Relevance in Machine Learning

- Linear transformations are used in both abstract mathematics, as well as computer science. Linear transformations within calculus are used as way of tracking change, also known as derivatives.
- Linear transformations are often used in machine learning applications. They are useful in the modeling of 2D and 3D animation, where an objects size and shape needs to be transformed from one viewing angle to the next.
- An object can be rotated and scaled within a space using a type of linear transformations known as geometric transformations.
- Linear transformations also used to project the data from one space to another. Sometimes, a dataset is not linearly separable in its original space, therefore we need to transform (project) the data in another space with different dimensions. This can be done either using ‘ linear transformation’ or ‘kernels’.

Linear Transformation

Let V and W be vectors spaces over a field F of dimensions n and m, respectively. A linear transformation is a mapping $T : V^{(n)}(F) \longrightarrow W^{(m)}(F)$ such that

$$(1)\ \mathrm{T}(v_1 + v_2) = \mathrm{T}(v_1) + \mathrm{T}(v_2)\ \forall\ v_1 \text{ and } v_2 \in V$$
$$(2)\ \mathrm{T}(\alpha \mathrm{v}) = \alpha \mathrm{T}(\mathrm{v})\ \forall\ \mathrm{v} \in \mathrm{V} \text{ and } \alpha \in \mathrm{F}.$$

Example:

- $\mathrm{T} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that
$$\mathrm{T}(x_1, x_2) = (x_1, x_1 + x_2)$$
- $\mathrm{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that
$$\mathrm{T}(x_1, x_2, x_3) = (x_2, x_1, 0)$$

Remarks:

- A linear map from T to itself is called a linear operator.
- A linear map from a vector space to underlying field is called a linear functional.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ s/t $T(x_1, x_2) = (x_1, x_1 + x_2)$

Let $v_1 = (x_1, x_2)$ and $v_2 = (y_1, y_2) \in V = \mathbb{R}^2(\mathbb{R})$

$$T(v_1) = T(x_1, x_2) = (x_1, x_1 + x_2)$$

$$T(v_2) = T(y_1, y_2) = (y_1, y_1 + y_2)$$

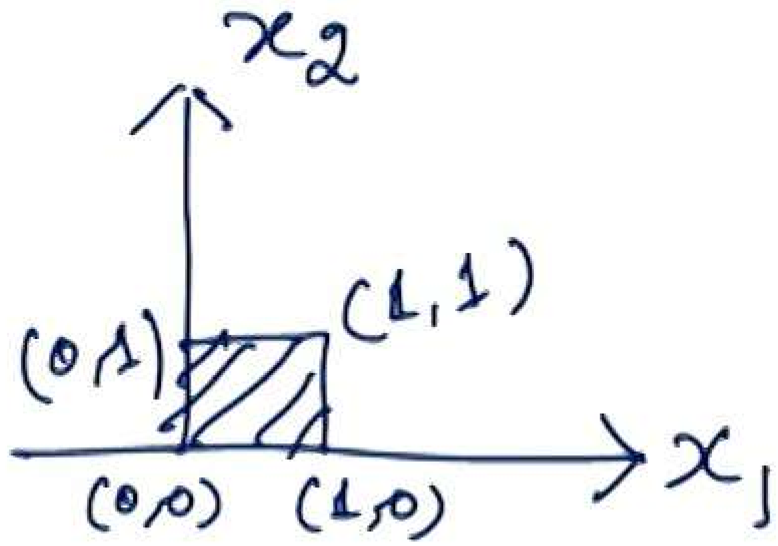
$$T(v_1 + v_2) = T(\underline{x_1 + y_1}, x_2 + y_2) = (x_1 + y_1, x_1 + y_1 + x_2 + y_2)$$

$$= T(v_1) + T(v_2) \checkmark$$

$$T(\alpha v_1) = T(\alpha x_1, \alpha x_2) = (\alpha x_1, \alpha(x_1 + x_2))$$

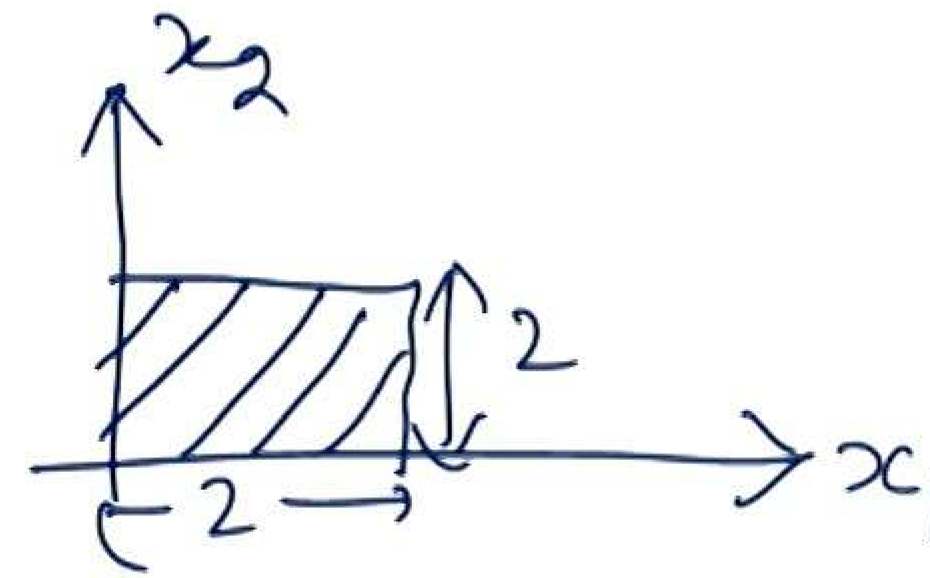
$$= \alpha T(v_1) \checkmark$$

T is a linear Transformation.

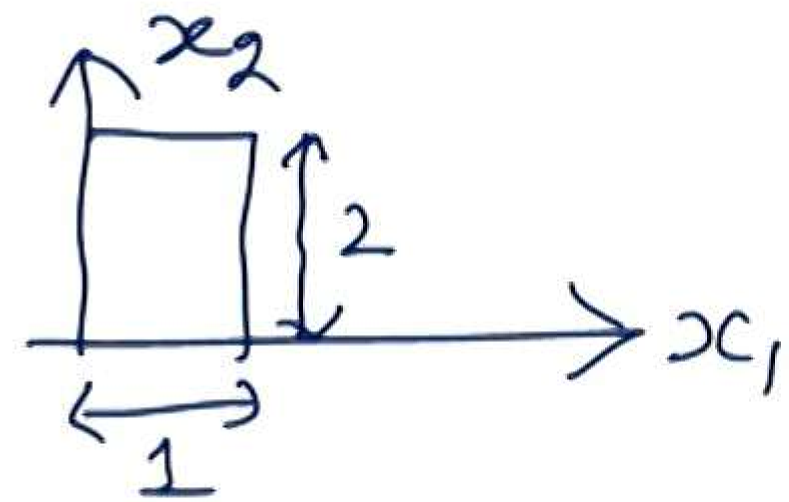


$$T: \mathbb{R}^2 \to \mathbb{R}^2$$

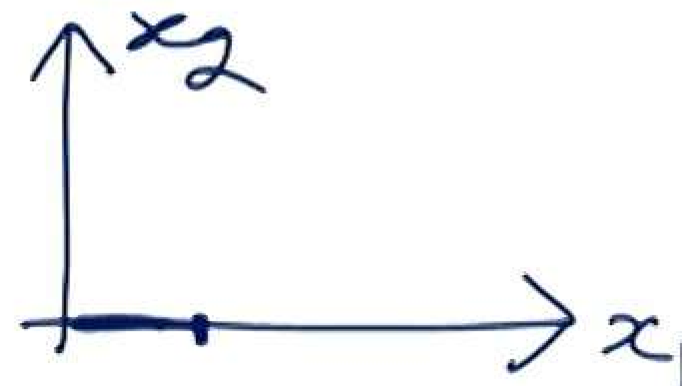
$$T(x_1, x_2) = (2x_1, 2x_2)$$

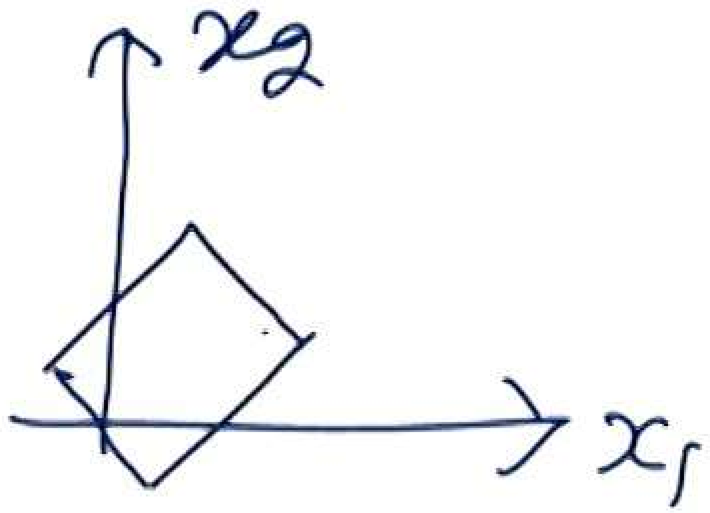


$$T(x_1, x_2) = (x_1, 2x_2)$$



$$T(x_1, x_2) = (x_1, 0)$$





$$T(x_1, x_2) = (x_1 \cos\theta - x_2 \sin\theta,\ x_1 \sin\theta + x_2 \cos\theta)$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$R \qquad\qquad X$$

Some more examples:

Check which of the followings are linear map?

1. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ s.t. $T(x_1, x_2) = (x_1 + x_2 + 1, 2x_1 - x_2, x_1 + x_2)$
2. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t. $T(x_1, x_2) = (x_1 - x_2, 2x_1^2 - x_2^2)$
3. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t. $T(x_1, x_2) = (x_1 - x_2, |x_1|)$
4. $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ s.t. $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 2x_1 + x_2, 3x_1 - 4x_2)$
5. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ s.t. $T(x_1, x_2, x_3) = (x_3, x_2, x_1)$
6. $T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ s.t. $T(A) = A^T$
7. $T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ s.t. $T(A) = I + A$
8. $T : M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$ s.t. $T(A) = BAB^{-1}$ where B is an invertible matrix.

$$T(x_1, x_2) = (x_1 - x_2,\ 2x_1^2 - x_2^2)$$

$(1,1)$ and $(2,2)$ in $\mathbb{R}^2$

$T(1,1) = (0,1)$ | $T(3,3) = (0,9)$

$T(2,2) = (0,4)$

$$T((1,1)+(2,2)) \neq T(1,1) + T(2,2)$$

The matrix of a linear map

Suppose V(F) and W(F) are finite dimensional vector spaces with bases $\{v_1, v_2, ..., v_n\}$ and $\{w_1, w_2, ..., w_m\}$ respectively. Let T : $V \longrightarrow$ W be a linear map. Then the matrix of T, with enteries $a_{ij} \in$ F is defined by

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m$$

for j = 1, 2, 3,. . . ,n

It shows that the j^{th} column of the matrix A of linear map T consists of the coordinates of $T(v_j)$ in the chosen basis for W.

Conversely, every matrix $A \in \mathbb{R}^{m\times n}$ induced a linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by

$$T(v) = Av$$

Examples of matrix of a linear map

* Consider a linear map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $T(x_1, x_2) = (2x_1 - 7x_2, 4x_1 + 3x_2)$.

  Find the matrix representation of T relative to basis B = {(1,3),(2,5)}.

  Solution: $T(1,3) = (-19,13) = a_{11}(1,3) + a_{21}(2,5)$

  $T(2,5) = (-31,23) = a_{12}(1,3) + a_{22}(2,5)$

  Gives $a_{11} = 121$: $a_{12} = 201$;

  $a_{21} = -70$; $a_{22} = -116$

  Hence $A = \begin{pmatrix} 121 & 201 \\ -70 & -116 \end{pmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$T(x_1, x_2) = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (2x_1 + x_2, 4x_1 + 3x_2)$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T : \mathbb{R}^n \to \mathbb{R}^m \longrightarrow \begin{bmatrix} & \\ & \end{bmatrix}_{m \times n}$$

Nullspace and Range of a linear map

Let T : V $\longrightarrow$ W be a linear map. Then the nullspace of T is defined as

$$\text{null}(T) = \{v \in V | T(v) = 0\}$$

and the range of T is defined as

$$\text{range}(T) = \{w \in W | \exists v \in V s.t. T(v) = w\}$$

Remarks:

- Nullspace of T is also called kernel of T denoted by (ker(T)).
- Null(T) is a subspace of V.
- Range(T) is a subspace of W.
- The dimension of Null(T) is called nullity of T.
- The dimension of Range(T) is called rank of T.

Rank-Nullity Theorem

For a linear map $T : V \longrightarrow W$, we have

$$\boxed{\dim(\mathrm{range}(T)) + \dim(\mathrm{null}(T)) = \dim(V)}$$

Example: Determine the range and nullspace of the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, x_2 - x_3, x_1, 2x_1 - 5x_2 + 5x_3)$.

Solution: Range(T) = {(1,0,1,2),(1,-1,0,5)}

Nullspace(T) = {(0,1,1)}

$T: \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3,\ x_2 - x_3,\ x_1,\ 2x_1 - 5x_2 + 5x_3)$$

Range(T):

$T(1,0,0) = (1,0,1,2)$

$T(0,1,0) = (-1,1,0,-5)$

$T(0,0,1) = (1,-1,0,5)$

$$\text{Range}(T) = L\{(1,0,1,2), (-1,1,0,-5)\}$$

$$\text{Rank}(T) = 2$$

Null(T): $X \mid T(X) = 0 \Rightarrow x_2 = x_1 + x_3;\ x_2 = x_3;\ x_1 = 0$

$5x_2 = 2x_1 + 5x_3$

$5x_2 = 5x_3$

$\{(x_1, x_2, x_3, \cancel{x_4}) \mid - - \}$

$\{(0, x_2, x_2)\} \to \{(0,1,1)\}$; Nullity$(T) = 1$