# Quiz Assignment 2: Solutions

1. Refer to the expression for the representation of a general qubit state on the Bloch sphere mentioned in the lecture. The values of $\theta$ and $\phi$ for a state $|\psi\rangle = \frac{i\sqrt{3}}{2}|1\rangle + \frac{1}{2}e^{i\pi/4}|0\rangle$ are

| | $\theta$ | $\phi$ |
|---|---|---|
| **A** | $\boldsymbol{\frac{2\pi}{3}}$ | $\boldsymbol{\frac{\pi}{4}}$ |
| B | $-\frac{\pi}{4}$ | $\frac{2\pi}{3}$ |
| C | $\frac{\pi}{3}$ | $-\frac{\pi}{4}$ |
| D | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ |

**Solution: A**

The expression for a general single-qubit state on the Bloch sphere is $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|0\rangle$ . Removing the global phase from the state provided, we get

$$|\psi\rangle = e^{\frac{i\pi}{4}}\left(\frac{1}{2}|0\rangle + e^{-\frac{i\pi}{4}}\frac{i\sqrt{3}}{2}|1\rangle\right) \approx \left(\frac{1}{2}|0\rangle + e^{i\left(\frac{\pi}{2}-\frac{\pi}{4}\right)}\frac{\sqrt{3}}{2}|1\rangle\right) = \left(\frac{1}{2}|0\rangle + e^{\frac{i\pi}{4}}\frac{\sqrt{3}}{2}|1\rangle\right)$$

$$\cos\left(\frac{\theta}{2}\right) = \frac{1}{2}, \sin\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\phi = \frac{\pi}{4}$$

2. The transformation $Y$ can be written as (according to the convention used in the slides)
   A. $-|1\rangle\langle 1| + |0\rangle\langle 0|$
   B. $|0\rangle\langle 1| + |1\rangle\langle 0|$
   **C.** $\boldsymbol{-|1\rangle\langle 0| + |0\rangle\langle 1|}$
   D. $-|0\rangle\langle 1| + |1\rangle\langle 1|$

**Solution: C**

$Y: \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -|1\rangle\langle 0| + |0\rangle\langle 1|$ (from lecture slides)

3. The transformation $X$ can be written as
   A. $-|1\rangle\langle 1| + |0\rangle\langle 0|$
   **B.** $\boldsymbol{|0\rangle\langle 1| + |1\rangle\langle 0|}$
   C. $-|1\rangle\langle 0| + |0\rangle\langle 1|$
   D. $-|0\rangle\langle 1| + |1\rangle\langle 1|$

**Solution: B**

$X: \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0| + |0\rangle\langle 1|$ (from lecture slides)

4. If we apply the Hadamard transform on $|0\rangle$ the result is
   A. $|1\rangle$
   B. $\frac{1}{2}(|0\rangle + |1\rangle)$
   **C.** $\boldsymbol{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}$
   D. $|0\rangle + |1\rangle$

**Solution: C**

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ by definition

5. Starting from $|0\rangle$,If we apply the Hadamard transform and then apply the $Z$ transform, the resulting single-qubit quantum state is
   A. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   **B.** $\boldsymbol{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}$
   C. $\frac{\sqrt{3}}{2}(|0\rangle - |1\rangle)$
   D. $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

**Solution: B**

$$ZH|0\rangle = Z[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)] = \frac{1}{\sqrt{2}}(Z|0\rangle + Z|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

6. Starting from $|1\rangle$, if we apply the Hadamard transform and then apply the $X$ transform, the resulting single-qubit quantum state is
   A. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   **B.** $\boldsymbol{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}$
   C. $\frac{\sqrt{3}}{2}(|0\rangle - |1\rangle)$
   D. $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

   **Solution: B**

   $XH|1\rangle = X[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)] = \frac{1}{\sqrt{2}}(X|0\rangle - X|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = \frac{-1}{\sqrt{2}}(|0\rangle - |1\rangle) \approx \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ (The last two expressions are the same state as they differ only by a global phase.)

For each pair consisting of a state and a measurement basis, describe the possible measurement outcomes and give the probability for each outcome.

7. Qubit state: $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$, Measurement basis $\{|0\rangle, |1\rangle\}$. Choices:

| | Probability of $|0\rangle$ | Probability of $|1\rangle$ |
|---|---|---|
| **A** | **3/4** | **1/4** |
| B | 1/4 | 3/4 |
| C | 1/2 | 1/2 |
| D | $\sqrt{3}/2$ | 1/4 |

**Solution: A**

$$P_0 = |\langle 0|\psi\rangle|^2 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$$

$$P_1 = |\langle 1|\psi\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

8. Qubit state: $|\psi\rangle = \frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle$, Measurement basis $\{|0\rangle, |1\rangle\}$.Choices:

| | Probability of $|0\rangle$ | Probability of $|1\rangle$ |
|---|---|---|
| **A** | $\mathbf{1/4}$ | $\mathbf{3/4}$ |
| B | $1/2$ | $1/2$ |
| C | $\sqrt{3}/2$ | $1/4$ |
| D | $3/4$ | $1/4$ |

**Solution: A**

$$P_0 = |\langle 0|\psi\rangle|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

$$P_1 = |\langle 1|\psi\rangle|^2 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}$$

9. Qubit state: $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$, Measurement basis $\{|+\rangle, |-\rangle\}$. Choices:

| | Outcome | Probability | Outcome | Probability |
|---|---|---|---|---|
| A | $|0\rangle$ | $\frac{2-\sqrt{3}}{4}$ | $|1\rangle$ | $\frac{2+\sqrt{3}}{4}$ |
| B | $|0\rangle$ | $\frac{2+\sqrt{3}}{4}$ | $|1\rangle$ | $\frac{2-\sqrt{3}}{4}$ |
| **C** | $|+\rangle$ | $\mathbf{\frac{2-\sqrt{3}}{4}}$ | $|-\rangle$ | $\mathbf{\frac{2+\sqrt{3}}{4}}$ |
| D | $|+\rangle$ | $\frac{2+\sqrt{3}}{4}$ | $|-\rangle$ | $\frac{2-\sqrt{3}}{4}$ |

**Solution: C**

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle = \frac{\sqrt{3}}{2\sqrt{2}}(|+\rangle + |-\rangle) - \frac{1}{2\sqrt{2}}(|+\rangle - |-\rangle)$$
$$= \frac{\sqrt{3}-1}{2\sqrt{2}}|+\rangle + \frac{\sqrt{3}+1}{2\sqrt{2}}|-\rangle)$$

$$P_+ = |\langle +|\psi\rangle|^2 = \left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right|^2 = \frac{2-\sqrt{3}}{4}$$

$$P_- = |\langle -|\psi\rangle|^2 = \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{2+\sqrt{3}}{4}$$

10. Qubit state: $|\psi\rangle = |-\mathbf{i}\rangle$, Measurement basis $\{|0\rangle, |1\rangle\}$.

| | Outcome | Probability | Outcome | Probability |
|---|---|---|---|---|
| A | $\|0\rangle$ | $\frac{1+\mathbf{i}}{2}$ | $\|1\rangle$ | $\frac{1-\mathbf{i}}{2}$ |
| B | $\|\mathbf{i}\rangle$ | $\frac{1-\mathbf{i}}{2}$ | $\|-\mathbf{i}\rangle$ | $\frac{1-\mathbf{i}}{2}$ |
| C | $\|\mathbf{i}\rangle$ | $\frac{1}{2}$ | $\|-\mathbf{i}\rangle$ | $\frac{1}{2}$ |
| **D** | $\|\mathbf{0}\rangle$ | $\boldsymbol{\frac{1}{2}}$ | $\|\mathbf{1}\rangle$ | $\boldsymbol{\frac{1}{2}}$ |

**Solution: D**

$$|\psi\rangle = |-\mathbf{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - \mathbf{i}|1\rangle)$$

$$P_0 = |\langle 0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

$$P_1 = |\langle 1|\psi\rangle|^2 = \left|\frac{-\mathbf{i}}{\sqrt{2}}\right|^2 = \frac{1}{2}$$