Course Code	Course Name	Credits
MEC401	**Engineering Mathematics-IV**	**04**

Pre-requisite: Engineering Mathematics-I, Engineering Mathematics-II, Engineering Mathematics-III, Binomial Distribution, Physical Interpretation of Vector differentiation, Vector differentiation operator, Gradient of scalar point function, Directional derivative, Divergence of vector point function, Curl of vector point function.

Objectives:

1. To study the concept of Vector calculus & its applications in engineering.
2. To study Line and Contour integrals and expansion of complex valued function in a power series.
3. To familiarize with the concepts of statistics for data analysis.
4. To acquaint with the concepts of probability, random variables with their distributions and expectations.
5. To familiarize with the concepts of probability distributions and sampling theory with its applications.

Outcomes: On successful completion of course learner/student will be able to:

1. Apply the concept of Vector calculus to evaluate line integrals, surface integrals using Green's theorem, Stoke's theorem & Gauss Divergence theorem.
2. Use the concepts of Complex Integration for evaluating integrals, computing residues & evaluate various contour integrals.
3. Apply the concept of Correlation, Regression and curve fitting to the engineering problems in data science.
4. Illustrate understanding of the concepts of probability and expectation for getting the spread of the data and distribution of probabilities.
5. Apply the concept of probability distribution to engineering problems & testing hypothesis of small samples using sampling theory.
6. Apply the concepts of parametric and nonparametric tests for analyzing practical problems.

Module	Detailed Contents	Hrs.
01	**Module : Vector Calculus** 1.1 Solenoidal and irrotational (conservative) vector fields. 1.2 Line integrals – definition and problems. 1.3 Green's theorem (without proof) in a plane, Stokes' theorem (without Proof), Gauss' Divergence theorem (without proof) and problems (only evaluation). **<u>Self Learning Topics</u>**: Identities connecting Gradient, Divergence and Curl, Angle between surfaces. Verifications of Green's theorem, Stoke's theorem & Gauss-Divergence theorem, related identities & deductions.	**07**
02	**Module: Complex Integration** 2.1 Line Integral, Cauchy's Integral theorem for simple connected and multiply connected regions (without proof), Cauchy's Integral formula (without proof). 2.2 Taylor's and Laurent's series (without proof). 2.3 Definition of Singularity, Zeroes, poles of $f(z)$, Residues, Cauchy's Residue Theorem (without proof) **Self-learning Topics:** Application of Residue Theorem to evaluate real integrations.	**07**

03	**Module: Statistical Techniques** 3.1 Karl Pearson's Coefficient of correlation (r) and related concepts with problems 3.2 Spearman's Rank correlation coefficient (R) (Repeated & non repeated ranks problems) 3.3 Lines of regression 3.4 Fitting of first and second degree curves. **Self-learning Topics:** Covariance, fitting of exponential curve.	06
04	**Module: Probability Theory:** 4.1 Conditional probability, Total Probability and Baye's Theorem. 4.2 Discrete and Continuous random variables, Probability mass and density function, Probability distribution for random variables, 4.3 Expectation, Variance, Co-variance, moments, Moment generating functions, (Four moments about the origin &about the mean). **Self- learning Topics:** Properties variance and covariance,	06
05	**Module: Probability Distribution and Sampling Theory-I** 5.1 Probability Distribution: Poisson and Normal distribution 5.2 Sampling distribution, Test of Hypothesis, Level of Significance, Critical region, One-tailed, and two-tailed test, Degree of freedom. 5.3 Students' t-distribution (Small sample). Test the significance of single sample mean and two independent sample means and paired t- test) **Self -learning Topics:** Test of significance of large samples, Proportion test, Survey based project.	07
06	**Module: Sampling theory-II** 6.1 Chi-square test: Test of goodness of fit and independence of attributes (Contingency table) including Yate's Correction. 6.2 Analysis of variance: F-test (significant difference between variances of two samples) **Self- learning Topics:** ANOVA: One way classification, Two-way classification (short-cut method).	06

Term Work:

General Instructions:

1. Students must be encouraged to write at least 6 class tutorials on entire syllabus.
2. A group of 4-6 students should be assigned a self-learning topic. Students should prepare a presentation/problem solving of 10-15 minutes. This should be considered as mini project in Engineering Mathematics. This project should be graded for 10 marks depending on the performance of the students.

The distribution of Term Work marks will be as follows –

1.	Attendance (Theory and Tutorial)	05 marks
2.	Class Tutorials on entire syllabus	10 marks
3.	Mini project	10 marks

Assessment:

Internal Assessment Test:

Assessment consists of two class tests of 20 marks each. The first class test (Internal Assessment I) is to be conducted when approx. 40% syllabus is completed and second class test (Internal Assessment II) when additional 35% syllabus is completed. Duration of each test shall be one hour.

End Semester Theory Examination:

1. Question paper will comprise of total 06 questions, each carrying 20 marks.
2. Total 04 questions need to be solved.
3. Question No: 01 will be compulsory and based on entire syllabus wherein 4sub-questions of 5 marks each will be asked.
4. Remaining questions will be randomly selected from all the modules.
5. Weightage of each module will be proportional to number of respective lecture hours as mentioned in the syllabus.

References:

1. Higher Engineering Mathematics, Dr. B. S. Grewal, Khanna Publication
2. Advanced Engineering Mathematics, Erwin Kreyszig, Wiley Eastern Limited,
3. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar, Narosa publication,
4. Vector Analysis, Murray R. Spiegel, Schaum Series
5. Complex Variables and Applications, Brown and Churchill, McGraw-Hill education
6. Probability, Statistics and Random Processes, T. Veerarajan, Mc. Graw Hill education.

Links for online NPTEL/SWAYAM courses:

1. https://www.youtube.com/watch?v=2CP3m3EgL1Q&list=PLbMVogVj5nJQrzbAweTVvnH6-vG5A4aN5&index=7
2. https://www.youtube.com/watch?v=Hw8KHNgRaOE&list=PLbMVogVj5nJQrzbAweTVvnH6-vG5A4aN5&index=8
3. https://nptel.ac.in/courses/111/105/111105041/

Chapter-11	Complex Integral
Topic-11.1	

Introduction

The complex line integral of $f(z)$ along the curve C from point A to point B is denoted as $\int_C f(z)dz$. To evaluate complex line integral, we express it in terms of two real line integrals and evaluate.

Let $f(z) = u(x, y) + iv(x, y)$, then

$$\int_C f(z)dz = \int_C (u+iv)(dx+idy)$$
$$= \int_C (udx - vdy) + i(vdx + idy)$$

Using the equation of given curve C, eliminate one variable (either x or y) and integrate w.r.t another variable.

Note: If path of integration is circle or arc of the circle, use polar form.

For the circle $|z-(a+ib)| = k$,

$z-(a+ib) = ke^{i\theta}$

$\therefore z = (a+ib)+ke^{i\theta}$ and $dz = kie^{i\theta}d\theta$

Solve Examples

1. Evaluate $\int_i^{2-i} (3xy + iy^2)dz$, along the straight line joining the points $z = i$ and $z = 2-i$

2. Evaluate $\int_0^{1+i} (x - y + ix^2)dz$ along
   a) the straight line joining $z = 0$ and $z = 1+i$
   b) the parabola $y = x^2$

3. Evaluate $\int_0^{1+i} (x^2 + iy)dz$, along the parabola (i) $y = x^2$ and (ii) $x = y^2$

4. Evaluate $\int_C (x^2 - iy^2)\,dz$ along the straight line from $(0,0)$ to $(0,1)$ and then from $(0,1)$ to $(2,1)$

5. Prove that $\int_C \log z\,dz = 2\pi i$, where C is the unit circle in the z-plane.

6. Evaluate $\int_C (z - z^2)\,dz$ along the upper half of the circle (i) $|z| = 1$

7. Evaluate $\int_C \frac{z+1}{z}\,dz$, where C is the semicircle $z = 2e^{i\theta}, 0 \le \theta \le \pi$

8. Evaluate $\int_0^{3+i} |z|^2\,dz$, along the straight line $3y = x$

9. Evaluate $\oint_C |z|^2\,dz$ around the square with vertices (0, 0), (1, 0), (1, 1), (0, 1).

10. Evaluate $\int_C (z+1)^2\,dz$ along

 (i) Straight line from $z = i$ to $z = 1$

 (ii) The imaginary axis from $z = i$ to $z = 0$ and then from $z = 0$ to $z = 1$ along the real axis.

 Is the integral independent of path? Justify.

Example 11.1.1: Evaluate $\int_i^{2-i}\left(3xy+iy^2\right)dz$, along the straight line joining the points $z=i$ and $z=2-i$

Solution: $z=i$ and $z=2-i$ represent points $(0,1)$ and $(2,-1)$ respectively.

Equation of straight line joining $(0,1)$ and $(2,-1)$ is

$$y-1=\frac{-1-1}{2-0}(x-0)\quad\left\{\text{Using } y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\right\}$$

That is, $y-1=-x$ or $x+y=1$

$z=x+iy \quad \therefore dz=dx+idy$

Along C; $x+y=1$

$\therefore \quad x=1-y$ and $dx=-dy$

$\therefore dz=dx+idy=-dy+idy=(-1+i)dy$

y varies from 1 to -1

$$\therefore I=\int_i^{2-i}\left(3xy+iy^2\right)dz$$

$$=\int_{y=1}^{y=-1}\left[3(1-y)\,y+iy^2\right](-1+i)dy$$

$$=\int_{y=1}^{y=-1}\left[3y-3y^2+iy^2\right](-1+i)dy$$

$$=(-1+i)\left[\frac{3y^2}{2}-\frac{3y^3}{3}+\frac{iy^3}{3}\right]_1^{-1}$$

$$=(-1+i)\left\{\left[\frac{3}{2}(-1)^2-(-1)^3+\frac{i(-1)^3}{3}\right]-\left[\frac{3}{2}(1)^2-(1)^3+\frac{i(1)^3}{3}\right]\right\}$$

$$=(-1+i)\left\{\left(\cancel{\frac{3}{2}}+1-\frac{i}{3}\right)-\left(\cancel{\frac{3}{2}}-1+\frac{i}{3}\right)\right\}$$

$$=(-1+i)\left(2-\frac{2i}{3}\right)$$

$$\therefore I = (-1+i)\left(\frac{6-2i}{3}\right)$$

$$= \frac{1}{3}[-6+2i+6i+2]$$

$$= \frac{1}{3}[-4+8i]$$

$$\therefore \boxed{\int_{i}^{2-i}\left(3xy+iy^2\right)dz = \frac{1}{3}[-4+8i]}$$

Example 11.1.2: Evaluate $\int_0^{1+i} \left(x - y + ix^2\right) dz$ along

(i) the straight line joining $z = 0$ and $z = 1 + i$

(ii) the parabola $y = x^2$

Solution: $z = 0$ and $z = 1 + i$ represent points $(0,0)$ and $(1,1)$ respectively.

Equation of straight line joining $(0,0)$ and $(1,1)$ is

$$y - 0 = \frac{1-0}{1-0}(x - 0)$$

That is, $y = x$

Along C; $y = x$ $\quad \therefore\ dy = dx$

$\therefore dz = dx + idy = dx + idx = (1+i)dx$

x varies from 0 to 1

$$\therefore I = \int_0^{1+i} \left(x - y + ix^2\right) dz$$

$$= \int_0^1 \left[\cancel{x} - \cancel{x} + ix^2\right](1+i)\,dx \quad \{\because y = x\}$$

$$= i(1+i)\int_0^1 x^2 dx$$

$$= (i-1)\left[\frac{x^3}{3}\right]_0^1 \quad \{i^2 = -1\}$$

$$= (-1+i)\left[\frac{1}{3} - 0\right] = \frac{1}{3}(-1+i)$$

Along straight line joining $z = 0$ and $z = 1 + i$, $\boxed{\int_0^{1+i} \left(x - y + ix^2\right) dz = \frac{1}{3}(-1+i)}$

(ii) Along parabola $y = x^2$, $dy = 2xdx$

$\therefore dz = dx + idy = dx + i2xdx = (1 + i2x)dx$ and x varies from 0 to 1

$$\therefore I = \int_0^{1+i} (x - y + ix^2) dz$$

$$= \int_0^1 [x - x^2 + ix^2](1 + i2x) dx$$

$$= \int_0^1 [(x - x^2) + i2x(x - x^2) + ix^2 - 2x^3] dx$$

$$= \int_0^1 [(x - x^2 - 2x^3) + i(3x^2 - 2x^3)] dx$$

$$= \left[\left(\frac{x^2}{2} - \frac{x^3}{3} - \frac{2x^4}{4}\right) + i\left(\frac{3x^3}{3} - \frac{2x^4}{4}\right)\right]_0^1$$

$$= \left\{\left[\frac{1}{2} - \frac{1}{3} - \frac{1}{2}\right] + i\left[1 - \frac{1}{2}\right]\right\} - \{0 + i0\}$$

$$= -\frac{1}{3} + \frac{i}{2}$$

$\therefore$ Along parabola $y = x^2$, $\boxed{\int_0^{1+i} (x - y + ix^2) dy = -\frac{1}{3} + \frac{i}{2}}$

Example 11.1.3: Evaluate $\int_0^{1+i}(x^2+iy)dz$, along the parabola (i) $y=x^2$ and (ii) $x=y^2$

Solution: $z=x+iy \quad \therefore dz=dx+idy$

(i) Along $y=x^2$, $dy=2xdx$ and x varies from 0 to 1.

$\therefore dz=dx+idy = dx+i2xdx = (1+i2x)dx$

$$\therefore I=\int_0^{1+i}(x^2+iy)dz$$

$$=\int_0^1(x^2+ix^2)(1+i2x)dx \quad \{\because y=x^2\}$$

$$=\int_0^1(1+i)x^2(1+i2x)dx$$

$$=(1+i)\int_0^1(x^2+i2x^3)dx$$

$$=(1+i)\left[\frac{x^3}{3}+i2\frac{x^4}{4}\right]_0^1$$

$$=(1+i)\left[\frac{1}{3}+\frac{i}{2}\right]$$

$$=(1+i)\frac{(2+i3)}{6}$$

$$=\frac{1}{6}[2+i3+i2-3]=\frac{-1+i5}{6}$$

$$\therefore \text{Along } y=x^2,\ \int_0^{1+i}(x^2+iy)dz=\frac{-1+i5}{6}$$

(ii) Along $x=y^2$, $dx=2ydy$ and y varies from 0 to 1

$\therefore dz=dx+idy = 2ydy+idy = (2y+i)dy$

$$\therefore I=\int_0^{1+i}(x^2+iy)dz$$

$$=\int_0^1(y^4+iy)(2y+i)dy$$

$$\therefore I=\int_{y=0}^{y=1}\left(2y^5+iy^4+i2y^2-y\right)dy$$

$$=\int_{y=0}^{y=1}\left[\left(2y^5-y\right)+i\left(y^4+2y^2\right)\right]dy$$

$$=\left\{\left[2\frac{y^6}{6}-\frac{y^2}{2}\right]+i\left[\frac{y^5}{5}+2\frac{y^3}{3}\right]\right\}_{y=0}^{y=1}$$

$$=\left\{\left[\frac{1}{3}-\frac{1}{2}\right]+i\left[\frac{1}{5}+\frac{2}{3}\right]\right\}-[0+i0]$$

$$=-\frac{1}{6}+i\frac{13}{15}$$

$$\therefore \text{Along } x=y^2,\quad \int_0^{1+i}\left(x^2+iy\right)dz=-\frac{1}{6}+i\frac{13}{15}$$

Example 11.1.4: Evaluate $\int_C (x^2 - iy^2)\,dz$ along the straight line from $(0,0)$ to $(0,1)$ and then from $(0,1)$ to $(2,1)$

Solution: Equation of straight line joining $(0,0)$ and $(0,1)$ is $x=0$ and Equation of straight line joining $(0,1)$ and $(2,1)$ is $y=1$

$$\therefore \int_C (x^2 - iy^2)\,dz = \int_{C_1} (x^2 - iy^2)\,dz + \int_{C_2} (x^2 - iy^2)\,dz \text{ ------(1)}$$

where C_1 is straight line $x=0$ and C_2 is straight line $y=1$

Along C_1, $x=0$ $\therefore dx=0$ and y varies from 0 to 1

$\therefore dz = dx + idy = idy$

$$\therefore \int_{C_1} (x^2 - iy^2)\,dz = \int_{y=0}^{y=1} (0 - iy^2)(0 + idy) = \int_0^1 y^2 dy = \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}$$

Along C_2, $y=1$ $\therefore dy=0$ and x varies from 0 to 2

$\therefore dz = dx + idy = dx$

$$\int_{C_2} (x^2 - iy^2)\,dz = \int_0^2 (x^2 - i)(dx) = \left[\frac{x^3}{3} - ix\right]_0^2 = \frac{8}{3} - 2i$$

$$\therefore \int_C (x^2 - iy^2)\,dz = \frac{1}{3} + \left(\frac{8}{3} - 2i\right) = 3 - i2$$

Example 11.1.5: Prove that $\int_C \log z\, dz = 2\pi i$, where C is the unit circle centred at $z=0$ in the z-plane.

Solution: Equation of the unit circle centred at $z=0$ in the z-plane is $|z-(0+i0)|=1$

That is, $|z|=1$

$\therefore |z|=1 \Rightarrow z=1\cdot e^{i\theta}=e^{i\theta} \qquad \left\{\because\ |z-(a+ib)|=k \Rightarrow z=(a+ib)+ke^{i\theta}\right\}$

$\therefore dz = e^{i\theta}d\theta$ and θ varies from 0 to 2π

$$\therefore I=\int_c \log z dz$$

$$=\int_0^{2\pi} \log e^{i\theta}\cdot\left(ie^{i\theta}d\theta\right)$$

$$=\int_0^{2\pi}(i\theta)\cdot ie^{i\theta}d\theta$$

$$=i^2\int_0^{2\pi}\theta\cdot e^{i\theta}d\theta$$

$$=-\left[\theta\cdot\left(\frac{e^{i\theta}}{i}\right)-(1)\cdot\left(\frac{e^{i\theta}}{i^2}\right)\right]_0^{2\pi} \quad \left\{\because \text{ Using } \int uvdx = uv_1-u'v_2+u''v_3-\cdots\right\}$$

$$=-\left[\theta\frac{e^{i\theta}}{i}+e^{i\theta}\right]_0^{2\pi} \qquad \because\left\{i^2=-1\right\}$$

$$=-\left\{\left(\frac{2\pi}{i}e^{i2\pi}+e^{i2\pi}\right)-\left(0+e^0\right)\right\}$$

$$=-\left\{\frac{2\pi}{i}(1)+1-1\right\} \qquad \left\{\because e^{i2\pi}=\cos 2\pi+i\sin 2\pi=1+i0=1\right\}$$

$$\therefore \boxed{\int_c \log z dz = -\frac{2\pi}{i}=2\pi i} \qquad \left\{\because \frac{-1}{i}=\frac{-i}{i^2}=\frac{-i}{-1}=i\right\}$$

Example 11.1.6: Evaluate $\int_C (z-z^2)\,dz$ along the upper half of the circle $|z|=1$

Solution: $|z|=1$ is a unit circle centred at $z=0$ in the z-plane.

$|z|=1 \;\Rightarrow\; z=e^{i\theta}$

$\therefore dz = e^{i\theta} i\,d\theta$ and θ varies from 0 to π $\{\because$ For upper half of the circle, $0 \le \theta \le \pi\}$

$$\therefore I = \int_C (z-z^2)dz$$

$$= \int_0^\pi \left[e^{i\theta} - e^{i2\theta}\right] ie^{i\theta} d\theta$$

$$= i\int_0^\pi \left[e^{i2\theta} - e^{i3\theta}\right] d\theta$$

$$= i\left\{\frac{e^{i2\theta}}{2i} - \frac{e^{i3\theta}}{3i}\right\}_0^\pi$$

$$= \left\{\frac{e^{i2\theta}}{2} - \frac{e^{i3\theta}}{3}\right\}_0^\pi$$

$$= \left[\frac{e^{i2\pi}}{2} - \frac{e^{i3\pi}}{3}\right] - \left[\frac{e^{0}}{2} - \frac{e^{0}}{3}\right]$$

$$= \left[\cancel{\frac{1}{2}} - \frac{(-1)}{3}\right] - \left[\cancel{\frac{1}{2}} - \frac{1}{3}\right] \qquad \{\because e^{i3\pi} = \cos 3\pi + i\sin 3\pi = -1 + i0 = -1\}$$

$$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

$$\therefore \boxed{\int_c (z-z^2)dz = \frac{2}{3}}$$

Example 11.1.7: Evaluate $\int_C \frac{z+1}{z}dz$, where C is the semicircle $z=2e^{i\theta}, 0\le\theta\le\pi$

Solution: Path of integration is the semicircle $z=2e^{i\theta}, 0\le\theta\le\pi$

$$z=2e^{i\theta} \Rightarrow dz=2ie^{i\theta}d\theta$$

$$\therefore I=\int_C \frac{z+1}{z}dz=\int_0^\pi\left(\frac{2e^{i\theta}+1}{\cancel{2e^{i\theta}}}\right)\cancel{2}i\cancel{e^{i\theta}}\,d\theta$$

$$=i\int_0^\pi\left(2e^{i\theta}+1\right)d\theta$$

$$=i\int_0^\pi\left[2\frac{e^{i\theta}}{i}+\theta\right]_0^\pi$$

$$=\left[2e^{i\theta}+i\theta\right]_0^\pi$$

$$=\left[2e^{i\pi}+i\pi\right]-\left[2e^0+0\right]$$

$$=2(-1)+i\pi-2$$

$$\therefore \boxed{\int_c \frac{z+1}{z}dz=-4+i\pi}$$

Example 11.1.8: Evaluate $\int_0^{3+i} |z|^2\, dz$, along the straight line $3y = x$

Solution: Along the straight line $x = 3y,\ dx = 3dy$

$dz = dx + idy = 3dy + idy = (3+i)dy$ and y varies from 0 to 1.

$$\therefore |z|^2 = |x+iy|^2 = x^2 + y^2 = (3y)^2 + y^2 = 9y^2 + y^2 = 10y^2$$

$$\therefore I = \int_0^{3+i} |z|^2\, dz$$

$$= \int_0^1 \left[10y^2\right](3+i)dy$$

$$= (3+i)\int_0^1 10y^2 dy$$

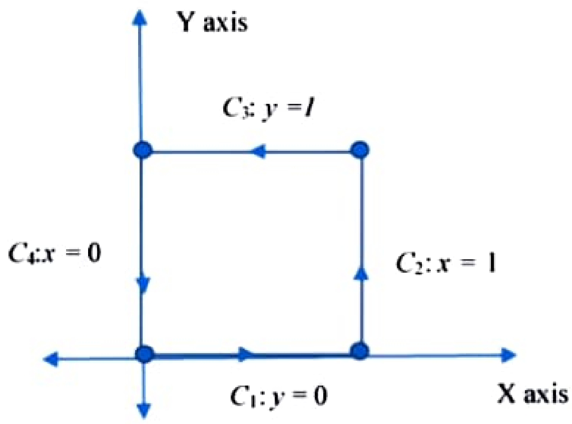
$$= 10(3+i)\left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{10}{3}(3+i)\left[1^3 - 0\right]$$

$$\therefore \boxed{\int_0^{3+i} |z|^2\, dz = \frac{10}{3}(3+i)}$$

Example 11.1.9: Evaluate $\oint_C |z|^2\, dz$ around the square with vertices (0, 0), (1, 0), (1, 1), (0, 1).

Solution: A square with vertices $(0,0), (1,0), (1,1), (0,1)$ is shown in the following figure



Boundary of the square consist of following four curves

$C_1 : y = 0$ (X axis) from $(0,0)$ to $(1,0)$

$C_2 : x = 1$ from $(1,0)$ to $(1,1)$

$C_3 : y = 1$ from $(1,1)$ to $(0,1)$

$C_4 : x = 0$ (Y axis) from $(0,1)$ to $(0,0)$

$$\therefore \oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)dz ---(1)$$

Let $z = x + iy \quad \therefore dz = dx + idy$

$f(z) = |z|^2 = x^2 + y^2$

Along C_1, $y = 0$ and x varies from 0 to 1

$y = 0 \Rightarrow dy = 0$

$$\therefore \int_{C_1} f(z)dz = \int_{C_1} (x^2 + y^2)(dx + idy)$$

$$= \int_0^1 (x^2 + 0)(dx + 0) \quad \{\because y = 0,\ dy = 0,\ x \text{ varies from 0 to 1 }\}$$

$$= \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\therefore \int_{C_1} f(z)dz = \frac{1}{3} ---------(2)$$

Along C_2, $x = 1$ and y varies from 0 to 1

$x = 1 \Rightarrow dx = 0$

$$\int_{C_2} f(z)dz = \int_{C_2} (x^2 + y^2)(dx + idy)$$

$$\therefore \int_{C_2} f(z)dz = \int_0^1 (1+y^2)(0+idy) \quad \{\because x=1,\ dx=0,\ y \text{ varies from 0 to 1}\}$$

$$= i\left[y+\frac{y^3}{3}\right]_0^1 = i\left[1+\frac{1}{3}\right] = \frac{4}{3}i$$

$$\therefore \int_{C_2} f(z)dz = \frac{4}{3}i \text{-------------(3)}$$

Along C_3, $y=1$ and x varies from 1 to 0

$y=1 \Rightarrow dy=0$

$$\int_{C_3} f(z)dz = \int_{C_3} (x^2+y^2)(dx+idy)$$

$$= \int_1^0 (x^2+1)(dx+0) \quad \{\because y=1,\ dy=0,\ x \text{ varies from 1 to 0}\}$$

$$= \left[\frac{x^3}{3}+x\right]_1^0 = (0+0)-\left(\frac{1}{3}+1\right) = -\frac{4}{3}$$

$$\therefore \int_{C_3} f(z)dz = -\frac{4}{3} \text{-----------(4)}$$

Along C_4, $x=0$ and y varies from 1 to 0

$x=0 \Rightarrow dx=0$

$$\int_{C_4} f(z)dz = \int_{C_4} (x^2+y^2)(dx+idy)$$

$$= \int_1^0 (0+y^2)(0+idy) \quad \{\because x=0,\ dx=0,\ y \text{ varies from 1 to 0}\}$$

$$= i\int_1^0 y^2 dy = i\left[\frac{y^3}{3}\right]_1^0 = i\left(0-\frac{1}{3}\right) = -\frac{i}{3}$$

$$\int_{C_4} f(z)dz = -\frac{i}{3} \text{-------------(5)}$$

Substituting from (2), (3), (4) and (5) in (1), we get

$$\oint_C f(z)dz = \frac{1}{3}+\frac{4}{3}i-\frac{4}{3}-\frac{i}{3} = -1+i$$

$$\therefore \boxed{\int_C |z|^2 dz = -1+i}$$

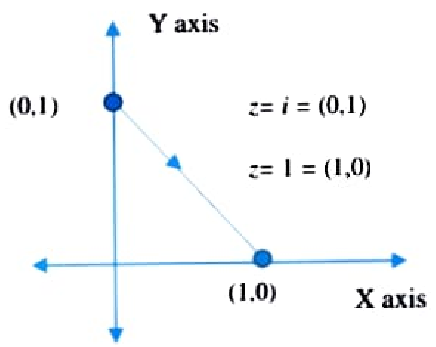
Example 11.1.10: Evaluate $\int_C (z+1)^2\, dz$ along

(i) Straight line from $z=i$ to $z=1$

(ii) The imaginary axis from $z=i$ to $z=0$ and then from $z=0$ to $z=1$ along the real axis.

Is the integral independent of path? Justify.

Solution: Complex numbers $z=i$ and $z=1$ are points $(0,1)$ and $(1,0)$ respectively.



Let $z = x+iy \quad \therefore dz = dx + idy$

$$I = \int_C (z+1)^2\, dz = \int_C (x+iy+1)^2 (dx+idy) \quad --(1)$$

(i) Along straight line from $z=i$ to $z=1$

That is, along straight line from $(0,1)$ to $(1,0)$

$\left\{\text{Equation of straight line joining } (x_1,y_1) \text{ to } (x_2,y_2) \text{ is } \dfrac{y_2-y_1}{x_2-x_1} = \dfrac{y-y_1}{x-x_1}\right\}$

$\therefore$ Equation of straight line joining $(0,1)$ to $(1,0)$ is $\dfrac{0-1}{1-0} = \dfrac{y-1}{x-0}$

$$\therefore -1 = \frac{y-1}{x}$$

$\therefore -x = y-1 \Rightarrow x+y=1$ i.e., $y = 1-x$

$\therefore$ Along this straight line $y = 1-x$,

$dy = -dx$ and x varies from 0 to 1

From (1), $I = \int_0^1 \left[x+i(1-x)+1\right]^2 (dx - idx)$

$$\therefore I = (1-i)\int_0^1 [x-ix+i+1]^2 dx$$

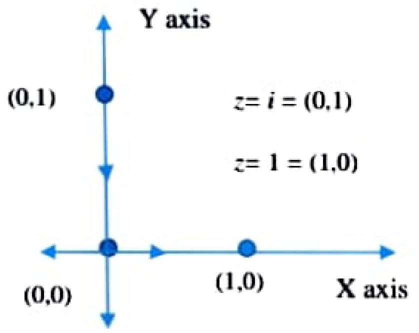
$$= (1-i)\int_0^1 \left[(1-i)x+(i+1)\right]^2 dx$$

$$= (1-i)\left[\frac{\left[(1-i)x+(i+1)\right]^3}{3(1-i)}\right]_0^1 \qquad \left\{\because \int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{(n+1)a}\right\}$$

$$= \frac{1}{3}\left[\left[(1-i)+(i+1)\right]^3 - (i+1)^3\right]$$

$$\therefore I = \frac{1}{3}\left[2^3 - (i+1)^3\right]$$
$$= \frac{1}{3}\left[8 - \left(i^3 + 3i^2 + 3i + 1\right)\right]$$
$$= \frac{1}{3}\left[8 - (-i - 3 + 3i + 1)\right]$$
$$= \frac{1}{3}(8 + i + 3 - 3i - 1) = \frac{1}{3}(10 - 2i)$$
$$\therefore \boxed{\int_C (z+1)^2\, dz = \frac{2}{3}(5-i)}$$

(ii) The imaginary axis from $z = i$ to $z = 0$ and then from $z = 0$ to $z = 1$ along the real axis.



Path of integration consist of following two curves

$C_1 : x = 0$ (Y axis) from $(0,1)$ to $(0,0)$

$C_2 : y = 0$ (X axis) from $(0,0)$ to $(1,0)$

$$\therefore \oint_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz \quad ---(1)$$

Let $z = x + iy \quad \therefore dz = dx + idy$

$$f(z) = (z+1)^2 = (x+iy+1)^2$$

Along C_1, $x = 0$

$\therefore dx = 0$ and y varies from 1 to 0

$$\int_{C_1} f(z)dz = \int_{C_1} (x+iy+1)^2 (dx + idy)$$
$$= \int_1^0 (iy+1)^2 (idy)$$
$$= i\left[\frac{(iy+1)^3}{3i}\right]_1^0$$
$$= \frac{1}{3}\left[1 - (i+1)^3\right]$$
$$= \frac{1}{3}\left[1 - \left(i^3 + 3i^2 + 3i + 1\right)\right]$$
$$= \frac{1}{3}\left[1 - (-i - 3 + 3i + 1)\right] = \frac{1}{3}[3 - 2i]$$
$$\therefore \int_{C_1} f(z)dz = \frac{1}{3}[3 - 2i] \quad --------(2)$$

Along C_2, $y=0$

$\therefore dy=0$ and x varies from 0 to 1

$$\int_{C_2} f(z)\,dz = \int_{C_2} (x+iy+1)^2 (dx+idy)$$

$$= \int_0^1 (x+0+1)^2 (dx+0)$$

$$= \int_0^1 (x+1)^2\, dx$$

$$= \left[\frac{(x+1)^3}{3}\right]_0^1 = \frac{2^3-1}{3} = \frac{7}{3}$$

$$\therefore \int_{C_2} f(z)\,dz = \frac{7}{3} \text{--------} (3)$$

Substituting from (2) and (3) in (1), we get

$$\therefore \int_C f(z)\,dz = \frac{1}{3}(3-2i) + \frac{7}{3} = \frac{1}{3}(10-2i)$$

$$\therefore \boxed{\int_C (z+1)^2\, dz = \frac{2}{3}(5-i)}$$

Value of Integral is independent of path since $f(z)$ is analytic function

Theorem 1: Cauchy's Integral Theorem OR Cauchy's Fundamental Theorem

If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points on and inside a simple closed curve C, then $\oint_C f(z)dz = 0$

Note: The above theorem can be proved without assuming continuity of $f'(z)$, as was done by French mathematician E. Goursat.

Theorem 2: Cauchy Goursat Theorem

If $f(z)$ is analytic at all points on and inside a simple closed curve C, then $\oint_C f(z)dz = 0$

Theorem 3: Cauchy's Integral Formula

If $f(z)$ is analytic inside and on a simple closed curve C that encloses a simply connected region R and if 'a' is any point R, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz \text{ ------------------ (1)}$$

Differentiating partially both sides of (1) w.r.t 'a' and performing Differentiation Under Integral Sign (DUIS) on R.H.S, we get

$$f'(a) = \frac{1!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}dz \qquad \therefore \boxed{\oint_C \frac{f(z)}{(z-a)^2}dz = 2\pi i f'(a)}$$

Proceeding further, we get

$$f''(a) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^3}dz \qquad \therefore \boxed{\oint_C \frac{f(z)}{(z-a)^3}dz = \frac{1}{2!}2\pi i f''(a)}$$

$$f'''(a)=\frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^4}dz \qquad \therefore \boxed{\oint_C \frac{f(z)}{(z-a)^4}dz=\frac{1}{3!}2\pi i f'''(a)}$$

Working Rule to Evaluate Complex Integral $\oint_C \phi(z)dz$ over a Closed Region.

1. If $\phi(z)$ is analytic at all points inside C then $\oint_C \phi(z)dz=0$ (by Cauchy Goursat Theorem)
2. If $\phi(z)$ is not analytic inside C, find its singular points.
3. If all singular points of $\phi(z)$ are outside C, then If $\phi(z)$ is analytic at all points inside C and $\oint_C \phi(z)dz=0$ (by Cauchy Goursat Theorem)
4. If there is only one singular point say z_1 of $\phi(z)$ which lies inside C, then write

   $\phi(z)=\dfrac{f(z)}{z-z_1}$ and

   $\boxed{\oint_C \phi(z)dz=\oint_C \frac{f(z)}{z-z_1}dz=2\pi i\left[f(z)\right]_{z=z_1}}$ by Cauchy's Integral Formula.
5. If there are two singular points say z_1 and z_2 of $\phi(z)$ which lie inside C, then write

   $$\phi(z)=\frac{f(z)}{(z-z_1)(z-z_2)}$$

   Resolve $\dfrac{1}{(z-z_1)(z-z_2)}$ into partial fraction as $\dfrac{A}{(z-z_1)}+\dfrac{B}{(z-z_2)}$

   $$\oint_C \phi(z)dz=\oint_C \frac{f(z)}{(z-z_1)(z-z_2)}=\oint_C f(z)\left[\frac{A}{z-z_1}+\frac{B}{z-z_2}\right]dz$$
   $$=A\oint_C \frac{f(z)}{z-z_1}dz+B\oint_C \frac{f(z)}{z-z_2}dz$$
   $$=A\left[2\pi i\left[f(z)\right]_{z=z_1}\right]+B\left[2\pi i\left[f(z)\right]_{z=z_2}\right]$$

Solved Examples (Using Cauchy Goursat Theorem or Cauchy's Integral Formula)

1. Evaluate $\oint_C \dfrac{e^{\pi z}}{z^2+4}dz$ where C is the positively oriented circle $|z-i|=2$.

2. Evaluate $\oint_C \dfrac{ze^z}{(z-a)^3}dz$, where $z=a$ lies inside the closed curve C.

3. Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z| = 2$ in the positive sense.

4. Evaluate $\int_c \frac{dz}{z^2(z^2+4)}$ where C is the circle $|z| = 1$

5. Evaluate $\int_c \frac{dz}{z^2+4}$ where C is $|z-i| = 2$ in the positive sense.

6. Evaluate $\oint_C \frac{z^2+1}{z^2-1} dz$, where C is the circle of unit radius with centre at $z = i$

7. Evaluate the integral $\int_c \frac{z}{z^4-1} dz$ where C is the circle $|z-2| = 2$

8. Evaluate $\oint_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $|z+1-i| = 2$

9. Evaluate $\oint_C \frac{4z-1}{z^2-3z-4} dz$, where C is the ellipse $x^2+4y^2 = 4$

10. Evaluate $\int_c \frac{\cos \pi z}{z^2-1} dz$ where C is the rectangle with vertices at $-i,\ 2-i,\ 2+i,\ i$.

11. Use Cauchy's Integral formula to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$, where $C: |z| = 4$

12. Use Cauchy's Integral formula to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$, where $C: |z| = 3$

Example 11.2.1: Evaluate $\oint_C \frac{e^{\pi z}}{z^2+4}dz$ where C is the positively oriented circle $|z-i|=2$

Solution: Let $f(z)=\frac{e^{z\pi}}{z^2+4}=\frac{e^{z\pi}}{(z+2i)(z-2i)}$

Singular points of $f(z)$ are $z=2i$ and $z=-2i$

Path of integration is $C:|z-i|=2$ or $|z-(0+i)|=2$

It is a circle having centre $=(0,1)$ and radius $=2$

Singular point $z=2i$ lies inside C, where as $z=-2i$ lies outside C

$\therefore$ By Cauchy's Integral formula,

$$\oint_C \frac{e^{z\pi}}{z^2+4}dz=\oint_C \frac{e^{z\pi}}{(z-i2)(z+i2)}dz$$

$$=\oint_C \frac{\frac{e^{z\pi}}{(z+i2)}}{(z-i2)}dz$$

$$=2\pi i\left[\frac{e^{z\pi}}{(z+i2)}\right]_{z=i2}$$

$$=2\pi i\left[\frac{e^{i2\pi}}{(i2+i2)}\right]=2\pi i\left[\frac{\cos 2\pi+i\sin 2\pi}{4i}\right]=\pi\left[\frac{1}{2}\right]$$

$$\therefore \boxed{\oint_c \frac{e^{z\pi}}{z^2+4}dz=\frac{\pi}{2}}$$

Example 11.2.2: Evaluate $\oint_C \frac{ze^z}{(z-a)^3}dz$, where $z=a$ lies inside the closed curve C.

Solution: Let $f(z)=\frac{ze^z}{(z-a)^3}$

Singular point of $f(z)$ is $z=a$ and it lies inside C

By Cauchy's Integral fromula

$$\oint_C \frac{ze^z}{(z-a)^3}dz = 2\pi i\left[\frac{d^2}{dz^2}(ze^z)\right]_{z=a}$$

$$=2\pi i\left[\frac{d}{dz}(ze^z+e^z)\right]_{z=a}$$

$$=2\pi i\left[ze^z+e^z+e^z\right]_{z=a}$$

$$=2\pi i\left[ae^a+2e^a\right]$$

$$\therefore \boxed{\oint_C \frac{ze^z}{(z-a)^3}dz = 2\pi i[a+2]e^a}$$

Example 11.2.3: Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$, where C is $|z|=2$ in the positive sense.

Solution: Let $f(z)=\frac{e^{2z}}{(z+1)^4}$

$C:|z|=2$ is a circle having centre at $(0,0)$ and radius = 2

Singular point of $f(z)$ is $z=-1$, which lies inside $C:|z|=2$

By Cauchy's Integral fromula,

$$\oint_C \frac{e^{2z}}{(z+1)^4} dz = 2\pi i\left[\frac{d^3}{dz^3}e^{2z}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{d^2}{dz^2}\left(2e^{2z}\right)\right]_{z=-1}$$

$$= 2\pi i\left[\frac{d}{dz}\left(4e^{2z}\right)\right]_{z=-1}$$

$$= 2\pi i\left[8e^{2z}\right]_{z=-1}$$

$$= 16\pi i\left[e^{-2}\right] = \frac{16\pi i}{e^2}$$

$$\therefore \oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{16\pi i}{e^2}$$

Example 11.2.4: Evaluate $\oint_C \frac{dz}{z^2\left(z^2+4\right)}$ where C is the circle $|z|=1$

Solution: Let $f(z)=\frac{1}{z^2\left(z^2+4\right)}=\frac{1}{z^2(z+2i)(z-2i)}$

$C:|z|=1$ is a circle having centre at $(0,0)$ and radius 1.

Singular point of $f(z)$ are $z=0,\ z=2i$ and $z=-2i$

Out of these three singular points only $z=0$ lies inside C.

$\therefore$ By Cauchy's Integral fromula,

$$\oint_C \frac{1}{z^2\left(z^2+4\right)}dz=\oint_C \frac{\frac{1}{z^2+4}}{z^2}dz$$

$$=2\pi i\left[\frac{d}{dz}\left(\frac{1}{z^2+4}\right)\right]_{z=0}$$

$$=2\pi i\left[-\frac{1}{\left(z^2+4\right)^2}2z\right]_{z=0}$$

$$=-4\pi i\left[-\frac{0}{(0+4)^2}\right]$$

$$=0$$

$$\therefore \boxed{\oint_C \frac{1}{z^2\left(z^2+4\right)}dz=0}$$

Example 11.2.5: Evaluate $\oint_C \frac{dz}{z^2+4}$ where C is $|z-i|=2$ in the positive sense.

Solution: Let $f(z)=\frac{1}{z^2+4}=\frac{1}{(z+2i)(z-2i)}$

$C:|z-i|=2$ is a circle having centre at $(0,1)$ and radius 2.

Singular point of $f(z)$ are $z=2i$ and $z=-2i$

Both the singular points lies outside C.

$\therefore$ Function $f(z)$ is analytic inside and on C.

By Cauchy Goursat theorem, $\oint_C \frac{1}{z^2+4}dz=0$

Example 11.2.6: Evaluate $\oint_C \frac{z^2+1}{z^2-1}dz$, where C is the circle of unit radius with centre at $z=i$

Solution: Let $f(z)=\frac{z^2+1}{z^2-1}=\frac{z^2+1}{(z+1)(z-1)}$

Equation of circle having centre at $z=i$ and radius =1 is $|z-i|=1$

Singular point of $f(z)$ are $z=-1$ and $z=1$

Both the singular points lie outside $C:|z-i|=1$.

$\therefore$ Function $f(z)$ is analytic inside and on C.

By Cauchy Goursat Theorem, $\oint_C \frac{z^2+1}{z^2-1}dz=0$

Example 11.2.7: Evaluate the integral $\int_c \frac{z}{z^4-1}dz$ where C is the circle $|z-2|=2$

Solution: Let $f(z)=\frac{1}{z^4-1}=\frac{1}{(z^2-1)(z^2+1)}=\frac{1}{(z-1)(z+1)(z-i)(z+i)}$

$C:|z-2|=2$ is a circle having centre at $(2,0)$ and radius $=2$.

Singular point of $f(z)$ are $z=1$, $z=-1$, $z=i$ and $z=-i$

Out of these four singular points, only $z=1$ lies inside C.

By Cauchy's Integral formula,

$$\oint_c \frac{1}{z^4-1}dz=\oint_c \frac{1}{(z-1)(z+1)(z^2+1)}$$

$$=\oint_c \frac{\frac{1}{(z+1)(z^2+1)}}{(z-1)}dz$$

$$=2\pi i\left[\frac{1}{(z+1)(z^2+1)}\right]_{z=1}$$

$$=2\pi i\left[\frac{1}{(1+1)(1^2+1)}\right]$$

$$=2\pi i\left[\frac{1}{4}\right]$$

$$=\frac{\pi i}{2}$$

$$\therefore \boxed{\oint_c \frac{1}{z^4-1}dz=\frac{\pi i}{2}}$$

Example 11.2.8: Evaluate $\oint_C \frac{z+4}{z^2+2z+5}dz$, where C is the circle $|z+1-i|=2$

Solution: Let $f(z)=\frac{z+4}{z^2+2z+5}$

$$z^2+2z+5=0 \Rightarrow z=\frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)}$$

$$\therefore \quad z=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm i4}{2}=-1\pm i2$$

Singular point of $f(z)$ are $z=-1+i2 \quad z=-1-i2$

Let $\alpha=-1+i2$ and $\beta=-1-i2$

$$\therefore f(z)=\frac{z+4}{(z-\alpha)(z-\beta)}$$

$C:|z+1-i|=2$ That is, $|z-(-1+i)|=2$ is a circle having centre at $(-1,1)$ and radius $=2$

Singular points $\alpha=-1+i2$ lies inside C, where $\beta=-1-i2$ lies outside C.

By Cauchy's integral formula,

$$\oint_C \frac{z+4}{z^2+2z+5}dz=\oint_C \frac{z+4}{(z-\alpha)(z-\beta)}dz=\oint_C \frac{\frac{z+4}{z-\beta}}{z-\alpha}dz$$

$$=2\pi i\left[\frac{z+4}{z-\beta}\right]_{z=\alpha}$$

$$=2\pi i\left[\frac{\alpha+4}{\alpha-\beta}\right]$$

$$\therefore \oint_C \frac{z+4}{z^2+2z+5}dz=2\pi i\left[\frac{-1+i2+4}{(-1+2i)-(-1-i2)}\right]$$

$$=2\pi i\left[\frac{3+i2}{4i}\right]=\frac{\pi}{2}(3+i2)$$

$$\therefore \boxed{\oint_C \frac{z+4}{z^2+2z+5}dz=\frac{\pi}{2}(3+i2)}$$

Example 11.2.9: Evaluate $\oint_C \frac{4z-1}{z^2-3z-4}dz$, where C is the ellipse $x^2+4y^2=4$

Solution: Let $f(z)=\frac{4z-1}{z^2-3z-4}=\frac{4z-1}{(z-4)(z+1)}$

Singular point of $f(z)$ are $z=4$ $z=-1$

$C: x^2+4y^2=4$

$\therefore \frac{x^2}{4}+\frac{y^2}{1}=1$

This is an ellipse.

Singular points $z=-1$ lies inside C, where $z=4$ lies outside C.

By Cauchy's integral formula,

$$\oint_C \frac{4z-1}{z^2-3z-4}dz=\oint_C \frac{4z-1}{(z-4)(z+1)}dz=\oint_C \frac{\frac{4z-1}{z-4}}{z+1}dz$$

$$=2\pi i\left[\frac{4z-1}{z-4}\right]_{z=-1}$$

$$=2\pi i\left[\frac{-4-1}{-1-4}\right]=2\pi i\left[\frac{-5}{-5}\right]=2\pi i(1)$$

$$\therefore \boxed{\oint_C \frac{4z-1}{z^2-3z-4}dz=2\pi i}$$

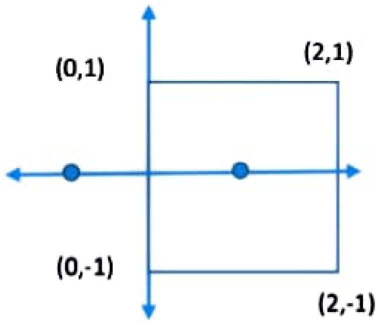
Example 11.2.10: Evaluate $\oint_C \frac{\cos \pi z}{z^2-1} dz$ where C is the square with vertices at $-i$, $2-i$, $2+i$, i.

Solution: Let $f(z) = \frac{\cos \pi z}{z^2-1} = \frac{\cos \pi z}{(z-1)(z+1)}$

Singular point of $f(z)$ are $z=1$ and $z=-1$

C is the rectangle with vertices at $-i, 2-i, 2+i, i$

Singular point $z=1$ lies inside C, where as $z=-1$ lies outside C.



$\therefore$ By Cauchy's integral formula,

$$\oint_C \frac{\cos \pi z}{z^2-1} dz = \oint_C \frac{\cos \pi z}{(z-1)(z+1)} dz = \oint_C \frac{\left(\frac{\cos \pi z}{z+1}\right)}{z-1} dz$$

$$= 2\pi i \left[\frac{\cos \pi z}{z+1}\right]_{z=1}$$

$$= 2\pi i \left[\frac{\cos \pi}{1+1}\right]$$

$$= 2\pi i \left(\frac{-1}{2}\right)$$

$$= -\pi i$$

$$\therefore \boxed{\oint_c \frac{\cos \pi z}{z^2 - 1} dz = -\pi i}$$

Example: 11.2.11: Use Cauchy's Integral formula to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz$,

where $C:|z|=4$

Solution: Let $f(z)=\frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-3)}$

$C:|z|=4$ is a circle having centre $=(0,0)$ and radius $=4$

$(z-2)(z-3)=0 \Rightarrow z=2,\ 3$

$\therefore$ Singular points of $f(z)$ are $z=2$ and $z=3$

Both $z=2$ and $z=3$ lie inside C

Let $\frac{1}{(z-2)(z-3)}=\frac{A}{z-2}+\frac{B}{z-3}=\frac{A(z-3)+B(z-2)}{(z-2)(z-3)}$ {Resolving into partial fractions}

$\therefore A(z-3)+B(z-2)=1$

Putting $z=2,\ A(2-3)+B(0)=1 \quad \therefore -A=1$ or $A=-1$

Putting $z=3,\ A(0)+B(3-2)=1 \quad \therefore B=1$

$$\therefore \frac{1}{(z-2)(z-3)}=\frac{-1}{z-2}+\frac{1}{z-3}=\frac{1}{z-3}-\frac{1}{z-2}$$

$$\therefore \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-3)}=\frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-3}-\frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}$$

$$\therefore I=\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-3)}dz$$

$$=\oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-3}dz - \oint_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z-2}dz$$

$$=2\pi i\left[\sin(\pi z^2)+\cos(\pi z^2)\right]_{z=3} - 2\pi i\left[\sin(\pi z^2)+\cos(\pi z^2)\right]_{z=2}$$

$$=2\pi i[\sin 9\pi+\cos 9\pi]-2\pi i[\sin 4\pi+\cos 4\pi]$$

$$=2\pi i\left[0+(-1)^9\right]-2\pi i\left[0+(-1)^4\right] \quad \{\because \cos n\pi=(-1)^n \text{ and } \sin n\pi=0\}$$

$$=2\pi i[-1]-2\pi i[1]$$

$$=-4\pi i$$

Example 11.2.12: Use Cauchy's Integral formula to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$, where $C : |z| = 3$

Solution: Let $f(z) = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$

$C : |z| = 3$ is a circle having centre $= (0,0)$ and radius = 3 $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z+1)(z+2)} dz$

$(z+1)(z+2) = 0 \Rightarrow z = -1, -2$

$\therefore$ Singular points of $f(z)$ are $z = -1$ and $z = -2$

Both $z = -1$ and $z = -2$ lie inside C

Let $\frac{1}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} = \frac{A(z+2) + B(z+1)}{(z+1)(z+2)}$ {Resolving into partial fractions}

$\therefore A(z+2) + B(z+1) = 1$

Putting $z = -1$, $A(-1+2) + B(0) = 1 \quad \therefore A = 1$

Putting $z = -2$, $A(0) + B(-2+1) = 1 \quad \therefore -B = 1$ or $B = -1$

$$\therefore \frac{1}{(z+1)(z+2)} = \frac{1}{z+1} - \frac{1}{z+2}$$

$$\therefore \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+1)(z+2)} = \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+1)} - \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+2)}$$

$$\therefore I = \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+1)(z+2)} dz$$

$$= \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+1)} dz - \oint_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z+2)} dz$$

$$= 2\pi i \left[\sin(\pi z^2) + \cos(\pi z^2)\right]_{z=-1} - 2\pi i \left[\sin(\pi z^2) + \cos(\pi z^2)\right]_{z=-2}$$

$$= 2\pi i [\sin \pi + \cos \pi] - 2\pi i [\sin 4\pi + \cos 4\pi]$$

$$= 2\pi i [0 - 1] - 2\pi i \left[0 + (-1)^4\right] \quad \{\because \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0\}$$

$$= 2\pi i [-1] - 2\pi i [1]$$

$$= -4\pi i$$

Taylor's Series (Taylor's Theorem)

If $f(z)$ is analytic inside a circle C_0 with centre at $'a'$ and radius r_0, then at each point z inside C_0,

$$f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\cdots$$

Above series as known as Taylor's Series

Singular point: A point at which $f(z)$ is not analytic is called a ***singular point*** or ***singularity*** of $f(z)$,

Note:

1. The largest circle with centre at 'a' such that $f(z)$ is analytic at every point inside it is the **circle of convergence** of the Taylor's series and its radius is the **radius of convergence** of the Taylor's series.

2. Putting $a=0$ in the Taylor's series, we get
$$f(z)=f(0)+\frac{f'(0)}{1!}z+\frac{f''(0)}{2!}z^2+\frac{f'''(0)}{3!}z^3+\cdots$$
This series is called the **Maclaurin's series** of $f(z)$.

3. Maclaurin's series of some elementary functions are given below

   (a) $e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots$ when $|z|<\infty$

   (b) $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$ when $|z|<\infty$

(c) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ when $|z| < \infty$

(d) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ when $|z| < \infty$

(e) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ when $|z| < \infty$

(f) $\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots$ when $|z| < 1$

(g) $\frac{1}{1+z} = (1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 + \cdots$ when $|z| < 1$

Laurent's Series

If C_1 and C_2 are two concentric circles with centre at 'a' and radii r_1 and $r_2 (r_1 > r_2)$, and if $f(z)$ is analytic on C_1 and C_2 and throughout the annular region R between them, then at each point z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{n+1}} dz,\ n = 0, 1, 2, \cdots$ and $b_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{(z-a)^{-n+1}} dz,\ n = 1, 2, 3, \cdots$

Note:-

(a) **In the analytic region, we get Taylor's series whereas in the annular region we get Laurent's series.**

(b) The part $\sum_{n=0}^{\infty} a_n (z-a)^n$, consisting of positive integral powers of $(z-a)$, is called the **analytic part of the Laurent's series**, while $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$, consisting of negative integral powers of $(z-a)$ is called the **principal part of the Laurent's series.**

Examples

1. Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{z-1}{z^2-2z-3}$ about $z=0$ indicating the region of convergence.

2. Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{1}{z^2(z-1)(z+2)}$ about $z=0$ indicating the region of convergence.

3. Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{1}{z(z+1)(z-2)}$ about $z=0$ indicating the region of convergence.

4. Find the Laurent's series for $f(z)=\dfrac{4z+3}{z(z-3)(z+2)}$ valid for $2<|z|<3$

5. Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{z-1}{z+1}$ about $z=0$ indicating the region of convergence.

6. Obtain Laurent's series expansions of $f(z)=\dfrac{1}{z^2+4z+3}$ when (*i*) $1<|z|<3$ (*ii*) $|z|>3$

7. Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{2z-3}{z^2-4z+3}$ about $z=4$ indicating the region of convergence. $\{$Expand in powers of $(z-4)\}$

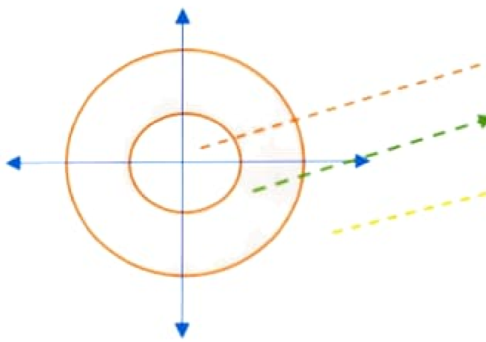
8. Find the Laurent's series exapansion of $f(z)=z^3e^{1/z}$ about $z=0$

Example 11.3.1: Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{z-1}{z^2-2z-3}$ about $z=0$ indicating the region of convergence.

Solution: $f(z)=\dfrac{z-1}{z^2-2z-3}=\dfrac{z-1}{(z-3)(z+1)}$

Singular Points of $f(z)$ are $z=3$ and $z=-1$.

To locate region of integration, draw a circle having centre at $z=0$ and passing through $z=3$ and $z=-1$.



Regions of Integration are

(i) $|z|<1$

(ii) $1<|z|<3$

(iii) $|z|>3$

Let $f(z)=\dfrac{z-1}{(z-3)(z+1)}=\dfrac{A}{(z-3)}+\dfrac{B}{(z+1)}$

$\therefore A(z+1)+B(z-3)=z-1$

Putting $z=3$, $A(3+1)+B(3-3)=3-1$

$\therefore 4A=2$ or $A=\dfrac{1}{2}$

Putting $z=-1$, $A(-1+1)+B(-1-3)=-1-1$

$\therefore -4B=-2$ or $B=\dfrac{1}{2}$

$$\therefore f(z)=\frac{\frac{1}{2}}{(z-3)}+\frac{\frac{1}{2}}{(z+1)}=\frac{1}{2}\left\{\frac{1}{(z-3)}+\frac{1}{(z+1)}\right\} \text{------(1)}$$

(i) For $|z|<1$, $\quad |z|<1<3$ and $|z|<3\Rightarrow\left|\dfrac{z}{3}\right|<1$

$$f(z)=\frac{1}{2}\left\{\frac{1}{(z-3)}+\frac{1}{(z+1)}\right\}=\frac{1}{2}\left\{\frac{1}{-3\left(1-\frac{z}{3}\right)}+\frac{1}{1+z}\right\} \quad \left\{\because |z|<1 \text{ and } \left|\frac{z}{3}\right|<1\right\}$$

$$\therefore f(z)=-\frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right]+\frac{1}{2}\left[1-z+z^2-z^3+\cdots\right]$$

(*ii*) For $1<|z|<3$

$1<|z|<3 \Rightarrow |z|>1$ and $|z|<3$

$|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$ and $|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$

$$f(z)=\frac{1}{2}\left\{\frac{1}{(z-3)}+\frac{1}{(z+1)}\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{-3\left(1-\frac{z}{3}\right)}+\frac{1}{z\left(1+\frac{1}{z}\right)}\right\} \quad \left\{\because \left|\frac{z}{3}\right|<1 \text{ and } \left|\frac{1}{z}\right|<1\right\}$$

$$\therefore f(z)=-\frac{1}{6}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right]+\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$

(*iii*) For $|z|>3$

$|z|>3>1$

$|z|>3 \Rightarrow \left|\frac{3}{z}\right|<1$ and $|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$

$$f(z)=\frac{1}{2}\left\{\frac{1}{(z-3)}+\frac{1}{(z+1)}\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{z\left(1-\frac{3}{z}\right)}+\frac{1}{z\left(1+\frac{1}{z}\right)}\right\} \quad \left\{\because \left|\frac{3}{z}\right|<1 \text{ and } \left|\frac{1}{z}\right|<1\right\}$$

$$\therefore f(z)=\frac{1}{2z}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right]+\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$

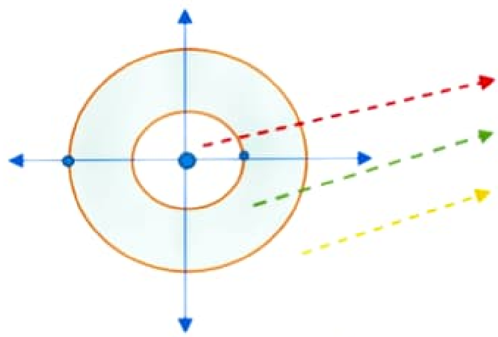
5

Example 11.3.2: Obtain all Taylor's and Laurent's series expansions of

$f(z)=\dfrac{1}{z^2(z-1)(z+2)}$ about $z=0$ indicating the region of convergence.

Solution: $f(z)=\dfrac{1}{z^2(z-1)(z+2)}$

Singular Points of $f(z)$ are $z=0,\ z=1$ and $z=-2$.
To locate region of integration, draw a circle having centre at $z=0$ and passing through $z=0,\ z=1$ and $z=-2$.



Regions of Integration are

(i) $0<|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$

Let $f(z)=\dfrac{1}{z^2(z-1)(z+2)}=\dfrac{A}{z}+\dfrac{B}{z^2}+\dfrac{C}{z-1}+\dfrac{D}{z+2}$

$\therefore Az(z-1)(z+2)+B(z-1)(z+2)+Cz^2(z+2)+Dz^2(z-1)=1$

Putting $z=0,\ A(0)+B(-1)(2)+C(0)+D(0)=1$

$\therefore -2B=1$ or $B=-\dfrac{1}{2}$

Putting $z=1,\ A(0)+B(0)+C(1)(3)+D(0)=1$

$\therefore 3C=1$ or $C=\dfrac{1}{3}$

Putting $z=-2,\ A(0)+B(0)+C(0)+D(4)(-3)=1$

$\therefore -12D=1$ or $D=-\dfrac{1}{12}$

Comparing coefficient of z^3, we get $A+C+D=0$

$A+\dfrac{1}{3}-\dfrac{1}{12}=0 \qquad \therefore A=\dfrac{1}{12}-\dfrac{1}{3}=-\dfrac{1}{4}$

$$\therefore f(z)=\frac{-\frac{1}{4}}{z}+\frac{-\frac{1}{2}}{z^2}+\frac{\frac{1}{3}}{z-1}+\frac{-\frac{1}{12}}{z+2}$$

$$\therefore f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3}\cdot\frac{1}{z-1} - \frac{1}{12}\cdot\frac{1}{z+2}$$

(i) For $0 < |z| < 1$, $\quad 0 < |z| < 1 < 2 \Rightarrow |z| < 1$ and $|z| < 2$

$$|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$$

$$f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3}\cdot\frac{1}{-(1-z)} - \frac{1}{12}\cdot\frac{1}{2\left(1+\frac{z}{2}\right)} \quad \left\{\because |z| < 1 \text{ and } \left|\frac{z}{2}\right| < 1\right\}$$

$$\therefore f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} - \frac{1}{3}\left(1+z+z^2+z^3+....\right) - \frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+...\right)$$

(ii) For $1 < |z| < 2$

$1 < |z| < 2 \Rightarrow |z| > 1$ and $|z| < 2$

$|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3}\cdot\frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{12}\cdot\frac{1}{2\left(1+\frac{z}{2}\right)} \left\{\because \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1\right\}$$

$$\therefore f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} - \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+....\right) - \frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+...\right)$$

(iii) For $|z| > 2$

$|z| > 2 > 1 \Rightarrow |z| > 2$ and $|z| > 1$

$|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $|z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} + \frac{1}{3}\cdot\frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{12}\cdot\frac{1}{z\left(1+\frac{2}{z}\right)} \quad \left\{\because \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{2}{z}\right| < 1\right\}$$

$$\therefore f(z) = -\frac{1}{4}\cdot\frac{1}{z} - \frac{1}{2}\cdot\frac{1}{z^2} - \frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}...\right) - \frac{1}{12z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+...\right)$$

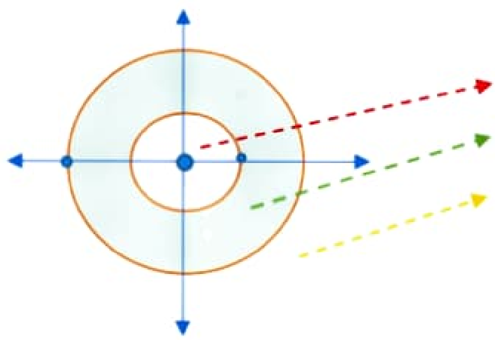
Example 11.3.3: Obtain all Taylor's and Laurent's series expansions of

$f(z)=\dfrac{1}{z(z+1)(z-2)}$ about $z=0$ indicating the region of convergence.

Solution: $f(z)=\dfrac{1}{z(z+1)(z-2)}$

Singular Points of $f(z)$ are $z=0$, $z=-1$ and $z=2$.

To locate region of integration, draw a circle having centre at $z=0$ and passing through $z=0$, $z=-1$ and $z=2$.



Regions of Integration are

(*i*) $0<|z|<1$

(*ii*) $1<|z|<2$

(*iii*) $|z|>2$

Let $f(z)=\dfrac{1}{z(z+1)(z-2)}=\dfrac{A}{z}+\dfrac{B}{z+1}+\dfrac{C}{z-2}$

$\therefore A(z+1)(z-2)+B(z-2)+Cz(z+1)=1$

Putting $z=0$, $A(1)(-2)+B(0)+C(0)=1$

$$\therefore -2A=1 \quad \text{or } A=-\frac{1}{2}$$

Putting $z=-1$, $A(0)+B(-1)(-3)+C(0)=1$

$$\therefore 3B=1 \quad \text{or } B=\frac{1}{3}$$

Putting $z=2$, $A(0)+B(0)+C(2)(3)=1$

$$\therefore 6C=1 \quad \text{or } C=\frac{1}{6}$$

$$\therefore f(z)=\frac{-\frac{1}{2}}{z}+\frac{\frac{1}{3}}{z+1}+\frac{\frac{1}{6}}{z-2}$$

$$\therefore f(z) = -\frac{1}{2z} + \frac{1}{3}\cdot\frac{1}{z+1} + \frac{1}{6}\cdot\frac{1}{z-2}$$

(i) For $0 < |z| < 1$,

$|z| < 1 < 2$

$|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = -\frac{1}{2z} + \frac{1}{3}\cdot\frac{1}{1+z} + \frac{1}{6}\cdot\frac{1}{-2\left(1-\frac{z}{2}\right)} \quad \left\{\because |z|<1 \text{ and } \left|\frac{z}{2}\right|<1\right\}$$

$$\therefore f(z) = \frac{-1}{2z} + \frac{1}{3}\left(1 - z + z^2 - z^3 + \ldots\right) - \frac{1}{12}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \ldots\right)$$

(ii) For $1 < |z| < 2$

$1 < |z| < 2 \Rightarrow |z| > 1$ and $|z| < 2$

$|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$f(z) = -\frac{1}{2z} + \frac{1}{3}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{1}{6}\cdot\frac{1}{-2\left(1-\frac{z}{2}\right)} \quad \left\{\because \left|\frac{1}{z}\right|<1 \text{ and } \left|\frac{z}{2}\right|<1\right\}$$

$$\therefore f(z) = -\frac{1}{2z} + \frac{1}{3z}\left(1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \ldots\right) - \frac{1}{12}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \ldots\right)$$

(iii) For $|z| > 2$

$\therefore |z| > 2 > 1 \Rightarrow |z| > 2$ and $|z| > 1$

$|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $|z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$

$$f(z) = -\frac{1}{2z} + \frac{1}{3}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{1}{6}\cdot\frac{1}{z\left(1-\frac{2}{z}\right)} \quad \left\{\because \left|\frac{1}{z}\right|<1 \text{ and } \left|\frac{2}{z}\right|<1\right\}$$

$$\therefore f(z) = -\frac{1}{2z} + \frac{1}{3z}\left(1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \ldots\right) - \frac{1}{6z}\left(1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \ldots\right)$$

Example 11.3.4: Find the Laurent's series for $f(z)=\dfrac{4z+3}{z(z-3)(z+2)}$ valid for $2<|z|<3$

Solution: $f(z)=\dfrac{1}{z(z+1)(z-2)}$

Singular Points of $f(z)$ are $z=0,\ z=-1$ and $z=2$.

Let $f(z)=\dfrac{4z+3}{z(z-3)(z+2)}=\dfrac{A}{z}+\dfrac{B}{z-3}+\dfrac{C}{z+2}$

$\therefore\ A(z-3)(z+2)+Bz(z+2)+Cz(z-3)=4z+3$

Putting $z=0,\ A(-3)(2)+B(0)+C(0)=0+3$

$$\therefore -6A=3 \quad \text{or} \quad A=-\frac{1}{2}$$

Putting $z=3,\ A(0)+B(3)(5)+C(0)=4(3)+3$

$$\therefore 15B=15 \quad \text{or} \quad B=1$$

Putting $z=-2,\ A(0)+B(0)+C(-2)(-5)=4(-2)+3$

$$\therefore 10C=-5 \quad \text{or } C=\frac{1}{2}$$

$$\therefore f(z)=\frac{\frac{-1}{2}}{z}+\frac{1}{z-3}+\frac{\frac{1}{2}}{z+2}=-\frac{1}{2z}+\frac{1}{z-3}+\frac{1}{2}\cdot\frac{1}{z+2}$$

For $2<|z|<3,\quad |z|>2$ and $|z|<3$

$|z|>2\Rightarrow\left|\dfrac{2}{z}\right|<1$ and $|z|<3\Rightarrow\left|\dfrac{z}{3}\right|<1$

$$f(z)=-\frac{1}{2z}+\frac{1}{-3\left(1-\frac{z}{3}\right)}+\frac{1}{2}\cdot\frac{1}{z\left(1+\frac{2}{z}\right)} \quad \left\{\because\left|\frac{2}{z}\right|<1 \text{ and } \left|\frac{z}{3}\right|<1\right\}$$

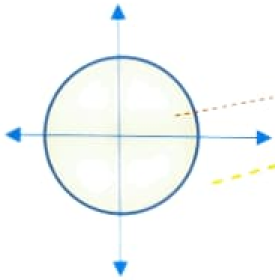
$$\therefore f(z)=-\frac{1}{2z}-\frac{1}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\ldots\right)-\frac{1}{2z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\ldots\right)$$

Example 11.3.5: Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{z-1}{z+1}$ about $z=0$ indicating the region of convergence.

Solution: $f(z)=\dfrac{z-1}{z+1}$

Singular Points of $f(z)$ is $z=-1$

To locate region of integration, draw a circle having centre at $z=0$ and passing through $z=-1$



Regions of Integration are

(*i*) $|z|<1$

(*ii*) $|z|>1$

$$f(z)=\frac{z-1}{z+1}=\frac{z+1-2}{z+1}=1-\frac{2}{z+1}$$

(*a*) For $|z|<1$

$$f(z)=1-\frac{2}{z+1}=1-\frac{2}{1+z}$$

$$=1-2\left[1-z+z^2-z^3+z^4+...\right]$$

$$=1-2+2z-2z^2+2z^3-2z^4+...$$

$$\therefore f(z)=1+2z-2z^2+2z^3-2z^4+...$$

(*b*) For $|z|>1$

$$|z|>1\Rightarrow\left|\frac{1}{z}\right|<1$$

$$f(z)=1-\frac{2}{z+1}=1-\frac{2}{z\left(1+\frac{1}{z}\right)}\quad\left\{\because\left|\frac{1}{z}\right|<1\right\}$$

$$\therefore f(z)=1-\frac{2}{z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+....\right]$$

Example 11.3.6: Obtain Laurent's series expansions of $f(z)=\dfrac{1}{z^2+4z+3}$ when

(*i*) $1<|z|<3$ (*ii*) $|z|>3$

Solution: $f(z)=\dfrac{1}{z^2+4z+3}=\dfrac{1}{(z+3)(z+1)}$

Let $f(z)=\dfrac{1}{(z+3)(z+1)}=\dfrac{A}{z+1}+\dfrac{B}{z+3}$

$\therefore A(z+3)+B(z+1)=1$

Putting $z=-1,\quad A(-1+3)+B(0)=1$

$$\therefore\ 2A=1 \text{ or } A=\frac{1}{2}$$

Putting $z=-3,\quad A(0)+B(-3+1)=1$

$$\therefore -2B=1 \text{ or } B=-\frac{1}{2}$$

$$\therefore f(z)=\frac{\frac{1}{2}}{z+1}+\frac{-\frac{1}{2}}{z+3}=\frac{1}{2}\cdot\frac{1}{z+1}-\frac{1}{2}\cdot\frac{1}{z+3}$$

(*i*) When $1<|z|<3$

$1<|z|<3 \Rightarrow |z|>1$ and $|z|<3$

$|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$ and $|z|<3 \Rightarrow \left|\frac{z}{3}\right|<1$

$$\therefore f(z)=\frac{1}{2}\cdot\frac{1}{z+1}-\frac{1}{2}\cdot\frac{1}{z+3}$$

$$=\frac{1}{2}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2}\cdot\frac{1}{3\left(1+\frac{z}{3}\right)}\quad\left\{\because \left|\frac{1}{z}\right|<1 \text{ and } \left|\frac{z}{3}\right|<1\right\}$$

$$\therefore f(z)=\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\dots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\left(\frac{z}{3}\right)^2-\left(\frac{z}{3}\right)^3+\dots\right]$$

(ii) When $|z|>3$

$|z|>3>1 \Rightarrow |z|>3$ and $|z|>1$

$|z|>3 \Rightarrow \left|\frac{3}{z}\right|<1$ and $|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$

$$\therefore f(z)=\frac{1}{2}\cdot\frac{1}{z+1}-\frac{1}{2}\cdot\frac{1}{z+3}$$

$$=\frac{1}{2}\cdot\frac{1}{z\left(1+\frac{1}{z}\right)}-\frac{1}{2}\cdot\frac{1}{z\left(1+\frac{3}{z}\right)} \quad \left\{\because \left|\frac{3}{z}\right|<1 \text{ and } \left|\frac{1}{z}\right|<1\right\}$$

$$\therefore f(z)=\frac{1}{2z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\ldots\ldots\right]-\frac{1}{2z}\left[1-\frac{3}{z}+\left(\frac{3}{z}\right)^2-\left(\frac{3}{z}\right)^3+\ldots\ldots\right]$$

Example 11.3.7: Obtain all Taylor's and Laurent's series expansions of $f(z)=\dfrac{2z-3}{z^2-4z+3}$ about $z=4$ indicating the region of convergence. {Expand in powers of $(z-4)$}

Solution: $f(z)=\dfrac{2z-3}{z^2-4z+3}=\dfrac{2z-3}{(z-3)(z-1)}$

To find Taylor's and Laurent's series expansion of $f(z)$ about $z = 4$,

put $z-4=v \quad \therefore z=v+4$

$$\therefore f(z)=\frac{2(v+4)-3}{(v+4-3)(v+4-1)}=\frac{2v+5}{(v+1)(v+3)}$$

Regions of integration are (i) $|v|<1$ (ii) $1<|v|<3$ and (iii) $|v|>3$

That is (i) $|z-4|<1$ (ii) $1<|z-4|<3$ and (iii) $|z-4|>3$

$$f(z)=\frac{2v+5}{(v+1)(v+3)}=\frac{\frac{3}{2}}{v+1}+\frac{\frac{1}{2}}{v+3} \quad \{\text{Resolving } f(z) \text{ into partial fractions}\}$$

$$\therefore f(z)=\frac{3}{2}\cdot\frac{1}{v+1}+\frac{1}{2}\cdot\frac{1}{v+3}$$

(i) For $|v|<1$, That is $|z-4|<1$

$|v|<1<3$

$|v|<3 \Rightarrow \left|\dfrac{v}{3}\right|<1$

$$f(z)=\frac{3}{2}\cdot\frac{1}{v+1}+\frac{1}{2}\cdot\frac{1}{v+3}$$

$$=\frac{3}{2}\cdot\frac{1}{1+v}+\frac{1}{2}\cdot\frac{1}{3\left(1+\frac{v}{3}\right)}$$

$$=\frac{3}{2}\left[1-v+v^2-v^3+\ldots..\right]+\frac{1}{6}\left[1-\frac{v}{3}+\left(\frac{v}{3}\right)^2-\left(\frac{v}{3}\right)^3+\ldots..\right]$$

$$\therefore f(z)=\frac{3}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+\ldots.\right]++\frac{1}{6}\left[1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2-\left(\frac{z-4}{3}\right)^3+\ldots..\right]$$

(ii) For $1<|v|<3$, That is, $1<|z-4|<3$

$1<|v|<3 \Rightarrow |v|>1$ and $|v|<3$

$|v|>1 \Rightarrow \left|\frac{1}{v}\right|<1$ and $|v|<3 \Rightarrow \left|\frac{v}{3}\right|<1$

$$f(z)=\frac{3}{2}\cdot\frac{1}{v+1}+\frac{1}{2}\cdot\frac{1}{v+3}$$

$$=\frac{3}{2}\cdot\frac{1}{v\left(1+\frac{1}{v}\right)}+\frac{1}{2}\cdot\frac{1}{3\left(1+\frac{v}{3}\right)} \quad \left\{\because \left|\frac{1}{v}\right|<1 \text{ and } \left|\frac{v}{3}\right|<1\right\}$$

$$=\frac{3}{2v}\left[1-\frac{1}{v}+\left(\frac{1}{v}\right)^2-\left(\frac{1}{v}\right)^3+.....\right]+\frac{1}{6}\left[1-\frac{v}{3}+\left(\frac{v}{3}\right)^2-\left(\frac{v}{3}\right)^3+.....\right]$$

$$\therefore f(z)=\frac{3}{2(z-4)}\left[1-\frac{1}{z-4}+\frac{1}{(z-4)^2}-\frac{1}{(z-4)^3}+....\right]+\frac{1}{6}\left[1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2-.....\right]$$

(iii) For $|v|>3$ That is, $|z-4|>3$

$|v|>3>1 \Rightarrow |v|>1$ and $|v|<3$

$|v|>1 \Rightarrow \left|\frac{1}{v}\right|<1$ and $|v|>3 \Rightarrow \left|\frac{3}{v}\right|<1$

$$f(z)=\frac{3}{2}\cdot\frac{1}{v+1}+\frac{1}{2}\cdot\frac{1}{v+3}$$

$$=\frac{3}{2}\cdot\frac{1}{v\left(1+\frac{1}{v}\right)}+\frac{1}{2}\cdot\frac{1}{v\left(1+\frac{3}{v}\right)} \quad \left\{\because \left|\frac{1}{v}\right|<1 \text{ and } \left|\frac{3}{v}\right|<1\right\}$$

$$=\frac{3}{2v}\left[1-\frac{1}{v}+\left(\frac{1}{v}\right)^2-\left(\frac{1}{v}\right)^3+.....\right]+\frac{1}{2v}\left[1-\frac{3}{v}+\left(\frac{3}{v}\right)^2-\left(\frac{3}{v}\right)^3+.....\right]$$

$$\therefore f(z)=\frac{3}{2(z-4)}\left[1-\frac{1}{z-4}+\frac{1}{(z-4)^2}-\frac{1}{(z-4)^3}+....\right]+\frac{1}{2v}\left[1-\left(\frac{3}{z-4}\right)+\left(\frac{3}{z-4}\right)^2-.....\right]$$

Example 11.3.8: Find the Laurent's series exapansion of $f(z)=z^3e^{1/z}$ about $z=0$

Solution: $f(z)=z^3e^{1/z}$

We know that $e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots$

$$\therefore e^{1/z}=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}\cdots\{\because 2!=2,\ \ 3!=6,\ \ 4!=24 \text{ and so on}\}$$

$$\therefore f(z)=z^3e^{1/z}=z^3\left[1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}\cdots\right]$$

$$\therefore f(z)=z^3+z^2+\frac{z}{2}+\frac{1}{6}+\frac{1}{24z}+\frac{1}{120z^2}\cdots\cdots$$

This is the required Laurent's series expansion for $f(z)$ at $z=0$

Cauchy's Residue Theorem

Singular point: A point at which $f(z)$ is not analytic is called a ***singular point*** or ***singularity*** of $f(z)$.

Pole: If $z=a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's series of $f(z)$ at $z=a$ valid in $0<|z-a|<r_1$ has only a ***finite number of terms***, then $z=a$ is called a pole. If in $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$

$b_m \neq 0, b_{m+1}=0=b_{m+2}\cdots$, then $z=a$ is called a ***pole of order m***.

A pole of order *ONE* is called a *simple pole*.

Note: If $f(z)$ has a pole of order m at $z=a$, then its Laurent's expansion is

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_m}{(z-a)^m}\text{ ,where } b_m\neq 0$$

$$=\frac{1}{(z-a)^m}\left[\sum_{n=0}^{\infty}a_n(z-a)^{n+m}+b_m+b_{m-1}(z-a)+\cdots+b_1(z-a)^{m-1}\right]$$

$$=\frac{1}{(z-a)^m}\phi(z)$$

Clearly, $\phi(z)$ is analytic everywhere that includes $z=a$ and $\phi(a)=b_m\neq 0$.

Thus for a function of the form $\dfrac{\phi(z)}{(z-a)^m}$, $z=a$ is a pole of order m, provided $\phi(z)$ is analytic everywhere and $\phi(a)\neq 0$

Essential Singularity: If $z=a$ is an isolated singularity of $f(z)$ such that the principal part of the Laurent's series of $f(z)$ at $z=a$ valid in $0<|z-a|<r_1$ has ***infinite number of terms***, then $z=a$ is called an ***essential singularity***.

Removable Singularity: If a single valued function $f(z)$ is not defined at $z = a$, but $\lim_{z \to a} f(z)$ exists then $z = a$ is called a ***removable singularity***.

Residue: If 'a' is an isolated singularity of any type for the function $f(z)$, then the **coefficient of** $\dfrac{1}{z-a}$ (That is, b_1) in the Laurent's expansion of $f(z)$ at $z = a$ valid in $0 < |z-a| < r_1$ is called **residue** of $f(z)$ at $z = a$.

Formula for the Evaluation of Residue

1. If $z = a$ is a simple pole of $f(z)$, then

$$[\text{Res.} f(z)]_{z=a} = \lim_{z \to a} \{(z-a) f(z)\}$$

2. If $z = a$ is a pole of order m of $f(z)$, then

$$[\text{Res.} f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \left[\frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right]$$

3. If $z = a$ is a simple pole of $f(z) = \dfrac{P(z)}{Q(z)}$, then

$$[\text{Res.} f(z)]_{z=a} = \lim_{z \to a} \left\{ \frac{P(z)}{Q'(z)} \right\}$$

Cauchy's Residue Theorem:

If $f(z)$ is analytic inside and on a simple closed curve C, except at finite number of singular points $z_1, z_2, \cdots z_n$ inside C, then

$$\oint_C f(z) dz = 2\pi i \left\{ \operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \operatorname*{Res}_{z=z_3} f(z) + \cdots + \operatorname*{Res}_{z=z_n} f(z) \right\}$$

Solved Examples

1. Determine the residue at each singular point of $f(z)=\dfrac{3z+1}{(z-2)^2(z^2-1)}$

2. Determine the residue at each singular point of $f(z)=\dfrac{z^2-z}{(z+1)^2(z^2+4)}$

3. Determine the residue at each singular point of $f(z)=\dfrac{z}{(z-1)(z+2)^2}$

4. Determine the residue at each singular point of $\dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-1)}$

5. Determine the residue of $f(z)=z^3e^{1/z}$ at $z=0$

6. Determine the residue of $f(z)=\dfrac{\sin z}{z^4}$ at $z=0$

7. Determine the residue of $f(z)=\dfrac{\sin^2 z}{z^3}$ at $z=0$

8. Evaluate $\oint_C \dfrac{z}{z^2-1}dz$ where C is the positively oriented circle $|z|=2$

9. Evaluate $\oint_C \dfrac{4z^2+1}{(2z-3)(z+1)^2}dz$ where $C:|z|=4$

10. Evaluate $\oint_C \dfrac{z-1}{(z-2)(z+1)^2}dz$ where $C:|z-i|=2$

11. Evaluate $\oint_C \dfrac{z+3}{2z^2+3z-2}dz$ where $C:|z-i|=2$

12. Evaluate $\oint_C \dfrac{1}{z^3+z^5}dz$ where $C:|z|=3$

13. Evaluate $\oint_C z^4 e^{1/z} dz$ where $C:|z|=1$

Practice Examples

(i) $\oint_C \frac{1-2z}{z(z-1)(z-2)} dz$, where C is the circle $|z|=1.5$

(ii) $\oint_C \frac{z^2}{(z-1)^2(z-2)} dz$, where C is the circle $|z|=2.5$

(iii) $\oint_C \tan z dz$, where C is the circle (i) $C:|z|=2$ (ii) $C:|z|=1$

(iv) $\oint_C \frac{\sin \pi z^2+\cos \pi z^2}{z-z^2} dz$ where C is the circle $|z-2|=4$

(v) Evaluate $\oint_C \frac{z^2}{z^4-1} dz$, where C is the circle

(a) $|z|=\frac{3}{4}$ (b) $|z-1|=1$ (c) $|z+i|=\frac{1}{2}$

Example 11.4.1: Determine the residue at each singular point of $f(z)=\dfrac{3z+1}{(z-2)^2(z^2-1)}$.

Solution: $f(z)=\dfrac{3z+1}{(z-2)^2(z^2-1)}=\dfrac{3z+1}{(z-2)^2(z-1)(z+1)}$

Singular points of $f(z)$ are $z=2$, $z=1$ and $z=-1$

(i) $z=2$ is a pole of order 2

$$\therefore \left[\text{Res}f(z)\right]_{z=2}=\frac{1}{(2-1)!}\cdot\lim_{z\to 2}\frac{d}{dz}\left[(z-2)^2 f(z)\right]$$

$$=\lim_{z\to 2}\frac{d}{dz}\left[(z-2)^2\cdot\frac{3z+1}{(z-2)^2(z^2-1)}\right]$$

$$=\lim_{z\to 2}\frac{d}{dz}\left(\frac{3z+1}{z^2-1}\right)$$

$$=\lim_{z\to 2}\frac{(z^2-1)\cdot 3-(3z+1)\cdot(2z)}{(z^2-1)^2}$$

$$=\frac{(4-1)\cdot 3-(6+1)(4)}{(4-1)^2}=\frac{9-28}{9}=-\frac{19}{9}$$

$$\left[\text{Res}f(z)\right]_{z=2}=-\frac{19}{9}$$

(ii) $z=1$ is a simple pole of $f(z)$

$$\therefore \left[\text{Res}f(z)\right]_{z=1}=\lim_{z\to 1}(z-1)f(z)$$

$$=\lim_{z\to 1}\cancel{(z-1)}\frac{3z+1}{(z-2)^2\cancel{(z-1)}(z+1)}$$

$$=\frac{3+1}{(1+2)^2(1+1)}=\frac{4}{2}=2$$

$$\left[\text{Res}f(z)\right]_{z=1}=2$$

(iii) $z=-1$ is a simple pole of $f(z)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=-1} = \lim_{z\to -1}(z+1)f(z)$$

$$= \lim_{z\to -1}\cancel{(z+1)}\frac{3z+1}{(z-2)^2(z-1)\cancel{(z+1)}}$$

$$= \frac{3(-1)+1}{(-1-2)^2(-1-1)} = \frac{-2}{9(-2)} = \frac{1}{9}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{1}{9}$$

Example 11.4.2: Determine the residue at each singular point of $f(z)=\dfrac{z^2-z}{(z+1)^2(z^2+4)}$.

Solution: $f(z)=\dfrac{z^2-z}{(z+1)^2(z^2+4)}=\dfrac{z(z-1)}{(z+1)^2(z+2i)(z-2i)}$

Singular points of $f(z)$ are $z=-1$, $z=2i$ and $z=-2i$

(i) $z=-1$ is a pole of order 2

$$\therefore \left[\text{Res} f(z)\right]_{z=-1}=\frac{1}{(2-1)!}\cdot\lim_{z\to-1}\frac{d}{dz}\left[(z+1)^2 f(z)\right]$$

$$=\lim_{z\to-1}\frac{d}{dz}\left[\frac{z^2-z}{(z^2+4)}\right]$$

$$=\lim_{z\to-1}\frac{(z^2+4)(2z-1)-(z^2-z)(2z)}{(z^2+4)^2}$$

$$=\frac{(1+4)(-2-1)-(1-(-1))(-2)}{(1+4)^2}$$

$$=\frac{5(-3)-(2)(-2)}{(5)^2}=\frac{-15+4}{25}=\frac{-11}{25}$$

$$\therefore \left[\text{Res} f(z)\right]_{z=-1}=\frac{-11}{25}$$

(ii) $z=2i$ is a simple pole of $f(z)$

$$\therefore \left[\text{Res} f(z)\right]_{z=2i}=\lim_{z\to2i}(z-2i)f(z)$$

$$=\lim_{z\to2i}\frac{z^2-z}{(z+1)^2(z+2i)}$$

$$=\frac{(2i)^2-2}{(2i+1)^2(2i+2i)}$$

$$=\frac{-4-2}{(1+2i)^2 4i}=\frac{-6}{(1+4i-4)4i}=\frac{-3}{2i(-3+4i)}$$

$$\therefore \left[\text{Res} f(z)\right]_{z=2i} = \frac{-6}{(1+4i-4)4i}$$

$$= \frac{-3}{2i(-3+4i)}$$

$$= \frac{-3}{-6i-8} = \frac{3}{8+6i}$$

$$\therefore \left[\text{Res} f(z)\right]_{z=2i} = \frac{3}{8+6i}$$

(ii) $z=-2i$ is a simple pole of $f(z)$

$$\therefore \left[\text{Res} f(z)\right]_{z=-2i} = \lim_{z\to -2i}(z+2i)f(z)$$

$$= \lim_{z\to -2i} \frac{z^2-z}{(z+1)^2(z-2i)}$$

$$= \frac{(-2i)^2-2}{(-2i+1)^2(-2i-2i)}$$

$$= \frac{-4-2}{(1-2i)^2(-4i)}$$

$$= \frac{6}{(1-4i-4)4i}$$

$$= \frac{3}{2i(-3-4i)}$$

$$= \frac{3}{-6i+8} = \frac{3}{8-6i}$$

$$\therefore \left[\text{Res} f(z)\right]_{z=-2i} = \frac{3}{8-6i}$$

Example 11.4.3: Determine the residue at each singular point of $f(z)=\dfrac{z}{(z-1)(z+2)^2}$.

Solution: $f(z)=\dfrac{z}{(z-1)(z+2)^2}$

Singular points of $f(z)$ are $z=1$ and $z=-2$

(i) $z=1$ is a simple pole

$$\therefore \left[\text{Res}f(z)\right]_{z=1}=\lim_{z\to 1}(z-1)f(z)$$

$$=\lim_{z\to 1}\frac{z}{(z+2)^2}=\frac{1}{(1+2)^2}=\frac{1}{3^2}=\frac{1}{9}$$

$$\therefore \left[\text{Res}f(z)\right]_{z=1}=\frac{1}{9}$$

(ii) $z=-2$ is a pole of order 2

$$\therefore \left[\text{Res}f(z)\right]_{z=-2}=\frac{1}{(2-1)!}\lim_{z\to -2}\frac{d}{dz}(z+2)^2 f(z)$$

$$=\lim_{z\to -2}\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$=\lim_{z\to -2}\frac{(z-1)(1)-z(1)}{(z-1)^2}$$

$$=\frac{(-2-1)-(-2)}{(-2-1)^2}=-\frac{1}{9}$$

$$\left[\text{Res}f(z)\right]_{z=-2}=-\frac{1}{9}$$

9

Example 11.4.4: Determine the residue at each singular point of $\dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-1)}$.

Solution: $f(z)=\dfrac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)(z-1)}$

Singular points of $f(z)$ are $z=1$ and $z=2$

(i) $z=1$ is a simple pole of $f(z)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=1}=\lim_{z\to 1}(z-1)f(z)$$

$$=\lim_{z\to 1}\frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-2)}$$

$$=\frac{\sin\pi+\cos\pi}{(1-2)}=\frac{0-1}{(-1)}=1$$

$$\left[\operatorname{Res} f(z)\right]_{z=1}=1$$

(ii) $z=2$ is a simple pole of order 2

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=2}=\lim_{z\to 1}(z-2)f(z)$$

$$=\lim_{z\to 2}\frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)}$$

$$=\frac{\sin(4\pi)+\cos(4\pi)}{(2-1)}=\frac{0+1}{1}=1$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=2}=1$$

Example 11.4.5: Determine the residue of $f(z)=z^3e^{1/z}$ at $z=0$

Solution: $f(z)=z^3e^{1/z}$

We know that $e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+....$ {Maclaurin's series expansion of e^z}

$$\therefore e^{1/z}=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}+\frac{1}{120z^5}+....$$

$$\therefore z^3e^{1/z}=z^3\left[1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}+\frac{1}{120z^5}+....\right]$$

$$\therefore f(z)=z^3e^{1/z}=z^3+z^2+\frac{z}{2}+\frac{1}{6}+\frac{1}{24z}+\frac{1}{120z^2}+..$$

Res $f(z)$ = coefficient $\frac{1}{z}$ in the Laurent's series expansion of $f(z)=\frac{1}{24}$

$$\therefore\left[\text{Res}f(z)\right]_{z=0}=\frac{1}{24}$$

Example 11.4.6: Determine the residue of $f(z)=\dfrac{\sin z}{z^4}$ at $z=0$

Solution: $f(z)=\dfrac{\sin z}{z^4}$

We know that $\sin z = z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+\dots$ {Maclaurin's series expansion of $\sin z$}

$$\therefore \frac{\sin z}{z^4}=\frac{1}{z^4}\cdot\sin z=\frac{1}{z^4}\cdot\left[z-\frac{z^3}{6}+\frac{z^5}{120}+\dots\right]$$

$$\therefore f(z)=\frac{\sin z}{z^4}=\frac{1}{z^3}-\frac{1}{6z}+\frac{z}{120}+\dots$$

$\operatorname{Res} f(z)=$ coefficient $\dfrac{1}{z}$ in the Laurent's series expansion of $f(z)=-\dfrac{1}{6}$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=0}=-\frac{1}{6}$$

Example 11.4.7: Determine the residue of $f(z)=\dfrac{\sin^2 z}{z^3}$ at $z=0$

Solution: $f(z)=\dfrac{\sin^2 z}{z^3}$

$$\sin^2 z=\frac{1-\cos 2z}{2}$$

$$=\frac{1}{2}\left\{1-\left[1-\frac{(2z)^2}{2!}+\frac{(2z)^4}{4!}-\frac{(2z)^6}{6!}+\cdots\right]\right\}$$

$$=\frac{1}{2}\left\{\frac{4z^2}{2}-\frac{16z^4}{24}+\frac{64z^6}{120}.+...\right\}$$

$$=\frac{1}{2}\left\{2z^2-\frac{2}{3}z^4+\frac{4}{45}z^6+....\right\}$$

$$\therefore \sin^2 z=z^2-\frac{1}{3}z^4+\frac{2}{45}z^6+...$$

$$\therefore \frac{\sin^2 z}{z^3}=\frac{1}{z^3}\cdot\sin^2 z=\frac{1}{z^3}\cdot\left[z^2-\frac{1}{3}z^4+\frac{2}{45}z^6+...\right]$$

$$f(z)=\frac{\sin^2 z}{z^3}=\frac{1}{z}+\frac{z}{3}+\frac{2}{45}z^3+..$$

$\operatorname{Res} f(z)=$ coefficient $\dfrac{1}{z}$ in the Laurent's series expansion of $f(z)=1$

$$\therefore\left[\operatorname{Res} f(z)\right]_{z=0}=1$$

Example 11.4.8: Evaluate $\oint_C \frac{z}{z^2-1}dz$ where C is the positively oriented circle $|z|=2$

Solution: Let $f(z)=\frac{z}{z^2-1}=\frac{z}{(z-1)(z+1)}$

Singular points of $f(z)$ are $z=1$ and $z=-1$

$C:|z|=2$ is a circle having centre at $(0,0)$ and radius $=2$

Both the singular points $z=1$ and $z=-1$ lie inside C.

$\therefore$ By Cauchy's Residue Theorem,

$$\oint_C f(z)dz = 2\pi i\left\{\left[\operatorname{Res} f(z)\right]_{z=1}+\left[\operatorname{Res} f(z)\right]_{z=-1}\right\}$$

$z=1$ is a simple pole of $f(z)$

$$\left[\operatorname{Res} f(z)\right]_{z=1}=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}\cancel{(z-1)}\frac{z}{\cancel{(z-1)}(z+1)}=\frac{1}{(1+1)}=\frac{1}{2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=1}=\frac{1}{2}$$

$z=-1$ is a simple pole of $f(z)$

$$\left[\operatorname{Res} f(z)\right]_{z=-1}=\lim_{z\to -1}(z+1)f(z)=\lim_{z\to 1}\cancel{(z+1)}\frac{z}{\cancel{(z+1)}(z-1)}=\frac{-1}{(-1-1)}=\frac{-1}{-2}=\frac{1}{2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-1}=\frac{1}{2}$$

$$\therefore \oint_c f(z)=2\pi i\left\{\frac{1}{2}+\frac{1}{2}\right\}=2\pi i(1)=2\pi i$$

$$\boxed{\oint_c \frac{z}{z^2-1}dz=2\pi i}$$

Example 11.4.9: Evaluate $\oint_C \frac{4z^2+1}{(2z-3)(z+1)^2}dz$ where $C:|z|=4$

Solution: $f(z)=\frac{4z^2+1}{(2z-3)(z+1)^2}$

Singular points of $f(z)$ are $z=\frac{3}{2}$ and $z=-1$

$C:|z|=4$ is a circle having centre $(0,0)$ and radius $=4$

Both the singular points of $f(z)$ lies inside C.

$\therefore$ By Cauchy's Residue Theorem,

$$\oint_C f(z)dz = 2\pi i\left\{\left[\operatorname{Res} f(z)\right]_{z=\frac{3}{2}}+\left[\operatorname{Res} f(z)\right]_{z=-1}\right\} \text{------}(1)$$

$z=\frac{3}{2}$ is a simple pole of $f(z)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\frac{3}{2}} = \lim_{z\to\frac{3}{2}}\left(z-\frac{3}{2}\right)\cdot f(z)$$

$$=\lim_{z\to\frac{3}{2}}\left[\frac{\cancel{(2z-3)}}{2}\cdot\frac{4z^2+1}{\cancel{(2z-3)}(z+1)^2}\right]$$

$$=\frac{1}{2}\cdot\frac{\cancel{4}\left(\frac{9}{\cancel{4}}\right)+1}{\left(\frac{3}{2}+1\right)^2}$$

$$=\frac{1}{2}\frac{10}{\left(\frac{5}{2}\right)^2}$$

$$=\frac{1}{2}\cdot\frac{10}{\frac{25}{4}}=\frac{20}{5}=2=\frac{4}{5}$$

$$\left[\operatorname{Res} f(z)\right]_{z=\frac{3}{2}}=\frac{4}{5} \text{-------}(2)$$

(*ii*) $z=-1$ is a pole of order 2

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z)$$

$$= \lim_{z\to -1}\frac{d}{dz}\left[\frac{4z^2+1}{(2z-3)}\right]$$

$$= \lim_{z\to -1}\frac{(2z-3)\cdot(8z)-(4z^2+1)\cdot 2}{(2z-3)^2}$$

$$= \frac{(-2-3)(-8)-2(4+1)}{(-2-3)^2}$$

$$= \frac{40-10}{25} = \frac{30}{25} = \frac{6}{5}$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=-1} = \frac{6}{5} \quad \text{-------}(3)$$

Substituting from (2) and (3) in (1)

$$\therefore \oint_C f(z)\,dz = 2\pi i\left[\frac{4}{5}+\frac{6}{5}\right] = 2\pi i(2) = 4\pi i$$

$$\therefore \boxed{\oint_C \frac{4z^2+1}{(2z-3)(z+1)^2}dz = 4\pi i}$$

Example 11.4.10: Evaluate $\oint_C \frac{z-1}{(z-2)(z+1)^2}dz$ where $C:|z-i|=2$

Solution: $f(z)=\frac{z-1}{(z-2)(z+1)^2}$

Singular points of $f(z)$ are $z=2$ and $z=-1$

$C:|z-i|=2$ That is ,$|z-(0+i)|=2$ is a circle having centre at $(0,1)$ and radius $=2$

Singular point $z=-1$ lies inside C where $z=2$ lies outside C.

$$\therefore \oint_C f(z)dz = 2\pi i\left\{\left[\text{Res} f(z)\right]_{z=-1}\right\}$$

$z=-1$ is a pole of order 2

$$\therefore \left[\text{Res} f(z)\right]_{z=-1} = \frac{1}{1!}\cdot \lim_{z\to -1}\frac{d}{dz}(z+1)^2 f(z)$$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z-1}{z-2}\right)$$

$$= \lim_{z\to -1}\cdot\frac{(z-2)\cdot 1-(z-1)\cdot 1}{(z-2)^2}$$

$$= \frac{(-1-2)-(-1-1)}{(-1-2)^2} = \frac{-3+2}{9} = \frac{-1}{9}$$

$$\therefore \left[\text{Res} f(z)\right]_{z=-1} = \frac{-1}{9}$$

$$\therefore \boxed{\oint_C \frac{z-1}{(z-2)(z+1)^2}dz = 2\pi i\left(\frac{-1}{9}\right) = \frac{-2\pi i}{9}}$$

Example 11.4.11: Evaluate $\oint_C \frac{z+3}{2z^2+3z-2}dz$ where $C:|z-i|=2$

Solution: $f(z)=\frac{z+3}{2z^2+3z-2}=\frac{z+3}{(2z-1)(z+2)}$

Singular points of $f(z)$ are $z=\frac{1}{2}$ and $z=-2$

$C:|z-i|=2$ That is, $|z-(0+i)|=2$ is a circle having centre at $(0,1)$ and radius $=2$

Singular points $z=\frac{1}{2}$ lies inside C where as $z=-2$ lies outside C.

$$\therefore \oint_C f(z)dz = 2\pi i\left\{\left[\operatorname{Res} f(z)\right]_{z=\frac{1}{2}}\right\}$$

$z=\frac{1}{2}$ is a simple pole of $f(z)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z)$$

$$= \lim_{z\to\frac{1}{2}}(2z-1)\frac{z+3}{(2z-1)(z+2)}$$

$$= \lim_{z\to\frac{1}{2}}\frac{z+3}{z+2}$$

$$= \frac{\frac{1}{2}+3}{\frac{1}{2}+2} = \frac{\frac{7}{2}}{\frac{5}{2}} = \frac{7}{5}$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\frac{1}{2}} = \frac{7}{5}$$

$$\therefore \oint_C f(z)dz = 2\pi i\left\{\frac{7}{5}\right\} = \frac{14\pi i}{5}$$

$$\therefore \boxed{\oint_C \frac{z+3}{2z^2+3z-2}dz = \frac{14\pi i}{5}}$$

Example 11.4.12: Evaluate $\oint_C \frac{1}{z^3+z^5}dz$ where $C:|z|=3$.

Solution: $f(z)=\frac{1}{z^3+z^5}=\frac{1}{z^3(z^2+1)}=\frac{1}{z^3(z-i)(z+i)}$

Singular points of $f(z)$ are $z=0$, $z=i$ and $z=-i$

$C:|z|=3$ is a circle having centre $(0,0)$ and radius $=3$

All the singular points of $f(z)$ lie inside C.

$\therefore$ By Cauchy's Residue Theorem,

$$\oint_C f(z)dz = 2\pi i\left\{\left[\operatorname{Res} f(z)\right]_{z=0}+\left[\operatorname{Res} f(z)\right]_{z=i}+\left[\operatorname{Res} f(z)\right]_{z=-i}\right\} ------(1)$$

(i) $z=0$ is a pole of order 3

$$\therefore\left[\operatorname{Res} f(z)\right]_{z=0}=\frac{1}{(3-1)!}\cdot\lim_{z\to 0}\frac{d^2}{dz^2}\left[z^3 f(z)\right]$$

$$=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\frac{1}{z^2+1}\right)$$

$$=\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[\frac{d}{dz}\left(\frac{1}{z^2+1}\right)\right]$$

$$=\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[-\frac{1}{(z^2+1)^2}2z\right]$$

$$=-\lim_{z\to 0}\frac{d}{dz}\left[\frac{z}{(z^2+1)^2}\right]$$

$$=-\lim_{z\to 0}\left[\frac{(z^2+1)^2\cdot(1)-z\cdot 2(z^2+1)\cdot 2z}{(z^2+1)^4}\right]$$

$$=-\left[\frac{(1)(1)-0}{1^4}\right]=-1$$

$$\therefore\left[\operatorname{Res} f(z)\right]_{z=0}=-1-------------(2)$$

(ii) $z=i$ is a simple pole of $f(z)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i}(z-i)f(z)$$

$$= \lim_{z\to i}\frac{1}{z^3(z+i)} = \frac{1}{i^3(i+i)} = \frac{1}{2i^4} = \frac{1}{2}$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=i} = \frac{1}{2} \text{---------}(3)$$

(iii) $z=-i$ is a simple pole of $f(z)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=-i} = \lim_{z\to -i}(z+i)f(z)$$

$$= \lim_{z\to -i}\frac{1}{z^3(z-i)}$$

$$= \frac{1}{(-i)^3(-i-i)}$$

$$= \frac{1}{(-i^3)(-2i)}$$

$$= \frac{1}{2i^4} = \frac{1}{2}$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=-i} = \frac{1}{2} \text{------}(4)$$

Substituting from (2), (3) and (4) in (1)

$$\oint_C f(z)\,dz = 2\pi i\left\{-1+\frac{1}{2}+\frac{1}{2}\right\} = 0$$

$$\therefore \oint_C \frac{1}{z^3+z^5}\,dz = 0$$

Example 11.4.13: Evaluate $\oint_C z^4 e^{1/z}\,dz$ where $C:|z|=1$

Solution: $f(z)=z^4 e^{1/z}$

Singular point of $f(z)$ is $z=0$.

We know that $e^z=1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dfrac{z^4}{4!}+\dfrac{z^5}{5!}+\cdots$ {Maclaurin's series expansion of e^z}

$$\therefore e^{1/z}=1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}+\frac{1}{120z^5}+\cdots$$

$$\therefore z^4 e^{1/z}=z^4\left[1+\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}+\frac{1}{24z^4}+\frac{1}{120z^5}+\cdots\right]$$

$$\therefore f(z)=z^4 e^{1/z}=z^4+z^3+\frac{z^2}{2}+\frac{z}{6}+\frac{1}{24}+\frac{1}{120z}+\frac{1}{720z^2}+\cdots$$

$\operatorname{Res} f(z)=$ coefficient $\dfrac{1}{z}$ in the Laurent's series expansion of $f(z)=\dfrac{1}{120}$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=0}=\frac{1}{120}$$

$$\oint_C f(z)\,dz=2\pi i\left\{\left[\operatorname{Res} f(z)\right]_{z=0}\right\}=2\pi i\left\{\frac{1}{120}\right\}=\frac{\pi i}{60}$$

$$\therefore \oint_C z^4 e^{1/z}\,dz=\frac{\pi i}{60}$$