Adaptive Quadrature: Application on Simpson, Midpoint and Trapezoid method

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I. THEORY

The word *Quadrature* refers to the process of determining the area. Adaptive quadrature refers to the process of approximating the integral of a given function to a specified precision by "adaptively" subdividing the integration interval into smaller sub-intervals over which a set of local quadrature rules are applied. Traditionally we used composite rules for numerical integration in involves division of the interval into various subdivisions and summing the area under curve for each subinterval. However we note that in this method the width of each subinterval remains same and depends on N(number of interval) provided as input. Value of N has to be analytically calculated for a required accuracy. However this process seems complicated and increases time complexity, also this may not work for functions, now we are ready to address the question **why we need *adaptive quadrature*?**

A. Motivation

We need an algorithm which decides the width of subintervals depending on the nature of the function and changes their width through out the process as we go through the interval. It will assign higher widths in the region where the function shows less variation in its values and low widths in the subintervals where the functions shows high variation in its values. So motivation is **to reduce computational cost and to make a better approximation within the required accuracy when we are unaware of the subtleties of function.**

B. Algorithm/Pseudocode

Adaptive quadrature is not a different method it is just a strategy to use standard quadrature rules like Simpsons, Midpoint, Trapezoid effectively. It leverages error estimators to achieve desired level of accuracy in minimal cost. Say we need to find integral of f in interval $[a, b]$ by applying adaptive quadrature to either Simpson, Midpoint or Trapezoid then the following pseudocode could be followed. Note that the factor K is a number which depends on what method out of Simpsons, Midpoint and Trapezoid is used.

Pseudocode: f : function, $[a, b]$: interval, ε : tolerance for error.
Compute $I_1[a, b]$ by using standard quadrature rule
and define $I_2 = I_1[a, c] + I_1[c, b]$
If $|I_2 - I_1| < K\varepsilon$
answer = Adaptive(f, a, c, ε)
    answer $= I_2 + (I_2 - I_1)/K$
else
    $c = (a + b)/2$
    L= Adaptive$(f, a, c, \varepsilon/2)$
    R=Adaptive$(f, c, b, \varepsilon/2)$
    answer=L+R
end

Value of K will be derived in the later section. For now one can note:

$$K = \begin{cases} 15 \rightarrow & Simpson \\ 3 \rightarrow & Midpoint \\ 3 \rightarrow & Trapezoid \end{cases}$$

II. DERIVATION

We use Simpson's $\frac{1}{3}$ rule which is the most widely used methods for numerical integration. Suppose that we want to approximate $\int_a^b f(x)dx$ within tolerance $\varepsilon > 0$. The first step is to apply Simpson's rule with step size $h = \frac{(b-a)}{2}$. This produces

$$\int_a^b f(x)dx = S(a, b) - \frac{h^5}{90} f^{(4)}(\xi) \tag{1}$$

where we denote the Simpson's rule approximation on $[a, b]$ by $S(a, b)$, and the second term gives the error in the quadrature.

$$S(a, b) = \frac{h}{3}[f(a) + 4f(a + h) + f(b)] \tag{2}$$

The next step is to determine an accuracy approximation that does not require $f^{(4)}(\xi)$. To do this, we apply the Composite Simpson's rule with $n = 4$ and step size $\frac{(b-a)}{4} = \frac{(h)}{2}$. This gives that : $\int_a^b f(x)dx$

$$= \frac{h}{6}\left[f(a) + 4f\left(a + \frac{h}{2}\right) + 2f(a + h) + 4f\left(a + \frac{3h}{2}\right) + f(b)\right] - \left(\frac{h}{2}\right)^4 \frac{(b-a)}{180} f^{(4)}(\bar{\xi}) \qquad \text{for some } \bar{\xi} \text{ in } (a, b) \tag{3}$$

The above equation can be given as:

$$\int_a^b f(x)dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\bar{\xi}) \quad (4)$$

Caveat: The error estimation is derived by assuming that $\xi \approx \bar{\xi}$ or, more precisely, that $\mathbf{f}^{(4)}(\xi) \approx \mathbf{f}^{(4)}(\bar{\xi})$, and the success of the technique depends on the accuracy of this assumption.

If it is accurate, then equating the integrals in Eqs. (1) and (4) gives

$$\begin{aligned} &S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\xi) \\ &\approx S(a,b) - \frac{h^5}{90} f^{(4)}(\xi) \end{aligned} \quad (5)$$

Hence

$$\frac{h^5}{90} f^{(4)}(\xi) \approx \frac{16}{15}\left[S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right] \quad (6)$$

Using this in Eq. (4) produces the error estimation

$$\begin{aligned} &\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \\ &\approx \frac{1}{16}\left(\frac{h^5}{90}\right) f^{(4)}(\xi) \\ &\approx \frac{1}{15}\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \end{aligned} \quad (7)$$

This implies that $S(a, \frac{(a+b)}{2}) + S(\frac{(a+b)}{2}, b)$ approximates $\int_a^b f(x)dx$ about 15 times better than it agrees with the computed value $S(a,b)$. Thus, if

$$\left|S(a,b) - S\left(a \cdot \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| < 15\varepsilon \quad (8)$$

we expect to have

$$\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| < \varepsilon \quad (9)$$

and $S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right)$ is assumed to be a sufficiently accurate approximation to $\int_a^b f(x)dx$.

If this does not work for any sub-interval, we again break that particular sub-interval into halves and then work out the same procedure again and so on. It could be clearly noted that if the above calculation is repeated for Trapezoid method and Midpoint method 15 in Eq.II would be replaced by 3. That is the reason we get K = 3 for Trapezoid and Midpoint as mentioned in the pseudocode.

III. ILLUSTRATION: π VALUE ESTIMATION

In this section we will solve an numerical integral example by using both adaptive quadrature applied to Simpson's method and Simpson rule without adaptive quadrature and we show that the adaptive quadrature applied Simpson has higher accuarcy and requires lesser number of subintervals: We will apply Simpson's rule to

$$\int_0^1 \frac{4}{1+x^2} dx = \pi$$

Here, S_{normal} denotes Simpson rule and $S_{adaptive}$ will denote Simpson rule with adaptive method. The values were calculated by using the python code attached with this report.

$$\boxed{S_{normal} = 3.14159265122482222}$$

$$\boxed{S_{adaptive} = 3.141592653708037}$$

Analytical Value of the integral is π = 3.141592653589793. Clearly Adaptive method is more accurate. The estimate for the error obtained:

$$\frac{1}{15}|S_{normal} - S_{adaptive}| = \boxed{1.6554766446574832 \times 10^{-10}}$$

which closely approximates the actual error

$$|\pi - 3.141592653708037| = \boxed{1.1824408119309737 \times 10^{-10}}$$

IV. CONCLUSION

This integral was also calculated using the program for Adaptive Quadrature in python and the number of subintervals divisions required for normal simpson method is **N =28** and for adaptive simpson method is **N=7** which is **four times small**. Thus Adaptive method reduces the time complexity and the amount of subintervals needed. Thus it is an efficient method.

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