

Tracking Realizable Trajectories via Incremental Exponential Stability

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1 Nonlinear System

Consider the control-affine system

$$\dot{x} = f_{ol}(x) + Gu.$$

Assume a state-feedback controller

$$u = k(x),$$

and define the closed-loop vector field

$$f_{cl}(x) := f_{ol}(x) + Gk(x).$$

Assumption 1 (Incremental Exponential Stability). *The system*

$$\dot{z} = f_{cl}(z)$$

is incrementally exponentially stable (IES), i.e., there exist constants $c \geq 1$ and $\lambda > 0$ such that for any two solutions $z_1(t)$, $z_2(t)$,

$$\|z_1(t) - z_2(t)\| \leq ce^{-\lambda t} \|z_1(0) - z_2(0)\|.$$

2 Reference Trajectory and Feedback–Feedforward Structure

Let $x^*(t)$ be a realizable reference trajectory generated by

$$\dot{x}^* = f_{ol}(x^*) + Gu^*.$$

Add and subtract $k(x^*)$:

$$\dot{x}^* = f_{cl}(x^*) + G(u^* - k(x^*)).$$

Defining

$$v^* := u^* - k(x^*),$$

the reference input admits a **feedback + feedforward decomposition**. We can thus use the following **feedback + feedforward control law** for the actual system:

$$u = k(x) + v^*,$$

where v^* is the **feedforward term induced by the reference trajectory** (x^*, u^*) as defined above.

This yields the closed-loop dynamics

$$\dot{x} = f_{cl}(x) + Gv^*, \quad \dot{x}^* = f_{cl}(x^*) + Gv^*.$$

3 General use in \mathcal{L}_1 -DRAC

As in the manuscript, we define

$$\begin{aligned} f(t, x) &:= f_{cl}(x) + Gv^*(t) = f_{ol}(x) + G(k(x) + u^*(t) - k(x^*(t))) \\ &= f_{ol}(x) + G(k(x) - k(x^*(t))) + Gu^*(t). \end{aligned} \tag{1}$$

If we assume that the nominal law k is Lipschitz, i.e., there exists $L_k > 0$ such that

$$\|k(x) - k(x^*)\| \leq L_k \|x - x^*\|, \quad \forall x, x^* \in \mathbb{R}^n,$$

then, the bound in the paper should be changed from

$$\|f(t, x)\| \leq \Delta_f \left(1 + \|x\|^2\right)^{\frac{1}{2}}, \quad \forall t, x,$$

to

$$\|f(t, x)\| \leq \Delta_{f_{ol}} \left(1 + \|x\|^2\right)^{\frac{1}{2}} + L_k \|x - x^*\| + \|G\| \|u^*(t)\|, \quad \forall t, x. \tag{2}$$

which we will further change to reduce the conservatism as follows:

$$\|f(t, x)\| \leq \left(\Delta_{f_1}^2 + \Delta_{f_2}^2 \|x\|^2\right)^{\frac{1}{2}} + L_k \|x - x^*\| \quad \forall t, x, \tag{3}$$

for some known constants $\Delta_{f_1}, \Delta_{f_2}, L_k \geq 0$, and where $\|G\| \|u^*(t)\|$ is absorbed into Δ_{f_1} .

4 Error Dynamics

Define the tracking error

$$e := x - x^*.$$

Then

$$\dot{e} = f_{cl}(x) - f_{cl}(x^*) = f_{cl}(x^* + e) - f_{cl}(x^*).$$

4.1 Interpretation

Since $x(t)$ and $x^*(t)$ satisfy the **same system**

$$\dot{z} = f_{cl}(z) + Gv^*(t),$$

and $\dot{z} = f_{cl}(z)$ is incrementally exponentially stable, the error dynamics represent the distance between two trajectories of an IES system.

5 Special Case: Linear Time-Invariant (LTI) Systems

5.1 LTI Plant

Consider the LTI system

$$\dot{x} = Ax + Bu,$$

with linear state-feedback

$$u = Kx,$$

and assume

$$A + BK \text{ is Hurwitz.}$$

Define

$$f_{cl}(x) := (A + BK)x.$$

5.2 Reference Trajectory and Feedback–Feedforward Structure

Let the reference trajectory satisfy

$$\dot{x}^* = Ax^* + Bu^*.$$

Defining

$$v^* := u^* - Kx^*,$$

the reference input admits a **feedback + feedforward decomposition**.

We can thus use the following **feedback + feedforward control law** for the actual system:

$$u = Kx + v^*,$$

where v^* is the **feedforward term induced by the reference trajectory** (x^*, u^*) .

This yields

$$\dot{x} = f(t, x) := (A + BK)x + Bv^*, \quad \dot{x}^* = (A + BK)x^* + Bv^*.$$

5.3 General use in \mathcal{L}_1 -DRAC

We have that

$$\begin{aligned}\|f(t, x)\| &= \|(A + BK)x + Bv^*(t)\| \\ &= \|(A + BK)x + B(u^*(t) - Kx^*(t))\| \\ &\leq \|A\|_F \|x\| + \|B\|_F \|K\|_F \|x(t) - x^*(t)\| + \|B\|_F \|u^*(t)\|, \quad \forall t, x.\end{aligned}$$

Then, as in (3), we will determine constants $\Delta_{f_1}, \Delta_{f_2}, L_k \geq 0$, such that

$$\|f(t, x)\| = \|(A + Bk)x + Bv^*(t)\| \leq \left(\Delta_{f_1}^2 + \Delta_{f_2}^2 \|x\|^2\right)^{\frac{1}{2}} + L_k \|x - x^*\| + \|G\| \|u^*(t)\|, \quad \forall t, x, \quad (4)$$

where Δ_{f_1} is defined to reflect $\|B\|_F \|u^*(t)\|$, $\Delta_{f_2} = \|A\|_F$, and $L_k = \|B\|_F \|K\|_F$. Note that u^* comes from a planner, in our case, covariance steering, so we can put constraints on its (expected) magnitude to help define Δ_{f_1} . Vivek has already established this.

5.4 Error Dynamics

Define $e := x - x^*$. Then

$$\dot{e} = (A + BK)e.$$

5.5 Interpretation (LTI)

Since $A + BK$ is Hurwitz, the system is incrementally exponentially stable, and the tracking error satisfies

$$\|e(t)\| \leq ce^{-\lambda t} \|e(0)\|,$$

for some $c \geq 1$, $\lambda > 0$, independent of the reference input u^* .

References

[1] Placeholder.