# Tracking Realizable Trajectories via Incremental Exponential Stability

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## Contents



## 1 Nonlinear System

Consider the control-affine system

$$\dot{x} = f_{ol}(x) + Gu.$$

Assume a state-feedback controller

$$u = k(x),$$

and define the closed-loop vector field

$$f_{cl}(x) := f_{ol}(x) + Gk(x).$$

**Assumption 1** (Incremental Exponential Stability)**.** *The system*

$$\dot{z} = f_{cl}(z)$$

*is **incrementally exponentially stable (IES)**, i.e., there exist constants $c \geq 1$ and $\lambda > 0$ such that for any two solutions $z_1(t)$, $z_2(t)$,*

$$\|z_1(t) - z_2(t)\| \leq ce^{-\lambda t}\|z_1(0) - z_2(0)\|.$$

## 2 Reference Trajectory and Feedback–Feedforward Structure

Let $x^\star(t)$ be a realizable reference trajectory generated by

$$\dot{x}^\star = f_{ol}(x^\star) + Gu^\star.$$

Add and subtract $k(x^\star)$:

$$\dot{x}^\star = f_{cl}(x^\star) + G\big(u^\star - k(x^\star)\big).$$

Defining

$$v^\star := u^\star - k(x^\star),$$

the reference input admits a **feedback + feedforward decomposition**. We can thus use the following **feedback + feedforward control law** for the actual system:

$$u = k(x) + v^\star,$$

where $v^\star$ is the **feedforward term induced by the reference trajectory** $(x^\star, u^\star)$ as defined above.

This yields the closed-loop dynamics

$$\dot{x} = f_{cl}(x) + Gv^\star, \qquad \dot{x}^\star = f_{cl}(x^\star) + Gv^\star.$$

## 3 General use in $\mathcal{L}_1$-DRAC

As in the manuscript, we define

$$\begin{aligned} f(t,x) := f_{cl}(x) + Gv^\star(t) =& f_{ol}(x) + G\left(k(x) + u^\star(t) - k\left(x^\star(t)\right)\right) \\ =& f_{ol}(x) + G\left(k(x) - k\left(x^\star(t)\right)\right) + Gu^\star(t). \end{aligned} \tag{1}$$

If we assume that the nominal law $k$ is Lipschitz, i.e., there exists $L_k > 0$ such that

$$\|k(x) - k\left(x^\star\right)\| \leq L_k \|x - x^\star\|, \quad \forall x, x^\star \in \mathbb{R}^n,$$

then, the bound in the paper should be changed from

$$\|f(t,x)\| \leq \Delta_f \left(1 + \|x\|^2\right)^{\frac{1}{2}}, \quad \forall t, x,$$

to

$$\|f(t,x)\| \leq \Delta_{f_{ol}} \left(1 + \|x\|^2\right)^{\frac{1}{2}} + L_k \|x - x^\star\| + \|G\| \|u^\star(t)\|, \quad \forall t, x. \tag{2}$$

which we will further change to reduce the conservatism as follows:

$$\|f(t,x)\| \leq \left(\Delta_{f_1}^2 + \Delta_{f_2}^2 \|x\|^2\right)^{\frac{1}{2}} + L_k \|x - x^\star\| \quad \forall t, x, \tag{3}$$

for some known constants $\Delta_{f_1}, \Delta_{f_2}, L_k \geq 0$, and where $\|G\| \|u^\star(t)\|$ is absorbed into $\Delta_{f_1}$.

# 4 Error Dynamics

Define the tracking error

$$e := x - x^\star.$$

Then

$$\dot{e} = f_{cl}(x) - f_{cl}(x^\star) = f_{cl}(x^\star + e) - f_{cl}(x^\star).$$

## 4.1 Interpretation

Since $x(t)$ and $x^\star(t)$ satisfy the **same system**

$$\dot{z} = f_{cl}(z) + Gv^\star(t),$$

and $\dot{z} = f_{cl}(z)$ is incrementally exponentially stable, the error dynamics represent the distance between two trajectories of an IES system.

# 5 Special Case: Linear Time-Invariant (LTI) Systems

## 5.1 LTI Plant

Consider the LTI system

$$\dot{x} = Ax + Bu,$$

with linear state-feedback

$$u = Kx,$$

and assume

$$A + BK \quad \text{is Hurwitz.}$$

Define

$$f_{cl}(x) := (A + BK)x.$$

## 5.2 Reference Trajectory and Feedback–Feedforward Structure

Let the reference trajectory satisfy

$$\dot{x}^\star = Ax^\star + Bu^\star.$$

Defining

$$v^\star := u^\star - Kx^\star,$$

the reference input admits a **feedback + feedforward decomposition**.

We can thus use the following **feedback + feedforward control law** for the actual system:

$$u = Kx + v^\star,$$

where $v^\star$ is the **feedforward term induced by the reference trajectory** $(x^\star, u^\star)$.

This yields

$$\dot{x} = f(t, x) := (A + BK)x + Bv^\star, \qquad \dot{x}^\star = (A + BK)x^\star + Bv^\star.$$

### 5.3 General use in $\mathcal{L}_1$-DRAC

We have that

$$\begin{aligned}
\|f(t,x)\| &= \|(A+BK)x + Bv^\star(t)\| \\
&= \|(A+BK)x + B\left(u^\star(t) - Kx^\star(t)\right)\| \\
&\leq \|A\|_F \|x\| + \|B\|_F \|K\|_F \|x(t) - x^\star(t)\| + \|B\|_F \|u^\star(t)\|, \quad \forall t, x.
\end{aligned}$$

Then, as in (3), we will determine constants $\Delta_{f_1}, \Delta_{f_2}, L_k \geq 0$, such that

$$\|f(t,x)\| = \|(A+Bk)x + Bv^\star(t)\| \leq \left(\Delta_{f_1}^2 + \Delta_{f_2}^2 \|x\|^2\right)^{\frac{1}{2}} + L_k \|x - x^\star\| + \|G\| \|u^\star(t)\|, \quad \forall t, x, \tag{4}$$

where $\Delta_{f_1}$ is defined to reflect $\|B\|_F \|u^\star(t)\|$, $\Delta_{f_2} = \|A\|_F$, and $L_k = \|B\|_F \|K\|_F$. Note that $u^\star$ comes from a planner, in our case, covariance steering, so we can put constraints on its (expected) magnitude to help define $\Delta_{f_1}$. Vivek has already established this.

### 5.4 Error Dynamics

Define $e := x - x^\star$. Then

$$\dot{e} = (A+BK)e.$$

### 5.5 Interpretation (LTI)

Since $A + BK$ is Hurwitz, the system is incrementally exponentially stable, and the tracking error satisfies

$$\|e(t)\| \leq ce^{-\lambda t}\|e(0)\|,$$

for some $c \geq 1$, $\lambda > 0$, independent of the reference input $u^\star$.

## References


[1] Placeholder.