Unit - III

Relations

Cartesian Products of sets

Let A and B be two sets. Then the set of all ordered pair (a, b), where $a \in A$, $b \in B$ is called the Cartesian product. (or Cross product or product set of A and B (in this order) and is denoted by $A \times B$. Thus

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Note:-

$$A \times B \neq B \times A$$

$$A \times A = A^2 = \{(a, b) \mid a \in A \text{ and } b \in A\}$$

If a set A has m elements and a set B has n elements, then a can be chosen from A in m ways and with every one of these choices (of a), b can be chosen from B in n ways. Accordingly (a, b) can be chosen in $m \times n$ ways. this means that $A \times B$ has exactly mn elements.

If A and B are finite sets with $|A| = m$, $|B| = n$, then $A \times B$ is a finite set with $|A \times B| = mn$,

i.e., $|A \times B| = |A| . |B|$

Also $|B \times A| = |B||A| = |A||B| = |A \times B|$.

∴ For any non-empty sets $A_1, A_2, \ldots, A_k$ the k-fold product $A_1 \times A_2 \times \ldots \times A_k$ is defined as the set of all ordered k-tuples $(a_1, a_2, \ldots, a_k)$ where $a_i \in A_i$, $i = 1, 2, \ldots k$.

i.e., $A_1 \times A_2 \times \ldots \times A_k = \{(a_1, a_2, \ldots, a_k) \mid a_i \in A_i, i = 1, \ldots k\}$

eg:- Let $A = \{1,3,5\}, B = \{2,3\}, C = \{4,6\}$.

Then $A \times B = \{(1,2),(1,3),(3,2),(3,3),(5,2),(5,3)\}$

$((A \times B) \cup C) = \{(1,2)(1,3)(3,2)(3,3),(5,2),(5,3),4,6\}$

= &

Relations:

Let A and B are two sets. Then a subset of $A \times B$ is called a binary relation or just a relation from A to B. Thus, if R is a relation from A to B, then R is a set of ordered pairs (a,b) where $a \in A$, $b \in B$, and conversely if R is a set of ordered pair (a,b) where $a \in A$ and $b \in B$, then R is a relation from A to B.

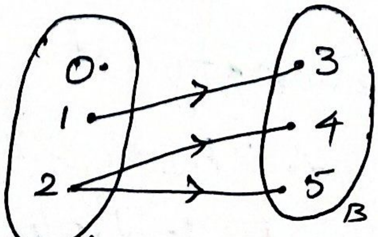
If $(a,b) \in R$, we say that "a is related to b by R" and denoted by aRb.

If R is a relation from A to A, ie., R is a subset of $A \times A$. Then we say that R is a binary relation on A.

eg:- consider the sets $A = \{0,1,2\}$, $B = \{3,4,5\}$.

Let $R = \{(1,3),(2,4),(2,5)\}$. Evidently R is a subset of $A \times B$. As such R is relation from A to B and

$1R3, 2R4, 2R5$



(arrow diagram).

Eg: If A is a set with m elements and B is a set with n elements, find the number of relations from A to B.

Sln:- Since a relation from A to B is a subset of $A \times B$. The set of all relations from A to B is the same as the set of all subsets of $A \times B$.

$\therefore$ No. of relations from $A \times B$ is equal to the no. of subsets of $A \times B$.

Given that $|A| = m$, $|B| = n$, Also we have $|A \times B| = mn$.

$\therefore$ $A \times B$ has 2^{mn} no. of subsets.

eg: Let A & B be finite sets, $|B| = 3$, If there are 4096 relations from A to B. what is $|A|$?

Sl: $n=3$, $2^{mn} = 4096 \Rightarrow m = |A| = 4$.

Zero - One Matrix :-

Consider the sets $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ of order m and n respectively. Then $A \times B$ consists of all ordered pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$, which are mn in number. Let R be a relation from A to B $\Rightarrow$ R is a subset of $A \times B$.

Let us take $m_{ij} = (a_i, b_j) \ni: m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$

This $m \times n$ matrix formed by m_{ij} is called matrix of the relation R or the relation matrix for R. It is denoted by M_R or $M(R)$. Also M_R is called as Zero-One Matrix for R.

eg: 1 Let $A = \{1, 2, 3, 4\}$, $B = \{a_1, a_2\}$, relation R is defined as

$R = \{(1, a_1), (1, a_2), (2, a_2), (3, a_1), (4, a_2)\}$

$$M_R = [m_{ij}] = \begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$$

Digraph of a Relation: -

Let V be a finite non-empty set. A directed graph or diagraph G on V

Digraph of a Relation:

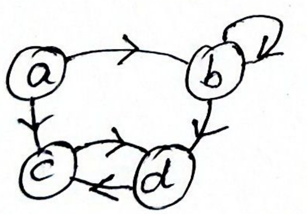
Let A be a finite set and R be a relation on A. Then R can be represented pictorially as follows.

* Draw a small circle for each element of A and label the circle with the corresponding element of A. (these circles are vertices).
* Draw an arrow from a_i to a_j if $a_i R a_j$. (these arrows are edges).

The resulting graph represent the relation R, that is Known as diagraph of R (or Directed graph of R).

An edge from a_i to a_i is called loop (at a_i).

eg:- Let $A = \{a, b, c, d\}$, $R = \{(a,b), (a,c), (b,b), (b,d), (c,d), (d,c)\}$



is the digraph of R.

	Indeg	outdeg
a	0	2
b	2	2
c	2	1
d	2	1

Here indeg = 0, 2, 2, 2 Out deg = 2, 2, 1, 1

1) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy iff "x divides y" (i.e., $x \mid y$)

(a) Write down R as a set of ordered pairs

(b) Draw the diagraph of R.

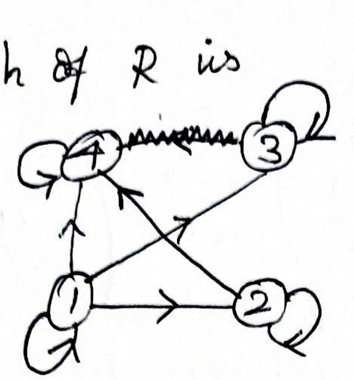
(c) Determine the in-degrees and out-degrees of the vertices in the digraph.

Sln :-

(a) $1|1,\ 1|2,\ 1|3,\ 1|4,\ 2|2,\ 2|4,\ 3|3,\ 4|4$

$\therefore R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(b) Diagraph of R is



Also,

& $$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

(c)

Vertices	In-deg	Out-deg
1	1	4
2	2	2
3	2	1
4	3	1

Composition of Relations :-

Consider a relation R from a set A to a set B and a relation S from the set B to a set C. With these relations, we can define a new relation called the product or the composition of R and S. Denoted by $R \circ S$.
(composition of relation is not commutative but associative).

1) Let $A = \{1,2,3,4\}$, $B = \{w, x, y, z\}$, $C = \{5,6,7\}$
Also let R_1 be a relation from A to B and R_2, R_3 be relations from B to C, defined by.

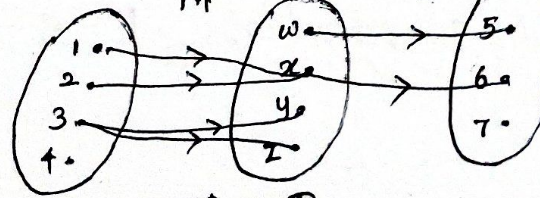
$R_1 = \{(1,x), (2,x), (3,y), (3,z)\}$

$R_2 = \{(w,5), (x,6)\}$, $R_3 = \{(w,5), (w,6)\}$. Find $R_1 \circ R_2$ and $R_1 \circ R_3$.

Sln :-

Given R_1, R_2, R_3 are the relations from A to B, B to C.

i)



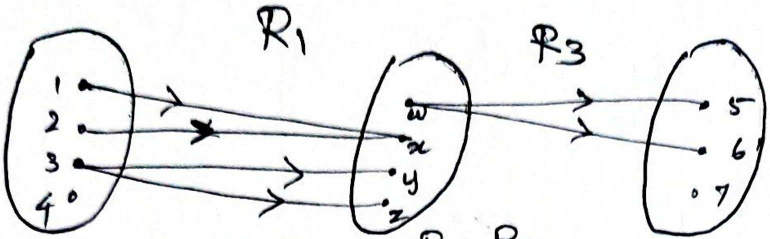
$R_1 \circ R_2$

$\Rightarrow (1,x) \in R_1$ and $(x,6) \in R_2$, $\therefore (1,6) \in R_1 \circ R_2$

$\Rightarrow (2,x) \in R_1$ and $(x,6) \in R_2$, $\therefore (2,6) \in R_1 \circ R_2$

Hence $R_1 \circ R_2 = \{(1,6),(2,6)\}$

ii)



$R_1 \circ R_3$

$\Rightarrow$ there is no element between $R_1 \circ R_3 \Rightarrow R_1 \circ R_3 = \phi$

2) For the relation R_1 and R_2 where $R_1 = \{(1,x),(2,x),(3,y),(3,z)\}$
$R_2 = \{(w,5),(x,6)\}$. Find $M(R_1)$, $M(R_2)$, $M(R_1 \circ R_2)$.
Also verify that $M(R_1 \circ R_2) = M(R_1).M(R_2)$. (Ref. previous Questn)

Sln:-

$$M(R_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \quad M(R_2) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$M(R_1 \circ R_2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore M(R_1).M(R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = M(R_1 \circ R_2)$$

3) Let $A = \{a,b,c\}$ and R, S be relations on A which matrices are given below,

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the composition relations $R \circ S$, $S \circ R$, R^2, S^2 and their matrices.

Sln:- From $M_R \Rightarrow R = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b)\}$ ④

$M_S \Rightarrow S = \{(a,a), (b,b), (b,c), (c,a), (c,c)\}$.

$R \circ S = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b), (c,c)\}$

$S \circ R = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,a), (c,c), (c,b)\}$

$R \circ R = R^2 = \{(a,a), (a,c), (a,b), (b,a), (b,c), (b,b), (c,a), (c,b), (c,c)\}$

$S \circ S = S^2 = \{(a,a), (b,b), (b,c), (b,a), (c,a), (c,c)\}$.

$M(R \circ S) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $M(S \circ R) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $M(R^2) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $M(S^2) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

4) Let $R = \{(1,2), (3,4), (2,2)\}$, $S = \{(4,2), (2,5), (3,1), (1,3)\}$ be a relations on the set $A = \{1, 2, 3, 4, 5\}$

Find $R \circ (R \circ S)$, $R \circ (S \circ R)$, $S \circ (R \circ S)$, $S \circ (S \circ R)$.

Sln:-

$R \circ S = \{(1,5), (3,2), (2,5)\}$

$S \circ R = \{(4,2), (3,2), (1,4)\}$

$R \circ (R \circ S) = \{(1,5), (2,5)\}$

$R \circ (S \circ R) = \{(3,2)\}$

$S \circ (R \circ S) = \{(4,5), (3,5), (1,2)\}$

$S \circ (S \circ R) = \{(3,4), (1,2)\}$.

Properties of Relations:-

Reflexive Relation: A relation R on a set A is said to be reflexive if $(a,a) \in R, \forall a \in A$.

In otherwords, a relation R on a set A is reflexive whenevery every element a of A is related to itself by R. (i.e., aRa, $\forall a \in A$).

R is not reflexive if there is some $a \in A \ni: (a,a) \notin R$.

eg: for a set $A = \{1, 2, 3\}$,
$R = \{(1,1), (2,2), (3,3)\}$ is a reflexive relation.

Symmetric Relation:
A relation R on a set is said to be symmetric if $(b,a) \in R$ whenever $(a,b) \in R$ $\forall$ $a,b \in A$.

A relation R on a set A is said to be antisymmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

Transitive Relation:-
A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$, $\forall a,b,c \in A$

eg: Let $A = \{1,2,3\}$, Find the nature of the relations on A given below,

(i) $R_1 = \{(1,2), (2,1), (1,3), (3,1)\}$ (iii) $R_3 = \{(1,1), (2,2), (3,3)\}$

(ii) $R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$ (iv) $R_4 = \{(1,1), (2,2) (3,3), (2,3) (3,2)\}$

(v) $R_5 = \{(1,3), (3,2)\}$ (vi) $R_6 = \{(2,3), (3,4), (2,4)\}$.

Sln:-
$R_1 \rightarrow$ Symmetric and irreflexive, but neither reflexive nor transitive.

$R_2 \rightarrow$ reflexive and transitive, but not symmetric.

$R_3, R_4 \rightarrow$ Reflexive and symmetric

$R_5 \rightarrow$ Irreflexive, but neither transitive nor symmetric

$R_6 \rightarrow$ Irreflexive, transitive, but not symmetric.

eg2 Let R be a relation on a set A. Prove that

(i) R is reflexive iff $\overline{R}$ is irreflexive

(ii) If R is reflexive so is R^c

(iii) If R is symmetric, so are R^c & $\bar{R}$

(iv) If R is transitive, so is R^c.

Sln:- (i) Suppose R is reflexive, then
$(a, a) \in R$ for every $a \in A$.
Consequently $(a, a) \notin \bar{R}$ for any $a \in A$.
This means that $\bar{R}$ is irreflexive when R is Reflexive

(ii) Suppose R is reflexive $\Rightarrow (a, a) \in R, \forall a \in R$
consequently, $(a, a) \in R^c$ as well.
$\therefore R^c$ is Reflexive.

(iii) Take any $(a, b) \in R^c$ then $(b, a) \in R$
Since R is symmetric $\Rightarrow (a, b) \in R$
$\Rightarrow (b, a) \in R^c \Rightarrow R^c$ is symmetric

(iv) Take any $(a, b), (b, c) \in R^c$
Then $(b, a), (c, b) \in R$
$\Rightarrow (c, a) \in R$ [$\because R$ is transitive]
$\therefore (a, c) \in R^c \Rightarrow R^c$ is transitive.

eg:3 Let R and S be relation on a set A. Prove that
(i) If R and S are reflexive, so are $R \cap S$ & $R \cup S$.
(ii) If R and S are symmetric, so are $R \cap S$ and $R \cup S$.
(iii) If R and S are antisymmetric, so is $R \cap S$.
(iv) If R and S are transitive, so is $R \cap S$.

Sln:- Suppose R and S are reflexive
(i) $\Rightarrow (a, b) \in R, (a, a) \in S \; \forall a \in A$.
Consequently, $(a, a) \in R \cap S$ & $(a, a) \in R \cup S$
$\therefore R \cap S$ & $R \cup S$ are Reflexive.

(ii) Suppose R and S are symmetric

Take any $(a,b) \in R \cap S$ then $(a,b) \in R$ & $(a,b) \in S$

$\therefore (b,a) \in R$ and $(b,a) \in S$.

Consequently, $(b,a) \in R \cap S$.

Hence $R \cap S$ is symmetric.

Now, let take any $(x,y) \in R \cup S$

$(x,y) \in R$ or $(x,y) \in S$

$(y,x) \in R$ or $(y,x) \in S$

$\Rightarrow (y,x) \in R \cup S \Rightarrow R \cup S$ is symmetric.

(iii) Suppose R and S are antisymmetric

Take any $(a,b), (b,a) \in R \cap S$

Then $(a,b), (b,a) \in R$ and $(a,b), (b,a) \in S$.

By the antisymmetric of R (or S) it follows that $R = a$.

Thus $R \cap S$ is anti-symmetric.

(iv) Suppose R and S are transitive,

take $(a,b), (b,c) \in R \cap S$

$\Rightarrow (a,b) \in R, (a,b) \in S, (b,c) \in R, (b,c) \in S$

These yield $(a,c) \in R$ and $(a,c) \in S. \Rightarrow (a,c) \in R \cap S$

$\therefore R \cap S$ is Transitive.

eg 4:- How many different reflexive relation can be defined on a set A containing n elements?

Sln:- Let $A = \{1, 2, 3, \ldots, n\}$

Universal Relation on $A = A \times A$

A total of n^2 ordered pairs lie in universal relation on A.

These ordered pairs can be classified into

Diagonal ordered pairs - total n

Non-diagonal ordered pairs - total n^2-n

Since, diagonal ordered pairs must lie in the Reflexive relation.

Hence, they can be selected in one way.

Remaining non-diagonal ordered pairs, which are (n^2-n) in number, may or may not be in the relation.

Since, each non-diagonal elements have 2 options, either it will be selected or not selected in relation.

$\Rightarrow 2 \times 2 \times 2 \times \dots \times 2$ (up to (n^2-n) times) $= 2^{n^2-n}$

$\therefore$ Total no. of Reflexive Relation on set A having n elements is equal to = No. of ways of selecting diagonal ordered pair

× No. of ways of selecting non-diagonal ordered pairs

$= 1 \times 2^{n^2-n}$

$= 2^{n^2-n}$

Note:- Let A be a set on which relation is defined $\ni$: $|A| = n$, then

	Relations	Number
1.	Relations	2^{n^2}
2.	Reflexive	2^{n^2-n}
3.	Irreflexive	2^{n^2-n}
4.	Neither Reflexive nor irreflexive	$2^{n-1} \times 2^{n^2-n}$
5.	Symmetric	$2^{(n^2+n)/2}$
6.	Asymmetric	$3^{(n^2-n)/2}$
7.	Antisymmetric	$2^n \times 3^{(n^2-n)/2}$
8.	Both symmetric & Asymmetric	1
9.	Both Reflexive and Antisymmetric	$3^{(n^2-n)/2}$

Operations on Relations:-

1) **Union & Intersection:-** Let R_1, R_2 be relations from set A to set B, the union of R_1 and R_2 $(R_1 \cup R_2)$ is defined by $(a,b) \in R_1 \cup R_2$ iff $(a,b) \in R_1$ or $(a,b) \in R_2$.

Similarly, the intersection of R_1 and R_2 $(R_1 \cap R_2)$ is defined with the property that $(a,b) \in R_1 \cap R_2$ iff $(a,b) \in R_1$ and $(a,b) \in R_2$.

[$R_1 \cup R_2$, $R_1 \cap R_2$ in the Universal set $A \times B$].

2) **Complement of a Relation:-** Given a relation R from A to B the complement of R $(\bar{R})$ is defined with the property that $(a,b) \in \bar{R}$ iff $(a,b) \notin R$.

In otherwords, $\bar{R}$ is the complement of R in Universal set $A \times B$.

3) **Converse of a Relation:-** (R^c), Let R be a relation from a set A to set B, the converse of R^c defined wit the property $(a,b) \in R^c$ iff $(b,a) \in R$.

eg:- $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $R = \{(a,1), (b,1), (c,2), (c,3)\}$, $S = \{(a,1), (a,2), (b,1), (b,2)\}$ from A to B, find $\bar{R}$, S^c, $R \cup S$, $R \cap S$.
$(\bar{S}, R^c)$

Sln:- $A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2), (c,3)\}$

$\bar{R} = (A \times B) - R = \{(a,2), (a,3), (b,2), (b,3), (c,1)\}$

$R \cup S = \{(a,1), (b,1), (c,2), (c,3), (a,2), (b,2)\}$

$R \cap S = \{(a,1), (b,1)\}$

$S^c = \{(1,a), (2,a), (1,b), (2,b)\}$

Equivalence Relations :-

A relation R on a set A is said to be an equivalence relation on A if R is reflexive, symmetric and transitive on A.

eg:-1 Let $R = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(3,1),(3,3),$
$(4,1),(4,4)\}$ be a relation on a set $A = \{1,2,3,4\}$.

(i) R is Reflexive as $(a,a) \in R$ for every $a \in A$
$(1,1),(2,2),(3,3),(4,4) \in R$.

(ii) R is symmetric for every $(a,b) \in R$, whenever $(a,b) \in R$ then $(b,a) \in R$ for $a,b \in A$.
$(1,2),(2,1) \in R$ and $(3,4),(4,3) \in R$.

(iii) R is transitive since for all $a,b,c \in R$, we have $(a,c) \in R$ whenever $(a,b) \in R$ & $(b,c) \in R$.
$(1,2),(2,1),(1,1) \in R$,
$(2,1),(1,2),(2,2) \in R$,
$(4,3),(3,4),(4,4) \in R$.

$\therefore$ R is equivalence relation.

eg:2 Let $A = \{1,2,3,4\}$ and
$R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$
be a relation on A. Verify that R is an equivalence relation.

Sln:- (i) Reflexive :- $\{(1,1),(2,2),(3,3),(4,4)\} \in R$
$\Rightarrow (a,a) \in R \ \forall \ a \in R$.
$\therefore$ R is reflexive relation.

(ii) Symmetric :- $\{(1,2),(2,1) \in R\}$ &
$\{(3,4),(4,3) \in R\}$
$\Rightarrow (a,b) \in R$ then $(b,a) \in R$ $\Rightarrow$ R is symmetric relation.

(iii) Transitive :-

$(1,2), (2,1), (1,1) \in R$,

$(2,1), (1,2), (2,2) \in R$,

$(4,3), (3,4), (4,4) \in R$

Whenever $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$

$\therefore$ R is transitive relation.

So R is reflexive, symmetric & transitive.

$\therefore$ R is an equivalence relation.

eg3:- If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$ and $A_3 = \{5\}$, define the relation R on by xRy iff x and y are in the same set A_i, $i = 1,2,3$. Is R is an equivalence relation?

Sln:- Reflexive :- xRx, $\forall x$ in A_i

because $(x,x) \in A_i \Rightarrow R$ is Reflexive.

Symmetric :- if $(x,y) \in A_i$, then $(y,x) \in A_i$ $\forall\, x,y \in A$

$\therefore$ R is symmetric.

Transitive :-

$(1,2) \in R$ [$\because$ 1 & 2 are in the same set A_1]

$(2,3) \in R$ [$\because$ 2 & 3 are in the same set A_2]

but $(1,3) \notin R$ [$\because$ 1 & 3 not in the same set].

Hence R is not transitive

$\therefore$ R is not an equivalence relation.

~~eg~~ congruence modulo ~~a~~ :-

eg 4:- For a fixed integer $n > 1$, Prove that the relation "congruent modulo n" is an equivalence relation on the set of all integers Z.

Sln:- Let us denote this relation by R so that aRb means $a \equiv b \pmod{n}$.

(i) Reflexive:- For every $a \in Z$

$a - a = 0$ is multiple of n

$\Rightarrow a \equiv 0 \pmod{n}$

$\Rightarrow a R a$

$\therefore R$ is Reflexive.

(ii) Symmetric:- $\forall a, b, \in Z \Rightarrow aRb \Rightarrow a \equiv b \pmod{n}$

$\Rightarrow (a-b)$ is a multiple of n

$\Rightarrow (b-a)$ is a multiple of n

$\Rightarrow b \equiv a \pmod{n}$

$\Rightarrow b R a$

$\therefore R$ is symmetric.

(iii) Transitive:- $\forall a, b, c \in Z$,

aRb and $bRc \Rightarrow a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

$\Rightarrow (a-b)$ and $(b-c)$ are multiples of n

$\Rightarrow (a-b) + (b-c) = a-c$ is multiples of n.

$\Rightarrow a \equiv c \pmod{n}$

$\Rightarrow aRc$.

$\therefore R$ is transitive

Hence R is an equivalence relation.

eg:5:- Let $A = \{1, 2, 3, 4, 5\}$. Define a relation R on $A \times A$ by $(x_1, y_1) R (x_2, y_2)$ iff $x_1 + y_1 = x_2 + y_2$. Verify that R is an equivalence relation on $A \times A$.

Sln:- Reflexive:- take $(x, y) \in A \times A$

we have, $x+y = x+y$

$\Rightarrow (x,y)\ R\ (x,y)$

$\therefore$ R is Reflexive.

(ii) Symmetric:- take $(x_1,y_1),(x_2,y_2) \in A\times A$

$\Rightarrow (x_1,y_1)\ R\ (x_2,y_2)$

then, $x_1+y_1 = x_2+y_2$.

This gives $x_2+y_2 = x_1+y_1 \Rightarrow (x_2,y_2)\ R\ (x_1,y_1)$

$\therefore$ R is symmetric.

(iii) Transitive:- take $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in A\times A$

$\Rightarrow (x_1,y_1)\ R\ (x_2,y_2)$ and $(x_2,y_2)\ R\ (x_3,y_3)$

then $x_1+y_1 = x_2+y_2$

$x_2+y_2 = x_3+y_3$

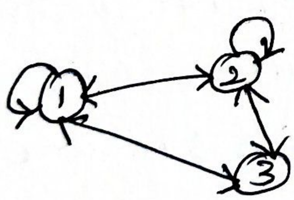
$\Rightarrow x_1+y_1 = x_3+y_3$

$\Rightarrow (x_1,y_1)\ R\ (x_3,y_3)$

$\therefore$ R is transitive.

Thus R is an equivalence relation.

eg.6:- The diagraph of a relation R on the set $A=\{1,2,3\}$ is given. Determine whether R is an equivalence relation.



Sln:- from the digraph, we note that the given relation is symmetric and transitive but not reflexive.

$\because (3,3)\notin R \Rightarrow$ R is not an equivalence relation.

Closure :-

If R is a relation on a set A, then the closure of R with respect to P, if it exists, is the relation S on A with property P that contains R and is a subset of every subset of $A \times A$ containing R with property P.

Reflexive closure : A relation R on A is obtained by adding (a,a) to R for each $a \in A$.

Symmetric closure : R is obtained by adding (b,a) to R for each $(a,b) \in R$.

Transitive closure : R is obtained by repeatedly adding (a,c) to R for each $(a,b) \in R$ and $(b,c) \in R$.

eg:1 Find Reflexive closure, given $R = \{(1,2),(2,3),(3,4)\}$ on set $A = \{1,2,3,4\}$.

Sln:- To find the reflexive closure, we need to add all pairs (a,a) that are missing from R for all $a \in A$.

Reflexive closure of $R = R \cup \{(1,1),(2,2),(3,3),(4,4)\}$

$= \{(1,1),(1,2),(2,2),(2,3),(3,3),(3,4)(4,4)\}$

eg:- $R = \{(1,2),(2,3),(1,3)\}$ on set $A = \{1,2,3\}$.

(we need to add (b,a) for every (a,b) in R for symme closure

$R = R \cup \{(2,1),(3,2),(3,1)\}$

$= \{(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)\}$

Equivalence classes :-

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements x of A which are related to a by R is called the equivalence class of a with respect to R. This equivalence class is denoted by $R(a)$ or $[a]$ or $\bar{a}$.

i.e., $\bar{a} = [a] = R(a) = \{x \in A \mid (x, a) \in R\}$

(equivalence class is the name given to a subset of some equivalence relation R which includes all the elements that are equivalent to each other. $[x] = \{y \mid (x, y) \in R\}$.

eg:- Let $A = \{1, 2, 3, 4, 5\}$

$R = \{(a, b) \mid a+b$ is even$\}$ [R is defined on A]

(i) Reflexive: $a + a = 2a$

(ii) Symmetric: $a+b$ is even $\rightarrow$ $b+a$ is even

(iii) Transitive: $a+b$ is even, $b+c$ is even $\rightarrow$ $a+c$ is even

Both a and b can be either even or odd.

If a is even and b is even.

b is even and c is even, then $a+c$ is even.

If a is odd and b is odd.

b is odd and c is odd, then $a+c$ is even.

Therefore, R is an equivalence relation.

$[1] = \{1, 3, 5\}$ because $1R1$, $1R3$, $1R5$

$[2] = \{2, 4\}$

$[3] = \{1, 3, 5\}$

$[4] = \{2, 4\}$

$[5] = \{1, 3, 5\}$

Equivalence class of elements 1, 3 and 5 are same and equivalence class of elements 2 and 4 are same.

Any element out of 1, 3 and 5 can be chosen as a representative of the equivalence class $\{1, 3, 5\}$.

Also, any element out of 2 and 4 can be the representative of equivalence class $\{2, 4\}$.

Let say 1 is the representative of equivalence class $\{1, 3, 5\}$ and 2 is the representative of equivalence class $\{2, 4\}$

$\therefore\ [1] = \{1, 3, 5\}$

$[2] = \{2, 4\}$.

<u>Partition of a set</u> :-

Let A be a nonempty set. Suppose there exist nonempty subsets $A_1, A_2, \ldots, A_k$ of A $\ni$: the following two conditions hold.

1) A is the union of $A_1, A_2, \ldots, A_k$ that is

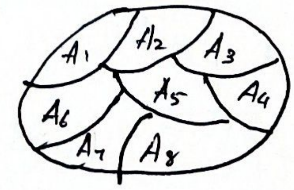
$A = A_1 \cup A_2 \ldots \cup A_k$.

2) Any two of the subsets $A_1, A_2, \ldots, A_k$ are disjoint

i.e. $A_i \cap A_j = \phi$ for $i \neq j$.

Then the set $P = \{A_1, A_2, \ldots, A_k\}$ is called <u>partition</u> of A. Also $A_1, A_2 \ldots, A_k$ are called the blocks or cells of the partition.

A partition of set



<u>Thrm</u>:- Let R be an equivalence relation on set A. These statements for elements a and b of A are equivalent (i) aRb

(ii) $[a] = [b]$; (iii) $[a] \cap [b] \neq \phi$.

Thrm:- Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i | i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$ as its equivalence classes.

eg1:- Consider the set $A = \{1, 2, 3, 4, 5\}$ and the equivalence relation $R = \{(1,1)\ (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$ defined on A. Find the partition of A induced by R.

Sln:- From the given R, we get,

$[1] = \{1\}$, $[2] = \{2,3\}$, $[3] = \{2,3\}$, $[4] = \{4,5\}$, $[5] = \{4,5\}$.

From these equivalence classes, only $[1]$, $[2]$ and $[4]$ are distinct.

$\Rightarrow$ the partition P of A determined by R.

$\therefore P = \{[1], [2], [4]\}$ is partition induced by R.

$\Rightarrow A = [1] \cup [2] \cup [4] = \{1\} \cup \{2,3\} \cup \{4,5\}$.

eg2 Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and R be the equivalence relation on A that induces the partition $A = \{1,2\} \cup \{3\} \cup \{4,5,7\} \cup \{6\}$

Find R.

Sln:- Given partition of A has 4 blocks

$\{1,2\}$, $\{3\}$, $\{4,5,7\}$, $\{6\}$.

Let R be the equivalence relation inducing this partition

Since the elements 1, 2 are in the same block, we have

$1R1$, $1R2$, $2R1$, $2R2$.

Since $3 \in$ block $\{3\}$ which contains only 3, $3R3$.

Since 4, 5, 7 belong to the same block, we have

$4R4$, $4R5$, $4R7$, $5R4$, $5R5$, $5R7$, $7R4$, $7R5$, $7R7$

Since 6 belongs to $\{6\}$ which contains only 6, we have $6R6$.

$$\therefore R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(4,5),(4,7),(5,4),(5,5)(5,7),(7,4),(7,5)(7,7),(6,6)\}.$$

Partial Orders :-

A relation R on set A is said to be a partial ordering relation or a partial order on A if

(i) R is reflexive

(ii) R is antisymmetric,

(iii) R is transitive on A.

A set A with a partial order R defined on it is called a partially ordered set or an ordered set or a poset, and is denoted by the pair (A, R).

Total Order :-

Let R be a partial order on a set A. Then R is called a total order on A if for all $x, y \in A$, either xRy or yRx. The poset (A, R) is called a totally ordered set.

Hasse Diagrams :-

The diagraph of a partial order by adopting the following conventions is called poset digram or Hasse diagram.

(i) Since the partial order is reflexive, at every vertex in the digraph of a partial order, there would be a cycle of length 1. Hence such cycle need not be exhibited explicitly.

(ii) If the digraph of a partial order, there is an edge from a vertex a to vertex b and there is an edge from vertex b to vertex c, then there should be an edge from a to c, since it is transitive.

(iii) To simplify the format of the diagraph of a partial order, the vertices are represented by dots and all edges are pointed upwards.

1) Let $A = \{1, 2, 3, 4, 6, 12\}$. On A, define the relation R by aRb iff a divides b. Prove that R is a partial order on A. Draw the Hass digram for this relation.

Sln:- $R = \{(a,b) \mid a, b \in A$ and a divides b$\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12),$
$(2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,12),$
$(4,4), (4,12), (6,6), (6,12), (12,12)\}$.

(i) Reflexive :- $(a,a) \in R$, $\forall a \in A$ $\Rightarrow$ R is reflexive.

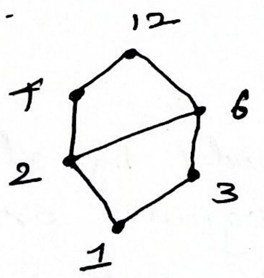
(ii) Antisymmetric :- $(a,b) \in A$, if a divides b and b divides a then $a \neq b$

$\Rightarrow$ R is antisymmetric.

(iii) Transitive :- $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$

$\Rightarrow$ R is transitive.

Hass diagram :-

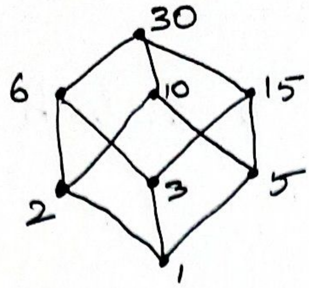


2) Consider the partial order of divisibility on the set A. Draw the Hass diagram for the poset and determine whether the poset is totally ordered or not.

(i) $A_1 = \{1, 2, 3, 5, 6, 10, 15, 30\}$

(ii) $A_2 = \{1, 3, 6, 12, 24\}$

(i)

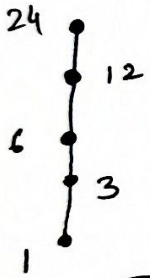


$R = \{(1,1)\ (1,2)\ (1,3)\ (1,5)$
$(1,6)\ (1,10), (1,15), (1,30),$
$(2,2), (2,6)\ (2,10), (2,30)$
$(3,3), (3,15)\ (3,30), (5,5)$
$(5,10)\ (5,15), (5,30), (6,6)$
$(6,30), (10,30), (10,10), (15,30)$
$(15,15), (30,30)\}$

Here neither 2 divides 3 nor 3 divides 2
3 " 5 " 5 " 3
6 " 10 " 10 " 6

By defn of total order, the poset (A_1, R) is not totally ordered set.

(ii) $A_2 = \{1, 3, 6, 12, 24\}$
$R = \{(1,1)\ (1,3)\ (1,6)\ (1,12), (1,24), (3,3)\ (3,6), (3,12), (3,24),$
$(6,6)\ (6,12), (6,12), (12,2), (12,24), (24,24)\}$



By defn, the poset (A_2, R) is totally ordered set.

eg 3:- Draw Hasse diagram (i) representing the positive divisors of 36 (ii) $A = \{1,2,3,4,6,8,9,12,18,24\}$ for the divisibility relation on set A.

Sln:- +ve Divisor of 36 $\Rightarrow D_{36} = \{1,2,3,4,6,9,12,18,36\}$

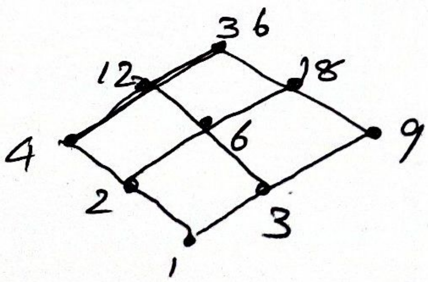
$1 \xrightarrow[\text{to all}]{\text{related}} D_{36}$; $2 \xrightarrow[\text{to}]{\text{related}} 2,4,6,12,18,36,$

$3 \longrightarrow 3,6,9,12,18,36$; $4 \longrightarrow 4,12,36$

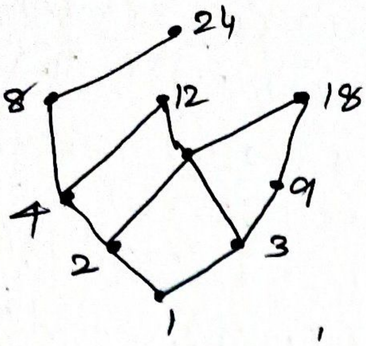
$6 \longrightarrow 6,12,18,36$; $9 \longrightarrow 9,18,36$; $12 \longrightarrow 12,36$

$18 \longrightarrow 18,36$; $36 \longrightarrow 36$

Hasse diagram :



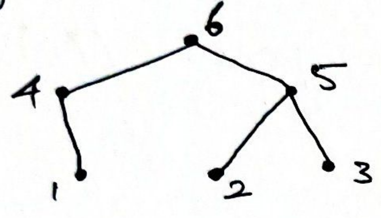
(ii)



aRb iff a divides b ; $a, b \in A$

Practice Questions :-

1) Hasse diagram of a partial order R on the set $A = \{1, 2, 3, 4, 5, 6\}$ is given. Write down R as a subset of $A \times A$. Construct its digraph.



2) Draw the Hasse diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$. whose matrix is $M_R = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$