Green's Theorem

If $M(x, y)$ and $N(x, y)$ are continuous functions having continuous partial derivatives in a region R which is bounded by a closed curve C, then

$$\oint_C [Mdx + Ndy] = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dxdy$$

where orientation along C is in anti-clockwise sense.

Solved Examples

1. Evaluate by Green's Theorem, $\oint_C (5x - 4y)dx + (3x + 8y)dy$ where C is the boundary of the parallelogram whose vertices are $(0,0)$, $(2,0)$, $(2,2)$ and $(4,2)$

2. Evaluate by Green's theorem, $\oint_C e^{-x}(\sin y dx + \cos y dy)$ where C is the rectangle with vertices $(0,0)$, $\left(\pi, \frac{\pi}{2}\right)$, $\left(0, \frac{\pi}{2}\right)$ and $(\pi, 0)$.

3. Using Green's theorem,evaluate $\oint_C (x^2 y dx + x^2 dy)$ where C is the boundary of the triangle with vertices $(0,0)$ $(1,0)$ & $(1,1)$.

4. Using Green's Theorem, evaluate $\oint_C (3x + 4y)dx + (2x - 3y)dy$ where C is the circle $x^2 + y^2 = a^2$

5. Using Green's Theorem, evaluate $\oint_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right]$ where C is the region bounded by the X – axis and the upper half of the circle $x^2 + y^2 = a^2$

3. Using Green's theorem,evaluate $\oint_C \left(x^2 y dx + x^2 dy\right)$

where C is the boundary of the triangle with vertices (0,0) (1,0) & (1,1).

4. Using Green's Theorem, evaluate $\oint_C (3x+4y)dx+(2x-3y)dy$ where C is the circle $x^2+y^2=a^2$

5. Using Green's Theorem, evaluate $\oint_C \left[(2x^2-y^2)dx+(x^2+y^2)dy\right]$ where C is the region bounded by the X – axis and the upper half of the circle $x^2+y^2=a^2$

6. Using Green's Theorem, Evaluate $\oint_C (x^2-2xy)dx+(x^2y+3)dy$, where C is the boundary of the region bounded by the parabola $y^2=8x$ and straight line $x=2$

7. Using Green's Theorem, evaluate $\oint_C (y-\sin x)dx+\cos x dy$, where C is the boundary of the triangle whose vertices are $(0,0)$, $\left(\frac{\pi}{2},0\right)$, $\left(\frac{\pi}{2},1\right)$

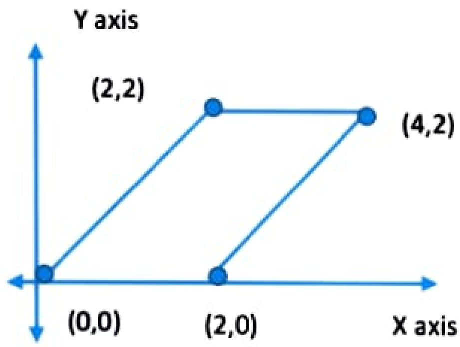
8. Verify Green's theorem, for $\oint_C \left[(xy+y^2)dx+x^2dy\right]$

where C is the closed curve bounded by the straight line $y=x$ and parabola $y=x^2$.

9. Verify Green's theorem,for $\oint_C \left[(3x^2-8y^2)dx+(4y-6xy)dy)\right]$ where C is bounded by the straight lines $x=0,\ y=0$ and $y+x=1$

Example 6B.4.1: Evaluate by Green's Theorem, $\oint_C (5x-4y)dx+(3x+8y)dy$ where C is the boundary of the parallelogram whose vertices are $(0,0)$, $(2,0)$, $(2,2)$ and $(4,2)$

Solution: Let $M=5x-4y$ and $N=3x+8y$



Let $M=5x-4y$ and $N=3x+8y$

$\frac{\partial M}{\partial y}=-4$ and $\frac{\partial N}{\partial x}=3$

$M,\ N,\ \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on parallelogram (closed curve).

$\therefore$ By Green's theorem, $\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

$$\oint_C[(5x-4y)dx+(3x+8y)dy]=\iint_R(3-(-4))\,dxdy$$

$$=7\iint_R dxdy$$

$$=7[\text{Area of the closed curve (parallelogram)}]$$

$$=7\times base\times height$$

$$=7\times 2\times 2=28$$

$$\therefore \boxed{\oint_C[(5x-4y)dx+(3x+8y)dy]=28}$$

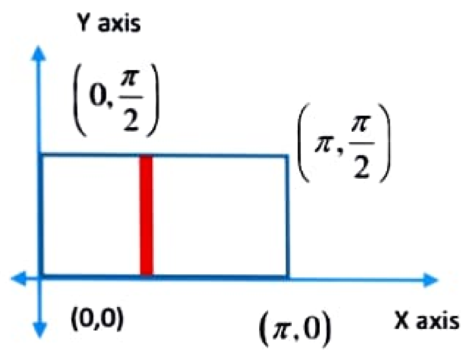
Example 6B.4.2: Evaluate by Green's theorem, $\oint_C e^{-x}(\sin y dx + \cos y dy)$

where C is the rectangle with vertices $(0,0), \left(\pi, \frac{\pi}{2}\right), \left(0, \frac{\pi}{2}\right)$ and $(\pi, 0)$.

Solution: Let $M = e^{-x}\sin y$ and $N = e^{-x}\cos y$

$\frac{\partial M}{\partial y} = e^{-x}\cos y$ and $\frac{\partial N}{\partial x} = -e^{-x}\cos y$

$M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on closed curve



$\therefore$ By Green's theorem, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$

$$\oint_C \left(e^{-x}\sin y dx + e^{-x}\cos y dy\right) = \iint_R \left(-e^{-x}\cos y - e^{-x}\cos y\right) dx dy$$

$$= \int_{x=0}^{x=\pi} \int_{y=0}^{y=\frac{\pi}{2}} \left(-2e^{-x}\cos y\right) dx dy$$

$$= -2\int_0^{\pi} e^{-x}\left[\sin y\right]_0^{\frac{\pi}{2}} dx$$

$$= -2\int_0^{\pi} e^{-x}[1-0] dx$$

$$= -2\int_0^{\pi} e^{-x} dx$$

$$= -2\left[-e^{-x}\right]_0^{\pi} = 2\left(e^{-\pi} - 1\right)$$

$$\therefore \boxed{\oint_C \left(e^{-x}\sin y dx + e^{-x}\cos y dy\right) = 2\left(e^{-\pi} - 1\right)}$$

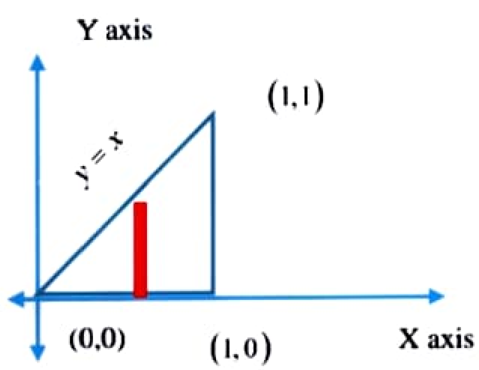
Example 6B.4.3: Using Green's theorem,evaluate $\oint_C \left(x^2 y dx + x^2 dy\right)$

where C is the boundary of the triangle with vertices (0,0) (1,0) & (1,1).

Solution: Let $M = x^2 y$ and $N = x^2$

$\frac{\partial M}{\partial y} = x^2$ and $\frac{\partial N}{\partial x} = 2x$

$M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on the triangle (closed curve) whose vertices are (0,0) (1,0) & (1,1).



Equation of straight line joining (0,0) and (1,1) is $y = x$
Equation of straight line joining (0,0) and (1,0) is $y = 0$ (X axis)
Equation of straight line joining (1,0) and (1,1) is $x = 1$

$\therefore$ By Green's theorem, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\therefore \oint_C \left[x^2 ydx + x^2 dy\right] = \iint_R \left[2x - x^2\right] dxdy$$

$$= \int_0^1 \int_{y=0}^{x} \left[2x - x^2\right] dxdy$$

$$= \int_0^1 \left[2x - x^2\right] [y]_0^x \, dx$$

$$= \int_0^1 \left[2x - x^2\right][x] dx$$

$$= \int_0^1 \left[2x^2 - x^3\right] dx$$

$$= \left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$

$$\therefore \boxed{\oint_C \left[x^2 ydx + x^2 dy\right] = \frac{5}{12}}$$

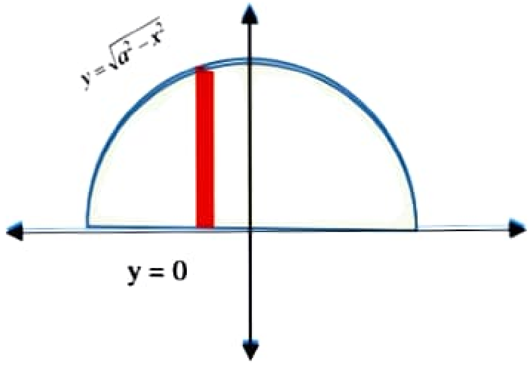
Example 6B.4.5: Using Green's Theorem, evaluate $\oint_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right]$ where C is the region bounded by the X – axis and the upper half of the circle $x^2 + y^2 = a^2$

Solution: Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$\frac{\partial M}{\partial y} = -2y$ and $\frac{\partial N}{\partial x} = 2x$

$M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on the given closed region .



Equation of the X axis is $y = 0$

$\therefore$ By Green's theorem, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\therefore \oint_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right] = \iint_R [2x + 2y]\,dxdy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} [2x + 2y]\,dxdy$$

$$= \int_0^a \left[2xy + y^2\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[2x\sqrt{a^2 - x^2} - (a^2 - x^2)\right] dx$$

$$= \int_0^a \left[2x\sqrt{a^2 - x^2} - a^2 + x^2\right] dx$$

$$= \left[-\frac{2}{3}(a^2 - x^2)^{3/2} - a^2 x + \frac{x^3}{3}\right]_0^a$$

$\left\{ \text{Put } a^2 - x^2 = t, \therefore -2xdx = dt, \quad \therefore \int 2x\sqrt{a^2 - x^2}\,dx = -\int \sqrt{t}\,dt = \frac{t^{3/2}}{3/2} = -\frac{2}{3}t^{3/2} = -\frac{2}{3}(a^2 - x^2)^{3/2} \right\}$

$$\therefore \oint_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right] = \left[0 - a^3 + \frac{a^3}{3}\right] - \left[-\frac{2}{3}(a^2)^{3/2} - 0 + 0\right]$$

$$= \frac{2}{3}a^3 + \frac{2}{3}a^3 = \frac{4}{3}a^3$$

$$\therefore \boxed{\oint_C \left[(2x^2 - y^2)dx + (x^2 + y^2)dy\right] = \frac{4}{3}a^3}$$

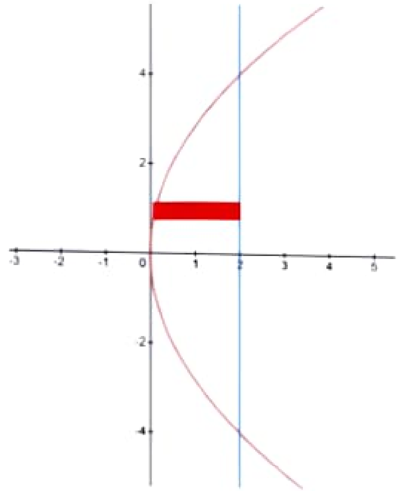
Example 6B.4.6: Using Green's Theorem, Evaluate $\oint_C (x^2-2xy)dx+(x^2y+3)dy$, where C is the boundary of the region bounded by the parabola $y^2=8x$ and straight line $x=2$

Solution: Let $M=x^2-2xy$ and $N=x^2y+3$

Let $M=x^2-2xy$ and $N=x^2y+3$

$\dfrac{\partial M}{\partial y}=-2x$ and $\dfrac{\partial N}{\partial x}=2xy$

M, N, $\dfrac{\partial M}{\partial y}$ and $\dfrac{\partial N}{\partial x}$ are continuous inside and on parallelogram (closed curve).



$\therefore$ By Green's theorem, $\oint_C Mdx+Ndy=\iint_R\left(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y}\right)dxdy$

$$\oint_C (x^2-2xy)dx+(x^2y+3)dy=\iint_R (2xy+2x)\,dxdy$$

$$=\int_{y=-4}^{y=4}\int_{x=\frac{y^2}{8}}^{x=2}(2xy+2x)\,dxdy$$

$$=\int_{y=-4}^{y=4}\left[yx^2+x^2\right]_{x=\frac{y^2}{8}}^{x=2}dy$$

$$=\int_{y=-4}^{y=4}\left\{[4y+4]-\left[\frac{y^5}{64}+\frac{y^4}{64}\right]\right\}dy$$

$$=\int_{y=-4}^{y=4}\left(4-\frac{y^4}{64}\right)dy \quad \left\{\because 4y \text{ and } \frac{y^5}{64} \text{ are odd functions and } \int_{y=-4}^{y=4}4ydy=0,\ \int_{y=-4}^{y=4}\frac{y^5}{64}dy=0\right\}$$

$$=2\int_{y=0}^{y=4}\left(4-\frac{y^4}{64}\right)dy \quad \left\{\because 4-\frac{y^4}{64} \text{ is an even function}\right\}$$

$$=2\left[4y-\frac{1}{64}\left(\frac{y^5}{5}\right)\right]_0^4$$

$$=2\left[16-\frac{1}{64}\left(\frac{1024}{5}\right)\right]=2\left[16-\frac{16}{5}\right]=\frac{128}{5}$$

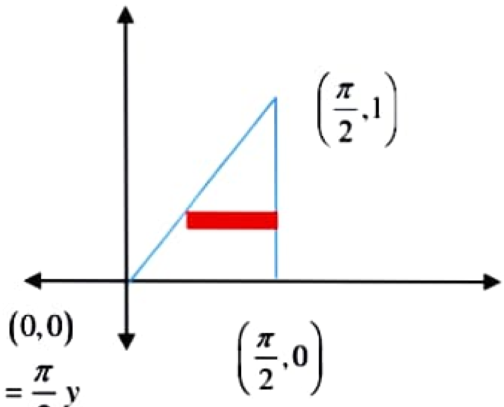
$$\therefore \boxed{\oint_C (x^2-2xy)dx+(x^2y+3)dy=\frac{128}{5}}$$

Example 6B.4.7: Using Green's Theorem, Evaluate $\oint_C (y-\sin x)dx+\cos x dy$, where C is the boundary of the triangle whose vertices are $(0,0)$, $\left(\frac{\pi}{2},0\right)$, $\left(\frac{\pi}{2},1\right)$

Solution: Let $M = y-\sin x$ and $N=\cos x$

$\frac{\partial M}{\partial y}=1$ and $\frac{\partial N}{\partial x}=-\sin x$

M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on triangle (closed curve).



Equation of straight line joining (0,0) and $\left(\frac{\pi}{2},1\right)$ is $x=\frac{\pi}{2}y$

Equation of straight line joining $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{\pi}{2},1\right)$ is $x=\frac{\pi}{2}$

Equation of straight line joining (0,0) and $\left(\frac{\pi}{2},1\right)$ is $y=0$

$\therefore$ By Green's theorem, $\oint_C Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

$$\oint_C[(y-\sin x)dx+\cos xdy]=\iint_R(-\sin x-1)\,dxdy$$

$$=-\int_{y=0}^{y=1}\int_{x=\frac{\pi}{2}y}^{x=\frac{\pi}{2}}(1+\sin x)\,dydx$$

$$=-\int_{y=0}^{y=1}[x-\cos x]_{x=\frac{\pi}{2}y}^{x=\frac{\pi}{2}}\,dy$$

$$=-\int_0^1\left[\left(\frac{\pi}{2}-0\right)-\left(\frac{\pi}{2}y-\cos\frac{\pi}{2}y\right)\right]dy$$

$$=-\int_0^1\left[\frac{\pi}{2}-\frac{\pi}{2}y+\cos\left(\frac{\pi}{2}y\right)\right]dy$$

$$=-\left[\frac{\pi}{2}y-\frac{\pi}{2}\frac{y^2}{2}+\frac{2}{\pi}\sin\frac{\pi}{2}y\right]_0^1$$

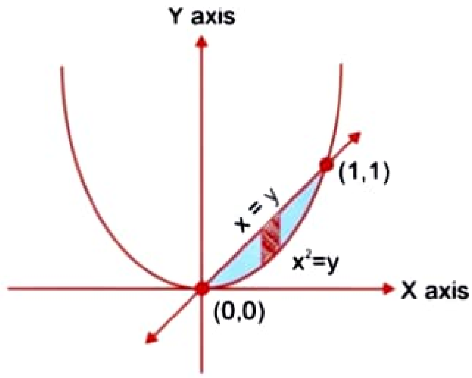
$$=-\left\{\left[\frac{\pi}{2}-\frac{\pi}{4}+\frac{2}{\pi}\right]-[0]\right\}=-\frac{\pi}{2}+\frac{\pi}{4}-\frac{2}{\pi}=-\frac{\pi}{4}-\frac{2}{\pi}$$

$$\therefore \boxed{\oint_C[(y-\sin x)dx+\cos xdy]=-\frac{\pi}{4}-\frac{2}{\pi}}$$

Example 6B.4.8: Verify Green's theorem, for $\oint_C \left[(xy+y^2)dx + x^2dy\right]$

where C is the closed curve bounded by the straight line $y = x$ and parabola $y = x^2$.

Solution: Let $M = xy + y^2$ and $N = x^2$



Let $M = xy + y^2$ and $N = x^2$

$\frac{\partial M}{\partial y} = x + 2y$ and $\frac{\partial N}{\partial x} = 2x$

M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on closed curve bounded by the straight lines $y = x$ and parabola $y = x^2$

By Green's Theorem, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

That is, $\oint_C \left[(xy+y^2)dx + x^2dy\right] = \iint_R [2x - x - 2y]dxdy$

$$\text{R.H.S} = \iint_R \left(\frac{\partial N}{dx} - \frac{\partial M}{dy}\right)dxdy = \iint_R [x - 2y]dxdy$$

$$= \int_0^1 \int_{y=x^2}^{y=x} (x-2y)dxdy$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x^2 - x^2\right) - \left(x^3 - x^4\right)\right]dx$$

$$= \int_0^1 \left(x^4 - x^3\right)dx = \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

$$\therefore \iint_R \left(\frac{\partial N}{dx} - \frac{\partial M}{dy}\right)dxdy = -\frac{1}{20} \text{------------} (1)$$

$$\text{L.H.S} = \oint_C Mdx + Ndy = \int_{\substack{C_1 \\ \text{Along } y=x^2}} Mdx + Ndy + \int_{\substack{C_2 \\ \text{Along } y=x}} Mdx + Ndy \text{-----} (2)$$

Along C_1; $y = x^2$

$\therefore$ $dy = 2xdx$ and x varies from 0 to 1.

$$\therefore \oint_{C_1} Mdx + Ndy = \oint_{C_1} \left[(xy + y^2)dx + x^2 dy \right]$$

$$= \int_0^1 \left[\left(x(x^2) + (x^2)^2 \right) dx + x^2 (2xdx) \right]$$

$$= \int_0^1 \left[x^3 + x^4 + 2x^3 \right] dx$$

$$= \int_0^1 \left[3x^3 + x^4 \right]$$

$$= \left[\frac{3}{4} x^4 + \frac{x^5}{5} \right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

$$\therefore \oint_{C_1} Mdx + Ndy = \frac{19}{20} \text{---------(3)}$$

Along C_2; $y = x$

$\therefore$ $dy = dx$ and x varies from 1 to 0.

$$\therefore \oint_{C_2} Mdx + Ndy = \oint_{C_2} \left[(xy + y^2)dx + x^2 dy \right]$$

$$= \int_1^0 \left[\left(x(x) + x^2 \right) dx + x^2 (dx) \right]$$

$$= \int_1^0 \left[x^2 + x^2 + x^2 \right] dx$$

$$= \int_1^0 \left[3x^2 \right] dx$$

$$= \left[x^3 \right]_1^0 = -1$$

$$\therefore \oint_{C_2} Mdx + Ndy = -1 \text{---------(4)}$$

Substituting from (3) and (4) in (2)

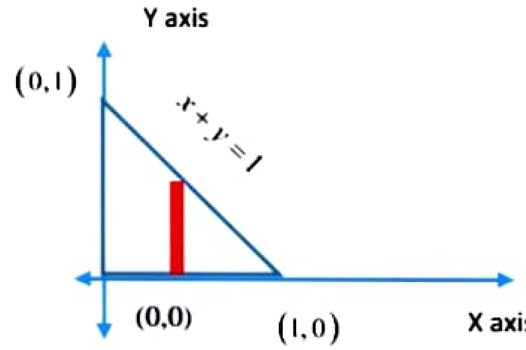
$$\oint_C Mdx + Ndy = \frac{19}{20} - 1 = -\frac{1}{20}$$

Thus, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{dx} - \frac{\partial M}{dy} \right) dxdy$

Hence Green's theorem in the plane is verified.

Example 6B.4.9: Verify Green's theorem,for $\oint_C \left[(3x^2-8y^2)dx+(4y-6xy)dy)\right]$ where C is bounded by the straight lines $x=0,\ y=0$ and $y+x=1$

Solution: Let $M=3x^2-8y^2$ and $N=4y-6xy$



$\frac{\partial M}{\partial y}=-16y$ and $\frac{\partial N}{\partial x}=-6y$

$M,\ N,\ \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous inside and on closed curve bounded by the straight lines $x=0,\ y=0$ and $x+y=1$

By Green's Theorem, $\oint_C Mdx+Ndy=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

That is, $\oint_C \left[(3x^2-8y^2)dx+(4y-6xy)dy\right]=\iint_R \left(-6y-(-10y)\right)dxdy$

$$\text{R.H.S}=\iint_R \left(\frac{\partial N}{dx}-\frac{\partial M}{dy}\right)dxdy=\iint_R \left(-6y-(-10y)\right)dxdy$$

$$=\int_0^1\int_0^{1-x} 10y\,dydx$$

$$=\int_0^1 5\left[y^2\right]_0^{1-x}dx$$

$$=5\int_0^1 (1-x)^2\,dx=5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$=-\frac{5}{3}(0-1)=\frac{5}{3}$$

$$\therefore \iint_R \left(\frac{\partial N}{dx}-\frac{\partial M}{dy}\right)dxdy=\frac{5}{3}-------------(1)$$

$$\text{L.H.S}=\oint_C Mdx+Ndy=\underset{\substack{C_1\\\text{Along } y=0}}{\oint} Mdx+Ndy+\underset{\substack{C_2\\\text{Along } x+y=1}}{\oint} Mdx+Ndy+\underset{\substack{C_3\\\text{Along } x=0}}{\oint} Mdx+Ndy---(2)$$

Along C_1; $y=0$

$\therefore$ $dy=0$ and the limits of x are from 0 to 1.

$$\therefore \oint_{C_1} Mdx+Ndy=\oint_{C_1}\left[(3x^2-8y^2)dx+(4y-6xy)dy\right]=\int_0^1 3x^2dx=\left[x^3\right]_0^1=1$$

$$\therefore \oint_{C_1} Mdx + Ndy = 1 \text{--------} (3)$$

Along C_2; $y = 1 - x$

$\therefore$ $dy = -dx$ and x varies from 1 to 0.

$$\therefore \oint_{C_2} Mdx + Ndy = \oint_{C_2}\left[(3x^2 - 8y^2)dx + (4y - 6xy)dy\right]$$

$$= \int_1^0 \left[\left(3x^2 - 8(1-x)^2\right)dx + \left(4(1-x) - 6x(1-x)\right)(-dx)\right]$$

$$= \int_1^0 \left[3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2\right]dx$$

$$= \int_1^0 (-12 + 26x - 11x^2)dx$$

$$= \left[-12x + 13x^2 - \frac{11}{3}x^3\right]_1^0 = -\left[-12 + 13 - \frac{11}{3}\right] = \frac{8}{3}$$

$$\therefore \oint_{C_2} Mdx + Ndy = \frac{8}{3} \text{----------} (4)$$

Along C_3; $x = 0$.

$\therefore$ $dx = 0$ and y varies from 1 to 0.

$$\oint_{C_3} Mdx + Ndy = \int_1^0 4y\,dy = \left[2y^2\right]_1^0 = -2 \text{--------} (5)$$

Substituting from (3), (4) and (5) in (2)

$$\oint_C Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$$

Thus, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{dx} - \frac{\partial M}{dy}\right)dxdy$

Hence Green's theorem in the plane is verified.

Stoke's Theorem

Statement: If $\overline{F}(x, y, z)$ is a continuous vector point function having continuous partial derivatives on an open surface S bounded by a closed curve C, then

$$\iint_S \left(\nabla \times \overline{F}\right) \cdot n\, ds = \oint_C \overline{F} \cdot d\bar{r}$$

Note: Curve C is always positively oriented (anti-clockwise sense) and n is the unit outward normal vector to the open surface S.
LHS is the surface integral and RHS is the line integral.

To evaluate **Surface Integral**, convert it into **Double Integral** by taking projection of the surface to **XY Plane, XZ Plane or YZ Plane.**

(i) When projection of the surface is taken to XY Plane, $ds = \dfrac{dxdy}{|n \cdot k|}$

(ii) When projection of the surface is taken to XZ Plane, $ds = \dfrac{dxdz}{|n \cdot j|}$

(iii) When projection of the surface is taken to YZ Plane, $ds = \dfrac{dydz}{|n \cdot \hat{i}|}$

Solved Examples

1. Prove that $\oint_C \bar{r}\cdot d\bar{r}=0$, where C is any closed curve.

2. Verify Stoke's Theorem for $\overline{F}=(x^2+y^2)\hat{i}-2xy\,j$ taken around the rectangle bounded by the lines $x=-a,\ x=a,\ y=0,\ y=b$

3. Evaluate $\iint_S \left(\nabla\times\overline{F}\right)\cdot nds$, where $\overline{F}=2y(1-x)\hat{i}+(x-x^2+y^2)\,j+(x^2+y^2+z^2)k$ and S is the surface of the hemispherical cap $x^2+y^2+z^2=a^2,\ z\ge 0$ above XY plane.

4. Evaluate $\iint_S \left(\nabla\times\overline{F}\right)\cdot nds$, where $\overline{F}=(x^2+y-4)\hat{i}+3xy\,j+(2xz+z^2)k$ and S is the surface of the paraboloid $z=4-\left(x^2+y^2\right)$ above XY plane.

5. Evaluate $\iint_S \left(\nabla\times\overline{A}\right)\cdot nds$, where $\overline{A}=(x-z)\hat{i}+(x^3+yz)\,j-3xy^2k$ and S is the surface of the cone $z=2-\sqrt{x^2+y^2}$ above XY plane.

Example 6B.5.1: Prove that $\oint_C \bar{r}\cdot d\bar{r}=0$, where C is any closed curve.

Solution: $\bar{r}=\hat{i}x+jy+kz$ is continuous and C is any closed curve.

By Stoke's Theorem, $\oint_C \bar{r}\cdot \overline{dr}=\iint_S(\nabla\times\bar{r})\cdot nds$, where n is unit outward normal vector to the surface

$$\nabla\times\bar{r}=\begin{vmatrix}\hat{i} & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$=\hat{i}\left[\frac{\partial}{\partial y}(z)-\frac{\partial}{\partial z}(y)\right]-j\left[\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial z}(x)\right]+k\left[\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial y}(x)\right]$$

$$=\hat{i}[0-0]-j[0-0]+k[0-0]$$

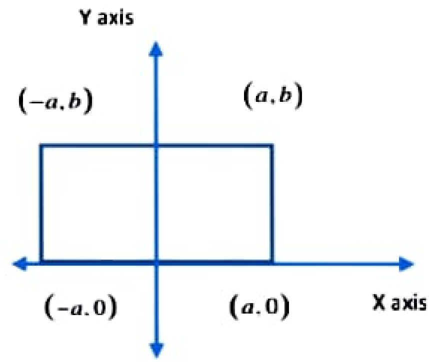
$$\therefore \nabla\times\bar{r}=\bar{0}$$

$$\therefore \oint_C \bar{r}\cdot\overline{dr}=\iint_S(\bar{0})\cdot\widehat{nds}=0$$

$$\therefore \boxed{\oint_C \bar{r}\cdot\overline{dr}=0}$$

Example 6B.5.2: Verify Stoke's Theorem for $\overline{F} = (x^2 + y^2)\hat{i} - 2xy\, j$ taken around the rectangle bounded by the lines $x = -a,\ x = a,\ y = 0,\ y = b$

Solution: Rectangle bounded by the lines $x = -a,\ x = a,\ y = 0,\ y = b$ is a closed curve and $\overline{F} = (x^2 + y^2)\hat{i} - 2xy\, j$ is continuous inside and on closed curve



$\therefore$ By Stoke's Theorem, $\oint_C \overline{F}\cdot\overline{dr} = \iint_S \left(\nabla\times\overline{F}\right)\cdot nds$

$$\nabla\times\overline{F} = \begin{vmatrix} \hat{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix}$$

$$\therefore \nabla\times\overline{F} = \hat{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy)\right] - j\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}\left(x^2+y^2\right)\right] + k\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}\left(x^2+y^2\right)\right]$$

$$= \hat{i}[0-0] - j[0-0] + k[-2y-2y]$$

$$\therefore \nabla\times\overline{F} = -4yk$$

Rectangle is in XY plane. Therefore, unit outward normal vector is $n = k$

$$\therefore \iint_S \left(\nabla\times\overline{F}\right)\cdot nds = \iint_S \left(-4yk\right)\cdot k\,\frac{dxdy}{\left|\widehat{n\cdot k}\right|}$$

$$= \iint_S -4y\,dxdy \qquad \left\{\because \widehat{n\cdot k} = \widehat{k\cdot k} = 1\right\}$$

$$= \int_{x=-a}^{x=a}\int_{y=0}^{y=b} -4y\,dxdy$$

$$= -4\int_{x=-a}^{x=a}\left[\frac{y^2}{2}\right]_{y=0}^{y=b} dx$$

$$= -2\int_{x=-a}^{x=a}\left[b^2-0\right]dx$$

$$= -2b^2\left[x\right]_{x=-a}^{x=a}$$

$$= -2b^2\left[a-(-a)\right] = -4ab^2$$

$$\therefore \boxed{\iint_S \left(\nabla\times\overline{F}\right)\cdot\widehat{nds} = -4ab^2} \text{-----------}(1)$$

$$\text{L.H.S} = \oint_C \overline{F}\cdot\overline{dr} = \int_{\substack{C_1\\ \text{Along } y=0}} \overline{F}\cdot\overline{dr} + \int_{\substack{C_2\\ \text{Along } x=a}} \overline{F}\cdot\overline{dr} + \int_{\substack{C_3\\ \text{Along } y=b}} \overline{F}\cdot\overline{dr} + \int_{\substack{C_4\\ \text{Along } x=-a}} \overline{F}\cdot\overline{dr}$$

$$\overline{F}\cdot\overline{dr} = \left[(x^2+y^2)\hat{i} - 2xy\,\hat{j}\right]\cdot\left[\hat{i}dx + \hat{j}dy\right] = \left(x^2+y^2\right)dx - (2xy)\,dy$$

Along C_1; $y=0$

$\therefore\ dy=0$ and x varies from $-a$ to a.

$\overline{F}\cdot\overline{dr}=\left(x^2+0\right)dx-0=x^2dx$

$$\therefore \int_{C_1}\overline{F}\cdot\overline{dr}=\int_{-a}^{a}x^2dx=\left[\frac{x^3}{3}\right]_{-a}^{a}=\frac{1}{3}\left[a^3-\left(-a^3\right)\right]=\frac{2a^3}{3}$$

Along C_2; $x=a$

$\therefore\ dx=0$ and y varies from 0 to b.

$\overline{F}\cdot\overline{dr}=0-(2ay)dy=-2aydy$

$$\therefore \int_{C_2}\overline{F}\cdot\overline{dr}=\int_{0}^{b}-(2ay)dy=-2a\left[\frac{y^2}{2}\right]_{0}^{b}=-ab^2$$

Along C_3; $y=b$

$\therefore\ dy=0$ and x varies from a to $-a$.

$\overline{F}\cdot\overline{dr}=\left(x^2+b^2\right)dx-0=\left(x^2+b^2\right)dx$

$$\therefore \int_{C_1}\overline{F}\cdot\overline{dr}=\int_{a}^{-a}\left(x^2+b^2\right)dx=\left[\frac{x^3}{3}+b^2x\right]_{a}^{-a}$$

$$\therefore \int_{C_1}\overline{F}\cdot\overline{dr}=\left[\frac{-a^3}{3}-ab^2\right]-\left[\frac{a^3}{3}+ab^2\right]=-\frac{2a^3}{3}-2ab^2$$

Along C_4; $x=-a$

$\therefore\ dx=0$ and y varies from b to 0.

$\overline{F}\cdot\overline{dr}=0-(-2ay)dy=2aydy$

$$\therefore \int_{C_4}\overline{F}\cdot\overline{dr}=\int_{b}^{0}(2ay)dy=2a\left[\frac{y^2}{2}\right]_{b}^{0}=a\left[0-b^2\right]=-ab^2$$

$$\therefore \oint_{C}\overline{F}\cdot\overline{dr}=\left[\frac{2a^3}{3}\right]+\left[-ab^2\right]+\left[-\frac{2a^3}{3}-2ab^2\right]+\left[-ab^2\right]=-4ab^2$$

$$\therefore \boxed{\oint_{C}\overline{F}\cdot\overline{dr}=-4ab^2}\text{ ---------(2)}$$

From (1) and (2), $\oint_{C}\overline{F}\cdot\overline{dr}=\iint_{S}\left(\nabla\times\overline{F}\right)\cdot\widehat{n}ds$

Hence, Stoke's Theorem is verified.

Example 6B.5.3: Evaluate $\iint_S \left(\nabla\times\overline{F}\right)\cdot nds$, where

$\overline{F}=2y(1-x)\hat{i}+(x-x^2+y^2)j+(x^2+y^2+z^2)k$ and S is the surface of the hemispherical cap $x^2+y^2+z^2=a^2,\ z\geq 0$ above XY plane.

Solution: surface of the hemispherical cap $x^2+y^2+z^2=a^2,\ z\geq 0$ above XY plane is an open surface, bounded by the closed curve $C:x^2+y^2=a^2,\ z=0$. It is a circle in the XY plane having centre at $(0,0)$ and radius $=a$

By Stoke's Theorem, $\iint_S \left(\nabla\times\overline{F}\right)\cdot nds=\oint_C \overline{F}\cdot\overline{dr}$

$$\overline{F}\cdot\overline{dr}=\left[2y(1-x)\hat{i}+(x-x^2+y^2)j+(x^2+y^2+z^2)k\right]\cdot\left[\hat{i}dx+jdy+kdz\right]$$

$$\therefore \overline{F}\cdot\overline{dr}=2y(1-x)dx+(x-x^2+y^2)dy+(x^2+y^2+z^2)dz$$

Along $C:x^2+y^2=a^2,\ z=0,$

$$x=a\cos\theta,\ \ y=a\sin\theta,\ \ z=0$$

$$dx=-a\sin\theta d\theta,\ \ dy=a\cos\theta d\theta,\ \ dz=0$$

$$\therefore \overline{F}\cdot\overline{dr}=2(a\sin\theta)(1-a\cos\theta)(-a\sin\theta d\theta)+\left(a\cos\theta-(a\cos\theta)^2+(a\sin\theta)^2\right)(a\cos\theta d\theta)+0$$

$$=\left(-2a^2\sin^2\theta+2a^3\sin^2\theta\cos\theta\right)d\theta+\left(a^2\cos^2\theta-a^3\cos^3\theta+a^3\sin^2\theta\cos\theta\right)d\theta$$

$$=\left(-2a^2\sin^2\theta+a^2\cos^2\theta-a^3\cos^3\theta+3a^3\sin^2\theta\cos\theta\right)d\theta$$

$$\therefore \oint_C \overline{F}\cdot\overline{dr}=\int_{\theta=0}^{\theta=2\pi}\left(-2a^2\sin^2\theta+a^2\cos^2\theta-a^3\cos^3\theta+3a^3\sin^2\theta\cos\theta\right)d\theta$$

$$=\int_{\theta=0}^{\theta=2\pi}\left(-2a^2\left(\frac{1-\cos 2\theta}{2}\right)+a^2\left(\frac{1+\cos 2\theta}{2}\right)-a^3\left(\frac{3\cos\theta+\cos 3\theta}{4}\right)+3a^3\sin^2\theta\cos\theta\right)d\theta$$

$$=\left[-a^2\left(\theta-\frac{\sin 2\theta}{2}\right)+\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)-\frac{a^3}{4}\left(3\sin\theta+\frac{\sin 3\theta}{3}\right)+3a^2\frac{\sin^3\theta}{3}\right]_{\theta=0}^{\theta=2\pi}$$

$$=\left[-a^2(2\pi-0)+\frac{a^2}{2}(2\pi+0)-\frac{a^3}{4}(0+0)+3a^2(0)\right]-[0]$$

$$=-2\pi a^2+\pi a^2=-\pi a^2$$

$$\therefore \boxed{\iint_S\left(\nabla\times\overline{F}\right)\cdot\widehat{nds}=-\pi a^2}$$

Example 6B.5.4: Evaluate $\iint_S (\nabla \times \overline{F}) \cdot n ds$, where $\overline{F} = (x^2 + y - 4)\hat{i} + 3xy\, j + (2xz + z^2)k$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above XY plane.

Solution: surface of the paraboloid $z = 4 - (x^2 + y^2)$ above XY plane is an open surface, bounded by the closed curve $C: x^2 + y^2 = 4$. It is a circle in the XY plane having centre at $(0,0)$ and radius = 2

By Stoke's Theorem, $\iint_S (\nabla \times \overline{F}) \cdot n ds = \oint_C \overline{F} \cdot \overline{dr}$

$$\overline{F} \cdot \overline{dr} = (x^2 + y - 4)\hat{i} + 3xy\, j + (2xz + z^2)k \cdot \left[\hat{i}dx + jdy + kdz\right]$$

$$\therefore \overline{F} \cdot \overline{dr} = (x^2 + y - 4)dx + 3xydy + (2xz + z^2)dz$$

Along $C: x^2 + y^2 = 4,\ z = 0,$

$$x = 2\cos\theta,\quad y = 2\sin\theta,\quad z = 0$$

$$dx = -2\sin\theta d\theta,\quad dy = 2\cos\theta d\theta,\quad dz = 0$$

$$\therefore \overline{F} \cdot \overline{dr} = \left((2\cos\theta)^2 + 2\sin\theta - 4\right)(-2\sin\theta d\theta) + \left(3(2\cos\theta)(2\sin\theta)\right)(2\cos\theta d\theta) + 0$$

$$= \left(-8\cos^2\theta\sin\theta - 4\sin^2\theta + 8\sin\theta\right)d\theta + \left(24\cos^2\theta\sin\theta\right)d\theta$$

$$= \left(16\cos^2\theta\sin\theta - 4\sin^2\theta + 8\sin\theta\right)d\theta$$

$$\therefore \oint_C \overline{F} \cdot \overline{dr} = \int_{\theta=0}^{\theta=2\pi} \left(16\cos^2\theta\sin\theta - 4\sin^2\theta + 8\sin\theta\right)d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \left(16\cos^2\theta\sin\theta - 4\left(\frac{1-\cos 2\theta}{2}\right) + 8\sin\theta\right)d\theta$$

$$= \left[16\left(-\frac{\cos^3\theta}{2}\right) - 2\left(\theta - \frac{\sin 2\theta}{2}\right) - 8\cos\theta\right]_{\theta=0}^{\theta=2\pi}$$

$$= \left[8(-1) - 2(2\pi - 0) - 8\right] - \left[8(-1) - 2(0-0) - 8\right]$$

$$= -4\pi$$

$$\therefore \boxed{\iint_S (\nabla \times \overline{F}) \cdot \widehat{nds} = -4\pi}$$

Example 6B.5.5: Evaluate $\iint_S (\nabla \times \overline{A}) \cdot n ds$, where $\overline{A} = (x-z)\hat{i} + (x^3 + yz) j - 3xy^2 k$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above XY plane.

Solution: surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above XY plane is an open surface, bounded by the closed curve $C : x^2 + y^2 = 4$. It is a circle in the XY plane having centre at $(0,0)$ and radius = 2

By Stoke's Theorem, $\iint_S (\nabla \times \overline{A}) \cdot n ds = \oint_C \overline{A} \cdot \overline{dr}$

$$\overline{A} = (x-z)\hat{i} + (x^3 + yz) j - 3xy^2 k$$

$$\overline{A} \cdot \overline{dr} = \left[(x-z)\hat{i} + (x^3 + yz) j - 3xy^2 k\right] \cdot \left[\hat{i}dx + jdy + kdz\right]$$

$$\therefore \overline{A} \cdot \overline{dr} = (x-z)dx + (x^3 + yz)dy - 3xy^2 dz$$

Along $C : x^2 + y^2 = 4,\ z = 0,$

$$x = 2\cos\theta,\quad y = 2\sin\theta,\quad z = 0$$

$$dx = -2\sin\theta d\theta,\quad dy = 2\cos\theta d\theta,\quad dz = 0$$

$$\therefore \overline{A} \cdot \overline{dr} = (2\cos\theta - 0)(-2\sin\theta d\theta) + \left((2\cos\theta)^3 + 0\right)(2\cos\theta d\theta) + 0$$

$$= \left(-4\sin\theta\cos\theta + 16\cos^4\theta\right) d\theta$$

$$\therefore \oint_C \overline{A} \cdot \overline{dr} = \int_{\theta=0}^{\theta=2\pi} \left(-4\sin\theta\cos\theta + 16\cos^4\theta\right) d\theta ----(1)$$

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left[1 + \cos 2\theta + \cos^2 2\theta\right]$$

$$= \frac{1}{4}\left[1 + \cos 2\theta + \left(\frac{1+\cos 4\theta}{2}\right)\right]$$

$$\therefore \cos^4\theta = \frac{1}{8}[3 + 2\cos 2\theta + \cos 4\theta]$$

$$\therefore \oint_C \overline{A} \cdot \overline{dr} = \int_{\theta=0}^{\theta=2\pi} \left(-4\sin\theta\cos\theta + \frac{16}{8}[3 + 2\cos 2\theta + \cos 4\theta]\right) d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} (-4\sin\theta\cos\theta + 6 + 4\cos 2\theta + 2\cos 4\theta) d\theta$$

$$\therefore \oint_C \overline{A}\cdot\overline{dr} = \int_{\theta=0}^{\theta=2\pi}\left(-4\sin\theta\cos\theta + \frac{16}{8}\left[3+2\cos 2\theta+\cos 4\theta\right]\right)d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi}\left(-2\sin 2\theta + 6 + 4\cos 2\theta + 2\cos 4\theta\right)d\theta$$

$$= \left[\cos 2\theta + 6\theta + 2\sin 2\theta + \frac{\sin 4\theta}{2}\right]_0^{2\pi}$$

$$= \left[1+12\pi+0+0\right] - \left[1+0+0+0\right] = 12\pi$$

$$\therefore \oint_C \overline{A}\cdot\overline{dr} = 12\pi$$

$$\therefore \boxed{\iint_S \left(\nabla\times\overline{A}\right)\cdot\widehat{nds} = 12\pi}$$