CSM
Fall 2017

Electrical Engineering 16A
Worksheet 1

1. Inverses!

(a) Find the inverse of:

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Answer: We know that for 2×2 matrices, we can simply use the formula for the inverse. The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$$

(b) Find the inverse of:

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix}$$

(Do not use co-factors. If you don't know what a co-factor is then don't worry about it.)

Answer: To find the inverse of a matrix, start with the equation

$$\mathbf{A} = \mathbf{IA}$$

Do row operations on the left hand side of this equation, to end up with

$$\mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}$$

Row2 = Row2 - 2Row1. Row3 = Row3 - 3Row1.

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -2 & -4 \\ 0 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \mathbf{A}$$

Row2 = Row2 * -0.5

$$\begin{bmatrix}1 & 3 & 5\\ 0 & 1 & 2\\ 0 & -3 & -7\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 1 & \frac{-1}{2} & 0\\ -3 & 0 & 1\end{bmatrix}\mathbf{A}$$

Row1 = Row1 - 3Row2. Row3 = Row3 + 3Row2.

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & -1\end{bmatrix} = \begin{bmatrix}-2 & \frac{3}{2} & 0\\ 1 & \frac{-1}{2} & 0\\ 0 & \frac{-3}{2} & 1\end{bmatrix}\mathbf{A}$$

Row3 = -1Row3

$$\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & \frac{3}{2} & 0\\ 1 & \frac{-1}{2} & 0\\ 0 & \frac{3}{2} & -1\end{bmatrix}\mathbf{A}$$

Row1 = Row1 + Row3. Row2 = Row2 - 2Row3

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}-2 & 3 & -1\\ 1 & \frac{-7}{2} & 2\\ 0 & \frac{3}{2} & -1\end{bmatrix}\mathbf{A}$$

Now this equation looks like $\mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$

$$\begin{bmatrix}-2 & 3 & -1\\ 1 & -3.5 & 2\\ 0 & 1.5 & -1\end{bmatrix}$$

A rotation matrix is a matrix that takes a vector and rotates it by some number of degrees. That matrix looks like:

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

for some angle θ. For example, if we had a rotation matrix with $\theta = 45^{circ}$, and we multiplied it with the vector [.5, .5], what would you expect?

Solution: Go over a couple of examples of what this matrix does to a vector. Consider the x-axis and the y-axis for instance.

(c) find the inverse of this matrix:

$$\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{-1}{2}\\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{bmatrix}$$

Answer:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(d) i) will a rotation matrix always have an inverse? Why or why not?
ii) consider a matrix that inverts a vector across the x-axis. Will it always have an inverse?
iii) consider a matrix that inverts only vectors that are beneath the x-axis across the x-axis, and does nothing if the vector is already above the x-axis. Will it have an inverse?

Answer: i) Yes, you can always rotate in the opposite direction.
ii) Yes, you can always invert back (In fact, the inverse would be itself)
iii) No, since the transformation loses information, there probably is NOT an inverse.

2. Are you linear?

(a) Consider a matrix $\mathbf{S}$ that transforms a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\vec{y} = \begin{bmatrix} a-b-c \\ a-b-c \\ a-b+c \end{bmatrix}$. Note that a, b, c can take on any values in $\mathbb{R}$. In other words, $\mathbf{S}\vec{x} = \vec{y}$. Is this transformation linear?

Solution: Please note that this might be the first time students are thinking of matrices as transformations. Let this settle in. The fact that a matrix is essentially a function that takes one vector and makes it a different vector. This is no different from a real-valued function like $f(x) = x^2$, except the only difference is that x is a vector, and f is a matrix.

Answer: To prove whether a transformation is linear, we must check whether it preserves scalar multiplication, addition and the zero vector.

Scalar multiplication
Let $\alpha \in \mathbb{R}$. Is $\mathbf{S}(\alpha\vec{x}) = \alpha\vec{y}$?
$\mathbf{S}\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix} = \alpha \begin{bmatrix} a-b-c \\ a-b-c \\ a-b+c \end{bmatrix}$. Try it!

Addition
Is $\mathbf{S}(\vec{x}_1 + \vec{x}_1) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$?
Let $\vec{x}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$. Then $\mathbf{S}(\vec{x}_1 + \vec{x}_2) = \mathbf{S}\vec{x}_1 + \mathbf{S}\vec{x}_2$. Try it out!
Zero vector
Is $\mathbf{S}\vec{0} = \vec{0}$? Yes.

This proves that $\mathbf{S}$ is indeed a linear transformation.

Solution: At the end of this, get the students to ask you "well... but... since matrix-vector multiplication is linear, of course every matrix is a linear operator!!" This should be the next question they ask.

(b) Now let's consider another matrix $\mathbf{Q}$ which takes a vector $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to $\begin{bmatrix} a+5 \\ b+c \end{bmatrix}$. Is this matrix a linear operator?

Solution: There might be a couple of students who immediately see the answer that the zero vector won't be preserved, but try to make sure they don't just blurt out the answer. Let everyone realize this by themselves.

Answer: Let's try the preservation of the zero vector first. Is $\mathbf{Q}\vec{0} = \vec{0}$? Nope, it is $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$. This matrix is not a linear operator! Notice that even though matrix-vector multiplication is generally linear

(c) Let's dive deeper. Write out the matrix $\mathbf{S}$ and $\mathbf{Q}$. Are they invertible?

Answer: $\mathbf{S} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. This matrix is not invertible, but it was still linear!

Writing out the matrix for $\mathbf{Q}$ is actually a trick question. There is no easy way to do this. In fact, you cannot simply write it without considering the context in which it is being used. Say that the context is multiplication with a vector.

$$\mathbf{Q} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+5 \\ b \\ c \end{bmatrix}$$

In this case,

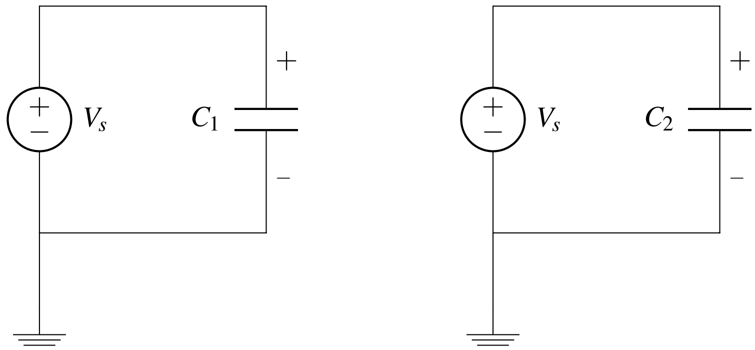
$$\mathbf{Q} = \begin{bmatrix} 1 & \frac{5}{b} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice how it is impossible to write the matrix out entirely using just numbers and that we need to use either a, b or c inside the matrix itself. Finally, note that the first row of this matrix could be written in other ways too. It could be $\begin{bmatrix} 1+\frac{5}{a} & 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & \frac{3}{b} & \frac{2}{c} \end{bmatrix}$ too. But the essential idea is that a **non-linear transformation matrix cannot be expressed using just scalars**.
So it is invertible? We can't say.

3. Series equivalence... or not?

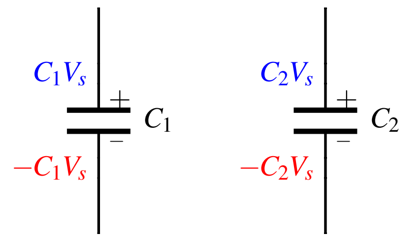
Solution: Prereq: The fact that Q=CV, and the series and parallel equivalence formula for capacitors.
Description: Problem shows that capacitors sometimes look like they're in series but they're not.

(a) Consider the following 2 circuits. What is the charge on the positive and negative plates of the two capacitors?

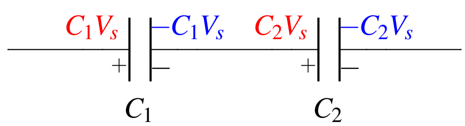


Answer: The charge on the positive plate of C_1 is C_1V_s. The charge on the negative plate is then $-C_1V_s$.
The charge on the positive plate of C_2 is C_2V_s. The charge on the negative plate is then $-C_2V_s$.

(b) Now consider that we first cut the capacitors off from their voltage sources and the ground nodes as such:



Next, we will connect these two capacitors as such:



Question: Can the charges on the positive plate of the capacitor C_1 move?

Answer: No, because charges cannot jump across the plates of a capacitor and there is no path for these charges to escape.

(c) What about the charges on the negative plate of C_1 and the positive plate of C_2?

Answer: In theory, these charges could redistribute... but look at the answer to the parts below.

(d) What about the charges on the negative plates of C_2?

Answer: These also cannot move similar to the charges on the positive plate of C_1.

(e) Here is a fundamental fact: If a capacitor's positive plate has x charge, then the negative plate must have $-x$ charge!
Question: These two capacitors look like they're in series. So they must have the same charge. How is this possible?

Answer: The capacitors 'look' like they're in series, and they are if looked at as electrical components. However, let's go through the derivation of the 'series equivalence formula' for capacitors.

Derivation:

$$V = V_1 + V_2 + \ldots + V_n$$

where V is the voltage drop across the branch of capacitors in series, and V_i are the individual voltage drops.

$$\frac{Q_{eq}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \ldots + \frac{Q_n}{C_n}$$

where Q_{eq} is the charge on the equivalent capacitor, C_{eq} is the equivalent capacitance, Q_i is the charge on each individual capacitor and C_i is the capacitance of each individual capacitor.
Since we know that the charge on each individual capacitor is the same and the charge on the equivalent capacitor is also equal to this charge, $Q_{eq} = Q_1 = Q_2 = \ldots = Q_n = Q$ **(say)**

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

Notice that in this derivation we *assume* that the charge on capacitors in series is the same, which leads to the formula. So capacitors having the same charge $\implies$ capacitors in series equivalence formula holds, not capacitors as components in series $\implies$ the series equivalence formula holds. The formula holds if and only if the capacitors are either discharged to begin with, or have the same charge on them to begin with!!

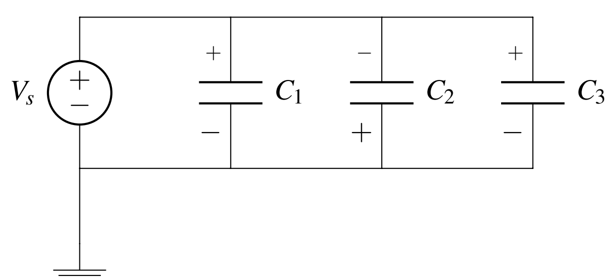
Intuitively, in this case, the capacitors do not have the same charge because the positive charges from C_1 and the negative charges from C_2 cannot move. This **forces** the negative charges of C_1 to stay where they are the positive charges of C_2 to stay where they are!!

Solution: Mentors: please make sure to go through the derivation. This part is so important that it must be drilled in.

4. Pitfall Problem

Solution: Prereq: Best placed right after the failure_cap_series_equivalence.tex
Description: Once students can do this, they can do every capacitor charge sharing problem (not necessarily those that include op amps too though)

(a) Consider the following circuit in ϕ_1. Assume that all capacitors are initially discharged. Find out the charge on each capacitor in this phase.



Answer:

$$Q_{C_1,\phi_1} = C_1 V_s$$
$$Q_{C_2,\phi_1} = -C_2 V_s$$
$$Q_{C_3,\phi_1} = C_3 V_s$$

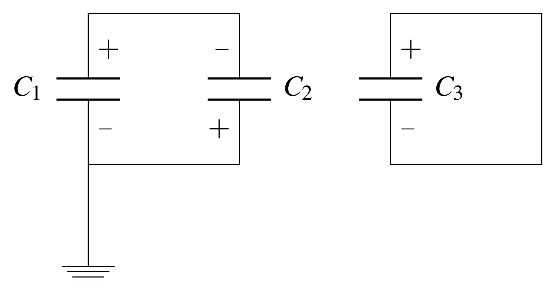
Note: What does it mean when we say that the charge on a capacitor is *negative*? It means that **on the positive plate of the capacitor, there is negative charge.** And since a capacitor has equal and opposite charge on each plate, the negative plate then has positive charge.

Solution: Make sure to stress the 'Note'. This is super important and is a misconception that students carry through the course.

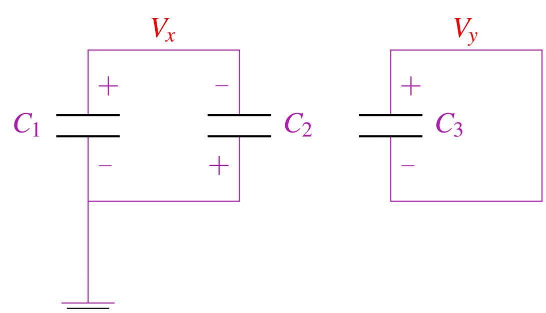
$$Q = CV = C(V_+ - V_-)$$

where V_+ and V_- are plates you arbitrarily mark as $+$ and $-$. It is obviously possible that you mark them incorrectly, and that there is actually negative charge on the positive plate and vice versa. If you use $Q = CV$ to calculate the charge on the capacitor, this always calculates the charge on the **positive** plate of the capacitor. If this quantity is negative, then there is negative charge on the plate you arbitrarily marked positive, and positive charge on the plate you arbitrarily marked positive. **This does not mean that you need to fix something in your circuit.**
This is fine. Just be consistent in the next phases.

(b) Assume that ϕ_1 has taken place, and that the capacitors are then moved to the following configuration in ϕ_2. Calculate the charge across each capacitor in ϕ_2.



Answer: **Step I: Write the voltage drop across each capacitor.**
If it cannot be determined, create variables till it can be determined.
Circuit redrawn with unknown voltages marked.



$$V_{C_1,\phi_2} = V_x - 0 = V_x$$

$$V_{C_2,\phi_2} = 0 - V_x = -V_x$$

$$V_{C_3,\phi_2} = V_y - V_y = 0$$

Step 2: Write the charge on each capacitor.
Just use $Q = CV$ on the first step

$$Q_{C_1,\phi_2} = C_1 V_x$$

$$Q_{C_2,\phi_2} = -C_2 V_x$$

$$Q_{C_3,\phi_2} = 0$$

Step 3: Write the charge sharing equations on floating nodes.
Floating nodes are those where charges cannot escape or enter.
Node marked V_x is the only floating node

$$Q_{C_1,\phi_2} - Q_{C_2,\phi_2} = Q_{C_1,\phi_1} - Q_{C_2,\phi_1} \tag{1}$$

$$C_1 V_x - (-C_2 V_x) = C_1 V_s - (-C_2 V_2) \implies V_x = V_s$$

Final Step 4: Plug variable voltage into charge equations

$$Q_{C_1,\phi_2} = C_1 V_x = C_1 V_s$$

$$Q_{C_2,\phi_2} = -C_2 V_x = -C_2 V_s$$

$$Q_{C_3,\phi_2} = 0$$

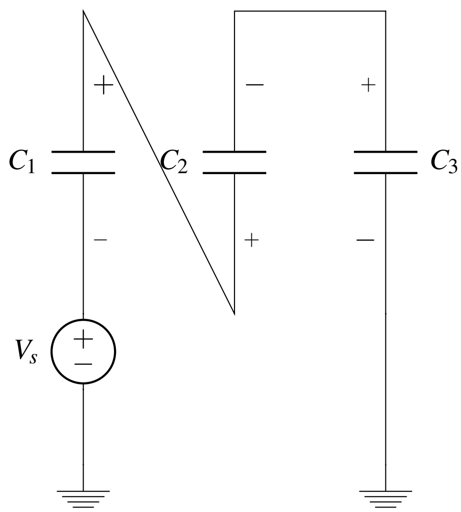
Solution: Note: Make sure to explain the charge sharing equations properly.

Pitfall number 1: Students think that charge sharing equations always have positive signs on them. Something along the lines of $Q_{1,1} + Q_{2,1} = Q_{1,2} + Q_{2,2}$. Equation 1 shows that this is not the case. Explain clearly why there is a negative sign. This is because the positive plate of C_1 is connected to the negative plate of C_2. Charge is being redistributed by Q_{C_1} and $-Q_{C_2}$ in the two phases.

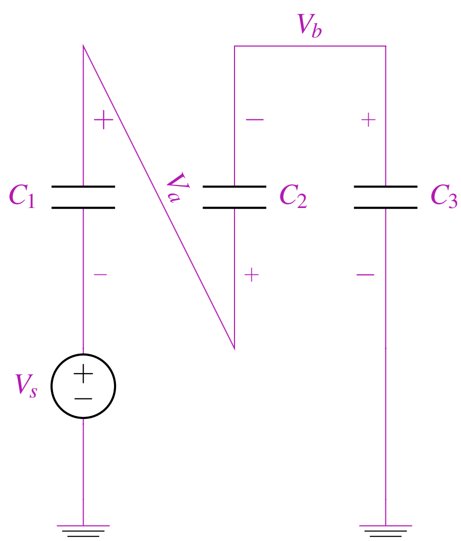
Pitfall number 2: Students think that charge sharing always happens between all capacitors. When you ask them for charge sharing equations, their gut reaction will be to say $Q_{C_1,\phi_1} + Q_{C_2,\phi_1} + Q_{C_3,\phi_1} = Q_{C_1,\phi_2} + Q_{C_2,\phi_2} + Q_{C_3,\phi_2}$. Show that them their gut reaction is wrong – because clearly C_3 is not involved in charge sharing. Also show them that the plates affect the signs for charge sharing, it isn't always a plug sign as shown in Equation 1.

Pitfall number 3: Many students don't realize that the voltage drop across C_3 is 0. They think that it will be the same as before and it will hold the same charge as before. This is not the case, as the math shows. Intuitively, this is not the case because + charges flow to recombine with the - charges resulting in 0 charge on the capacitor.

(c) Assume that ϕ_2 has taken place, and that the capacitors are then moved to the following configuration in ϕ_3. Calculate the charge across each capacitor in ϕ_3.



Answer: Step I: Write the voltage drop across each capacitor.
If it cannot be determined, create variables till it can be determined.
Circuit redrawn with unknown voltages marked.



$$V_{C_1,\phi_3} = V_a - V_s$$
$$V_{C_2,\phi_3} = V_a - V_b$$
$$V_{C_3,\phi_3} = V_b$$

Step 2: Write the charge on each capacitor.
Just use $Q = CV$ on the first step

$$Q_{C_1,\phi_3} = C_1(V_a - V_s)$$
$$Q_{C_2,\phi_3} = C_2(V_a - V_b)$$
$$Q_{C_3,\phi_3} = C_3(V_b)$$

Step 3: Write the charge sharing equations on floating nodes.
Floating nodes are those where charges cannot escape or enter.
Nodes marked V_a and V_b are the only floating nodes

$$Q_{C_1,\phi_3}+Q_{C_2,\phi_3} = Q_{C_1,\phi_2}+Q_{C_2,\phi_2} \tag{2}$$

$$-Q_{C_2,\phi_3}+Q_{C_3,\phi_3} = -Q_{C_2,\phi_2}+Q_{C_3,\phi_2} \tag{3}$$

Plugging in values to Equation 2

$$C_1(V_a - V_s) + C_2(V_a - V_b) = C_1 V_s + (-C_2 V_s)$$

Plugging in values to Equation 3

$$-C_2(V_a - V_b) + C_3 V_b = -(-C_2 V_s) + 0$$

Final Step 4: Plug variable voltage into charge equations
We now have two equations in two variables (V_a, V_b), and so we can solve for them. After that, we can plug those into Step 2, and find the charges on each capacitor. There is no need to do this step as the expressions aren't very neat.

Solution: **Pitfall number 4**: Make sure to explain equations 2 and 3 very carefully. Students get confused specially on 2 because in ϕ_2, the positive plate of C_1 is connected to the negative plate of C_2. In ϕ_3, the positive plates of both are connected to each other. The thing is that we only need to look at the current phase ϕ_3's configuration to write the charge sharing equations. Previous configs don't matter. One way to explain this is by talking about phases as a 2 step process – disconnection and connection. First disconnect all capacitors, and all positive charges stay where they are and negative stay where they are. Now, if we connect the positive plate of two capacitors and if originally the positive and negative plates of two separate capacitors were connected earlier, the previous config doesn't matter. It is the two positive plate's charges that will need to stay conserved between phases. This is why the signs on 2 will be the same on both the left hand side of the equation and the right hand side, and you get the signs by simply looking at ϕ_3.

5. Invertibility and equations

Solution: Prereq: Understanding of how to convert equations to matrix form, and knowing what inverses are and figuring our invertibility using gaussian elimination.
Description: Problem shows that pattern matching = failure. Basically, non-invertibility doesn't always mean no solutions.

(a) Consider the following system of equations

$$2x - 2y = -6$$

$$x - y + z = 1$$

$$3y - 2z = -5$$

Write these equations in matrix form. Then, write an expression for the solution to the equations using inverses, but don't compute the inverse.

Answer:

$$\begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix}$$

This can be rewritten as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 1 \\ -5 \end{bmatrix}$$

(b) Let the system of equations be $\mathbf{A}\vec{x} = \vec{y}$. What does it mean if $\mathbf{A}$ is not invertible?
Hint: The solution to the previous part.

Answer: If **A** is not invertible, then the system cannot be solved uniquely. We may have infinite solutions or no solutions. **Make sure you read through the entire problem. We talk more in depth about what invertibility means in later parts!**

Solution: Mentors – at this point, get your students to nod. It's important that they think this is always the case, because we're going to trick them soon :-)
Also, don't read the "make sure..." part in section. This is mainly so that when students are looking over the solutions online, they don't look at this part and think this is always true. The whole point of this problem is to show that sometimes **A** can be non-invertible, but the system can still have solutions.

(c) Consider the matrix

$$\begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 1 & -3 \end{bmatrix}$$

. Is it invertible?

Answer: We don't actually need to do gaussian elimination on this matrix to check whether it is invertible. It is not. An R^{NxM} matrix, where $N \neq M$ is **never** invertible. Why? Because when we do gaussian elimination, for invertible matrix, we must get 3 pivots since we have 3 rows. But we cannot get 3 pivots because we have only 2 columns.

Solution: Be careful here. The solution is worded as if it this is obvious. This might be obvious to you having taken the class and studied linear algebra, but this is not immediately obvious to students! If students are confused by this, and please check if they are, then do guassian elimination, and show that you cannot possible get 3 pivots.

(d) Does the system of equations that is represented by the following have any solutions?

$$\underbrace{\begin{bmatrix} -1 & 1 \\ -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{B}\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 9 \\ -7 \end{bmatrix}}_{\vec{y}}$$

Answer: From part (b), we want to say that the system doesn't have any solutions. We saw in the previous part that **B** is not invertible. But... let's convert this system to actual equations.

$$-1x_1 + x_2 = 5$$

$$-2x_1 + x_2 = 9$$
$$1x_1 - 3x_2 = -7$$

These are 3 equations in 2 variables. Surely, they could have a solution. If we solve them, we get $(x_1 = -4, x_2 = 1)$ as a solution. How could this happen? **B** was not invertible!

The invertibility test actually only holds well for the case when $\mathbf{B}_{NxN}\vec{x}_{Nx1} = \vec{y}_{Nx1}$ are the dimensions of the matrices and vectors in question. In this case, **B** being non-invertible, means that gaussian elimination gives you a row of zeros.

- If you have infinite solutions, then in your augmented matrix, you will have a row of 0s and the corresponding element from $\vec{y}$ will also be 0. (Why does this mean infinite solutions?)
- If you have no solutions, then in your augmented matrix, you will have a row of 0s, but the corresponding element from $\vec{y}$ will be non-zero. (Why does this mean zero solutions?)

However, if you have an equation of the form $\mathbf{B}_{MxN}\vec{x}_{Nx1} = \vec{y}_{Mx1}$, then you have M equations in N variables.

- If $M > N$ (like in this example), then you have more equations than variables.
  - This could certainly have a solution if the equations are linearly dependent. For instance, $x+y=1, 2x+2y=2, x-y=4$ definitely has a solution. The first 2 equations are linearly dependent, so we can remove one of them. Notice how Gaussian Elimination would help you realize this and find the one solution!
  - You could also have no solution, if the equations are all linearly independent. $x+y=3, x-y=-1, x+2y=0$ does not have any solutions. Notice how Gaussian Elimination would help you realize there are no solutions! *If you don't see it, try it out. Do you get a row of 0s on the left part of the augmented matrix, but not a corresponding zero element?*
  - You could also have infinite solutions. Consider $x+y=1, 2x+2y=2, 3x+3y=3$. Gaussian elimination helps here too!
- If $M < N$, then you have more variables than equations.
  - This could have many solutions. Consider 1 equation: $x+y=3$.
  - This could have no solutions. Consider $x+y+z=1$ and $x+y+z=2$
  - This cannot possibly have just one solution. You have more variables than you have equations!

  Notice though that Gaussian Elimination would help you realize all 3 of these cases.

  Big take away: Do not pattern match. If you hear it once "no inverse means no solutions", don't pattern match. That holds in a *particular* case. Remember that at the end, these are all equations, and use your intuition about equations.

Solution: A few notes

i. let students solve the equations themselves, don't just give them $-4, 1$

ii. There is a lot of 'answer' in here, but that does not mean that you as a mentor need to do all the talking. This material is very discussion-y, so discuss with them. Ask what happens in the case of $M > N$, and $M < N$. Let them come up with examples and ideas.

6. Solutions of linear equations

Solution: Prereq: Nothing really.
Description: Simple mechanical gaussian elimination problem + some insight about free variables

(a) Consider the following set of linear equations:

$$2x+3y+5z=0$$
$$-1x-4y-10z=0$$
$$x-2y-8z$$

Place these equations into a matrix, and row reduce the matrix.

Solution: Note to mentors: When you do Gaussian Elimination – start by making $a_{2,1}$ = 0 using some multiple of $a_{1,1}$. Next, make $a_{3,1}$ = 0 using some multiple of $a_{1,1}$. Next, make $a_{3,2}$ = 0 by using some multiple of $a_{2,2}$. In this last step, when you use row 2's pivot to subtract out row 3, the first element of row 3 will not be affected (it will remain 0). This is because in the previous steps, we got rid of the first element of row 2 as well. This is what I like to call the zig zag method of doing Gaussian Elimination. Start at the top left, move down the column. Then start again at the top of the second column and move down.

Answer:

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

$$R_2 = R_2 + \frac{1}{2}R_1$$
$$R_2 = R_2 - \frac{1}{2}R_1$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & -2.5 & -7.5 \\ 0 & -3.5 & -10.5 \end{bmatrix}$$

Make the numbers nicer by dividing row 2 by -2.5, and multiplying row 3 by -2. This is always a good thing to do if you realize your numbers are getting messy!

$$R_2 = \frac{1}{-2.5}R_2$$
$$R_3 = -2R_3$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 7 & 21 \end{bmatrix}$$

$$R_3 = R_3 - 7R_2$$

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) Convert the row reduced matrix back into equation form.

Answer:

$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x + 3y + 5z = 0$$

$$0x + 1y + 3z = 0$$

$$0x + 0y + 0z = 0$$

(c) Intuitively, what does the last equation from the previous part tell us?

Answer: It tells us that there are infinite solutions to the equations. $0x + 0y + 0z = 0$ is satisfied by **any** x, y, z.

Solution: If students are confused at this point about why we can infer this, their confusion is well justified. Suppose that there were **4** equations in 3 variables – 3 of them were linearly independent, and the fourth one was $0x + 0y + 0z = 0$, then the system still has just 1 solution. The last equation is never *used* in some sense. Feel free to talk about this with students. Present it as: what if you had 4 equations, you wrote them in matrix form, got pivots in all rows except for one where you got a row of all 0s – are there still infinite solutions? The answer is no.

(d) What is the general form of the infinite solutions to the system? Clearly, x, y, z cannot **actually** take on any values. The values $x = 1, y = 1, z = 1$ don't satisfy the first equation, so they don't work.

Solution: Explain why z is the free variable. (Because it is the one that doesn't have a pivot in the corresponding column).

Answer: z is a free variable. If $z = t$, then

$$y = -3z = -3t$$

$$2x + 3y + 5z = 0 \implies 2x - 3t + 5t = 0 \implies 2x = -2t \implies x = -t$$

The general solution is then $t \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$. What this means is that any multiple of the vector $\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$ will satisfy the equations. Try it!

7. First Proof

Solution: Prereq: Knowing that linear independence and dependence are
Description: A very simple and basic proof about linear independence.

Prove that a subset of a finite linear independent set of vectors is linearly independent

Solution: This is probably pretty early for when students will see proof. Very carefully introduce general proving techniques. Take the question, write down what is given in mathematical notation, and write out what needs to be proven in mathematical notation. A proof is essentially going from the 'given' to the 'to prove'.
Another note is that remember to assume that students have not taken CS70. Assume that they do not know proof techniques such as proof by contradiction, direct proof, induction, etc. This question is a proof by contradiction, so introduce it as such.
Final note: explain the 'without loss of generality' in the 'To Prove' section. Why

Answer:

$$\textbf{Given: } \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n = 0 \implies \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

$$\textbf{To Prove: } \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0 \implies \beta_1 = \beta_2 = \ldots = \beta_k = 0$$

Assume that $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0$ is true but not $\beta_1 = \beta_2 = \ldots = \beta_k = 0$.
Consider $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n$. If $\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k = 0$ then

$$\beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \ldots + \beta_k\vec{v}_k + 0\vec{v}_{k+1} + 0\vec{v}_{k+2} + \ldots + 0\vec{v}_n = 0$$

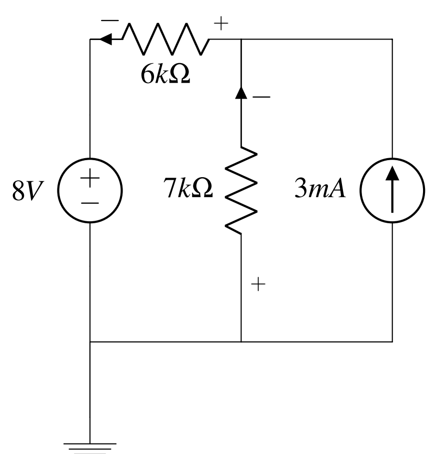
However, since we assumed that not all $\beta_1, \beta_2, \ldots, \beta_k$ are 0, this means that the set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is not linearly independent, which is a contradiction because it is given that the set is linearly independent. Therefore, $\beta_1 = \beta_2 = \ldots = \beta_k = 0$ must have been true.

8. You're grounded for no damn reason

Solution: Prereq: supernode.tex or understanding of what to do when voltage sources are in the way of doing nodal analysis.
Brief description: The problem walks through one circuit first with nodal analysis and then with superposition using different grounds to show that where you mark your ground makes no difference.

In this class, we always say "choose your ground wherever you want". In this question, we will explore how our choice of ground can change our answer drastically!

(a) Consider the following circuit. In this, we have explicitly marked a ground node for you. We have also marked a direction for the polarities on the resistor. Using nodal analysis, find the voltage drop across each resistor.



Answer: We know the potential of the node marked ground. We also know the potential of the node connected to the positive terminal of the $8V$ source, it is $8V$. There is only one unknown potential, which is the top-left node. Let's write the equation for that node, and call the potential there V_x.

$$\frac{V_x - 8V}{6k\Omega} = 3mA + \frac{0 - V_x}{7k\Omega}$$

Multiply by $42k\Omega$ on both sides

$$\implies 7V_x - 56 = 126V + -6V_x$$

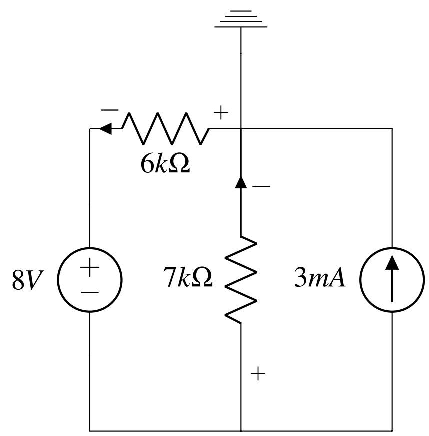
$$\implies 13V_x = 182V$$

$$V_x = 14V$$

.

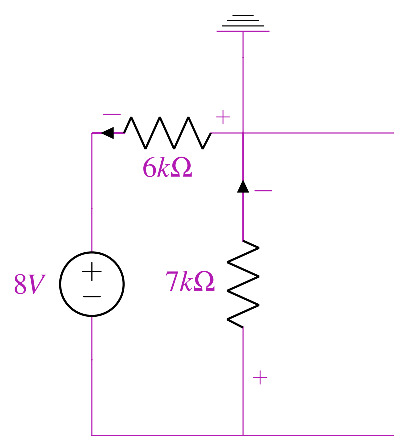
Voltage drop across $R_{7k} = V_+ - V_- = 0 - 14V = -14V$. All this means is that the way I marked my polarities was incorrect, and that current actually flows the other way.
Voltage drop across $R_{6k} = V_+ - V_- = 14V - 8V = 6V$.

(b) This time, we will mark our ground somewhere else and we will solve the circuit by superposition.



Answer: This time, we have two unknown potentials to solve for. One at the + end of the voltage source, and one at the negative. Let's get to it. Let's call them V_a and V_b respectively.

For superposition, draw the circuit with only the voltage source, by open circuiting the current source. Let the potentials from this case be V_{a_1} and V_{b_1} respectively.



Consider the V_a node. We don't know the current through the voltage source, so we use our super node trick. Consider V_{a_1} and V_{b_1} together. One equation is that

$$V_{a_1} - V_{b_1} = 8V$$

The second equation is that **Sum I into** V_{a_1}**,** V_{b_1} **= Sum I out of** V_{a_1}**,** V_{b_1}

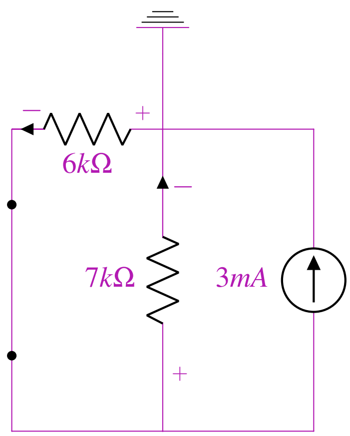
$$\frac{0 - V_{a_1}}{6k\Omega} = \frac{V_{b_1} - 0}{7k\Omega}$$

We know that $V_{a_1} = V_{b_1} + 8V$, so multiplying by $42k\Omega$

$$-7V_{a_1} = 6V_{b_1} \implies -7V_{b_1} - 6V_{b_1} - 56V = 0 => V_{b_1} = \frac{-56}{13}V$$

$$V_{a_1} = 8V + \frac{-56}{13}V = \frac{48}{13}V$$

Now, short circuit the voltage source. Let the potentials here be V_{a_2} and V_{b_2} respectively.



First, observe that $V_{a_2} = V_{b_2}$. Let's solve for the unknown node.

$$3mA + \frac{V_{a_2} - 0}{7k\Omega} = \frac{0 - V_{a_2}}{6k\Omega}$$

$$126V + 6V_{a_2} = -7V_{a_2}$$

$$V_{a_2} = \frac{-126}{13} = V_{b_2}$$

Finally,

$$V_a = V_{a_1} + V_{a_2} = \frac{48}{13} + \frac{-126}{13} = \frac{-78}{13} = -6V$$

$$V_b = V_{b_1} + V_{b_2} = \frac{-56}{13}V + \frac{-126}{13}V = \frac{-182}{13}V = -14V$$

What is the voltage drop across R_{6k} in the original circuit? $V_+ - V_- = 0 - V_a = 0 - (-6)V = 6V$. What is the votlage drop across R_{7k} in the original circuit $V_+ - V_- = V_b - 0 = -14V - 0 = -14V$.

Solution: Feel free to go ahead and mark the third node that we haven't marked as ground yet and do the problem once again. You can do it quickly.

(c) In an exam, you're solving a mechanical problem with resistors, voltage sources and current sources, and it asks you to find the voltage drop across one of the resistors. The TAs forgot to mark a ground on the circuit. Do you ask for a clarification and point out that they've made a mistake?

Answer: The TAs haven't made a mistake. Where you mark your ground **DOES NOT MATTER**.

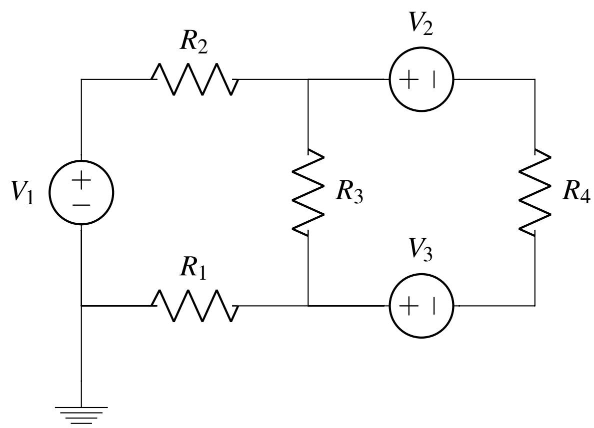
Solution: Make sure that students realize **the ground DOES NOT MATTER**. Why? Because you measure the DROP, i.e., DIFFERENCE IN POTENTIAL. Yes, the potentials at each node are different, but that doesn't matter, we never measure it. Show them that we used a **different technique and a different ground**, and our answer still didn't change. Say it a million times if you have to...

Fun fact: I wrote this problem on the day after Summer 17 MT2. One student legit asked me this during the exam, which motivated me to write this whole problem.

9. Supernode

Solution: Prereq: Best placed after a super easy nodal analysis problem.
Description: This shows how to deal with multiple voltage sources that don't share a ground when doing nodal analysis since we don't know how much current goes through a voltage sources.

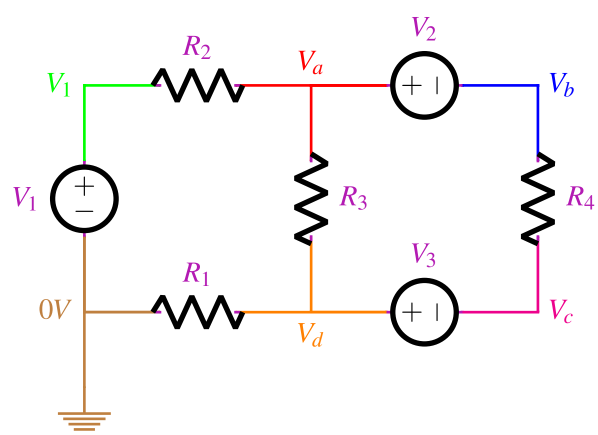
In this question, we will explore how to deal with multiple voltage sources when doing nodal analysis.



(a) Mark all the nodes. If you know the potential at the node, write down the value next to the node. If you don't know the value, then assign a variable for the potential.

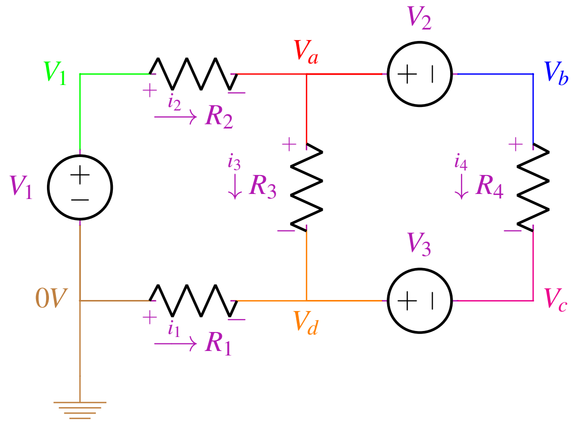
Answer:

V_a, V_b, V_c, V_d are variables. We know the potentials at the node marked ground and at the node marked V_1.



(b) Mark current directions arbitrarily and corresponding polarities on each resistor. Note that if current goes from left to right, then the left side of the resistor is to be marked + and the right side must be marked -. This is the passive sign convention.

Answer:



Solution: Might be a good idea to mark the nodes in the same fashion as the solutions for later parts!

(c) Note that we define 4 nodes with unknown potentials. So we need 4 equations. Each of these nodes with unknown potential should give us one equation.
Write the equation for the first node with unknown potential.

Answer: For the V_a node, we have a problem. The problem is that we don't know what current enters the voltage source V_2. The fix here is to treat V_a and V_b combined as a super-node. Since we are considering 2 nodes at the same time, this 'supernode' better give us 2 equations!
We can get equation from the fact that the voltage source exists.

$$V_a - V_b = V_2 \tag{4}$$

For the other equation, let's consider: (Sum of current entering nodes V_a and V_b) = (Sum of current leading nodes V_a and V_b).

Consider V_a:

- i_2 enters V_a.
- i_3 leaves V_a.

Consider V_b:

- i_4 leaves V_b.

$$i_2 = i_3 + i_4$$
$$\frac{V_1 - V_a}{R_2} = \frac{V_a - V_d}{R_3} + \frac{V_b - V_c}{R_4} \tag{5}$$

Solution: For equation 5, make sure students know that in the numerator, you have to write it as $V_+ - V_-$, depending on how you marked the polarities. But it is super important to mark the + where the current starts and - where it ends. This is passive sign convention. Make sure they do this right! Also, if students ask why (Sum of current entering nodes V_a and V_b) = (Sum of current leading nodes V_a and V_b) is something that we can use, show on the circuit that current cannot stay between V_a and V_b. Any current that enters V_a or V_b must leave through V_a or V_b

(d) Write equations for all remaining unknown nodes.

Answer: Nodes V_c and V_d remain. Let's start at V_c. Again, we don't know what current enters V_3, so we do the same thing. Treat the 2 ends of the voltage source, V_c and V_d as a super-node. One equation we get is

$$V_d - V_c = V_3 \tag{6}$$

The other equation comes from (Sum of current entering nodes V_a and V_b) = (Sum of current leading nodes V_a and V_b).

Consider V_c:

- i_4 enters.

Consider V_d:

- i_3 enters
- i_1 enters.

$$i_4 + i_3 + i_1 = 0$$

$$\frac{V_b - V_c}{R_4} + \frac{V_a - V_d}{R_3} + \frac{0 - V_d}{R_1} = 0 \tag{7}$$

Equations (4), (5), (6), (7) are four equations for the four variables V_a, V_b, V_c, V_d and can be solved uniquely. One fun fact is that these equations are guaranteed to be linearly independent as long as the circuit is not an impossible circuit (such as 2 current sources of differing values in series, or 2 voltage sources of different values in parallel).

10. Null space drill

Solution: Prereq: Introduction to nullspaces. A mini-lecture gets you ready.
Description: First a proof about nullspaces, and then lots of mechanical practice on nullspaces.

In this question, we explore intuition about null spaces and a recipe to compute them. Recall that the nullspace of a matrix $\mathbf{M}$ is the set of all vectors, $\vec{x}$ such that $\mathbf{M}\vec{x} = \vec{0}$.

(a) First, we begin by proving that a null space is indeed a subspace. Show that any nullspace of a matrix $\mathbf{M}$ with n rows and n columns is a subspace.
Steps for reference:

i. claim subset of X
ii. claim X is known vector space
iii. closures and 0
   i. closure under addition
   ii. closure under scalar multiplication
   iii. existence of the zero element.

Answer:

i. A nullspace of a matrix with n rows and n columns must contain vectors of n elements. These vectors clearly form a subset of $\mathbb{R}^n$.
ii. $\mathbb{R}^n$ is a known vector space.
iii. Closures and 0
   i. Consider two elements, $\vec{x}_1$ and $\vec{x}_2$ in the nullspace of $\mathbf{M}$. By definition, we know that $\mathbf{M}\vec{x}_1 = \vec{0}$ and $\mathbf{M}\vec{x}_2 = \vec{0}$. Now consider the vector $\vec{x}_1 + \vec{x}_2$. Is $\mathbf{M}(\vec{x}_1 + \vec{x}_2) \stackrel{?}{=} \vec{0}$. $\mathbf{M}\vec{x}_1 + \mathbf{M}\vec{x}_2 = \vec{0} + \vec{0} = \vec{0}$. Done.
   ii. Consider another element, $\vec{x}_3$ in the nullspace of $\mathbf{M}$. Consider a scalar $a \in \mathbb{R}$. Is $a\vec{x}_2$ in the nullspace of $\mathbf{M}$, i.e., is $\mathbf{M}(a\vec{x}_3) \stackrel{?}{=} \vec{0}$. Yes, because $a\mathbf{M}\vec{x}_3 = 0$ since $\vec{x}_3$ is in the nullspace.

iii. Is $\vec{0}$ in the nullspace of $\mathbf{M}$? Yes, because $\mathbf{M}\vec{0} = \vec{0}$.

Therefore, a nullspace is indeed a subspace.

(b) Now we will explore a recipe to compute null spaces. Let's start with some 3x3 matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix}$$

$\mathbf{A}'$ is the row reduced matrix $\mathbf{A}$.

$$\mathbf{A}' = \begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix}$$

Compute the nullspace of $\mathbf{A}$.

Answer: Since the row reduced matrix $\mathbf{A}'$ has a pivot in every column, the matrix has a trivial nullspace. The nullspace is the vector $\vec{0}$.

Let's look at this in more depth, however. Remember that a nullspace is the set of vectors such that $\mathbf{A}\vec{x} = \vec{0}$. Let's solve this as linear equations.

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To solve this, we row reduce. This results in

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let's convert this back to linear equations:

$$x_1 - 3x_2 + x_3 = 0$$
$$-x_2 + x_3 = 0$$
$$-18x_3 = 0$$

The third equation is only satisfied by $x_3 = 0$. The second equation implies that $x_2 = x_3 = 0$. And finally, the first equation is also only satisfied by $x_1 = 0$. Therefore, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the only vector which satisfies these equations

(c) Consider another matrix

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 4 & -2 \\ -2 & 2 & -4 \end{bmatrix}$$

$\mathbf{B}'$ is row reduced $\mathbf{B}$.

$$\mathbf{B}' = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 8 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the null space of **B**? What is the dimension of the row space of **B**?

Answer: Think of this as linear equations once again. Let the first column correspond to x, the second to y and the third to z. In equation form, the row reduced matrix becomes

$$x - y + 2z = 0 \tag{8}$$

$$8y - 10z = 0 \tag{9}$$

$$0x + 0y + 0z = 0 \tag{10}$$

Equation 10 gives us no information – it is always true. So we ignore it.

Equation 9 says that $4y = 5z$. Let's set $z = t$ (let z be a free variable that can take on any value). Then $y = \frac{5}{4}t$.

Equation 8 is then $x - \frac{5}{4}t + 2t = 0 \implies x = \frac{-3}{4}t$. The nullspace is then all vectors of the form $t\begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix}$, where t is any real number. Another way to say this is that the nullspace is spanned by the vector

$$\begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \\ 1 \end{bmatrix} \tag{11}$$

The dimension of the nullspace, i.e., the minimum number of vectors required to span it is 1.

From the rank-nullity theorem, we know that Dim(Rowspace(**B**)) + Dim(Nullspace(**B**)) = Number of columns in **B**. Therefore, the dimension of the rowspace of **B** is 2.

Solution: Mentors: State the rank nullity theorem without proof. For any matrix, **A**, Rank(**A**) + Nullity(**A**) = number of columns in **A**. Rank(**A**) = dim(colspace(**A**)) = dim(rowspace(**A**)). Nullity(**A**) = dim(nullspace(**A**))

(d) In the previous part, we chose one of the variables and set it to be a free variable. Can we choose any variable as our free variable?

Answer: Let's investigate this question by choosing each variable as a free variable. We know z works from the solution to the previous part.

Let's consider y. If we set $y = t$ instead, then we get, from Equation (9) $5z = 4t \implies z = \frac{4}{5}t$.

Equation (8) then gives us $x - t + 2\frac{4}{5}t = 0 \implies x - \frac{5t}{5} + \frac{8t}{5} = 0 \implies x = \frac{-3t}{5}$.

The nullspace is then spanned by the vector $\begin{bmatrix} \frac{-3}{5} \\ 1 \\ \frac{4}{5} \end{bmatrix}$

Note that this vector is $\frac{4}{5}$ times the vector we found in (11).

Solution: At this point, stress that the choice of free variable doesn't change the null space. Since subspaces are closed under scalar multiplication, the fact that this vector is a multiple of the previous shouldn't be a surprise to students. If it is, explain why this is the case.

Answer:

Now, let's see what happens if we set $x = t$ instead. We can't use Equation (9) yet, so let's try using Equation (8). $t - y + 2z = 0$. Now what? How do we find the value for y or z in terms of t? ...We can't. So x does not work...

(e) How can we know which variables can be used as free variables?

Answer: Pick your free variables are by looking at columns with no pivots. Although, sometimes, other variables might work (like y above), the variables with no pivots will always work!

(f) Now consider another matrix, $\mathbf{C} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 2 & 4 & 12 & -17 \\ 1 & -4 & -12 & 22 \end{bmatrix}$ Without doing any math, will this matrix have a trivial nullspace, i.e. consisting of only $\vec{0}$?

Answer: No! A 3x4 matrix can simply not have 4 pivots. So at least one of the variables will need to be free!

(g) Consider another matrix, $\mathbf{D} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find vector(s) that span the nullspace.

Answer: In terms of equations, let the variables for cols 1-4 be a to d respectively. Column 3 does not have a pivot. So c is free. Let $c = t$. At this point, we should feel comfortable reading the matrix as its equations without explicitly writing the equations!

Row 3 of the matrix says that $-d = 0$, or that $d = 0$.

Row 2 says that $-2b - 6c + 10d = 0 \implies -2b - 6t = 0 \implies b = -3t$.

Row 1 says that $a - 2b - 6c + 12d = 0 \implies a + 6t - 6t = 0 \implies a = 0$.

The vector that spans this nullspace is $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$

Solution: Students could be confused about 'pivots'. Column 3 doesn't have a pivot because it has a 0 in the place of the 'diagonal' instead. Also, ask them which column doesn't have a pivot in this case: $\begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & -1 & 9 \end{bmatrix}$ (note that the last row is different.) Basically, column 4 doesn't a pivot, because that pivot would have been in the 4th row which doesn't exist. *Make sure these concepts about pivots settle in.*

(h) Consider one final matrix, $\mathbf{E} = \begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. What are the vector(s) that span this nullspace?

Answer: Again, let the variables for the columns be a to d respectively. Columns 3 and 4 don't have pivots. So let's set both of them to be free!

Let $c = t, d = s$.

Row 2 says $-2b - 6c + 10d = 0 \implies -2b = 6t - 10s \implies b = -3t + 5s$.

Row 1 says $a - 2b - 6c + 10d = 0 \implies a = 0$.

The general form of vectors in the nullspace is then $\begin{bmatrix} 0 \\ -3t+5s \\ t \\ s \end{bmatrix}$. This needs to be rewritten by splitting the free variables $s\begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

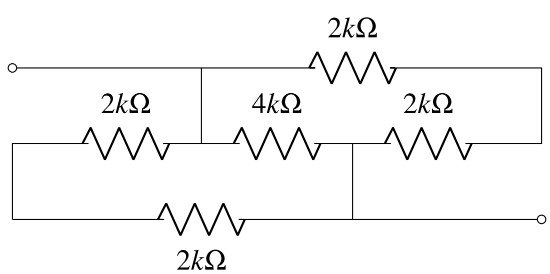
Finally we conclude that the vectors that span the nullspace are $\begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$.

Observation: notice that the number of free variables = number of columns without pivot = number of vectors required to span the nullspace = dimension of the nullspace!

11. Never fail at resistor equivalence

Solution: Prereq: Nothing except knowing the very basic idea about series and parallel equivalence for resistors.
Description: walks through a slowish 'recipe' for recognizing series and parallel resistors...

(a) Mark all nodes on the following circuit.

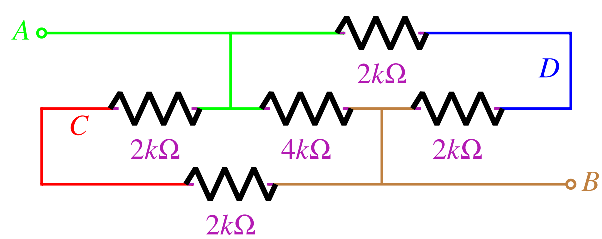


Solution: Mentors: at this point, explain the concept of nodes. Students may be confused as to what constitutes a nod and what the properties of a node are.

Here is how I like to teach it: keep a pencil (marker) at a wire, and keep going in all directions till you hit any components. Having multiple colors of markers helps a ton here! Otherwise draw little symbols. Forward slashed lines on one node, backward slashed on another, circles on another and so on.

The property of a node is that the potential on it is the same. The convention is to mark a node with an alphabet, and then call the potential at that point V_a for instance.

Answer:



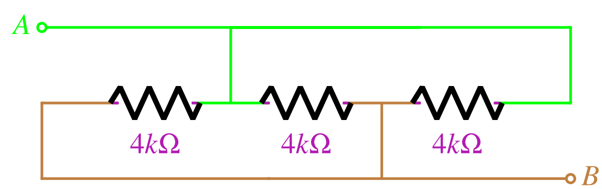
(b) Mark which nodes are 2-nodes and multi-nodes. 2 nodes are connected to only 2 components, and multi-nodes are connected to 3 or more components.
Note: *Whenever nodes are marked across which equivalent resistance must be found, those are considered 'components' because something could be connected there.*

Answer: Nodes A and B are multi-nodes. Nodes C and D are 2-nodes.

(c) Resistors that are connected to 2-nodes are considered to be in series. Redraw the circuit, and find the 2 and multi-nodes again after combining the resistors connected to 2 nodes.

Answer:

Nodes C and D disappear and become nodes A and B.



Solution: Mentors please explain how to combine resistors in series. The simple idea is that since they are on a 2-node, you can drag one till you reach the other, and just write $R_1 + R_2$ on that.

(d) Now we should be left with only multi-nodes. So far, we have seen that any resistors connected to 2-nodes are in series. We will see what happens to resistors connected to multi-nodes. Begin by writing out the 2 nodes that each resistor is connected to.

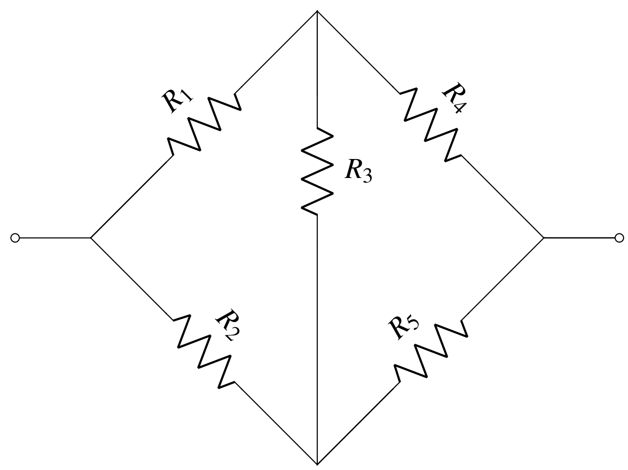
Answer: The resistor on the left is connected to nodes A and B. The resistor on the middle is connected to nodes A and B. The resistor on the right is connected to nodes A and B.

(e) If you have 2 or more resistors that are connected to the same 2 nodes, then they are in parallel. What does this mean for the 3 remaining resistors?

Answer: Since all 3 resistors are connected to nodes A and B, they are all in parallel. The equivalent resistance is $\frac{4}{3}k\Omega$. Quick tip: if you have 2 resistors in parallel, both of whose resistance is R, then the equivalent resistance is $\frac{R}{2}$. Similarly, if you have 3 resistors in parallel, all three of whose resistance is R, then the equivalent resistance is $\frac{R}{3}$, and so on with 4 or more resistors of the same value.

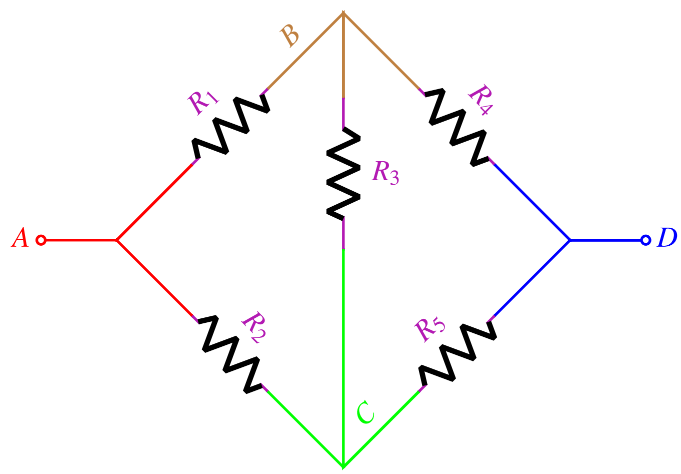
Solution: If students don't realize the tip, show them by deriving it with the formula.

12. Another one for mastery



(a) Mark and label the nodes on the circuit above.

Answer:



(b) Which nodes are 2 nodes? Which nodes are multi-nodes?

Answer: All nodes are multi-nodes. Remember that the terminals across which equivalent resistance will be measured are also considered components.

Solution: If students probe for why the terminals are considered components, draw a voltage source in between A and B. Now tell them that the question is actually to find the equivalent resistance in the resulting circuit to find the total current flowing. Now they will see that the resistors R_1 and R_2 are not in series. (They would have been if A was a 2-node instead, for instance).

(c) Combine 2-node resistors are they are in series, similar to the previous question.

Answer: Trick question! There are none!

(d) Write down each resistor and the nodes that it is connected to.

Answer: R_1 is connected to A and B.
R_2 is connected to A and C.
R_3 is connected to B and C.
R_4 is connected to B and D.
R_5 is connected to C and D.

(e) What do the nodes that the resistors are connected to tell you?

Solution: Let this be at least slightly discussion based. Let students try to come up with things.

Answer: This says that no 2 resistors are in parallel with each other. Recall that we need R_x and R_y to both be connected to some node M and N for them to be in parallel. Since this is not the case with any pair of resistors, none of them are in parallel or series. **We cannot simplify this circuit any further using series and parallel equivalences.**

Solution: As a parting note, tell students that in this class, they will see circuits with many many resistors. This recipe is intended to be a starting point for them to understand equivalences or for them to fall back on it during hard times. But don't always start off by doing this – it takes very long. Make sure to show them some circuit examples where you can easily identify series and parallel without this recipe. The circuit right at the top of this link for instance looks complicated, but if you start on the right end and find nodes as you go it is easy. R8 and R10 in series because a 2-node, then that combined in parallel with R9 and so on. This recipe is merely a ...fallback option. As days go by, they should get quicker at noticing nodes; please tell them this.

13. Explore Subspace

Solution: Prereqs: What are vector spaces and subspaces?
Description: Explains how to read set notation, tries to make students really realize that the notation means a set of vectors, and that a subspace is also a set of vectors. And what a subspace intuitively means.

(a) Consider the set $W = \{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, a_1, a_2, a_3 \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 0\}$. Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ an element of the set W?

Answer: The way set notation works is that for an element to be a part of the set, $a_1 + 2a_2 - 3a_3$ must be satisfied. We can plug in numbers for a_1, a_2, a_3 from the vector and see if the equation is satisfied. $1 + 4 - 9 \neq 0$, so this element is not a member of the set.

(b) Write any 3 elements from this set

Solution: The purpose here is really to make sure the students realize that W is indeed a set. It has infinite number of elements, but it is still a set. Let the students come up with whatever they want. Ultimately though, it will be super helpful if the final 3 elements you get are the same as the ones in the answer below. We will be using these again! So verify a couple of the elements that the students put forth, but ultimately write these on the board.

Answer: There are many possibilities. For instance, we could set the values of a_2 and a_3 to be 0 and then see what value of a_1 works. $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in the set.

Another element can be obtained by setting the values of a_1 and a_2 to be 1, and then we get $1 + 2 - 3a_3 = 0$, or $a_3 = 1$. Therefore $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is also in the set.

Another element can be obtained by setting the values of $a_1 = 3$ and $a_3 = 2$ and then we get $3 + 2a_2 - 6 = 0 \implies a_2 = \frac{3}{2}$. Therefore $\vec{v}_3 = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ is in the set too.

(c) Is the set W a subspace?

Answer: **Step 1: Claim that W is a sub*set* of, say, X.**
W is clearly a subset of $\mathbb{R}^3$. This can be seen because the elements of W contain 3 elements, but W is not equal to $\mathbb{R}^3$ since some elements from $\mathbb{R}^3$ are not in W.

Step 2: Claim that X is a vector space
$\mathbb{R}^3$ is a vector space that we have seen in lecture.

Step 3: If X is a known vector space, and W is a subset of X, only 3 axioms must be proven.

a: Prove closure under addition
Consider two arbitrary elements from the set $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. Since these elements are a part of the set, it is true that $b_1 + 2b_2 - 3b_3 = 0$ and that $c_1 + 2c_2 - 3c_3 = 0$.
Consider the sum of these elements. $\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$. Is this element a part of the set too? In other words, is $(b_1 + c_1) + 2(b_2 + c_2) - 3(b_3 + c_3) = 0$?

$$(b_1 + c_1) + 2(b_2 + c_2) - 3(b_3 + c_3) \stackrel{?}{=} 0$$
$$\implies (b_1 + 2b_2 - 3b_3) + (c_1 + 2c_2 - 3c_3) \stackrel{?}{=} 0$$

Clearly, the left hand side equals the right hand side. Therefore,

$$0 + 0 = 0$$

. Thus, the element $\begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix}$ is a part of the set and we have proven closure under addition.

b: Prove closure under scalar multiplication
Consider an arbitrarily element from the set $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$. This means that $d_1 + 2d_2 - 3d_3 = 0$ is true.
Consider some scalar s. Is $s \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ in the set W? I.e., is $\begin{bmatrix} sd_1 \\ sd_2 \\ sd_3 \end{bmatrix}$ in the set W? I.e. is $sd_1 + 2sd_2 - 3sd_3 = 0$?

$$sd_1 + 2sd_2 - 3sd_3 \stackrel{?}{=} 0$$
$$\implies s \cdot (d_1 + 2d_2 - 3d_3) \stackrel{?}{=} 0$$

Indeed the left hand side of the equation equals the right hand side, i.e.,

$$s \cdot 0 = 0$$

Therefore, $\begin{bmatrix} sd_1 \\ sd_2 \\ sd_3 \end{bmatrix}$ is in the set W and the set W is closed under scalar multiplication.

c: Prove existence of 0 element

We need to check whether $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ exists in the set. This is easy to check because we just need to check whether $0 + 2 \cdot 0 - 3 \cdot 0 = 0$, which it is. So the 0 element exists in the set.

Therefore, the set W is a subspace of $\mathbb{R}^3$.

(d) How can we now quickly find more elements of this set?

Answer: Since we know the set is closed under scalar multiplication and under addition, we can easily find more elements. Previously, we found $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to be an element. Now we know that any multiple of this is in the set too!

We also found that $\vec{v}_2 = \begin{bmatrix} 3 \\ \frac{3}{2} \\ 2 \end{bmatrix}$ was in the set. Now we can add the 2 elements we found, and $\begin{bmatrix} 4 \\ \frac{5}{2} \\ 3 \end{bmatrix}$ is also in the set. In fact, any $s\vec{v}_1 + r\vec{v}_2$, where $s, r \in \mathbb{R}$ are in the set!

14. Eigenvalues everywhere

Solution: Prereq: All of linear algebra basically, including page rank et al.
Description: Meant to be an intuition problem on eigenvalues and eigenvectors.

In this problem, when asked for eigenvectors, you may simply state that the eigenvector comes from a set. For instance, you could state that any $\vec{x} \in Colspace(\mathbf{A})$ is an eigenvector. Also, note that when asked to find eigenvalues, only consider real eigenvalues for this problem.

(a) What is one eigenvalue and eigenvector of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Answer: Since this matrix is clearly not-invertible, it must have an eigenvalue 0.

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$
$$\mathbf{A}\vec{x} = 0\vec{x}$$
$$\mathbf{A}\vec{x} = \vec{0}$$

This equation is precisely the equation for computing the nullspace of $\mathbf{A}$. Therefore, any $\vec{x} \in Nullspace(\mathbf{A})$ works.

Solution: The point of this problem is not to find the eigenvalues mechanically, but instead use properties of the matrix that you can eyeball to figure out some eigenvalues and eigenvectors. Don't spend time mechancially computing the eigenvalues.

(b) What are the eigenvalues and eigenvectors of the matrix

$$\mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Answer: This is a scaling matrix. It scales any vector by a factor of 3. What this means is that any vector $\vec{x} \in \mathbb{R}^3$ when post-multiplied by $\mathbf{A}$ will output $3\vec{x}$. This matrix has only one eigenvalue, $\lambda = 3$ and any $\vec{x} \in \mathbb{R}^3$ is an eigenvector.

(c) What are the eigenvalues of

$$\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 3 & 4 \\ 1 & 3 \end{bmatrix}?$$

Answer: This is a trick question. Eigenvalues are defined only for square matrices.

(d) Consider a matrix that rotates a vector in $\mathbb{R}^2$ by 45° counterclockwise. For instance, it rotates any vector along the x-axis to orient towards the $y = x$ line. Find its eigenvalues and corresponding eigenvectors. This matrix is given as

$$\mathbf{D} = \begin{bmatrix} cos45 & -sin45 \\ sin45 & cos45 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Solution: Please draw a picture to show what the matrix does to a vector. Also remember we are only considering real eigenvalues, as written in the prompt of the problem.

Answer: Remember that the equation $\mathbf{A}\vec{x} = \lambda\vec{x}$ geometrically means that for the matrix $\mathbf{A}$, there exist some special vectors $\vec{x}$ that are merely scaled by λ when post-multiplied by $\mathbf{A}$. For a matrix that takes a vector and rotates it by 45°, there are no real vectors that it can simply scale. This means that there are no real eigenvalues for this matrix either.

(e) What are the eigenvalues of the following matrix?

$$\mathbf{E} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Answer: Remember that for upper triangular matrices, the eigenvalues can be read from the diagonal. $1, \frac{1}{2}, \frac{1}{3}$ are the three eigenvalues.

(f) What is one eigenvalue of the following matrix?

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Answer: This is a matrix whose rows sum to 1, therefore, it has an eigenvalue 1.

This is proven by letting $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be a potential eigenvector of the matrix $\mathbf{F}$. Looking at the column view of matrix-vector multiplication –

$$\mathbf{F}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

$$\mathbf{F}\vec{x} = 1 \cdot \vec{x}$$

since the rows sum to one.

Therefore, 1 is an eigenvalue with corresponding eigenvector $\begin{bmatrix}1\\1\\1\end{bmatrix}$

Solution: Make sure students see why this works generally. Essentially $\mathbf{A}\begin{bmatrix}1\\1\\1\end{bmatrix} = 1\cdot\vec{v}_1 + 1\cdot\vec{v}_2 + 1\cdot\vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, where $\vec{v}_i$ are the columns of $\mathbf{A}$, and the sum equals $\begin{bmatrix}1\\1\\1\end{bmatrix}$ because each row sums to one.

(g) Show that a matrix and its transpose have the same eigenvalues

Solution: For any matrix $\mathbf{M}$,

$$det(\mathbf{M}) = det(\mathbf{M}^T)$$

Eigenvalues are found by solving the equation $det(\mathbf{M}-\lambda\mathbf{I}) = 0$.

Note that $(\mathbf{M}-\lambda\mathbf{I})^T = \mathbf{M}^T - \lambda\mathbf{I}^T = \mathbf{M}^T - \lambda\mathbf{I}$.

Let $\mathbf{M}-\lambda\mathbf{I} = \mathbf{G}$.

$$det(G) = det(G^T)$$

$$det(\mathbf{M}-\lambda\mathbf{I}) = det(\mathbf{M}^T - \lambda\mathbf{I})$$

If we set the left hand side to 0 to solve for the lambdas, we also extract the lambdas corresponding to the right hand side. Therefore, $\mathbf{M}$ and its transpose have the same eigenvalues.

(h) Consider a matrix whose columns sum to one. What is one possible eigenvalue of this matrix?

Solution: We showed that for any matrix like $\mathbf{F}$ whose rows sum to 1, one eigenvalue is 1. We also showed that a matrix and its transpose have the same eigenvalues. Consider $\mathbf{F}^T$. It has columns summing to 1. Therefore, 1 is an eigenvalue of $\mathbf{F}^T$ too, and by extension of all matrices whose columns sum to one.