Wasserstein Distributionally Robust Adaptive Covariance Steering

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***Abstract*— We present a methodology for predictable and safe covariance steering control of uncertain nonlinear stochastic processes. The systems under consideration are subject to general uncertainties, which include unbounded random disturbances (*aleatoric* uncertainties) and incomplete model knowledge (state-dependent *epistemic* uncertainties). These general uncertainties lead to temporally evolving state distributions that are entirely unknown, can have arbitrary shapes, and may diverge unquantifiably from expected behaviors, leading to unpredictable and unsafe behaviors. Our method relies on an $\mathcal{L}_1$-adaptive control architecture that ensures robust control of uncertain stochastic processes while providing Wasserstein metric certificates in the space of probability measures. We show how these distributional certificates can be incorporated into the high-level covariance control steering to guarantee safe control. Unlike existing distributionally robust planning and control methodologies, our approach avoids difficult-to-verify requirements like the availability of finite samples from the true underlying distribution or an *a priori* knowledge of time-varying ambiguity sets to which the state distributions are assumed to belong.**

I. INTRODUCTION

From the inception of feedback control in the form of centrifugal governors to the contemporary highly complex black-box autonomy algorithms for robotics, the safe operation of dynamical systems is an omnipresent requirement for the adoption of any technology by the general society. Guaranteeing predictable and safe operation in an inherently uncertain world environment is challenging, further exacerbated by incomplete knowledge of the systems themselves. In response to such challenges, one usually appeals to methodologies that can ensure robustness against epistemic uncertainties (lack of knowledge but learnable) and aleatoric uncertainties (inherently random disturbances). However, even before the control synthesis, one must consider various representations of the uncertainties we desire robustness against. Assuming the uncertainties lie in deterministic bounded subsets of the state space offers a relatively easier solution to the robust control synthesis problem. However, attempting robust worst-case analysis assuming bounded disturbances and deterministic uncertainties often leads to overly conservative results, see e.g. [1]–[3] and references therein. Instead, one can study average-case (or high-probability) stochastic safety guarantees to alleviate the conservativeness since then one analyzes the distributions and their associated statistical properties instead of purely their supports. Using stochastic representations of uncertainties can lead to safety guarantees with possible reduced conservatism. Furthermore, representing the stochastic systems' states using the associated distributions (probability measures) allows the use of statistical learning theory [4] to incorporate data-driven learned components [5].

Control of stochastic systems, stochastic optimal control (SOC) in particular, is a rich field with early examples such as the celebrated linear quadratic Gaussian (LQG) [6], [7], to the modern approaches like model predictive path integral (MPPI) control [8], covariance steering [9]–[11], and even reinforcement learning (RL) [12], [13] since it can be argued that RL combines learning of models/reward functions and SOC synthesis. To ensure predictability and safety for stochastic systems, e.g., collision avoidance, a designer alters the SOC synthesis problem by incorporating binary evaluations (safe vs. unsafe) in the constraint set and continuous evaluations (e.g., the farther from an obstacle, the better) within the cost function [14], [15]. Given the stochastic nature of the systems, the notions of safety can be represented using *risk metrics* like chance constraints (probability or moments of state $\in$ safe set) [16], value at risk (VaR), and conditional value at risk (CVaR) [17]. An in-depth investigation into risk metrics can be found in [18].

Evaluating risk metrics requires the knowledge of the state distribution (probability measure induced by the random state). Except for overly simple systems like linear plants initialized by Gaussian random variables, analytical descriptions of the time-varying state distributions are seldom available. Hence, one usually resorts to approximating the distributions numerically, which is computationally expensive since this approach requires multiple samples from the true time-varying distributions. A well-known example of approximating distributions via samples is sample average approximation [19]. An approach to circumvent the limitation is to assume that the uncertainties and disturbances induce distributions with compact supports, thus allowing for a deterministic worst-case robust synthesis, see e.g. [1]–[3]. As we discussed, this approach leads to overly conservative results and ignores the statistical information in the stochastic representation.

Distributionally robust optimization (DRO) allows one to take an alternate approach to robustness by directly hedging

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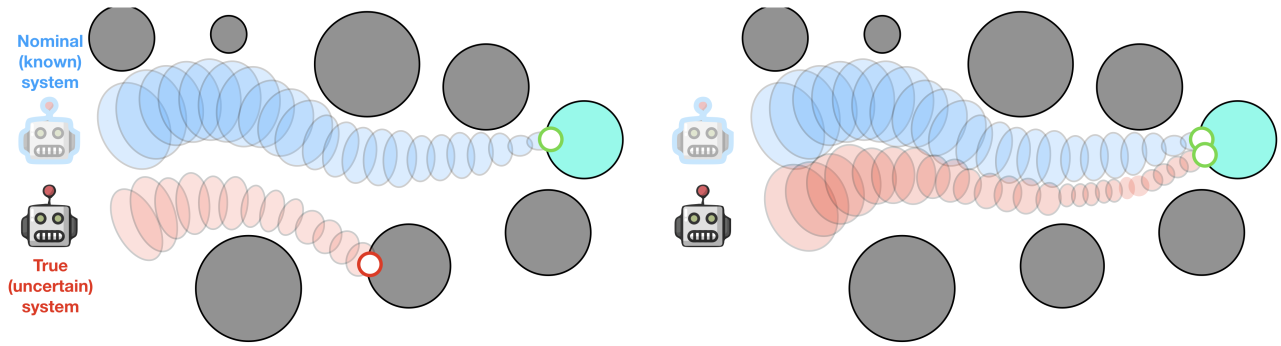

(a) (b)

Fig. 1. Consider the problem of safely navigating an uncertain system to a goal set (green circle), avoiding unsafe subsets (gray circles). One constructs a control law/policy for the uncertainty-free nominal (known) system as it is the best knowledge available for the true (uncertain) system. (a) While the control policy successfully guides the nominal (known) system to the goal set, illustrated with temporal state distributions in light blue, applying the same policy to the true (uncertain) system leads to unquantifiable and undesirable deviation due to the presence of uncertainties (illustrated with light-red temporal state-distributions). (b) Thus, it is critical to design an augmentation feedback policy to handle uncertainties such that the original policy can still guide the true (uncertain) system predictably and safely.

against the uncertainty due to the mismatch between the available *nominal* (estimated/approximated) and the true distributions [20]–[22]. Since, in this case, the uncertainties are quantified by some notion of 'distance' between distributions (probability measures), one considers an *ambiguity set* - a set of probability measures, usually chosen as a ball centered around the nominal distribution. The radius of the ambiguity set is chosen such that the true distribution belongs to the ambiguity set with a desired probability. The use of DRO within SOC problems is usually referred to as distributionally robust control in the literature. The area of distributionally robust control has seen considerable recent research effort since it allows one to cast the robust synthesis problem in the natural space of probability measures without requiring restrictive assumptions. See [22], [23] for a few recent examples. One may define the ambiguity sets using various notions of 'distances' between probability measures, e.g., moment-based ambiguity sets [24], ambiguity sets based on f-divergence [25], and Wasserstein metric-based ambiguity sets [26]–[28]. Moment-based ambiguity sets are often conservative and require a large sample size to produce reliable moment estimates. The use of f-divergence, in particular, KL-divergence, is popular to quantify the dissimilarity between probability distributions, but suffers from lack of symmetry in their arguments, the requirement of connected support of probability measures, and is not applicable between different types of probability measures, e.g., discrete and continuous probability measures. Wasserstein distance-based approaches offer superior advantages since the Wasserstein distance defines a metric on the space of probability measures [29]. This allows one to quantify the dissimilarity between distributions in a rigorous fashion that accounts for the differences in the shapes of distributions as well [30]. Moreover, Wasserstein ambiguity sets contain a richer set of relevant distributions, and the corresponding Wasserstein DRO provides a superior sample performance guarantee [31].

While using Wasserstein DRO is attractive, establishing ambiguity sets containing the true distributions is challenging since the Wasserstein metric is the optimum of an infinite-dimensional optimization problem [31]. An approach to construct Wasserstein ambiguity sets is to leverage the finite sample guarantees to construct balls around empirical distributions such that the true distributions lie inside the ambiguity set with a high probability [32]. The benefits of this approach stem from it depending purely on finite samples and has thus seen its application for a multitude of distributionally robust controller synthesis approaches [33]–[35]. However, the assumption that samples are available from the true distributions can prove to be unverifiable for various applications of interest. For example, state distributions are time-varying by their nature of describing the evolution of dynamic processes. Hence, requiring the availability of samples from the true distribution would translate to requiring multiple samples at each point in time, considering the system is described in a discrete-time fashion. Moreover, the number of samples required at each point corresponds directly to the size of the high-probability ambiguity sets via the finite sample guarantees [31], [36]. Another major hurdle that is relevant to the control of uncertain systems, as we consider in the manuscript, is that a control law will have access to either state or output measurements to compute the input to the system. Hence, the controller has access to precisely a *single sample* from the true distribution at each point in time, rendering the Wasserstein finite-sample guarantees unusable.

We provide Wasserstein distributionally robust adaptive covariance steering - a methodology for distributionally robust control of uncertain nonlinear systems subject to both epistemic and aleatoric uncertainties. The features of our approach are as follows:

1) We do not require the availability of any samples from the true distribution to construct the Wasserstein ambiguity sets;

2) We instead rely on our recently developed $\mathcal{L}_1$-distributionally robust adaptive control ($\mathcal{L}_1$-DRAC) [37] as a state-feedback control augmentation, that provides certificates of Wasserstein ambiguity sets with probability 1. The $\mathcal{L}_1$-DRAC controller is based on the architecture of the $\mathcal{L}_1$-adaptive control [38]. We have previously successfully applied $\mathcal{L}_1$-adaptive control to model-based reinforcement learning [39];
3) The overall controller constitutes the coupling of the $\mathcal{L}_1$-DRAC controller with a high-level control law (covariance steering) in a feedback-motion planning approach [40, Chp. 8];
4) The high-level covariance steering (CS) controller generates reference trajectories using only the *nominal plant* knowledge, but incorporates Wasserstein DRO constraints using CVaR as enabled via the guaranteed ambiguity sets ensured by the low-level $\mathcal{L}_1$-DRAC;
5) The proposed approach thus ensures safety of the system via the Wasserstein DRO while retaining the solvability of the CS controller;
6) Since the high-level CS controller is altered only in way to incorporate the additional Wasserstein DRO constraint, the proposed approach can be easily adapted to be used with most SOC controllers.

The manuscript is organized as follows: Section II provides the problem statement that we aim to address, along with a brief discussion of the constituent covariance steering control and $\mathcal{L}_1$-DRAC. The main contribution is presented in Section III, which is then followed with the conclusions and exploratory directions in the final section.

A. Notation

We denote by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ the set of positive and non-negative reals, respectively. $\mathbb{N}_0$ is the set of natural numbers starting at 0. $\mathcal{C}(\mathbb{R}^n;\mathbb{R}^m)$ denotes the set of continuously differentiable maps $\mathbb{R}^n \to \mathbb{R}^m$. We denote by $\mathcal{B}(F)$, the Borel σ-algebra generated by F and the measures on $\mathcal{B}(F)$ by $\mathcal{M}(F)$. $\mathbb{1}_S$ denotes the indicator set of the set S. Furthermore, let $(S, \Sigma, \mathcal{M})$ be a measure space and $1 \leq p < \infty$, $\|f\|_{L_p}$ denotes the L_p norm given by $\|f\|_{L_p} := \left(\int_S |f|^p d\mathcal{M}\right)^{1/p}$ and $\mathbb{W}_{2\mathsf{p}}(\mathbb{X}_t, \mathbb{X}_t^\star)$ denotes the 2 Wasserstein metric between probability measures $\mathbb{X}_t$ and $\mathbb{X}_t^\star$ (see [37] for the complete definition).

II. Problem Statement and Preliminaries

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, an adapted Brownian motion $W_t \in \mathbb{R}^{n_w}$, and a random variable $x_0 \sim \xi_0$ with finite second moment which is independent of $\sigma(\cup_{t\geq 0}\mathcal{W}_t)$, where $\mathcal{W}_t = \sigma(W_s \ : \ s \leq t)$. We consider an uncertain system for which the evolution of dynamics are governed by the following nonlinear Itô process:

$$\begin{aligned} dX_t =& F_\mu(X_t, U_t)\,dt + F_\sigma(X_t)\,dW_t \\ =& \left[A_\mu X_t + B\left(U_t + H_\mu(X_t)\right)\right]dt \\ &+ \left[A_\sigma + BH_\sigma(X_t)\right]dW_t, \quad X_0 = x_0, \end{aligned} \tag{1}$$

where $X_t \in \mathbb{R}^n$ is the system state, $U_t \in \mathbb{R}^m$ is the control input, and $B \in \mathbb{R}^{n\times m}$ is the input operator. The matrices $A_\mu \in \mathbb{R}^{n\times n}$ and $A_\sigma \in \mathbb{R}^{n\times n_w}$ represent the *known* components of the dynamics, whereas, the functions $H_\mu \in \mathcal{C}(\mathbb{R}^n;\mathbb{R}^m)$ and $H_\sigma \in \mathcal{C}(\mathbb{R}^n;\mathbb{R}^{m\times n_w})$ represent the *uncertainties* in the drift and diffusion, respectively.

Assumption 1: The functions H_μ and H_σ are globally Lipschitz continuous with known constants L_μ and L_σ, respectively. Moreover, there exist known positive constants Δ_μ and Δ_σ such that $\|H_\mu(a)\|^2 \leq \Delta_\mu^2\left(1+\|a\|^2\right)$ and $\|H_\sigma(a)\|_F^2 \leq \Delta_\sigma^2\left(1+\|a\|^2\right)^{\frac{1}{2}}$, $\forall a \in \mathbb{R}^n$.

Remark 1: Under the assumption above, and the independence of the initial condition from the driving Brownian motion W_t, we assume that the input process U_t is adapted to the underlying filtration and is regular enough so that X_t exists as the unique strong solution of (1)*, see [41, Thm. 5.2.9] and [42, Thm. 5.2.1]. Furthermore, see [37] for the well-posedness under the $\mathcal{L}_1$-DRAC feedback.*

For the unique strong solution X_t, we define its distribution as

$$X_t \sim \mathbb{X}_t, \quad \mathbb{X}_t \in \mathcal{M}(\mathbb{R}^n), \quad t \geq 0. \tag{2}$$

Next, assuming there are $n_o \in \mathbb{N}$ obstacles to be avoided, we consider each of the obstacles defined via the union of $n_l \in \mathbb{N}$ half-spaces[1] which allows us to define the *safe region* as follows:

$$\mathbb{R}^n \supseteq \mathcal{X}_{safe} := \bigcap_{j=1}^{n_o}\bigcup_{l=1}^{n_l}\left\{z \in \mathbb{R}^n \mid c_{j,l}^\top z \geq d_{j,l}\right\}, \tag{3}$$

where $c_{j,l} \in \mathbb{R}^n$ and $d_{j,l} \in \mathbb{R}$ are known *a priori*. A representation of the safe region via a set of affine constraints is helpful in maintaining the numerical tractability of the high-level planner, and is thus common in the literature [43], [44]. We further define the notion of safety via the *safety map* $\Phi(\cdot, \mathcal{X}_{safe}) : \mathbb{R}^n \to \mathbb{R}$ by establishing the following requirement:

$$\text{Safe operation of (1)} \Leftarrow \Phi(X_t, \mathcal{X}_{safe}) \leq \delta_s, \ \forall t \in [0, T], \tag{4}$$

for a chosen horizon length $T \in (0, \infty)$ and a given user specified *risk tolerance parameter* $\delta_s \in \mathbb{R}_{>0}$ of acceptable risk. As stated previously, we will consider two instances of Φ, with the first being the distributionally robust conditional value at risk (CVaR) that encodes safety via the Wasserstein metric, and the second instance is defined using the chance constraints. We now state the general form of the problem statement that we address. The complete details of the problem will be provided later.

Problem Statement: Given Gaussian initial and terminal probability measures $\mathcal{N}(\mu_0, \Sigma_0)$ and $\mathcal{N}(\mu_T, \Sigma_T)$, respectively, for any $T \in (0, \infty)$, the safety map Φ, and the risk tolerance parameter δ_s: compute the input process $(U_t)_{t\in[0,T]}$, such that the uncertain process X_t in (1) is optimal with

[1] We keep each obstacle's description uniform in n_l (the number of half-spaces) for simplicity of exposition. The extension to differing number of half-spaces for each obstacle is straightforward.

respect to a user designed optimal control objective J, while satisfying the safety requirement (4).

We next provide a brief description of covariance steering control which is the chosen high-level planner, and the $\mathcal{L}_1$-DRAC control that enables the distributional robustness of the entire scheme.

A. *Covariance Steering Control*

As discussed briefly before, covariance steering control methodology is an approach to SOC that considers both mean and covariance as optimization variables [9], [45]. Controlling the first two moments can be interpreted as effectuating the average behavior of the system and the *shape* of the uncertainties. Since CS is a model-based approach to SOC, it relies on the availability of a known (nominal) model of the system dynamics. We define the *nominal (known)* system which is the uncertainty-free version of (1) by dropping the unknown $H_{\{\mu,\sigma\}}$ to obtain

$$dX_t^\star = \left[A_\mu X_t^\star + BU_t^\star\right] dt + A_\sigma dW_t^\star, \quad X_0^\star = x_0^\star, \tag{5}$$

where $W_t^\star \in \mathbb{R}^{n_w}$ is another Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and is independent of W_t. Additionally, the initial condition $x_0^\star \sim \xi_0^\star$ is a random variable with finite second moment and is independent of the filtration induced by $W_t^\star$. Similar to (2), we denote by $\mathbb{X}_t^\star \in \mathcal{M}(\mathbb{R}^n)$ to be the distribution of $X_t^\star$, i.e. $X_t^\star \sim \mathbb{X}_t^\star$. By applying the Euler-Maruyama scheme [46] to (5), we obtain the following discrete-time version of the nominal (known) plant:

$$X_{k+1}^\star = X_k^\star + \left[A_\mu X_k^\star + BU_k^\star\right] \Delta T + A_\sigma \Delta W_k^\star, \tag{6}$$

with $X_0^\star = x_0^\star$, where $k \in \mathbb{N}_0 \Delta T$, $\Delta T \subset \mathbb{R}_{\geq 0}$ is the interval of temporal discretization, and $\Delta W_k^\star = W_{k+1}^\star - W_k^\star \overset{i.i.d.}{\sim} \mathcal{N}\left(0_{n_w}, \Delta T \mathbb{I}_{n_w}\right)$ are the increments of $W_t^\star$ [42] for each $k \in \mathbb{N}_0 \Delta T$. We may re-write (6) as

$$X_{k'}^\star = \mathcal{A}_\mu X_0^\star + \hat{B}\mathcal{U}_T^\star + \mathcal{A}_\sigma \mathcal{W}_T^\star, \tag{7}$$

where $\mathcal{U}_T^\star$ and $\mathcal{W}_T^\star$ denote the input and disturbance sequences, respectively, as $\mathcal{U}_T^\star = [U_0^{\star\top}, U_1^{\star\top}, \ldots, U_{k'-1}^{\star\top}]^\top$, and $\mathcal{W}_T^\star = [W_0^{\star\top}, W_1^{\star\top}, \ldots, W_{k'-1}^{\star\top}]^\top$, and matrices $\mathcal{A}_\mu$, $\hat{B}$, and $\mathcal{A}_\sigma$ are defined accordingly, see [10], [47]. The variable $k' \in \mathbb{N}$ defines the horizon as $T = k'\Delta T$. Given some desired positive definite $\mathcal{Q}$ and $\mathcal{R}$ of appropriate dimensions, the CS problem can be cast as

$$\boldsymbol{\mathcal{U}_T^\star} := \boldsymbol{U_{0\ldots k'-1}^\star} = \arg\min_{\mathcal{X}_T^\star, \mathcal{U}_T^\star} J\left(\mathcal{X}_T^\star, \mathcal{U}_T^\star\right) \tag{8a}$$

$$J\left(\mathcal{X}_T^\star, \mathcal{U}_T^\star\right) := \left(\mathcal{X}_T^\star\right)^\top \mathcal{Q}\mathcal{X}_T^\star + \left(\mathcal{U}_T^\star\right)^\top \mathcal{R}\mathcal{U}_T^\star, \tag{8b}$$

subject to the constraints

$$X_{k+1}^\star = X_k^\star + \left[A_\mu X_k^\star + BU_k^\star\right] \Delta T + A_\sigma \Delta W_k^\star, \tag{9a}$$

$$\mathbb{E}\left[X_0^\star\right] = \mu_0, \quad \mathbb{E}\left[X_T^\star\right] = \mu_T, \tag{9b}$$

$$\mathbb{E}\left[X_0^\star \left(X_0^\star\right)^\top\right] - \mathbb{E}\left[X_0^\star\right]\mathbb{E}\left[X_0^\star\right]^\top = \Sigma_0, \tag{9c}$$

$$\mathbb{E}\left[X_T^\star \left(X_T^\star\right)^\top\right] - \mathbb{E}\left[X_T^\star\right]\mathbb{E}\left[X_T^\star\right]^\top = \Sigma_T. \tag{9d}$$

The reader is referred to [47] for further details. Covariance steering can also be applied to nonlinear nominal models as in [48]. Note that we have not included the safety constraints in the CS problem (8)- (9), even though the aforementioned references explicitly consider the risk measures. The reason, as we discussed in the introduction, is that safety considerations based on the CS problem will not be valid for the uncertain (true) system since the state distributions of each are dissimilar owing to the presence of uncertainties. Thus, we wish to formulate constraints that satisfy the distributionally robust notions of safety. For this purpose, we need a method to obtain the Wasserstein ambiguity sets, which in our approach, is provided by the $\mathcal{L}_1$-DRAC control, which we discuss next.

B. *$\mathcal{L}_1$-Distributionally Robust Adaptive Control ($\mathcal{L}_1$-DRAC)*

Recall that the nominal (known) system in (5) is the uncertainty free version of the uncertain (true) system in (1). The input $U_t^\star$ is designed such that the nominal state $X_t^\star$ satisfies the desired control objectives, e.g., optimality while remaining safe via CS control discussed above. Due to the presence of the state-dependent nonlinear uncertainties $H_{\{\mu,\sigma\}}$ in the true system, along with their interaction with the driving Brownian motion W_t, the behavior of the true state X_t under the input $U_t^\star$ will be unquantifiably different to that of $X_t^\star$. In order to quantify and control the divergence between the nominal and true system states, we consider an input of the form

$$U_t = U_t^\star + U_{\mathcal{L}_1,t}, \tag{10}$$

where $U_{\mathcal{L}_1,t} \in \mathbb{R}^m$ is the $\mathcal{L}_1$-DRAC input and is defined as the output of the following *low pass filter*:

$$U_{\mathcal{L}_1,t} = -\omega \int_0^t e^{-\omega(t-\nu)} \hat{\Lambda}(\nu)\, d\nu, \tag{11}$$

where $\omega \in \mathbb{R}_{>0}$ is the **filter bandwidth**. The *adaptive estimate* $\hat{\Lambda} \in \mathbb{R}^m$ is obtained via the following *adaptation law*:

$$\hat{\Lambda}(t) = 0_m \mathbb{1}_{[0,T_s)}(t) + \lambda_s \left(1 - e^{\lambda_s T_s}\right)^{-1} \sum_{i=1}^{\lfloor \frac{t}{T_s} \rfloor} \tilde{X}_{iT_s} \mathbb{1}_{[iT_s,(i+1)T_s)}(t), \tag{12}$$

where $\tilde{X}_{iT_s} = \hat{X}_{iT_s} - X_{iT_s}$, and $T_s \in \mathbb{R}_{>0}$ is the **sampling period**, $\Theta_{ad}(t) = \begin{bmatrix}\mathbb{I}_m & 0_{m,n-m}\end{bmatrix} \bar{B}^{-1} \in \mathbb{R}^{m \times n}$, with $\bar{B} \in \mathbb{R}^{n \times n}$ is defined using B as in [37].

The parameter $\lambda_s \in \mathbb{R}_{>0}$ contributes to the solution $\hat{X}_t$ of the *process predictor* given by:

$$\hat{X}_t = x_0 + \int_0^t \left(-\lambda_s \mathbb{I}_n \tilde{X}_\nu + f(\nu, X_\nu)\right) d\nu + \int_0^t \left(g(\nu)U_{\mathcal{L}_1,\nu} + \hat{\Lambda}(\nu)\right) d\nu, \tag{13}$$

where $\tilde{X}_t = \hat{X}_t - X_t$. We collectively refer to $\{\omega, T_s, \lambda_s\}$ as the **control parameters** for the $\mathcal{L}_1$-DRAC input (11) - (13). Next, we place a regular assumption on the stability of the nominal diffusion free (deterministic) dynamics, i.e., (5) with $A_\sigma \equiv 0$.

Assumption 2: There exist $P, Q \in \mathbb{S}^n_{\succ 0}$, and known scalars $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$ such that

$$\alpha_1 \|x\|^2 \leq x^\top P x \leq \alpha_2 \|x\|^2, \quad \forall x \in \mathbb{R}^n,$$

and $A_\mu^\top P + P A_\mu = -Q$.
The assumption implies that the deterministic (diffusion free) nominal system is exponentially stable with $P \in \mathbb{S}^n_{\succ 0}$ defining the quadratic Lyapunov function certificate. Usually, a static feedback operator K is constructed such that the closed loop $A_\mu + BK$ is Hurwitz. In such cases, we can simply consider $A_\mu \leftarrow A_\mu + BK$. The following result from [37] establishes the guarantees of the $\mathcal{L}_1$-DRAC controller.

Theorem 1 ([37]): Suppose that the strong solution $X_t^\star$ of the nominal system (5) under the input $U_t^\star$ satisfies $\|X_t^\star\|_{L_{2\mathsf{p}}} \leq \Delta_\star$, $\forall t \in [0, T]$ for some known $\mathsf{p} \in \mathbb{N}_{\geq 1}$ and $\Delta_\star \in \mathbb{R}_{>0}$. Furthermore, for arbitrarily chosen scalars ρ_a and ϵ, define

$$\rho_r = \sqrt{\frac{\alpha_2}{\alpha_1}} \|x_0 - x_0^\star\|_{L_{2\mathsf{p}}} + \Delta_{A_\sigma} + \epsilon,$$

and $\rho = \rho_r + \rho_a + \Delta_{A_\sigma}$, where $\Delta_{A_\sigma} \in \mathbb{R}_{>0}$ is a known constant that depends on $\|A_\sigma\|_F$. We can construct strictly positive functions $\Theta_1(\omega) \propto 1/\omega$ and $\Theta_2(T_s) \propto T_s$, and positive scalars β_1 and β_2 that depend on the continuity and growth bounds in Assumption 1, and set $\omega, T_s \in \mathbb{R}_{>0}$ to satisfy

$$\beta_1 > \Theta_1(\omega), \quad \beta_2 > \Theta_2(T_s).$$

Then, under the input (10), the uncertain state X_t satisfies

$$\|X_t - X_t^\star\|_{L_{2\mathsf{p}}} \leq \rho, \quad \forall t \in [0, T]. \tag{14}$$

Furthermore, the distribution $\mathbb{X}_t$ of the uncertain state satisfies

$$\mathbb{W}_{2\mathsf{p}}(\mathbb{X}_t, \mathbb{X}_t^\star) \leq \rho, \quad \forall t \in [0, T], \tag{15}$$

where $\mathbb{X}_t^\star$ is the distribution of the nominal (known) system.

From (15), it is straightforward to see how the $\mathcal{L}_1$-DRAC controller ensures that the uncertain state distribution $\mathbb{X}_t$ belongs to the Wasserstein ambiguity set of order 2p centered around the nominal state distribution $\mathbb{X}_t^\star$. Moreover, using the moment bounds (14) guaranteed by $\mathcal{L}_1$-DRAC, one is at a liberty to obtain high-probability bounds by invoking the Markov inequality.

III. Distributionally Robust $\mathcal{L}_1$-DRAC Covariance Steering

As we presented in the preceding section, the $\mathcal{L}_1$-DRAC controller ensures that the uncertain system state $X_t^\star \sim \mathbb{X}_t^\star$ remains *uniformly bounded* with respect to the nominal state $X_t^\star \sim \mathbb{X}_t^\star$ in terms of (14) and (15). Thus, in order to ensure the safety of the uncertain system, we are now able to design inputs for the nominal system with the additional constraint imposed by the either the moment bound (14) or the distributional bound (15). Indeed, this is the idea behind our design, and is illustrated in Fig. 2.

Recall from Sec. II-A that the covariance steering (CS) control algorithm is setup for discrete-time dynamics. Thus,

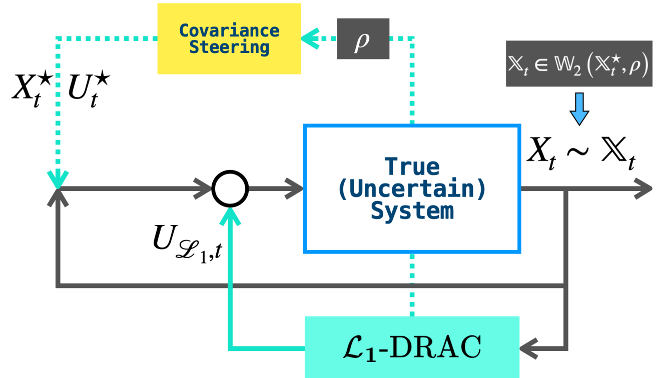

Fig. 2. Architecture of the distributionally robust adaptive covariance steering methodology. The low-level $\mathcal{L}_1$-DRAC provides guarantees in terms of Wasserstein ambiguity sets of radius ρ to the high-level covariance steering (CS) controller. The CS controller utilizes the ambiguity sets to generate nominal input robust to the distributional uncertainties by appending an additional constraint that does not sacrifice solvability of the entire scheme.

in order to modify the CS control algorithm to incorporate the distributionally robust safety constraints, we once again use the Euler-Maruyama scheme [46] on the uncertain dynamics (1) to obtain the following discrete-time dynamics

$$X_{k+1} = X_k + F_\mu(X_k, U_k)\Delta T + F_\sigma(X_k)\Delta W_k, \tag{16}$$

with $X_0 = x_0$, where $k \in \{\mathbb{N}_0\}\Delta T$, $\Delta T \subset \mathbb{R}_{\geq 0}$ is the interval of temporal discretization and $\Delta W_k = W_{k+1} - W_k$ are i.i.d. $\mathcal{N}(0_{n_w}, \Delta T \mathbb{I}_{n_w})$ random variables for each $k \in \{\mathbb{N}_0\}\Delta T$.

It is important to note that the CS control algorithm applies to discrete-time dynamics, while the $\mathcal{L}_1$-DRAC control is designed for continuous-time stochastic systems. Thus, the bounds in (14) and (15), may not translate identically to their discrete counterparts. Hence, one needs to account for the temporal-discretization errors. Fortunately, the Euler-Maruyama scheme allows the explicit dependence of the strong error as a function of the temporal-discretization resolution ΔT [49]. Then, to account for the discretization error, one needs to simply inflate the constant ρ by the strong error. We do not account for the strong error so as to not overly complicate the exposition, especially given the trivial nature of the solution.

Now, let us define Wasserstein ambiguity sets of radius $\rho \in \mathbb{R}_{>0}$ centered around the nominal state probability measure $\mathbb{X}_k^\star \in \mathcal{M}(\mathbb{R}^n)$:

$$\mathbb{B}_\rho(\mathbb{X}_k^\star) = \{\nu \in \mathcal{M}(\mathbb{R}^n) \mid \mathbb{W}_2(\nu, \mathbb{X}_k^\star) \leq \rho.\} \tag{17}$$

From (15) in Theorem 1 with $\mathsf{p} = 1$, we see that the input (10) ensures that the state distribution $\mathbb{X}_k$ of (16) (or (1) equivalently) satisfies $\mathbb{X}_k \in \mathbb{B}_\rho(\mathbb{X}_k^\star)$, $\forall k \in \{1, \ldots, k'\}$. Using the guaranteed membership of the uncertain state distribution within the ambiguity set, and a specified risk tolerance parameter $\delta_s \in \mathbb{R}_{>0}$, we specify the safety constraint as the following Wasserstein distributionally robust

Algorithm 1: Distributionally Robust $\mathcal{L}_1$-DRAC Covariance Steering

Input:
Initial and terminal conditions: $\mu_0, \Sigma_0, \mu_T, \Sigma_T, x_0^*$,
Known dynamics: A_μ, A_σ, B
Safe set: $\mathcal{X}_{safe}$
Risk tolerance parameters: δ, α
Discretization parameter: $\Delta T, k' := T/\Delta T$

Plan For:
$\boldsymbol{\mathcal{U}_T^*}$ by solving (8) *subject to*
(9) and (18) for all $k \in \{1, 2, \ldots, k'\}$

Operate With:
$U_t = \boldsymbol{U_t^\star} + U_{\mathcal{L}_1,t}$, where $U_{\mathcal{L}_1,t}$ as in (11)
and $\boldsymbol{U_t^\star}$ is as defined in Theorem 2.

CVaR constraint:

$$\max_{\nu \in \mathbb{B}_\rho(\mathbb{X}_k^\star)} \mathrm{CVaR}_\alpha^{X_k \sim \nu}\left[\mathrm{dist}(X_k, \mathcal{X}_{safe})\right] \leq \delta_s, \tag{18}$$

where $\alpha \in \mathbb{R}_{>0}$ represents the tail threshold. Given the representation of the safe region $\mathcal{X}_{safe}$ in (3) using half-planes, the distance function dist is a straightforward point-to-plane distance, see [44] for an explicit characterization. By appending the constraint above to the CS control problem in (8)-(9), we arrive at the desired control scheme.

Theorem 2: Let $\boldsymbol{\mathcal{U}_T^\star} := \boldsymbol{U^\star_{0\ldots k'-1}}$ be the solution of the covariance steering optimal control problem (8)-(9), subject to the additional constraint (18), for all $k \in \{1, \ldots, k'\}$. Then, with the input $U_t = \boldsymbol{U_t^\star} + U_{\mathcal{L}_1,t}$, where $\boldsymbol{U_t^\star}$ is the piecewise constant signal generated by $\boldsymbol{U^\star_{0\ldots k'-1}}$, the state X_t of the uncertain system (1) (or (16) equivalently), satisfies the desired safety margin:

$$\mathrm{CVaR}_\alpha^{X_k \sim \mathbb{X}_k}\left[\mathrm{dist}(X_k, \mathcal{X}_{safe})\right] \leq \delta_s, \tag{19}$$

for all $k \in \{1, \ldots, k'\}$.

The proof is trivial by observing that the satisfaction of the distributionally robust constraint (18) implies that (19) holds. The equivalence of the two constraints is due to the inclusion $\mathbb{X}_K \in \mathbb{B}_\rho(\mathbb{X}_k^\star)$, $\forall k$, enforced by the $\mathcal{L}_1$-DRAC control evidenced by the uniform bound (15).

The computation of the distributionally robust CVaR in (18) requires the solution to an infinite-dimensional optimization problem over the set of all probability measures within the ambiguity set $\mathbb{B}_\rho(\mathbb{X}_k^\star)$. One may formulate an equivalent finite-dimensional optimization problem by using the Kantorovich duality [32]. We, however, given the sample-free nature of the ambiguity sets, and the fact that the nominal distributions are normally distributed at each time step k, propose the use of recently developed Gelbrich bound based algorithms that can run significantly faster [50], [51]. We provide a pseudo-code for the proposed methodology in Algorithm 1.

Finally, if one wishes to disregard the statistical information contained in the Wasserstein ambiguity sets as a tradeoff for fast computation, the moment bounds in (14), in conjunction with the Markov inequality, can be used to formulate CS control with high-probability chance constraints.

IV. Conclusion and Future Work

We present a novel approach to Wasserstein distributionally robust safe control of uncertain stochastic systems. The key innovation lies in the use of our distributionally robust adaptive $\mathcal{L}_1$-DRAC controllers that enforce the inclusion of the uncertain system's distributions in uniform ambiguity sets around any nominal distributions generated by the high-level covariance steering (CS) controller. Thus, without the need for samples from the true distribution, and avoiding the need for empirical distributions, our approach can be implemented in a data-free regime while ensuring the satisfaction of safety constraints with probability 1.

Our approach is amenable to multiple directions of further development; for example, the use of alternate high-level stochastic optimal controllers, robustness against the added uncertainty in the environment, and the use of learned components with the loop.

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