06) [Ans. : πa^2]

7. Gauss Divergence Theorem

The surface integral of the normal component of a vector over a closed surface S is equal to the volume integral of the divergence of $\overline{F}$ throughout the volume bounded by S.

$$\boxed{\iint_S \overline{N} \cdot \overline{F}\, ds = \iiint_V \nabla \cdot \overline{F}\, dv}$$

where $\overline{N}$ is the unit <u>outward</u> normal.

We shall accept this important theorem without proof.

Example 5 : Use Gauss's Divergence Theorem to evaluate $\iint\limits_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = x^2 i + zj + yzk$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. **(M.U. 2015)**

Sol. : By the Gauss's Divergence Theorem $\iint\limits_S \overline{N} \cdot \overline{F}\, ds = \iiint\limits_V \nabla \cdot \overline{F}\, dV$

$$\text{But } \nabla \cdot \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(yz) = 2x + y$$

$$\therefore \iiint\limits_V \nabla \cdot \overline{F}\, dV = \int_0^1\int_0^1\int_0^1 (2x + y)\, dx\, dy\, dz = \int_0^1\int_0^1 [2xz + yz]_0^1\, dx\, dy$$

$$= \int_0^1\int_0^1 (2x + y)\, dx\, dy = \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \left(2x + \frac{1}{2}\right) dx$$

$$\therefore \iiint\limits_V \nabla \cdot \overline{F}\, dV = \left[x^2 + \frac{1}{2}x\right]_0^1 = \frac{3}{2} \qquad \therefore \iint\limits_S \overline{N} \cdot \overline{F}\, ds = \frac{3}{2}.$$

Example 6 : Use Gauss's Divergence Theorem to evaluate $\iint\limits_S \overline{N} \cdot \overline{F}\, ds$ where $\overline{F} = 4x\,i + 3y\,j - 2z\,k$ and S is the surface bounded by $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$. **(M.U. 2014)**

Sol. : By Divergence Theorem

$$\iint\limits_S \overline{N} \cdot \overline{F}\, ds = \iiint\limits_V \nabla \cdot \overline{F}\, dv$$

Now, $\overline{F} = 4x\,i + 3y\,j - 2z\,k \qquad \therefore \nabla \cdot \overline{F} = 4 + 3 - 2 = 5$

$$\text{Now, } \iiint\limits_V \nabla \cdot \overline{F}\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{z=4-2x-2y} 5\, dx\, dy\, dz$$

$$= \int_0^2\int_0^{2-x} 5(4 - 2x - 2y)\, dx\, dy$$

$$= 5\int_0^2 \left[4y - 2xy - y^2\right]_0^{2-x} dx$$

$$= 5\int_0^2 \left[4(2 - x) - 2x(2 - x) - (2 - x)^2\right] dx$$

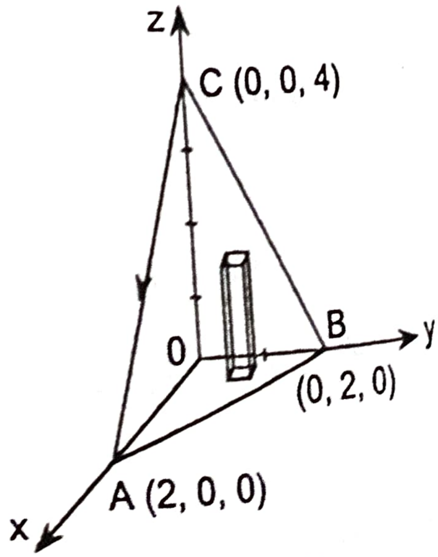

Fig. 4.30

$$= 5\int_0^2 \left[4 - 4x + x^2\right] dx = 5\left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 = 5\left[8 - 8 + \frac{8}{3}\right] = \frac{40}{3}$$

$$\therefore \quad \iint_S \bar{N} \cdot \bar{F}\, ds = \frac{40}{3}.$$

Example 7 : Use Gauss's Divergence Theorem to evaluate $\iint_S \bar{N} \cdot \bar{F}\, ds$ where $\bar{F} = 4x\,i - 2y^2\,j + z^2\,k$ and S is the region bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

(M.U. 1992, 94, 98, 2001, 06, 15)

Sol. : By Divergence Theorem $\quad \iint_S \bar{N} \cdot \bar{F}\, ds = \iiint_V \nabla \cdot \bar{F}\, dv$

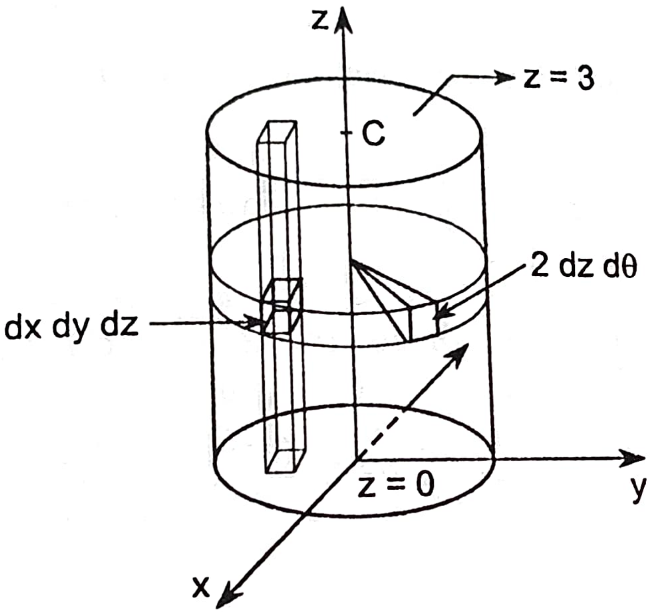

Fig. 4.31 (a)

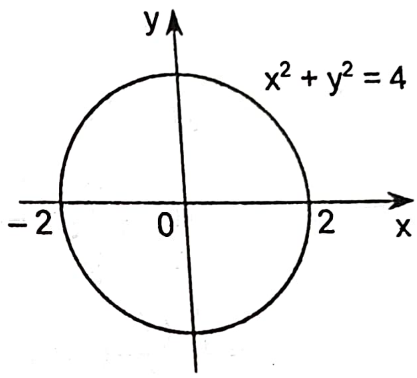

Fig. 4.31 (b)

Now, $\quad \bar{F} = 4x\,i - 2y^2\,j + z^2\,k \qquad \therefore \quad \nabla \cdot \bar{F} = 4 - 4y + 2z$

$$\therefore \quad \iiint_V \nabla \cdot \bar{F}\, dv = \iiint_V (4 - 4y + 2z)\, dx\, dy\, dz$$

For the whole volume z varies from 0 to 3, y varies from $-\sqrt{4 - x^2}$ to $+\sqrt{4 - x^2}$ and x varies from -2 to 2.

$$\therefore \quad \iiint_V \nabla \cdot \bar{F}\, dv = \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 dx\, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dxdy = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dx\, dy$$

[$\because \int_{-a}^{a} 12y\,dy = 0$ as $12y$ is an odd function and $\int_{-a}^{a} 21\,dy = 2 \times 21 \int_{0}^{a} dy$ as $21\,dy$ is an even function.]

$$\therefore \iiint_V \nabla \cdot \bar{F}\,dv = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} 2 \times 21\,dx\,dy = \int_{-2}^{2} [42\,y]_0^{\sqrt{4-x^2}}\,dx$$

$$= 42 \int_{-2}^{2} \sqrt{4-x^2}\,dx = 42 \times 2 \int_{0}^{2} \sqrt{4-x^2}\,dx$$

$$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2} \right]_0^2$$

$$= 84 \cdot 2 \cdot \frac{\pi}{2} = 84\pi \qquad \text{.................. (1)}$$

Alternatively, changing to cylindrical polar coordinates, we put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ and $dx\,dy\,dz = r\,dr\,d\theta\,dz$.

$$\therefore \iiint_V \nabla \cdot \bar{F}\,dv = \iiint (4 - 4y + 2z)\,dx\,dy\,dz = \int_0^a \int_0^{2\pi} \int_0^b (4 - 4r\sin\theta + 2z)\,r\,dr\,d\theta\,dz$$

$$= 4\int_0^a r\,dr \int_0^{2\pi} d\theta \int_0^b dz - 4\int_0^a r^2\,dr \int_0^{2\pi} \sin\theta\,d\theta \int_0^b dz + 2\int_0^a r\,dr \int_0^{2\pi} d\theta \int_0^b z\,dz$$

$$= 4 \cdot \frac{(a^2)}{2}(2\pi)(b) - 4 \cdot \frac{(a^3)}{2}[-\cos\theta]_0^{2\pi}\,b + 2 \cdot \frac{(a^2)}{2}(2\pi)\left(\frac{b^2}{2}\right)$$

$$= 4\pi a^2 b + 0 + \pi a^2 b^2$$

Putting $a = 2$, $b = 3$

$$\iiint_V \nabla \cdot \bar{F}\,dv = 4\pi \cdot 4 \cdot 3 + \pi \cdot 4 \cdot 9 = 84\,\pi.$$

$$\therefore \iint_S \bar{N} \cdot \bar{F}\,ds = 84\,\pi.$$

Example 8 : Use Gauss's Divergence Theorem to evaluate $\iint_S \bar{N} \cdot \bar{F}\,ds$ where $\bar{F} = 2xi + xyj + zk$ over the region bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$, $z = 6$.

(M.U. 1996, 201

Sol. : By Divergence Theorem $\iint_S \bar{N} \cdot \bar{F}\,ds = \iiint_V \nabla \cdot \bar{F}\,dv$

Here, $\bar{F} = 2xi + xyj + zk$

$$\therefore \nabla \cdot \bar{F} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(z) = 2 + x + 1 = 3 + x \qquad \text{..........}$$

$$\therefore \iiint_V \nabla \cdot \bar{F}\,dv = \iiint_V (3 + x)\,dv = \iiint_V (3 + x)\,dx\,dy\,dz$$

Now, to cover the whole volume bounded by the cylinder $x^2 + y^2 = 4$, $z = 0$ and $z = 6$, z varies from 0 to 6, y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$, and x varies from -2 to 2 (as in the previous example).

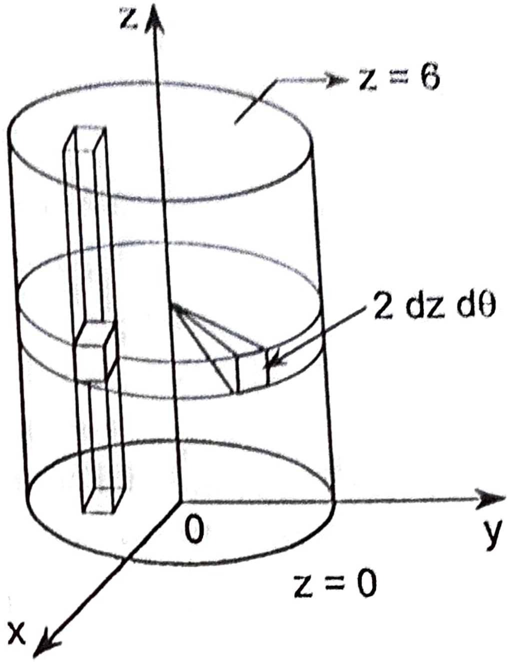

Fig. 4.32

$$\therefore \iiint_V (3+x)\,dx\,dy\,dz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{6}(3+x)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[3z + xz\right]_0^6 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(18+6x)\,dx\,dy$$

$$= \int_{-2}^{2}\left[18y + 6xy\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2}\left[\left(18\sqrt{4-x^2} + 6x\sqrt{4-x^2}\right) - \left(-18\sqrt{4-x^2} - 6x\sqrt{4-x^2}\right)\right]dx$$

$$= \int_{-2}^{2}\left(36\sqrt{4-x^2} + 12x\sqrt{4-x^2}\right)dx$$

$$= \left[36\left\{\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right\} - 4(4-x^2)^{3/2}\right]_{-2}^{2}$$

$$= 36\left\{2\cdot\frac{\pi}{2} + 2\frac{\pi}{2}\right\} = 72\pi$$