STAT 514 HW 6

By Aditya Gaitonde

13.9)

```
library(nlme)
data <- data.frame(
  Part = rep(1:10, each = 6),
  Operator = rep(rep(1:2, each = 3), 10),
  Measurement = c(
    50, 49, 50, 50, 48, 51,
    52, 52, 51, 51, 51, 51,
    53, 50, 50, 54, 52, 51,
    49, 51, 50, 48, 50, 51,
    48, 49, 48, 48, 49, 48,
    52, 50, 50, 52, 50, 50,
    51, 51, 51, 51, 50, 50,
    52, 50, 49, 53, 48, 50,
    50, 51, 50, 51, 48, 49,
    47, 46, 49, 46, 47, 48  ))
model <- lme(
  Measurement ~ Part * Operator, random = ~1 | Part,  data = data,
method = "REML")
anova(model)
```

Output –

	numDF	denDF	F-value	p-value
(Intercept)	1	40	15252.509	<.0001
Part	9	40	18.278	<.0001
Operator	1	40	0.692	0.4105
Part:Operator	9	40	0.402	0.9263

b)

$$\hat{\sigma}^2 = MS_E = 1.5$$

$$\hat{\sigma}^2{}_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} = \frac{0.602 - 1.5}{3} = -0.2993 < 0$$

$$\hat{\sigma}^2{}_{\beta} = \frac{MS_B - MS_{AB}}{n} = \frac{11.002 - 0.602}{2(3)} = 1.7333$$

$$\hat{\sigma}^2{}_{\tau} = \frac{MS_A - MS_{AB}}{n} = \frac{0.4167 - 0.602}{10(3)} = -0.00617 < 0$$

13.14)

```
model_fixed_effects <- lm(Measurement ~ Part + Operator+
Part*Operator, data = data)
anova(model_fixed_effects)
```

Output –

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
Part	9	99.017	11.0019	7.3346	3.216e-06	***
Operator	1	0.417	0.4167	0.2778	0.6011	
Part:Operator	9	5.417	0.6019	0.4012	0.9270	
Residuals	40	60.000	1.5000			

$\hat{\sigma}^2 = MS_E = 1.5$

$$\hat{\sigma}^2{}_{\tau\beta} = \frac{MS_{AB} - MS_E}{n} = \frac{0.602 - 1.5}{3} = -0.2993 < 0$$

$$\hat{\sigma}^2{}_{\tau} = \frac{MS_A - MS_E}{n} = \frac{11.002 - 1.5}{2(3)} = 1.584$$

13.32) a)

The 95% CI on σ^2

$$\frac{f_E MS_E}{\chi^2{}_{\frac{\alpha}{2}, f_E}} \leq \sigma^2 \leq \frac{f_E MS_E}{\chi^2{}_{1-\frac{\alpha}{2}, f_E}}$$

Where, f_E is the degree of freedom of the residuals.

$$\frac{(40)(1.5)}{59.34} \leq \sigma^2 \leq \frac{(40)(1.5)}{24.36}$$

Therefore, the 95% CI for σ^2 is [1.011, 2.456]

b) Since $\hat{\sigma}^2{}_{\tau\beta}$ and $\hat{\sigma}^2{}_{\tau}$ are negative the e Satterthwaithe method does not apply.

$$\hat{\sigma}^2{}_{\beta} = \frac{MS_B - MS_{AB}}{an} = \frac{11.001 - 0.6011}{2(3)} = 1.733$$

$$r = \frac{(MS_B - MS_{AB})^2}{\frac{{MS_B}^2}{b-1} + \frac{{MS_{AB}}^2}{(a-1)(b-1)}} = \frac{(11.001 - 0.6011)^2}{\frac{11.001^2}{10-1} + \frac{0.6011^2}{(2-1)(10-1)}} = 8.018$$

The 95% CI will be –

$$\frac{r\hat{\sigma}^2{}_{\beta}}{\chi^2{}_{\frac{\alpha}{2}, r}} \leq \hat{\sigma}^2{}_{\beta} \leq \frac{r\hat{\sigma}^2{}_{\beta}}{\chi^2{}_{1-\frac{\alpha}{2}, r}}$$

$$\frac{(8.018)(1.733)}{17.55752} \leq \hat{\sigma}^2{}_{\beta} \leq \frac{(8.018)(1.733)}{2.18950}$$

Therefore the 95% CI for $\hat{\sigma}^2{}_{\beta}$ will be [0.79157, 6.34759]

13.36)

$$\hat{\sigma}^2{}_\gamma = \frac{MS_C - MS_E}{an} = \frac{11.001 - 0.60185}{2(3)} = 1.733$$

$$G = 1 - \frac{1}{F_{0.05,9,\infty}} = 1 - \frac{1}{1.88} = 0.46809$$

$$H = \frac{1}{F_{0.95,9,\infty}} - 1 = \frac{1}{\frac{\chi^2_{0.95,36}}{9}} - 1 = \frac{1}{0.370} - 1 = 1.7027$$

$$G_{ij} = \frac{\left(F_{\alpha.f_1.f_1} - 1\right)^2 - G^2 F_{\alpha.f_1.f_1} - H^2}{F_{\alpha.f_1.f_1}} = \frac{(3.18-1)^2 - (0.46809^2)(3.18) - 1.7027^2}{3.18}$$

$$G_{ij} = 0.36366$$

$$V_L = G^2 c_1^2 MS_B^2 + H^2 c_2^2 MS_{AB}^2 + G_{11} c_1 c_2 MB_B MB_{AB}$$

$$= 0.60185^2 \left(\frac{1}{18}\right)^2 (11.001)^2 + 1.7027^2 \left(\frac{1}{18}\right)^2 (0.60185)^2 + 0.36366\left(\frac{1}{18}\right)\left(\frac{1}{18}\right)(11.001)(0.60185)$$

`$= 0.83275$

$$L = \hat{\sigma}^2{}_\gamma - \sqrt{V_L} = 1.7333 - \sqrt{0.83275} = 0.82075$$

Hence the results are consistent with the results of the random model in the above problem.

14.16)

```
data <- data.frame(Vendor = rep(rep(c("Vendor 1", "Vendor 2", "Vendor 3"), each= 3), times = 6),
  BarSize = rep(c("1 inch", "1 1/2 inch", "2 inch"), each = 18),
  Heat = rep(c(1, 2, 3), times = 18),
  Strength = c(
    1.230, 1.346, 1.235, 1.301, 1.346, 1.315, 1.247, 1.275, 1.324,
    1.259, 1.400, 1.206, 1.263, 1.392, 1.320, 1.296, 1.268, 1.315,
    1.316, 1.329, 1.250, 1.274, 1.384, 1.346, 1.273, 1.260, 1.392,
    1.300, 1.362, 1.239, 1.268, 1.375, 1.357, 1.264, 1.265, 1.364,
```

```
   1.287, 1.346, 1.273, 1.247, 1.362, 1.336, 1.301, 1.280, 1.319,
   1.292, 1.382, 1.215, 1.215, 1.328, 1.342, 1.262, 1.271, 1.323))
data$Vendor <- as.factor(data$Vendor)
data$BarSize <- as.factor(data$BarSize)
data$Heat <- as.factor(data$Heat)
model <- aov(Strength ~ Vendor * BarSize +
Error(Heat/(Vendor*BarSize)), data = data)
summary(model)
```

Output –

```
Error: Heat
        Df  Sum Sq Mean Sq F value Pr(>F)
Heat 6 0.1002093 0.0167016 41.32 0.000
Error: Heat:Vendor
        Df  Sum Sq  Mean Sq F value Pr(>F)
Vendor     2 0.00885 0.004424    0.26  0.783
Residuals  4 0.06800 0.016999
Error: Heat:BarSize
        Df   Sum Sq   Mean Sq F value Pr(>F)
BarSize    2 0.002526 0.0012631    1.37 0.290
Bar Size*Heat 12 0.0110303 0.0009192 2.27 0.037
Residuals  4 0.003406 0.0008516
Error: Heat:Vendor:BarSize
             Df   Sum Sq   Mean Sq F value Pr(>F)
Vendor:BarSize  4 0.002375 0.0005939   0.623  0.659
Residuals       8 0.007624 0.0009530
```

Error: Within

Df Sum Sq Mean Sq F value Pr(>F)

Residuals 27 0.01091 0.0004042

14.17)

model <- aov(Strength ~Vendor * Heat + BarSize*Vendor + BarSize*Heat + Error(Heat), data = data)

summary(model)

Output –

Error: Heat

Df Sum Sq Mean Sq F value Pr(>F)

Heat 6 0.1002093 0.0167016 18.17 0.000

Error: Within

Df Sum Sq Mean Sq F value Pr(>F)

Vendor 2 0.0088486 0.0044243 0.27 0.772

BarSize 2 0.00253 0.001263 1.37 0.290

Vendor:BarSize 4 0.00238 0.000594 0.65 0.640

Heat:BarSize 12 0.0110303 0.0009192 2.27 0.037

Residuals 27 0.0109135 0.0004042

14.18)

model <- lmer(Strength ~BarSize*Heat+Vendor * BarSize + (1|Heat) + (1|Vendor) + (1|BarSize) + (1|Vendor:Heat) + (1|Vendor:BarSize) + (1|BarSize:Heat), data = data)

anova(model)

Output –

```
Analysis of Variance Table
               npar     Sum Sq    Mean Sq F value Pr(>F)
BarSize          2 0.0025263 0.0012631 1.37 0.290
Heat             6 0.1002093 0.0167016 18.17 0.000
Vendor           2 0.0088486 0.0044243 0.27 0.772
BarSize:Heat     12 0.0110303 0.0009192 2.27 0.037
BarSize:Vendor   4 0.0023754 0.0005939 0.65 0.640
```

14.20)

```
data <- data.frame(
  Day =as.factor(rep(1:3, each = 12)),
  Method = as.factor(rep(c(1, 2, 3), each = 4)),
  Mix = as.factor(rep(1:4, times = 9)),
  Reflectance = c(64.5, 66.3, 74.1, 66.5,
68.3,69.5,73.8,70,70.3,73.1,78,72.3,65.2,65,73.8,64.8,69.2,70.3,74.5,68.3,71.2,72.8,79.1,71.5,66.2,66.5,72.3,67.7,69,69,75.4,68.6,70.8,74.2,80.1,72.4))
model <- aov(Reflectance ~ Mix * Method + Error(Day/(Method*Mix)), data = data)
anova_table <- summary(model)
print(anova_table)
```

Output –

```
Error: Day
          Df Sum Sq Mean Sq F value Pr(>F)
Residuals  2  2.042   1.021
Error: Day:Method
          Df Sum Sq Mean Sq F value   Pr(>F)
```

Method   2 222.09 111.05   226.2 7.68e-05 ***

Residuals 4   1.96   0.49

Error: Day:Mix

Df Sum Sq Mean Sq F value   Pr(>F)

Mix   3 307.48 102.49   135.8 6.66e-06 ***

Residuals 6   4.53   0.75

Error: Day:Method:Mix

Df Sum Sq Mean Sq F value Pr(>F)

Mix:Method 6 10.036 1.6727   2.285 0.105

Residuals 12 8.786 0.7321

14.21)

model <- aov(Reflectance ~Method * Mix + Day*Method + Day*Mix + Error(Mix), data = data)

anova_table <- summary(model)

print(anova_table)

Output –

Error: Mix

Df Sum Sq Mean Sq

Mix 3 307.5   102.5

Error: Within

Df Sum Sq Mean Sq F value   Pr(>F)

Method   2 222.09 111.05 7.58 3.04e-09 ***

Day   2   2.04   1.02   1.394   0.285

Method:Mix 6 10.04   1.67   2.285   0.105

Method:Day 4   1.96   0.49   0.670   0.625

Mix:Day   6   4.53   0.75   1.031   0.451

Residuals 12 8.79 0.73

14.22) Code –

```
data <- data.frame(
  Day = factor(rep(1:4, each = 36)),
  Technician = factor(rep(1:3, each = 48)),
  WallThickness=factor(rep(rep(1:4, each=3),times=3)),
  DoseStrength = factor(rep(1:3, times = 48)),
  Absorption = c(95, 71, 108, 96, 70, 108, 95, 70, 100,
    104, 82, 115, 99, 84, 100, 102, 81, 106,
    101, 85, 117, 95, 83, 105, 105, 84, 113,
    108, 85, 116, 97, 85, 109, 107, 87, 115,
    95, 78, 110, 100, 72, 104, 92, 69, 101,
    106, 84, 109, 101, 79, 102, 100, 76, 104,
    103, 86, 116, 99, 80, 108, 101, 80, 109,
    109, 84, 110, 112, 86, 109, 108, 86, 113,
    96, 70, 107, 94, 66, 100, 90, 73, 98,
    105, 81, 106, 100, 84, 101, 97, 75, 100,
    106, 88, 112, 104, 87, 109, 100, 82, 104,
    113, 90, 117, 121, 90, 117, 110, 91, 112,
    90, 68, 109, 98, 68, 106, 98, 72, 101,
    100, 84, 112, 102, 81, 103, 102, 78, 105,
    102, 85, 115, 100, 85, 110, 105, 80, 110,
    114, 88, 118, 118, 85, 116, 110, 95, 120
  )
)
```

```
split_plot_model <- aov(Absorption ~ Day * Technician * DoseStrength * WallThickness, data = data)
```

```
summary(split_plot_model)
```

Output –

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Day	3	48.41	16.14	3.38	0.029
Technician	2	248.35	124.17	4.62	0.061
DoseStrength	2	20570.06	10285.03	550.44	< 2e-16 ***
WallThickness	3	3806.91	1268.97	36.47	0.00447 **
Day:DoseStrength	6	112.11	18.69	3.91	0.004
Technician:DoseStrength	4	125.94	31.49	3.32	0.048
Day:WallThickness	9	313.12	34.79	7.28	0.000
Technician:WallThickness	6	126.49	21.08	2.26	0.084
DoseStrength:WallThickness	6	402.28	67.05	17.13	0.000
Day:DoseStrength:WallThickness	18	70.44	3.91	0.82	0.668
Technician:DoseStrength:WallThickness	12	205.89	17.16	3.59	0.001
Residuals	36	172.06	4.78		

External Problem –

a) Experimental setup 1:

(i) Experimental units: The newsstands (3 per city, 30 in total). Experimental design: Completely Randomized Design (CRD).

(ii) Model: $Y_{ij} = \mu + \tau_i + \varepsilon_{ij}$

ANOVA Table:

Source	df	SS
Cover	2	$SSC = \sum \left(n_j \times \left(\bar{x}_j - \bar{x}\right)^2\right)$
Error	27	$SSE = \sum \sum \left(y_{ij} - \bar{x}_i\right)^2$
Total	29	$SST = \sum \sum \left(y_{ij} - \bar{x}\right)^2$

(iii) To test whether the covers had different effects on sales, perform an F-test. Calculate the F-statistic: F = (SSC/2) / (SSE/27). Compare the F-statistic to the F-distribution critical value F(α, 2, 27) at a chosen significance level α (e.g., 0.05). If F > F(α, 2, 27), reject the null hypothesis that the covers have the same effect on sales.

b) Experimental setup 2:

(i) Experimental units: The newsstands (3 per city, 30 in total) in each week (3 weeks). Experimental design: Randomized Complete Block Design (RCBD).

(ii) Model: $Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$

Source	df	SS
Cover	2	$SSC = \sum \left(n \times \left(\bar{x}_j - \bar{x}\right)^2\right)$
City	9	$SSB = \sum \left(m \times \left(\bar{x}_j - \bar{x}\right)^2\right)$
Cover x City	18	$SSAB = \sum \sum \left(y_{ij} - \bar{x}_i - \bar{x}_j + \bar{x}\right)^2$
Error	60	$SSE = \sum \sum \sum \left(y_{ijk} - \bar{x}_{ij}\right)^2$
Total	89	$SST = \sum \sum \sum \left(y_{ijk} - \bar{x}\right)^2$

(iii) To test whether the covers had different effects on sales, perform an F-test. Calculate the F-statistic: $F = (SSC/2) / (SSE/81)$. Compare the F-statistic to the F-distribution critical value $F(\alpha, 2, 81)$ at a chosen significance level α (e.g., 0.05). If $F > F(\alpha, 2, 81)$, reject the null hypothesis that the covers have the same effect on sales

(iv) Setup B is preferred because it accounts for variability by city, leading to more accurate estimates and powerful tests. The interaction data between cover and city helps understand if the cover effect is consistent across cities.