Unit-III

$\beta(m,n)$

Beta & Gamma functions

Beta function

Let $m>0$, $n>0$ be positive numbers then

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

~~Beta~~ Gamma function

Let $n>0$ then

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$$

$\Gamma(0) =$ undefined

Properties of Beta funcⁿ

(1) $B(m,n) = B(n,m)$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

let $x = 1-y \Rightarrow dx = -dy$

$$= -\int_1^0 (1-y)^{m-1}(1-1+y)^{n-1}(-dy)$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= B(n,m)$$

(2) $B(m,n) = \int_0^\infty \frac{x^{n-1}\,dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$

Proof:

$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}$

Put $x = \frac{1}{1+y}$ $\left(\Rightarrow 1+y = \frac{1}{x},\ y = \frac{1}{x} - 1\right)$

$\Rightarrow dx = \frac{-1}{(1+y)^2}dy$

$$= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1 - \frac{1}{1+y}\right)^{n-1} \times \frac{-1}{(1+y)^2}dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1}} \times \frac{y^{n-1}}{(1+y)^{n-1}} \times \frac{1}{(1+y)^2}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

(3) $$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\cancel{\int_0^{\frac{\pi}{2}} \sin^{p}\theta \cos^{q}\theta \, d\theta}$$

Pf:

$$\because \quad B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

$$B(m,n) = \int_0^{\frac{\pi}{2}} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \times 2\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta \times 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

Properties of Gamma function

(i) $\Gamma(n+1) = n!$ if n is +ve integer

$= n\Gamma(n)$ otherwise

$\lim_{x\to\infty} e^{-x} x^n \quad (0\times\infty)$

$= \lim_{x\to\infty} \frac{x^n}{e^x} \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{n!}{e^x} = \frac{n!}{e^\infty} = 0$

Pf: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$ — (1)

$$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx$$

$$= \left\{-e^{-x} x^n\right\}_0^\infty + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= 0 - 0 + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= n\int_0^\infty e^{-x} x^{n-1}\, dx \quad \text{— (ii)}$$

$$= n\Gamma(n)$$

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n \, dx$$

$$\left[-e^{-x}\right]_0^\infty = 0+1$$
$$= 1$$

$$= n \int_0^\infty e^{-x} x^{n-1} \, dx$$

$$= n(n-1) \int_0^\infty e^{-x} x^{n-2} \, dx$$

$$= n(n-1)(n-2) \int_0^\infty e^{-x} x^{n-3} \, dx$$

$$= n(n-1)(n-2) \cdots (n-(n-1)) \int_0^\infty e^{-x} x^{n-n} \, dx$$

$$= n!$$

$$\therefore \quad \Gamma(n+1) = n\Gamma(n)$$
$$= n!$$

$\Gamma 1 = 0! = 1$

$\Gamma(2) = 1!$

$\Gamma(6) = 5!$

$\Gamma(11) = 10!$ and so on.

As we know that

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\tfrac{3}{2}\right) = \Gamma\left(\tfrac{1}{2}+1\right)$$

$$= \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{5}{2}\right) = \Gamma\left(\tfrac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\tfrac{3}{2}\right)$$

$$= \frac{3}{4}\sqrt{\pi}$$

$$\Gamma\left(\tfrac{7}{2}\right) = \Gamma\left(\tfrac{5}{2}+1\right)$$

$$= \frac{5}{2}\Gamma(5/2) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{15}{8}\sqrt{\pi}.$$

(2) $\frac{\Gamma(n)}{k^n} = \int_0^\infty e^{-kx} x^{n-1}\, dx$

Pf: $\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

put $x = ky \Rightarrow dx = k\,dy$

$\Gamma(n) = \int_0^\infty e^{-ky} (ky)^{n-1} (k\,dy)$

$= \int_0^\infty e^{-ky} k^{n-1} y^{n-1} k\, dy$

$= k^n \int_0^\infty e^{-ky} y^{n-1}\, dy$

$= k^n \int_0^\infty e^{-kx} x^{n-1}\, dx$

$\therefore \int_0^\infty e^{-kx} x^{n-1}\, dx = \frac{\Gamma(n)}{k^n}$

(3) $$\boxed{\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = n\Gamma(n)}$$

$$\because\ \Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$$

put $x = y^{\frac{1}{n}}$

$$dx = \frac{1}{n}\,y^{\frac{1}{n}-1}\,dy$$

$$= \int_0^\infty e^{-y^{\frac{1}{n}}}\left(y^{\frac{1}{n}}\right)^{n-1}\frac{1}{n}\,y^{\frac{1}{n}-1}\,dy$$

$$= \int_0^\infty e^{-y^{\frac{1}{n}}}\,y^{1-\frac{1}{n}} \times \frac{1}{n}\,y^{\frac{1}{n}-1}\,dy$$

$$= \frac{\int_0^\infty e^{-y^{\frac{1}{n}}}\,dy}{n}$$

$$= \frac{1}{n}\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^{\frac{1}{n}}}\,dx = n\Gamma(n)}$$

If $n = \frac{1}{2}$

$$\boxed{\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}}$$

Relationship b/w Beta & Gamma func^n

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Proof: $\Gamma(m) = \int_0^\infty e^{-x} x^{m-1}\,dx$, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx$

Also

$$\frac{\Gamma(n)}{z^n} = \int_0^\infty e^{-zx} x^{n-1}\,dx$$

$$\Rightarrow \Gamma(n) = \int_0^\infty e^{-zx} z^n x^{n-1}\,dx$$

multiplying both sides by $z^{m-1} e^{-z}$ and integrating w.r.t 'z' from 0 to ∞, we get

$$\Gamma(m)\Gamma(n) = \int_0^\infty e^{-z} z^{m-1} \left[\int_0^\infty e^{-zx} z^n x^{n-1}\,dx\right] dz$$

$$= \int_0^\infty \int_0^\infty e^{-z-zx} z^{m+n-1} x^{n-1}\,dx\,dz$$

$$= \int_0^\infty \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1}\,dx\,dz$$

Let $z(1+x) = t$

$$\Rightarrow dz = \frac{dt}{1+x}$$

$$= \int_0^\infty \int_0^\infty e^{-t} \left(\frac{t}{1+x}\right)^{m+n-1} x^{n-1} \times dx \times \frac{dt}{1+x}$$

$$= \int_0^\infty \left\{ \int_0^\infty e^{-t}\, t^{m+n-1}\, dt \right\} \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \Gamma(m+n) \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$= \Gamma(m+n)\, B(m,n)$$

$$\therefore \quad \Gamma(m)\,\Gamma(n) = \Gamma(m+n)\, B(m,n)$$

$$\therefore \boxed{B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}$$

Deduction

(1) Let $m+n=1$ in

$$B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \int_0^\infty \frac{x^{n-1}\, dx}{1+x} = \Gamma(n)\,\Gamma(1-n)$$

As $\displaystyle\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}$

$$\therefore \boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

Put $n = \frac{1}{2}$

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$\Rightarrow \left\{\Gamma\left(\tfrac{1}{2}\right)\right\}^2 = \pi$$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

If $n = 1/3$

$$\Gamma(1/3)\Gamma(2/3) = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\sqrt{3}/2}$$

$$\Rightarrow \boxed{\Gamma\left(\tfrac{1}{3}\right)\Gamma(2/3) = \frac{2\pi}{\sqrt{3}}}$$

If $n = \frac{1}{4}$

$$\Gamma\left(\tfrac{1}{4}\right)\Gamma(3/4) = \frac{\pi}{\sin\frac{\pi}{4}} = \pi\sqrt{2}$$

$$\therefore \boxed{\Gamma\left(\tfrac{1}{4}\right)\Gamma(3/4) = \pi\sqrt{2}}$$

If $n = \frac{1}{6}$

$$\Gamma(1/6)\Gamma(5/6) = \frac{\pi}{\sin\frac{\pi}{6}} = 2\pi$$

$$\Gamma(1/6)\Gamma(5/6) = 2\pi$$

(2) we know that

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} = \int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

let $2m-1 = p$, $2n-1 = q$

$\Rightarrow m = \frac{p+1}{2}$, & $n = \frac{q+1}{2}$

$$\therefore \boxed{\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta}$$

If $q = 0$, $p = \frac{1}{2}$

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}}\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(5/4)}$$

$$= \frac{\Gamma(3/4)\sqrt{\pi}}{2\Gamma\left(\frac{1}{4}+1\right)}$$

$$= \frac{\Gamma(3/4)\sqrt{\pi}}{2\times\frac{1}{4}\times\Gamma(1/4)}$$

$$= \frac{2\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)}$$

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Legendre's Duplication formula

$B(m,n) = B(n,m)$

$$\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}} \times \Gamma(2n)$$

Pf: We know that

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)} \quad \text{— (1)}$$

put $m = \frac{1}{2}$ in (1)

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \, d\theta = \frac{\Gamma(1/2)\,\Gamma(n)}{2\Gamma(n+1/2)} \quad \text{— (2)}$$

If $m = n$ in (1)

$$\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(n)\,\Gamma(n)}{2\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin\theta \cos\theta)^{2n-1} \, d\theta = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} 2\theta \, d\theta = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

Let $2\theta = t$

$\Rightarrow d\theta = \frac{dt}{2}$

$$\Rightarrow \int_0^{\pi} \frac{1}{2^{2n-1}} \sin^{2n-1} t \, \frac{dt}{2} = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{2\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} t \, \frac{dt}{2} = \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\left\{ \because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(2a-x)\,dx \right\}$$

$$\Rightarrow \cancel{2} \int_0^{\frac{\pi}{2}} \frac{1}{2^{2n-1}} \sin^{2n-1} \theta \, \frac{d\theta}{\cancel{2}}$$

$$= \frac{1}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2n-1} \theta \, d\theta$$

$$= \frac{2^{2n-1}}{2} \frac{\{\Gamma(n)\}^2}{\Gamma(2n)} \quad — (3)$$

By (2) & (3)

$$\frac{1}{\cancel{2}} \frac{\cancel{\Gamma(n)}\Gamma(1/2)}{\Gamma(n+1/2)} = \frac{2^{2n-1}}{\cancel{2}} \frac{\{\Gamma(n)\}^{\cancel{2}}}{\Gamma(2n)}$$

$$\Rightarrow \boxed{\Gamma(n)\,\Gamma(n+1/2) = \frac{\Gamma(2n) \times \sqrt{\pi}}{2^{2n-1}}}$$

Note:

① $$\Gamma(1/n)\,\Gamma(2/n) \times \cdots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

② $$\int_0^\infty e^{-ax} x^{n-1} \cos bx \, dx = \frac{\Gamma(n) \cos n\theta}{(a^2+b^2)^{n/2}}$$

$$\int_0^\infty e^{-ax} x^{n-1} \sin bx \, dx = \frac{\Gamma(n) \sin n\theta}{(a^2+b^2)^{n/2}} \quad \text{when } \theta = \tan^{-1}(b/a)$$

Q Evaluate

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^n}}$$

$\boxed{x^n = \sin^2\theta}$ $\quad \boxed{\int_0^1 \underbrace{t^{n-1}(1-t)^{m-1}}_{} = B(m,n)}$

B $\quad$ ~~$x^n = t$~~ $\quad \int_0^\infty$

sol: let $x^n = t$

$\Rightarrow x = t^{1/n}$

$\Rightarrow dx = \frac{1}{n} t^{\frac{1}{n}-1} dt$

$$I = \int_0^1 \frac{1}{n} \frac{t^{\frac{1}{n}-1} dt}{(1-t)^{1/2}}$$

$$= \frac{1}{n}\int_0^1 t^{\frac{1}{n}-1}(1-t)^{\frac{1}{2}-1} dt$$

$$= \frac{1}{n} B(1/n, 1/2) = \frac{1}{n} \frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(\frac{1}{n}+1/2)} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n+1/2)}$$ Ans.

Q Evaluate

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

① $I = \int_0^1 \left(1 - x^{\frac{1}{n}}\right)^m dx$

② $I = \int_0^1 x^m (1-x^n)^p \, dx$

③ $I = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

④ $I = \int_0^{\pi/2} \tan^p \theta \, d\theta = \frac{\pi}{2} \sec \frac{p\pi}{2}$

⑤ $I = \int_0^1 \frac{dx}{\sqrt{-\log x}}$

⑥ $I = \int_0^\infty e^{-h^2 x^2} dx$

⑦ $I = \int_0^\infty x^4 e^{-x^2} dx$

- You Tube Link:
- 1. https://youtu.be/EUV1kpKS24c
- 2. https://youtu.be/zY9yf1N5Vbs
- 3. https://youtu.be/McJFWZVvBvw