Unit-II

(Matrix - Algebra)

$A=$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$$= (a_{ij})_{3\times 3}$$

Symmetric matrix

$A=(a_{ij})_{m\times n}$ is said to be symmetric if

$$A' = A$$

$$\boxed{a_{ji} = a_{ij}}$$

e.g: $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$, $A'=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

$\Rightarrow A' = A \Rightarrow A$ is symmetric.

Skew symmetric matrix

A matrix $A=(a_{ij})_{m\times n}$ is said to be skew symmetric if $A' = -A$

$$\boxed{a_{ji} = -a_{ij}}$$

$$\left(\Rightarrow \begin{array}{l} a_{21} = -a_{12} \\ a_{31} = -a_{13} \end{array}\right)$$

$$\begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$$

if $j = i$, then

$$a_{ii} = -a_{ii}$$

$$\Rightarrow 2a_{ii} = 0$$

$$\Rightarrow \boxed{a_{ii} = 0}$$

$\Rightarrow$ diagonals of the skew symmetric matrix are 0.

e.g. $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix}$, $A' = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & -3 \\ 1 & 3 & 0 \end{bmatrix}$

$$\Rightarrow A' = -\begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & 3 \\ -1 & -3 & 0 \end{bmatrix} = -A$$

Hermitian & skew Hermitian matrix

A matrix A is said to Hermitian if

$$(\bar{A})' = A \quad \underline{\text{OR}} \quad A^{\theta} = A, \qquad A^{\theta} = (\bar{A})'$$

$\underline{\text{OR}}$ $\quad \boxed{\bar{a}_{ji} = a_{ij}} \Rightarrow a_{12} = \bar{a}_{21}$ ✓

$a_{13} = \bar{a}_{31}$ ✓

$$A = \begin{bmatrix} 1-i & i \\ 1 & -i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1+i & -i \\ 1 & i \end{bmatrix}$$

$$A^{\theta} = (\bar{A})' = \begin{bmatrix} 1+i & 1 \\ -i & i \end{bmatrix}$$

If $j = i$,

$$\bar{a}_{ii} = a_{ii}$$

If $a_{ii} = x + iy$, then

$$\cancel{x} - iy = \cancel{x} + iy$$

$$\Rightarrow 2iy = 0$$

$$y = 0$$

$$\therefore \boxed{a_{ii} = x}$$

A is said to be skew hermitian if

$(\bar{A})' = -A$ or $A^\theta = -A$ where $A^\theta = (\bar{A})'$

$a_{ij} = \cancel{x}\, iy$

$\Rightarrow \bar{a}_{ji} = -a_{ij}$

If $j = i$,

$\bar{a}_{ii} = -a_{ii}$

If $a_{ii} = x + iy$, then

$x - iy = -(x + iy)$

$\Rightarrow x - \cancel{iy} = -x - \cancel{iy}$

$\Rightarrow 2x = 0$

$x = 0$

$\therefore$ $\boxed{a_{ii} = iy}$ or $a_{ii} = 0$ (if $y = 0$)

$\therefore$ Diagonals are either 0 or pure imaginary.

Idempotent matrix $\rightarrow$ $A^2 = A$

Nilpotent matrix $\rightarrow$ $A^k = 0$, k is known as index of nilpotent matrix.

Involutary matrix $\rightarrow$ $A^2 = I$

Singular matrix $\rightarrow$ $|A| = 0$ otherwise non-singular. $(|A| \neq 0)$

Q Every square matrix can be written as a sum of two matrices one symmetric (or Hermitian) and the other skew-symmetric (or skew Hermitian)

Sol: (i) Let $A = \underbrace{\frac{1}{2}(A + A')} + \frac{1}{2}(A - A')$

$= X + Y$

$A = X + Y$

$A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$

$X = \frac{1}{2}(A + A')$

$Y = \frac{1}{2}(A - A')$

where $X = \frac{1}{2}(A + A')$, $Y = \frac{1}{2}(A - A')$

$X' = \frac{1}{2}(A' + A'')$, $Y' = \frac{1}{2}(A' - A'')$

$= \frac{1}{2}(A' + A)$, $Y' = \frac{1}{2}(A' - A)$

$= -\frac{1}{2}(A - A') = -Y$

$= X$,

$\Rightarrow X' = X$, $Y' = -Y$

$\Rightarrow$ X is symmetric and Y is skew symmetric.

$$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta), \quad A^\theta = (\bar{A})' \qquad \cancel{(\bar{\bar{A}})'' = A}$$

$$= X + Y$$

$$X^\theta = \frac{1}{2}(A^\theta + A^{\theta\theta}), \quad Y^\theta = \frac{1}{2}(A^\theta - A^{\theta\theta})$$

$$= \frac{1}{2}(A^\theta + A), \quad Y^\theta = \frac{1}{2}(A^\theta - A) = -\frac{1}{2}(A - A^\theta) = -Y$$

$$= X$$

$$\Rightarrow X^\theta = X, \quad Y^\theta = -Y$$

$\Rightarrow$ X is Hermitian and Y is skew Hermitian.

Elementary operations

The following operations are known as Elementary operations

(i) Interchange of two rows or column

$R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(ii) multiplication of ith row/column by a constant

$R_i \to kR_i$ or $C_i \to kC_i$

(iii) Addition of k times of ith row (column) in jth row (column)

$R_j \to kR_i + R_j$ or $C_j \to kC_i + C_j$

$\begin{bmatrix} 1 & 2 \\ 3k & 4k \end{bmatrix}$

(1) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \sim R_2 \to 2R_1 + R_2$

$|A| \neq 0$

$A^{-1} = \frac{\text{Adj} A}{|A|}$

$x+y+z=1$

$x-y+z=2$

$2x+3y+4z=5$

$AX=B$

$X = A^{-1}B$

{ Gauss-Jordan Method)

Inverse of the matrix using E-row operations

$|A-\lambda I|=0$

$\lambda = 1, 2, 3$

$A = AI$

$AB = I$

$\Rightarrow A^{-1} = B$

$A \sim I$

$[\ ] \sim [\ ]$

$A = AI$

Applying E-row operations we get

$I = AB$

$\Rightarrow \boxed{A^{-1} = B}$

$A\cancel{A} = I$

Q Ex Find the inverse of the following matrix using E-row operation

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$A = AI$

$I = AB$

$\boxed{A^{-1} = B}$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$A = AI$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to 2R_3 - 5R_1$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$$R_3 \to 2R_3 + R_2$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -10 & 1 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$$R_2 \to R_2 + R_3$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$R_1 \to R_1 - R_3$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 11 & -1 & -4 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$R_1 \to 2R_1 - R_2$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 32 & -4 & -12 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$I = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$$-6+5$$

$$A = AI$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to 2R_3 - 5R_1$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & \underline{-1} & \underline{-1} \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$$R_3 \to 2R_3 + R_2$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -10 & 1 & 4 \end{bmatrix}$$

$$R_1 \to R_1 + R_2$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -10 & 1 & 4 \end{bmatrix}$$

$$R_2 \to R_2 + R_3$$

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 1 & 0 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$R_1 \to 2R_1 - 3R_2$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 32 & -4 & -12 \\ -10 & 2 & 4 \\ -10 & 1 & 4 \end{bmatrix}$$

$$I = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 15 & -1 & -4 \end{bmatrix} = AB$$

$$\boxed{B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 15 & -1 & -4 \end{bmatrix}}$$

Sol:-

$$A = AI$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to 2R_3 - 5R_1$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ 0 & 1 & 0 \\ -5 & 0 & 2 \end{bmatrix}$$

$4-5$

$-6+5$

$R_1 \to R_1 - R_3$

$R_1 \to R_1 - 2R_3$

$R_2 \to R_2 + R_3$

$R_3 \to 2R_3 + R_2$

$A = I$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_3$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ -5 & 1 & 2 \\ -5 & 0 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & 0 \\ \underline{0} & \underline{0} & -1 \end{bmatrix} = A \begin{bmatrix} -4 & 0 & 2 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A \begin{bmatrix} 16 & -2 & -6 \\ -5 & 1 & 2 \\ -10 & 1 & 4 \end{bmatrix}$$

$A A^{-1} = I$ ✓

$R_1 \to \frac{1}{2} R_1$, $R_3 \to R_3 \times -1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$\because \quad I = AB$

$$\Rightarrow A^{-1} = B = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$ Ans.

$$\left\{ \because A A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

Q Find the inverse of the matrix using E-row operations

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:-

$$A = AI$$

$$\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix} = A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} 4 & 3 & 1 \\ 4 & 8 & 16 \\ \hline 0 & -5 & -15 \end{array}$$

$$R_1 \longrightarrow R_3$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & -1 & -4 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & 0 & -2 \end{bmatrix}$$

-20

$R_3 \to 5R_3 - R_2$

-10

4

$5 - 24$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -15 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 5 & -1 & -6 \end{bmatrix}$$

$R_1 \to 5R_1 + 4R_3$

$$\begin{bmatrix} 5 & 10 & 0 \\ 0 & -5 & -15 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} 20 & -4 & -19 \\ 0 & 1 & -4 \\ 5 & -1 & -6 \end{bmatrix}$$

$R_2 \to R_2 - 3R_3$

$$\begin{bmatrix} 5 & 10 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} 20 & -4 & -19 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$\begin{array}{r} 28 \\ -19 \\ \hline \end{array}$$

$R_1 \to R_1 + 2R_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} = A \begin{bmatrix} -10 & 4 & 9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$R_1 \to \frac{1}{5} R_1, \quad R_2 \to \frac{-1}{5} R_2, \quad R_3 \to \frac{-1}{5} R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A \times \frac{1}{5} \begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}$$

$I = AB$

$$\Rightarrow \boxed{A^{-1} = B = \frac{1}{5}\begin{bmatrix} -10 & 4 & 9 \\ 15 & -4 & -14 \\ -5 & 1 & 6 \end{bmatrix}}$$

$\{ A A^{-1} = I \}$

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \quad 4 \times 5$$

$\boxed{|A| \neq 0}$

$|A| = 0 \rightarrow$

$\rho(A) < 4$

$A = \begin{bmatrix} \cancel{1} & \cancel{2} \\ 1 & 2 \end{bmatrix} \quad \rho(A) < 2$

$|A| = 2 - 2 = 0$

$\boxed{A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \rightarrow \rho(A) = 2$

$\underline{\underline{|A|}} = 4 - 6 = \underline{\underline{-2 \neq 0}}$

Unit II(Matrix Theory)

- L1: https://youtu.be/jBukOH3HxhU
- L2: https://youtu.be/J-QMiJwXT9Q
- L3: https://youtu.be/XNLWHDhGDiA
- L4: https://youtu.be/OCXWrlMWK6g
- L5: https://youtu.be/Y58bx36-z2c