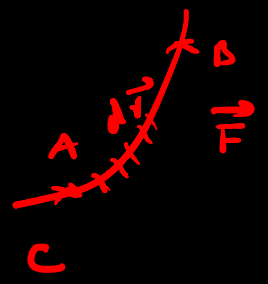
Vector Integration

(i) Line integral

Any integral which is evaluated over a curve is said to be line integral.



A line integral is denoted by

$$I = \int_C \vec F \cdot d\vec r$$

If $\vec F = F_1\hat i + F_2\hat j + F_3\hat k$, $d\vec r = dx\,\hat i + dy\,\hat j + dz\,\hat k$ then

$$I = \int_C \vec F \cdot d\vec r = \int_C [F_1\,dx + F_2\,dy + F_3\,dz]$$

→ If the curve is closed then

$\int_C \vec{F}\cdot d\vec{r}$ is said to be circulation of $\vec{F}$ round the curve c.

→ If circulation is zero, then $\vec{F}$ is said to be irrotational and ~~any force~~ the force $\vec{F}$ in this case is said to be conservative force.

→ Also work done by a particle during the displacement from A to B is given by $W = \int_C \vec{F}\cdot d\vec{r} = \int_A^B \vec{F}\cdot d\vec{r}$

we know that if $\vec{F}$ is irrotational | conservative $\vec{F}\cdot d\vec{r} = d\phi$

hence

$$W = \int_A^B d\phi = [\phi]_B - [\phi]_A$$

Thus ϕ depends on the value of ϕ at the ends points ~~and~~ and not on the path joining A and B.

Ques: If $\vec{F} = 3xy\,\hat{i} - y^2\hat{j}$ then evaluate $\int_C \vec{F}\cdot d\vec{r}$ for the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Sol: $\int_C \vec{F}\cdot d\vec{r} = \int_C (3xy\,\hat{i} - y^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$

$$= \int_C [3xy\,dx - y^2\,dy]$$

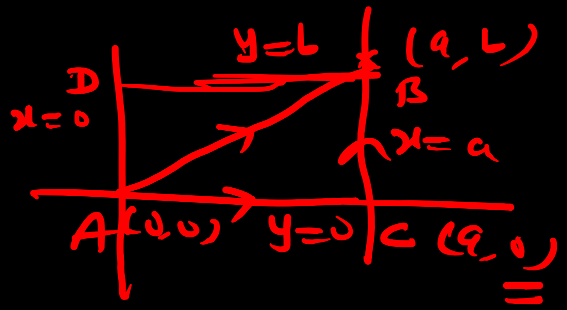
As $y = 2x^2 \Rightarrow dy = 4x\,dx$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C [6x^3\,dx - 4x^4 \times 4x\,dx]$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx = \frac{6}{4} - \frac{16}{6}$$

$$= \frac{3}{2} - \frac{8}{3} = \frac{-7}{6}.$$

Q Find the total work done by a force $\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ in moving a point from $(0,0)$ to (a,b) along the rectangle bounded by the lines $x=0$, $x=a$, $y=0$ and $y=b$.



Sol: $\int_C \vec{F}\cdot d\vec{r} = \int_{AC} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r}$

Along AC

$y=0 \Rightarrow dy=0$

$$\int_{AC} \vec{F}\cdot d\vec{r} = \int_{AC} \vec{F}\cdot d\vec{r} = \int_{AC} (x^2\hat{i})\cdot(dx\,\hat{i}+dy\,\hat{j})$$

$$= \int x^2\,dx = \int_0^a x^2\,dx = \frac{a^3}{3}$$

$\iint_S \vec{F}\cdot\hat{n}\,dS$

Along CB $\quad x = a \Rightarrow dx = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C [(a^2+y^2)\hat{i} - 2ay\hat{j}] \cdot (dx\hat{i} + dy\hat{j})$$

$$= \int_C (-2ay\, dy)$$

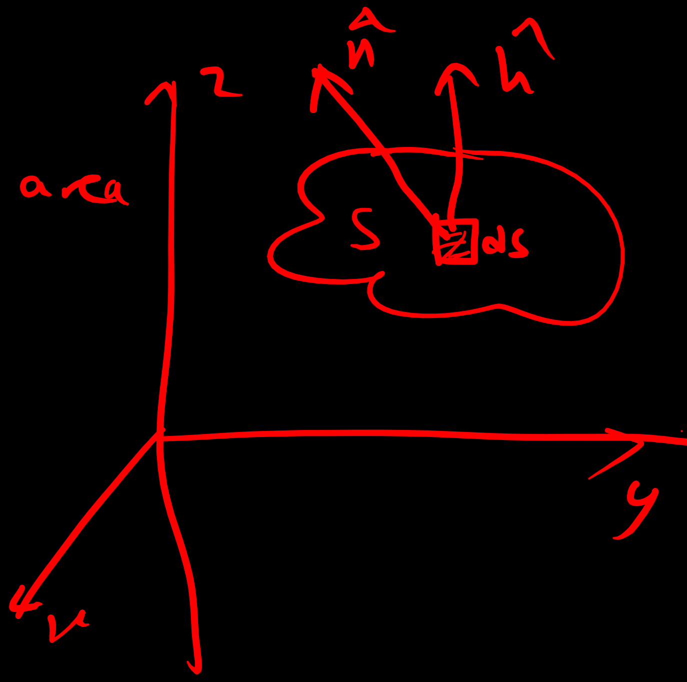
$$= -2a\int_0^b y\, dy = -2a\frac{b^2}{2} = -ab^2$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 \text{ Ans.}$$

Surface integral

Any integral which is evaluated over a surface is known as surface integral.

If $d\vec{s}$ is small element area of the surface area S and $\hat{n}$ be the outward normal to this surface then



$$\iint_S \vec{F}\cdot\hat{n}\,ds \quad \text{or} \quad \iint_S \vec{F}\cdot d\vec{s}$$

is known surface integral, where

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|} = \frac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$$

and

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

<u>volume integral</u>

Any integral which is evaluated over a volume is known as volume integral and it is given by

$$\iiint_V \vec{F}\, dv \quad \text{or} \quad \iiint_V \phi\, dV.$$

Gauss divergence theorem

If $\vec{F}$ is a vector point function having continuous first order partial derivative in the region V bounded by a closed surface S

then $$\boxed{\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \operatorname{div} \vec{F}\, dv}$$

where $\hat{n}$ is the outward unit normal drawn to the surface S.

Stokes theorem:

If S is any open surface bounded by a closed curve C and $\vec{F}$ be the vector point function having continuous first order partial derivatives then

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds}$$

where $\hat{n}$ is the outward unit normal.

Green's theorem:

If c is a regular closed curve in the xy-plane and R is the region bounded by c, then

$$\int_c (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Note: If $M = -y$, $N = x$ then $\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = 1$

$$\therefore \int_c (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\Rightarrow \int_c (-y\,dx + x\,dy) = \iint_R (1+1)\,dx\,dy$$

$$\Rightarrow \boxed{\iint_K dx\,dy = \frac{1}{2}\int_C (x\,dy - y\,dx)}$$

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$\therefore$ Area bounded by a simple closed curve C is

$$= \frac{1}{2}\int_C (x\,dy - y\,dx).$$

- L4: https://youtu.be/ecvZqe0YEOc
- L5: https://youtu.be/CpBPEwEMSs4
- L6: https://youtu.be/WxkyjLJ-w4s