Solved Example Problems

Practical Quantum Computation using Qiskit and IBMQ

Introduction

The following document contains some solved problems that shall illustrate the concepts discussed during the lectures. The solutions to these problems are also intended to provide hints for solving the quiz questions. Readers are requested to go through slides to clarify notations

1 Qubit states and Basis representations

It should be remembered that for a given basis $\{|v_0\rangle, |v_1\rangle\}$ for the space $\mathbb{C}^2$, a vector $|\psi\rangle = a|v_0\rangle + b|v_1\rangle$ is said to be in a **superposition** of the basis vectors if $a, b \neq 0$.

In the computational basis $\{|0\rangle, |1\rangle\}$, the vectors $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\frac{1}{\sqrt{3}}|0\rangle + |1\rangle$ and $\sqrt{2}|0\rangle + |1\rangle$ are in a superposition of the basis vectors. The vectors $|0\rangle$, $2|1\rangle$ and $|+\rangle - |\mathbf{i}\rangle$ are not in a superposition of the computational basis vectors.

Next, it is important to ensure that a given vector in $\mathbb{C}^2$ is a **valid qubit state**. A vector $|\psi\rangle$ is a valid qubit state if and only if $\langle\psi|\psi\rangle = 1$.

For example, the vector $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is a valid qubit state. This can be seen as follows,

$$\begin{aligned}
\langle+|+\rangle &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)(|0\rangle + |1\rangle) \\
&= \frac{1}{2}\langle 0|0\rangle + \frac{1}{2}\langle 0|1\rangle + \frac{1}{2}\langle 1|0\rangle + \frac{1}{2}\langle 1|1\rangle \\
&= \frac{1}{2} + \frac{1}{2} = 1
\end{aligned} \tag{1}$$

The vector $|\phi\rangle = 2|0\rangle + |1\rangle$ is not a valid qubit state since,

$$
\begin{aligned}
\langle\phi|\phi\rangle &= (2\langle 0| + \langle 1|)(2|0\rangle + |1\rangle) \\
&= 4\langle 0|0\rangle + 2\langle 0|1\rangle + 2\langle 1|0\rangle + \langle 1|1\rangle \\
&= 4 + 1 = 5
\end{aligned} \tag{2}
$$

However, the vector $|\tilde{\phi}\rangle = \frac{1}{\sqrt{\langle\phi|\phi\rangle}}|\phi\rangle = \frac{1}{\sqrt{5}}|\phi\rangle$ is a valid qubit state, this process is called normalization. The normalized state $|\tilde{\phi}\rangle$ is given as,

$$
|\tilde{\phi}\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle
$$

2 Measurement of Qubit states

When a state $|\psi\rangle$ is measured with respect to the basis $\{|v_0\rangle, |v_1\rangle\}$, then the outcome of measurement will **either be $|v_0\rangle$ or $|v_1\rangle$**.

The probability of the outcome being $|v_0\rangle$ is given by $|\langle v_0|\psi\rangle|^2$ and the probability of the outcome being $|v_1\rangle$ is given by $|\langle v_1|\psi\rangle|^2$

For instance, measuring the state$|\phi\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$ with respect to the computational basis $\{|0\rangle, |1\rangle\}$ gives the outcome $|0\rangle$ with probability $|\langle 0|\phi\rangle|^2$, this can be evaluated as,

$$
\begin{aligned}
\langle 0|\phi\rangle &= \langle 0|\left(\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\right) \\
&= \frac{2}{\sqrt{5}}\langle 0|0\rangle + \frac{1}{\sqrt{5}}\langle 0|1\rangle \\
&= \frac{2}{\sqrt{5}} \\
\therefore\ |\langle 0|\phi\rangle|^2 &= \frac{4}{5}
\end{aligned} \tag{3}
$$

The probability of observing $|1\rangle$ is given by $|\langle 1|\phi\rangle|^2$

$$
\begin{aligned}
\langle 1|\phi\rangle &= \langle 1|\left(\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle\right) \\
&= \frac{2}{\sqrt{5}}\langle 1|0\rangle + \frac{1}{\sqrt{5}}\langle 1|1\rangle \\
&= \frac{1}{\sqrt{5}} \\
\therefore\ |\langle 0|\phi\rangle|^2 &= \frac{1}{5}
\end{aligned} \tag{4}
$$

The outcome of the measurement yields the results $|0\rangle$ or $|1\rangle$ with probabilities $\frac{4}{5}$ and $\frac{1}{5}$ respectively.

3 Single qubit operators

The important quantities to remember here are the outer products $|0\rangle\langle 0|$, $|0\rangle\langle 1|$, $|1\rangle\langle 0|$ and $|1\rangle\langle 1|$. These quantities are evaluated as

$$\begin{aligned} |0\rangle\langle 0| &= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \\ |0\rangle\langle 1| &= \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \\ |1\rangle\langle 0| &= \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \\ |1\rangle\langle 1| &= \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \end{aligned} \tag{5}$$

The Z gate, $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ can be represented as $|0\rangle\langle 0| - |1\rangle\langle 1|$. The Hadamard gate $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ can be represented $\frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$

The action of operators on the state is obtained through matrix multiplication. Consider the $|0\rangle$ state to which the X gate is applied first and then the Hadamard gate is applied. The action of these operations is given by:

$$\begin{aligned} X\,|0\rangle &= \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \end{aligned} \tag{6}$$

Therefore, the action of the X gate followed by the Hadamard gate gives the state $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$.

4 Photons and Polarizers

The first point to remember while solving these problems is that the transmission probability is $|\cos\theta|^2$ where θ is the angle between the polarizer and the incident light polarization.

Secondly, The light that is transmitted by a polarizer has its polarization aligned to the direction of the polarizer through which it was transmitted . As an example, Consider

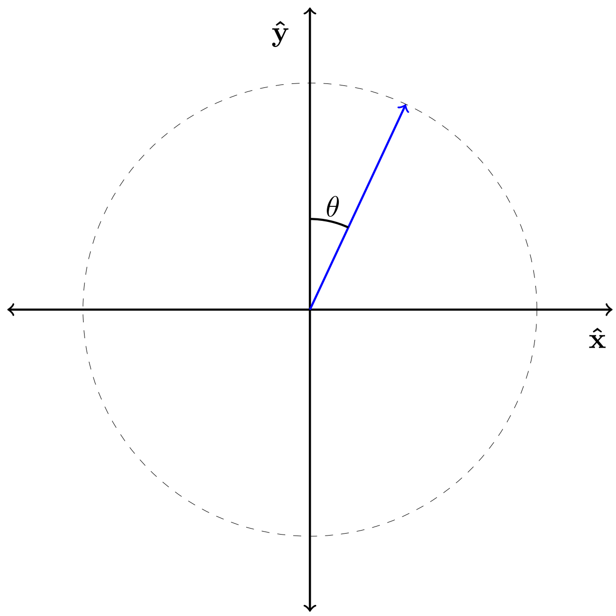

Figure 1: Diagram showing the positive angle convention.

a vertically polarized light beam incident on a polarizer (P1) oriented at 30°, the light transmitted from P1 is incident on a polarizer (P2) oriented at 60°

The transmission probability for P1 is given by $|\cos 30°|^2 = 3/4$. The light that is transmitted is now oriented at 30° and therefore the angle between the light polarization and the polarizer P2 is once again 30° with a transmission probability of 3/4 the transmission probability for the combined P1-P2 system is given by $3/4 \times 3/4 = 9/16$.

In addition to these rules, knowledge of the following trigonometric identities may be helpful:

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta) \\
\cos(2\theta) &= 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)
\end{aligned}$$