Quiz Assignment 1: Solutions

1. Consider the following states:
   a. $|+\rangle$
   b. $\frac{1}{\sqrt{2}}(\ |+\rangle + \ |-\rangle\ )$

   Which of the following is true:

   A. a. and b. both are in superposition with respect to standard basis.
   B. a. is in superposition with respect to standard basis, but b. is not.
   C. a. is not in superposition with respect to standard basis, but b. is in superposition with respect to standard basis.
   D. Neither a. nor b. is in superposition with respect to standard basis.

Solution: B

$|+\rangle = \frac{1}{\sqrt{2}}(\ |0\rangle + \ |1\rangle\ )$ which is a superposition in the standard basis. $\frac{1}{\sqrt{2}}(\ |+\rangle + \ |-\rangle\ ) = |0\rangle$ which is not a superposition in the standard basis.

2. Consider the following states:
   a. $\frac{1}{\sqrt{2}}(\ |\mathbf{i}\rangle - \ |-\mathbf{i}\rangle\ )$
   b. $\frac{1}{\sqrt{2}}(\ |0\rangle - \ |1\rangle\ )$

   Which of the following is true?

   A. a. and b. both are in superposition with respect to standard basis.
   B. a. is in superposition with respect to standard basis, but b. is not.
   C. a. is not in superposition with respect to standard basis, but b. is in superposition with respect to standard basis.
   D. Neither a. nor b. is in superposition with respect to standard basis.

Solution: C

$\frac{1}{\sqrt{2}}(\ |\mathbf{i}\rangle - \ |-\mathbf{i}\rangle\ ) = |1\rangle$ which is not a superposition in the standard basis. $\frac{1}{\sqrt{2}}(\ |0\rangle - \ |1\rangle\ )$ is clearly a superposition in the standard basis.

3. Consider the bases
   a. $\{|0\rangle,\ |1\rangle\}$, and
   b. $\left\{|0\rangle, \frac{1}{\sqrt{2}}(\ |0\rangle + |1\rangle\ )\right\}$.

   Which of the following is true?
   A. a. and b. both are orthonormal bases.
   B. a. is an orthonormal basis, but b. is not.
   C. a. is not an orthonormal basis, but b. is an orthonormal basis.
   D. Neither a. nor b. is an orthonormal basis.

Solution: B

The option a. is the standard basis. It is known to be orthonormal. b. is $\left\{|0\rangle, \frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle\,)\right\} = \{|0\rangle, |+\rangle\}$. Now, $\langle 0|+\rangle = \frac{1}{\sqrt{2}} \neq 0$. b. is not orthogonal and hence not an orthonormal basis.

4. Find a matrix X that leaves both basis vectors of the Hadamard basis $\mathcal{H} = \{|+\rangle, |-\rangle\}$ unchanged upon multiplication except by a multiplicative factor. i.e., the matrix X should satisfy the following equations: $X|\psi\rangle = c|\psi\rangle$ for all $|\psi\rangle \in \{|+\rangle, |-\rangle\}$, where c is a number

A. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ B. $\begin{pmatrix} 2 & i \\ 1 & 0 \end{pmatrix}$ **C.** $\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$ D. $\begin{pmatrix} -1 & 0 \\ i & 0 \end{pmatrix}$

Solution: C

Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We can see by matrix multiplication (the form of the kets for the Hadamard basis are given in the lecture slides), that $X|+\rangle = +1|+\rangle$ and $X|-\rangle = -1|-\rangle$. This is the only option for which this is true

5. In the Python 3 programming language, which of the following expressions would evaluate to an integer? (One or more options may be correct)

A. 7/3 B. 8/2 **C. 25//3** D. 25.0//2

Solution: C

All other options evaluate to floating point numbers.

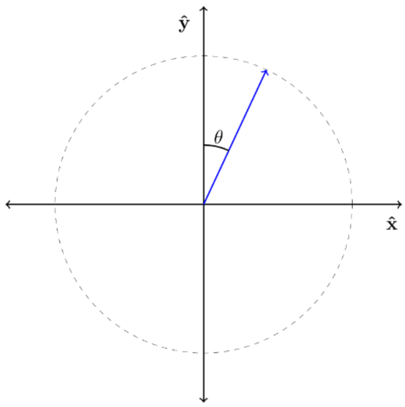


Diagram showing positive angle convention

6. A vertically polarized beam of light is incident on a polarizer P1 (oriented at an angle θ) and it is observed that the beams intensity is halved. Another polarizer P2 (oriented at an angle 30°) is placed such the beam must pass through P2 before P1. The fraction of the intensity of light transmitted through both these polarizers

   A. <u>Increases</u>
   B. Decreases

Solution: A

With only P1, transmitted beam intensity is half $\Rightarrow \theta_1 = 45°$ (angle of polarizer P1).

$\theta_2 = 30°$(angle of polarizer P1). Relative angle between P1 and P2 is $\theta_{12} = 15°$.

Final beam intensity = Fraction of original beam intensity transmitted through P2 × Fraction of beam intensity from P2 which is transmitted through P1

$$= |\cos 30°|^2 \times |\cos 15°|^2 = \left|\frac{\sqrt{3}}{2}\right|^2 \times \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{3}{4} \times \frac{2+\sqrt{3}}{4} = \frac{6+3\sqrt{3}}{16} > \frac{1}{2}$$

So, the intensity of the final beam increases when the polarizer P2 in inserted.

7. Two polarizers P1 and P2 are oriented at angles 45° and 60°. The ratio of the probabilities of a vertically polarized photon being transmitted by the combinations P1-P2 and P2-P1 is
   A. 1:4
   B. 1:2
   C. 2:1
   D. 1:1

Solution: C

Incoming photon is vertical $\Rightarrow \theta = 0°$
Relative angle between incoming photon and first polarizer θ_{i1}
Relative angle between first polarizer and second polarizer θ_{12}
P1-P2: $\theta_{i1} = 45°;\ \theta_{12} = 15°$

Final intensity: $I_1 = |\cos\theta_{i1}|^2 \times |\cos\theta_{12}|^2 = |\cos 45°|^2 \times |\cos 15°|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 \times \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{1}{2} \times \frac{2+\sqrt{3}}{4} = \frac{2+\sqrt{3}}{8}$
P2-P1: $\theta_{i1} = 60°;\ \theta_{12} = -15°$

Final intensity: $I_2 = |\cos\theta_{i1}|^2 \times |\cos\theta_{12}|^2 = |\cos 60°|^2 \times |\cos -15°|^2 = \left|\frac{1}{2}\right|^2 \times \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = \frac{1}{4} \times \frac{2+\sqrt{3}}{4} = \frac{2+\sqrt{3}}{16}$
Ratio: $I_1 : I_2 = \frac{2+\sqrt{3}}{8} : \frac{2+\sqrt{3}}{16} = 2:1$

8. Two polarizers are P1 and P2 are oriented at angles 0° and 90° respectively. A third polarizer in P3 is inserted in between P1 and P2 for what orientation of P3 will the total transmission probability of this system be maximum?
   A. 60°
   B. 45°
   C. depends on the polarization of the incident photon.
   D. The transmission probability will always be 0.

Solution: B

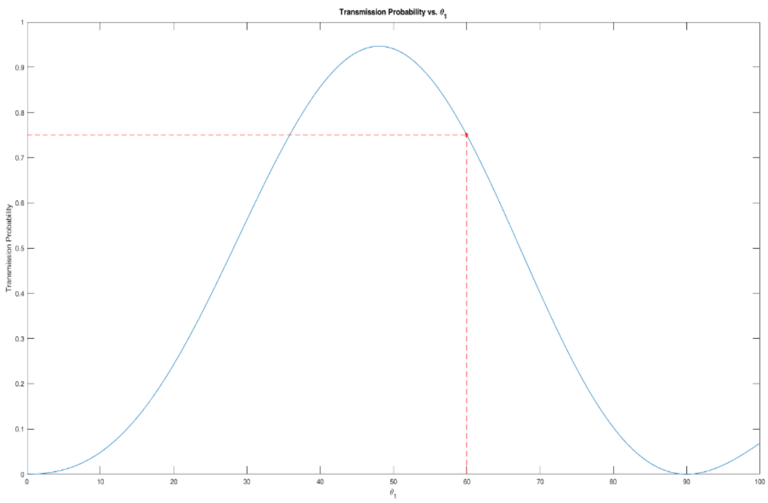
Final transmission probability: $I = |\cos\theta|^2 \times \left|\cos(\frac{\pi}{2} - \theta)\right|^2 = 1 \times |\cos\theta|^2 \times |\sin\theta|^2 = \frac{1}{4} \times |2\sin\theta\cos\theta|^2 = \frac{1}{4}|\sin 2\theta|^2 = \frac{1}{4}\sin^2(2\theta)$
To find the maximum, $\frac{dI}{d\theta} = \frac{1}{4} \times 2 \times \sin 2\theta \times \cos 2\theta \times 2 = 0 \Rightarrow \theta = 45°$

Alternatively, we know that the maximum for $\sin^2 \alpha$ and $\sin \alpha$ is the same and occurs at $\alpha = \frac{\pi}{2}$.
So, the maximum for $\sin^2(2\theta)$ occurs at $2\theta = \frac{\pi}{2}, \theta = \frac{\pi}{4}$.

9. Plane polarized light with polarization at 60° is incident on 2 polarizers oriented at angle θ_1 and θ_2 respectively such that $\theta_1 + \theta_2 = 90°$ and θ_1 & θ_2 are both positive. Maximum resultant intensity is achieved for $\theta_1 = 60°$.
    A. True
    B. <u>False</u>

Solution: B

Incoming photon is vertical $\Rightarrow \theta = 60°$
Relative angle between incoming photon and first polarizer: $60° - \theta_1$
Relative angle between first polarizer and second polarizer: $\theta_2 - \theta_1 = 90° - 2\theta_1$
(since $q_1 + q_2 = 90°$)
Final transmission probability: $I = |\cos(60° - \theta_1)|^2 \times |\cos(90° - 2\theta_1)|^2$
$= |\cos(60° - \theta_1) \times \cos(90° - 2\theta_1)|^2 = |\cos(60° - \theta_1) \times \sin(2\theta_1)|^2$
The transmission probability is a function of the variable θ_1 which can vary from 0° to 90°. One may now plot this function using any graph plotting tool/software. The graph would look like shown and the point of interest is marked.



This point does not correspond to the maximum value of transmission probability. The answer is therefore false.

10. A polarizer is oriented at an angle of 45° the polarization for which a photon has a 75% probability of transmission is:
    A. 60°
    B. <u>75°</u>
    C. <u>15°</u>
    D. 90°

Solution: B, C
Transmission probability = $|\cos\theta|^2 = 0.75 = \frac{1}{4}$ (75%)

$$\Rightarrow \cos\theta = \pm\frac{1}{2} \Rightarrow \theta = 15°, 75°$$