Single Qubit states and their visualization

Practical Quantum Computing using Qiskit and IBMQ

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Outline

The vector space $\mathbb{C}^2$

- A vector in the space $\mathbb{C}^2$ is represented as follows:

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \; ; \; a, b \in \mathbb{C}$$

- If $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$, the inner products $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$ are defined as:

$$\langle\phi|\psi\rangle = \begin{pmatrix} \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a\bar{c} + b\bar{d}$$

  and

$$\langle\psi|\phi\rangle = \left\{ \begin{pmatrix} \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\}^\dagger = \bar{a}c + \bar{b}d = \overline{\langle\phi|\psi\rangle}$$

Normalization and the Qubit state

- Consider the quantity $\langle\psi|\psi\rangle$ which has a value
$$\langle\psi|\psi\rangle = |a|^2 + |b|^2$$
- Therefore a vector $|\psi\rangle$ is a qubit state if $\langle\psi|\psi\rangle = 1$.
- A general vector $|\phi\rangle$ can be converted to a qubit state $|\tilde{\phi}\rangle$ as follows:
$$|\tilde{\phi}\rangle = \frac{1}{\langle\phi|\phi\rangle^{\frac{1}{2}}}|\phi\rangle$$
- The above process is referred to as normalization and the quantity $\langle\phi|\phi\rangle^{\frac{1}{2}}$ is called the norm of the vector $|\phi\rangle$.
- Qubit states can now be formally defined as vectors in $\mathbb{C}^2$ with a unit norm.

Basis States

- Defining the vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the vector $|\psi\rangle$ is now expressed as:

  $$|\psi\rangle \;=\; a\,|0\rangle + b\,|1\rangle$$

  It is possible to represent any vector in $\mathbb{C}^2$ is the above manner.
- The set $\{|0\rangle\,, |1\rangle\}$ is called the standard or the computational basis and is said to span $\mathbb{C}^2$.
- The inner products have the values; $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.
- A basis satisfying the above property is known as an orthonormal basis. It is worth noting that orthonormal basis vectors are valid qubit states.

Coordinates and Projections

- Considering the vector $|\psi\rangle$ and the computational basis $\{|0\rangle, |1\rangle\}$, the following is true

$$\langle 0|\psi\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a \,;\; \langle 1|\psi\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = b$$

- The coordinates of the vector $|\psi\rangle$ can be defined in terms of the inner product with the basis vectors.
- The inner product of $|\psi\rangle$ with a basis vector is known as the projection of $|\psi\rangle$ along that basis vector.
- The vector $|\psi\rangle$ can now be represented in terms of the projections as follows:

$$|\psi\rangle = \langle 0|\psi\rangle |0\rangle + \langle 1|\psi\rangle |1\rangle$$

"Non-standard" Basis

- The idea of coordinates and projections is true for any orthonormal basis.
- Consider the basis orthonormal basis $\{|+\rangle, |-\rangle\}$, where $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$.
- In this basis, the vector $|\psi\rangle$ is represented as:

$$|\psi\rangle = \langle+|\psi\rangle |+\rangle + \langle-|\psi\rangle |-\rangle$$

evaluating the inner products give the result

$$|\psi\rangle = \frac{a+b}{\sqrt{2}} |+\rangle + \frac{a-b}{\sqrt{2}} |-\rangle$$

Projections and Photons

- The state of an obliquely polarized photon $|\chi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, this state can be represented in the standard basis as:

$$|\chi\rangle \; = \; \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

  where the basis vectors $|0\rangle$, $|1\rangle$ represent the vertical and horizontal polarization states.
- The probability that this photon is transmitted by a polarized aligned along the $|0\rangle$ is given by $|\cos\theta|^2$.
- The transmission probability can now be correctly reinterpreted as $|\langle 0|\chi\rangle|^2$ and this result maybe used to calculate the transmission probabilities for circularly polarized light as well.
- The above result can be generalised to transmission probabilities for a polarizer oriented along any direction using the same method as described before.

Polar Coordinates

- A complex number, $z = x + \mathbf{i}\, y$ can be represented in the polar form as, $z = \mathrm{r}e^{\mathbf{i}\phi}$ where

$$\mathrm{r} = \sqrt{x^2 + y^2}\ ;\ \phi = \arctan\left(\frac{y}{x}\right)$$

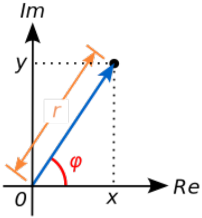

Figure 1: Figure showing the representations of a complex number. Source: Wikipedia[2]

[2] Complex_number_illustration.svg: The original uploader was Wolfkeeper at English Wikipedia. derivative work: Kan8eDie (talk) `https://commons.wikimedia.org/wiki/File:Complex_number_illustration_modarg.svg`, "Complex number illustration modarg", `https://creativecommons.org/licenses/by-sa/3.0/legalcode`

The global phase

- The unit complex number $e^{\mathbf{i}\phi}$ is also referred to as a phase factor.
- Multiplying a photon state $|\psi\rangle$ with a phase factor gives a state $e^{\mathbf{i}\phi}|\psi\rangle$. The phase factor is now referred to as a global phase factor.
- The projection of this new vector with respect to a basis state (say $|0\rangle$) is given by $e^{\mathbf{i}\phi}\langle 0|\psi\rangle$.
- While this is different from $\langle 0|\psi\rangle$, it should be noted that this new state will have the same transmission probabilities as that of $|\psi\rangle$.
- It is therefore not possible to distinguish $e^{\mathbf{i}\phi}|\psi\rangle$ from $|\psi\rangle$ by performing polarization measurements.
- Therefore, states that differ from each other by only a global phase are considered to be equivalent.

Qubit state in polar coordinates

- Consider the state $|\psi\rangle$ in the standard basis with the coordinates represented in polar coordinates. $a = \mathrm{r}_0 e^{\mathbf{i}\phi_0}$ and $b = \mathrm{r}_1 e^{\mathbf{i}\phi_1}$
- The state may now be expressed as follows:

$$\begin{aligned} |\psi\rangle &= \mathrm{r}_0 e^{\mathbf{i}\phi_0} |0\rangle + \mathrm{r}_1 e^{\mathbf{i}\phi_1} |1\rangle \\ &= e^{\mathbf{i}\phi_0} \left(\mathrm{r}_0 |0\rangle + \mathrm{r}_1 e^{\mathbf{i}(\phi_1 - \phi_0)} |1\rangle \right) \\ &\equiv \mathrm{r}_0 |0\rangle + \mathrm{r}_1 e^{\mathbf{i}(\phi_1 - \phi_0)} |1\rangle \end{aligned}$$

setting $\phi_1 - \phi_0 = \phi$,

$$|\psi\rangle = \mathrm{r}_0 |0\rangle + \mathrm{r}_1 e^{\mathbf{i}\phi} |1\rangle$$

additionally,

$$\langle\psi|\psi\rangle = 1 \Rightarrow \mathrm{r}_0^2 + \mathrm{r}_1^2 = 1$$

Parameter Selection

- Since, $0 \leq \mathrm{r}_0, \mathrm{r}_1 \leq 1$ and $\mathrm{r}_0^2 + \mathrm{r}_1^2 = 1$ it is possible to represent $\mathrm{r}_0 = \cos(\theta/2)$ and $\mathrm{r}_1 = \sin(\theta/2)$ where, $\theta \in [0, \pi]$
- Since, $e^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi \;\Rightarrow\; \phi \in [0, 2\pi)$. This angle is known as the relative phase.
- These parameters are identical to the angle variable in spherical polar coordinates. Therefore, each qubit state defined using these parameters corresponds to a point on a unit sphere.
- The qubit state in terms of this parameter is represented as

$$|\psi\rangle \;=\; \cos(\theta/2)\,|0\rangle + e^{\mathbf{i}\phi}\sin(\theta/2)\,|1\rangle$$

- The sphere on which the point corresponding to the state is present is known as the *Bloch Sphere*.

Qubit states on the Bloch Sphere

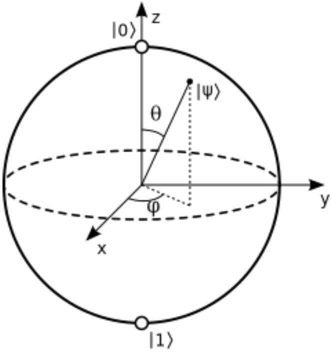

Figure 2: Figure showing a point on the Bloch Sphere. Source: Wikipedia[3]

[3]Smite-Meister (https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg), "Bloch sphere", (https://creativecommons.org/licenses/by-sa/3.0/legalcode)

Qubit states on the Bloch Sphere

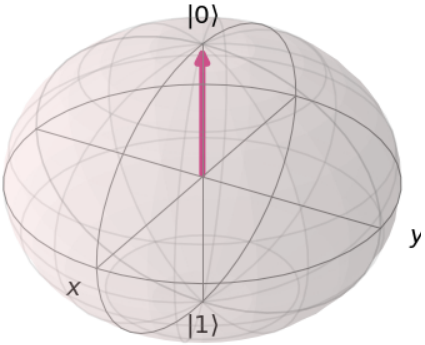

Figure 3: Figure showing the the point corresponding to $|0\rangle$

Qubit states on the Bloch Sphere

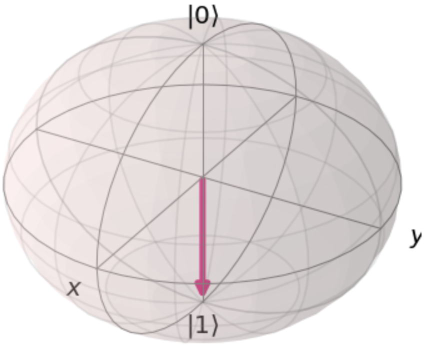

Figure 4: Figure showing the point corresponding to $|1\rangle$

Qubit states on the Bloch Sphere

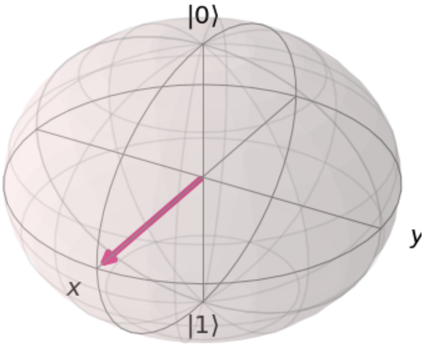

Figure 5: Figure showing the point corresponding to $|+\rangle$

Quantum Computing Course

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Module 1

Lecture 5a: Single-qubit measurement

- Learn the meaning of single-qubit measurement
- Given single-qubit state and a measurement basis describe the possible measurement outcomes and compute the probability for each of them.

Single qubit measurement

$|\psi\rangle = a|0\rangle + b|1\rangle$

$|a|^2 + |b|^2 = 1$.

- A single-qubit measurement, M is associated to an orthonormal basis

$$\{|\Phi_1\rangle, |\Phi_2\rangle\}$$

$\langle\Phi_1|\Phi_2\rangle = 0 = \langle\Phi_2|\Phi_1\rangle$

$\langle\Phi_1|\Phi_1\rangle = 1 = \langle\Phi_2|\Phi_2\rangle$

- Measuring $|\Psi\rangle = a|0\rangle + b|1\rangle$ by M outputs either $|\Phi_1\rangle$ or $|\Phi_2\rangle$.

- The probability of outcome $|\Phi_1\rangle$ is $|\langle\Phi_1|\Psi\rangle|^2$

- The probability of outcome $|\Phi_2\rangle$ is $|\langle\Phi_2|\Psi\rangle|^2$

Example 1

$$\langle 0| \left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right) = \frac{1}{\sqrt{2}}\langle 0|0\rangle + i\frac{1}{\sqrt{2}}\langle 0|1\rangle = \frac{1}{\sqrt{2}}$$

$$\langle 1| \left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right) = \frac{1}{\sqrt{2}}\langle 1|0\rangle + i\frac{1}{\sqrt{2}}\langle 1|1\rangle = \frac{i}{\sqrt{2}}$$

- Consider the single-qubit state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathbf{i}|1\rangle)$ and the measurement basis $\underbrace{\{|0\rangle, |1\rangle\}}_{M}$.

- The measurement outcome is $|0\rangle$ with probability

$$|\langle 0|\Psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

- The measurement outcome is $|1\rangle$ with probability

$$|\langle 1|\Psi\rangle|^2 = \left|\mathbf{i}\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Calculations

- $\langle 0|\Psi\rangle = \langle 0|\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}|1\rangle\right) = \frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}\langle 0|1\rangle = \frac{1}{\sqrt{2}}.$

- $\langle 0|\Psi\rangle = \langle 1|\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}|1\rangle\right) = \frac{1}{\sqrt{2}}\langle 1|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}\langle 1|1\rangle = \frac{1}{\sqrt{2}}\mathbf{i}.$

Example 2

Hadamard basis $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

- Consider the single-qubit state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathbf{i}|1\rangle)$ and the measurement basis $\{|+\rangle, |-\rangle\}$.

$\langle + | \psi \rangle =$

- The measurement outcome is $|+\rangle$ with probability

$$|\langle + | \Psi \rangle|^2 = \left| \frac{1}{2}(1 + \mathbf{i}) \right|^2 = \frac{1}{2}.$$

$\langle - | \psi \rangle =$

- The measurement outcome is $|-\rangle$ with probability

$$|\langle - | \Psi \rangle|^2 = \left| \frac{1}{2}(1 - \mathbf{i}) \right|^2 = \frac{1}{2}.$$

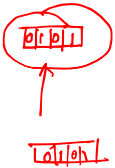
Calculations

- $\langle + | \Psi \rangle = \left(\frac{1}{\sqrt{2}} (\langle 0 | + \langle 1 |) \right) \left(\frac{1}{\sqrt{2}} (|0\rangle + \mathbf{i} |1\rangle) \right) = \frac{1}{2}(1 + \mathbf{i})$.

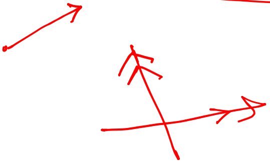
- $\langle - | \Psi \rangle = \left(\frac{1}{\sqrt{2}} (\langle 0 | - \langle 1 |) \right) \left(\frac{1}{\sqrt{2}} (|0\rangle + \mathbf{i} |1\rangle) \right) = \frac{1}{2}(1 - \mathbf{i})$.

- $|\langle + | \Psi \rangle|^2 = \left| \frac{1}{2}(1 + \mathbf{i}) \right|^2 = \frac{1}{2}$.

- $|\langle - | \Psi \rangle|^2 = \left| \frac{1}{2}(1 - \mathbf{i}) \right|^2 = \frac{1}{2}$.



$|\psi\rangle = a|0\rangle + b|1\rangle$



Module 1

Lecture 5b: Single-qubit operations

- The Pauli Transformations
- The Hadamard Transformation

Outer product

$\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$

$\boxed{\begin{pmatrix} x & y \\ z & t \end{pmatrix}} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ax+by \\ az+bt \end{pmatrix}$

2×1

$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{f} \begin{pmatrix} ax+by \\ az+bt \end{pmatrix}$

$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} \bar{x} & \bar{z} \\ \bar{y} & \bar{t} \end{pmatrix} = I$

- Let $|\psi\rangle$ and $|\Phi\rangle$ be two vector.
- $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\Phi\rangle = c|0\rangle + d|1\rangle$.

$\langle \psi | \Phi \rangle$

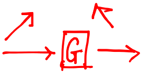
$\begin{pmatrix} a \\ b \end{pmatrix}$ $\begin{pmatrix} c \\ d \end{pmatrix}$

- The outer product of $|\psi\rangle$ and $|\Phi\rangle$ is

$$|\Psi\rangle\langle\Phi| = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}^{\dagger} = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \bar{c} & \bar{d} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} a\bar{c} & a\bar{d} \\ b\bar{c} & b\bar{d} \end{pmatrix}}$$

Quantum state transformations



- Quantum computers have the capability of transforming one quantum state to another by applying unitary transformations on the former.

$TT^\dagger = I$

- A linear transformation T is said to be unitary if

$$T\,T^\dagger = I$$

where I is the identity operator.

The Pauli Transformations

$0 \mapsto |0\rangle$

$1 \mapsto |1\rangle$

$0 \to \boxed{\neg} \to 1 \qquad 1 \to \boxed{\neg} \to 0$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$|\psi\rangle = a|0\rangle + b|1\rangle \to \begin{pmatrix}a\\b\end{pmatrix}$

- I : $|0\rangle\langle 0| + |1\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}(1\quad 0) + \begin{pmatrix}0\\1\end{pmatrix}(0\quad 1)$

$\begin{pmatrix}0\\1\end{pmatrix}$

$$= \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \boxed{\begin{pmatrix}1&0\\0&1\end{pmatrix}}$$

$|0\rangle \to \boxed{X} \to |1\rangle$

$|1\rangle \to \boxed{X} \to |0\rangle$

$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix}$

- X : $|1\rangle\langle 0| + |0\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}(1\quad 0) + \begin{pmatrix}1\\0\end{pmatrix}(0\quad 1)$

$b|0\rangle + a|1\rangle$

$$= \begin{pmatrix}0&0\\1&0\end{pmatrix} + \begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

$a|0\rangle + b|1\rangle \longmapsto b|0\rangle + a|1\rangle$

The Pauli Transformations

$\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\left(a|0\rangle + b|1\rangle\right)$

$= a|0\rangle\underline{\langle 0|0\rangle} + b|0\rangle\underline{\langle 0|1\rangle}$

$- a|1\rangle\underline{\langle 1|0\rangle} - b|1\rangle\langle 1|1\rangle$

- Y : $-|1\rangle\langle 0| + |0\rangle\langle 1| = -\begin{pmatrix}0\\1\end{pmatrix}(1 \quad 0) + \begin{pmatrix}1\\0\end{pmatrix}(0 \quad 1)$

$$= -\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$$

$= a|0\rangle - b|1\rangle$

$a|0\rangle + b|1\rangle \xmapsto{Z} a|0\rangle - b|1\rangle$

- Z : $\underline{|0\rangle\langle 0| - |1\rangle\langle 1|} = \begin{pmatrix}1\\0\end{pmatrix}(1 \quad 0) - \begin{pmatrix}0\\1\end{pmatrix}(0 \quad 1)$

$$= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Action of the Pauli Transformations

- I = identity transformation
- X = negation, it is similar to the classical not operation
- Z = changing the relative phase of a superposition in the standard basis.
- $Y = ZX$.

The Hadamard Transformation

- $H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$

$|0\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|1\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\underset{\{|0\rangle, |1\rangle\}}{|0\rangle} \overset{H}{\longmapsto} \underbrace{\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}_{\{|0\rangle, |1\rangle\}}$