Derivations inspired from Lukin notes, Ran et al. [1], Perplexity and Bankim's code.

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1 Two-Level Atom Coupled to Light: Lindblad Equation Solution

1.1 System Definition

The Hamiltonian in the rotating wave approximation (RWA):

$$H = \frac{\hbar\Delta}{2}\sigma_z + \hbar g(\sigma_+ + \sigma_-) \tag{1}$$

Where:

- $\Delta = \omega_0 - \omega_L$ is the detuning
- g is the coupling strength
- $\sigma_z, \sigma_+, \sigma_-$ are Pauli matrices

Lindblad operator for spontaneous emission:

$$L = \sqrt{\gamma}\sigma_- \tag{2}$$

This hamiltonian assumes that the states are at $\frac{-\hbar\Delta}{2}$ and $\frac{\hbar\Delta}{2}$. Some references derive results by adding constant $\frac{\hbar\Delta}{2}$ so that the ground state comes at 0 energy and the upper state at $\hbar\Delta$.

1.2 Lindblad Equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H, \rho] + \gamma\left(\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right) \tag{3}$$

1.3 Expanded Equations

Density matrix:

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \tag{4}$$

Coupled differential equations:

$$\frac{d\rho_{ee}}{dt} = -\gamma\rho_{ee} + ig(\rho_{eg} - \rho_{ge}) \tag{5}$$

$$\frac{d\rho_{gg}}{dt} = \gamma\rho_{ee} - ig(\rho_{eg} - \rho_{ge}) \tag{6}$$

$$\frac{d\rho_{eg}}{dt} = -\left(\frac{\gamma}{2} + i\Delta\right)\rho_{eg} + ig(\rho_{ee} - \rho_{gg}) \tag{7}$$

$$\frac{d\rho_{ge}}{dt} = -\left(\frac{\gamma}{2} - i\Delta\right)\rho_{ge} - ig(\rho_{ee} - \rho_{gg}) \tag{8}$$

1.4 Steady-State Solution

Setting time derivatives to zero:

$$0 = -\gamma\rho_{ee} + ig(\rho_{eg} - \rho_{ge}) \tag{9}$$

$$0 = \gamma\rho_{ee} - ig(\rho_{eg} - \rho_{ge}) \tag{10}$$

$$0 = -\left(\frac{\gamma}{2} + i\Delta\right)\rho_{eg} + ig(\rho_{ee} - \rho_{gg}) \tag{11}$$

$$0 = -\left(\frac{\gamma}{2} - i\Delta\right)\rho_{ge} - ig(\rho_{ee} - \rho_{gg}) \tag{12}$$

1.5 Solution

Steady-state populations:

$$\rho_{ee} = \frac{g^2}{2g^2 + \frac{\gamma^2}{4} + \Delta^2} \tag{13}$$

$$\rho_{gg} = 1 - \rho_{ee} \tag{14}$$

Coherences:

$$\rho_{eg} = \rho_{ge}^* = \frac{ig(\frac{\gamma}{2} - i\Delta)}{2g^2 + \frac{\gamma^2}{4} + \Delta^2} \tag{15}$$

Usually in our QNLO group we tend to call $\gamma' = \frac{\gamma}{2}$, so watchout for different conventions. Here in this derivation I have not used our group convention.

To learn trick to write χ for any level refer to [1]. For Susceptibility, we work in low probe power regime, hence we work with linear susceptibility which allows us to use following equations:

$$\alpha = k\mathrm{Im}(\chi(0)) \tag{16}$$

$$n = 1 + \frac{\mathrm{Re}(\chi(0))}{2} \tag{17}$$

Note that these formulae are only valid when the slowly varying envelope approximation holds, i.e.

$$|\,\mathrm{Im}[\chi(0)]|, |\,\mathrm{Re}[\chi(0)]| \ll 1.$$

Now lets understand how to include OD in these calculations.

$$\chi = \frac{OD \times \text{Coherence}}{\Omega_{probe}} \tag{18}$$

$$\text{Transmission} = e^{-kl\times\mathrm{Im}(\chi)} \tag{19}$$

$$\text{Phase} = \mathrm{Re}(\chi) \tag{20}$$

1.6 Involving OD and susceptibity by using Beer Lambert's law

We start with the Beer-Lambert law:

$$I = I_0 e^{-\alpha l} \tag{21}$$

where α is the absorption coefficient.

Now, let's express α as:

$$\alpha = 2k\,\mathrm{Im}(n) \tag{22}$$

The refractive index n can be written as:

$$n = \sqrt{1+\chi} \tag{23}$$

For weak χ during Electromagnetically Induced Transparency (EIT), we approximate:

$$n \approx 1 + \frac{\chi}{2} \tag{24}$$

Thus, the imaginary part of n is:

$$\mathrm{Im}(n) = \frac{\mathrm{Im}(\chi)}{2} \tag{25}$$

Substituting this into the expression for α, we get:

$$\alpha = k\,\mathrm{Im}(\chi) \tag{26}$$

Therefore, the intensity I can be written as:

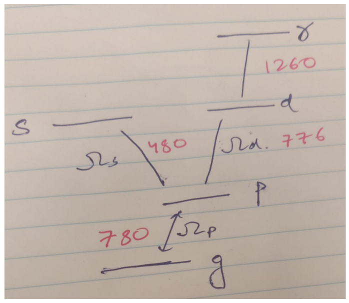

Figure 1: Five level system

$$\boxed{I = I_0 e^{-kl\,\mathrm{Im}(\chi)}} \tag{27}$$

So transmission can be plotted from here and phase is just the real part of χ. The kl factor is normalized by putting $\frac{1}{kl}$ in χ. For example consider the χ example from [1]

The expression for χ_p is given by:

$$\chi_p = \frac{\alpha_0 \gamma}{\gamma - i\Delta_p + \frac{|\Omega_c|^2}{\gamma_{21} - i\Delta_{21}}} \tag{28}$$

, after comparing from other resources including Bankim's code and Lee's thesis I found that

$$\alpha_0 = \frac{i\,OD}{KL} \tag{29}$$

For five level system as shown in Fig.1, the analytical expression after following the trick in [1] for χ is

$$\chi = \frac{i \cdot od \cdot \Gamma}{\Gamma - i(\Delta_p) + \frac{\Omega_s^2}{\gamma_{gs} - i(\Delta_p + \Delta_s)} + \frac{\Omega_d^2}{\gamma_{gd} - i(\Delta_p + \Delta_d) + \frac{\Omega_r^2}{\gamma_{gr} - i(\Delta_p + \Delta_d + \Delta_r)}}} + \tag{30}$$

We normalize medium length of medium by considering $kl = 1$

```
    import numpy as np
import qutip as qt
import matplotlib.pyplot as plt
from ipywidgets import interactive, FloatSlider
import ipywidgets as widgets

def create_hamiltonian(Delta, g):
    """Create the Hamiltonian for the system."""
    return 0.5 * Delta * qt.sigmaz() + g * (qt.sigmap() + qt.sigmam
        ↪ ())   # two level system

def calculate_coherence(Delta, g, gamma):
```

```
    """Calculate coherence using QuTiP and analytical methods."""
    # QuTiP calculation
    H = create_hamiltonian(Delta, g)
    c_ops = [np.sqrt(gamma) * qt.sigmam()]
    rho_ss = qt.steadystate(H, c_ops)
    rho_eg_qutip = rho_ss[0, 1]

    # Analytical calculation
    rho_eg_analytical = -1j * g * ((gamma / 2) - 1j * Delta) / (2 *
        ↪ g**2 + (gamma**2) / 4 + Delta**2)

    return rho_eg_qutip, rho_eg_analytical

def plot_coherences(g, gamma, OD):
    # System parameters
    Delta_values = np.linspace(-5, 5, 100)

    # Arrays to store coherence values
    coherence_qutip = []
    coherence_analytical = []

    # Calculate coherences for each Delta
    for Delta in Delta_values:
        qutip_result, analytical_result = calculate_coherence(Delta
            ↪ , g, gamma)
        coherence_qutip.append(qutip_result)
        coherence_analytical.append(analytical_result)

    # Convert lists to numpy arrays for plotting
    coherence_qutip = np.array(coherence_qutip)
    coherence_analytical = np.array(coherence_analytical)

    chi_analytical = OD*coherence_analytical/g
    chi_numerical = OD*coherence_qutip/g
    transmission_analytical = np.exp(-np.real(1j*chi_analytical))
    transmission_numerical = np.exp(-np.real(1j*chi_numerical))

    phase_analytical = np.real(chi_analytical)
    phase_numerical = np.imag(1j*chi_numerical)

    fig, axes = plt.subplots(1, 3, figsize=(15, 5))

    # Plot on each subplot
    axes[0].plot(Delta_values, transmission_analytical, label='
        ↪ Analytical transmission', color='blue')
    axes[0].plot(Delta_values, transmission_numerical, '--', label=
        ↪ 'QuTiP transmission', color='red')
    axes[0].set_title("Transmission")
    axes[0].set_xlabel("Detuning Delta")
    axes[0].set_ylabel("Transmission")
    axes[0].legend()

    axes[1].plot(Delta_values, phase_analytical, label='Analytical
        ↪ Phase', color='blue')
    axes[1].plot(Delta_values, phase_numerical, '--', label='QuTiP
        ↪ Phase', color='red')
    axes[1].set_title("Phase")
```

```python
    axes[1].set_xlabel("Detuning Delta")
    axes[1].set_ylabel("Phase")
    axes[1].legend()

    axes[2].plot(Delta_values, abs(coherence_analytical), label='
        ↪ Analytical coherence rho_12', color='blue')
    axes[2].plot(Delta_values, abs(coherence_qutip), '--', label='
        ↪ QuTiP coherence rho_12', color='red')
    axes[2].set_title(r"coherence rho_12")
    axes[2].set_xlabel("Detuning Delta")
    axes[2].set_ylabel("coherence rho_12")
    axes[2].legend()

    plt.suptitle("Always consider weak probe limit that is very low
        ↪  g", fontsize=16)
    plt.tight_layout()
    plt.show()

# Create sliders
g_slider = FloatSlider(value=0.1, min=0.001, max=.5, step=0.1,
    ↪ description='g:')
gamma_slider = FloatSlider(value=0.1, min=0.05, max=1.0, step=0.01,
    ↪  description='gamma:')
OD = FloatSlider(value=50, min=0, max=150, step=5, description='OD:
    ↪ ')
# Create interactive plot
interactive_plot = interactive(plot_coherences, g=g_slider, gamma=
    ↪ gamma_slider, OD=OD)

# Display the interactive plot
display(interactive_plot)
```

References

[1] Ran Finkelstein and Ofer Firstenberg. "A practical guide to electromagnetically induced transparency in atomic vapor". In: *New Journal of Physics* 25.6 (2023), p. 063016. DOI: 10.1088/1367-2630/acbc40.