HOW DOES THE LAUNCH POSITION ANGLE OF AN ANTI-BALLISTIC MISSILE (ABM) AFFECT THE MAGNITUDE OF ITS VELOCITY TO BE GAINED SO THAT IT CAN INTERCEPT AN INTERCONTINENTAL BALLISTIC MISSILE (ICBM)?

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Background Research

Importance of Study

Intercontinental Ballistic Missiles (ICBM) are missiles that can strike at targets more than 5,500 kilometres away, typically armed with nuclear warheads (Cochran and Norris). It is thus crucial to defend against them, especially as more countries gain the capability to design and produce ICBMs, such as Iran or North Korea ("Fact Sheet: U.S. Ballistic Missile Defense").

Anti-ballistic missiles (ABM) are missiles designed to intercept and destroy incoming ballistic missiles ("Antiballistic Missile (ABM)"). However, advanced ICBMs with small warheads travelling over 15,000 miles per hour, are particularly challenging to intercept (Karlis).

This paper proposes a novel algorithm using the concept of Velocity to Be Gained to guide these ABMs towards ICBMs within a computer simulation. We will also explore how the launch angles of the ABM, effects the velocity to be gained for interception. This helps figure out at what positions ABM's can be placed, to best ensure successful interception.

Simulation

Before analysing an ABM intercepting an ICBM, we will touch on the Physics simulation this scenario is based in. The physics is a 3D software simulation programmed in Python in C++, simulating a spherical, rotating Earth.

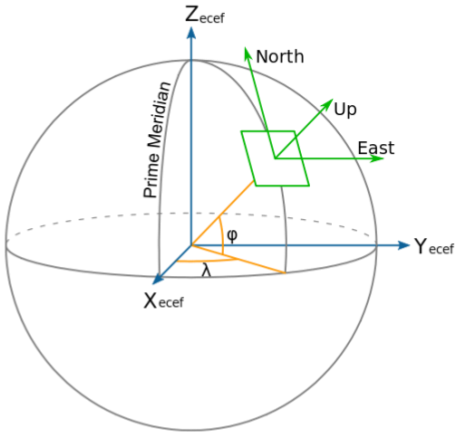

Figure 1, visual representation of the Earth that is modelled by the simulation ("Coordinate Systems"). The latitude is represented by φ and the longitude is represented by λ.

As seen in Figure 1, the Earth is modelled as a sphere with mass $M = 5.972168 \times 10^{24}\ kg$ and radius $R = 6371000\ m$ (Wikipedia Contributors). It rotates a full 2π radians in 0.99726968 days around the z-axis, from the perspective of an infinitely distant point (Wikipedia Contributors) ("1904PA.....12..649B Page 649"). The Earth's angular velocity thus is:

$$\omega = \frac{2\pi}{0.99726968 \ \times 86400}$$

$$\omega = 0.00007292115\, rad\ s^{-1}$$

The Earth's rotation angle thus is:

$$\theta = \omega t \text{ (Equation 1)}$$

Where t is the elapsed simulation time in s.

All position vectors are represented with their origin at the Earth's centre.

Note: Vectors will be bolded and capitalised, otherwise it's a scalar. The vector's magnitude is denoted by ‖…‖. The dot product operation between two vectors is signified with · and gives a scalar. While the cross-product operation between two vectors is signified with × and gives a vector.

Missiles have an initial position vector with components in m that is found from the latitude φ and longitude λ:

$$\boldsymbol{P_{initial}} = \begin{bmatrix} Rcos(\varphi)cos(\lambda) \\ Rcos(\varphi)sin(\lambda) \\ Rsin(\varphi) \end{bmatrix} \text{ (Equation 2)}$$

Missile positions **while on the ground** are rotated using a rotation matrix accounting for Earth's rotation:

$$\boldsymbol{P} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{P_{initial}} \text{ (Equation 3)}$$

("Rotation Matrix for Rotations around Z-Axis - MATLAB Rotz")

When the **missiles are flying**, their position is updated using equations of motion. At launch, their velocity vector $\boldsymbol{V}$ due to the Earth's rotation with components in ms^{-1} start off as:

$$\boldsymbol{V} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times \boldsymbol{P} \text{ (Equation 4)}$$

(Physics Stackexchange)

Equations of Motion in Freefall

The motion of missiles over Earth is primarily determined by the gravitational acceleration. At position $\boldsymbol{P}$ the magnitude of gravitational acceleration is given by $\frac{GM}{\|\boldsymbol{P}\|^2}$, with universal gravitational constant $G = 6.6743015 \times 10^{-11}\ Nm^2kg^{-1}$. While the gravitational acceleration's direction is given by a vector with length 1, that points opposite to $\boldsymbol{P}$: $-\frac{\boldsymbol{P}}{\|\boldsymbol{P}\|}$. Combining magnitude and direction, the function providing the gravitational acceleration vector in ms^{-2}:

$$\boldsymbol{A_g}(\boldsymbol{P}) = -\frac{\boldsymbol{P}}{\| \boldsymbol{P} \|}\frac{GM}{\| \boldsymbol{P} \|^2}$$

$$\boldsymbol{A_g}(\boldsymbol{P}) = -\boldsymbol{P}\frac{GM}{\|\boldsymbol{P}\|^3} \text{ (Equation 5)}$$

To model freefall under gravity over a timestep Δt s we can calculate the new position $\boldsymbol{P_{new}}$ like so:

$$\boldsymbol{P_{new}} = \boldsymbol{P_{old}} + \boldsymbol{V_{old}}\Delta t + \frac{1}{2}\boldsymbol{A_g}(\boldsymbol{P_{old}})\Delta t^2 \text{ (Equation 6)}$$

This aligns with the kinematic equation $s = vt + \frac{1}{2}at^2$.

After computing $\boldsymbol{P_{new}}$, we can calculate the new velocity $\boldsymbol{V_{new}}$ like so:

$$\boldsymbol{V_{new}} = \boldsymbol{V_{old}} + \frac{1}{2}(\boldsymbol{A_g}(\boldsymbol{P_{old}}) + \boldsymbol{A_g}(\boldsymbol{P_{new}}))\Delta t \text{ (Equation 7)}$$

Unlike the kinematic equation $v = u + at$, this equation averages the old and new gravitational accelerations, accounting for their continuous change in direction and magnitude. This method known as leapfrog integration, ensures far superior simulation accuracy even for **larger timesteps** (Young).

Velocity to Be Gained

The ABM's velocity to be gained $\boldsymbol{V_{TBG}}$ is the velocity that would need to be added to the ABM so that under freefall, where the ABM is only accelerated by gravity, it would intercept the ICBM ("6.5: The Correlated Velocity and Velocity-To-Be-Gained Concepts | GlobalSpec"). The components of $\boldsymbol{V_{TBG}}$ are in ms^{-1}.

Error Function

We can frame the problem of finding $\boldsymbol{V_{TBG}}$ as an optimisation problem that minimizes the error function:

$$f(\boldsymbol{V_{TBG}}) = 10^{-6} \times d_{closest}(\boldsymbol{V_{TBG}}) + 10^{-3}\max(0, v_{closing}(\boldsymbol{V_{TBG}})) \text{ (Equation 8)}$$

Where $d_{closest}(\boldsymbol{V_{TBG}})$ is a function representing the closest distance reached by the ABM and ICBM under freefall in m. And $v_{closing}(\boldsymbol{V_{TBG}})$ is a function that represents the rate at which the ABM is approaching the ICBM in ms^{-1} if $\boldsymbol{V_{TBG}}$ is added to the ABM's current velocity.

There are two main careful considerations that went into creating this function:

Firstly, velocities are on the order of 1,000 ms^{-1}, while $d_{closest}(\boldsymbol{V_{TBG}})$ can be on the order of 1,000,000 meters. To ensure efficient optimisation by software, these are scaled to have the same range ("8.2 Addressing Numerical Issues — MOSEK Optimizer API for Python 10.2.13"). $\max(0, v_{closing}(\boldsymbol{V_{TBG}}))$ is multiplied by 10^{-3} and $d_{closest}(\boldsymbol{V_{TBG}})$ is multiplied by 10^{-6}.

Secondly, when the optimization routine is initiated, the ICBM might be moving away from the ABM's launch point. The closest distance reached by the ABM and ICBM will thus be the initial distance, and that will not change for as long as when $\boldsymbol{V_{TBG}}$ is added ABM's current velocity, it still moves away from the ICBM. This will result in the optimization routine making no progress as $d_{closest}(\boldsymbol{V_{TBG}})$ will be unchanged, and so, to force progress, $\boldsymbol{V_{TBG}}$ must ensure that the ABM and ICBM do not move apart. This is why the closing speed $v_{closing}(\boldsymbol{V_{TBG}})$ is included in the optimization function, to ensure that it initially be driven to 0. Once it is driven to 0, it will have no effect on the optimization function, and then we can focus on solely driving $d_{closest}(\boldsymbol{V_{TBG}})$ to 0.

Finding the Velocity to Be Gained

To determine the optimum velocity to be gained $\boldsymbol{V_{TBG}}$ for the ABM to intercept the ICBM that minimizes the error function $f(\boldsymbol{V_{TBG}})$ (defined in Equation 8), f is approximated as a linear function with 3 variables:

$$\hat{f}(\boldsymbol{V_{TBG}}) = a_1x + a_2y + a_3z + f(\boldsymbol{V_{TBG_{current}}}) \text{ (Equation 9)}$$

Where x, y, and z are $\frac{V_{TBG_x}-V_{TBG_{current_x}}}{10^3}$, $\frac{V_{TBG_x}-V_{TBG_{current_y}}}{10^3}$, $\frac{V_{TBG_x}-V_{TBG_{current_z}}}{10^3}$ respectively. This scaling is again done for numerical stability so that the linear regression can be done with small zero-centred values.

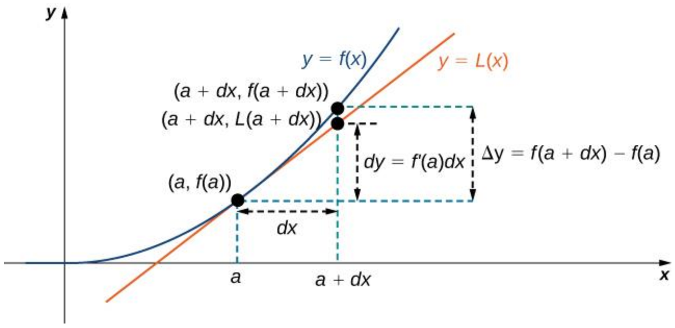

Figure 2, an example of linear approximation (shown by the orange function) that approximates the blue function. Notice that the linear approximation is only accurate for values in a **region around the x-coordinate *a*** ("4.5: Linear Approximations and Differentials")

As seen in Figure 2, the linear approximation $\hat{f}$ of Equation 8 is only accurate around $\boldsymbol{V_{TBG_{current}}}$ within a **trust region**, where the values of f and $\hat{f}$ are trusted to closely align with each other. Mathematically, the **trust region** is the set of all $\boldsymbol{V_{TBG}}$ values, within a distance of $10^3\delta_{current}$ to $\boldsymbol{V_{TBG_{current}}}$, where $\delta_{current}$ is the **trust region's radius**.

The coefficients a_1 through a_3 in Equation 9 are determined through a linear regression using 30 random $\boldsymbol{V_{TBG}}$ vectors and their associated error values $f(\boldsymbol{V_{TBG}})$ as data points, with $\boldsymbol{V_{TBG}}$ vectors that are within the trust region.

Now we can find an improved solution, $\boldsymbol{V_{TBG_{sol}}}$, like so:

$$\boldsymbol{V}_{\boldsymbol{TBG}_{sol}} = \boldsymbol{V}_{\boldsymbol{TBG}_{current}} - \frac{10^3 \delta_{current}}{\sqrt{(a_1{}^2 + a_2{}^2 + a_3{}^2)}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{(Equation 10)}$$

The actual error of the improved solution $f(\boldsymbol{V}_{\boldsymbol{TBG}_{sol}})$ can be compared with the predicted error of the improved solution $\hat{f}(\boldsymbol{V}_{\boldsymbol{TBG}_{sol}})$ using a scalar k defined as:

$$k = \frac{\hat{f}(\boldsymbol{V}_{\boldsymbol{TBG}_{sol}}) - f(\boldsymbol{V}_{\boldsymbol{current}})}{f(\boldsymbol{V}_{\boldsymbol{TBG}_{sol}}) - f(\boldsymbol{V}_{\boldsymbol{TBG}_{\boldsymbol{current}}})} \text{(Equation 11)}$$

If $k < 0.1$, the improvement is negligible. For the next iteration, we keep the old solution for velocity to be gained: $\boldsymbol{V}_{\boldsymbol{TBG}_{next}} = \boldsymbol{V}_{\boldsymbol{TBG}_{current}}$ and reduce the trust region radius as we are pessimistic in its ability to provide improvements: $\delta_{next} = 0.5 \times \delta_{current}$ (Stanford).

Otherwise, if $k \geq 0.1$ we can accept the improved solution as our new velocity to be gained: $\boldsymbol{V}_{\boldsymbol{TBG}_{next}} = \boldsymbol{V}_{\boldsymbol{TBG}_{sol}}\ \boldsymbol{V}_{\boldsymbol{TBG}_{current}}$ and expand the size of the trust region, as we are confident in its ability to provide improvements: $\delta_{next} = 1.1 \times \delta_{current}$ (Stanford).

Now we can repeat all of this, using the new velocity to be gained and trust region radius as the current ones. For more repeats, or iterations, of the algorithm, it can find better values of $\boldsymbol{V}_{\boldsymbol{TBG}}$.

Initially the ABM starts by calculating $\boldsymbol{V}_{\boldsymbol{TBG}}$ from an initial value of $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, a trust region radius δ of 0.25 (Stanford), and runs for 100 iterations.

While in flight, the velocity to be gained is updated by using the most recent $\boldsymbol{V_{TBG}}$ as a starting value, a smaller trust region radius δ of 2^{-12} and just 1 iteration. This ensures **rapid adjustments** for real time interceptions.

Using Velocity to Be Gained for ABM Guidance

To determine the thrust direction for the ABM, a simple strategy would be to apply thrust in the direction of $\boldsymbol{V_{TBG}}$, aiming for an acceleration towards $\boldsymbol{V_{TBG}}$. However, this approach neglects the influence of gravity, which can pull the ABM of its trajectory. Instead, the thrust direction must also compensate for gravitational acceleration, to ensure the **net acceleration** is in the direction of $\boldsymbol{V_{TBG}}$.

This is represented by the equation:

$$\boldsymbol{A_g}(\boldsymbol{P_{ABM}}) + a_{thrust}\boldsymbol{N} = k\boldsymbol{V_{TBG}} \ (k > 0) \ \text{(Equation 12)}$$

Where a_{thrust} is the magnitude of the acceleration provided by thrust in ms^{-2}, $\boldsymbol{N}$ is a vector of magnitude 1 indicating the thrust direction, and k is a dimensionless scale factor.

Rearranging:

$$k\boldsymbol{V_{TBG}} - \boldsymbol{A_g}(\boldsymbol{P_{ABM}}) = a_{thrust}\boldsymbol{N}$$

On each side taking the dot product of itself:

$$(k\boldsymbol{V_{TBG}} - \boldsymbol{A_g}(\boldsymbol{P_{ABM}})) \cdot (k\boldsymbol{V_{TBG}} - \boldsymbol{A_g}(\boldsymbol{P_{ABM}})) = a_{thrust}\boldsymbol{N} \cdot a_{thrust}\boldsymbol{N}$$

$$k^2 \parallel \boldsymbol{V_{TBG}} \parallel^2 - k(2\boldsymbol{A_g} \cdot (\boldsymbol{P_{ABM}})) + \parallel \boldsymbol{A_g}(\boldsymbol{P_{ABM}}) \parallel^2 = a_{thrust}{}^2 \parallel \boldsymbol{N} \parallel^2$$

Since $\parallel \boldsymbol{N} \parallel^2 = 1$:

$$k^2 \parallel \boldsymbol{V_{TBG}} \parallel^2 - k(2\boldsymbol{A_g} \cdot (\boldsymbol{P_{ABM}})) + \parallel \boldsymbol{A_g}(\boldsymbol{P_{ABM}}) \parallel^2 = a_{thrust}{}^2$$

$$k^2 \parallel \boldsymbol{V_{TBG}} \parallel^2 - k(2\boldsymbol{A_g} \cdot (\boldsymbol{P_{ABM}})) + (\parallel \boldsymbol{A_g}(\boldsymbol{P_{ABM}}) \parallel^2 - a_{thrust}{}^2) = 0 \text{ (Equation 13)}$$

We can thus solve for k by using the quadratic formula $\frac{-b+\sqrt{b^2-4ac}}{2a}$, taking a as $\parallel \boldsymbol{V_{TBG}} \parallel^2$, b as $2\boldsymbol{A_g} \cdot (\boldsymbol{P_{ABM}})$ and c as $\parallel \boldsymbol{A_g}(\boldsymbol{P_{ABM}}) \parallel^2 - a_{thrust}{}^2$.

With k found, the gravity compensated thrust direction vector $\boldsymbol{N}$ is:

$$\boldsymbol{N} = \frac{k\boldsymbol{V_{TBG}} - \boldsymbol{A_g}(\boldsymbol{P_{ABM}})}{a_{thrust}} \text{ (Equation 14)}$$

Finding Closest Distance Between ABM and ICBM

A key part of the error function in Equation 8, is the function to find the closest distance ever reached by the ABM and ICBM $d_{closest}(\boldsymbol{V_{TBG}})$.

Calculating this closest distance involves simulating the missile's motions forward in time using the free-fall Equations 6 and 7. The ABM's initial velocity when calculating the minimum distance is: $\boldsymbol{V_{ABM_d}} = \boldsymbol{V_{ABM}} + \boldsymbol{V_{TBG}}$.

As the missile's positions and velocities are played out, **the closest distance occurs when the missiles first start moving apart**. We can check when this happens with the following condition (Mathematics Stackexchange):

$$(\boldsymbol{P}_{\boldsymbol{ICBM_d}} - \boldsymbol{P}_{\boldsymbol{ABM_d}}) \cdot (\boldsymbol{V}_{\boldsymbol{ICBM_d}} - \boldsymbol{V}_{\boldsymbol{ABM_d}}) > 0 \text{ (Equation 15)}$$

Using leapfrog integration and a **large** timestep Δt of 8 seconds, the future positions and velocities are computed efficiently. However, this timestep introduces uncertainty about the exact time the missiles start moving apart. For example, if separation occurs in 10 seconds, the range is initially between 8 and 16 seconds. To refine this range, the equations of motion are reversed to **play the simulation backward in time** by Δt:

$$\boldsymbol{P}_{\boldsymbol{new}} = \boldsymbol{P}_{\boldsymbol{old}} - \boldsymbol{V}_{\boldsymbol{old}}\Delta t - \frac{1}{2}\boldsymbol{A}_{\boldsymbol{g}}(\boldsymbol{P}_{\boldsymbol{old}})\Delta t^2$$

$$\boldsymbol{V}_{\boldsymbol{new}} = \boldsymbol{V}_{\boldsymbol{old}} - \frac{1}{2}(\boldsymbol{A}_{\boldsymbol{g}}(\boldsymbol{P}_{\boldsymbol{old}}) + \boldsymbol{A}_{\boldsymbol{g}}(\boldsymbol{P}_{\boldsymbol{new}}))\Delta t$$

By iteratively halving the time range using binary search (Khan Academy), the exact moment of closest distance is pinpointed, giving us an accurate value for $d_{closest}(\boldsymbol{V}_{\boldsymbol{TBG}})$.

Current Time (seconds)	Current Range (seconds)	Middle Time of Range (seconds)	How Much to Play Back or Forward (seconds)	Are Missiles Moving Apart at Middle Time
16	8 to 16	12	Back 4	Yes

12	8 to 12	10	Back 2	Yes
10	8 to 10	9	Back 1	No
9	9 to 10	9.5	Forward 1	No
9.5	9.5 to 10	9.75	Forward 0.25	No

Table 1, example of pinpointing the time when missiles start moving apart. The time range is narrowed from the initial 8–16 seconds to 9.5–10 seconds.

Variables

Independent Variable

The launch position angle of the Anti-Ballistic Missile (0°, 18°, 36°, 54°, 72°, 90°, 108°, 126°, 144°, 162°, 180°). The launch position angle is measured as the angle between the displacement between the ABM and ICBM, and the velocity vector of the ICBM.

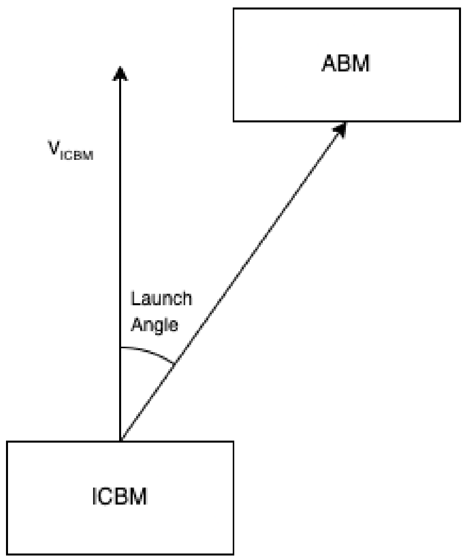

Figure 3, a representation of the meaning of the Launch Angle. The positions of the ICBM and ABM are from a top-down view.

According to Figure 3, a minimum increment of 0° was chosen to signify an ABM that was placed right in front of the ICBM, and a maximum increment of 180° was chosen to signify an ABM that was placed right behind an ICBM. Increments of 18° were chosen so that 11 increments can be obtained between the minimum and maximum increments of 0° and 180°.

Dependant Variable

The magnitude of the velocity to be gained by the ABM in ms^{-1}.

Control Variables

Controlled Variable	Reason for Control	Method of Control
Launch and target coordinates of ICBM	If the launch and target coordinates of the ICBM vary, it will approach the target at different velocities. This will require different $\boldsymbol{V_{TBG}}$s by the ABM for successful interception of the ICBM.	Choose fixed launch coordinates corresponding to North Korea (0.70406407492 rad N, 2.22547107455 rad E), and target coordinates, corresponding to Chicago, USA (0.73091072947 rad N, -1.52942853286 rad E). Then, calculate the $\boldsymbol{V_{TBG}}$ for all launch angles and corresponding launch points, once the ICBM has fell below a certain altitude.
Distance from ABM to ICBM	If the distance between the ABM and ICBM increases, the ABM would need to gain a greater	Set the distance between the ABM and ICBM of 2,400 km, when the ABM is launched, this was chosen to represent

	velocity to move over the greater distance to reach the ICBM. Increasing the $\boldsymbol{V_{TBG}}$ from what was expected. However, the increase in $\boldsymbol{V_{TBG}}$ due to increased horizontal distance, won't be as significant as that of increased altitude, since the ABM isn't **directly opposing gravity** by going vertically.	the maximum range of the Arrow 3, the longest-range ballistic missile interceptor in service today (Potomac Officers Club).
Altitude of ICBM	If the altitude of the ICBM increases, the ABM would need to gain a greater velocity to overpower gravity, so that it can travel a greater vertical distance to intercept the ICBM. Increasing the $\boldsymbol{V_{TBG}}$ from what was expected.	Set the altitude to $\frac{2400}{\sqrt{2}}$ km, which results in an equal horizontal and vertical distance between the ABM and the ICBM. This ensures a high altitude, meaning that there will be enough time to intercept the ICBM as it will take longer to fall to the target. So, the $\boldsymbol{V_{TBG}}$ doesn't need to be excessively large to force a quick interception.

Hypothesis

The rate at which the distance between the ABM and ICBM changes is represented by:

$$\frac{(\boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}}) \cdot (\boldsymbol{V_{ICBM}} - \boldsymbol{V_{ABM}})}{\| \boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}} \|}$$

For the sake of simplicity, $\boldsymbol{V_{ABM}}$ can be assumed to be $\begin{bmatrix}0\\0\\0\end{bmatrix}$ as the ABM is on the ground

(though the ABM has a small amount of velocity due to the Earth's rotation) giving:

$$\frac{(\boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}}) \cdot (\boldsymbol{V_{ICBM}})}{\| \boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}} \|}$$

This can be rewritten as ("Dot Product of Two Vectors | Definition, Properties, Formulas and Examples"):

$$\frac{\| \boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}} \| \| \boldsymbol{V_{ICBM}} \| \; cos(180° - \theta)}{\| \boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}} \|}$$

$$-\| \boldsymbol{V_{ICBM}} \| \; cos(\theta)$$

Where θ is the launch angle, and $180° - \theta$ is the angle between vectors $\boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}}$ and $\boldsymbol{V_{ICBM}}$. The angle of $180° - \theta$ can be proven using the below diagram:

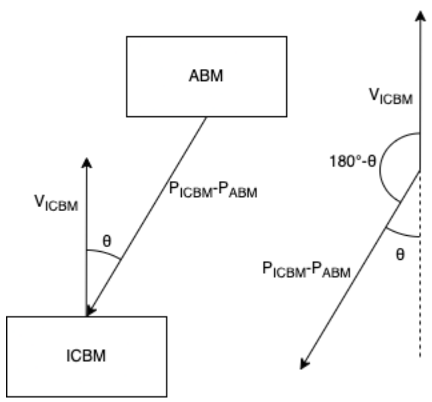

Figure 4, representing the angles between $\boldsymbol{P_{ICBM}} - \boldsymbol{P_{ABM}}$ and $\boldsymbol{V_{ICBM}}$.

Since the ICBM's trajectory is controlled, $\| \boldsymbol{V_{ICBM}} \|$ can be assumed to be a constant positive coefficient a ms^{-1}. Thus, the rate at which the distances between the missiles are changing initially is $-acos(\theta)$.

We can also say the rate at which we desire for the distances between the missiles to decrease is b ms^{-1}, so the desired rate of change of distance is $-b$. The magnitude of the velocity to be gained can be the difference between the initial rate of change of distance and desired rate of change of distance between the ABM and ICBM, having the form:

$$\| \boldsymbol{V_{TBG}} \| = -acos(\theta) - (-\mathrm{b})$$

$$\| \boldsymbol{V_{TBG}} \| = -acos(\theta) + \mathrm{b} \text{ (Equation 16)}$$

Thus, we can easily make the hypothesis that $cos(\theta)$ is negatively linearly correlated with the magnitude of the velocity to be gained.

Simulated Methodology

1. Place the ICBM on the coordinates of North Korea.
2. Launch the ICBM towards the target coordinates of Chicago, USA with an initial velocity inherited from the rotation of the Earth.
    a. While the ICBM is flying, continuously calculate the thrust direction needed for the ICBM to hit the target.
3. Wait until the ICBM's altitude decreases below $\frac{2400}{\sqrt{2}}$ km.

4. Calculate all positions for the ABMs such that they have launch angles (0°, 18°, 36°, 54°, 90°, 108°, 126°, 144°, 162°, 180°), a 2,400 km distance from the ICBM, and are on the ground (the process to calculate these position vectors is not as relevant to the core investigation so it is omitted).
5. For all the positions calculated for the ABM, calculate the velocity it inherited from the rotation of the Earth.
    a. Use this to calculate the $\boldsymbol{V_{TBG}}$s for all the ABM positions such that an ABM placed there, can intercept the ICBM.
6. Record the magnitudes of the $\boldsymbol{V_{TBG}}$s for the purpose of data collection.

Data

Raw Data

Since the method of collecting data was through a simulation, most values have no measurement uncertainty. So, they are rounded to as many decimals as provided by the simulation. However, due to the magnitude of the velocities to be gained varying between repeated calculations (because of random sampling to form linear approximations), they do have an uncertainty calculated through their standard deviation.

Launch Angles (degrees)	Magnitude of the Velocity to Be Gained by the ABM (ms^{-1})									
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6	Trial 7	Trial 8	Trial 9	Trial 10
0	2201.128875	2372.482107	1782.509130	1900.004182	2342.339207	1905.980244	1883.814758	1873.734926	2065.762497	1867.767620
18	2496.398638	2393.420889	2902.233825	2481.431761	2533.153284	2768.798516	2412.819600	2514.004980	2471.793959	2496.706013
36	3608.852580	3475.021859	3442.733014	3498.258001	3462.169914	4121.662493	3437.837529	3511.577344	3381.928420	3504.879531
54	4457.521894	4457.798034	4458.074023	4500.258983	4457.870181	4457.241267	4457.795645	4457.521931	4458.074150	4457.521868

72	5481.884940	5481.883752	5481.768531	5621.906159	5515.802159	5481.883347	5519.717678	5508.997030	5911.317032	5507.377792
90	6407.013853	6411.257202	6410.902820	6574.541674	6438.098759	6471.750862	6407.723553	6405.597692	6423.684446	6405.088069
108	7126.224046	6978.115426	7192.775795	7165.112460	7192.566333	7203.123198	7117.555783	7216.027137	7132.360805	7193.413061
126	7542.197174	7734.199374	7995.657670	7563.627494	7722.242251	7674.247168	7659.879673	7785.899453	7330.490425	7664.614930
144	8220.920429	7867.468876	7836.390472	8261.627675	7765.354609	7982.631646	8037.536611	8011.677007	7476.747716	7940.964841
162	7983.214801	8024.098013	8134.626767	7483.426636	6986.405367	7884.406397	8013.997888	7936.859584	7875.370204	7917.679660
180	7788.820523	7963.292497	7599.429339	7423.822409	7971.163130	8023.680003	8015.949668	8112.892487	8695.584390	7828.436019

Table 2, magnitudes of velocity to be gained by the ABM for various launch angles.

Processed Data

Cosine of Launch Angle	Average Velocity to Be Gained Magnitude (ms^{-1})	Standard Deviation of Velocity to Be Gained Magnitude (ms^{-1})	Percentage Uncertainty of Velocity to Be Gained Magnitude
1.00000000	2019.55235	213.427603	10.6
0.95105652	2547.07615	161.004502	6.3
0.80901699	3544.49207	211.2971000	6.0
0.58778525	4461.96780	13.4567590	0.3
0.30901699	5551.25384	133.2766860	2.4
0.00000000	6435.56589	53.1118204	0.8
-0.309017000	7151.72740	70.2218157	1.0
-0.587785300	7667.30556	172.9896260	2.3
-0.809017000	7940.13199	226.0602760	2.8
-0.951056500	7824.00853	340.4788670	4.4
-1.000000000	7942.30705	338.7493620	4.3

Table 3, average magnitudes of velocity to be gained by the ABM for various cosines of the launch angle.

Graphs

In Figure 5 which shows the interception trajectories, the ICBM is simulated to be launched from North Korea to target Chicago, USA, while the ABM is simulated to be launched from Anchorage, Alaska, USA.

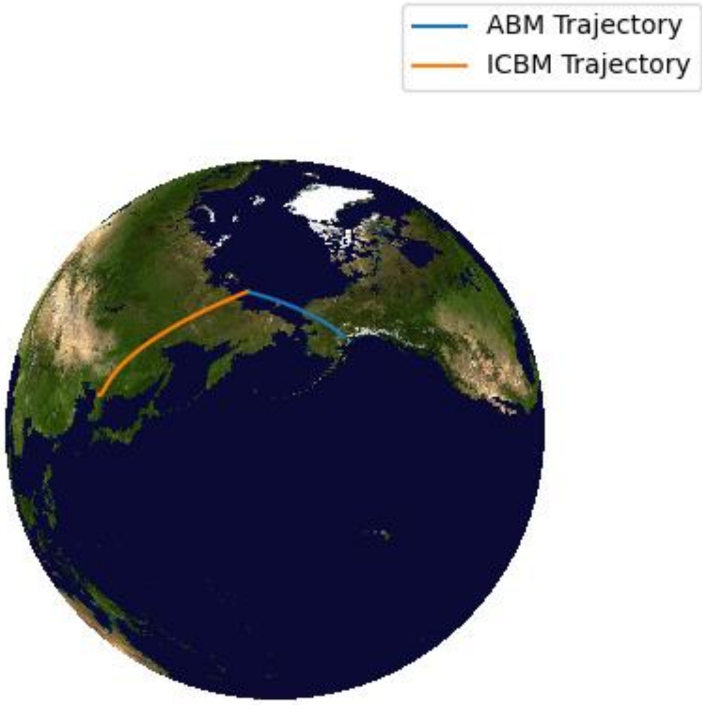

Figure 5, trajectories taken by ABM launched from Anchorage, USA (blue) to intercept ICBM launched from North Korea targeting Chicago, USA (orange).

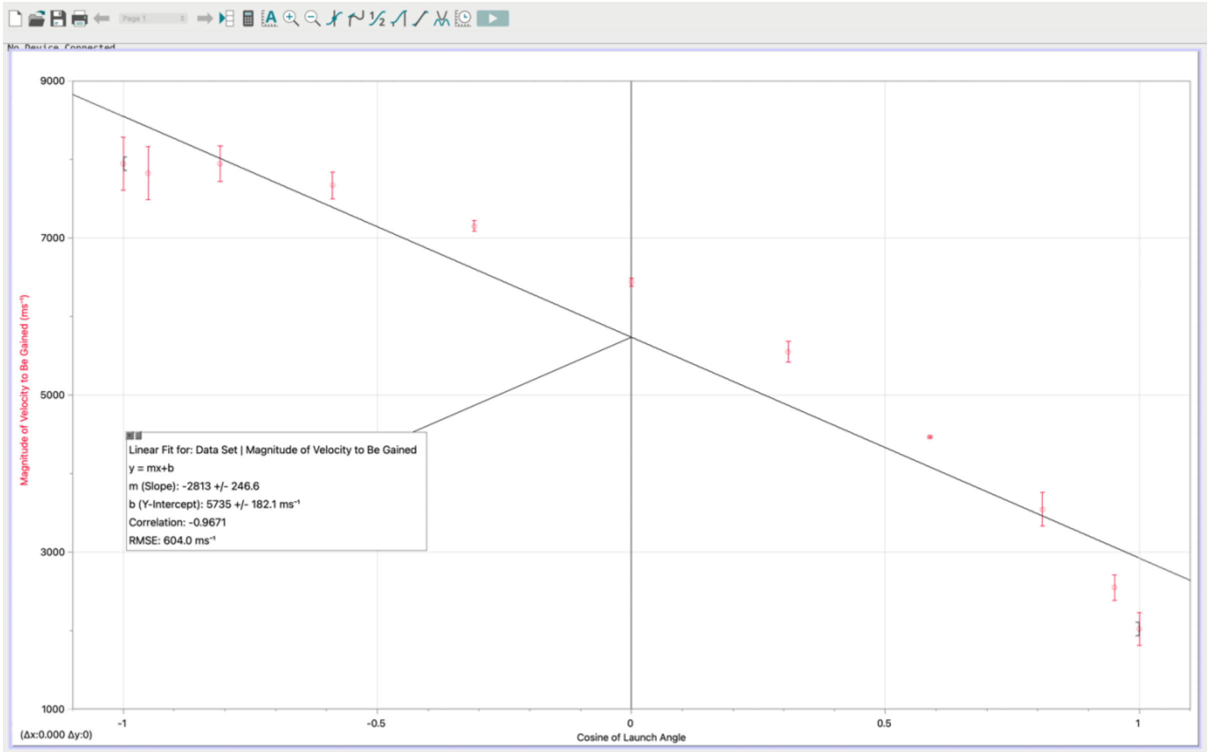

Figure 6, Graph showing linearised relationship between the Cosine of Launch Angle and Magnitude of the Velocity to Be Gained for successful interception.

Sample Calculations for First Row Processed Data

Cosine of Launch Angle

$\cos(0 \times \frac{\pi}{180})$ (Cosine in coding language takes input only in radians)

1.000000000

Average Velocity to Be Gained

$$\frac{2201.128875 + 2372.482107 + 1782.50913 + 1900.004182 + 2342.339207 + 1905.980244 + 1883.814758 + 1873.734926 + 2065.762497 + 1867.767620}{10}$$

2019.5523546 ms^{-1}

Standard Deviation of Velocity to Be Gained

Numerator:

$$(2201.128875 - 2019.55235)^2 + (2372.482107 - 2019.55235)^2 + (1782.509130 - 2019.55235)^2 + (1900.004182 - 2019.55235)^2 + (2342.339207 - 2019.55235)^2 + (1905.980244 - 2019.55235)^2 + (1883.814758 - 2019.55235)^2 + (1873.734926 - 2019.55235)^2 + (2065.762497 - 2019.55235)^2 + (1867.767620 - 2019.55235)^2$$

409962.075711

Standard Deviation:

$$\sqrt{\frac{409962.075711}{10}}$$

202.475202361 ms^{-1}

Percentage Uncertainty

$$100 \times \frac{202.475202361}{2019.5523546}$$

10.6%

Conclusion

As the cosine of the launch angle of the ABM (θ) increases, the magnitude of the velocity to be gained by the ABM ($\| \boldsymbol{V_{TBG}} \|$) decreases. This aligns perfectly with the hypothesis that the magnitude of the velocity to be gained is negatively correlated with $cos(\theta)$ as outlined in the following equation:

$$\| \boldsymbol{V_{TBG}} \| = -a cos(\theta) + \mathrm{b}$$

Where $\| \boldsymbol{V_{TBG}} \|$ is the magnitude of the velocity to be gained, θ is the ABM's launch angle, a is magnitude of the ICBM's velocity, and b is desired rate at which the ABM should approach the missile.

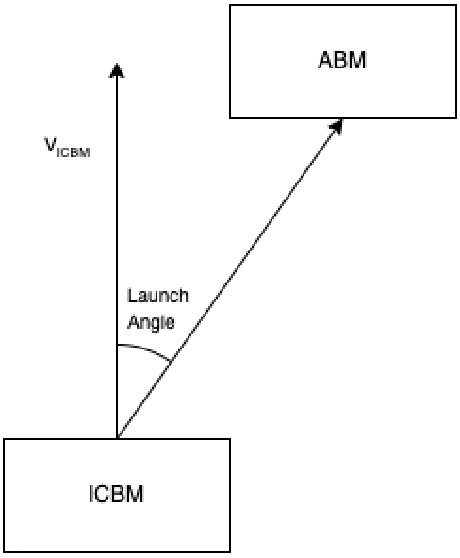

Figure 7, a representation of the meaning of the Launch Angle. The positions of the ICBM and ABM are from a top-down view.

The predicted negative correlation from the hypothesis between $\| \boldsymbol{V_{TBG}} \|$ and $cos(\theta)$ is shown through the close to -1 correlation coefficient of -0.9671 on the graph in Figure 6, indicating the trend predicted by the hypothesis fits the data well, proving the hypothesis' accuracy.

Logically, this makes sense as a negative correlation between $\| \boldsymbol{V_{TBG}} \|$ and $cos(\theta)$ implies a positive correlation between $\| \boldsymbol{V_{TBG}} \|$ and θ itself. Considering when θ is 0 degrees, the ABM is right in front of the ICBM, the ABM only must gain a little velocity so that it can gain altitude and essentially wait for the ICBM to hit it. However, when θ is 180 degrees, the

ABM is directly behind the ICBM. So, the ABM must gain a lot more velocity so that it can catch up with the ICBM, and then some to ensure an energetic collision. This explains why as θ increases, the data shows $\| \boldsymbol{V_{TBG}} \|$ as increasing as well.

However, two notable anomalies do appear within Figure 6:

On the leftmost four data points where $\theta \approx 180°$, $\| \boldsymbol{V_{TBG}} \|$ decreases unusually slowly as $cos(\theta)$ increases. At these angles the ABM is behind the ICBM, and so the optimisation algorithm struggles to compute the high $\boldsymbol{V_{TBG}}$s required for interception, as the error function starts to prioritise decreasing the change in velocity more and closing the distance less. Thus, for large angles, $\| \boldsymbol{V_{TBG}} \|$ represents the velocity needed to close the gap from thousands to tens of kilometres, rather than ensuring direct collision. This explains the limited variation in $\| \boldsymbol{V_{TBG}} \|$, as similar values suffice to simply reduce the distance between the two missiles rather than ensure a collision. In practice, since $\boldsymbol{V_{TBG}}$ is continuously refined by the ABM in flight, an initial inaccuracy in $\boldsymbol{V_{TBG}}$ would not result in a miss.

On the rightmost three data points where $\theta \approx 0°$, $\| \boldsymbol{V_{TBG}} \|$ decreases unusually quickly as $cos(\theta)$ increases. At small angles where the ABM is ahead of the ICBM, slight variations in θ can significantly affect the intercept geometry, transitioning from a head-on collision, to a side-on collision. This is also reflected in the larger error bars, showing uncertainty in the value of $\boldsymbol{V_{TBG}}$ calculated, but in this case due to the sporadic interception points, rather than difficulty in calculating $\boldsymbol{V_{TBG}}$ as for the previous anomaly. This sensitivity for small θ results in larger changes of $\| \boldsymbol{V_{TBG}} \|$ when comparing to that of larger angles, which have lesser change as they result in side-on interceptions regardless of θ.

Within the real world, the data highlights the importance of ABM launch site placements. In the explored scenario of an ICBM launched from North Korea targeting Chicago, USA, the required speed to be gained by the ABM ranges from around 2,000 to 8,000 ms^{-1} depending on the launch angle, as indicated by Table 3. Current ABM missiles such as the Arrow 3 can reach hypersonic speeds to intercept their target (at a maximum of 3,430 ms^{-1} (Potomac Officers Club) ("Hypersonic Speed")). From the line of best fit in Figure 3 we can figure out the maximum launch angle θ the Arrow 3 can be launched from:

$$3430 = -2813cos(\theta) + 5735$$

$$\theta = cos^{-1}(\frac{3430 - 5735}{-2813})$$

$$\theta \approx 35.0°$$

The maximum launch angle for the Arrow 3 of 35.0° is quite small, forcing it to be placed to a few locations that are along the path of a possible North Korean ICBM's trajectory. Given the unpredictability of future threats and the USA's vast size, depending on fixed ABM placements is impractical. To ensure an interception across a wider range of launch angles, larger ABMs are needed that can reach a higher speed. This motivates the development of new missiles such as the Ground Based Interceptor, which can reach speeds of up to 11,000 ms^{-1} ("Groundbased Midcourse Defense -- the Ultimate Smart Weapon"), capable of handling the toughest interceptions requiring up to around 8,000 ms^{-1}, as indicated by Table 3.

Evaluation

Strengths

The method and simulation were successfully able to answer the research question: "How does the launch angle of an ABM effect the magnitude of its velocity to be gained to intercept an ICBM?" The data revealed a strong negative correlation of -0.9671 between the cosine of the launch angle and the magnitude of the velocity to be gained, aligning with the theoretical expectations outlined in the hypothesis, and thus confirming the data's accuracy. The data was also precise, with the optimisation algorithm calculating $\| \boldsymbol{V_{TBG}} \|$ with an at most percentage uncertainty of 10.6%. This precision was achieved by calculating $\| \boldsymbol{V_{TBG}} \|$ across all ABM launch point increments, for a single ICBM trajectory, eliminating the uncertainty caused by calculating $\| \boldsymbol{V_{TBG}} \|$ for multiple, slightly different ICBM trajectories.

The techniques used to enable the interception were also highly effective, with Figures 5 and 6 demonstrating an ABM launched from Anchorage Alaska, intercepting a simulated North Korean ICBM heading to Chicago with an accuracy of under 10 meters, from an initial horizontal distance of 4,000 km. This can be attributed to several factors:

First, the use of linear approximation, where the algorithm simplified the minimisation of an unoptimizable error function into multiple optimisable linear approximations, ensured fast (on the order of a few milliseconds) and accurate refinements of $\boldsymbol{V_{TBG}}$ mid-flight. This quickly corrected initial errors in the $\boldsymbol{V_{TBG}}$ that could have resulted in the ABM missing the ICBM by tens of kilometres.

Secondly, were the use of leapfrog integration, binary search and trust region techniques. Leapfrog integration provided equations of motion that could accurately predict the future motion of the ABM, while being efficient to calculate by using large timesteps. These were refined by binary search to find the exact closest distance between the ABM and ICBM. While the trust region technique ensured the linear approximations were always accurate. When combined these techniques allowed for an **accurate** $\boldsymbol{V_{TBG}}$ to be found **quickly** that minimizes the error function.

Lastly, the gravity compensation to find the thrust direction for the ABM, ensured that the acceleration of the ABM always aligned with $\boldsymbol{V_{TBG}}$, avoiding trajectory deviations due to gravity and thus minimizing the corrections needed to be done by the ABM mid-flight.

Weaknesses and Extensions

The main weakness of the investigation came from the error function that was chosen to be minimized when finding $\boldsymbol{V_{TBG}}$. For interceptions that required smaller velocities to be gained, the error function was able to balance reducing the minimum distance between the ABM and ICBM, as well as minimising $\| \boldsymbol{V_{TBG}} \|$, to ensure it is within the ABM missile's limits. However, for interceptions requiring larger velocities, the error function could not balance this as successfully. This was seen when the algorithm tried to keep $\| \boldsymbol{V_{TBG}} \|$ as low as possible by compromising on accuracy through preventing the closest distance from falling below a few kilometres (in contrast to the desired few meters for interception).

For large launch angles in which high ABM velocities are required, this had the effect of reducing the $\| \boldsymbol{V_{TBG}} \|$ from what was expected, as it did not correspond to a complete intercept (see to the anomalies discussed in the conclusion for more). This can be mitigated in the future by experimenting with different error functions, where the minimum distance term in Equation 8 is for example, squared, to more severely penalize compromising on interception accuracy in favour of reducing velocity requirements. Despite this, the investigation can still be considered overall successful, as this defect in the error function only **slightly impacts a small subset of the data** where the ABM would need to gain an unusually high amount of velocity.

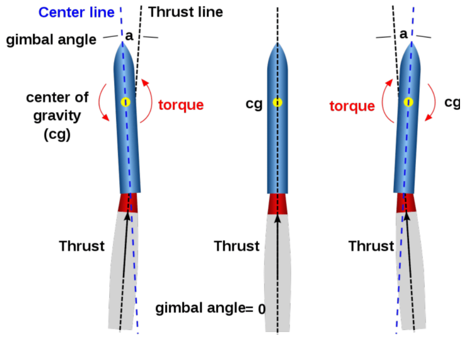

Figure 8, a missile as a rigid body, where changing the direction of the thrust not only provides translational motion, but also rotational motion (Catonoso).

In the future, I plan to extend the simulation from modelling the ABM as a point particle with a combined thrust and gravitational acceleration, to modeling it as a fully-fledged rigid body. This change introduces a new challenge in that the ABM will no longer be able to simply point its thrust in the direction of the gravity compensated thrust vector from Equation 14. Instead, the ABM will need to direct its thrust in a way that induces a torque that would rotate it toward the desired thrust vector, like Figure 8.

By limiting the angle between the ABM's thrust direction and its current orientation, we can more accurately model the ABM's ability to rotate and align itself with the desired thrust vector. This approach differs from the current scenario, where the ABM can instantly point its thrust in any direction to accelerate toward the target. Implementing this change will provide a clearer understanding of how effective the promising Velocity to Be Gained guidance technique is in enabling realistic ICBM interception scenarios.

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