Example 6·1. *Let X be a random variable with the following probability distribution :*

x	:	-3	6	9
$P_r(X=x)$	:	$1/6$	$1/2$	$1/3$

Find $E(X)$ and $E(X^2)$ and using the laws of expectation, evaluate $E[(2X+1)^2]$

imp

(Gauhati Univ. B.Sc., 1992)

Solution. $E(X) = \Sigma x \cdot p(x)$

$$= (-3)\times\frac{1}{6}+6\times\frac{1}{2}+9\times\frac{1}{3} = \frac{11}{2}$$

$$E(X^2) = \Sigma x^2 p(x)$$

$$= 9\times\frac{1}{6}+36\times\frac{1}{2}+81\times\frac{1}{3} = \frac{93}{2}$$

$$\therefore \quad E[(2X+1)^2] = E[4X^2+4X+1] = 4E(X^2)+4E(X)+1$$

$$= 4\times\frac{93}{2}+4\times\frac{11}{2}+1 = 209$$

$$= [2(-3)+1]^2\left\{\frac{1}{6}\right\} + [2(6)+1]^2\left\{\frac{1}{2}\right\} + [2(9)+1]^2\left\{\frac{1}{3}\right\}$$

$$= \frac{(-5)^2}{6} + \frac{13^2}{2} + \frac{19^2}{3} = \frac{25}{6} + \frac{169}{2} + \frac{361}{3} = 209$$

Expected Value for Binomial Discrete Random Variable

The Expected Value for a binomial random variable is calculated using the formula P(x) * X.
P(x) is the probability of success, and X is the number of trials.

successes or failures in ; $\boxed{p(x) = {}^nC_x\, p^x (1-p)^{n-x}}$

X measures the num of successes in 'n' trials of an event

$$\mathbb{E}(X) = \underline{\sum x \cdot p(x)} = \sum_{x=0}^{n} {}^nC_x\, p^x (1-p)^{n-x} \cdot x = \boxed{np(1-p)}$$

Example: You work as a financial analyst for a development company, for instance. Your manager just instructed you to determine which future development initiatives are most likely to succeed and pick that one. Upon completion, Project A is predicted to have a 0.4 ~~percent~~ chance of reaching a value of $2 million and a 0.6 ~~percent~~ chance of reaching a value of $500,000. Upon completion, Project B has a 0.3 ~~percent~~ chance of being valued at $3 million and a 0.7 ~~percent~~ chance of being valued at $200,000.

Answer:
You must determine the expected value of each project and contrast the values to choose the best project. The following formula can be used to get the EV:
EV (Project A) equals [0.4 x $2,000,000] plus [0.6 x $500,000] to arrive at $1,100,000.
EV (Project B) equals [0.3 $3,000,000] plus [0.7 $200,000] for a total of $1,040,000.
Project A's EV is higher than Project B's EV. As a result, Project A should be chosen by your company.

X 40% → 2,000,000 60% → 500,000 $\mathbb{E}(X) =$

Y 30% → 3,000,000 70% → 200,000 $\mathbb{E}(Y) =$

$$\mathbb{E}(X) : \sum x \cdot p(x) = 2\,000\,000\,(0.4) + 500\,000\,(0.6) = \boxed{1,100,000}$$

$$\mathbb{E}(Y) = \sum y \cdot p(y) = 3\,000\,000\,(0.3) + 200\,000\,(0.7) = \boxed{1,040,000}$$

Probability Density Function

In probability theory, the probability that a random variable will fall into a specific range of values rather than taking on a single value is defined by a **probability density function** (PDF).

The probability function expressing the density of a continuous random variable between a particular range of values is defined by the probability density function (PDF). In other words, the probability density function yields the likelihood that a continuous random variable will have a particular value.

Expected Value for Multiple events Continuous Random Variable

The expected value of a random variable is just the ~~mean~~ probability weighted average of the random variable. You can calculate the EV of a continuous random variable using this formula:

$$\boxed{E(x) = \int_{-\infty}^{\infty} x f(x) dx}$$

Where f(x) is the probability density function, representing a function for the density curve.
The "∫" symbol is called an integral, equivalent to finding the area under a curve.

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value of E(X).

$X \to$ continuous r.v. : $X \sim \exp(\lambda)$ distribution.
$f(x) = \lambda e^{-\lambda x}$ range : $x \geq 0$

$$\mathbb{E}(X) = \int_0^\infty f(x) \cdot x \, dx = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx$$

$$\lambda \int_0^\infty \underbrace{x}_{I} \underbrace{e^{-\lambda x}}_{II} dx = \lambda \left[x \int_0^\infty e^{-\lambda x} dx - \int_0^\infty 1 \cdot \frac{e^{-\lambda x}}{-\lambda} dx \right]_0^\infty$$

$$= \lambda \left[\left[x \frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty + \frac{1}{\lambda} \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^\infty \right]$$

$$= \cancel{\lambda} \left[\left\{ 0 - \left(\cancel{\frac{0}{\infty}} \right) \right\} + \frac{1}{\cancel{\lambda}} \left(\frac{1}{\lambda} \right) \right]$$

$$= \quad = 0 + \boxed{\frac{1}{\lambda}} = \boxed{\frac{1}{\lambda}}$$

$$\boxed{\mathbb{E}(X) = \frac{1}{\lambda}}$$

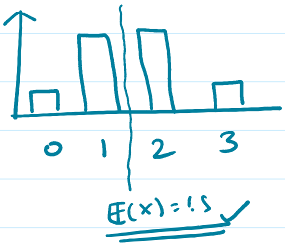
Expected Value Problem - 2

Question: You toss a fair coin three times. X is the number of heads which appear. What is the Expected Value?

$\mathbb{E}(X) = 1.5$

X	P(x)	
0	1/8	TTT
1	3/8	HTT THT TTH
2	3/8	THH HTH HHT

1	3/8	HTT THT TTH
2	3/8	THH HTH HHT
3	1/8	HHH



Solution:

Step 1: Figure out the possible values for X. For a three coin toss, you could get anywhere from 0 to 3 heads. So your values for X are 0, 1, 2 and 3.

Step 2: Figure out your probability of getting each value of X. You may need to use a sample space (The sample space for this problem is: {HHH TTT TTH THT HTT HHT HTH THH}). The probabilities are: 1/8 for 0 heads, 3/8 for 1 head, 3/8 for two heads, and 1/8 for 3 heads.

Step 3: Multiply your X values in Step 1 by the probabilities from step 2.
E(X) = 0(1/8) + 1(3/8) + 2(3/8) + 3(1/8) = 3/2.
The EV is 3/2.

Question: Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 3/x^4 & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the expected value.

$$f(x) = \frac{3}{x^4} \quad ; \quad x \geq 1$$

$$\mathbb{E}(X) = \int_1^\infty \frac{3}{x^4} \cdot x \, dx = \int_1^\infty \frac{3}{x^3} dx = 3\int_1^\infty x^{-3} dx$$

$$= 3\left[\frac{x^{-3+1}}{-3+1}\right]_1^\infty = \frac{3}{-2}\left[x^{-2}\right]_1^\infty = -\frac{3}{2}\{0 - 1\} = \boxed{\frac{3}{2}}$$

{A, B, C, 1, 2, 9, 24, 31.2} A set is a collection of elements.

Introduction to Set Theory

In the area of mathematical logic known as **set theory**, we study sets and their characteristics. A set is a grouping or collection of objects. These things are frequently referred to as elements or set members.

A set is, for instance, a team of cricket players. We can say that this set is finite because a cricket team can only have 11 players at a time.
A collection of English vowels is another illustration of a finite set.

However, many sets, including the sets of whole numbers, imaginary numbers, real numbers, and natural numbers, have an unlimited number of members.

$\{x : x \in (0,1)\}$ ( ), [ ]

↳ $0 \in x$ Yes No ✓

$\{x : x \in \mathbb{N}\}$ $\mathbb{N} : \{1, 2, 3, 4, \ldots\}$

Definition of Set

A set is a clearly defined group of things or people. Numerous examples from everyday life, including the number of rivers in India and the hues in a rainbow, can be used to relate sets.

Examples of set:

- Set representing all the leap years between 1995 and 2015

A ={1996,2000,2004,2008,2012}

- Finite set: The number of items is finite in a finite set. eg. {A, B, 34, 68.1}
- Infinite set: Set with an unlimited number of elements: eg. $\mathbb{N}$
- Null set → Empty set: A set that is empty has no elements. eg { } → empty set is denoted by ϕ
- Singleton set: It only contains one component. eg {A}
- Equal set: If two sets contain the same elements, they are equivalent. eg. X = {3, 1, 2} Y = {1, 2, 3}
- Equivalent set: If two sets have the same amount of elements, they are ~~equal~~ equivalent : X = {A, B, C} Y={1,2,8}
- Power set: A collection of all conceivable subsets.
- Universal set: Any set that contains every set being considered is referred to as a universal set. → contains all possible elements of the analysis
- Subset: A is a subset of B when all of set A's elements are members of set B.

A = {5, 10, 15, 20, 25, 30, 35, 40, 45, 50}

B = {10, 30, 50} B is a subset of A

C : {2, 5, 10, 15} C is not a subset of A.

POWER SET

A set containing all possible subsets.

X = {A, B, 1, 2} : ϕ, {A}, {B}, {C}, {D}, {A, B}, {A, 1}
{A, 2}, {B, 1}, {B, 2}, {1, 2}, {A, B, 1}, {A, B, 2}
{A, 1, 2}, {B, 1, 2}, {A, B, 1, 2}

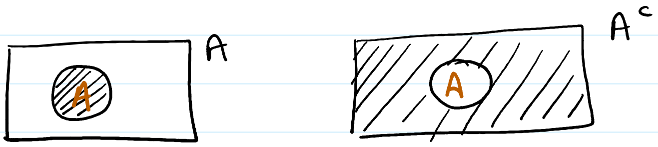
Power set of X = { ϕ, {A}, {B}, {C}, {D}, {A, B}, {A, 1}
{A, 2}, {B, 1}, {B, 2}, {1, 2}, {A, B, 1}, {A, B, 2}
{A, 1, 2}, {B, 1, 2}, {A, B, 1, 2} }

elements = 16 = n^2 ; n → # elements in your original set

Symbol	Symbol Name	Meaning	Example
{ } $\{ \quad \}$	set	a collection of elements	A = {1, 7, 9, 13, 15, 23}, B = {7, 13, 15, 21}
A ∪ B $A \cup B$	union	Elements that belong to set A or set B	A ∪ B = {1, 7, 9, 13, 15, 21, 23}
A ∩ B $A \cap B$	intersection	Elements that belong to both the sets, A and B	A ∩ B = {7, 13, 15 }
A ⊆ B $\subseteq$	subset	subset has few or all elements equal to the set	{7, 15} ⊆ {7, 13, 15, 21}
A ⊄ B	not subset	left set is not a subset of right set	{1, 21} ⊄ B
A ⊂ B $\subset$	proper subset / strict subset	subset A has fewer elements than the set B	{7, 13, 15} ⊂ {1, 7, 9, 13, 15, 23}
A ⊃ B $\supset$	proper superset / strict superset	set A has more elements than set B	{1, 7, 9, 13, 15, 23} ⊃ {7, 13, 15}
A ⊇ B $\supseteq$	superset	set A has more elements or equal to the set B	{1, 7, 9, 13, 15, 23} ⊃ {7, 13, 15, 21}
Ø	empty set	Ø = { }	C = {Ø}

If $X = \{A, B, 1, 2\}$; $\{A, B, 1, 2\} \subseteq \{A, B, 1, 2\}$

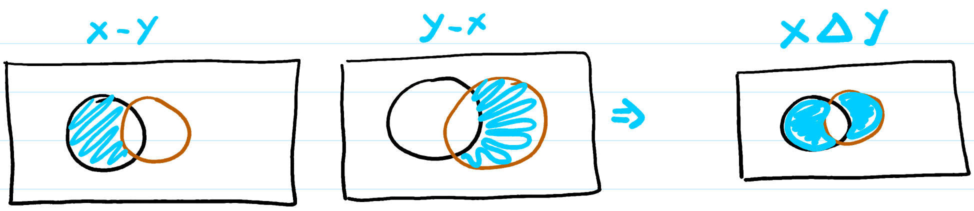
Symbol	Symbol Name	Meaning	Example
P (C)	power set	all subsets of C	C = {4,7}, P(C) = {{}, {4}, {7}, {4,7}} Given by 2^s, s is number of elements in set C
A ⊅ B	not superset	set X is not a superset of set Y	{1, 2, 5} ⊅ {1, 6}
A = B	equality	both sets have the same members	{7, 13,15} = {7, 13, 15}
A \ B or A-B	relative complement	objects that belong to A and not to B	{1, 9, 23}
A^c	complement	all the objects that do not belong to set A but belong to the universal set	We know, U = {1, 2, 7, 9, 13, 15, 21, 23, 28, 30} A^c = {2, 21, 28, 30}
A Δ B	symmetric difference	objects that belong to A or B but not to their intersection	A Δ B = {1, 9, 21, 23}
a∈B	element of	set membership	B = {7, 13, 15, 21}, 13 ∈ B



$X = \{1, 7, 9, 13, 15, 23\}$ $\quad Y = \{7, 15, 13, 21\}$

$X - Y = \{1, 9, 23\}$ $\quad Y - X = \{21\}$

$X \Delta Y = (X-Y) \cup (Y-X) = \{1, 9, 21, 23\}$



Symbol	Symbol Name	Meaning	Example
(a,b)	ordered pair	collection of 2 elements	(1, 2)
x∉A $\notin$	not element of	no set membership	A = {1, 7, 8, 13, 15, 23}, 5 ∉ A
|B|, #B	cardinality	the number of elements of set B	B = {7, 13, 15, 21}, |B|=4
A×B	cartesian product	set of all ordered pairs from A and B	{3,5} × {7,8} = {(3,7), (3,8), (5,7), (5, 8) }
N_1	natural numbers / whole numbers set (without zero)	N_1 = {1, 2, 3, 4, 5,...}	6 ∈ N_1
N_0	natural numbers / whole numbers set (with zero)	N_0 = {0, 1, 2, 3, 4,...}	0 ∈ N_0
Q	rational numbers set	Q= {x | x=a/b, a, b∈Z}	2/6 ∈ Q
Z	integer numbers set	Z= {...-3, -2, -1, 0, 1, 2, 3,...}	-6 ∈ Z
R	real numbers set	R= {x | -∞ < x <∞}	6.343434 ∈ R

SET OPERATIONS:
1. Union
2. Intersection
3. Difference
4. Symmetric difference

Question: If U = {a, b, c, d, e, f}, A = {a, b, c}, B = {c, d, e, f}, C = {c, d, e}, find (A ∩ B) U (A ∩ C)'.

{c}

$(A \cap B) = \{c\}$ $\qquad$ $(A \cap C) = \{c\}$

union

$\{c\}$

HW

Question: If U = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11}, A = {3, 5, 7, 9, 11} and B = {7, 8, 9, 10, 11}, Then find (A – B)'.

HW

Question: Class XI is made up of 200 students in total. One hundred and twenty of them study mathematics, fifty study commerce, and thirty studies. Find the number of students who

i) Study mathematics but not commerce

ii) Study commerce but not mathematics

iii) Study mathematics or commerce