# Quantum Computing Course

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# Module 1

Lecture 1: Introducing qubits

- Learn the definition of a qubit
- Decide whether a complex vector represents a qubit-state or not
- Learn the difference between a bit and a qubit

# Qubits

- A quantum bit or a qubit is a fundamental unit of quantum information processing just as a bit is a fundamental unit of classical information processing.

- A single qubit state is represented by a pair of complex numbers $\begin{pmatrix} a \\ b \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$.

- So a single qubit can exist in an infinite number of states whereas a bit can exist in either in $0$ state or $1$ state.

# Writing conventions

- It is customary to write $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- Then, a single qubit state is

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a|0\rangle + b|1\rangle$$

- We must not forget that

$$|a|^2 + |b|^2 = 1$$

# Recalling complex numbers

- A complex number is written as $z = x + \mathbf{i}\, y$ where $x, y$ are real numbers, and $\mathbf{i}^2 = -1$.

- The conjugate of $z$ is $\bar{z} = x - \mathbf{i}y$.

- The modulus of a complex number is $|z|$ where

$$|z|^2 = z\bar{z} = x^2 + y^2$$

# A single-bit system

- A single-bit system can exist in one of the two states: $0$ and $1$. Such a system can be visualized as


- In a classical computer it is possible to set a bit to the $0$ or $1$ state. It is also possible to read (measure) that state, and reading from a bit does not change its state.
- On a quantum computer it is possible to create a single-qubit state, but it is not possible to measure it without changing the state.

# Physical realization of a Qubit

## Practical Quantum Computing using Qiskit and IBMQ

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# Outline

# Information and Computation

- The classical bit may exist in one of two states, these states are labelled as “0” and “1”. This is the most fundamental unit of information.
- A physical realization of the bit is required for performing computation. In a classical computer, bits are realised in the states of a register.
- By the same token, a physical realization of a qubit is required for performing quantum computation.

# Polarization of light waves

- The polarization of an electromagnetic wave propagating along the $z$-axis is in the $xy$-plane.
- Any vector in the $xy$-plane can be represented in terms of its $x$ and $y$ components.

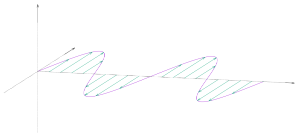

Figure 1: Horizontally Polarized Wave

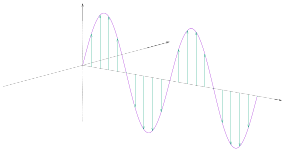

Figure 2: Vertically Polarized Wave

# A little bit of Vector Algebra

- The linearly independent vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to represent vertical and horizontal polarizations respectively.
- A polarization vector inclined at an angle $\theta$ to the vertical is represented as follows:

$$\cos\theta \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad (1)$$

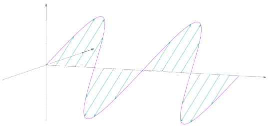

Figure 3: An obliquely polarized wave

# Fun with polarizers

- Polarizers are optical filters that allow light waves of a particular polarization to pass through them.
- A vertically aligned polarizer will block all horizontally polarized light from passing through and vice versa.
- But what will happen when an obliquely polarized light wave is incident on a vertically (or horizontally) aligned polarizer?
- The following video answers the above question through demonstrations: `https://www.youtube.com/watch?v=6N3bJ7Uxpp0`

# A classical result

- When a light wave with polarization as defined in equation (1) is incident on a vertically aligned polarizer, only a fraction of this wave emerges.
- The fraction of the intensity that is transmitted is given by $|\cos\theta|^2$. In the case of a horizontally aligned polarizer, the same fraction is $|\sin\theta|^2$.
- In general, the fraction of the intensity that is transmitted is given by $|\cos\alpha|^2$ where $\alpha$ is the angle between the polarization vector and the direction along which the polarizer is aligned.
- It should be remembered that the polarization of the transmitted light wave is along the alignment of the polarizer.

# The Quantum Perspective

- One of the earliest conclusions of quantum theory was that light waves exhibited particle like behaviour and these particles were named photons.
- Light waves discussed thus far may be visualised as a stream of a very large number of photons.
- The reduction of intensity when a light wave passes through a polarizer implies that only a fraction of the incident photons are transmitted. This fraction as mentioned before is $|\cos\alpha|^2$.
- It is once again emphasized that the polarization of the photons emerging is along the direction in which the polarizer is aligned.

# The Photon as a Qubit

- Based on the ideas discussed so far, it is possible to state the following about single photons incident on a polarizer.
- It is in general not possible to state with certainty if an incident photon will be transmitted by a polarizer.
- A photon polarized along the direction of a polarizer will certainly be transmitted. A photon with a polarization that is orthogonal to a polarizer's alignment will certainly not be transmitted.
- The probability of a photon being transmitted by a polarizer is once again given by $|\cos \alpha|^2$.
- The aforementioned facts have all been verified through experiments.

# Concluding Remarks

- The photon is a particle that can exist simultaneously in both (orthogonal) polarization states.
- The polarization of the photon may be represented as a linear combination of these orthogonal states.
- The outcome of an experiment to estimate the polarization of a photon can only be interpreted statistically. The experiment also leaves the state of the photon changed.
- Such photon based qubits are used extensively in Quantum Information and Communication, most notably in Quantum Key Distribution (QKD).

# But what of the complex numbers?

- The photon states described so far have all been real linear combinations of the vertical and the horizontal polarization states.
- Complex linear combination of these states are used to define polarization states such as circular and elliptical polarization.

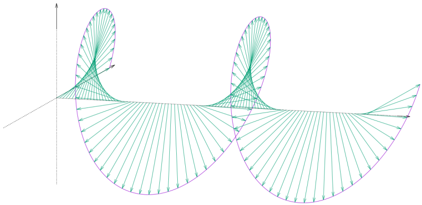

Figure 4: A circularly polarized wave

# Module 1

Lecture 2: Mathematical Preliminaries

- Definition of a basis of $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$
- Perform the inner product operation on $\mathbb{C}^2$
- Orthonormal basis
- Dirac notation
- Superposition of states

# Quantum bits – Qubits

- Quantum bits – Qubits :

  A qubit is the fundamental unit of quantum information just as a bit is the fundamental unit of classical information.

  - A bit can exist in two states: $0$ and $1$.

  - A qubit is a vector having two complex components.

    Consider the vector space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{C} \right\}$.

    A vector of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ defines a state of a qubit if and only if

$$|a|^2 + |b|^2 = 1.$$

# Basis of $\mathbb{C}^2$

- The set of vectors $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is said to be a basis is $\mathbb{C}^2$ since any element in $\mathbb{C}^2$ can be written uniquely as a linear combination
$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}.$$

- Any set of vectors with this property is said to be a basis of $\mathbb{C}^2$.

  For example: $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$, $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\mathbf{i}\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathbf{i}\end{pmatrix}\right\}$

  where $\mathbf{i}^2 = -1$.

# Inner product on $\mathbb{C}^2$

- Inner product of two vectors $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{C}^2$ is

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d.$$

- Two vector are said to be orthogonal if

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d = 0.$$

# Orthonormal basis of $\mathbb{C}^2$

- Suppose $\left\{\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right\}$ is a basis such that

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d = 0$$

and

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} a \\ b \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} a \\ b \end{pmatrix} = \bar{a}a + \bar{b}b = |a|^2 + |b|^2 = 1$$

$$\begin{pmatrix} c \\ d \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{c} \quad \bar{d}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{c}c + \bar{d}d = |c|^2 + |d|^2 = 1$$

# Orthonormal basis of $\mathbb{C}^2$

- Computational basis: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\bar{1} \quad \bar{0}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{1}0 + \bar{0}1 = 0$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\dagger} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\bar{1} \quad \bar{0}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bar{1}1 + \bar{0}0 = 1$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\dagger} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\bar{0} \quad \bar{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{0}0 + \bar{1}1 = 1$$

# Orthonormal basis of $\mathbb{C}^2$: Examples

- Hadamard basis: $\mathcal{H} = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$.

- Nega-Hadamard basis: $\mathcal{N} = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\mathbf{i}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathbf{i}\end{pmatrix}\right\}$.

- Verify that $\mathcal{H}$ and $\mathcal{N}$ are orthonormal bases.

# Dirac's bra/ket notation

- A vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ is written as $|\psi\rangle$ read as "*ket psi".*

- The vector $\begin{pmatrix} a \\ b \end{pmatrix}^\dagger = (\bar{a} \quad \bar{b})$ is written as $\langle\psi|$.

- Inner product of two vectors $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$, and $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ is

  $\langle\psi|\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d.$

  *The order in the which $|\phi\rangle$ and $|\psi\rangle$ appear matters. This is the inner product of $|\phi\rangle$ and $|\psi\rangle$ and not $|\psi\rangle$ and $|\phi\rangle$.*

# Computational, Hadamard and Nega-Hadamard Bases in Dirac's notation

- Computational basis: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- Hadamard basis: $|+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

- Nega-Hadamard basis:

  $|\mathbf{i}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} = \frac{|0\rangle + \mathbf{i}|1\rangle}{\sqrt{2}}, |-\mathbf{i}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} = \frac{|0\rangle - \mathbf{i}|1\rangle}{\sqrt{2}}$
- Verify that all the above bases are orthonormal.

# Superposition of states

- The state of a single-qubit is of the form

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a|0\rangle + b|1\rangle$$

where $|a|^2 + |b|^2 = 1$.

- If $a \neq 0$ and $b \neq 0$ the qubit is said to be in the superposition of two states $|0\rangle$ and $|1\rangle$.

# Once a superposition, always a superposition?

NO

- $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ is a superposition of two states $|0\rangle$, and $|1\rangle$.
- We say that $|\psi\rangle$ is in superposition with respect to the basis $\{|0\rangle, |1\rangle\}$.
- However, the representation of $|\psi\rangle$ with respect to the basis $\mathcal{H} = \{|+\rangle, |-\rangle\}$ is $|\psi\rangle = |+\rangle$.
- Therefore, $|\psi\rangle$ is not in superposition with respect to the basis $\mathcal{H}$.

# Changing a Qubit representation from computational to Hadamard basis

- $|\psi\rangle = a|0\rangle + b|1\rangle$ is a single-qubit state written in computational basis.
- The Hadamard basis vectors in terms of computational basis vectors are:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

- Solving for $|0\rangle$ and $|1\rangle$ yields:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \qquad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}.$$

- $|\psi\rangle = a\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) + b\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) = \frac{a+b}{\sqrt{2}}\ |+\rangle + \frac{a-b}{\sqrt{2}}\ |-\rangle.$

# Global phase versus relative phase

- Two single-qubit states $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$ are said to differ by the global phase $\theta$ if

$$|\psi\rangle = a|0\rangle + b|1\rangle = e^{\mathbf{i}\theta}(c|0\rangle + d|1\rangle) = e^{\mathbf{i}\theta}\,|\phi\rangle.$$

- If two quantum states differ by a global phase, they are considered to be same. We write $|\psi\rangle \sim |\phi\rangle$.

- The relative phase of a single-qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ is a number $\varphi$ which satisfies the equation

$$\frac{a}{b} = e^{\mathbf{i}\varphi}\,\frac{|a|}{|b|}.$$

- Two quantum states with different relative phases are not the same quantum state.

# Examples of qubits differing by a global phase

- Consider: $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{\mathbf{i}\pi}{4}}|1\rangle\right)$ and $\frac{1}{\sqrt{2}}\left(e^{-\frac{\mathbf{i}\pi}{4}}|0\rangle + |1\rangle\right)$
- The qubit state $\frac{1}{\sqrt{2}}\left(e^{-\frac{\mathbf{i}\pi}{4}}|0\rangle + |1\rangle\right) = \frac{e^{-\frac{\mathbf{i}\pi}{4}}}{\sqrt{2}}\left(|0\rangle + e^{\frac{\mathbf{i}\pi}{4}}|1\rangle\right)$
- Therefore, these two quantum states are the same.

# Examples of qubits differing by relative phases

- Consider: $\frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(-|0\rangle + \mathbf{i}|1\rangle)$

- Let $$a|0\rangle + b|1\rangle = \frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle)$$
  and $$a'|0\rangle + b'|1\rangle = \frac{1}{\sqrt{2}}(-|0\rangle + \mathbf{i}|1\rangle).$$
  $\frac{a}{b} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{1} = e^{0\mathbf{i}}\frac{|a|}{|b|}$, and $\frac{a'}{b'} = -\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\mathbf{i}} = -\frac{1}{\mathbf{i}} = \mathbf{i} = e^{\frac{\pi\mathbf{i}}{2}}\frac{|a'|}{|b'|}$.

  By definition the relative phase of the first qubit is $0$ and the relative phase of the second qubit is $\frac{\pi}{2}$. Since they have different relative phases they are different quantum states.