Modeling Volatility using Stochastic Processes: Ornstein-Uhlenbeck and Cox-Ingersoll-Ross Models

1 Introduction

Modeling volatility is a central challenge in financial mathematics. Two stochastic processes that have proven to be particularly useful are the Ornstein-Uhlenbeck (OU) process and the Cox-Ingersoll-Ross (CIR) process. Both models are mean-reverting, capturing the intuition that market volatility does not wander indefinitely but rather fluctuates around some long-term level.

In this report, we carefully build up the theory of these processes, their distributional properties, and how one may estimate their parameters using maximum likelihood estimation (MLE). We then turn our attention to simulation, paying special attention to how Euler-type discretizations are constructed and why they are widely used in practice. Finally, we broaden the discussion by connecting these stochastic models to the concepts of realized and implied volatility, which are widely used in empirical finance and option pricing.

2 The Ornstein-Uhlenbeck Process

2.1 Definition

The Ornstein-Uhlenbeck process is a stationary Gauss-Markov process given by:

$$dX_t = -\kappa X_t \, dt + \sigma \, dW_t, \tag{1}$$

where $\kappa > 0$ governs the strength of mean reversion, $\sigma > 0$ is the volatility parameter, and W_t is a standard Wiener process.

A common extension introduces a long-term mean θ, producing what is often referred to as the Vasicek model:

$$dX_t = \kappa(\theta - X_t) \, dt + \sigma \, dW_t. \tag{2}$$

2.2 Distributional Properties

Rewriting and integrating the SDE using an integrating factor $e^{\kappa t}$, one obtains:

$$X_T = X_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}) + \sigma \int_0^T e^{-\kappa(T-t)} dW_t. \tag{3}$$

From this, the mean and variance follow:

$$E[X_T] = X_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}), \tag{4}$$

$$Var[X_T] = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa T}). \tag{5}$$

Hence, X_T is Gaussian:

$$X_T \sim \mathcal{N}\left(E[X_T], Var[X_T]\right). \tag{6}$$

2.3 Maximum Likelihood Estimation

The conditional distribution is normal:

$$X_{t+\Delta t}|X_t \sim \mathcal{N}(\mu, \overline{\sigma}^2), \tag{7}$$

with

$$\mu(x_t, \kappa, \theta) = x_t e^{-\kappa \Delta t} + \theta(1 - e^{-\kappa \Delta t}), \tag{8}$$

$$\overline{\sigma}^2(\kappa, \sigma) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa \Delta t}). \tag{9}$$

The likelihood function becomes:

$$L(\kappa, \theta, \sigma|X) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \exp\left(-\frac{(x_{i+1} - \mu(x_i, \kappa, \theta))^2}{2\overline{\sigma}^2}\right). \tag{10}$$

This can be maximized numerically to estimate the parameters.

3 The Cox-Ingersoll-Ross Process

3.1 Definition

The Cox-Ingersoll-Ross (CIR) model is given by:

$$dV_t = \kappa(\theta - V_t)\, dt + \xi \sqrt{V_t}\, dW_t, \tag{11}$$

where $\kappa > 0$ is the speed of mean reversion, $\theta > 0$ is the long-term mean, and $\xi > 0$ is the volatility of volatility.

3.2 Distribution and MLE

The conditional distribution of $V_{t+\Delta t}$ given V_t is a scaled non-central chi-squared distribution. The log-likelihood for a sequence $V_1, \ldots, V_n$ is:

$$l(\kappa, \theta, \xi|V) = \sum_{i=1}^{n} \Big[-\ln(c) + (q-1)\ln\left(\frac{v_i}{u_i}\right) - u_i - v_i + \ln(I_q(2\sqrt{u_i v_i})) \Big], \tag{12}$$

where

$$c = \frac{2\kappa}{\xi^2(1 - e^{-\kappa\Delta t})}, \qquad u_i = cV_i e^{-\kappa\Delta t}, \tag{13}$$

$$v_i = cV_{i+1}, \qquad q = \frac{2\kappa\theta}{\xi^2} - 1. \tag{14}$$

Here $I_q(\cdot)$ is the modified Bessel function of the first kind. Parameter estimation requires numerical optimization.

4 Discretization and Simulation: The Euler–Maruyama Scheme

Up to this point, we have worked in continuous time. However, when it comes to simulation and numerical work, we must discretize these stochastic differential equations. The natural first step is the Euler–Maruyama scheme.

4.1 From SDE to Discretization

Consider a general SDE:

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t. \tag{15}$$

In discrete time, we approximate dt by a small step Δt and dW_t by $\sqrt{\Delta t}Z$, where $Z \sim \mathcal{N}(0, 1)$. This yields the Euler approximation:

$$X_{t+\Delta t} = X_t + a(X_t, t)\Delta t + b(X_t, t)\sqrt{\Delta t}\, Z. \tag{16}$$

4.2 Application to OU

For the OU process:

$$dX_t = \kappa(\theta - X_t)\, dt + \sigma\, dW_t, \tag{17}$$

we identify

$$a(X_t, t) = \kappa(\theta - X_t), \quad b(X_t, t) = \sigma.$$

Thus, the Euler approximation becomes:

$$X_{t+\Delta t} = X_t + \kappa(\theta - X_t)\Delta t + \sigma\sqrt{\Delta t}Z. \tag{18}$$

4.3 Application to CIR

For the CIR process:

$$dV_t = \kappa(\theta - V_t)\, dt + \xi\sqrt{V_t}\, dW_t, \tag{19}$$

we have

$$a(V_t, t) = \kappa(\theta - V_t), \quad b(V_t, t) = \xi\sqrt{V_t}.$$

Hence the Euler approximation is:

$$V_{t+\Delta t} = V_t + \kappa(\theta - V_t)\Delta t + \xi\sqrt{V_t\Delta t}Z. \tag{20}$$

4.4 Remarks on Practical Use

For the OU process, this discretization works smoothly because the process is Gaussian and can even be simulated exactly. The Euler scheme simply provides a convenient numerical method.

For the CIR process, the situation is more delicate because the diffusion term depends on $\sqrt{V_t}$. A naive Euler step may occasionally produce negative values for V_t, which are not admissible since V_t represents a variance. In practice, one may apply modified schemes such as the *full truncation Euler method*, or alternatively exploit the exact transition distribution of CIR (non-central chi-squared) when feasible.

5 Realized and Implied Volatility

So far we have studied continuous-time models that attempt to describe the dynamics of volatility. To better understand their practical relevance, it is useful to contrast two key market concepts: realized (or actual) volatility and implied volatility.

5.1 Realized Volatility

Realized volatility (RV) is the actual, observed variability in asset prices over time. It is computed from past data and is typically expressed as the standard deviation of log returns, annualized. If S_t denotes the price at time t, the log-return is:

$$r_t = \ln \frac{S_t}{S_{t-1}}. \tag{21}$$

Then realized volatility is given by:

$$rv_t \approx \mathrm{StDev}(r_t) \times \sqrt{252}, \tag{22}$$

where the factor $\sqrt{252}$ annualizes daily volatility, assuming 252 trading days in a year.

5.2 Implied Volatility

Implied volatility (IV), in contrast, is forward-looking. It is extracted from option prices using an option pricing model such as Black–Scholes. For a European call option:

$$C = S_t \Phi(d_1) - K e^{-r\tau} \Phi(d_2), \tag{23}$$

where

$$d_1 = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \tag{24}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}. \tag{25}$$

Here σ is the volatility parameter. The implied volatility is the value of σ that makes the model price C equal to the observed market price. Thus, IV encodes the market's expectation of future volatility.

5.3 Comparing the Two

The distinction between realized and implied volatility is subtle yet crucial. Realized volatility is backward-looking, summarizing past fluctuations. Implied volatility is forward-looking, reflecting market expectations of future uncertainty. Naturally, the two differ, and the gap between them creates both trading opportunities and challenges in risk management.

6 Conclusion

Both the Ornstein-Uhlenbeck and Cox-Ingersoll-Ross processes offer elegant frameworks for modeling volatility. Understanding their exact distributions provides theoretical clarity, while the Euler–Maruyama discretization builds a bridge to numerical implementation and simulation.

Finally, placing these models in the context of realized and implied volatility highlights the practical connection between stochastic theory and financial markets: while models capture mean-reversion and randomness in volatility, traders and risk managers navigate the ever-present tension between past data (realized volatility) and future expectations (implied volatility).