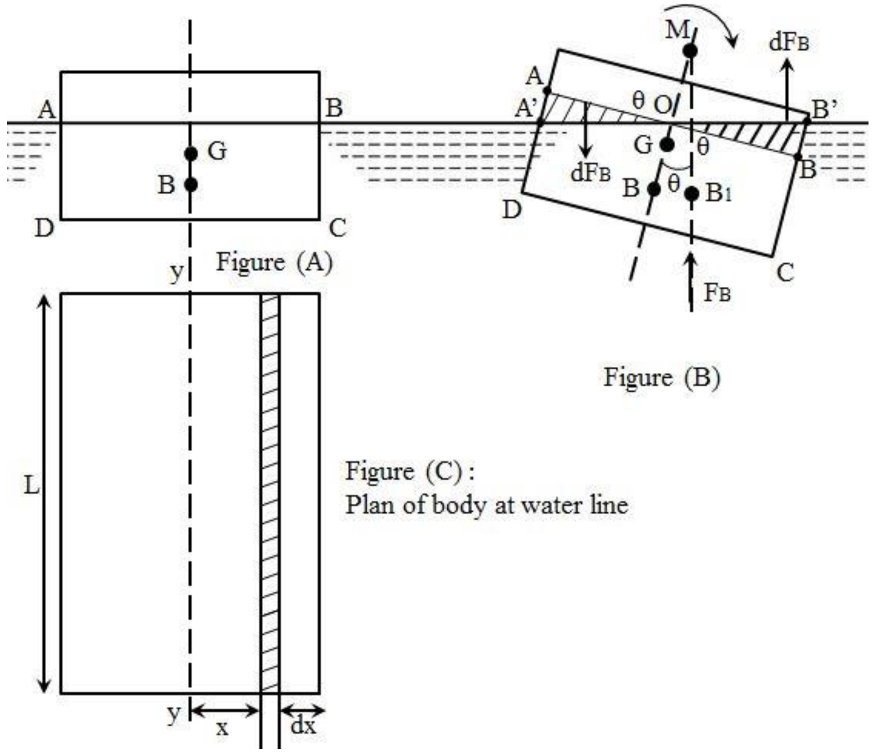
Let the body is given a small angular displacement in a clockwise direction as shown in Figure (B). The centre of buoyancy, which is the centre of gravity of the displaced a liquid or centre of gravity of the portion of the body submerged in a liquid, will now be shifted towards the right from the normal axis. Let it is at B1 as shown in Figure (B). The line of action of force of buoyancy in this new position will intersect the normal axis of the body at some point say M. This point M is called Meta centre.

Meta Centric Height

The distance MG, i.e. the distance between the meta centre of floating body and the centre of gravity of the body is called Meta centric height.

Analytical Method for Meta Centre Height

Figure (A) shows the position of floating body in the equilibrium. The location of centre of gravity and the centre of buoyancy in this position is at G and B. The floating body is given a small angular displacement in a clockwise direction. This is shown in Figure (B). The new centre of buoyancy is at B_1. The vertical line through B_1 cut the normal axis at M and hence M is the meta centre and GM is the metacentric height.



The angular displacement of the body in the clockwise direction causes the wedge shaped prism BOB' on the right of the axis to go inside the water while the identical wedge shaped prism represented by AOA' emerges out of the water on the left of the axis. These wedges represent a gain in buoyant force on the right side and a corresponding loss of buoyant force on the left side. The gain is represented by vertical force dF_B acting through the CG of the prism BOB' while the loss is represented by an equal and opposite force dF_B acting vertically downward through the centroid of AOA'. The couple due to this buoyant force dF_B tends to rotate the ship in the counter clockwise direction. Also the moment caused by the displacement of centre of buoyancy from B to B_1 is also in the counter clockwise direction. Thus these two couples must be equal.

Couple due to Wedges:

Consider towards the right of the axis a small strip of thickness 'dx' at a distance of 'x' from 'O' as shown in Figure (B).

The height of strip $x \times \angle BOB' = x \times \theta$ $\quad\because (\angle BOB' = \angle AOA' = BMB_1' = \theta)$

Area of strip = Height × Thickness $= x \times \theta \times dx$

If 'L' is the length of the floating body, then

Volume of strip = Area × L $= x \times \theta \times dx \times L$

Weight of strip $= \rho\, g \times$ *volume* $= \rho\, g\, x\, \theta\, L\, dx$

Similarly, if a small strip of thickness 'dx' at a distance 'x' from 'O' towards the left of the axis is considered, the weight of strip will be $\rho\, g\, x\, \theta\, L\, dx$*. The two weights are acting in the opposite direction and hence constitute a couple.*

Moment of this couple = Weight of each strip × Distance between these two weights

$$= \rho\, g\, x\, \theta\, L\, dx \times [x + x] \qquad = 2\, \rho\, g\, x^2\, \theta\, L\, dx$$

Moment of the couple for the whole wedge

$$= \int 2\,\rho\, g\, x^2\, \theta\, L\, dx \quad ....(18)$$

Moment of couple due to shifting of centre of buoyancy from B to B_1

$$= F_B \times BB_1$$

$$= F_B \times BM \times \theta \qquad \because\ (BB_1 = BM \times \theta,\ \text{as } \theta \text{ is very small})$$

$$= W \times BM \times \theta \qquad \because (F_B = W) \quad ....(19)$$

But these two couples are the same. Hence equating equation (18) and (19), we get,

$$W \times BM \times \theta = \int 2\,\rho\, g\, x^2\, \theta\, L\, dx$$

$$W \times BM \times \theta = 2\,\rho\, g\,\ \theta \int x^2\, L\, dx$$

$$W \times BM = 2\,\rho\, g \int x^2\, L\, dx$$

Now, L.dx = Elemental area on the water line shown in Figure (C) and = dA

$$W \times BM = 2\,\rho\, g \int x^2 dA$$

But from Figure (C) it is clear that $2\int x^2 dA$ is the second moment of area of the plan of the body at water surface about the axis y-y. Therefore,

$$W \times BM = \rho\, g\, I \qquad \because (I = 2\int x^2 dA)$$

$$BM = \frac{\rho\, g\, I}{W}$$

But $\quad$ W = weight of the body = weight of the fluid displaced by the body

= $\rho\, g$ × Volume of the fluid displaced by the body

= $\rho\, g$ × Volume of the body submerged in water = $\rho\, g \times \forall$

$$BM = \frac{\rho\, g\, I}{\rho\, g\, \forall} = \frac{I}{\forall}$$

$$GM = BM - BG = \frac{I}{\forall} - BG$$

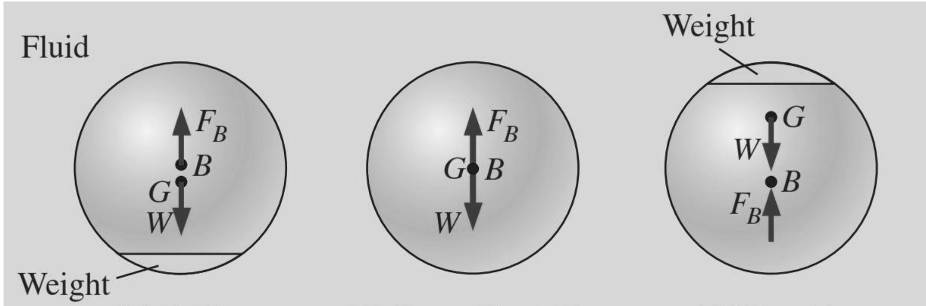
Meta centric height = GM = $\frac{I}{\forall} - BG$

Conditions of Equilibrium of Floating and Submerged Bodies

A submerged or a floating body is said to be stable if it comes back to its original position after a slight disturbance. The relative position of the centre of gravity (G) and centre of buoyancy (B_1) of a body determines the stability of a submerged body.

Stability of a submerged body

The position of centre of gravity and the centre of buoyancy in case of a completely submerged body are fixed. Consider a balloon, which is completely submerged in air. Let the lower portion of the balloon contains heavy material, so that its centre of gravity is lower than its centre of buoyancy as shown in Figure (a). Let the weight of the balloon is W. The weight W is acting through G, vertically in the downward direction, while the buoyant force F_B is acting vertically up, through B. For the equilibrium of the balloon W = F_B. If the balloon is given an angular displacement in the anticlockwise direction as shown in the Figure (a), then W and F_B constitute a couple acting in the anticlockwise direction and brings the balloon in the original position. Thus the balloon in the position shown, by Figure (a) is in stable equilibrium.



(*a*) Stable (*b*) Neutrally stable (*c*) Unstable

a) Stable equilibrium

When W =F_B and point B is above G, the body is said to be in stable equilibrium

b) Neutrally equilibrium

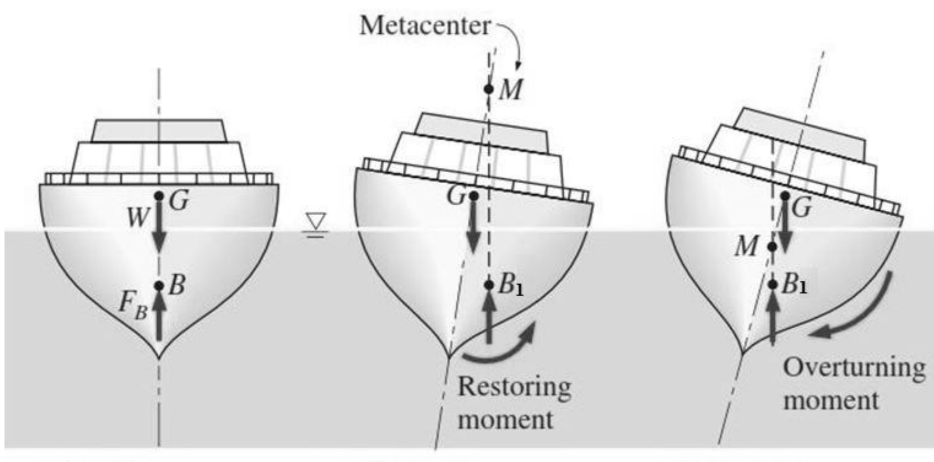
If W =F_B, and B and G are at the same point, as shown in Figure (b), the body is said to be in neutral equilibrium.

c) Unstable equilibrium

If W =F_B, but the centre of buoyancy (B) is below centre of gravity (G), the body is in unstable equilibrium as shown in Figure (c). A slight displacement to the body, in the clockwise direction, gives the couple due to W and F_B also in the clockwise direction. Thus the body does not return to its original position and hence the body is in unstable equilibrium.

Stability of Floating Body

The stabiliy of a floating body is determined from the position of Meta centre (M). In case of floating body, the weight of the body is equal to the weight of liquid displaced.



(*a*) Stable (*b*) Stable (*c*) Unstable

a) Stable equilibrium

If the point M is above G, the floating body will be in stable equilibrium as shown in Figure (b). If a slight angular displacement is given to the floating body in the clockwise direction, the centre of buoyancy shifts from B to B_1 such that the vertical line through B_1 cuts at M. Then the buoyant force F_B through B_1 and weight W through G constitute a couple acting in the anti - clockwise direction and thus bringing the floating body in the original position.

b) Unstable equilibrium

If the point M is below G, the floating body will be in unstable equilibrium as shown in Figure (c). The disturbing couple is acting in the clockwise direction. The couple due to buoyant force F_B and W is also acting in the clockwise direction and thus overturning the floating body.

c) Neutrally equilibrium

If the point M is at the centre of gravity of the body, the floating body will be in neutral equilibrium.

Experimental Method of Determination of Meta Centric Height

The meta centric height of a floating vessel can be determined, provided we know the centre of gravity of the floating vessel. Let 'w_1' is a known weight placed over the centre of the vessel as shown in Figure (A) and the vessel is floating.

Let,

W = weight of vessel including w_1

G = Centre of gravity of the vessel

B = Centre of buoyancy of the vessel

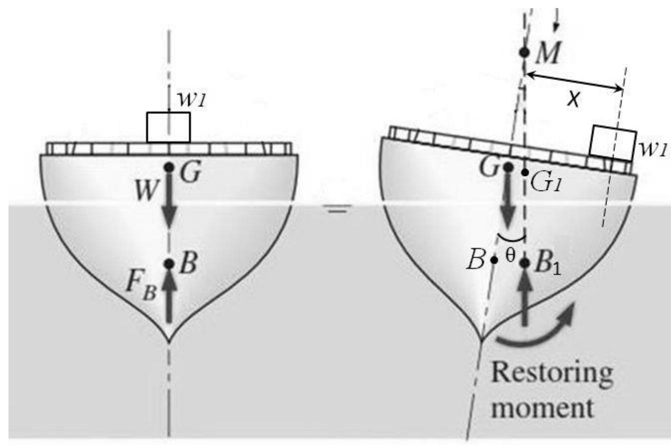


Figure (A): Floating Body Figure (B): Tilted Body

The weight w_1 is moved across a vessel towards right through a distance 'x' as shown in Figure (B). The vessel will be tilted. The angle of heel 'θ' is measured by means of plumb line and a protector attached on the vessel. The new centre of gravity of vessel will shift to G_1 as the weight w_1 one has been moved towards right. Also the centre of buoyancy will change to B_1 as the vessel has tilted. Under equilibrium, the moment caused by the movement of the load w_1 through a distance 'x' must be equal to the moment caused by the shift of the centre of gravity from G to G_1.

Thus,

The moment due to change of G = $GG_1 \times W = W \times GM \tan\theta$

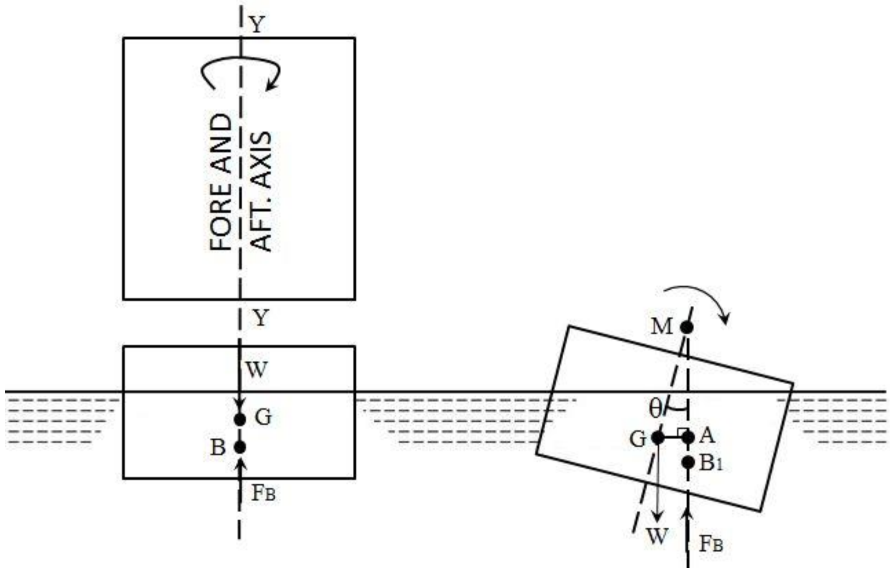
The moment due to movement of $w_1 = w_1 \times x$

$$w_1 x = W \times GM \tan\theta$$

Hence $$GM = \frac{w_1 x}{W \tan\theta}$$

Oscillation (Rolling) of a Floating Body

Consider a floating body, which is tilted through an angle by an overturning couple as shown in Figure. Let the overturning couple is suddenly removed. The body will start oscillating. Thus, the body will be in a state of oscillation as if suspended at the Meta centre M. This is similar to the case of a pendulum. The only force on the body is due to the restoring couple due to the weight W of the body force of buoyancy F_B.



Restoring couple $= W \times$ *Distance GA*

$= W \times GM \sin \theta$ ....(20)

This couple tries to decrease the angle

Angular acceleration of the body, $\alpha = -\frac{d^2\theta}{dt^2}$

Negative sign has been introduced as the restoring couple tries to decrease the angle θ.

Torque due to inertia = Moment of Inertia about Y-Y × Angular acceleration

$= I_{Y\text{-}Y} \times \left(-\frac{d^2\theta}{dt^2}\right)$

But $I_{Y\text{-}Y} = \frac{W}{g} K^2$

Where W = weight of body *K = radius of gyration about Y-Y*

$\because$ *Inertia torque* $= \frac{W}{g} K^2 \left(-\frac{d^2\theta}{dt^2}\right) = -\frac{W}{g} K^2 \left(\frac{d^2\theta}{dt^2}\right)$ ....(21)

Equating (20) and (21), we get

$$W \times GM \sin \theta = -\frac{W}{g} K^2 \left(\frac{d^2\theta}{dt^2}\right)$$

Or

$$GM \sin \theta = -\frac{K^2}{g} \left(\frac{d^2\theta}{dt^2}\right)$$

For a small angle θ, $\sin \theta \cong \theta$

$$GM \times \theta = -\frac{K^2}{g} \left(\frac{d^2\theta}{dt^2}\right)$$

Dividing by $\frac{K^2}{g}$, *we get*

$$\left(\frac{d^2\theta}{dt^2}\right) + \frac{GM \times \theta \times g}{K^2} = 0$$

The above equation is a differential equation of second degree. The solution is

$$\theta = C_1 \sin \sqrt{\frac{GM.\, g}{K^2}} \times t + C_2 \cos \sqrt{\frac{GM.\, g}{K^2}} \times t \qquad ....(22)$$

Where, C_1 *and* C_2 *are constants of integration.*

The values of C_1 and C_2 are obtained from boundary conditions which are,

1) *at $t = 0$, $\theta = 0$*
2) *at $t = \frac{T}{2}$, $\theta = 0$*

Where T is the time period of one complete oscillation, we get

Substituting the 1st boundary condition in equation (22)

$$0 = C_1 \times 0 + C_2 \times 1 \qquad \because (\sin\theta = 0, \cos\theta = 1)$$

$$C_2 = 0$$

Substituting the 2nd boundary condition in equation (22), we get

$$0 = C_1 \sin\sqrt{\frac{GM.\ g}{K^2}} \times \frac{T}{2}$$

But C_1 cannot be equal to zero and so the other alternative is

$$\sin\sqrt{\frac{GM.\ g}{K^2}} \times \frac{T}{2} = 0 = \sin\pi \qquad \because (\sin\pi = 0)$$

$$\sqrt{\frac{GM.\ g}{K^2}} \times \frac{T}{2} = \pi \qquad Or \qquad T = 2\pi\sqrt{\frac{K^2}{GM.\ g}}$$

Time period of oscillation is given by above equation.