Conditional probability and introduction to statistical inference 13oct22

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Set theory elements:
1. Union
2. Intersection
3. Difference
4. Complement
5. P(AUB) formula
6. And = x or = +

Set : is a collection of objects or elements.

$A1 = \{A, B, 1, 2, 3, \}$

$A2 = \{x : x \in \mathbb{N}\}$ $\mathbb{N} \to$ natural nos.

$= \{1, 2, 3, 4, \ldots\}$ infinite elements.

Union of sets.

$X1 = \{1, 3, 5, 7, 9\}$

$X2 = \{2, 4, 6, 8, 10\}$

$X_1 \cup X_2 = \{1, 3, 5, 7, 9, 2, 4, 6, 8, 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

union

Intersection of sets

$X_1 \cap X_2 = \{\ \}$ null since there are no common elements

a null set is denoted by $\emptyset$ (greek letter phi)

Difference of sets.

$X3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ $X_1 = \{1, 3, 5, 7, 9, 11\}$

$X3 - X1 = \{2, 4, 6, 8, 10\}$

$X1 - X3 = \{11\}$

now we will use set theory operations in solving probability questions

If E_1 & E_2 are two events, then $P(E_1 \cup E_2)$ says that prob. of either E_1 occurs or E_2 occurs.

If E_1 & E_2 are 2 events then $P[E_1 \cap E_2]$ says that prob. of E_1 and E_2 both occur

What is conditional?

Conditional in this context means reducing sample space by having extra information about an event.

Conditional Probability

- Introduction to Set theory ✓
- Probability of two events happening at the same time
- Conditional Probability

$E \to$ getting a red card

- Introduction to Set theory ✓
- Probability of two events happening at the same time
- Conditional Probability
- Bayes Theorem

$E_1 \rightarrow$ getting a red card

$E_2 \rightarrow$ getting a face card $\{K, Q, J, A\}$

$P(E_1 \cap E_2) = \frac{8}{52}$ ✓

P[redcard and face card] = $\frac{8 \text{ red face card}}{52 \text{ total}}$

$P(E_1 \cup E_2) =$ P(red card or face card) $= \frac{26+8}{52} = \frac{34}{52}$

Extra information that we have cards example

P(having a face card GIVEN THAT I HAVE A BLACK CARD)

$= 8/26$

fav. outcomes

13 ♠

13 ♣

13 ♦

13 ♥

Symbol of conditional is

given extra inf.

$P(X|Y)$ = Prob. of event X when Y has already occured.

Formula for conditional probability

DEFINITION For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has occurred is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad (2.3)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

eg. P(face card | black card) $= 8/26$ from above.

FORMULA VERIFY

$P(A \cap B) =$ P(black face card) $= 8/52$

$P(B) =$ P(black card) $= 1/2$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{8}{52} \div \frac{1}{2} = \frac{8}{52} \times 2 = \frac{8}{26}$

4·7. Multiplication Law of Probability and Conditional Probability

Theorem 4·8. *For two events A and B*

$$P(A \cap B) = P(A) \cdot P(B \mid A), \; P(A) > 0$$
$$= P(B) \cdot P(A \mid B), \; P(B) > 0 \quad \ldots(4 \cdot 8)$$

where $P(B \mid A)$ represents the conditional probability of occurrence of B when the event A has already happened and $P(A \mid B)$ is the conditional probability of happening of A, given that B has already happened.

$P(A \cap B) = P(A)\, P(B|A) \Rightarrow P(B|A) = \frac{P(A \cap B)}{P(A)}$

$P(A \cap B) = P(B) \cdot P(A|B)$

$\Rightarrow P(A|B) = \frac{P(A \cap B)}{P(B)}$

Example 2.25 Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include an extra battery, and 30% include both a card and battery. Consider randomly selecting a buyer and let $A = \{\text{memory card purchased}\}$ and $B = \{\text{battery purchased}\}$. Then $P(A) = .60$, $P(B) = .40$, and $P(\text{both purchased}) = P(A \cap B) = .30$. Given that the selected individual purchased an extra battery, the probability that an optional card was also purchased is

0.75

extra info already occ.

$P(O) = 0.6 \qquad P(B) = 0.4 \qquad P(O \cap B) = 0.3$

$$P(O|B) = \frac{P(O \cap B)}{P(B)} = \frac{0.3}{0.4} = \frac{3}{4} = 0.75$$

Example 2.27 Four individuals have responded to a request by a blood bank for blood donations. None of them has donated before, so their blood types are unknown. Suppose only type O+ is desired and only one of the four actually has this type. If the potential donors are selected in random order for typing, what is the probability that at least three individuals must be typed to obtain the desired type?

HW

try it

Making the identification $B = \{\text{first type not O+}\}$ and $A = \{\text{second type not O+}\}$, $P(B) = \frac{3}{4}$. Given that the first type is not O+, two of the three individuals left are not O+, so $P(A|B) = \frac{2}{3}$. The multiplication rule now gives

$$\begin{aligned} P(\text{at least three individuals are typed}) &= P(A \cap B) \\ &= P(A|B) \cdot P(B) \\ &= \frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12} \\ &= .5 \end{aligned}$$

→ answer

~~The multiplication rule is most useful when the experiment consists of several stages in succession. The conditioning event B then describes the outcome of the first stage and A the outcome of the second, so that $P(A|B)$—conditioning on what occurs first—will often be known. The rule is easily extended to experiments involving more than two stages. For example,~~

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P(A_3 | A_1 \cap A_2) \cdot P(A_1 \cap A_2) \\ &= P(A_3 | A_1 \cap A_2) \cdot P(A_2 | A_1) \cdot P(A_1) \end{aligned} \quad (2.4)$$

~~where A_1 occurs first, followed by A_2, and finally A_3.~~

4·7·3. Independent Events. *An event B is said to be independent (or statistically independent) of event A, if the conditional probability of B given A i.e., P (B | A) is equal to the unconditional probability of B, i.e., if*

$$P(B|A) = P(B)$$

prob. of B is unaffected by happening of event A or not

then B is indep of A, They are indep. events.

A cannot effect the occurance of B.

Independent Events

An independent event is **unrelated** to the likelihood of another event occurring (or not happening). In other words, the occurrence has no bearing on the probability that a subsequent event will occur. There is no difference between independent events in probability and independent happenings in daily life. What colour car you drive has nothing to do with where you work. No lottery ticket purchase will result in a child with blue eyes.
One occurrence does not affect the likelihood of another event when the two events are independent.

no effect

Simple examples of independent events:

- Having a dog and maintaining a herb garden.
- Having enough milk and winning the lotto.
- Finding a penny on the ground after purchasing a lottery ticket (your odds of Finding a penny does not depend on you buying a lottery ticket).
- Finding your favourite movie on cable while taking a cab home.
- Getting a parking ticket and going to the casino to play craps.

$P(\text{win lott} \mid \text{more milk}) = P(\text{win lott})$ w/o condition

$\Rightarrow$ taking cab can't effect movies on cable

Dependent Events

When two events are reliant on one another, the likelihood of one event affects the probability of the other. An event is said to be dependent if it **needs another event to occur first**. Probabilistic dependent events are identical to dependent actual-world occurrences: If you want to go to a performance, it may rely on whether your employer will grant you time off, and if you want to visit family overseas next month, it may depend on your ability to obtain a passport in time. Formally, we can state that when two events are interdependent, one event's likelihood influences the other's likelihood.

Simple examples of dependent events:

- Going to jail after robbing a bank.
- Your power will be turned off if you don't pay your electricity payment on time.
- Seeking a decent seat before getting on a bus.
- Receiving a parking ticket for doing so. If you park illegally, your chances of receiving a ticket increase.
- Winning the lottery after purchasing ten tickets. Your chances of winning increase as you buy more tickets.
- Driving a car and being involved in a collision.

$P(B \mid A) \neq P(B)$

occurance of A has affected B
so B is dependent on A

$P(\text{going to jail} \mid \text{robbed}) > P(\text{going})$

$P(\text{collision} \mid \text{driving}) > P(\text{collision})$

Statistical independence

2 events are Statistically independent if

$P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$

mutually independent

Dependent or Independent Event Formula

P(A|B) = P(A) ✓
P(B|A) = P(B) ✓

A: getting a diamond card

B: getting a red card

$P(A) = 1/4$
$P(B) = 1/2$

$P(A \cap B) = 1/4$

$P(A \mid B) = \dfrac{1/4}{1/2} = 1/2$

$P(A \mid B) \neq P(A)$

DEPENDENT

$\ldots \cap C) = \frac{1}{52}$

C

getting an ace

$P(C) = \frac{1}{13}$

$P(A \cap C) = \frac{1}{52}$

$P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{1}{52}}{1/4} = \frac{4}{52} = \frac{1}{13}$

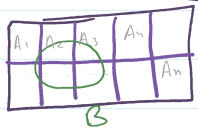
$P(C|A) = P(C)$ INDEPENDENT

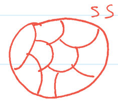
The Law of Total Probability

Let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k)$$
$$= \sum_{i=1}^{k} P(B|A_i)P(A_i) \qquad (2.5)$$

sample space





$P(A_i \cap A_j) = 0$

$A_1, A_2, \cdots, A_n$ mutually exclusive events & exhaustive

→ EXCLUSIVE MEANS INTERSECTION IS NULL

→ EXHAUSTIVE MEANS TOGETHER THEY COVER FULL SAMPLE SPACE

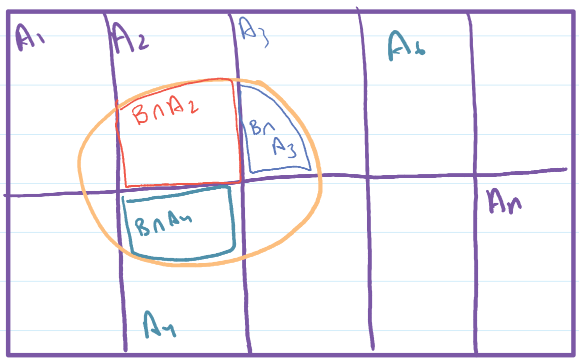
then for any other event B

$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$

$P(B) = \frac{P(B \cap A_1)}{P(A_1)} \cdot P(A_1) + \frac{P(B \cap A_2)}{P(A_2)} \cdot P(A_2) \cdots + \frac{P(B \cap A_n)}{P(A_n)} \cdot P(A_n)$

$$\boxed{P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)}$$

B



$P(B \cap A_6) = 0$

Bayes Theorem

The Bayes theorem establishes the likelihood of an event occurring given any condition. It is considered for the case of conditional probability. Also, this is known as the formula for the likelihood of "causes".

$$P(A|B) = \frac{P(B|A).P(A)}{P(B)}$$

A,B = Events
P(A|B) = Probability of A given B is true
P(B|A) = Probability of B given A is true
P(A), P(B) = The independent probabilities of A and B

$$P(A_1|B) = \frac{P(B|A_1)\cdot P(A_1)}{P(B)} = \frac{\frac{P(B\cap A_1)}{\cancel{P(A_1)}}\cdot \cancel{P(A_1)}}{P(B)}$$

$$= \frac{P(B\cap A_1)}{P(B)}$$

$$= P(A_1|B)$$

useful Baye's thm formula.

$$\boxed{P(A_1|B) = \frac{P(B|A_1)\cdot P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)}}$$

using above law of total prob.

Bayes Theorem: Example-2

Question: Think about the three machines. Each machine can produce one thousand pins at once. A defective pin will be produced at a rate of 10% from Machine 1, 20% from Machine 2, and 5% from Machine 3. What is the likelihood that ~~[illegible] would be defective~~?

when a defective pin comes it's from M_1

Solution:
Let us take the probability of choosing a faulty pin randomly be represented by P(A);
Pin choose from the first machine be represented by M1;
Pin choose from the second machine be represented by M2;
Pin choose from the third machine be represented by M3;
Chance of choosing pin any one of the three machines = P(M1) = P(M2) = P(M3) = 1/3
The probability of choosing a faulty pin from 1st machine is

$$P(M_1|A) = \frac{P(M_1)P(A|M_1)}{P(M_1)P(A|M_1)+P(M_2)P(A|M_2)+P(M_3)P(A|M_3)}$$

$$= \frac{(1/3)\times 0.1}{(1/3)\times 0.2 + (1/3)\times 0.1 + (1/3)\times 0.05}$$

$$= 2/7$$

let
$D \to$ defective
$P(M_1) = 1/3 = P(M_2) = P(M_3)$
$P(D|M_1) = 0.1$
$P(D|M_2) = 0.2$
$P(D|M_3) = 0.05$
$P(M_1|D) = ?$

$$P(M_1|D) = \frac{P(D|M_1)P(M_1)}{P(D|M_1)P(M_1) + P(D|M_2)P(M_2) + P(D|M_3)P(M_1)}$$

$$= \frac{0.1 \times 1/3}{0.1\times\frac{1}{3} + 0.2\times\frac{1}{3} + 0.05\times\frac{1}{3}}$$

$$= \frac{0.1}{0.35} = \frac{10}{35} = \frac{2}{7}$$

$$= \frac{0.1}{0.2 + 0.05}$$

$0.1 \times \frac{1}{3} +$

$= \frac{0.1}{0.1 + 0.2 + 0.05} = \frac{0.1}{0.35} = 35$ ✓

Example 2.30 An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2 and the remaining 10% into account #3. Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively. What is the probability that a randomly selected message is spam?

HW

To answer this question, let's first establish some notation:

$A_i = \{\text{message is from account \# } i\}$ for $i = 1, 2, 3$, $\quad B = \{\text{message is spam}\}$

[illegible]

Then the given percentages imply that

$$P(A_1) = .70,\ P(A_2) = .20,\ P(A_3) = .10$$
$$P(B|A_1) = .01,\ P(B|A_2) = .02,\ P(B|A_3) = .05$$

Now it is simply a matter of substituting into the equation for the law of total probability:

$$P(B) = (.01)(.70) + (.02)(.20) + (.05)(.10) = .016$$

In the long run, 1.6% of this individual's messages will be spam. ■

INTRO TO STATISTICAL INFERENCE

Sample Vs Population

A **population** in research is the complete group that you are interested in examining. This can refer to a group of individuals (such as all US adults or all employees of a corporation), but it can also refer to a group that includes other kinds of elements, such as things, occasions, organisations, nations, species, and organisms.