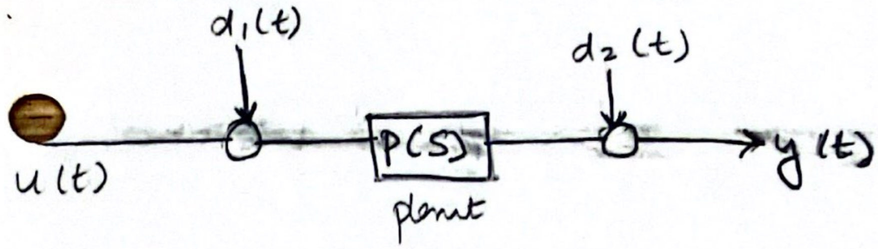
Control Systems - EECS 460



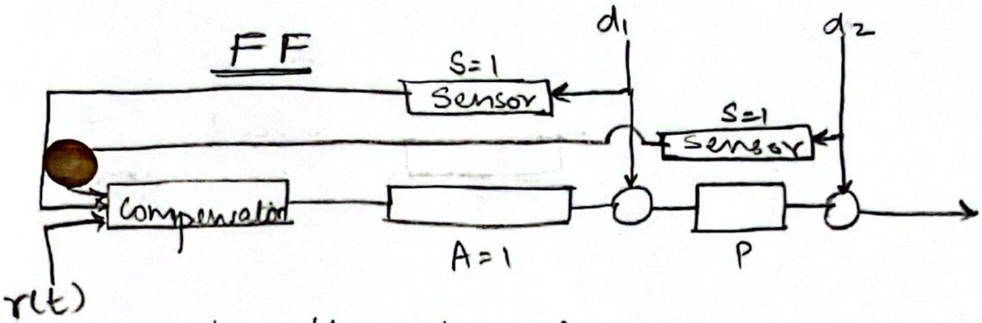
Goals of control

- Reference tracking
- Disturbance rejection
- Stability
- Robustness.

Types :- Feedforward vs. Feedback.
(Compensation) (deviation)

FF :- Need to measure disturbances
- Need to know plant precisely
- Cannot stabilize an unstable system. (Does not change Plant dynamics)

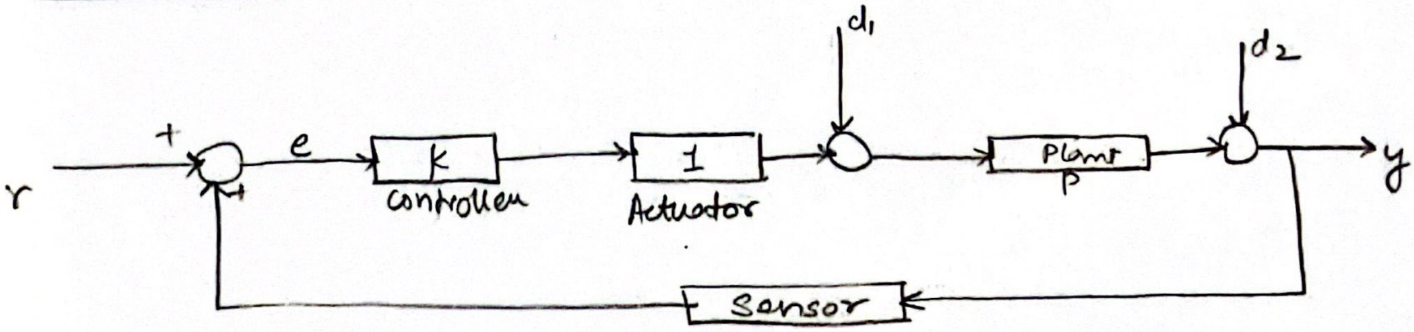
FF



- $y = P(u+d_1) + d_2 = r \Rightarrow u = \dfrac{r - d_2 - Pd_1}{P}$.

To define a u so that we can track the reference signal we need to measure d_1, d_2 & also know 'P' well.

Feedback.



$$y = \underbrace{\frac{PK}{1+PK}\cdot r}_{\text{reference tracking}} + \underbrace{\frac{P}{1+PK}\,d_1 + \frac{1}{1+PK}\,d_2}_{\text{disturbance rejection}}$$

For large K :

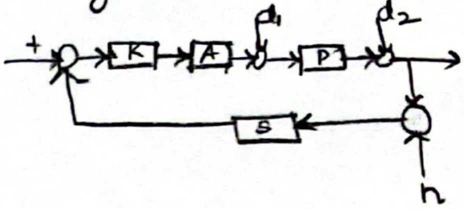
Also changes plant dynamics $\Rightarrow$ Be careful of stability.

$c = r - y = \frac{1}{1+pK}(r - d_2 - pd_1) \Rightarrow \boxed{S = \frac{1}{1+pK}}$ Sensitivity function. → 0 when $K \to \infty$

→ Stability ⟷ sensitivity tradeoff. $\boxed{T = \frac{PK}{1+pK}}$ Complimentary sensitivity function.

→ <u>Conservation laws</u>

→ ① Measurement noise hurts feedback systems.



$$y = G_r \cdot r + G_{d_1} \cdot d_1 + G_{d_2} \cdot d_2 - \frac{PK}{1+pK} \cdot n$$

$K \to \infty \Rightarrow$ Tracking ↓

→ ② Reference tracking ⟷ Sensor noise rejection tradeoff (since $S+T=1$)

→ ③ Conservation of dirt : For dynamic stable system.

- $\int_0^\infty \log_{10}|S(j\omega)|\, d\omega = 0$ ⇒ If we make sensitivity low at some frequencies it will become high elsewhere.

High sensitivity & low noise ⇒ open loop
Low sensitivity & high noise ⇒ closed loop.

→ Feedback moves openloop poles but not zeroes!

$P(s) = \frac{n_p(s)}{d_p(s)}$ (plant) $\quad K(s) = \frac{n_K(s)}{d_K(s)} \Rightarrow G_s(s) = \frac{P(s)K(s)}{1+P(s)K(s)} = \frac{n_p(s)\, n_K(s)}{d_p(s) d_K(s) + n_p(s) n_K(s)}$

Zeroes are same.

poles are different

Math preliminaries.

① Differential equation form

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = b_0 u^{(m)} + b_1 u^{(m-1)} + \dots + b_m u$$

② Transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n}$$

finite zeroes. (numerator)

Characteristic equation → poles (denominator)

- n = order
- n−m = relative degree.
- $n \geq m$.

→ Going from T.F to Diff. eqn:- Assume $s = \frac{d}{dt}$ (operator), $s^2 = \frac{d^2}{dt^2} \dots s^n = \frac{d^n}{dt^n}$ & write in terms of $y(t), u(t)$.

$$\Rightarrow \frac{y(t)}{u(t)} = \frac{b_0 s^m + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n} \Rightarrow (s^n + a_1 s^{n-1} + \dots + a_n) y(t) = (b_0 s^m + \dots + b_m) u(t)$$

→ m is the number of finite zeroes ⇒ no. of times the input is differentiated

⇒ if $u = \epsilon \sin \omega t$ and $\epsilon \ll 1$ but $\omega \gg 1$ ⇒ small signal @ high frequencies

⇒ $\dot{u} = \epsilon \omega \sin \omega t$ ⇒ it starts becoming significant.

③ Impulse Response

$$g(t) = \mathcal{L}^{-1}\{G(s)\} ; \quad y(t) = g(t) * u(t) = \int_0^t u(\tau) g(t-\tau) \, d\tau$$

Step Response

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \underline{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}} \quad \text{and} \quad h(t) = \int_0^t g(t) \, dt$$

④ Frequency Characteristics

@ $s = j\omega$ $\quad G(j\omega) = |G(j\omega)| \cdot e^{j \angle G(j\omega)}$ ⟶ Bode.

$= Re(G(j\omega)) + j \, Im(G(j\omega))$ ⟶ Nyquist

Bode plots :- Find poles and zeroes. LHP pole $\Rightarrow$

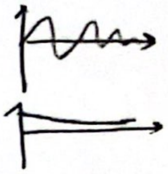
(For more info look at general Control notes)

LHP zero $\Rightarrow$

Bode plots with complex poles or zeroes

> $as^2+bs+c=0 \Rightarrow$ when $b^2<4ac$ we get complex roots.

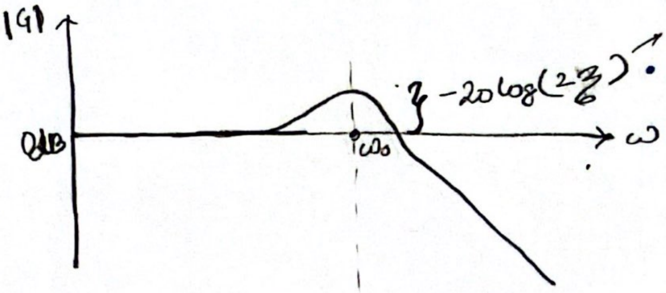
> Damping ratio :- $\zeta = \sqrt{\frac{b^2}{4ac}}$ (Beta) $\zeta<1 \Rightarrow$ oscillations, $\zeta>1 \Rightarrow$ exponential.

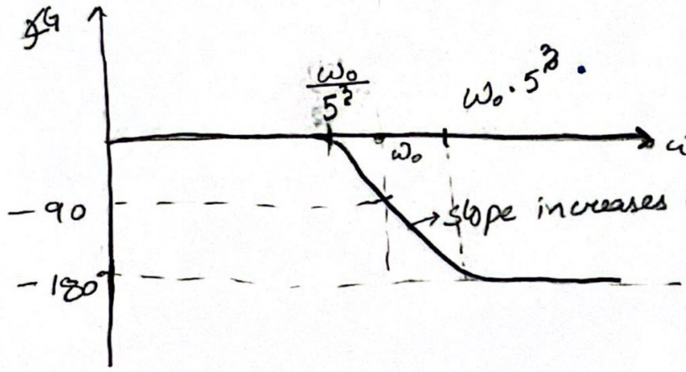


$$G(s) = \frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2} = \frac{1}{\left(\frac{s}{\omega_0}\right)^2+2\zeta\left(\frac{s}{\omega_0}\right)+1}$$

$G(j\omega) = Re(G(j\omega)) + j\,Im(G(j\omega))$ where, Real $= \dfrac{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2}$

Imag $= \dfrac{-\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)}{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_0}\right)\right)^2}$





Case ①: $\omega \ll \omega_0 \Rightarrow$ Real $=1$, $Im=0$
$\Rightarrow |G| = 0dB$
$\measuredangle G = 0°$

Case ②: $\omega \gg \omega_0 \Rightarrow$ Real $= -\left(\frac{\omega}{\omega_0}\right)^{-2}$
$Im = -2\zeta\left(\frac{\omega}{\omega_0}\right)^{-3}$
$\Rightarrow |G| = -40dB/dec$
$\measuredangle G = -180°$

Case ③: $\omega = \omega_0 \Rightarrow Re = 0$, $Im = -\frac{1}{2\zeta}$
$\Rightarrow |G| = -20\log(2\zeta)$
$\measuredangle G = -90°$

→ $\zeta = 0 \Rightarrow$ Peak goes to $-\infty$
→ Peak is only drawn when $\zeta < 0.5$.
→ when $\zeta = 0.5 \Rightarrow$ no peak.
→ when $\zeta > 0.5 \Rightarrow$ negative peak but

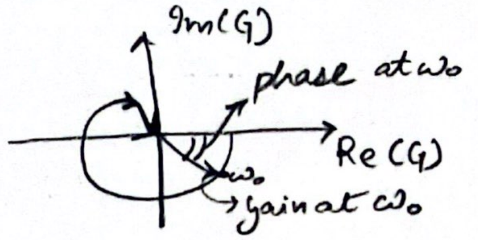
→ For complex zeroes just reflect the response w.r.t ω-axis.

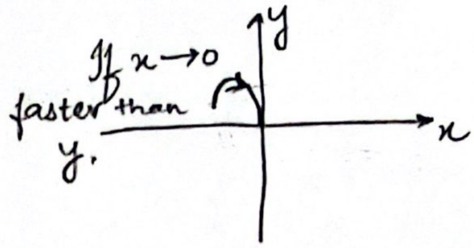
Nyquist plots

① Find Re(G) and Im(G)

② Substitute $\omega=0$, $\omega=\infty$, etc and find Re, Im parts asymptotically!

③ Connect the asymptotes





3^{rd} order $\Rightarrow$ 270° of phase at $\omega=\infty$

Example on page (14)

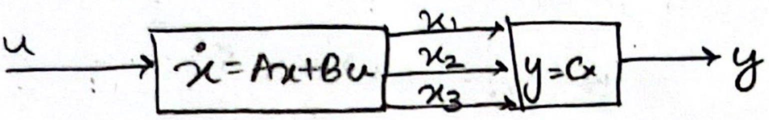
⑤ State space representation

→ A way to look at system states using first order diff. eq.

→ A → matrix relating how each state changes due to other states.

→ B → state vs. inputs.

→ C → Decide which state is op or which state we are interested in



$$\boxed{\begin{aligned}\dot{x} &= Ax+Bu\\ y &= Cx\end{aligned}}$$

A is $n\times n$
B is $n\times 1$
C is $1\times n$

A, B, C are used to characterize the system.

Going from G(s) to Statespace

$$G(s) = \frac{1}{s^3+3s+4} \Rightarrow \dddot{y} + 3\dot{y} + 4y = u$$

Third order $\Rightarrow$ 3 states

$$y = x_1 \qquad \dot{y} = x_2 \qquad \ddot{y} = x_3$$

$$\Rightarrow \quad \dot{x}_1 = x_2 \qquad \dot{x}_2 = x_3 \qquad \dot{x}_3 = -3x_2 - 4x_1 + u$$

Output $y = x_1$

$$A = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & 0 \end{bmatrix} \end{array}$$

$$B = \begin{array}{c c} & u \\ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{array} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \qquad C = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ y & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array}$$

If there are finite zeroes use an intermediate stage $\bar{y}$.

Example $\quad G(s) = \dfrac{5s+2}{s^3+3s+4}$ $\qquad \dddot{y} + 3\dot{y} + 4y = 5\dot{u} + 2$

$$u \longrightarrow \boxed{\frac{1}{s^3+3s+4}} \xrightarrow{\bar{y}} \boxed{5s+2} \longrightarrow y$$

$\dfrac{y}{\bar{y}} = 5s+2 \Rightarrow y = 5\dot{\bar{y}} + 2\bar{y}$

$\bar{y} = x_1$	$\dot{x}_1 = x_2$	$\bar{y} = x_1$
$\dot{\bar{y}} = x_2$	$\dot{x}_2 = x_3$	$y = 5\dot{x}_1 + 2x_1 = 5x_2 + 2x_1$
$\ddot{\bar{y}} = x_3$	$\dot{x}_3 = -3x_2 - 4x + u$	

$$\Rightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & 0 \end{bmatrix} ; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ; C = \begin{bmatrix} 2 & 5 & 0 \end{bmatrix}$$

Is A, B, C for a given $G(s)$ unique? No

Let $\dot{x} = Ax + Bu$; $y = Cx$ $\Rightarrow$ If $x = Tz$ given $\det T \neq 0$

$\Rightarrow T\dot{z} = ATz + Bu$; $y = CTz$ $\Rightarrow A_1 = T^{-1}AT$; $B_1 = T^{-1}B$; $C_1 = CT$ } Not unique

However eigenvalues of A, B, C and A_1, B_1, C_1 are same!

$\rightarrow$ Going from A, B, C to $G(s)$. $\Rightarrow$ $\boxed{G(s) = C(sI - A)^{-1}B}$

$\Rightarrow$ Eigen values of A = poles of $G(s)$ and given by $|sI - A| = 0$

Linearization

→ A nonlinear system can be decomposed into one or more linear systems around equilibrium points.

→ However information about stability could be lost so be aware.

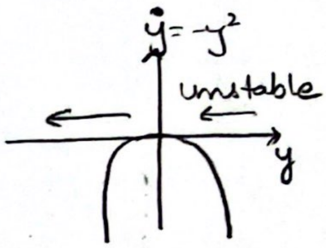
Example :- $\dot{y} = -y^2$; $\dot{y} = -y^3$

↓ linearizing. ↓

$\dot{y} = 0$; $\dot{y} = 0$

↓ ↓

unstable stable.

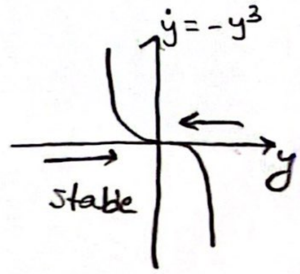


$y = +ve \Rightarrow \dot{y} = -ve$

$\Rightarrow \leftarrow$

$y = -ve \Rightarrow \dot{y} = -ve$

$\Rightarrow \leftarrow$



$y = +ve \Rightarrow \dot{y} = -ve$

$y = -ve \Rightarrow \dot{y} = +ve$

Steps of linearization.

Step① :- Write state space form : $\dot{x}_1 = f_1(x_1, x_2 \ldots x_n)$

$\dot{x}_2 = f_2(x_1, x_2 \ldots x_n)$

$\vdots$

Step② :- Find equilibrium points by substituting $f_1, f_2, \ldots$ to 0.

Eg :- equilibrium points $x^{*1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x^{*2} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$ for 2nd order

Step③ : Find Jacobian of $f(x)$

$$J = \frac{df}{dx} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_2}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}$$

Step④ : Evaluate Jacobian at equilibrium points.

$A_1 = J|_{x^{*1}}$; $A_2 = J|_{x^{*2}}$

Step⑤ Write linearized equation: $\dot{x} = A_1 x$; $\dot{x} = A \ldots$

Generalizing to when u is present

Step1: State space : $\dot{x} = f(x, u^*)$; u^* is just a notation, $u^* = u$

Step2: Find equilibria $f(x, u^*) = 0$

Step3: Jacobians $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$; $\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}$

Step4: Evaluate Jacobian at equilibrium.

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x = x^{*i} \\ u = u^{*i}}} = A_i \qquad \left.\frac{\partial f}{\partial u}\right|_{\substack{x = x^{*i} \\ u = u^*}} = B_i$$

Step: Linearized model: $\dot{x} = Ax + Bu$.

Example: $\dddot{y} + \sin\ddot{y} + \dot{y}^2 + y = \cos u$. Linearize at $u^* = 0$

Step1:

$$\dot{x}_1 = x_2 = f_1$$
$$\dot{x}_2 = x_3 = f_2$$
$$\dot{x}_3 = -\sin x_3 - x_2^2 - x_1 + \cos u = f_3$$

Step2: $x_2^* = 0$; $x_3^* = 0$; $x_1^* = 1$ $\Rightarrow$ $x^* = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Step 3: $f(x, u) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \Rightarrow A_x = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2x_2 & -\cos x_3 \end{bmatrix}$ $B_x = \frac{\partial f}{\partial u} = \begin{bmatrix} 0 \\ 0 \\ -\sin u \end{bmatrix}$

Step4: $A = \left.\frac{\partial f}{\partial x}\right|_{x^* = \begin{bmatrix}1\\0\\0\end{bmatrix} u^* = 0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$; $B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x^* = \begin{bmatrix}1\\0\\0\end{bmatrix} \\ u^* = 0}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Step5: $\dot{x} = Ax + Bu \Rightarrow$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\boxed{\dot{x}_3 = -x_1 - x_3} \rightarrow$$

System is not controllable at equilibrium since u dissappear.

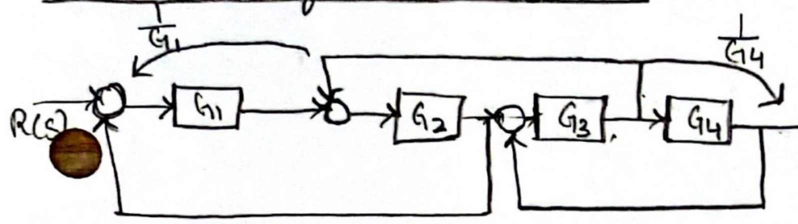
Hartman & Grobman theorem

→ If A has all eigenvalues in OLHP, then the equilibrium point x^* is also stable.

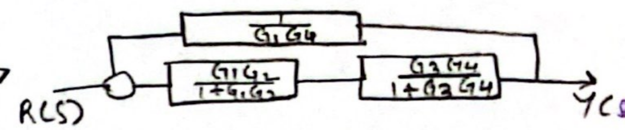
→ If A has atleast one eigenvalue in ORHP, x^* is unstable.

→ If A has some eigenvalues on $j\omega$ axis and rest in OLHP, no conclusion can be made (need to look at trajectories, $\dot{x}_1$ or 2 for +ve & -ve x).

Block diagram algebra (Manipulation) → compensate for new blocks that the signal sees when you change path

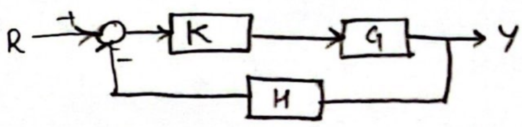


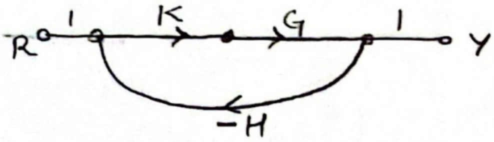
⇒



Signal Flow graph

→ Block diagram → SFG
Lines → nodes
Blocks → lines





→ **Mason's formula**

$$\frac{Y(s)}{R(s)} = \sum_{K=1}^{S \rightarrow \text{no. of direct paths}} \frac{DP_K(s)\,\Delta_K(s)}{\Delta(s)}$$

$DP_K(s)$ → Gain of K^{th} direct path from R to Y

$\Delta(s)$ → 1 − sum of loop gains
+ sum of gain products of all possible pairs of non touching loops
− sum of gain products of all possible triplets of non touching loops
+ . . .

$\Delta_K(s)$ → Δ with all loops touching DP_K being

Stability.

Types: ① BIBO : A causal LTI system is BIBO stable if and only if it's impulse response is absolutely integrable. (Stable, unstable, marginally stable)

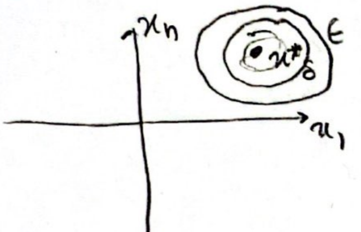
$$\int_0^\infty |g(t)|\,dt < \infty$$

Lyapunov ② Internal stability:- Defined only for equilibriums.

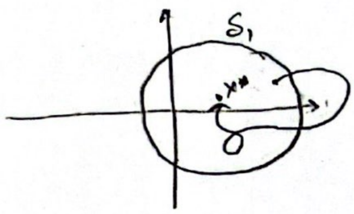
Input must be zero. { If initial pt. is away from equilibrium will it return to equilibrium? (Stable, asymptotically stable, unstable)
marginally

→ Definition

Stable ⇒ i) Equilibrium point x^* is stable if $\forall \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that $\|x(t) - x^*\| < \epsilon$, $\forall\, t \geq 0$ if $\|x(0) - x^*\| < \delta$.



Asymptotically stable ⇒ ii) Equilibrium pt. x^* is asymptotically stable if it is stable and in addition there exists ($\exists$) δ_1 such that $\|x(t) - x^*\| \to 0$ as $t \to \infty$ if $\|x(0) - x^*\| < \delta_1$



Globally asymptotically stable iii) If $\delta_1 = \infty$, globally asymptotically stable.

→ A system with 2 eq. points cannot be globally asymptotically stable.

→ Eigenvalues of A → OLHP ⇒ asymptotically stable.
→ CLHP ⇒ stable (not asymptotically)
↳ Here also the poles must be simple ⇒ only one pole at a point
Or else we get polynomial instability.

→ Internal stability or equilibrium stability is given by eigenvalues of A.

→ BIBO stability is given by $G(s)$.

→ Asymptotic stability ⇒ BIBO stable too since all terms will be $e^{+\lambda_i t}$
↓ eigen values $Re(\lambda_i) < 0$.

→ Stable but not asymptotic ⇏ BIBO stable
unless poles on imaginary axis are cancelled.

Summary

→ Asymptotically stable ⇒ All eigenvalues of A lie in OLHP
→ Stable but not asymptotic ⇒ All λ_i of A lie in CLHP and are simple.

→ BIBO stable ⇒ $g(t)$ is absolutely integrable
→ BIBO marginally stable ⇒ $g(t)$ is oscillatory.
} Cannot look at poles of $|C(sI-A)B|$ & conclude since some poles may cancel & make it stable.
→ Asymptotic ⇒ BIBO
→ BIBO ⇒ Asymptotic only if no unstable poles have been cancelled.

→ Transfer function loses some information since poles and zeroes may cancel

→ You could have stable input-output with one of the internal states blowing up since the coefficients A, B, C do not let it affect the O/P.
→ However if internal states are all stable ⇒ I/P O/P will be stable

Routh Hurwitz Criteria.

→ A system is asymptotically stable if it's characteristic equation is Hurwitz. ⇒ all coefficients have same sign and no sign changes in first column of RH table.

→ No. of sign changes = no. of poles in ORHP

→ If an entire row is 0, there will be symmetry. Need to take previous row as the equation, differentiate it and fill the row of zeros. See example.

→ If a single element in the row is 0, assume it is $\epsilon \to 0$.

→ If last row is zeroes ⇒ pole at origin.

Routh Hurwitz Table.

$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0$ → Char. eq.

s^4	a_0	a_2	a_4
s^3	a_1	a_3	0
s^2	$\frac{a_1 a_2 - a_0 a_3}{a_1} = A$	$\frac{a_3 a_4 - a_2 \cdot 0}{a_3} = a_4$	0
s^1	$\frac{A a_3 - a_1 a_4}{A} = B$	0	0
s^0	a_4	0	0

Example $s^5 + 4s^4 + 8s^3 + 8s^2 + 7s +$ [illegible]

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	0
s^1	$\cancel{0}$ 8	0	0
s^0	4	0	0

$4s^2 + 4 = 0 \Rightarrow 8s = 0$

Roots of $s^2 + 1 = 0$ give the poles on $j\omega$ axis at $\pm j$.

⇒ 3 poles in OLHP & 2 on $j\omega$ axis

General observations

① For a static plant with positive feedback we can track reference by using -ve of reference at input

$$y = \frac{pk}{1-pk} r \Rightarrow \text{at } k \to \infty \Rightarrow y = -r$$

② First order stable linear dynamic plant with dc gain = 1

$$\Rightarrow P = \frac{1}{1+Ts}, \quad T > 0$$

Using positive feedback

$$Y(s) = \frac{K}{1-K+Ts} R(s)$$

} K < 1 for stability. Could use $-\infty$ gain for stability & tracking

Positive feedback with -ve controller can track a stable linear plant.

③ First order unstable linear dynamic plant (dc gain = 1)

$$\Rightarrow P = \frac{1}{1-Ts}, \quad T > 0$$

Using positive feedback

$$Y(s) = \frac{k}{1-k-Ts} \Rightarrow$$

$K > 1 \Rightarrow$ Stable and good tracking when $K \to \infty$

→ Positive feedback can stabilize an unstable plant!

Nyquist plot example

$$G(s) = \frac{s+1}{s^2+s+1} \overset{\text{at } s=j\omega}{=} \frac{j\omega+1}{-\omega^2+j\omega+1} = \frac{(1+j\omega)(1-\omega^2-j\omega)}{(-\omega^2+1)^2+\omega^2}$$

$$= \frac{(1-\cancel{\omega^2}+\cancel{\omega^2})+j(\cancel{\omega}-\omega^3-\cancel{\omega})}{\omega^4-\omega^2+1} = \frac{1-j\omega^3}{\omega^4-\omega^2+1}$$

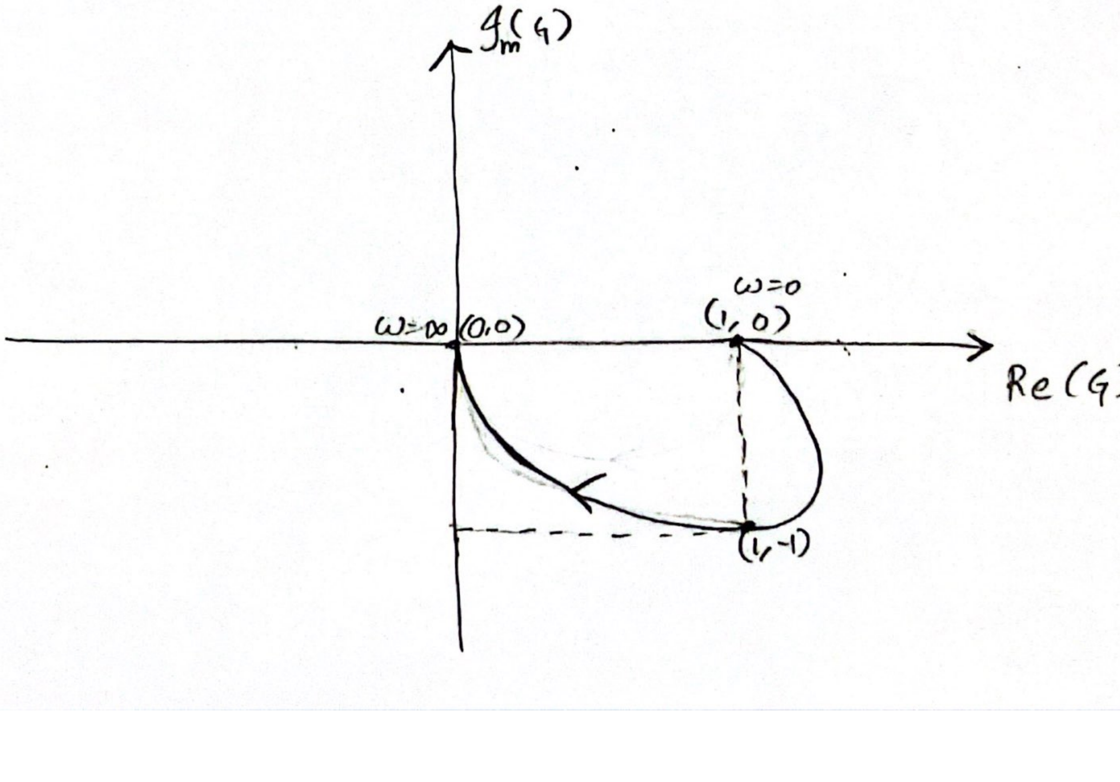
$$Re(G(j\omega)) = \frac{1}{\omega^4-\omega^2+1} \quad ; \quad Im(G(j\omega)) = \frac{-\omega^3}{\omega^4-\omega^2+1}$$

① At $\omega \to 0 \Rightarrow Re(G) \to 1$; $Im(G) \to 0^-$ (asymptotically from -ve side)

$$Re(G(j\omega)) = \frac{\frac{1}{\omega^4} \to \text{changes fast}}{1-\frac{1}{\omega^2}+\frac{1}{\omega^4}} \quad ; \quad Im(G(j\omega)) = \frac{-\frac{1}{\omega} \to \text{changes slowl}}{1-\frac{1}{\omega^2}+\frac{1}{\omega^4}}$$

② At $\omega \to \infty \Rightarrow Re(G) \to 0^+$; $\underset{(\text{slow})}{Im(G)} \to 0^-$ {Approach (0,0) from slower side}

③ At $\omega = 1 \Rightarrow Re(G) = 1$; $Im(G) = -1$

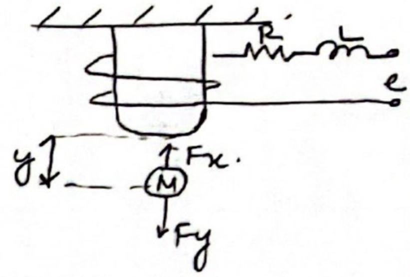


Mathematical modelling.

① Plant modelling. → First principles based modelling.
→ Phenomenological modelling.
→ Identification approach to modelling.

i) First principles based modelling.

Example: Magnetic ball levitation system.



$$\boxed{\begin{aligned} M\ddot{y} &= Mg - \frac{\alpha i^2}{y^2} \\ Ri + L\frac{di}{dt} &= e \end{aligned}}$$

$\Rightarrow$ $i\uparrow \Rightarrow \ddot{y}\downarrow$
$y\uparrow \Rightarrow \ddot{y}\uparrow$

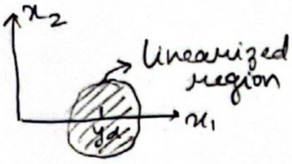
→ Nonlinear model.

Linearizing gives:
$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{2\alpha x_3^{*2}}{M x_1^{*3}} = a_{21} & 0 & -\frac{2\alpha}{M} \cdot \frac{x_3^*}{x_1^{*2}} = a_{23} \\ 0 & 0 & -\frac{R}{L} = a_{33} \end{bmatrix}$$

x^* is the equilibrium value of x.

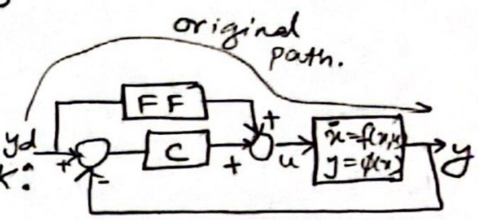
$\dot{x} = Ax + Bu$

$y = Cx.$



$$B = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

→ If R, L, M are not well defined we need feedback: and controller 'C'.



$$\frac{d\Delta x}{dt} = A\Delta x + B\Delta u$$

where $\Delta x = x - x^*$
$\Delta u = u - u^*$

ii) Phenomenological modelling. (Volterra Lotka modelling)

x_1 = # prey x_2 = # predators.

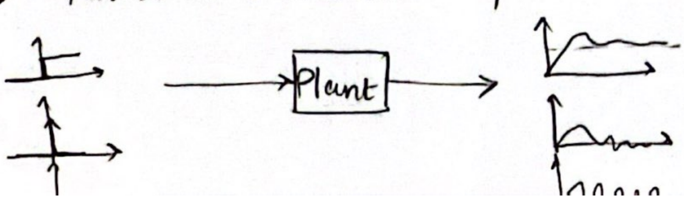
$\dot{x}_1 = ax_1 - bx_1x_2$ → how many times do they meet

$\dot{x}_2 = -cx_2 + dx_1x_2$

} could add more terms as a "controller" to control population.

iii) System identification

Hit it with standard inputs to see how the system behaves.



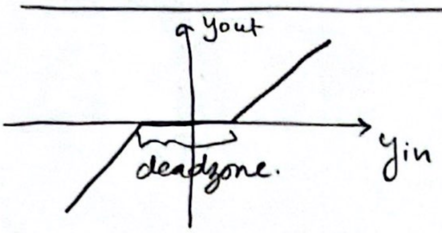
② Sensor modelling

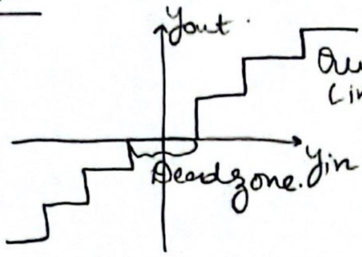
$$G_{sensor}(s) = \begin{cases} 1 \text{ or } K & \text{} \} \text{ Static} \\ \frac{1}{TS+1} \text{ or } \frac{K}{TS+1} & \} \text{ First order dynamic} \\ K\frac{\omega_n^2}{S^2+2\zeta\omega_n S+\omega_n^2} & \} \text{ Second order dynamic.} \end{cases}$$

$\to$ most ideal (remember the midterm question)

$\frac{S}{TS+1}$ (Tachometer) } Rate of change of output is measured. (want angle but measures ang. velocity)

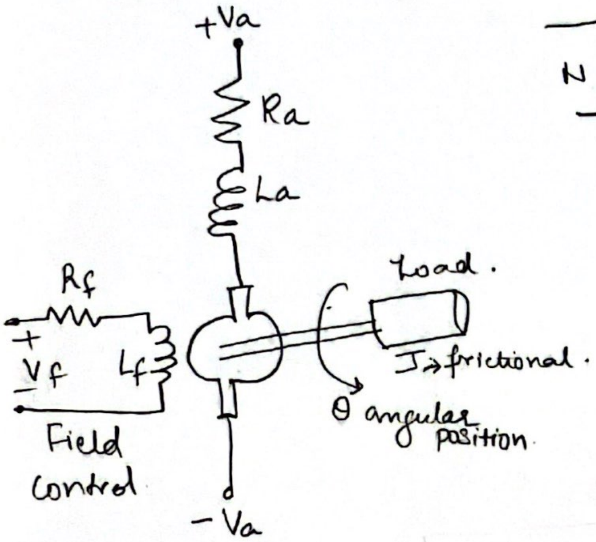
Practical limitations.

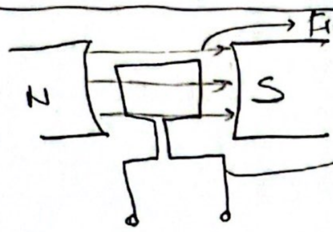




③ Actuator modelling

Armature control vs. Field control (DC motor)





$\to$ Field changes to induce currents & turn coil. (field control)

$\to$ Current changes to turn coil (armature control).

Ideally we want

$$G(s) = \frac{\theta(s)}{V(s)} = \frac{K}{S}$$

$\to$ In field control case: $\frac{\theta(s)}{V_a(s)} = \frac{K_{M_1}}{S(JS+B)(LS+R)}$

$K_{M_1} \to$ fn. of material & I_a.

B $\to$ friction, J $\to$ Inertia.

J, L are large & ∴ not used.

$\to$ In armature case

$$\frac{\theta(s)}{V_a(s)} = \frac{K_{M_2}}{S[R_a(JS+B)+\underbrace{K_b K_{M_2}}]}$$

$\to$ This term creates feedback & changes dynamics.

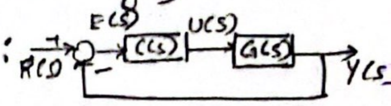
$$= \frac{\frac{K_{M_2}}{R_aB+K_bK_{M_2}}}{S\left(\frac{R_a J}{R_aB+K_bK_{M2}}+1\right)} = \frac{K}{S(TS+1)} \quad \text{where } T = \frac{R_a J}{R_aB+K_bK_{M_2}}$$

We can design it to make $T \ll 1$

$\Rightarrow G(s) \simeq \frac{K}{S}$

Modelling Controllers (P, I, D, Lead, Lag, Lead Lag)

> Standard feedback system used for the discussion:



1. P controller $C(s) = K \Rightarrow u(t) = K e(t)$.

> Changes the free term (term w/o s) of characteristic polynomial (for all pole system)

> Reduces sensitivity function & S.S error.

---

2. PI controller $C(s) = K_P + \frac{K_I}{s} \Rightarrow$ pole at 0, zero at $-\frac{K_I}{K_P}$

↳ Bad (amplifies high frequency noise).

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau) d\tau$$

> Even when $e(t)$ is zero feedback is not 'lost' since it integrates previous values of $e(t)$.
> $\Rightarrow$ Steady state error $= 0 \Rightarrow$ High precision.

> Increases order of system by 1 ($\Rightarrow$ can't use high 'k')

> Introduces finite zero.

> May stabilize or destabilize plant.

$G(s) = \frac{1}{Ts-1}$; $C(s) = K_P + \frac{K_I}{s}$

$$\Rightarrow \frac{Y(s)}{R(s)} = \frac{K_P s + K_I}{Ts^2 + (K_P - 1)s + K_I}$$

Stable if $K_P > 1$, $K_I > 0$

$$e_{ss} = \lim_{t \to \infty} e(t)$$
$$= \lim_{s \to 0} sE(s).$$
$= 0$ where;
$$E(s) = \frac{1}{s} \cdot \frac{s(Ts-1)}{Ts^2 + (K_P - 1) + K_I}$$
step i/p

---

3. PD Controller. $C(s) = K_P + K_D s$ → not causal since numerator order is greater. In reality $\frac{K_D s}{\tau s + 1}$

↳ very small

Introduces dissipation!

> Zero at $-\frac{K_P}{K_D}$ (Bad)

> Introduces dissipation. (stabilizes plant)

> Results in nonzero steady state error.

Used for plants like: $-\frac{1}{s^2}$, $\frac{1}{s^2+1}$, $\frac{1}{s^2 - a_1 s + a_2}$ that are not asymptotically stable.

---

4. PID controller: $C(s) = K_P + K_I/s + K_D$ (Best of both worlds)

---

5. PDDD... controller $\Rightarrow$ more dissipation.

If $G(s) = \frac{1}{s^5 + a_1 s^4 - a_2 s^3 - a_3 s^2 - a_4 s - a_5}$ $\Rightarrow$ PDDD controller is needed to stabilize. But very noisy.

$\underbrace{a_2 s^3}_{PDDD}$ $\underbrace{a_3 s^2}_{PDD}$ $\underbrace{a_4 s}_{PD}$ $\underbrace{a_5}_{P}$

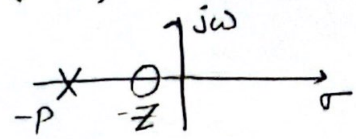
6) Not all pole systems.

→ Finite zero plays role of one 'D' eg: $\frac{s+1}{s^2+2}$ is $G(s)$ plant ⇒ P is enough to make it asymptotically stable.

→ Pole at origin plays role of one 'I' eg: $\frac{1}{s(s+2)}$ " " ⇒ P is enough for $e_{ss}=0$.

7) Phase Lead ⇒ (PD)

$$C(s) = \frac{s+z}{s+p}$$



⇒ Bode phase plot has +ve phase!

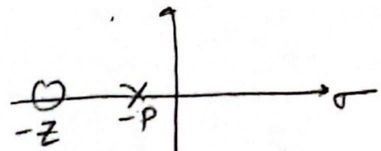
$0 < z < p$

$$C(s) = \frac{z}{p} \cdot \frac{\frac{s}{z}+1}{\frac{s}{p}+1} = \frac{z}{p}\left[\frac{s}{z}+1\right] \Rightarrow PD$$

∴ P part is fixed → small so use a pre-compensator to get large gain.

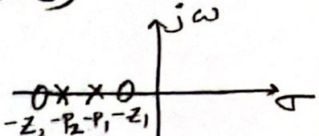
8) Phase Lag : (PI)

$$C(s) = \frac{s+z}{s+p} \simeq 1 + \frac{z}{s} \text{ if } p \ll 1$$



9) Lead Lag : (PID)

$$C(s) = \frac{s+z_1}{s+p_1} \cdot \frac{s+z_2}{s+p_2}$$

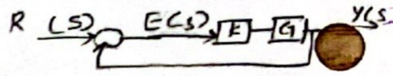


Time Domain Analysis (Transient & Steady state response)

1) Error coefficients

$E(s) = S(s) \cdot R(s)$

Sensitivity fn $\rightarrow \frac{1}{1+K(s)G(s)}$



Use $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$

Std. I/ps

γ/s	step
γ/s^2	ramp
γ/s^3	parabolic

$\gamma = 1 \Rightarrow$ unit.

K_p (step) $= \lim_{s\to 0} K(s)G(s)$. ~ DC gain of loop.

K_v (ramp) $= \lim_{s\to 0} sK(s)G(s)$.

K_a (parabolic) $= \lim_{s\to 0} s^2K(s)G(s)$

2) Steady state error

$$e_{ss\,step} = \frac{\gamma}{1+K_p}$$

$$e_{ss\,ramp} = \frac{\gamma}{K_v}$$

$$e_{ss\,para} = \frac{\gamma}{K_a}$$

System Type

[No. of poles at origin of loop gain]

$$K(s)G(s) = \frac{k}{s^2(Ts+1)} \Rightarrow \text{Type 2.}$$

→ As Type ↑ the system can handle faster & faster I/p (step → parabolic) to get 0 s.s error.

→ If Type is insufficient we can increase it by adding 'I' since it gives pole at origin in loop gain

	K_p	e_{ss} step	K_v	e_{ss} ramp	K_a	e_{ss} para.
Type 0	$K_{dc} = \frac{b_m}{a_n}$ → dc gain of loop	$\frac{\gamma}{1+K_{dc}}$	0	∞	0	∞
Type 1	∞	0	$\frac{b_m}{a_{n-1}}$	$\frac{\gamma a_{n-1}}{b_m}$	0	∞
Type 2	∞	0	∞	0	$\frac{b_m}{a_{n-2}}$	$\frac{\gamma a_{n-2}}{b_m}$

→ All this is only true if closed loop system is asymptotically stable.

$$K(s)\,G(s) = \frac{b_0 s^m + \dots + b_m}{s^{\gamma}(s^{n-j} + a_1 s^{n-j-1} + \dots + a_{n-j})}$$

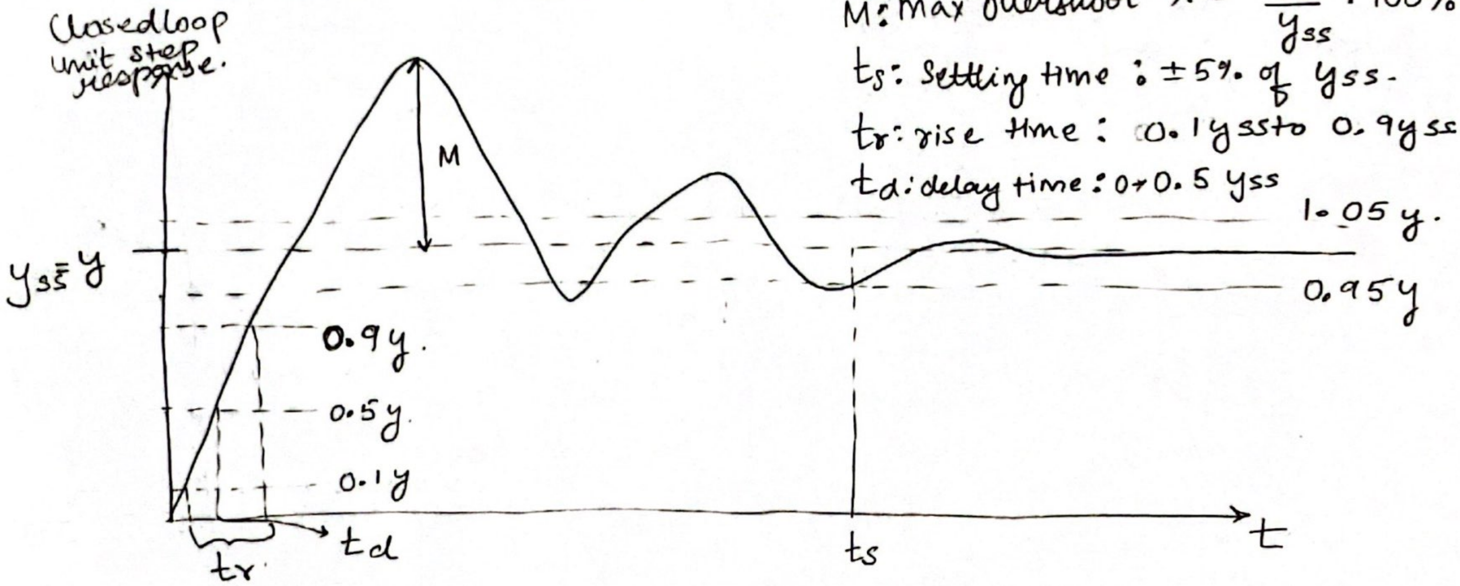
Transient response specifications.

M: Max overshoot % $= \frac{M}{y_{ss}} \cdot 100\%$

t_s: Settling time : ±5% of y_{ss}.

t_r: rise time : $0.1 y_{ss}$ to $0.9 y_{ss}$.

t_d: delay time : 0 → $0.5 y_{ss}$



① Transients of first order system

$$R(s) \longrightarrow \boxed{\frac{1}{\tau s+1}} \longrightarrow Y(s)$$

closed loop

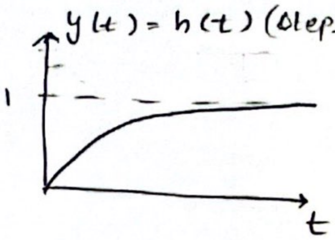
Could be open or closed loop.

Input step $\Rightarrow y(t) = h(t) = 1 - e^{-t/\tau}$

$h(\tau) = 0.632$

$h(2\tau) = 0.865$

$h(3\tau) = 0.95$



$$\boxed{\begin{array}{c} \%OS = 0\% \\ t_s = 3\tau \\ t_r < 3\tau \\ t_d \simeq \tau \end{array}}$$

② Transients of second order system.

$$R(s) \longrightarrow \boxed{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \longrightarrow Y(s)$$

closed loop

$\omega_n \to$ natural freq

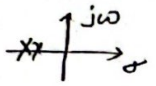
$\zeta \to$ damping ratio

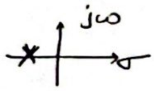
$\zeta\omega_n \to$ damping factor

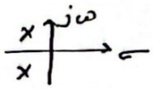
If given $G(s) = \dfrac{b}{s^2+a_1 s+a_2} \xrightarrow[\text{to}]{\text{Convert}} \dfrac{K \cdot \omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ where $\zeta = \dfrac{a_1}{2\sqrt{a_2}}$

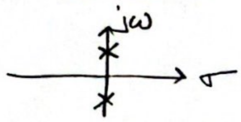
$\omega_n = \sqrt{a_2}$

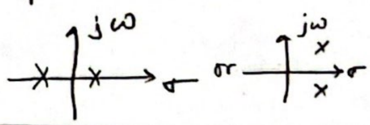
$K = \dfrac{b}{a_2}$

$\zeta > 1 \Rightarrow$ overdamped 

$\zeta = 1 \Rightarrow$ critically damped 

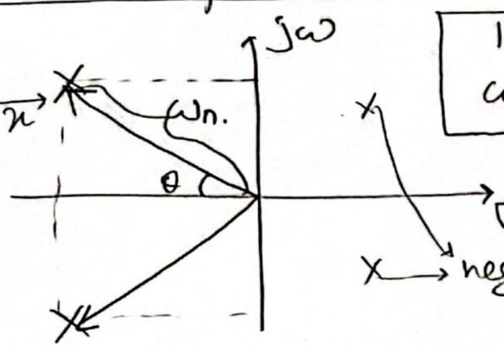
$0 < \zeta < 1 \Rightarrow$ underdamped (desirable). 

$\zeta = 0 \Rightarrow$ undamped 

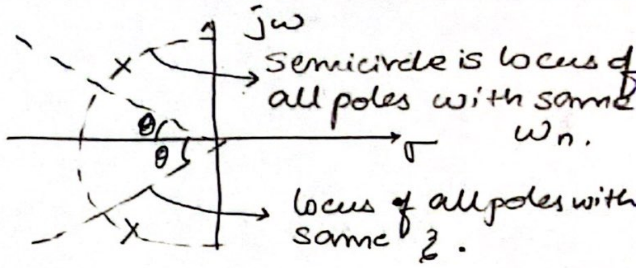
$\zeta < 0 \Rightarrow$ negatively damped 

Notice here the model used for study has no finite zeroes $\Rightarrow$ Closed loop is an all pole system. If not so we need to pre compensate it.

Underdamped case



$$\boxed{\begin{array}{c} |\vec{x}| = \omega_n. \\ \cos\theta = \zeta \end{array}}$$



Unit step response - 2nd order transients.

$$h(t) = \mathcal{L}^{-1}\left\{\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\cdot\frac{1}{s}\right\}$$

> Although it doesn't make much sense we define a time constant

$$\boxed{T = \frac{1}{\zeta\omega_n}}$$

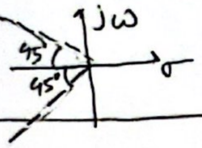
$\Rightarrow$ t_s $\longrightarrow$ $t_s = \dfrac{3}{\zeta\omega_n}$ $\longrightarrow$ In reality: $t_s = \begin{cases} \boxed{\dfrac{3.2}{\zeta\omega_n}} & 0<\zeta<0.7 \\ \dfrac{4.5\zeta}{\omega_n} & 0.7<\zeta<1 \end{cases}$ (use this often)

t_{peak} $\longrightarrow$ $t_{peak} = \dfrac{\pi}{\omega_d}$; $\omega_d = \omega_n\sqrt{1-\zeta^2}$

%OS $\longrightarrow$ $\%OS = 100\exp\left(\dfrac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ (Depends only on ζ)

We want ζ b/w 0.6 to 1!

ζ of 0.707 gives %OS < 5%.
$\Rightarrow \pm\theta \leq 45°$



$t_r \cong \dfrac{0.8+2.5\zeta}{\omega_n}$ $\quad (0<\zeta<1)$

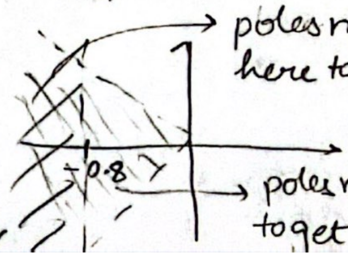
t_r $\longrightarrow$ t_r (better) $\cong \dfrac{1-0.4167\zeta+2.917\zeta^2}{\omega_n}$ $\quad (0<\zeta<1)$

$t_d = \dfrac{1+0.7\zeta}{\omega_n}$

t_d $\longrightarrow$ t_d (better) $= \dfrac{1+0.125\zeta+0.469\zeta^2}{\omega_n}$

→ If characteristic eq. of C.L system is $s^2+2\zeta\omega_n s+\omega_n^2$ the poles are at $S_{1,2} = \underbrace{-\zeta\omega_n}_{\alpha} \pm \sqrt{\zeta^2\omega_n^2-\omega_n^2}$

→ If t_s = 4secs, $t_s = \dfrac{3.2}{\zeta\omega_n} = \dfrac{3.2}{\alpha} \Rightarrow \alpha = +0.8$

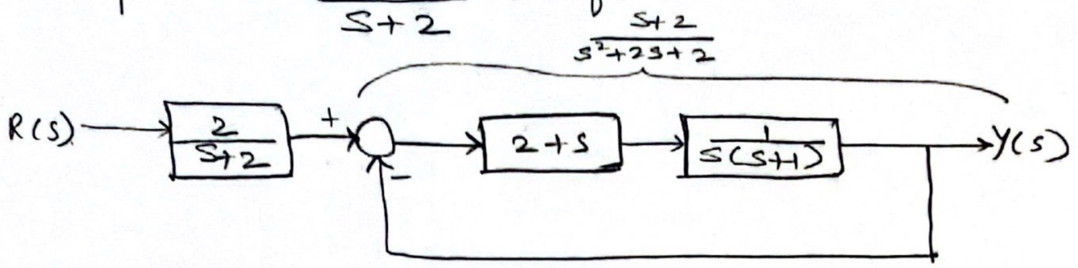


poles must lie here to get t_s=4sec

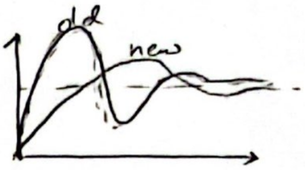
poles must lie here to get ≤5% OS.

→ 'D' controller pushes poles to left → more damping or

> If the final CL system that meets specs is $\frac{S+2}{S^2+2S+2}$, use a Precompensator $\frac{2}{S+2}$ to fit the model used.

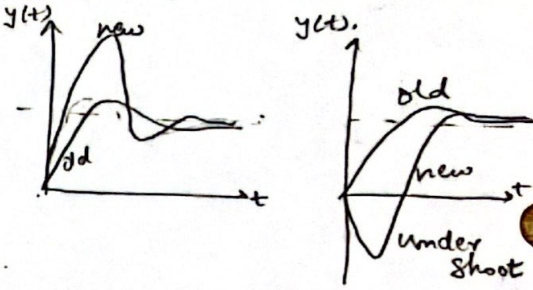


Adding Poles and Zeroes.



> Adding poles makes system more "sluggish"

> Adding zeroes makes system "faster" => creates overshoot & sometimes undershoot



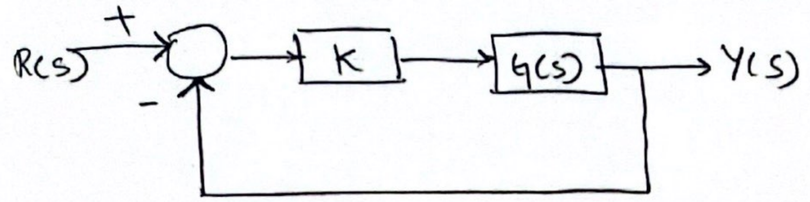
→ In higher order system, design to have all poles except 2 to be at very high frequencies => insignificant. But make sure dc gain remains same when modelling. => (Delete insignificant poles if their dc gain is 1.)

Eg: $G(s) = \frac{11}{(s+10)(s^2+2s+2)}$

$$\xrightarrow{\times} \frac{11}{(s^2+2s+2)\cancel{(s+10)}}$$

$$\xrightarrow{\checkmark} \frac{11}{10(s^2+2s+2)\cancel{(\frac{s}{10}+1)}}$$

Root locus



$G(s) = \frac{1}{s+1}$

Given $G(s)$, find the <u>locus of closed loop poles</u> as a function of K.

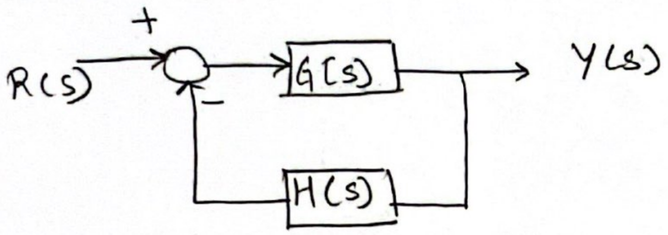
Basic definitions

$K \in [0, \infty)$ ⇒ Root locus → practically used

$K \in (-\infty, 0]$ ⇒ Complimentary root locus → -ve controller gain

$K \in (-\infty, \infty)$ ⇒ Complete root locus.

> Generic Block diagram



Closed loop poles given by
$\underbrace{1 + G(s) \cdot H(s) = 0}$
Called return difference
→ not the characteristic eqn

Rewriting ⇒ $1 + K\, G_1(s)\, H_1(s) = 0$

⇒ $G_1(s)\, H_1(s) = \frac{-1}{K}$

Magnitude condition : $|G_1(s)\, H_1(s)| = \frac{1}{K}$

Angle condition : $\angle G_1(s)\, H_1(s) = (2k+1)\pi$ $\quad k = 0, \pm 1, \pm 2 \ldots$

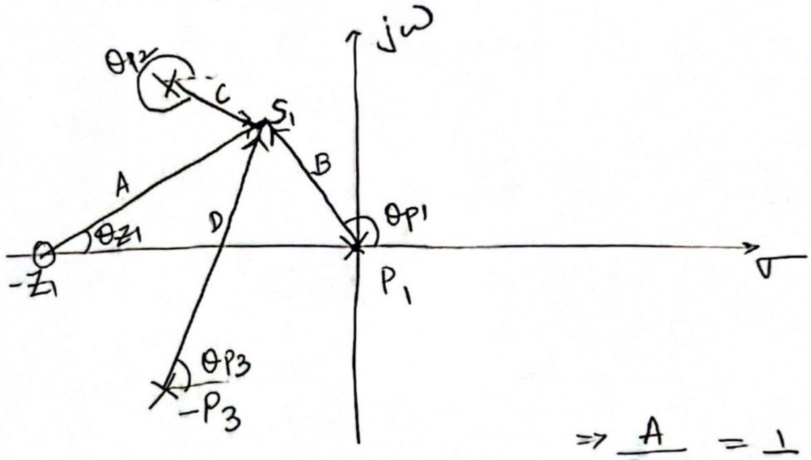
If $G_1(s)H_1(s) = \dfrac{(S+Z_1)(S+Z_2)\dots(S+Z_n)}{(S+P_1)(S+P_2)\dots(S+P_n)}$

$$|G_1(s)H_1(s)| = \frac{\prod_{i=1}^{m} |s+z_i|}{\prod_{i=1}^{n} |s+p_i|} = \frac{1}{K}$$ → used for calibration.

used for sketching.

$$\angle G_1(s)H_1(s) = \sum_{i=1}^{m} \angle s+z_i - \sum_{i=1}^{n} \angle s+p_i = (2K+1)\pi, \quad K=0,\pm1,\pm$$

> Take a random point S_1 and see if it satisfies the $||$ & $\angle$ conditions to see if it lies on RL.



$$\Rightarrow \frac{A}{BCD} = \frac{1}{K}$$

$$\theta_{Z_1} - \theta_{P_1} - \theta_{P_2} - \theta_{P_3} = (2K+1)\pi$$

$G(S)H(S) = K \cdot \dfrac{Q(S)}{P(S)}$ ⇝ $P(s)=0$ → open loop poles.

$Q(s)=0$ → open loop zeroes.

RD

↳ ⇒ $1 + G(s)H(S) = 1 + K\dfrac{Q(s)}{P(S)} = \dfrac{P(S) + KQ(S)}{P(S)}$

> Zeroes of return difference are given by $P(S)+KQ(S)=0$
> poles of return difference = open loop poles → $P(S)=0$

> Poles of RD $\rightarrow$ open loop poles.
> Zeroes of RD $\rightarrow$ closed loop poles.

10 Rules of R.L construction.

#1 :- Root locus originates in open loop poles.

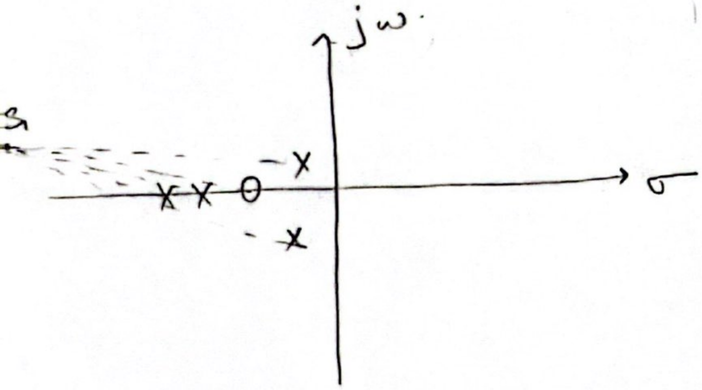
$$P(s) + K\,Q(s) = 0\ ; \quad K = 0 \Rightarrow P(s) = 0 \longrightarrow \text{open loop poles.}$$

#2 :- Root locus terminates in open loop zeroes.

$$P(s) + K\,Q(s) = 0 \rightarrow \frac{1}{K} = 0 \Rightarrow Q(s) = 0 \rightarrow \text{open loop zeroes.}$$

#3 :- No. of branches = no. of poles.

#4 :- Angle of R.L asymptotes is given by $\theta_K = \dfrac{(2K+1)\pi}{\underbrace{n-m}_{\text{relative degree}}}$



When $|s_1| \rightarrow \infty$ all angles are same.

$$\Rightarrow m\theta - n\theta = (2K+1)\pi \rightarrow \text{Angle condition.}$$

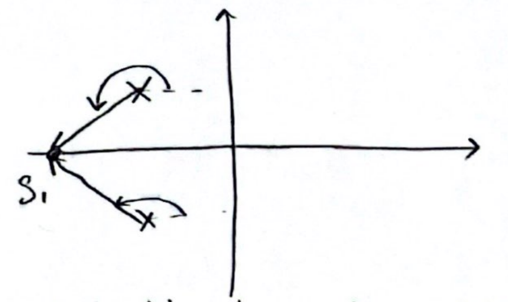
$$\Rightarrow \boxed{\theta_K = \frac{(2K+1)\pi}{n-m}} \rightarrow (\text{Since } K = \pm\text{ve anyway})$$

Where $K = 0, 1, \ldots n-m-1$

#5:- Asymptotes intersect on the real axis at centroid σ_1

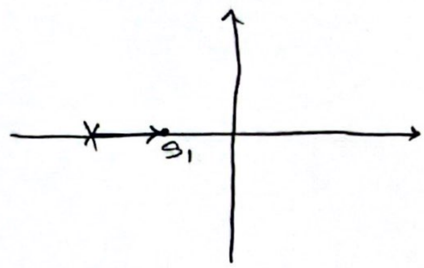
$$\sigma_1 = \frac{\sum \text{all open loop poles} - \sum \text{all open loop zeroes}}{n-m}$$

#6: RL is on a segment of real axis if and only if the number of OL poles & zeroes to the right of each test point on this segment is odd.



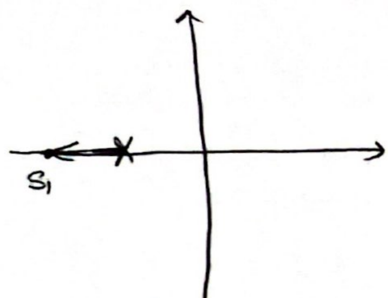
contribution to angle condition $= 2\pi$

$\Rightarrow$ RL cannot exist



angle contribution $= 0$

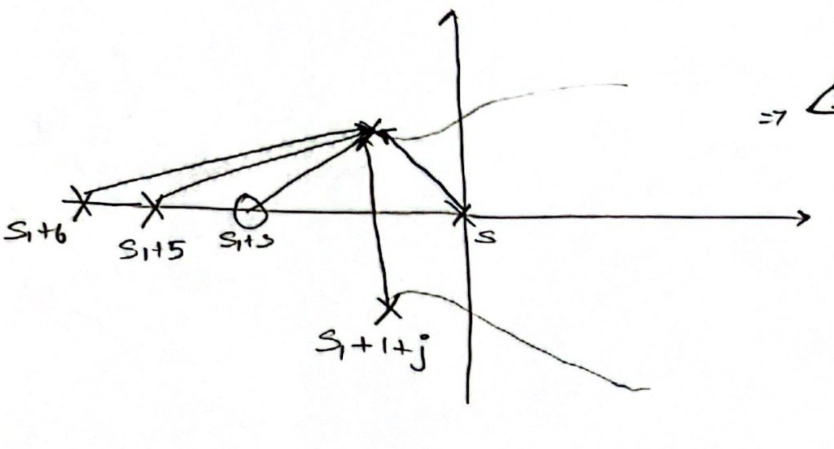
$\Rightarrow$ RL cannot exist



angle contribution $= \pi$

$\Rightarrow$ RL exists

#7 Angle of departure from a pole or arrival to a zero is given by satisfying the angle condition at the pole or zero of interest.



$$\Rightarrow \angle S_1+3 - \angle S - \angle S+1+j - \angle S_1+5 - \angle S_1+6 - \theta = (2k+1)\pi$$

Eg: $26.6 - 135 - 90 - 14 - 11.4 - \theta = (2k+1)\pi$

$\Rightarrow \theta = -43.8°$

#8 Intersection of R.L with $j\omega$ axis is given by the RH criteria, by find k that gives a row of zeroes.

Eg: CL ch. eq = $s^5 + 13s^4 + 54s^3 + 82s^2 + (60+K)s + 3K = 0$

s^5	1	54	60+K
s^4	13	82	3K
s^3	47	60+0.769K	0
s^2	65.6-0.212K	3K	0
s^1	$\frac{3940-105K-0.163K^2}{65.6-0.212K}$	0	0
s^0	3K	0	0

$\Rightarrow 3940 - 105K - 0.163K^2 = 0 \Rightarrow \boxed{K = 35.56}, -679.7$

Auxilliary equation: $[65.6 - 0.212(35)]s^2 + 3[35]s^0 = 0$

$\Rightarrow \boxed{s_{1,2} = \pm j(1.34)}$

#9 Value of Breakaway (or break in) point for multiple poles is given by $\frac{d}{ds}[1 + K G_1(s) H_1(s)] = 0$. Must lie on real axis.

$s^5 + 13s^4 + 66s^3 + 142s^2 + 123s + 45 = 0$

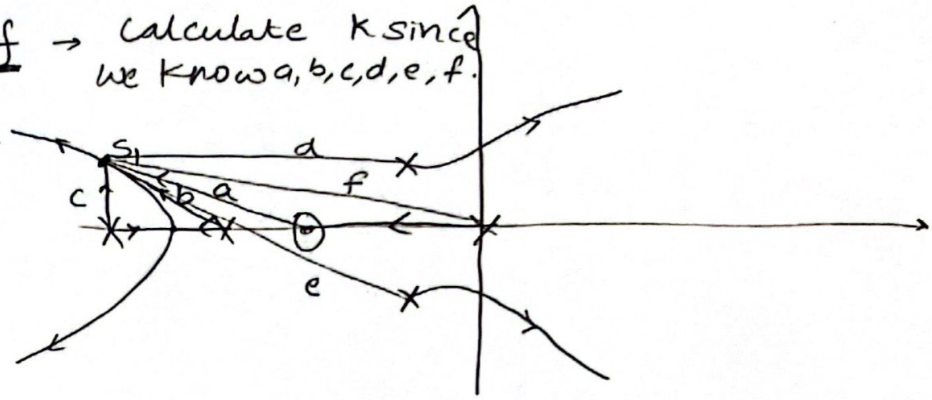
$\Rightarrow s_{1,2} = 3.33 \pm j\,1.204$
$s_{3,4} = -0.656 \pm j\,0.468$ } Not on Real axis
$s_5 = -5.53 \longrightarrow$ Break away pt.

Angle of departure is $\frac{180°}{r}$ where $r \rightarrow$ no. of poles breaking away. (or breaking in)

#10 To find K at a desired point on RL, use the magnitude condition.

$$K = \frac{1}{|G_1(s)H_1(s)|} = \frac{\prod_{i=1}^{n} |s_1 + p_i|}{\prod_{i=1}^{m} |s_1 + z_i|}$$

$$K = \frac{bcdef}{a}$$ → calculate K since we know a, b, c, d, e, f.



→ **Conclusion**

- Root locus analysis the location of closed loop poles as a function of K.
- When $K = 0$, $TF_{CL} = \dfrac{G(s)}{1 + KG_1(s)H_1(s)} = G(s) = TF_{OL}$

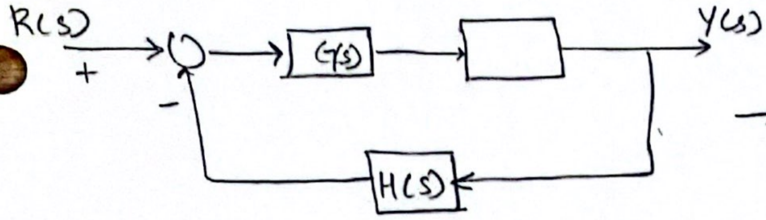
  closed loop system = open loop system ⇒ No feedback.
- When $K = \infty$, $TF_{C.L} = 0$ ⇒ no poles or zeroes exist.

Eg: If $G(s)H(s) = K \cdot \dfrac{Q(s)}{P(s)}$ where degree of $Q(s)$ > degree of $P(s)$

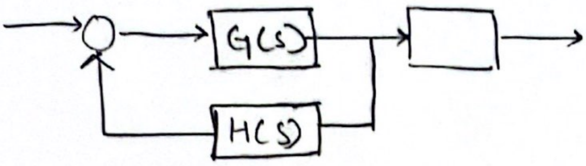
⇒ $KQ(s) + P(s) = 0$ ⇒ $Q(s) + \frac{1}{K}P(s) = 0$ ⇒ $1 + \dfrac{K' P(s)}{Q(s)} = 0$

- Construct R.L w.r.t K_1
- Replace poles with zeroes & zeroes with poles & K_1 by $1/K$,
- Change direction of arrows

> Adding poles and zeroes



→ Does affect R.L

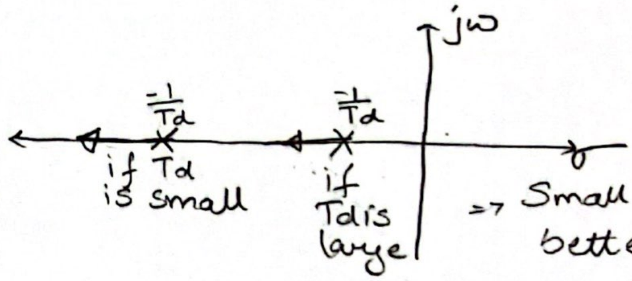


→ Does not affect the RL. since it does not change with k. Will have additional poles & zeroes that don't move.

- Adding pole to OL bends RL to the right.
- Adding zero to left bends RL to left ⇒ good
- Adding zero to right bends RL to right ⇒ bad.
- Systems with time delay are more difficult to control.

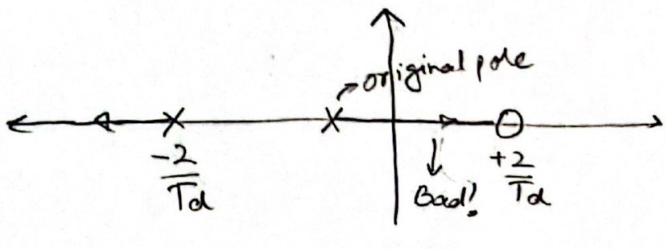
Eg:- add delay of $e^{-T_d s}$.

approx ① $e^{-T_d s} \simeq \dfrac{1}{T_d s+1}$

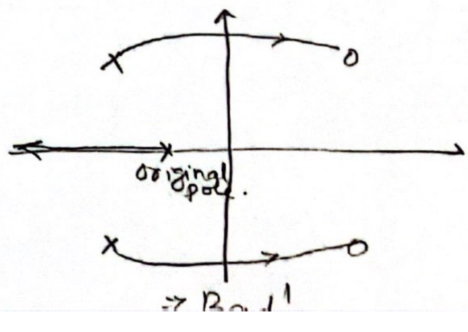


⇒ Small Td is better.

approx ② Order one Pade approximation $e^{-T_d s} \simeq \dfrac{\frac{T_d s}{2} - 1}{\frac{T_d s}{2} + 1}$



approx ③ Order 2 Pade approximation $e^{-T_d s} \simeq \dfrac{1 - \frac{T_d s}{2} + \frac{(T_d s)^2}{12}}{1 + \frac{T_d s}{2} + \left(\frac{T_d s}{12}\right)^2}$



⇒ Bad!

Design of Control Systems in the Time Domain

- Trial and error is the way to go!

Design approach ①

- Select the "right" controller (based on specs and educated guess)
- Choose all controller parameters, except for one → use this to drive RL to admissible domain.
- If RL enters admissible domain, select the right gain.
- Try again if fails.

Design approach ②

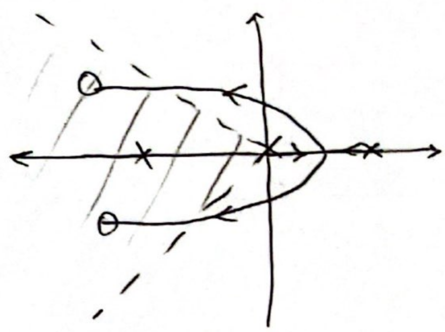
- Choose positions of additional zeroes inside admissible domain.
- Select parameters of controller, leaving one free for RL.

use this for RL.

Select these to put zeroes in the admissible domain

$$K(s) = K_D \cdot \frac{s^2 + \frac{K_P}{K_D} s + \frac{K_I}{K_D}}{s}$$

- Construct RL w.r.t K_D
- Choose K_P, K_I to keep zeroes in good place & choose appropriate K_D.
- Use a precompensator!



Example. Design of PID controller.

$$K(s) = K_P + K_D s + \frac{K_I}{s} = \frac{s^2 K_D + s K_P + K_I}{s}$$

Idea: Split PID into PI & PD.

$$K(s) = \underbrace{(1 + K_{D1} s)}_{(P)D} \underbrace{\left(K_{P2} + \frac{K_{I2}}{s}\right)}_{PI}$$

$K_P = K_{P2} + K_{D1} K_{I2}$

$K_D = K_{D1} K_{P2}$

$K_I = K_{I2}$

Design procedure

1) Design first PI part to meet steady state & few transient specs.
2) Design PD to meet remaining transient specs.
3) Calculate K_P, K_D, K_I
4) Design precompensator to cancel zero of (P)D part.

Eg: Given, $G_1(s)H_1(s) = \dfrac{1}{s(0.1s+1)(0.2s+1)}$

design to have $K_v \geq 100 \Rightarrow e_{ss_{ramp}} \leq 0.01$

$\% OS \leq 5\%$

$t_r \leq 0.25$ sec.

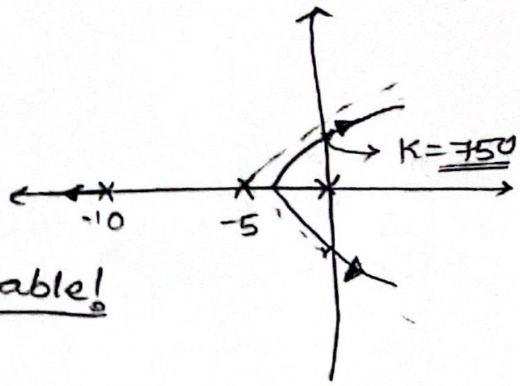
$t_s \leq 0.5$ sec.

Sol: Try P :- $G_1(s)H_1(s) = \dfrac{50}{s(s+5)(s+10)}$.

$$\Rightarrow G(s)H(s) = \frac{K_P\, 50}{s(s+5)(s+10)} = \frac{K}{s(s+5)(s+10)}$$

To get $K_v = 100$ we need.

$$K_v = \lim_{s\to 0} s\,G(s)\,H(s) = \frac{K}{50}$$



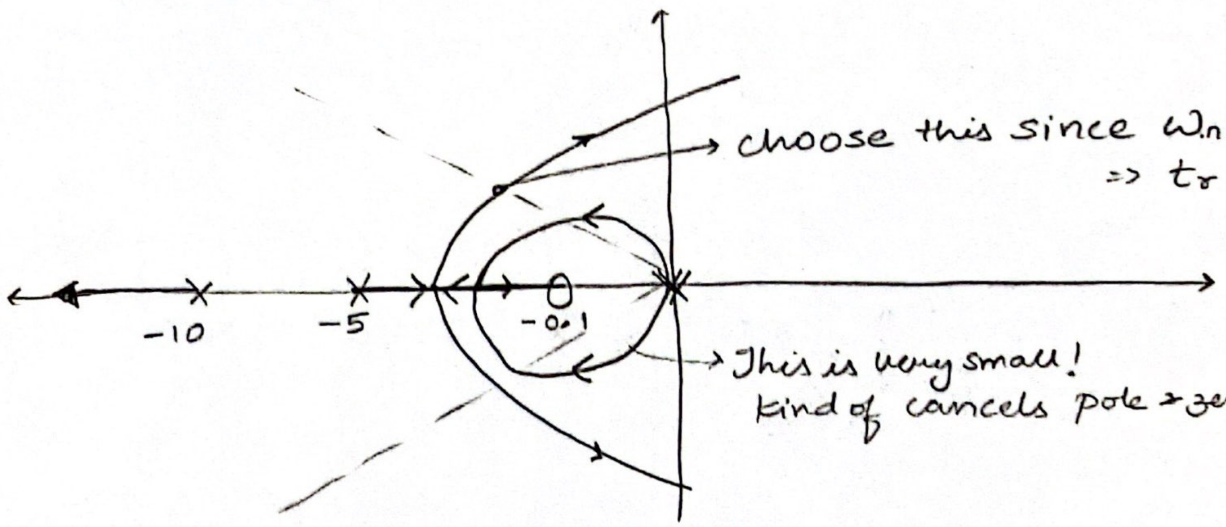
$\Rightarrow$ $K = 5000$ $\Rightarrow$ $K > 750 \Rightarrow$ unstable!

> Use PI $\Rightarrow$ $K_v = \infty$

$$G(s)\,H(s) = \frac{50\,K_{P_2}\left(s + \frac{K_{I_2}}{K_{P_2}}\right)}{s^2(s+5)(s+10)}$$

choose $\frac{K_{I_2}}{K_{P_2}} \simeq 0$

$$= K \cdot \frac{s+0.1}{s^2(s+5)(s+10)}$$



choose this since ω_n is $\uparrow$.
$\Rightarrow$ t_r is $\downarrow$

This is very small!
Kind of cancels pole & zero!

$K_{P_2} = \frac{81}{50} \simeq 1.63$ $\Rightarrow$ $K_{I_2} = 0.163$.

$\Rightarrow$ $\boxed{PI = 1.63 + \frac{0.163}{s}.}$

Check tr and ts

$$t_r = \frac{0.8 + (2.5)(0.707)}{\underset{\rightarrow \omega_n}{3}} \simeq 0.885$$

$$t_s = \frac{3.2}{\alpha} = \frac{3.2}{2.1} = 1.52$$

Design PD part

$$K(s) = \underbrace{\left(K_{P_2} + \frac{K_{I_2}}{s}\right)}_{\text{have.}} \underbrace{\left(1 + K_{D_1} s\right)}_{\text{find}}$$

$$1 + G(s)H(s) = 1 + \frac{81\,\cancel{(s+0.1)}\,(1 + K_{D_1} s)}{s^{\cancel{2}}(s+5)(s+10)}$$

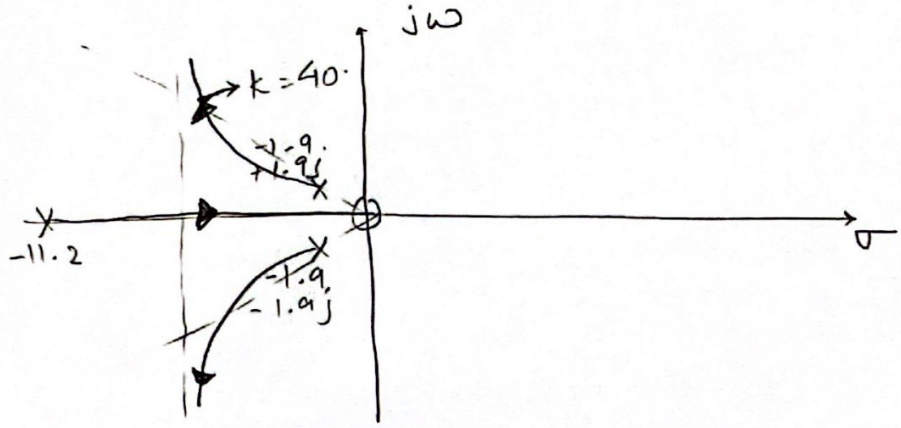
$\Rightarrow$

$$s(s+5)(s+10) + 81 + 81 K_{D_1} s = 0$$

poles : $= -11.2,\ -1.9 \pm j1.9$

New $G(s)H(s)$ for RL w.r.t K_{D_1} $= K_{D_1} \cdot \dfrac{81 \cdot s}{s(s+5)(s+10) + 81}$

$$= K \cdot \frac{s}{s(s+5)(s+10) + 81}$$



$K_{D1} = 1 + 0.49s$

$$t_r = \frac{0.8 + (2.5)(0.707)}{10} \simeq 0.25$$

$$t_s = \frac{3.2}{7.5} = 0.5$$

meets spec!

$$\Rightarrow K(s) = \left(1.63 + \frac{0.163}{s}\right)(1 + 0.49s)$$

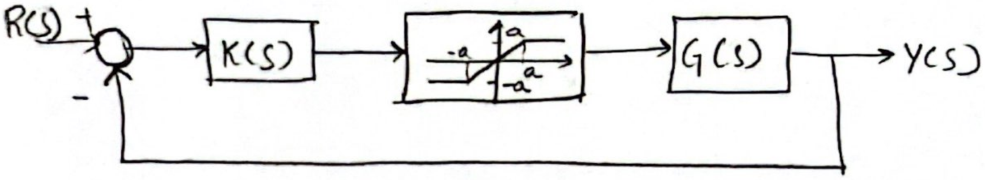
$$\Rightarrow K_P = K_2 + K_D,\ K_{I_2} = 1.71$$

$$K_D = (0.49)(0.163) = 0.8$$

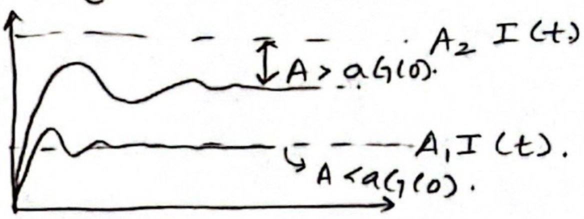
$$K_{I_2} = 0.163$$

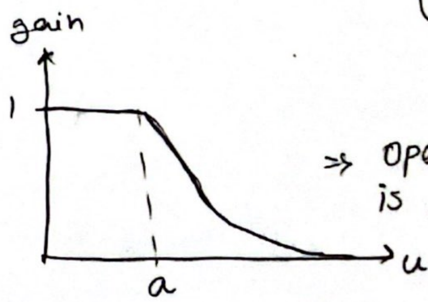
$$G_{precomp} = \frac{81}{50(1 + 0.49s)}$$

Effects of saturating actuator



Largest $y(t) = a \cdot G(0)$ If i/p $r(t) = A\,I(t)$. and $\begin{cases} A < aG(0) \Rightarrow \text{s.s is OK.} \\ A > aG(0) \\ \text{S.S is not OK} \\ \text{Truncation.} \end{cases}$

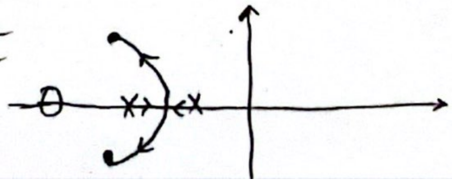




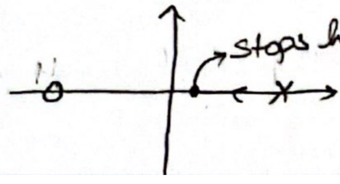
$$\text{gain:} = \begin{cases} 1 & \text{if } |u| < a \\ \dfrac{a}{u} & \text{if } |u| > a \end{cases} \Rightarrow$$

$\Rightarrow$ open loop gain is limited!

$\Rightarrow$ Root locus does not terminate in OL zeroes.



or

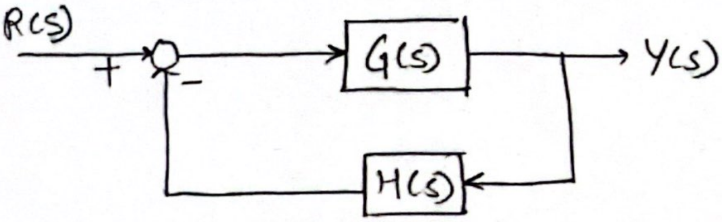


stops here $\Rightarrow$ cannot stabilize this system!

Analysis of Control systems in the Frequency Domain.

Steady state specs

- K_p, K_v, K_a.



Dynamic specs

- Resonance peak
- Bandwidth.
- Gain and Phase Margins.

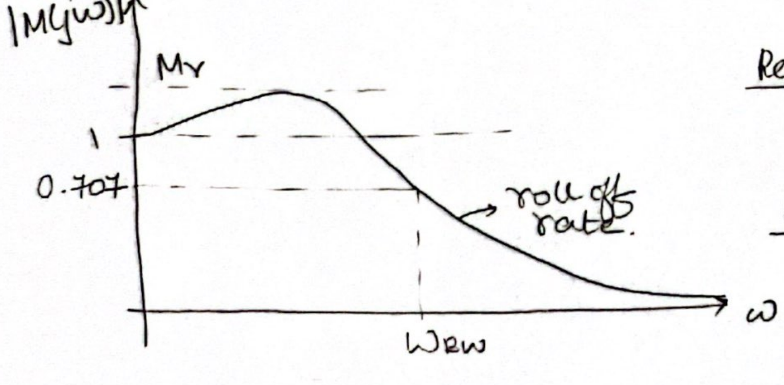
$$\frac{Y(s)}{R(s)} = M(s) = \frac{G(s)}{1+G(s)H(s)}$$

$$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)H(j\omega)}$$

$$= |M(j\omega)|\, e^{j\angle M(j\omega)}$$

$$|M(j\omega)| = \frac{|G(j\omega)|}{|1+G(j\omega)H(j\omega)|}$$

$$\angle M(j\omega) = \angle G(j\omega) - \angle G(j\omega)H(j\omega)$$



Resonance peak

$M_r = \max |M(j\omega)|$; $\omega \in [0,\infty]$

Resonance frequency

$\omega_r = \arg.\max |M(j\omega)|$; $\omega \in [0,\infty]$

Bandwidth (or 3dB BW)

BW = Frequency where $|M(j\omega)|$ drops to 70.7%.

Roll off rate

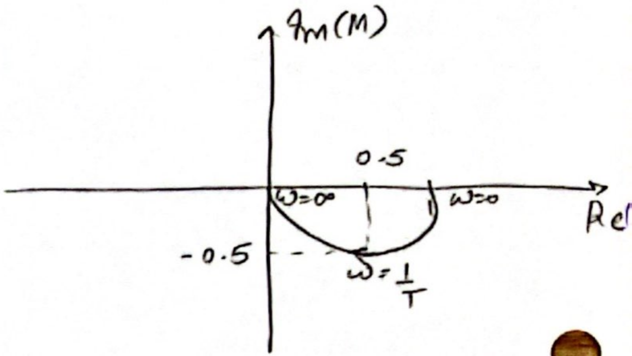
How fast $|M|$ drops off (-20dB/dec ... etc.)

Gain and Phase Margin

> Done before.

---

1) First order systems

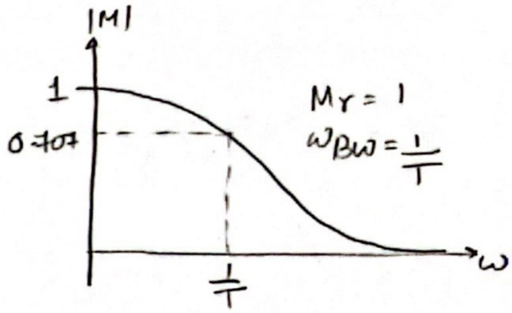


$$M(s) = \frac{1}{Ts+1}$$

$$M(j\omega) = \frac{1}{Tj\omega+1} = \frac{1-Tj\omega}{1+T^2\omega^2}$$

$$|M(j\omega)| = \frac{1}{\sqrt{1+T^2\omega^2}}$$

$$\angle M(j\omega) = \tan^{-1}(-T\omega) = -\tan^{-1}(T\omega)$$



$$\boxed{\begin{array}{l} M_r = 1 \\ \omega_{BW} = \frac{1}{T} \end{array}}$$

$$M(s) = \frac{BW}{s+BW}$$

2) Second order systems

$$M(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad 0<\zeta<1$$

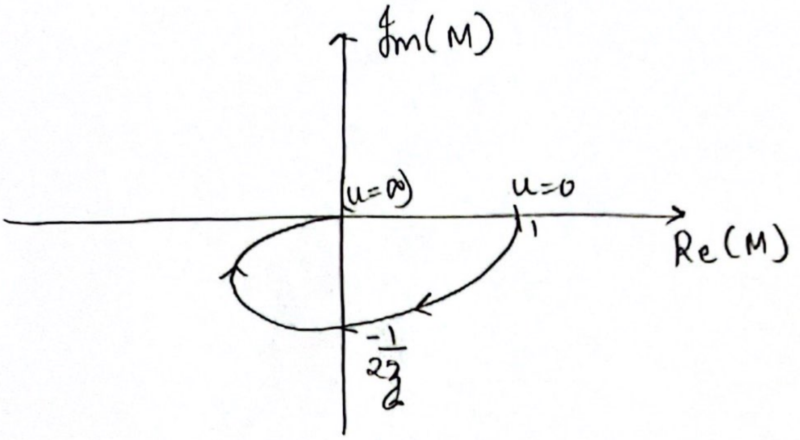
$$M(j\omega) = \frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n j\omega+\omega_n^2}$$

$$= \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2+2j\left(\frac{\omega}{\omega_n}\right)\zeta+1}$$

$\longrightarrow$ let $\omega/\omega_n = u$ dimentionless

$$M(j\omega) = \frac{1}{(1-u^2) + j2u\zeta}$$

$$M(ju) = \frac{(1-u^2) - j2u\zeta}{(1-u^2)^2 + (2u\zeta)^2} \quad \text{—— (1)}$$



$$|M(ju)| = \frac{1}{\left[(1-u^2)^2 + 4u^2\zeta^2\right]^{1/2}}$$

$$\angle M(ju) = \tan^{-1}\left(\frac{-2u\zeta}{1-u^2}\right)$$

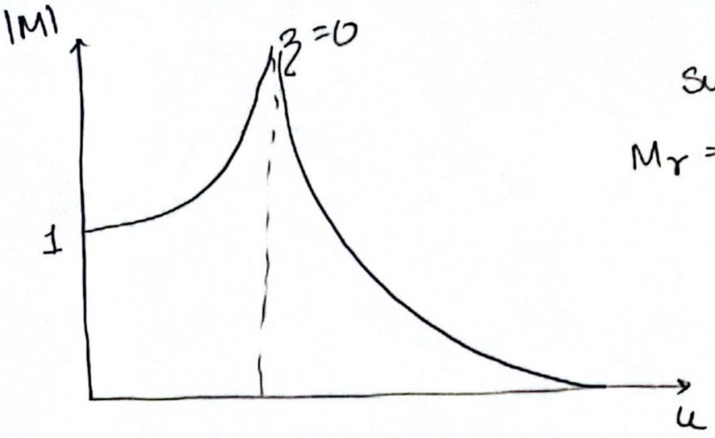
$$= -\tan^{-1}\left[\frac{2u\zeta}{1-u^2}\right]$$

$$\frac{d}{du}|M(ju)| = -\frac{1}{2}\left[(1-u^2)^2 + (2\zeta u)^2\right]^{-3/2} \cdot \left[4u^3 - 4u + 8u\zeta^2\right]$$

$= 0$ set to zero & solve.

$$\Rightarrow \boxed{u_{res} = \begin{cases} \sqrt{1-2\zeta^2} & 2\zeta^2 < 1 \\ 0 & 2\zeta^2 > 1 \end{cases}}$$

$$\Rightarrow \boxed{\omega_{res} = \begin{cases} \omega_n\sqrt{1-2\zeta^2} & ; 0<\zeta<\frac{1}{\sqrt{2}} \\ 0 & ; 1>\zeta>\frac{1}{\sqrt{2}} \end{cases}}$$

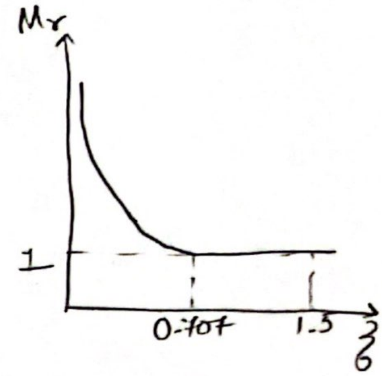


Sub u_{res} in 1.

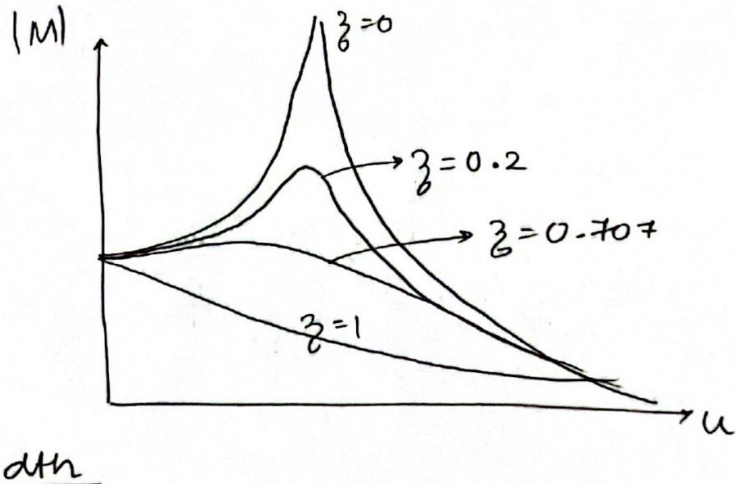
$M_r = |M(\omega_r)|$

$$\boxed{M_r = \begin{cases} \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}, & 0<\zeta<0.707 \\ 1, & 1>\zeta>0.707 \end{cases}}$$

Depends only on ζ



$M_r \in [1, 1.3] \rightarrow$ common.

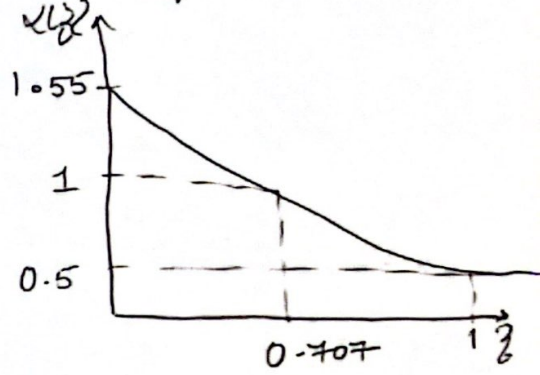


Bandwidth

$$|M| = \frac{1}{\left[(1-u^2) + (2\zeta u)^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\omega_{BW} = \omega_n \underbrace{\left[(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{1/2}}_{\alpha(\zeta)}}$$

$\Rightarrow \boxed{\omega_{BW} = \omega_n \cdot \alpha(\zeta)}$ Mostly a function of ω_n.



> Select ζ to meet M_r specs and then ω_n to meet ω_{BW} specs.

Conclusion

M_r depends only on ζ

$\zeta \uparrow \Rightarrow M_r \downarrow \Rightarrow M_r \sim$ overshoot

> $BW = \alpha(\zeta) \cdot \omega_n$.

$\Rightarrow \omega_n \uparrow \Rightarrow BW \uparrow \Rightarrow BW \sim$ temporal characteristics.

$\Rightarrow BW \uparrow \Rightarrow t_r, t_d, t_s \downarrow \downarrow \downarrow$.

Additional poles and zeroes

Add poles $\xrightarrow{\text{usually}}$ $BW \downarrow$ & $M_r \uparrow$

Add zeroes $\xrightarrow{\text{usually}}$ $BW \uparrow$
M_r may $\uparrow$ or $\downarrow$.

Stability Analysis in Freq. Domain

$$C.L.TF = M(s) = \frac{G(s)}{1+G(s)H(s)}.$$

$$L.G = G(s)H(s) = \frac{n(s)}{d(s)} \longrightarrow d(s) = 0 \text{ gives OL poles.}$$

Return difference

$$\Delta(s) = 1 + G(s)H(s)$$

$$= 1 + \frac{n(s)}{d(s)}$$

$$= \frac{d(s) + n(s)}{d(s)}$$

$d(s) = 0 \Rightarrow$ poles of $\Delta(s)$ = open loop poles.

$d(s) + n(s) = 0 \Rightarrow$ zeroes of $\Delta(s)$ = closed loop poles.

> We care about zeroes of $\Delta(s)$ for closed loop stability.

Cauchy's argument principle.

Given a transfer function $TF(s) = fn(s)$. We can form a "mapping" from s plane to 'W' plane using this transfer function.

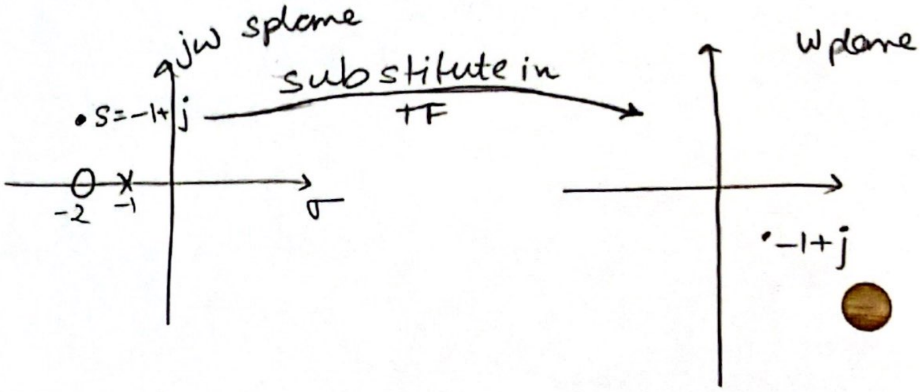
Eg: $TF = \dfrac{s+2}{s+1}$

$s = -1+j$

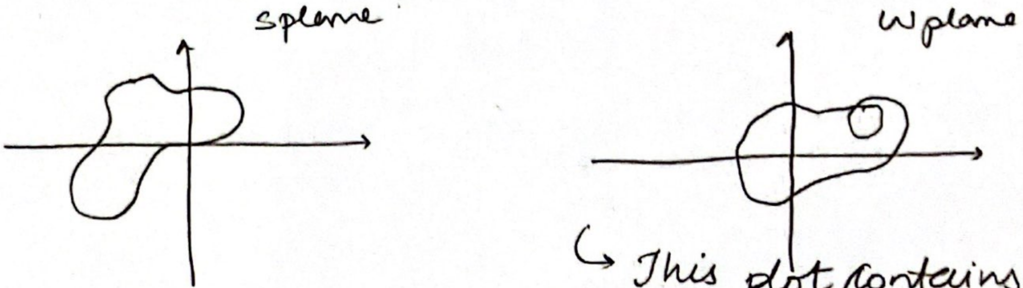
$\Rightarrow TF = \dfrac{-1+j+2}{-1+j+1}$

$= \dfrac{1+j}{j}$

$TF = 1-j$

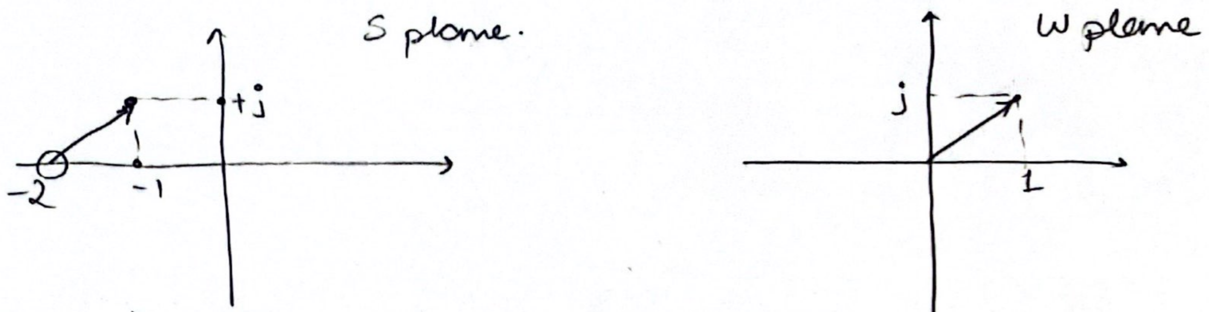


→ If we map a 'contour' from s plane to 'W' plane it has some interesting properties based on the number of poles and zeroes it encloses in splane.



↳ This plot contains the phase & magnitude information of the poles & zeroes of the TF used for this mapping.

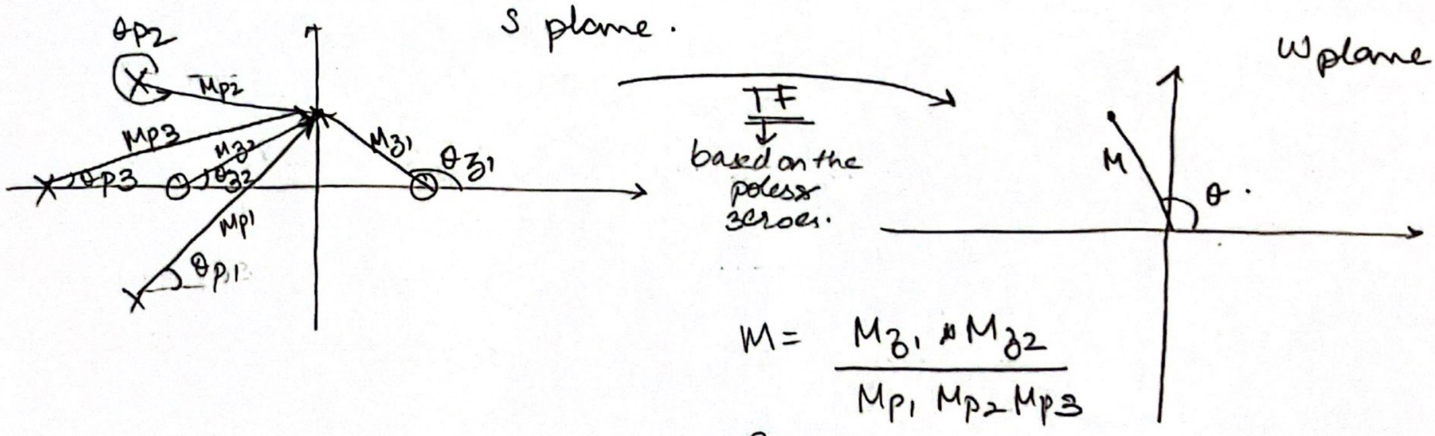
Eg: TF = $\frac{S+2}{1}$ eg: $S = -1+j \Rightarrow TF(S) = 1+j$



phasor b/w zero & pt. in splane is same as phasor between origin & pt in w plane.

> What about multiple poles & zeroes?

Steps
1) Pick your point in s plane.
2) Draw phasors
3) For magnitude multiply all the zero phasor magnitudes & divide all pole phasor magnitudes.
4) For phase add phase of zeroes & subtract phase of poles.



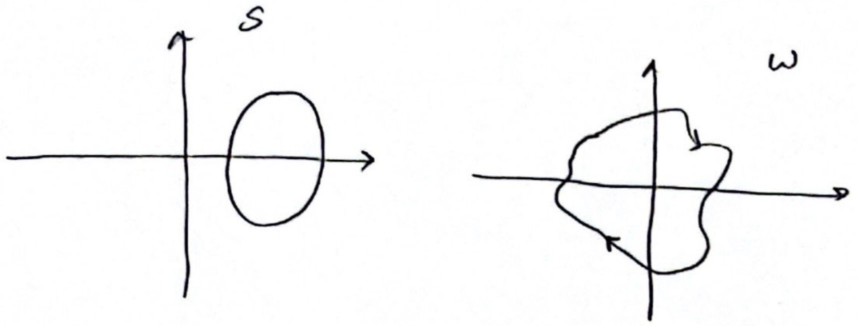
$$M = \frac{M_{Z1} * M_{Z2}}{M_{P1}\, M_{P2}\, M_{P3}}$$

$$\theta = \theta_{Z1} + \theta_{Z2} - \theta_{P1} - \theta_{P2} - \theta_{P3}$$

→ If contour of points in splane circles a pole, we need to subtract 360° of phase ⇒ contour encircles origin in counterclockwise direction. 2 poles in contour ⇒ plot rotates around origin twice!

→ Cauchy's argument principle allows us to tell how many poles or zeroes exist inside a contour by looking at the no. of times the w plane plot encircles the origin and in which direction.

Eg:

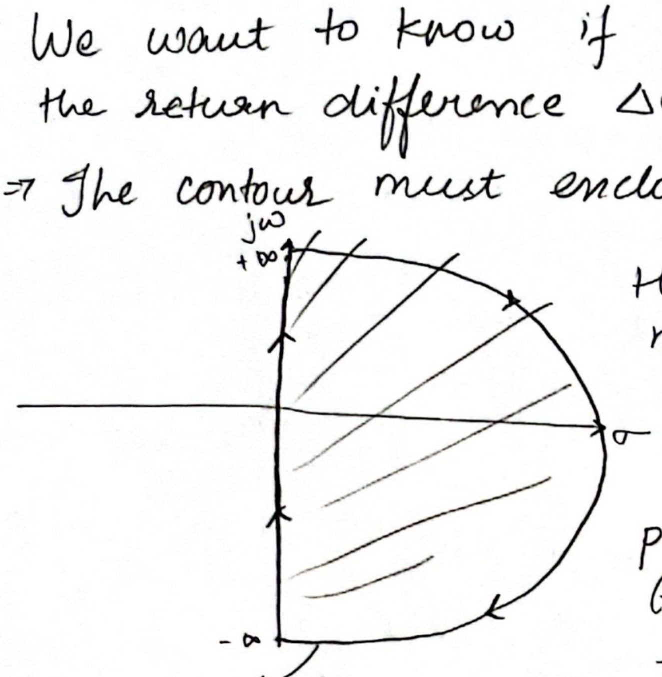


⇒ no. of zeroes > no. of poles by 1 in the contour.

⇒ Need to know no. of poles to determine no. of zeroes within the contour (after looking at w plane plot).

→ We want to know if there any zeroes in ORHP for the return difference $\Delta(s) = 1 + G(s)H(s)$.

⇒ The contour must enclose the entire right half plane.



Aka Nyquist contour or Nyquist path. Ts

However this contour in s plane maps to the Nyquist plot in the W plane if the mapping TF was GH. But we want to see the plot for the mapping of 1+GH & not GH that Nyquist gives. So we move the origin to (−1,0)! This let's us look at the no. of encirclements of (−1,0) and determine the no. of zeroes of 1+GH in the ORHP, which gives stability of CL system.

Steps

1) Nyquist plot of GH, (open loop)
2) Count encirclements of -1, & note direction. $\circlearrowright \Rightarrow$ zeroes. $\circlearrowleft \Rightarrow$ poles.
3) Figure out how many poles or zeroes are in ORHP.

BUT

To know how many zeroes are in ORHP we need to know how many poles of $\Delta(s)$ are in ORHP. What do the poles of $\Delta(s)$ represent? OL poles!!

⇒ "For a system with no OL poles in ORHP, if the Nyquist plot encircles $(-1,0)$ atleast once in the clockwise direction, the CL system is unstable"

↳ Nyquist Stability Criteria!

Therefore,

$$Z = \overset{\text{no. of clockwise encirclements of } -1}{N} + P$$

Z: no. of zeroes in ORHP of $1+GH$ or no. of CL poles in ORHP

P: no. of open loop RHP poles. also no. of poles of $\Delta(s)$ in ORHP

⇒ To get $Z=0$ we need $\boxed{N=-P}$

⇒ If the system has P open loop poles in RHP and we want the CL system to be stable, we need the Nyquist plot to encircle $(-1,0)$ (exactly?) P times!! ⇒ Unstable CL system

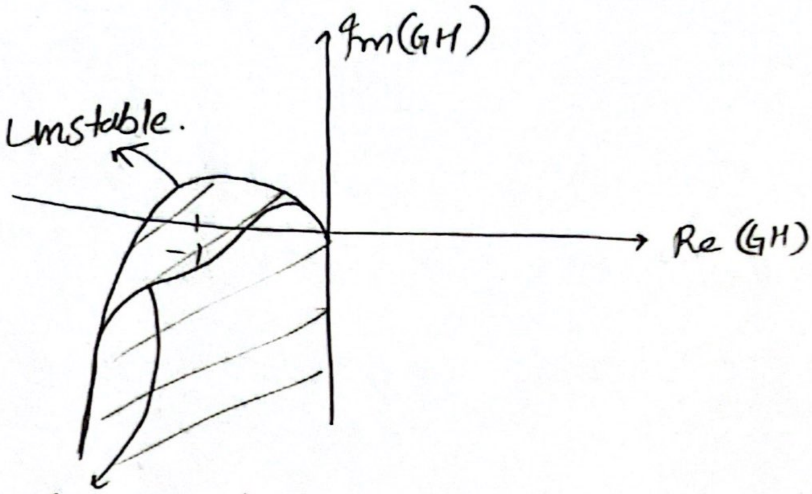
> Nyquist plot + open loop poles locations are both needed to give full information of CL stability.

> Nyquist > Bode => can stabilize unstable openloop systems with Nyquist but not with Bode.

<u>NSC</u>

↳ When system has no poles & zeroes of OL gain in ORHP.

Closed loop system is asymptotically stable iff (-1, 0) is not encircled by the Nyquist plot of open loop system.



<u>Gain & Phase margins.</u>

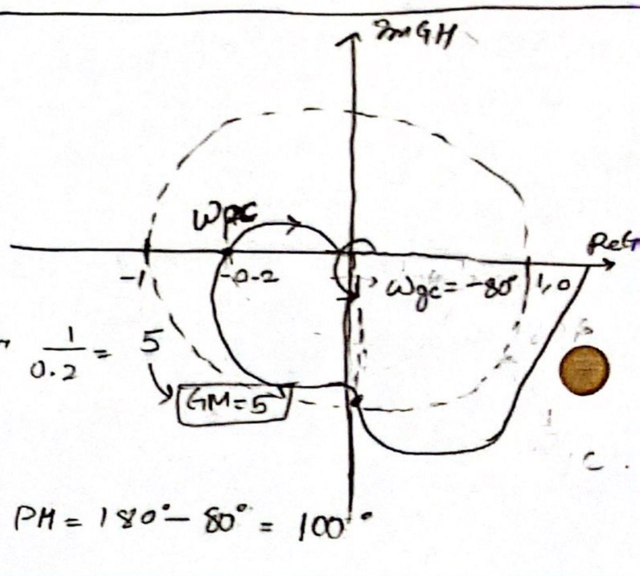
$GM = 20 \log K_{additional}$.

where, $K_{additional} = \dfrac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}$

$PM = \phi$

$= 180° + \angle G(j\omega_{gc})H(j\omega_{gc})$

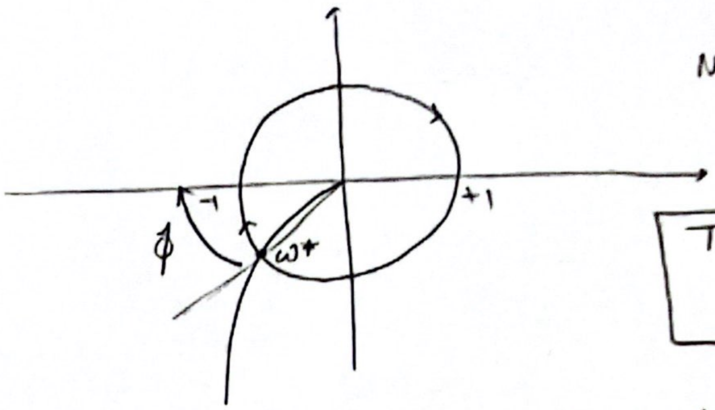
=> For stability $\boxed{\omega_{gc} < \omega_{pc}}$



Stability analysis in systems with Time Delay

$$G(s)\,H(s) = G_1(s)\,H_1(s)\,e^{-T_d s}$$

$$\Delta(s) = 1 + G_1(s)\,H_1(s)\,e^{-T_d s}$$



Nyquist plot of time delay.

$$\boxed{T_d = \frac{\phi}{\omega^*}}$$

$T_d \rightarrow$ maximum time delay the system cantolerate at ω^*

$\phi \rightarrow$ Phase shift at ω^* that makes system unstable.

Control systems - Poles and zeroes

L1

Let $G(s) = \dfrac{Y(s)}{X(s)}$ where $G(s) = \mathcal{L}\{g(t)\}$

$g(t) = y(t) * x(t)$

$G(s) = Y(s) . X(s)$.

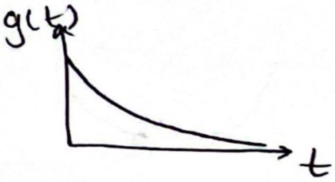
Poles :- s values at which $s \Rightarrow \infty$

Zeroes :- s values at which $s \Rightarrow 0$

Stability from poles and time domain equivalence

→ Let $G(s) = \dfrac{1}{s+2}$ at $s=-2$ we have a pole.

$\Rightarrow g(t) = e^{-2t}u(t) \longrightarrow$



STABLE

→ Let $G(s) = \dfrac{s+2}{s^2+5s} \Rightarrow$ at $s = 0, -5$ where poles exist

$g(t) =$ decaying sinusoidal exponential. [Recall Laplace transforms].

→ ∴ poles $< 0 \Rightarrow$ stable.

poles $= 0 \Rightarrow$ marginally stable.

poles $> 0 \Rightarrow$ unstable.

Control systems - Frequency response

→ When discussing frequency response we only care about $s = j\omega$ term.

→ We can represent the transfer function frequency response on the Re-Im plane of $G(s)$.

Let $G(s) = \frac{1}{s+1}$

$G(j\omega) = \frac{1}{1+j\omega}$

① At $\omega = 0 \Rightarrow s = 0 \Rightarrow G = \frac{1}{1}$

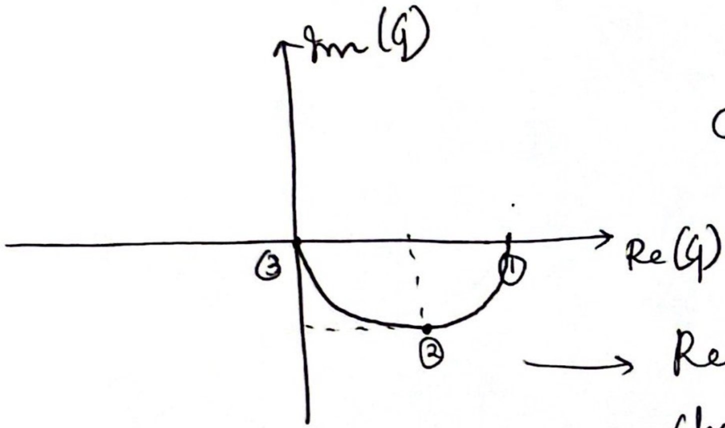
$\therefore \angle G = 0°$ & $|G| = 1$

② At $\omega = 1 \Rightarrow s = j \Rightarrow G = \frac{1}{1+j}$
$= \frac{1}{2} - j\frac{1}{2}$

$\angle G = -45°$ & $|G| = \frac{1}{\sqrt{2}}$

③ At $\omega = \infty \Rightarrow s = j\infty \Rightarrow G = \frac{1}{1+j\infty}$

$\angle G = -90°$ $|G| = 0$



→ Represents how transfer function changes in both Mag & Phase over all frequencies but here we can't visualise the frequency so we use Bode plots.

Control systems - Bode plots

Linear time invariant systems

→ Frequency of input and output is ALWAYS same.

→ For sinusoidal functions only amplitude and phase can change.

$$B \sin \omega t \longrightarrow \boxed{LTI} \longrightarrow A \sin(\omega t + \phi)$$

magnitude & phase change

→ We want to see how magnitude & phase change over the whole frequency spectrum.

∴ Bode plots.

Gain dB vs log scale f.

power gain

Phase vs. log f.

(In degrees) ⇒ $2\pi = 360°$

→ Why use $20 \log \frac{V_o}{V_i}$?

Bell telephone labs was trying to quantify loss in audio levels in early 20th century & used log scales since the ear perceives sound on a log scale.

$10 \log_{10} \Delta power$ was approx. the smallest power attenuation detectable.

To honour Alexander Graham bell they called it deci-Bell (since Bell was too large a unit we use deciBell)

→ Bode plot is for all frequencies. So sweep across all the frequencies or use Laplace transform.

→ When multiple systems are cascaded, body plots provide an additive property for both mag. & phase.

→ $X(s) \longrightarrow \boxed{H(S)} \longrightarrow Y(s)$

Solve for $H(S) = \dfrac{Y(S)}{X(S)}$

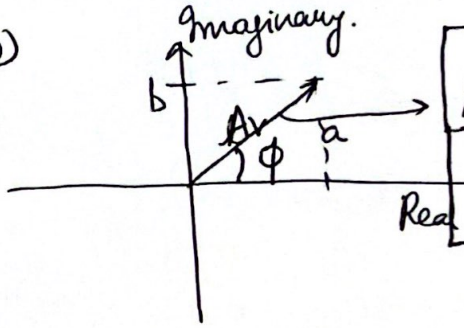
Here $S = \sigma + j\omega$

→ When talking about frequency response of a system we only care about steady state response where the exponential term 'σ' is 0.

→ <u>Steady state $S = j\omega$</u>

$\therefore H(j\omega) = a + jb.$

→ Plotting $H(j\omega)$



$\boxed{Av = \sqrt{a^2 + b^2}}$

$\boxed{\phi = \arctan \frac{b}{a}}$

→ Only Imaginary part changes with ω.

→ We use a system of poles and zero to draw bode plots by hand.

- $\rightarrow$ Transfer function $= H(s)$
- $\rightarrow$ S.S freq. response $= H(j\omega)$
- $\rightarrow$ $H(j\omega) =$ real + imaginary.
- $\rightarrow$ gain $= |H(j\omega)|$
- $\rightarrow$ Phase $= \angle H(j\omega)$

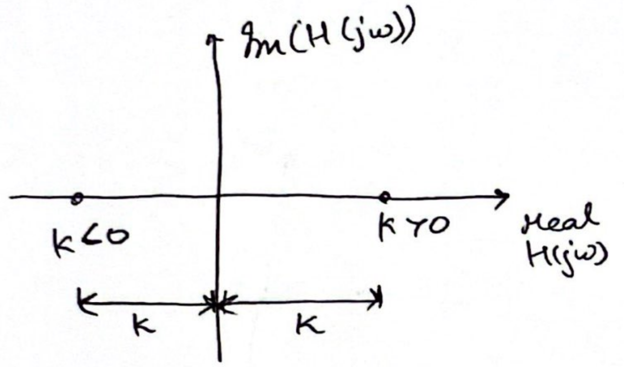
Example

$$H(s) = K \rightarrow \text{constant}$$

$$\xrightarrow[u(s)]{\text{input}} \boxed{K} \xrightarrow[y(s)]{\text{output}}$$

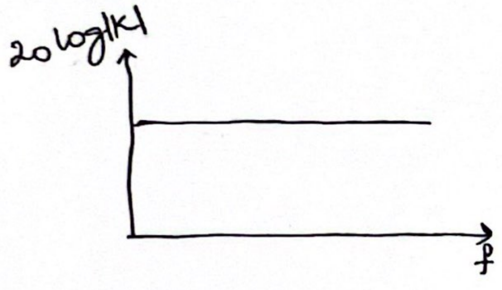
$$|H(s)| = K$$

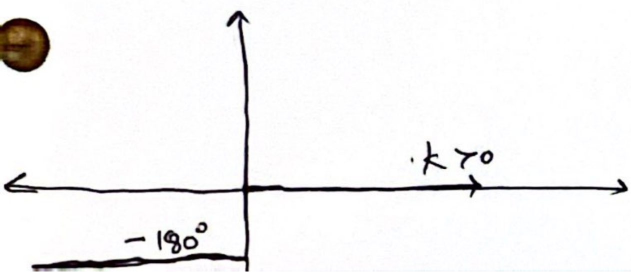
$$\arg(H(s)) = \tan^{-1}\left(\frac{0}{K}\right)$$



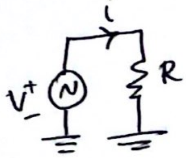
gain is always K

Phase can be 0 or $-180°$
(K>0) (K<0)





Electrical example



$V(t) = i(t)\,R.$

$V(s) = i(s)\,R.$

$$\therefore \frac{i(s)}{V(s)} = H(s) = \frac{1}{R} \rightarrow |H(j\omega)| = \frac{1}{R}$$

$$\angle H(j\omega) = 0^\circ$$

Poles and zeroes

→ A pole is a value of 's' for which $H(s) \rightarrow \infty$

→ A zero is a value of 'S' for which $H(s) \rightarrow 0$

→ A pole at origin $\Rightarrow \frac{1}{s}$

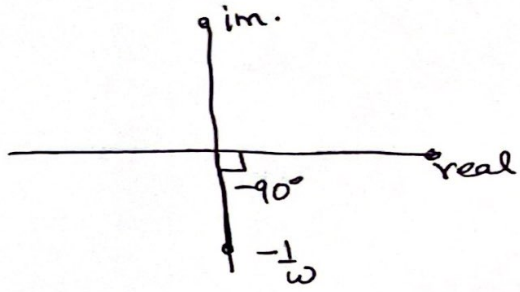
→ A zero at the origin $\Rightarrow$ s.

Pole at origin

→ Notice it is laplace transform of $\int$.

$H(s) = \frac{1}{s}$

$H(j\omega) = \frac{1}{j\omega} = -\frac{j}{\omega}$



$\Rightarrow$ real $= 0$

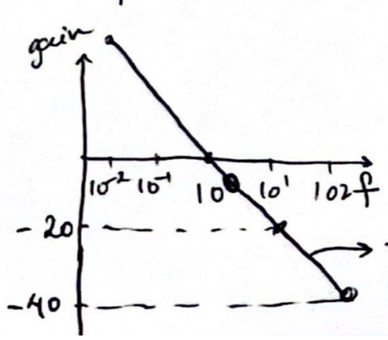
im $= -\frac{1}{\omega}$

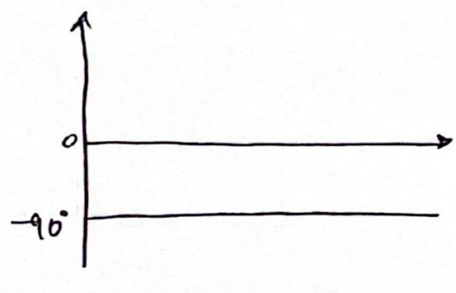
gain $= \frac{1}{\omega}$

phase $= -90^\circ$

Notice

$\sin\omega t \longrightarrow \boxed{\int dt} \longrightarrow \underbrace{\frac{-1}{\omega}}_{\text{gain}} \overbrace{\cos\omega t}^{\text{phase}}$





Real non zero pole

$G(s) = \frac{1}{s+1}$ pole at $s = -1$

Let $G(j\omega) = \frac{1}{j\omega + 1}$

at low frequencies $G(j\omega) \simeq \frac{1}{0+1} = 1$

$\Rightarrow \angle G = 0° \; ; \; |G| = 1 \rightarrow 0dB$

at high frequencies $G(j\omega) \cong \frac{1}{j\omega} = \frac{-j}{\omega}$

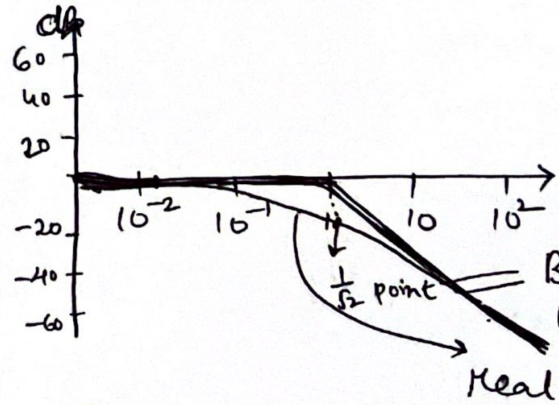
$\Rightarrow \angle G = -90° \; ; \; |G| = \frac{1}{\omega}$

Notice this is -20 dB/dec like a zero pole.

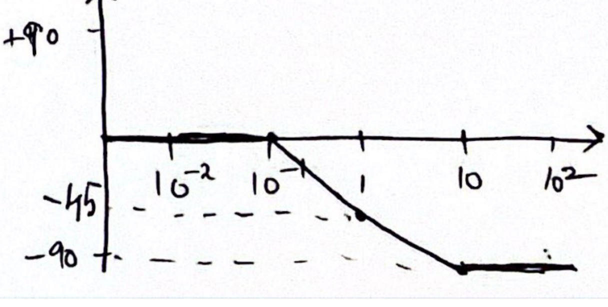
at $\omega = 1$ $G(j\omega) = \frac{1}{1+j}$

$|G| = \frac{1}{\sqrt{2}} \cdot \angle G = -45°$

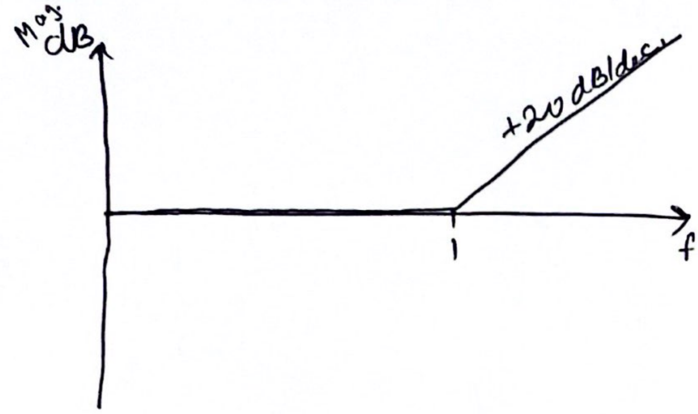
Approximating Bode plot we see asymptotes at $\omega = 0, \infty$

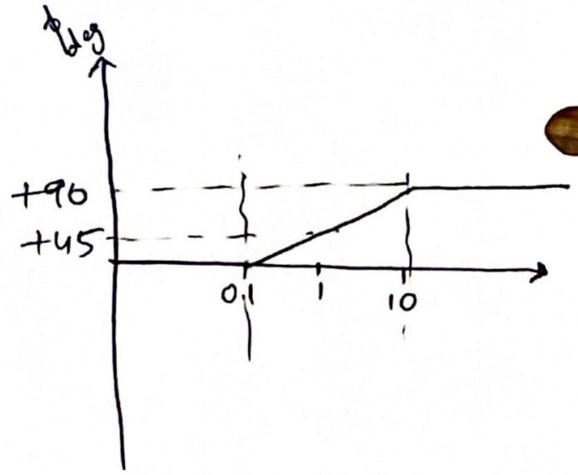


Because $\frac{1}{\sqrt{2}}$ pt. coincides with pole $|s_p|$ we can call it our ω_{-3dB} → freq when $20\log A_v$ → $\frac{1}{\sqrt{2}}$



Real non zero - zero





NOTE

→ Therefore for finding the point at which magnitude plot turns we use $|S_z|$ & $|S_p|$
To find 0.1× pt. at which phase plot turns and 10× pt. at which it turns we use $|S_z|$, [illegible]).
This is just the consequence of how poles and zeroes affect the log plots and does not represent how transfer function goes to ∞ because these points are NOT the poles. we just found these points from the poles.

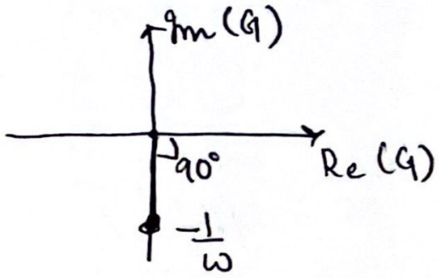
→ To visualize the T.F going to ∞ we also need to consider the σ term. Here we are ignoring it and just looking at ω. Not ignoring σ & looking in time domain we would see the T.F → ∞/0 at poles & zeroes.

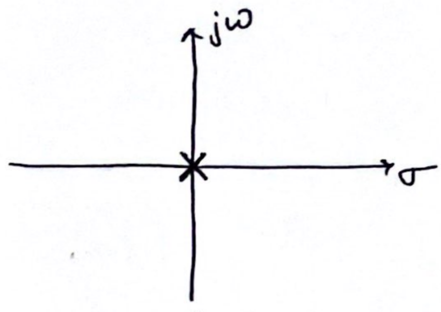
Pole Zero Plot ⟷ Bode plot

Case ① :- Pole at origin → Integrator

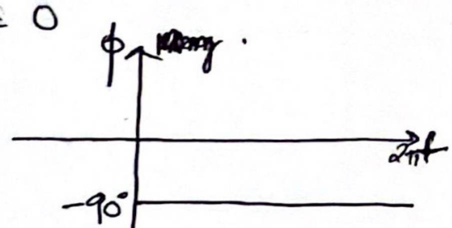
$G(s) = \frac{1}{s} \Rightarrow G(j\omega) = \frac{1}{j\omega}$

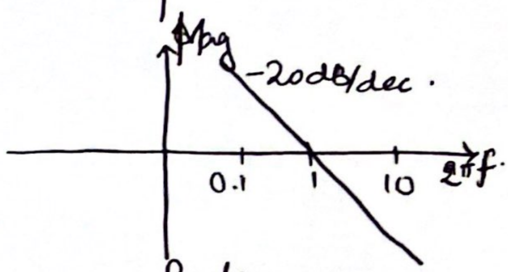
$s_p = 0$





pole zero plot



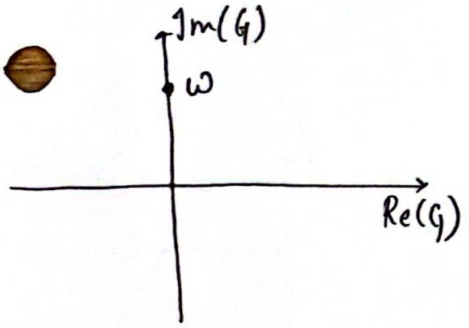


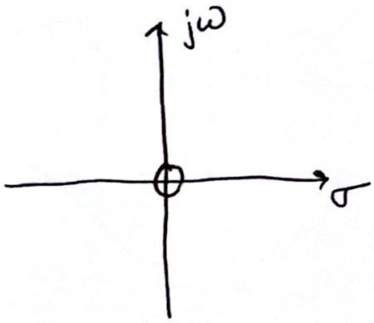
Bode

Case ② Zero at origin → Differentiator.

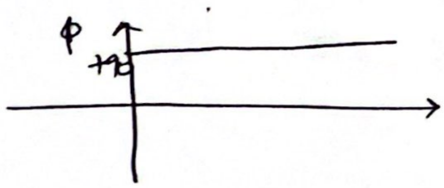
$G(s) = s \Rightarrow G(j\omega) = j\omega$

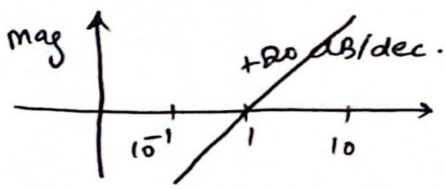
$s_z = 0$





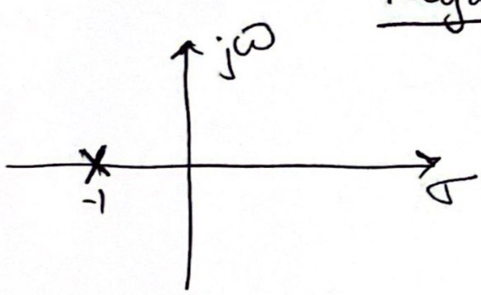
pole zero plot

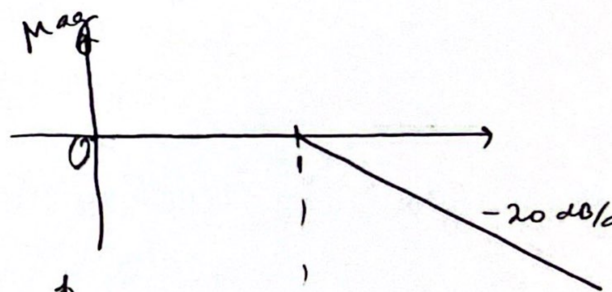


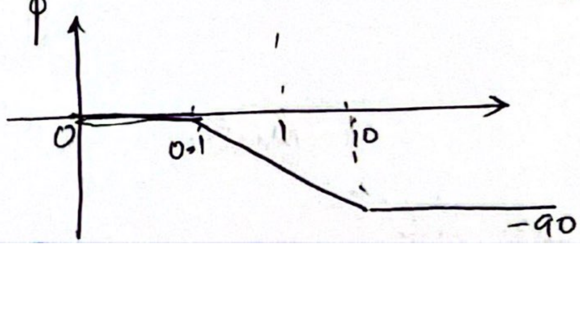


Case ③ Real pole negative

$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{1+j\omega}$

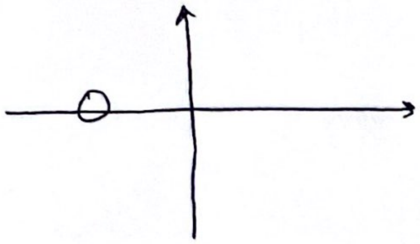


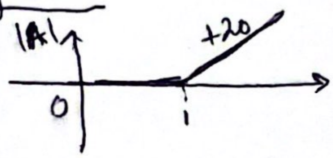


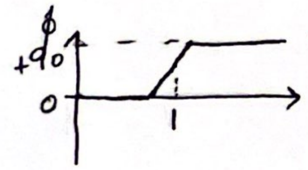


Case ④ Real zero negative

LHP

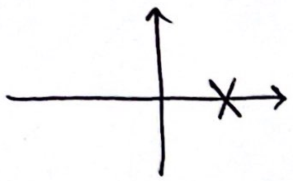


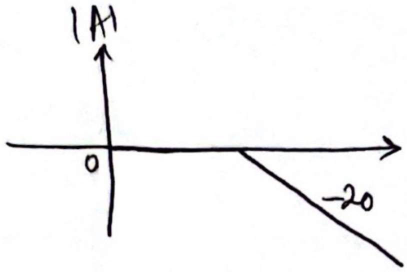


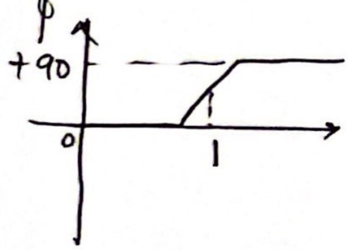


Exactly opposite of a real negative pole.

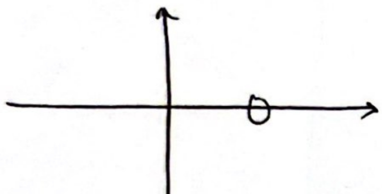
Case ⑤ Real positive pole (Phase is flipped)

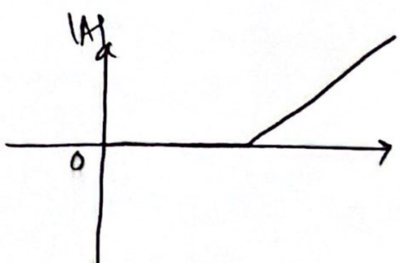


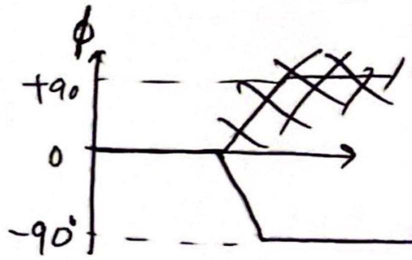




Case ⑥ Real positive zero







Rules for drawing Bode Plot

Step① Rewrite T.F such that numerator & denominator have 0 constant value. Even $H(s) = M \cdot \frac{Ax^2+Bx+1}{\alpha x^2+\beta x+1}$ "

for values inside factors of terms. → $M \cdot \frac{(Ax+1)(Bx+1)}{(Cx+1)(Dx+1)}$ "

Step② Find multiplier ('m') & (S_P & S_Z) poles and zeroes.

Step③ Draw Bode plots for each pole & zero. and also for multiplier M. Mark the pointers from S_Z, S_P

Hint:- This has 0 phase (+ve) or 180° phase (-ve)

Step④ Add up all the plots since in log scale multiplication is addition

∴ Let's look at an example

$$G(s) = \frac{s+1}{(s+0.1)(s+10)}$$

$S_z = -1$

$S_{p1} = -0.1$

$S_{p2} = -10$

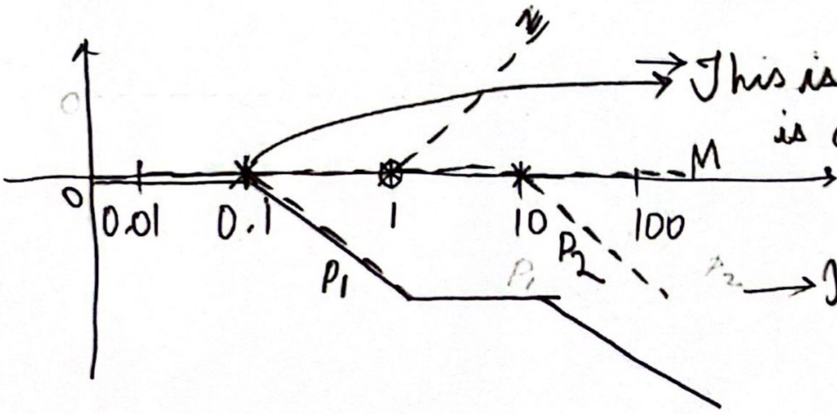
Step ①

$$G(s) = \frac{s+1}{\left(s+\frac{1}{10}\right)(s+10)} = \frac{s+1}{\frac{\cancel{10}}{\cancel{10}}(s\times 10+1)\left(\frac{s}{10}+1\right)}$$

$$= \frac{s+1}{(10s+1)\left(\frac{s}{10}+1\right)}$$

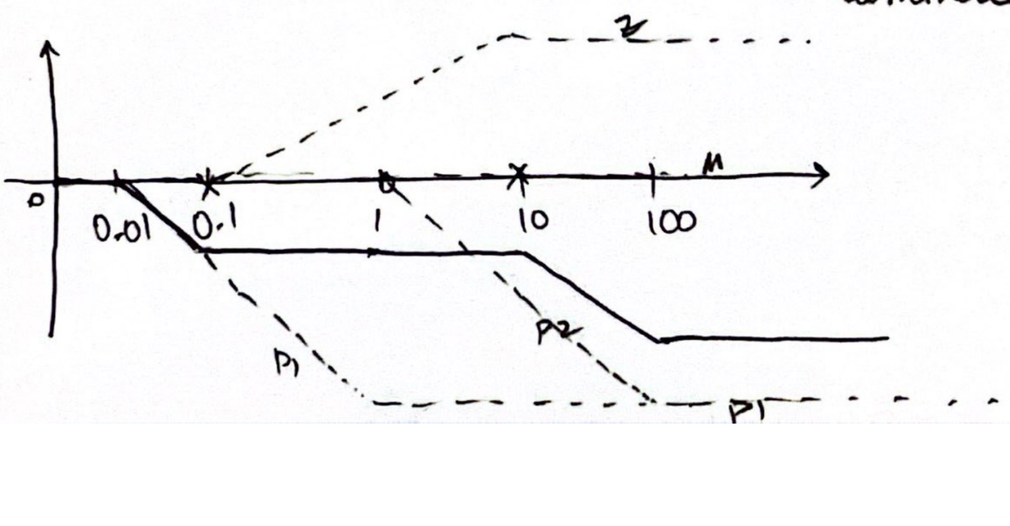
$$= \frac{\frac{s}{s_z}+1}{\left(\frac{s}{s_{p_1}}+1\right)\left(\frac{s}{s_{p_2}}+1\right)}$$

Step ② $M=1$, $S_z=-1$, $S_{p_1}=-0.1$, $S_{p_2}=-10$



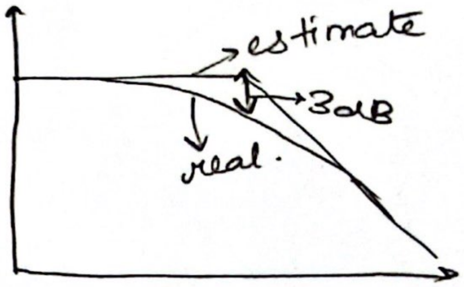
→ This is the small pole and ∴ is dominating the freq response.

→ In case of rising edge, the larger pole would dominate.

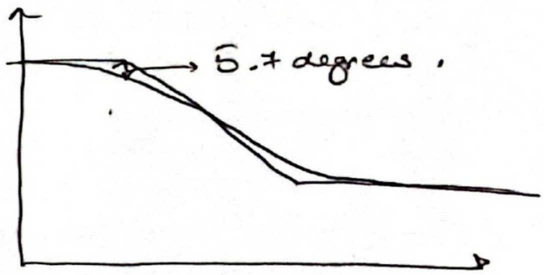


Trivia

- decade is a factor of 10. Whereas, octave is a factor of 2. It comes from western music where freq. of each octave (do re mi fa so la ti do) is double that of the previous octave → (From do to do)
- Division into 7 notes is arbitrary to western music.
- 20 dB per decade = 6 dB per octave.
- Magnitude plot max error is 3 dB



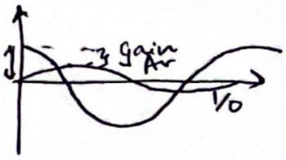
- Phase plot max error is 5.7 degrees.



Gain and Phase Margins

Review :-

I/p → LTI → G(s) → O/p



→ Gain & phase depend on f.
∴ Bode plots.

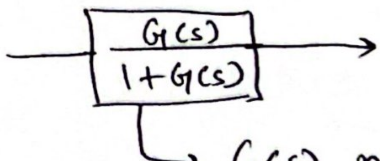
→ Gain margin & phase margin

The extra gain we can use before the system oscillates and becomes unstable.

→ Less margin = less stable since small variants in system can cause instability.

→ But what makes a system unstable?

⇒

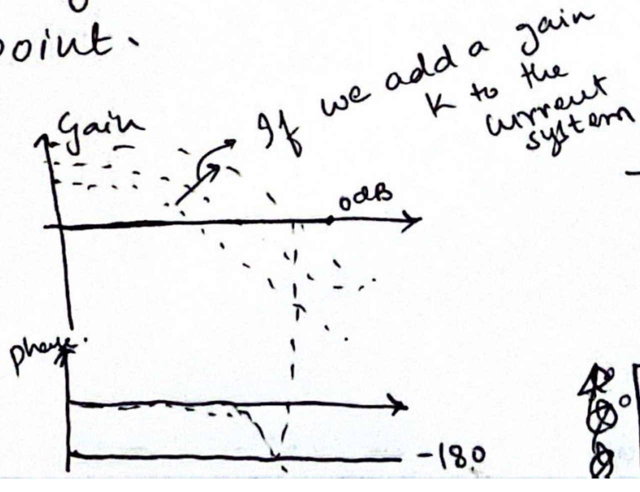


→ G(s) must become −1 for gain of ∞.

If any one frequency hits −1, the whole system becomes unstable.

⇒ Gain = 1 → (0dB)
phase = −180°

→ Margin tells us how far we really are from the −1 point.

→ 

→ This moves the "cross over" point of the system closer to the −1 point in gain plot. ⇒ RISKY. Becomes less stable.

Margin is how much we can move around the gain & phase before

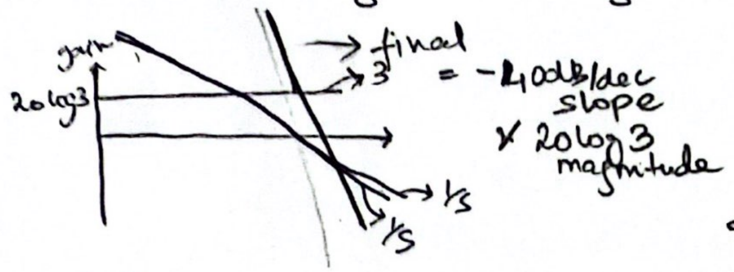
→ Since we use a log scale we can just add separate plots drawn for each of the multiplicand:

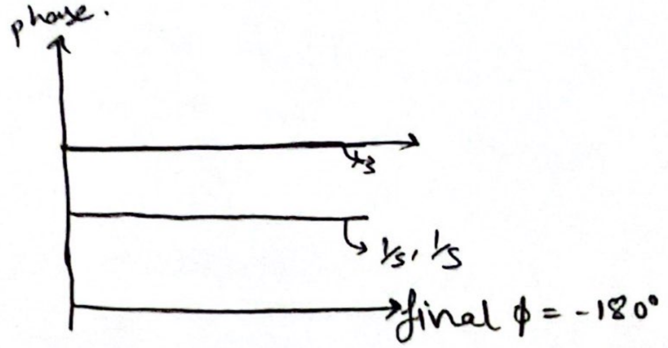
Example

$H(s) = \frac{5}{s} \Rightarrow 5 \cdot \frac{1}{s}$; $H(s) = \frac{3}{s^2} = 3 \cdot \frac{1}{s} \cdot \frac{1}{s}$

↓

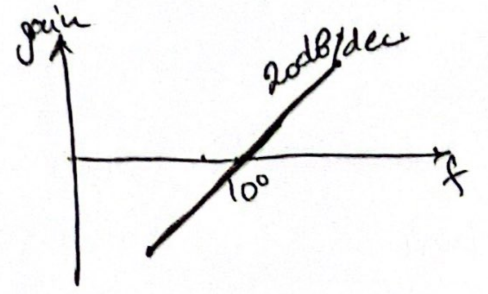
response of 3
+
response of $\frac{1}{s}$
+
response of $\frac{1}{s}$

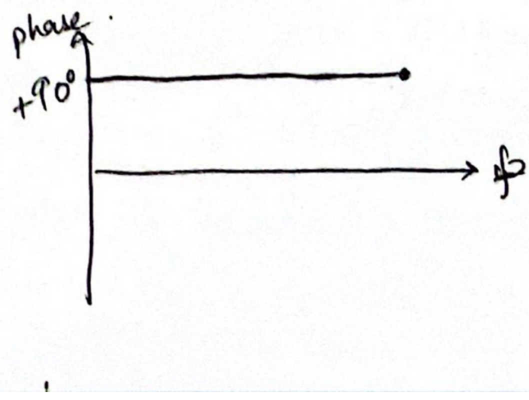




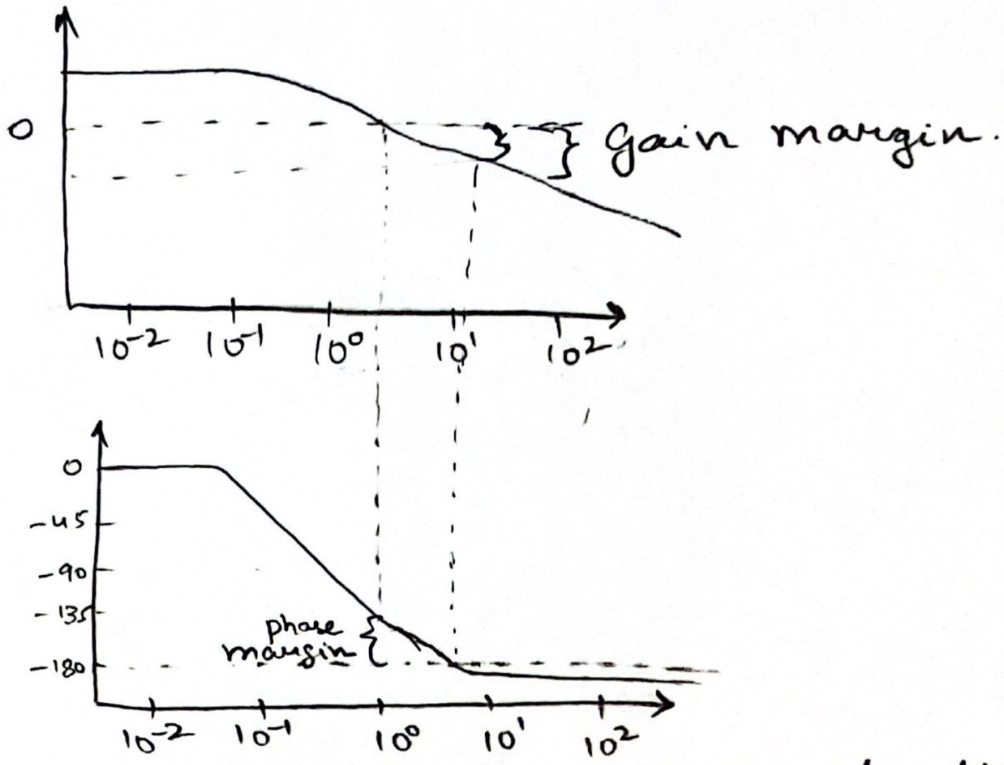
→ Zero at the origin.

$s = \frac{1}{\frac{1}{s}}$ → pole ⇒ response of 1 − response of $\frac{1}{s}$.





Gain & Phase margins on the Bode plot



- One parameter being changed in the transfer function can affect both margins since both plots change.