Metric: A metric on a set S is a function

$$d : S \times S \to \mathbb{R}$$

satisfying

① $d(x, y) \geq 0 \ \forall x, y \in S$ and $d(x, y) = 0$ iff $x = y$

② $d(x, y) = d(y, x)$

③ $d(x, z) \leq d(x, y) + d(y, z) \ \forall x, y, z \in S$

(S, d) is called a metric space.

Ex:- Take $S \subseteq \mathbb{R}$ and define

$$d(x, y) = |x - y|$$

then (S, d) is a metric space.

Norm: A norm on a real vector space V is a function

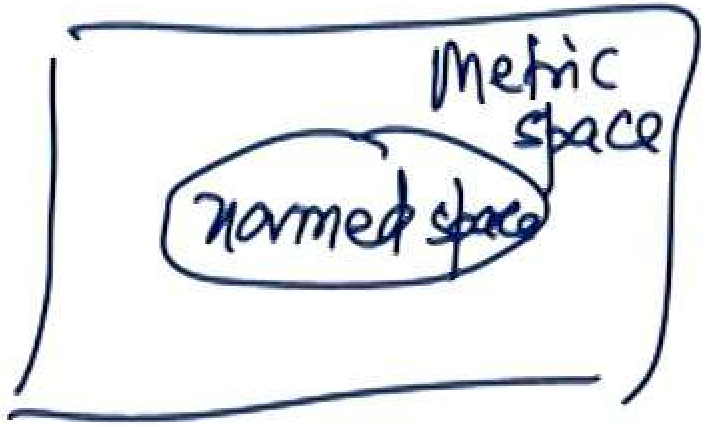
$$\|\cdot\| : V \longrightarrow \mathbb{R}$$

such that

1. $\|X\| \geq 0 \ \forall \ X \in V$ and $\|X\| = 0$ iff $X = \vec{0}$
2. $\|\alpha X\| = |\alpha| \, \|X\| \ \forall \ X \in V$ and $\alpha \in \mathbb{R}$
3. $\|X + Y\| \leq \|X\| + \|Y\| \ \forall \ X, Y \in V$

$(V, \|\cdot\|)$ is called a normed space

Relation :



→ every normed space is a metric space

→ converse is not true

Ex: Let $X = \{0, 1\}$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

(X, d) is a metric space, but not a normed space.

Examples of some norms on $\mathbb{R}^n$

- $||X||_1 = \sum_{i=1}^{n} |x_i|$
- $||X||_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$
- $||X||_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$
- $||X||_\infty = \max_{1 \leq i \leq n} |x_i|$

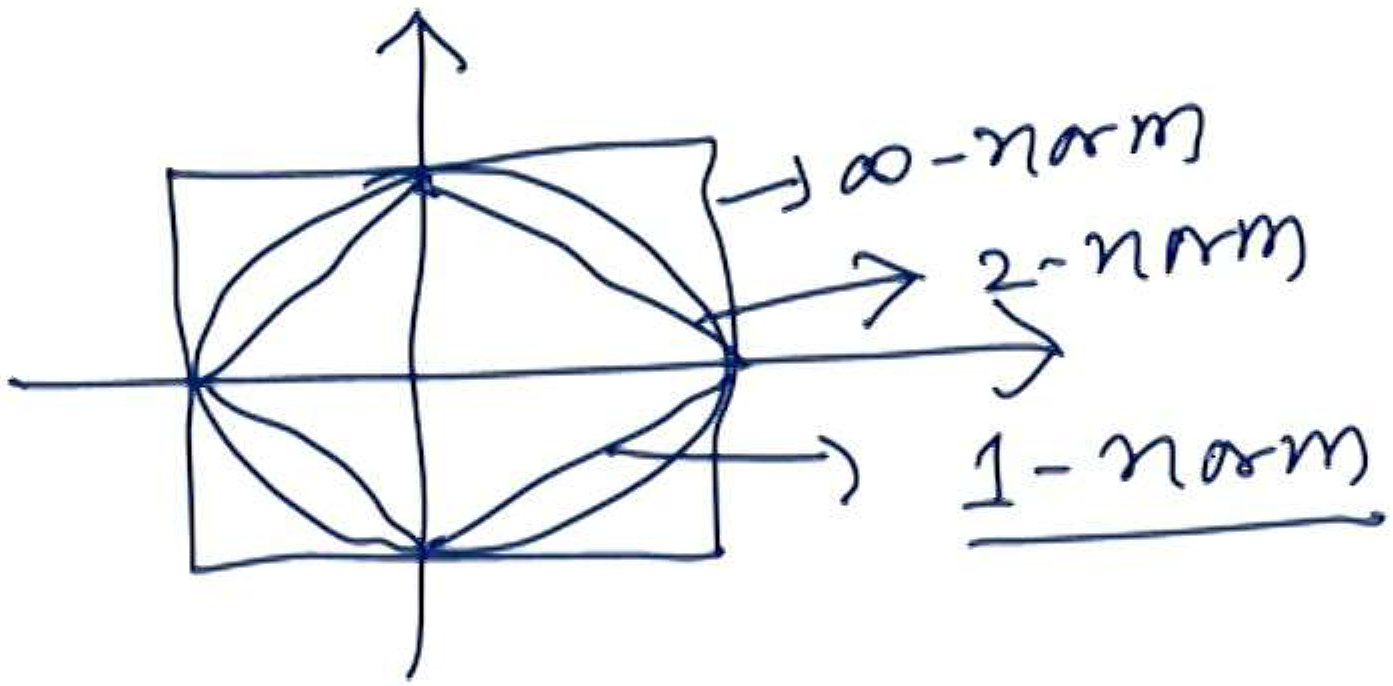
Examples: Let $V = \mathbb{R}^3(\mathbb{R})$

$X = (1, 0, -2)$ (Manhattan norm)

$\|X\|_1 = |1| + 0 + |-2| = 3$

$\|X\|_2 = (1^2 + 0^2 + (-2)^2)^{1/2} = \sqrt{5}$

$\|X\|_\infty = 2$

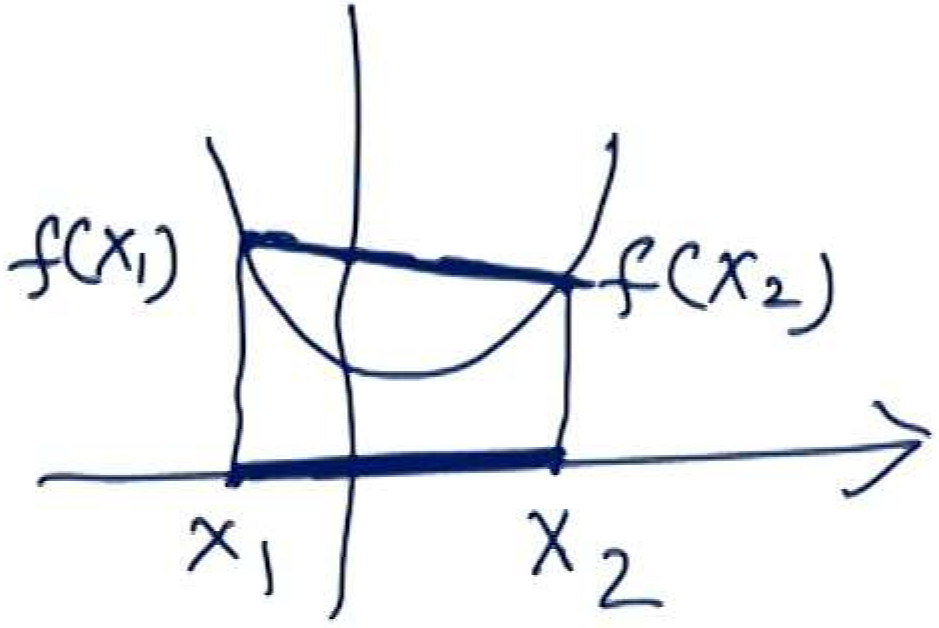


Convex function:- A function $f: S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be convex, if for $x_1, x_2 \in S$, we have

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

where, $0 \leq \lambda \leq 1$

Convex set:- A set is said to be convex, if the line joining any two points of the set lies entirely in the set S.



$x_1, x_2 \in S$

$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$

Not convex

Inner Product Spaces

An inner product on a real vector space V is a function

$$< \cdot , \cdot > : V \times V \longrightarrow \mathbb{R}$$

satisfying

(i) $< X, X > \geqslant 0 \quad \forall X \in V$ and $< X, X >= 0$ iff $X = 0$

(ii) $< X + Y, Z > = < X, Z > + < Y, Z >$ and $< \alpha X, Y > = \alpha < X, Y >$

$$\forall X, Y, Z \in V \text{ and } \alpha \in \mathbb{R}$$

(iii) $< X, Y > = < Y, X > \forall X, Y \in V$

A vector space together with an inner product is called an inner product space.

Examples of inner product spaces :

$V = \mathbb{R}^n(\mathbb{R})$

$\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$

$X = (x_1, x_2, \cdots x_n)$

$Y = (y_1, y_2, \cdots y_n)$

$\underline{\underline{X \cdot Y}}$

$\langle X, Y \rangle = \|X\| \, \|Y\| \cos \underline{\theta}$

Examples of inner product on $\mathbb{R}^n$

1. Let $V = \mathbb{R}^n$, let $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Then standard inner product on $\mathbb{R}^n$ is given as follows:

$$< \mathbf{x}, \mathbf{y} > = \sum_{i=1}^{n} x_i y_i$$

2. Let $V = \mathbb{R}^2$, $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$. Then inner product is defined as

$$< \mathbf{u}, \mathbf{v} > = 2u_1 v_1 - u_1 v_2 - v_1 u_2 + u_2 v_2$$

l_0-norm :- $\|x\|_0$ is the number of nonzero elements of x.

$x = (1, 2, 0, 0, 3, 0, 0, 4) \in \mathbb{R}^8$

then $\|x\|_0 = 4$

Compressed sensing

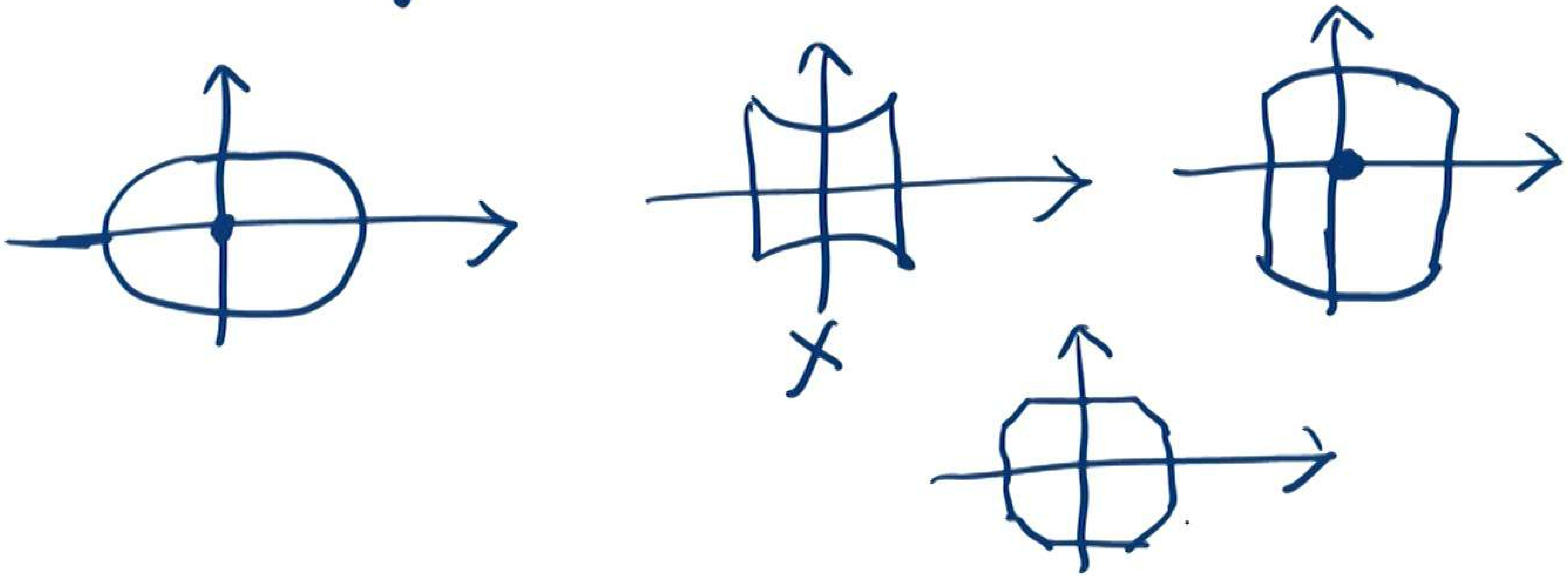
② $\|\alpha x\|_0 \neq |\alpha| \, \|x\|_0$ for $\alpha \neq 1$

$\alpha = 2$

L·H·S = 4

R·H·S = 2·4 = 8

In real finite dimensional vector spaces, any <u>symmetric</u>, <u>compact</u>, <u>convex</u> region centred at the origin defines a norm.



Metric Spaces

It is a generalization of the notion of distance from Euclidean space.

Definition

A metric on a set S is a function

$$d : S \times S \longrightarrow \mathbb{R}$$

such that

(i) $d(x, y) \geqslant 0 \; \forall x, y \in S$ and $d(x, y) = 0$ iff $x = y$

(ii) $d(x, y) = d(y, x) \; \forall \; x, y \in S$

(iii) $d(x, z) \leqslant d(x, y) + d(y, z) \; \forall \; x, y, z \in S$

Example: Take $S \subseteq \mathbb{R}$ and define

$$d(x, y) = |x - y|$$

then (S, d) forms a metric space.