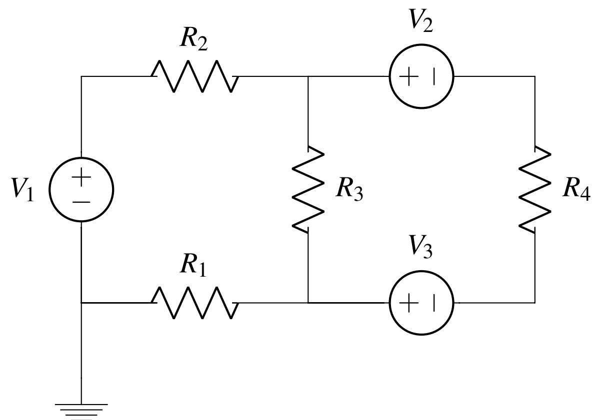
1. **Supernode**

   In this question, we will explore how to deal with multiple voltage sources when doing nodal analysis.



   (a) Mark all the nodes. If you know the potential at the node, write down the value next to the node. If you don't know the value, then assign a variable for the potential.

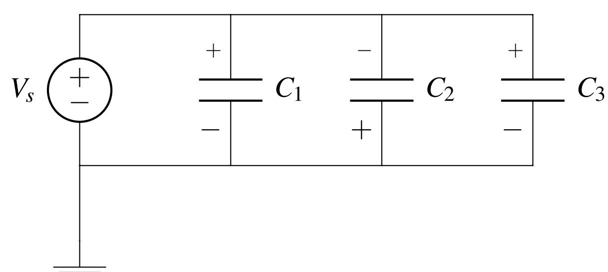
   (b) Mark current directions arbitrarily and corresponding polarities on each resistor. Note that if current goes from left to right, then the left side of the resistor is to be marked + and the right side must be marked -. This is the passive sign convention.

   (c) Note that we define 4 nodes with unknown potentials. So we need 4 equations. Each of these nodes with unknown potential should give us one equation.
   Write the equation for the first node with unknown potential.

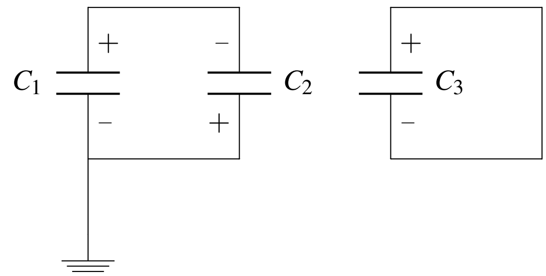
   (d) Write equations for all remaining unknown nodes.

2. **Pitfall Problem**

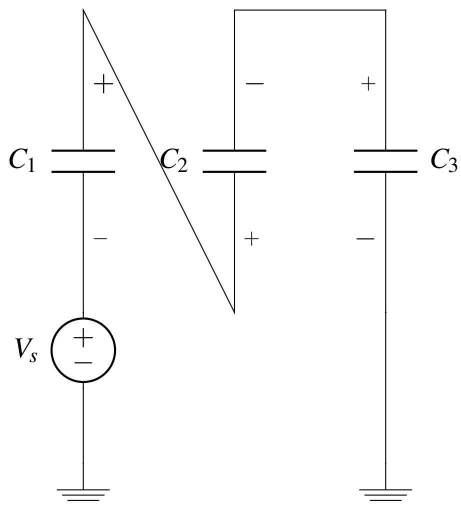
   (a) Consider the following circuit in ϕ_1. Assume that all capacitors are initially discharged. Find out the charge on each capacitor in this phase.



(b) Assume that ϕ_1 has taken place, and that the capacitors are then moved to the following configuration in ϕ_2. Calculate the charge across each capacitor in ϕ_2.

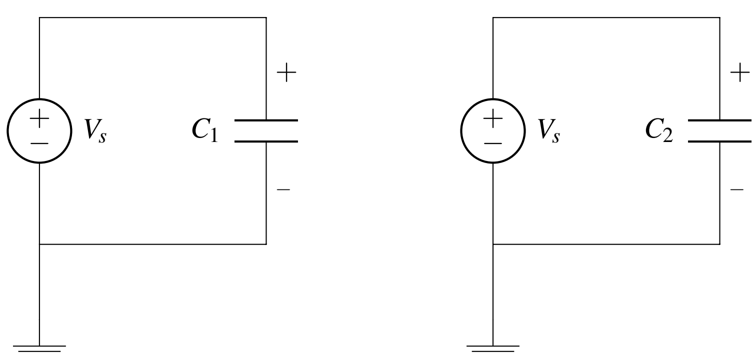


(c) Assume that ϕ_2 has taken place, and that the capacitors are then moved to the following configuration in ϕ_3. Calculate the charge across each capacitor in ϕ_3.

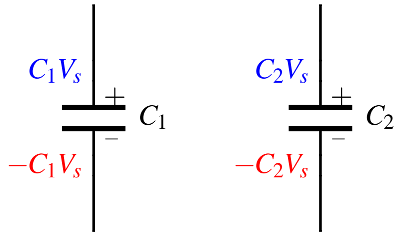


3. Series equivalence... or not?

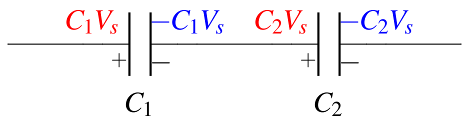
(a) Consider the following 2 circuits. What is the charge on the positive and negative plates of the two capacitors?



(b) Now consider that we first cut the capacitors off from their voltage sources and the ground nodes as such:



Next, we will connect these two capacitors as such:



Question: Can the charges on the positive plate of the capacitor C_1 move?

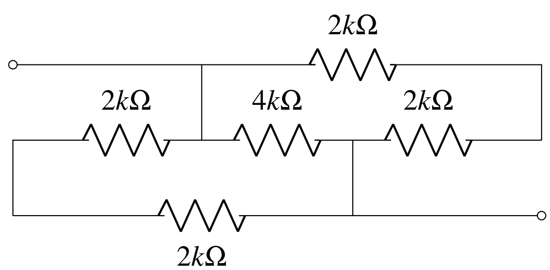
(c) What about the charges on the negative plate of C_1 and the positive plate of C_2?

(d) What about the charges on the negative plates of C_2?

(e) Here is a fundamental fact: If a capacitor's positive plate has x charge, then the negative plate must have $-x$ charge!
Question: These two capacitors look like they're in series. So they must have the same charge. How is this possible?

4. Never fail at resistor equivalence

(a) Mark all nodes on the following circuit.



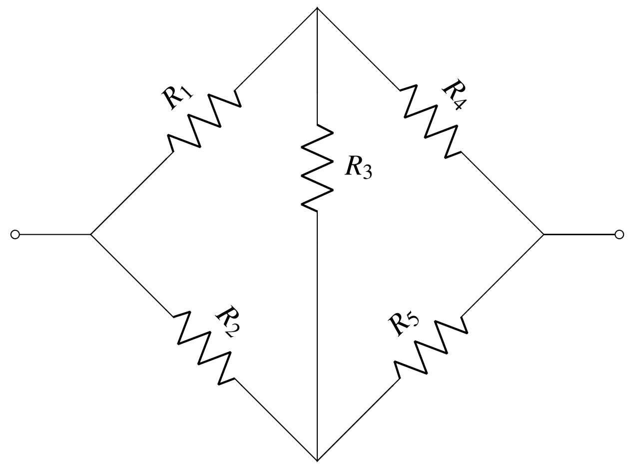
(b) Mark which nodes are 2-nodes and multi-nodes. 2 nodes are connected to only 2 components, and multi-nodes are connected to 3 or more components.
Note: *Whenever nodes are marked across which equivalent resistance must be found, those are considered 'components' because something could be connected there.*

(c) Resistors that are connected to 2-nodes are considered to be in series. Redraw the circuit, and find the 2 and multi-nodes again after combining the resistors connected to 2 nodes.

(d) Now we should be left with only multi-nodes. So far, we have seen that any resistors connected to 2-nodes are in series. We will see what happens to resistors connected to multi-nodes. Begin by writing out the 2 nodes that each resistor is connected to.

(e) If you have 2 or more resistors that are connected to the same 2 nodes, then they are in parallel. What does this mean for the 3 remaining resistors?

5. Another one for mastery



(a) Mark and label the nodes on the circuit above.

(b) Which nodes are 2 nodes? Which nodes are multi-nodes?

(c) Combine 2-node resistors are they are in series, similar to the previous question.

(d) Write down each resistor and the nodes that it is connected to.

(e) What do the nodes that the resistors are connected to tell you?

6. First Proof Prove that a subset of a finite linear independent subset of vectors is linearly independent

7. Solutions of linear equations

(a) Consider the following set of linear equations:

$$2x+3y+5z=0$$
$$-1x-4y-10z=0$$
$$x-2y-8z$$

Place these equations into a matrix, and row reduce the matrix.

(b) Convert the row reduced matrix back into equation form.

(c) Intuitively, what does the last equation from the previous part tell us?

(d) What is the general form of the infinite solutions to the system? Clearly, x, y, z cannot **actually** take on any values. The values $x=1, y=1, z=1$ don't satisfy the first equation, so they don't work.

8. Null space drill

In this question, we explore intuition about null spaces and a recipe to compute them. Recall that the nullspace of a matrix $\mathbf{M}$ is the set of all vectors, $\vec{x}$ such that $\mathbf{M}\vec{x}=\vec{0}$.

(a) First, we begin by proving that a null space is indeed a subspace. Show that any nullspace of a matrix $\mathbf{M}$ with n rows and n columns is a subspace.
Steps for reference:

i. claim subset of X
ii. claim X is well defined space
iii. closures and 0
  i. closure under addition
  ii. closure under scalar multiplication
  iii. existence of the zero element.

(b) Now we will explore a recipe to compute null spaces. Let's start with some 3x3 matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix}$$

$\mathbf{A}^{"}$ is the row reduced matrix $\mathbf{A}$.

$$\mathbf{A}^{"} = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix}$$

Compute the nullspace of $\mathbf{A}$.

(c) Consider another matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 4 & -2 \\ -2 & 2 & -4 \end{bmatrix}$$

$\mathbf{B}^{"}$ is row reduced $\mathbf{B}$.

$$\mathbf{B}^{"} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the null space of $\mathbf{B}$?

(d) In the previous part, we chose one of the variables and set it to be a free variable. Can we choose any variable as our free variable?

(e) How can we know which variables can be used as free variables?

(f) Now consider another matrix, $\mathbf{C} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{bmatrix}$ Without doing any math, will this matrix have a trivial nullspace, i.e. consisting of only $\vec{0}$?

(g) Consider another matrix, $\mathbf{D} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find vector(s) that span the nullspace.

(h) Consider one final matrix, $\mathbf{E} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What are the vector(s) that span this nullspace?

9. Explore Subspace

(a) Consider the set $W = \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0\}$. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ an element of the set W?

(b) Write any 3 elements from this set

(c) Is the set W a subspace?

(d) How can we now quickly find more elements of this set?