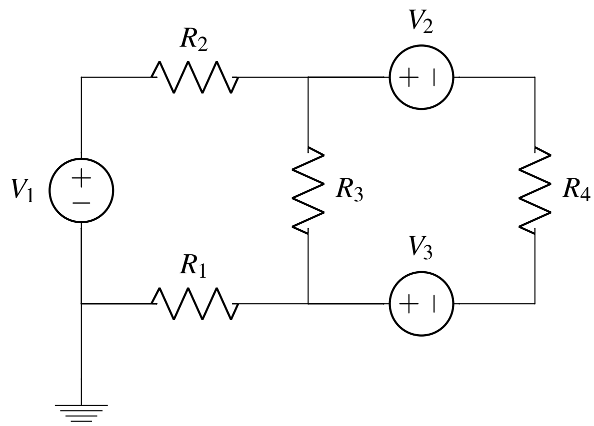
1. Supernode

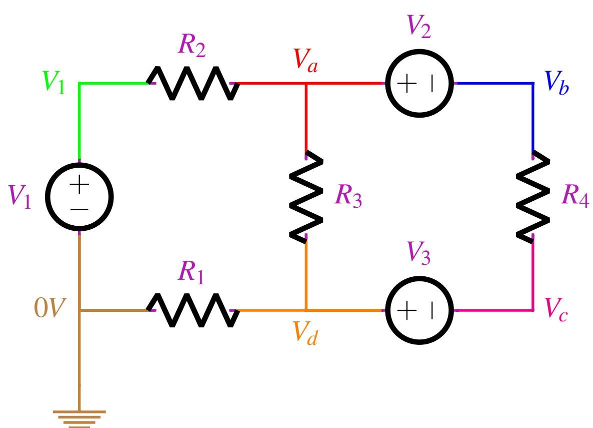
In this question, we will explore how to deal with multiple voltage sources when doing nodal analysis.



(a) Mark all the nodes. If you know the potential at the node, write down the value next to the node. If you don't know the value, then assign a variable for the potential.

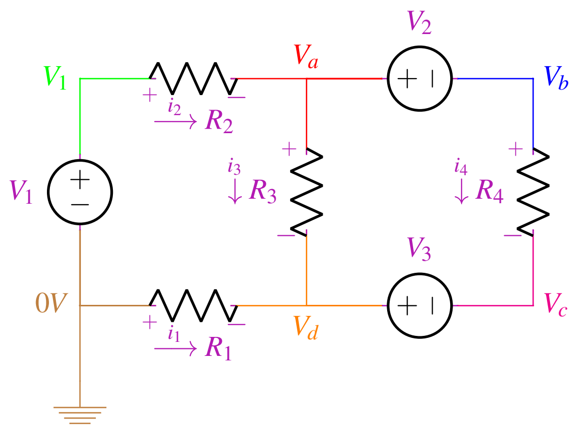
Answer:

V_a, V_b, V_c, V_d are variables. We know the potentials at the node marked ground and at the node marked V_1.



(b) Mark current directions arbitrarily and corresponding polarities on each resistor. Note that if current goes from left to right, then the left side of the resistor is to be marked + and the right side must be marked -. This is the passive sign convention.

Answer:



(c) Note that we define 4 nodes with unknown potentials. So we need 4 equations. Each of these nodes with unknown potential should give us one equation.
Write the equation for the first node with unknown potential.
Answer: For the V_a node, we have a problem. The problem is that we don't know what current enters the voltage source V_2. The fix here is to treat V_a and V_b combined as a super-node. Since we are considering 2 nodes at the same time, this 'supernode' better give us 2 equations!
We can get equation from the fact that the voltage source exists.

$$V_a - V_b = V_2 \tag{1}$$

For the other equation, let's consider: (Sum of current entering nodes V_a and V_b) = (Sum of current leading nodes V_a and V_b).

Consider V_a:

- i_2 enters V_a.
- i_3 leaves V_a.

Consider V_b:

- i_4 leaves V_b.

$$i_2 = i_3 + i_4$$
$$\frac{V_1 - V_a}{R_2} = \frac{V_a - V_d}{R_3} + \frac{V_b - V_c}{R_3} \tag{2}$$

(d) Write equations for all remaining unknown nodes.
Answer: Nodes V_c and V_d remain. Let's start at V_c. Again, we don't know what current enters V_3, so we do the same thing. Treat the 2 ends of the voltage source, V_c and V_d as a super-node. One equation we get is

$$V_d - V_c = V_3 \tag{3}$$

The other equation comes from (Sum of current entering nodes V_a and V_b) = (Sum of current leading nodes V_a and V_b).

Consider V_c:

- i_4 enters.

Consider V_d:

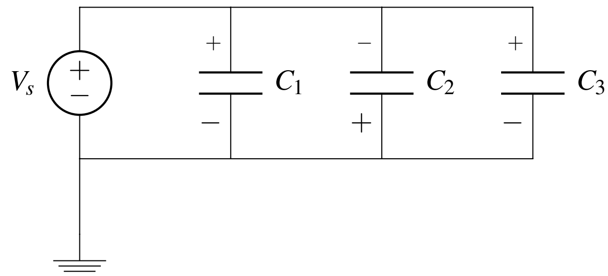
- i_3 enters
- i_1 enters.

$$i_4 + i_3 + i_1 = 0$$
$$\frac{V_b - V_c}{R_4} + \frac{V_a - V_d}{R_3} + \frac{0 - V_d}{R_1} = 0 \tag{4}$$

Equations 1, 2, 3, 4 are 4 equations for the 4 variables V_a, V_b, V_c, V_d and can be solved uniquely. One fun fact is that these equations are guaranteed to be linearly independent as long as the circuit is not an impossible circuit (such as 2 current sources of differing values in series, or 2 voltage sources of different values in parallel).

2. Pitfall Problem

(a) Consider the following circuit in ϕ_1. Assume that all capacitors are initially discharged. Find out the charge on each capacitor in this phase.

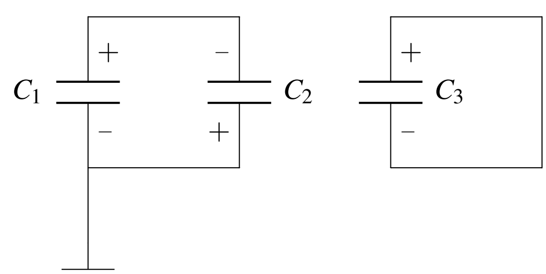


Answer:

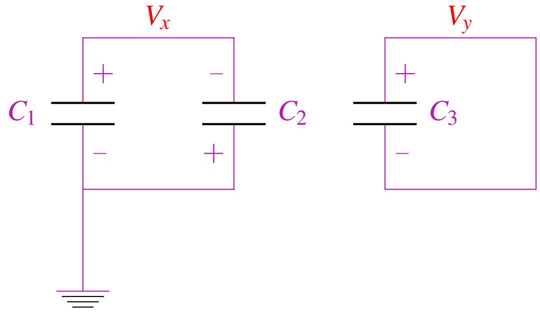
$$Q_{C_1,\phi_1} = C_1 V_s$$
$$Q_{C_2,\phi_1} = -C_2 V_s$$
$$Q_{C_3,\phi_1} = C_3 V_s$$

Note: What does it mean when we say that the charge on a capacitor is *negative*? It means that **on the positive plate of the capacitor, there is negative charge.** And since a capacitor has equal and opposite charge on each plate, the negative plate then has positive charge.

(b) Assume that ϕ_1 has taken place, and that the capacitors are then moved to the following configuration in ϕ_2. Calculate the charge across each capacitor in ϕ_2.



Answer: **Step I: Write the voltage drop across each capacitor.**
If it cannot be determined, create variables till it can be determined.
Circuit redrawn with unknown voltages marked.



$$V_{C_1,\phi_2} = V_x - 0 = V_x$$
$$V_{C_2,\phi_2} = 0 - V_x = -V_x$$
$$V_{C_3,\phi_2} = V_y - V_y = 0$$

Step 2: Write the charge on each capacitor.
Just use $Q = CV$ on the first step

$$Q_{C_1,\phi_2} = C_1 V_x$$
$$Q_{C_2,\phi_2} = -C_2 V_x$$
$$Q_{C_3,\phi_2} = 0$$

Step 3: Write the charge sharing equations on floating nodes.
Floating nodes are those where charges cannot escape or enter.
Node marked V_x is the only floating node

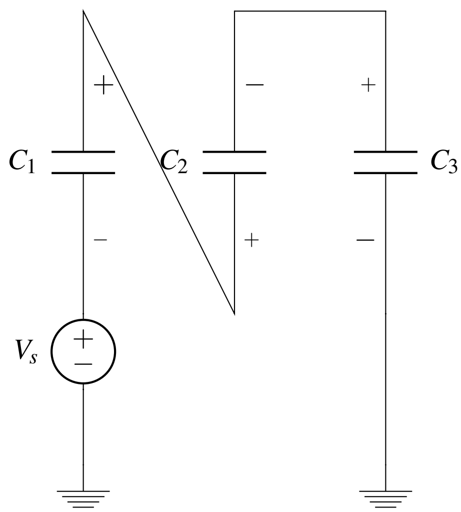
$$Q_{C_1,\phi_2} - Q_{C_2,\phi_2} = Q_{C_1,\phi_1} - Q_{C_2,\phi_1} \tag{5}$$

$$C_1 V_x - (-C_2 V_x) = C_1 V_s - (-C_2 V_2) \implies V_x = V_s$$

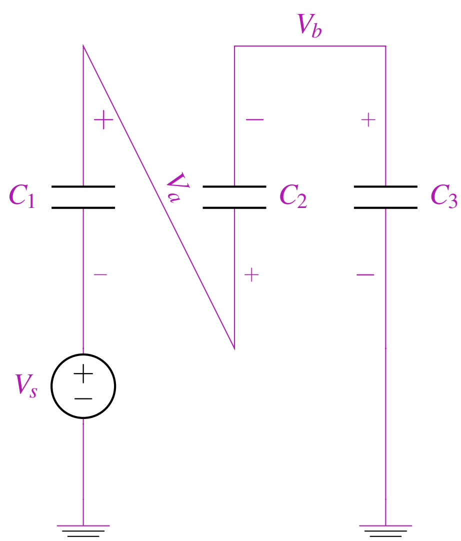
Final Step 4: Plug variable voltage into charge equations

$$Q_{C_1,\phi_2} = C_1 V_x = C_1 V_s$$
$$Q_{C_2,\phi_2} = -C_2 V_x = -C_2 V_s$$
$$Q_{C_3,\phi_2} = 0$$

(c) Assume that ϕ_2 has taken place, and that the capacitors are then moved to the following configuration in ϕ_3. Calculate the charge across each capacitor in ϕ_3.



Answer: **Step I: Write the voltage drop across each capacitor.**
If it cannot be determined, create variables till it can be determined.
Circuit redrawn with unknown voltages marked.



$$V_{C_1,\phi_3} = V_a - V_s$$
$$V_{C_2,\phi_3} = V_a - V_b$$
$$V_{C_3,\phi_3} = V_b$$

Step 2: Write the charge on each capacitor.
Just use $Q = CV$ on the first step

$$Q_{C_1,\phi_3} = C_1(V_a - V_s)$$
$$Q_{C_2,\phi_3} = C_2(V_a - V_b)$$
$$Q_{C_3,\phi_3} = C_3(V_b)$$

Step 3: Write the charge sharing equations on floating nodes.
Floating nodes are those where charges cannot escape or enter.
Nodes marked V_a and V_b are the only floating nodes

$$Q_{C_1,\phi_3}+Q_{C_2,\phi_3} = Q_{C_1,\phi_2}+Q_{C_2,\phi_2} \tag{6}$$

$$-Q_{C_2,\phi_3}+Q_{C_3,\phi_3} = -Q_{C_2,\phi_2}+Q_{C_3,\phi_2} \tag{7}$$

Plugging in values to Equation 6

$$C_1(V_a - V_s) + C_2(V_a - V_b) = C_1V_s + (-C_2V_s)$$

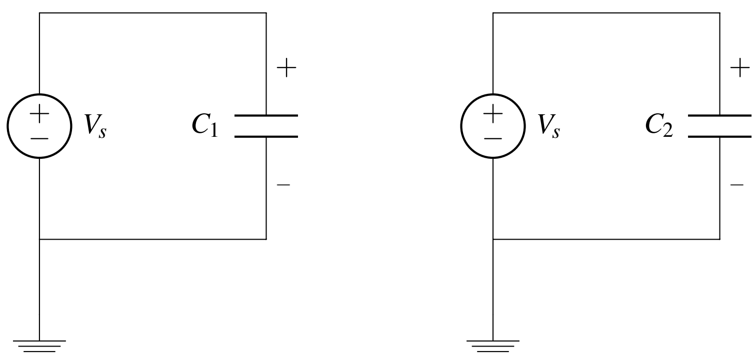
Plugging in values to Equation 7

$$-C_2(V_a - V_b) + C_3V_b = -(-C_2V_s) + 0$$

Final Step 4: Plug variable voltage into charge equations
We now have two equations in two variables (V_a, V_b), and so we can solve for them. After that, we can plug those into Step 2, and find the charges on each capacitor. There is no need to do this step as the expressions aren't very neat.

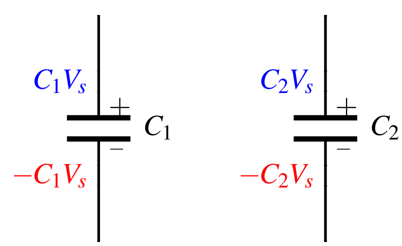
3. Series equivalence... or not?

(a) Consider the following 2 circuits. What is the charge on the positive and negative plates of the two capacitors?

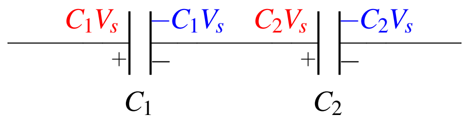


Answer: The charge on the positive plate of C_1 is C_1V_s. The charge on the negative plate is then $-C_1V_s$.
The charge on the positive plate of C_2 is C_2V_s. The charge on the negative plate is then $-C_2V_s$.

(b) Now consider that we first cut the capacitors off from their voltage sources and the ground nodes as such:



Next, we will connect these two capacitors as such:



Question: Can the charges on the positive plate of the capacitor C_1 move?

Answer: No, because charges cannot jump across the plates of a capacitor and there is no path for these charges to escape.

(c) What about the charges on the negative plate of C_1 and the positive plate of C_2?

Answer: In theory, these charges could redistribute... but look at the answer to the parts below.

(d) What about the charges on the negative plates of C_2?

Answer: These also cannot move similar to the charges on the positive plate of C_1.

(e) Here is a fundamental fact: If a capacitor's positive plate has x charge, then the negative plate must have $-x$ charge!
Question: These two capacitors look like they're in series. So they must have the same charge. How is this possible?

Answer: The capacitors 'look' like they're in series, and they are if looked at as electrical components. However, let's go through the derivation of the 'series equivalence formula' for capacitors.

Derivation:

$$V = V_1 + V_2 + \ldots + V_n$$

where V is the voltage drop across the branch of capacitors in series, and V_i are the individual voltage drops.

$$\frac{Q_{eq}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \ldots + \frac{Q_n}{C_n}$$

where Q_{eq} is the charge on the equivalent capacitor, C_{eq} is the equivalent capacitance, Q_i is the charge on each individual capacitor and C_i is the capacitance of each individual capacitor.
Since we know that the charge on each individual capacitor is the same and the charge on the equivalent capacitor is also equal to this charge, $Q_{eq} = Q_1 = Q_2 = \ldots = Q_n = Q$ **(say)**

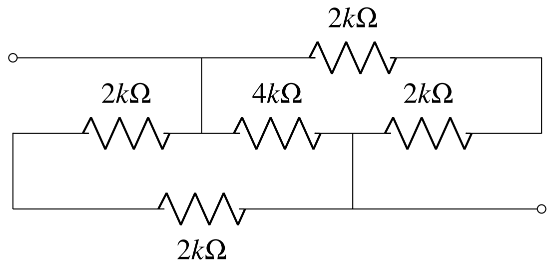
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

Notice that in this derivation we *assume* that the charge on capacitors in series is the same, which leads to the formula. So capacitors having the same charge $\implies$ capacitors in series equivalence formula holds, not capacitors as components in series $\implies$ the series equivalence formula holds. The formula holds if and only if the capacitors are either discharged to begin with, or have the same charge on them to begin with!!

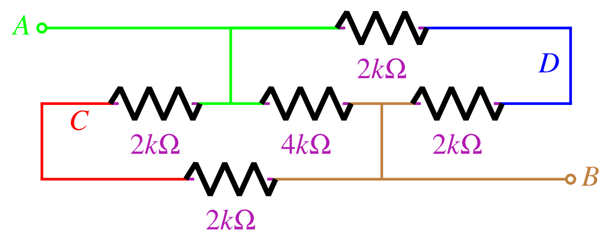
Intuitively, in this case, the capacitors do not have the same charge because the positive charges from C_1 and the negative charges from C_2 cannot move. This **forces** the negative charges of C_1 to stay where they are the positive charges of C_2 to stay where they are!!

4. Never fail at resistor equivalence

(a) Mark all nodes on the following circuit.



Answer:



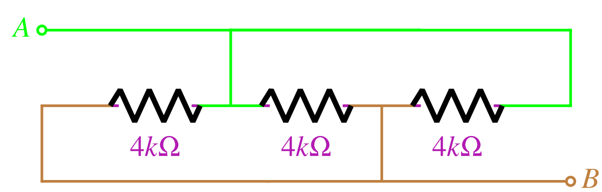
(b) Mark which nodes are 2-nodes and multi-nodes. 2 nodes are connected to only 2 components, and multi-nodes are connected to 3 or more components.
Note: *Whenever nodes are marked across which equivalent resistance must be found, those are considered 'components' because something could be connected there.*

Answer: Nodes A and B are multi-nodes. Nodes C and D are 2-nodes.

(c) Resistors that are connected to 2-nodes are considered to be in series. Redraw the circuit, and find the 2 and multi-nodes again after combining the resistors connected to 2 nodes.

Answer:

Nodes C and D disappear and become nodes A and B.



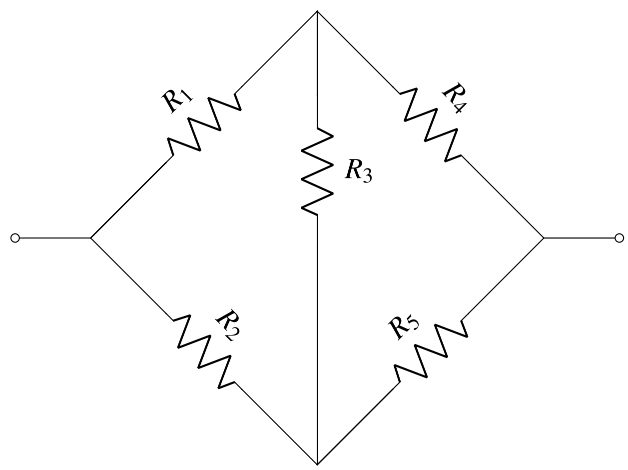
(d) Now we should be left with only multi-nodes. So far, we have seen that any resistors connected to 2-nodes are in series. We will see what happens to resistors connected to multi-nodes. Begin by writing out the 2 nodes that each resistor is connected to.

Answer: The resistor on the left is connected to nodes A and B. The resistor on the middle is connected to nodes A and B. The resistor on the right is connected to nodes A and B.

(e) If you have 2 or more resistors that are connected to the same 2 nodes, then they are in parallel. What does this mean for the 3 remaining resistors?

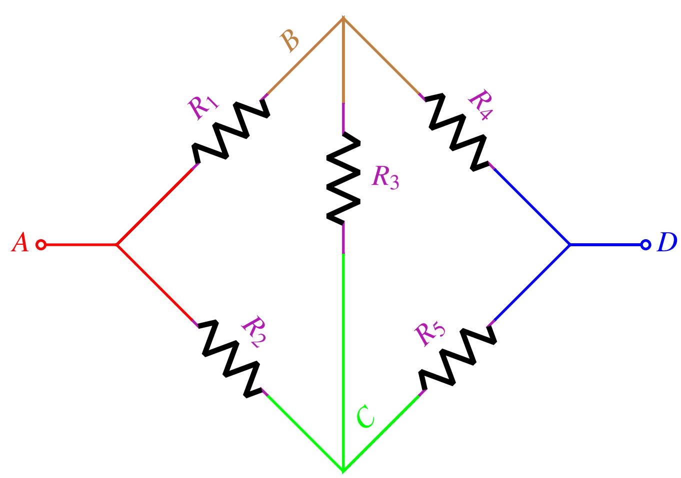
Answer: Since all 3 resistors are connected to nodes A and B, they are all in parallel. The equivalent resistance is $\frac{4}{3}k\Omega$. Quick tip: if you have 2 resistors in parallel, both of whose resistance is R, then the equivalent resistance is $\frac{R}{2}$. Similarly, if you have 3 resistors in parallel, all three of whose resistance is R, then the equivalent resistance is $\frac{R}{3}$, and so on with 4 or more resistors of the same value.

5. Another one for mastery



(a) Mark and label the nodes on the circuit above.

Answer:



(b) Which nodes are 2 nodes? Which nodes are multi-nodes?

Answer: All nodes are multi-nodes. Remember that the terminals across which equivalent resistance will be measured are also considered components.

(c) Combine 2-node resistors are they are in series, similar to the previous question.

Answer: Trick question! There are none!

(d) Write down each resistor and the nodes that it is connected to.

Answer: R_1 is connected to A and B.
R_2 is connected to A and C.
R_3 is connected to B and C.
R_4 is connected to B and D.
R_5 is connected to C and D.

(e) What do the nodes that the resistors are connected to tell you?

Answer: This says that no 2 resistors are in parallel with each other. Recall that we need R_x and R_y to both be connected to some node M and N for them to be in parallel. Since this is not the case with any pair of resistors, none of them are in parallel or series. **We cannot simplify this circuit any further using series and parallel equivalences.**

6. **First Proof** Prove that a subset of a finite linear independent subset of vectors is linearly independent

Answer:

$$\textbf{Given: } \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \ldots + \alpha_n\vec{v_n} = 0 \implies \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

$$\textbf{To Prove: } \beta_1\vec{v_1} + \beta_2\vec{v_2} + \ldots + \beta_k\vec{v_k} = 0 \implies \beta_1 = \beta_2 = \ldots = \beta_k = 0$$

Assume that $\beta_1\vec{v_1} + \beta_2\vec{v_2} + \ldots + \beta_k\vec{v_k} = 0$ is true but not $\beta_1 = \beta_2 = \ldots = \beta_k = 0$.
Consider $\beta_1\vec{v_1} + \beta_2\vec{v_2} + \ldots + \beta_k\vec{v_k} + 0\vec{v_{k+1}} + 0\vec{v_{k+2}} + \ldots + 0\vec{v_n}$. If $\beta_1\vec{v_1} + \beta_2\vec{v_2} + \ldots + \beta_k\vec{v_k} = 0$ then

$$\beta_1\vec{v_1} + \beta_2\vec{v_2} + \ldots + \beta_k\vec{v_k} + 0\vec{v_{k+1}} + 0\vec{v_{k+2}} + \ldots + 0\vec{v_n} = 0$$

However, since we assumed that not all $\beta_1, \beta_2, \ldots, \beta_k$ are 0, this means that the set $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$ is not linearly independent, which is a contradiction because it is given that the set is linearly independent. Therefore, $\beta_1 = \beta_2 = \ldots = \beta_k = 0$ must have been true.

7. **Solutions of linear equations**

(a) Consider the following set of linear equations:

$$2x + 3y + 5z = 0$$
$$-1x - 4y - 10z = 0$$
$$x - 2y - 8z$$

Place these equations into a matrix, and row reduce the matrix.

Answer:

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2}R_1$$
$$R_2 = R_2 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy!

$$R_2 = \frac{1}{-2.5}R_2$$

$$R_3 = -2R_3$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Convert the row reduced matrix back into equation form.

Answer:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 3y + 5z = 0$$

$$0x + 1y + 3z = 0$$

$$0x + 0y + 0z = 0$$

(c) Intuitively, what does the last equation from the previous part tell us?

Answer: It tells us that there are infinite solutions to the equations. $0x + 0y + 0z = 0$ is satisfied by **any** x, y, z.

(d) What is the general form of the infinite solutions to the system? Clearly, x, y, z cannot **actually** take on any values. The values $x = 1, y = 1, z = 1$ don't satisfy the first equation, so they don't work.

Answer: z is a free variable. If $z = t$, then

$$y = -3z = -3t$$

$$2x + 3y + 5z = 0 \implies 2x - 3t + 5t = 0 \implies 2x = -2t \implies x = -t$$

The general solution is then $t \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$. We can also say that the vector $\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$ forms a basis for the solution set of the system of equations defined in this question.

8. Null space drill

In this question, we explore intuition about null spaces and a recipe to compute them. Recall that the nullspace of a matrix $\mathbf{M}$ is the set of all vectors, $\vec{x}$ such that $\mathbf{M}\vec{x} = \vec{0}$.

(a) First, we begin by proving that a null space is indeed a subspace. Show that any nullspace of a matrix $\mathbf{M}$ with n rows and n columns is a subspace.

Steps for reference:

i. claim subset of X

ii. claim X is well defined space

iii. closures and 0

   i. closure under addition

   ii. closure under scalar multiplication

   iii. existence of the zero element.

Answer:

i. A nullspace of a matrix with n rows and n columns must contain vectors of n elements. These vectors clearly form a subset of $\mathbb{R}^n$.

ii. $\mathbb{R}^n$ is a well-defined subspace.

iii. Closures and 0

   i. Consider two elements, $\vec{x}_1$ and $\vec{x}_2$ in the nullspace of $\mathbf{M}$. By definition, we know that $\mathbf{M}\vec{x}_1 = \vec{0}$ and $\mathbf{M}\vec{x}_2 = \vec{0}$. Now consider the vector $\vec{x}_1 + \vec{x}_2$. Is $\mathbf{M}(\vec{x}_1 + \vec{x}_2) \stackrel{?}{=} \vec{0}$. $\mathbf{M}\vec{x}_1 + \mathbf{M}\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$. Done.

   ii. Consider another element, $\vec{x}_3$ in the nullspace of $\mathbf{M}$. Consider a scalar $a \in \mathbb{R}$. Is $a\vec{x}_2$ in the nullspace of $\mathbf{M}$, i.e., is $\mathbf{M}(a\vec{x}_3) \stackrel{?}{=} \vec{0}$. Yes, because $a\mathbf{M}\vec{x}_3 = 0$ since $\vec{x}_3$ is in the nullspace.

   iii. Is $\vec{0}$ in the nullspace of $\mathbf{M}$? Yes, because $\mathbf{M}\vec{0} = \vec{0}$.

Therefore, a nullspace is indeed a subspace.

(b) Now we will explore a recipe to compute null spaces. Let's start with some 3x3 matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix}$$

$\mathbf{A}''$ is the row reduced matrix $\mathbf{A}$.

$$\mathbf{A}'' = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix}$$

Compute the nullspace of $\mathbf{A}$.

Answer: Since the row reduced matrix $\mathbf{A}''$ has a pivot in every column, the matrix has a trivial nullspace. The nullspace the vector $\vec{0}$.

Let's look at this in more depth, however. Remember that a nullspace is the set of vectors such that $\mathbf{A}\vec{x} = \vec{0}$. Let's solve this as linear equations.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we row reduce. This results in

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's convert this back to linear equations:

$$x_1 - 3x_2 + x_3 = 0$$

$$-x_2 + x_3 = 0$$

$$-18x_3 = 0$$

The third equation is only satisfied by $x_3 = 0$. The second equation implies that $x_2 = x_3 = 0$. And finally, the first equation is also only satisfied by $x_1 = 0$. Therefore, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only vector which satisfies these equations

(c) Consider another matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 4 & -2 \\ -2 & 2 & -4 \end{bmatrix}$$

$\mathbf{B}^{"}$ is row reduced $\mathbf{B}$.

$$\mathbf{B}^{"} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the null space of $\mathbf{B}$?

Answer: Think of this as linear equations once again. Let the first column correspond to x, the second to y and the third to z. In equation form, the row reduced matrix becomes

$$x - y + 2z = 0 \tag{8}$$

$$8y - 10z = 0 \tag{9}$$

$$0x + 0y + 0z = 0 \tag{10}$$

Equation 10 gives us no information – it is always true. So we ignore it.

Equation 9 says that $4y = 5z$. Let's set $z = t$ (let z be a free variable that can take on any value). Then $y = \frac{5}{4}t$.

Equation 8 is then $x - \frac{5}{4}t + 2t = 0 \implies x = \frac{-3}{4}t$. The nullspace is then all vectors of the form $t\begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$, where t is any real number. Another way to say this is that the nullspace is spanned by the vector

$$\begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix} \tag{11}$$

The dimension of the nullspace, i.e., the minimum number of vectors required to span it is 1.

(d) In the previous part, we chose one of the variables and set it to be a free variable. Can we choose any variable as our free variable?

Answer: Let's investigate this question by choosing each variable as a free variable. We know z works from the solution to the previous part.

Let's consider y. If we set $y = t$ instead, then we get, from Equation (9) $5z = 4t \implies z = \frac{4}{5}t$.

Equation (8) then gives us $x - t + 2\frac{4}{5}t = 0 \implies x - \frac{5t}{5} + \frac{8t}{5} = 0 \implies x = \frac{-3t}{5}$.

The nullspace is then spanned by the vector $\begin{bmatrix} \frac{-3}{5} \\ 1 \\ \frac{4}{5} \end{bmatrix}$

Note that this vector is $\frac{4}{5}$ times the vector we found in (11).

Answer:
Now, let's see what happens if we set $x = t$ instead. We can't use Equation (9) yet, so let's try using Equation (8). $t - y + 2z = 0$. Now what? How do we find the value for y or z in terms of t? ...We can't. So x does not work...

(e) How can we know which variables can be used as free variables?

Answer: Pick your free variables are by looking at columns with no pivots. Although, sometimes, other variables might work (like y above), the variables with no pivots will always work!

(f) Now consider another matrix, $\mathbf{C} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{bmatrix}$ Without doing any math, will this matrix have a trivial nullspace, i.e. consisting of only $\vec{0}$?

Answer: No! A 3x4 matrix can simply not have 4 pivots. So at least one of the variables will need to be free!

(g) Consider another matrix, $\mathbf{D} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find vector(s) that span the nullspace.

Answer: In terms of equations, let the variables for cols 1-4 be a to d respectively. Column 3 does not have a pivot. So c is free. Let $c = t$. At this point, we should feel comfortable reading the matrix as its equations without explicitly writing the equations!

Row 3 of the matrix says that $-d = 0$, or that $d = 0$.

Row 2 says that $-2b - 6c + 10d = 0 \implies -2b - 6t = 0 \implies b = -3t$.

Row 1 says that $a - 2b - 6c + 12d = 0 \implies a + 6t - 6t = 0 \implies a = 0$.

The vector that spans this nullspace is $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

(h) Consider one final matrix, $\mathbf{E} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What are the vector(s) that span this nullspace?

Answer: Again, let the variables for the columns be a to d respectively. Columns 3 and 4 don't have pivots. So let's set both of them to be free!

Let $c = t, d = s$.

Row 2 says $-2b-6c+10d=0 \implies -2b=6t-10s \implies b=-3t+5s$.

Row 1 says $a-2b-6c+10d=0 \implies a=0$.

The general form of vectors in the nullspace is then $\begin{bmatrix} 0 \\ -3t+5s \\ t \\ s \end{bmatrix}$. This needs to be rewritten by splitting the free variables $s\begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Finally we conclude that the vectors that span the nullspace are $\begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Observation: notice that the number of free variables = number of columns without pivot = number of vectors required to span the nullspace = dimension of the nullspace!

9. Explore Subspace

(a) Consider the set $W = \{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0\}$. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ an element of the set W?

Answer: The way set notation works is that for an element to be a part of the set, $a_1+2a_2-3a_3$ must be satisfied. We can plug in numbers for a_1, a_2, a_3 from the vector and see if the equation is satisfied. $1+4-9 \neq 0$, so this element is not a member of the set.

(b) Write any 3 elements from this set

Answer: There are many possibilities. For instance, we could set the values of a_2 and a_3 to be 0 and then see what value of a_1 works. $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in the set.

Another element can be obtained by setting the values of a_1 and a_2 to be 1, and then we get $1+2-3a_3=0$, or $a_3=1$. Therefore $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is also in the set.

Another element can be obtained by setting the values of $a_1=3$ and $a_3=2$ and then we get $3+2a_2-6=0 \implies a_2=\frac{3}{2}$. Therefore $\vec{v}_3 = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ is in the set too.

(c) Is the set W a subspace?

Answer: **Step 1: Claim that W is a sub*set* of, say, X.**
W is clearly a subset of $\mathbb{R}^3$. This can be seen because the elements of W contain 3 elements, but W is not equal to $\mathbb{R}^3$ since some elements from $\mathbb{R}^3$ are not in W.

Step 2: Claim that X is a well-defined space
$\mathbb{R}^3$ is a well-defined space.

Step 3: If X is a well-defined space, and W is a subset of X, only 3 axioms must be proven.

a: Prove closure under addition
Consider two arbitrary elements from the set $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Since these elements are a part of the set, it is true that $b_1 + 2b_2 - 3b_3 = 0$ and that $c_1 + 2c_2 - 3c_3 = 0$.
Consider the sum of these elements. $\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$. Is this element a part of the set too? In other words, is $(b_1 + c_1) + 2(b_2 + c_2) - 3(b_3 + c_3) = 0$?

$$(b_1 + c_1) + 2(b_2 + c_2) - 3(b_3 + c_3) \stackrel{?}{=} 0$$
$$\implies (b_1 + 2b_2 - 3b_3) + (c_1 + 2c_2 - 3c_3) \stackrel{?}{=} 0$$
$$\implies 0 + 0 = 0$$

. Thus, the element $\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$ is a part of the set and we have proven closure under addition.

b: Prove closure under scalar multiplication
Consider an arbitrarily element from the set $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. This means that $d_1 + 2d_2 - 3d_3 = 0$ is true.

Consider some scalar s. Is $s \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ in the set W? I.e., is $\begin{bmatrix} sd_1 \\ sd_2 \\ sd_3 \end{bmatrix}$ in the set W? I.e. is $sd_1 + 2sd_2 - 3sd_3 = 0$?

$$sd_1 + 2sd_2 - 3sd_3 \stackrel{?}{=} 0$$
$$\implies s * (d_1 + 2d_2 - 3d_3) \stackrel{?}{=} 0$$
$$\implies s * 0 = 0$$

Therefore, $\begin{bmatrix} sd_1 \\ sd_2 \\ sd_3 \end{bmatrix}$ is in the set W and the set W is closed under scalar multiplication.

c: Prove existence of 0 element
We need to check whether $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ exists in the set. This is easy to check because we just need to check whether $0 + 2 * 0 - 3 * 0 = 0$, which it is. So the 0 element exists in the set.

Therefore, the set W is a subspace of $\mathbb{R}^3$.

(d) How can we now quickly find more elements of this set?

Answer: Since we know the set is closed under scalar multiplication and under addition, we can easily find more elements. Previously, we found $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to be an element. Now we know that any multiple of this is in the set too!

We also found that $\vec{v}_2 = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ was in the set. Now we can add the 2 elements we found, and $\begin{bmatrix} 4 \\ \frac{5}{2} \\ 3 \end{bmatrix}$ is also in the set. In fact, any $s\vec{v}_1 + r\vec{v}_2$, where $s, r \in \mathbb{R}$ are in the set!