Problem 1

'Leap frog scheme preserves Energy as long as CFL condition is satisfied'

$$\frac{f(t+dt,x) - f(t-dt,x)}{2\,dt} = -v \cdot \frac{f(t, x+dx) - f(t, x-dx)}{2 \cdot dx}$$

Given that

$$f(x,t) = \xi^{t} e^{ikx} \longrightarrow ①$$

For ① to be stable, $|\xi|^2 = 1$. Let's prove this: Apply $f(x,t)$ in leap frog scheme.

$$f(t+dt, x) = \xi^{t+dt} \cdot e^{ikx}$$

$$f(t-dt, x) = \xi^{t-dt} \cdot e^{ikx}$$

$$f(t, x+dx) = \xi^{t} \cdot e^{ik(x+dx)}$$

$$f(t, x-dx) = \xi^{t} \cdot e^{ik(x-dx)}$$

$$\frac{\left(\xi^{t+dt} \cdot e^{ikx}\right) - \left(\xi^{t-dt} \cdot e^{ikx}\right)}{2 \cdot dt} = -v \frac{\xi^{t} e^{ik(x+dx)} - \xi^{t} e^{ik(x-dx)}}{2\,dx}$$

$$\cancel{\xi^t}\left(\frac{\xi^{dt}\cdot e^{ikx} - \xi^{-dt}\cdot e^{ikx}}{2dt}\right) =$$

$$-v\,\cancel{\xi^t}\left(\frac{e^{ik(x+dx)} - e^{ik(x-dx)}}{2dx}\right)$$

$$\cancel{e^{ikx}}\left(\frac{\xi^{dt} - \xi^{-dt}}{2dt}\right) = -v\,\cancel{e^{ikx}}\left(\frac{e^{ikdx} - e^{-ikdx}}{2dx}\right)$$

From $\dfrac{e^{ix} - e^{-ix}}{2i} = \sin x,$

$$\frac{\xi^{dt} - \xi^{-dt}}{2dt} = \frac{-vi}{dx}\cdot \sin k\,dx$$

$$\xi^{dt} - \xi^{-dt} = \frac{-2dt}{dx}\cdot vi\sin k\,dx$$

$$\xi^{2dt} - 1 = \frac{-2dt}{dx}\cdot v\, i\sin k\,dx\cdot \xi^{dt}$$

(complex number raised 0 is 1)
(ξ)

Let's assume $dt = 1$

$$\xi^2 + \frac{2vi}{dx}\left(\sin kx\,dx\right)\xi - 1 = 0 \rightarrow (2)$$

Eq. (2) is a 2^{nd} order equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$

$b = \frac{2 v i}{dx} (\sin kx \, dx)$

$c = -1$

$$x = \frac{\left(-\frac{\cancel{2} v i}{dx} \sin kx \, dx\right) \pm \sqrt{\left(\frac{-\cancel{4} v^2}{dx^2} \sin^2 kx \, dx + \cancel{4}\right)}}{\cancel{2}1}$$

$$\xi = \frac{-v i}{dx} \sin kx \cdot dx \pm \sqrt{1 - \frac{v^2}{dx^2} \sin^2 kx \, dx}$$

$\xi = -ia \pm b$

Where, $a = \frac{v}{dx} \sin kx \, dx$

$$b = \sqrt{1 - \frac{v^2}{dx^2} \sin^2 kx \, dx}$$

$\xi^* = ia \pm b$

$|\xi|^2 = \xi \xi^* = a^2 + b^2$

$$= \cancel{\frac{v^2}{dx^2} \sin^2 kx \, dx} + 1 - \cancel{\frac{v^2}{dx^2} \sin^2 kx \, dx}$$

$|\xi|^2 = 1$

Hence it is proved that as long as CFL condition is true, Leap frog scheme is satisfied !

Problem 2 (a)

Given : $V(r) = \log(r)$ $\Rightarrow$ Since we are solving in 2D

$$\boxed{V(r) = n \log r + m}$$

We are given with

$V(0,0) = 1$ $\rho(0,0) = 1$

Let's work out the potential at origin

$$V(0,0) - \frac{1}{4}\left(V(0,1) + V(1,0) + V(-1,0) + V(0,-1)\right) = \rho(0,0)$$

This comes from the fact that,

$$\rho = V(x,y) - \frac{1}{4}\left(V(x+dx, y) + V(x-dy, y) + V(x, dy+y), + V(x, y-dy)\right)$$

Considering $dx = dy = 1$ and $r = x^2+y^2$,

$$\cancel{1} - \frac{1}{\cancel{4}}\left(\cancel{4}m + \cancel{4}n\log(1)\right) = \cancel{1}$$

$$m + n\log(1) = 0$$

$$n\log(1) = -m \rightarrow (1)$$

$$\log(1) = -\frac{m}{n}$$

Now let's find the potential at $V(1,0)$

$$v(1,0) = \rho(1,0) + \frac{1}{4}\Big(v(0,0) + v(1,-1) + v(2,0) + v(1,1)\Big)$$

$$= n\log(1) + m - \frac{1}{4}\Big(1 + (n\log\sqrt{2} + m) + (n\log 2 + m) + (n\log\sqrt{2} + m)\Big)$$

$$n\log(1) + m = \frac{1}{4}\left(n\log 2 + 1 + 3m + n\log 2\right)$$

$$n\log(1) - n\log(1) = \frac{1}{4}\left(2n\log 2 + 1 + 3m\right)$$

(From Eq. 1)

$$2n\log 2 + 1 - 3n\log 1 = 0$$

$$n\left(2\log 2 - 3\log 1\right) = -1$$

$$n = \frac{-1}{2\log 2 - 3\log 1} = -1.66$$

$$m = 0$$

For sanity check,

$$v(5,0) = n\log r + m$$

$$= -1.66(\log 5) + 0 = -1.16$$

which is closer to the given value -1.05

Hence, $v(1,0) = -1.66(\log 1) + 0 = 0$

$v(2,0) = -1.66\log(2) + 0 = 0.349$ //