Fibonocci Numbers

Q. Given a no. N, calc the n^{th} fibonocci

↳ # Matrix Exponentiation

↳ Linear Recurrence → It is a func^n in which each term of the seq is a linear combination of prev terms.

$$f_n = f_{n-1} + f_{n-2}$$

general eqⁿ

Matrix Expo

$$f_k = a f_{k-1} + b f_{k-2} + c f_{k-3} \cdots\cdots\cdots\cdots$$

Step 1 → Define the no. of prev terms you are dependent on. Let's call this $\underline{\underline{k}}$.

$$f_n = f_{n-1} + f_{n-2} \qquad \underline{\underline{k = 2}}$$

$$f_{k'} = 2 f_{k'-1} + f_{k'-2} + 3 f_{k'-4} \qquad \underline{\underline{K = 4}}$$

Step 2 → find the first K terms of the seq then store them in a column matrix.

$$f_n = f_{n-1} + f_{n-2}$$

$\underline{\underline{K = 2}}$

$$f_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_{\underline{\underline{K \times 1}}}$$

Step 3 Define transformation Matrix

$$\underbrace{\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}_{1 \times k}}_{T.M} \times \begin{bmatrix} - \\ - \\ - \end{bmatrix}_{k \times d} \longrightarrow \begin{bmatrix} \quad \end{bmatrix}_{k \times 1}$$

Transformation matrix of fib.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}_{k \times k} \times \begin{bmatrix} f_{i-1} \\ f_i \end{bmatrix} = \begin{bmatrix} f_i \\ f_{i+1} \end{bmatrix}$$

prev → next

$$f_n = T \times f_{n-1} = T(T f_{n-2}) = T(T(T f_{n-3}))$$

$$[\ ][\ ]$$

$$O(k^3 \log n)$$

$$\vdots$$

$$\boxed{f_n = T^{n-1} f_1}$$

$$T^n = T^{n/2} \times T^{n/2}$$

$$T^{n/2} = T^{n/4} \times T^{n/4}$$

$\log n$

$$f_i = f_{i-1} + 2f_{i-2} + 0f_{i-3} + 4f_{i-4}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 1 \end{bmatrix}}_{K \times K}^{T} \times \underbrace{\begin{bmatrix} f_{i-4} \\ f_{i-3} \\ f_{i-2} \\ f_{i-1} \end{bmatrix}}_{K \times 1}^{f_i} = \underbrace{\begin{bmatrix} f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \end{bmatrix}}_{K \times 1}^{f_{i+1}}$$

$$f_i = f_{i-1} + 2f_{i-2} + 0f_{i-3} + 4f_{i-4} + \underline{\underline{d}}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{K \times K} \times \begin{bmatrix} f_{i-4} \\ f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ d \end{bmatrix}_{K \times 1} = \begin{bmatrix} f_{i-3} \\ f_{i-2} \\ f_{i-1} \\ f_i \\ d \end{bmatrix}_{K \times 1}$$

Willson theorem

if p is a prime number, then

$$(p-1)!\ \%\ p = -1$$

or

$$p-1$$

what willson theorem state

Q -> $(n!) \% p$ p is a prime

① if $n \geq p \longrightarrow 0$

$\left(n \times (n-1) \times (n-2) \cdots (p) \cdots (2)(1) \right) \% p$

$n \geq p$

if $n < p$

$(p-1)! \mod p = -1$ or $p-1$

$(1 \times 2 \times 3 \cdots\cdots (n) \cdots\cdots (p-2)(p-1)) \% p = -1$ or $p-1$

$$\left(\underbrace{(1 \times 2 \times 3 \times \cdots\cdots n)}_{n!} \times ((n+1) \times (n+2) \cdots\cdots (p-1))\right) \% p = -1 \text{ or } p-1$$

$$(n!) \% p \times \underbrace{((n+1)(n+2) \cdots\cdots (p-1))}\% p = -1$$

$$n! \% p = (-1) \times (n+1)^{-1} \% p \times (n+2)^{-1} \% p \cdots$$

$$\times (p-1)^{-1} \% p$$

$$a^{-1} \% p = a^{p-2} \% p$$ using fermat theorem

$$n! \% p = (-1) \underbrace{(n+1)^{p-2} \% p \times (n+2)^{p-2} \% p \cdots\cdots (p-1)^{p-2}} \% p$$

modular exponentiation

$$n! \% p = \underline{-1 \times r} + p$$

$$f_n = f_{n-1} + f_{n-2} + f_{n-1} \times f_{n-2}$$

$$f_n = f_{n-1}(1 + f_{n-2}) + f_{n-2} + 1 - 1$$

$$f_n = (1 + f_{n-2})(1 + f_{n-1}) - 1$$

$$1 + f_n = (1 + f_{n-1}) \times (1 + f_{n-2})$$

$$1 + f_x = G_x$$

$$\boxed{G_n = G_{n-1} G_{n-2}}$$

$G_0 = 1 + F_0 = a$

$G_1 = b$

$G_2 = ab$

$G_3 = ab^2$

$G_4 = a^2 b^3$

$G_5 = a^3 b^5$

$G_6 = a^5 b^8$

$\vdots$

$G_n = \left(a^{f_{n-1}} \, b^{f_{n-2}}\right) \% \, mod$

$10^9 + 7$

$G_n \% mod$

Hint