Tracking Realizable Trajectories via Incremental Exponential Stability

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1 Nonlinear System

Consider the control-affine system

$$\dot{x} = f(x) + Gu.$$

Assume a state-feedback controller

$$u = k(x),$$

and define the closed-loop vector field

$$F(x) := f(x) + Gk(x).$$

Assumption 1 (Incremental Exponential Stability)**.** *The system*

$$\dot{z} = F(z)$$

is ***incrementally exponentially stable (IES)****, i.e., there exist constants $c \geq 1$ and $\lambda > 0$ such that for any two solutions $z_1(t)$, $z_2(t)$,*

$$\|z_1(t) - z_2(t)\| \leq ce^{-\lambda t}\|z_1(0) - z_2(0)\|.$$

2 Reference Trajectory and Feedback–Feedforward Structure

Let $x_r(t)$ be a realizable reference trajectory generated by

$$\dot{x}_r = f(x_r) + Gu_r.$$

Add and subtract $k(x_r)$:

$$\dot{x}_r = F(x_r) + G\big(u_r - k(x_r)\big).$$

Defining

$$v_r := u_r - k(x_r),$$

the reference input admits a **feedback + feedforward decomposition**. We can thus use the following **feedback + feedforward control law** for the actual system:

$$u = k(x) + v_r,$$

where v_r is the **feedforward term induced by the reference trajectory** (x_r, u_r) as defined above.

This yields the closed-loop dynamics

$$\dot{x} = F(x) + Gv_r, \qquad \dot{x}_r = F(x_r) + Gv_r.$$

3 Error Dynamics

Define the tracking error

$$e := x - x_r.$$

Then

$$\dot{e} = F(x) - F(x_r) = F(x_r + e) - F(x_r).$$

3.1 Interpretation

Since $x(t)$ and $x_r(t)$ satisfy the **same system**

$$\dot{z} = F(z) + Gv_r(t),$$

and $\dot{z} = F(z)$ is incrementally exponentially stable, the error dynamics represent the distance between two trajectories of an IES system.

4 Special Case: Linear Time-Invariant (LTI) Systems

4.1 LTI Plant

Consider the LTI system

$$\dot{x} = Ax + Bu,$$

with linear state-feedback

$$u = Kx,$$

and assume

$$A + BK \quad \text{is Hurwitz.}$$

Define

$$F(x) := (A + BK)x.$$

4.2 Reference Trajectory and Feedback–Feedforward Structure

Let the reference trajectory satisfy

$$\dot{x}_r = Ax_r + Bu_r.$$

Defining

$$v_r := u_r - Kx_r,$$

the reference input admits a **feedback + feedforward decomposition**.

We can thus use the following **feedback + feedforward control law** for the actual system:

$$u = Kx + v_r,$$

where v_r is the **feedforward term induced by the reference trajectory** (x_r, u_r).

This yields

$$\dot{x} = (A + BK)x + Bv_r, \qquad \dot{x}_r = (A + BK)x_r + Bv_r.$$

4.3 Error Dynamics

Define $e := x - x_r$. Then

$$\dot{e} = (A + BK)e.$$

4.4 Interpretation (LTI)

Since $A + BK$ is Hurwitz, the system is incrementally exponentially stable, and the tracking error satisfies

$$\|e(t)\| \leq ce^{-\lambda t}\|e(0)\|,$$

for some $c \geq 1$, $\lambda > 0$, independent of the reference input u_r.

References

[1] Placeholder.