Circuits and Systems

→ Maxwell's Equations

Gauss

① $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

$\oiint_V \vec{E} \cdot dA = \frac{Q(r)}{\epsilon_0}$

Gauss

② $\vec{\nabla} \cdot \vec{B} = 0$

$\oiint_V \vec{B} \cdot dA = 0$

Faraday

③ $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\oint_S \vec{E} \cdot dl = -\frac{\partial \phi_B}{\partial t}$

④ $\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

Ampere → without this term.

$\oint_S \vec{B} \cdot dl = \mu_0 I_S + \frac{\partial \phi_E}{\partial t}$

H → magnetic field
D → Electric flux density
E → Electric field
B → Magnetic flux density
ρ → charge density (charge / vol)
J → Current density (current / area)

$B = \mu H$
$D = \epsilon E$

ϵ_0 → permittivity
μ_0 → permeability
∇ - gradient
$\nabla \cdot$ - divergence
$\nabla \times$ - curl

Divergence
How much does a point bleed?

Curl
How much does it turn?
How much of a closed circular field does it generate?

→ Eq. ③ & ④ give EM wave behaviour!
Eq ③ → changing magnetic field produces E field. & Eq ④ is vice versa.

→ Taking divergence on Eq ④ :- $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) - \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot \vec{J}$

$\Rightarrow \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ → Conservation of charge.

$\Updownarrow$

$\oint \vec{J} \cdot \vec{ds} = -\frac{dQ}{dt}$ → Change in charge gives flow of current through a surface.

Lumped Circuits	Microwave	Optics
$D \ll \lambda$	$D \sim \lambda$	$D \gg \lambda$

→ Assumptions we make in circuits

① → Speed of light is ∞. ($\because D \ll \lambda$)

$\Rightarrow \vec{\nabla}.\vec{J} = 0 \Rightarrow$ No charge accumulation at nodes.

$\Rightarrow$ No propogation delays.

② → Charge of electron $e \simeq 0$. Think of it as a continuum

→ n = # nodes ; m = # meshes ; b = # branches.

$$\boxed{b = m + n - 1}$$

Voltage

" Work done per unit charge in moving from point a & b"

$$\boxed{V_{ab} = \frac{dW_{ab}}{dq}}$$

Power

$$P = \frac{dw}{dt} = \underbrace{\frac{dw}{dq}}_{V} \cdot \underbrace{\frac{dq}{dt}}_{i}$$

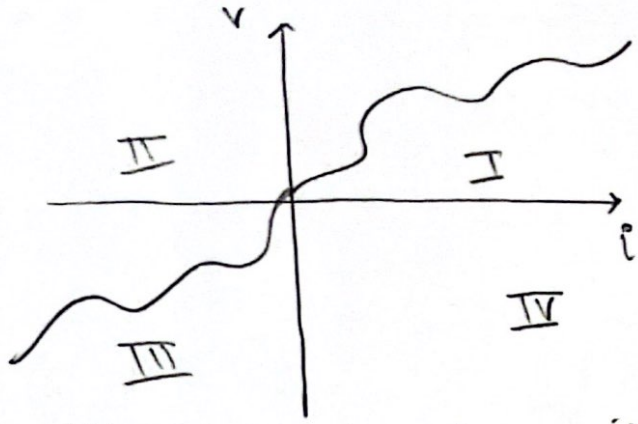
$\Rightarrow \boxed{P = Vi} \sim \boxed{P(t) = v(t)\, i(t)}$

P + ⇒ delivering/absorbing.

−ve ⇒ generating/producing.

$W_{t_1 t_2} = \int_{t_1}^{t_2} P(t)\,dt$

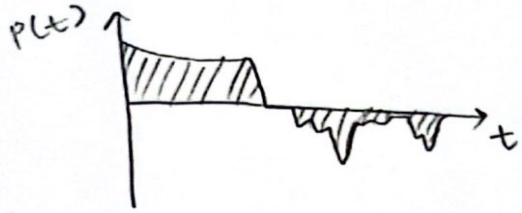
Passive elements $\Rightarrow$ IV characteristic always in Q(I) & Q(III)



- $$W(t) = \int_{-\infty}^{t} P(t')\,dt$$

for all

$\forall t,\ W(t) \geq 0$



$\longrightarrow$ When integrating for any 't' the area will always be ≥ 0.

Active element

> For some 't', $W(t)$ could be negative.

$\Rightarrow$ $\boxed{IV}$ characteristics could exist in Q I & Q IV as well.

-

For resistor

$\forall t,\ P(t) \geq 0$ & $\forall t,\ W(t) \geq 0$ } Reactive.

For capacitor

$\forall t,\ W(t) \geq 0$ but $\exists t,\ P(t) < 0$

There exists a time 't'

KCL

No net charge accumulation at any node!

Σ Currents at a node $=0$. Current going in is +ve.

KVL

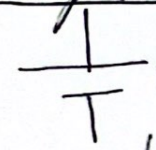
$\oint \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \iint \vec{B} \cdot \vec{ds} = 0$ $\longrightarrow$ Does not work for transformers

When there is no changing magnetic field/flux.

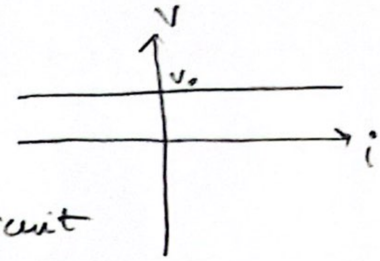
Sum of all voltages along a loop is 0.

Sources

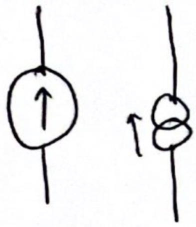
Independant sources



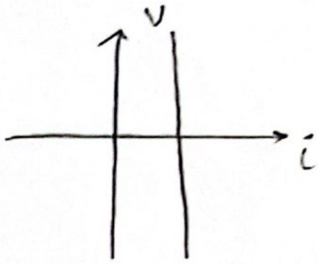
V_0 Voltage source



↳ nulling ⇒ short circuit

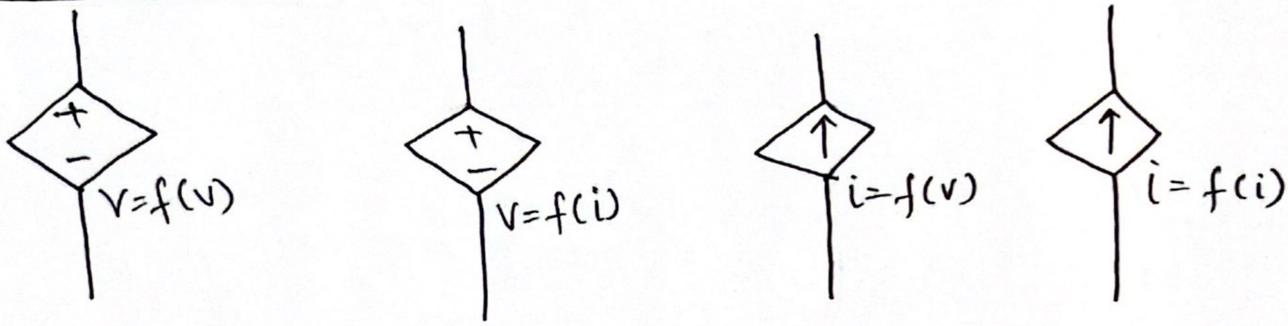


Current source



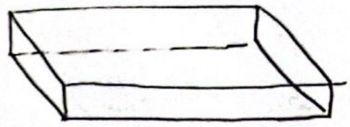
↳ nulling ⇒ open circuit

Dependant Sources



Resistance

$$R = \rho \frac{L}{A}$$



Equal partition theorem

→ At thermal equilibrium Every degree of freedom takes $\frac{1}{2}KT$.

$$\frac{1}{2} m_e v^2 \simeq \frac{3}{2} kT$$

⇒ $v \cong$ on the order of 10^5 m/s velocity of each electron.

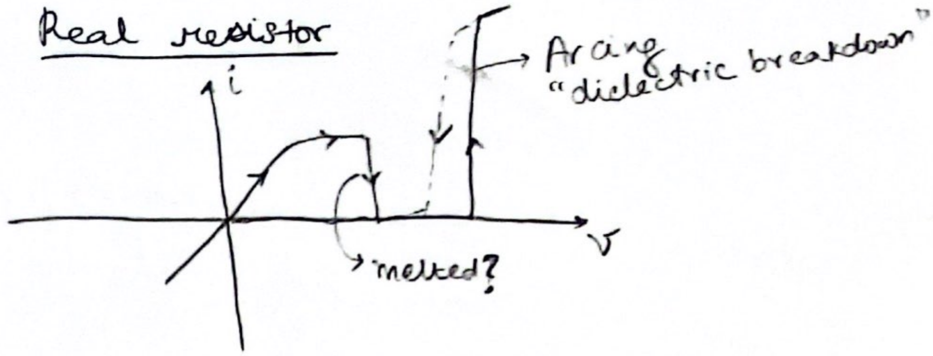
→ If 1Amp of current ⇒ $t = 10^{-5}$ sec & if 1m ⇒ $v = 10^{-5}$ m/s

$I \simeq 6\times10^{23} \times 1.6\times10^{-19}$ C $\simeq 10^5$ A If they were moving at 1 m/s.

Conductance

$$G = \frac{1}{R}$$ $\rightsquigarrow$ **Real resistor**

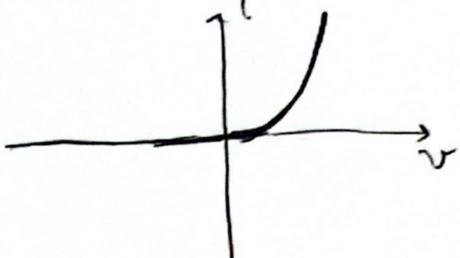
$$P = \frac{V^2}{R} = i^2 R$$



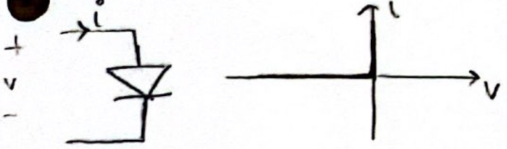
Diode

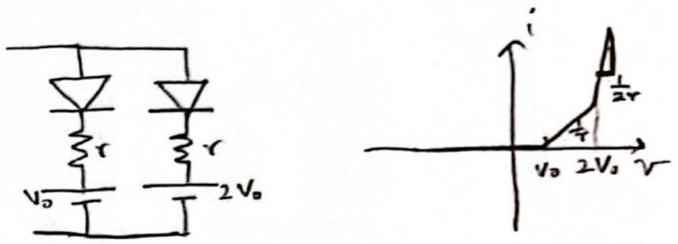
$$I = I_S \exp\left(\frac{V}{V_T}\right)$$

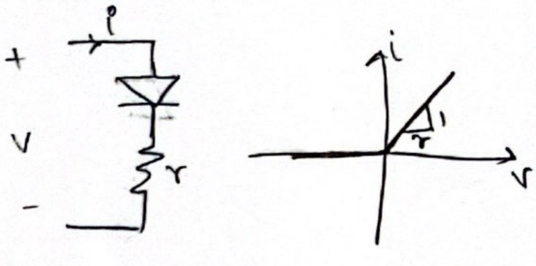
$$V_T = \frac{KT}{q} \approx 26\,mV.$$

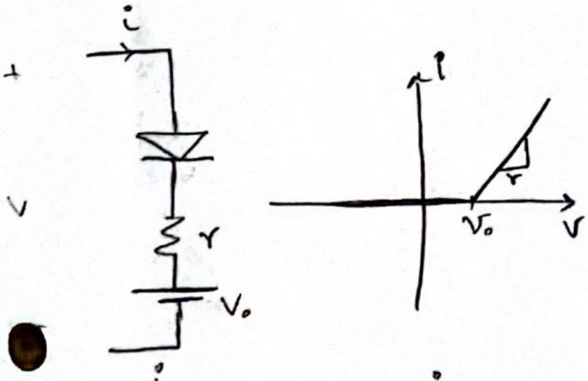


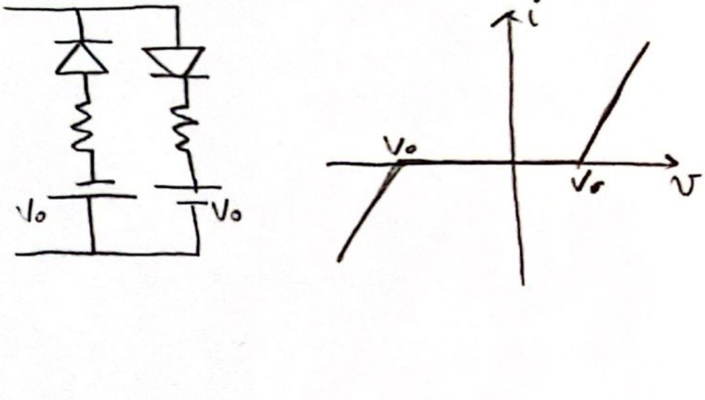
Ideal diode → Use piecewise component graphs & add them.

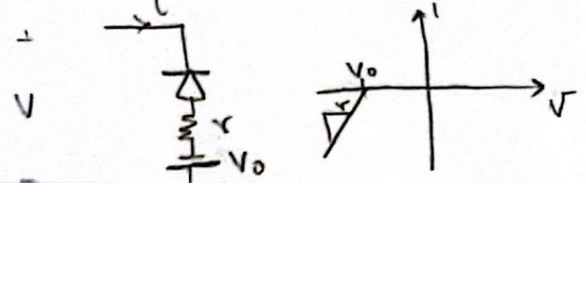












Nodal analysis

→ These analysis tools are only useful for linear circuits. For non linear circuit we need to linearize them using small signal analysis.

1. Select reference node (ground)
2. Define Voltages (v_1, v_2 ...) w.r.t gnd.
3. Apply KCL to node current expressed as node voltages.
4. Solve resulting linear algebra problem.

$\tilde{Y}$ matrix → matrix of conductances.

$$\tilde{Y}\vec{V} = \vec{I}$$

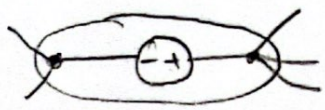
$$\vec{V} = \tilde{Y}^{-1}\vec{I}$$

$$Y^{-1} = \frac{C^T}{\det|Y|}$$

C^T → cofactor matrix.

$\det|Y|$ → It is a scaling factor! gives you the amount of area change in the polygon that Y is multiplied with. Here it is $\vec{V}$.

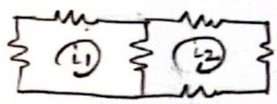
Supernode



Could apply KCL to this supernode.

Mesh analysis

1. Define currents (i_1, i_2 ....) in each mesh.

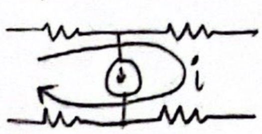


2. Apply KVL to meshes expressed as mesh currents.
3. Solve.

→ We could end up with very different meshes based on how we draw a circuit (Especially in 3D circuits)

→ Mesh analysis fails in 3D circuits. Sometimes we cannot planarize 3D circuits so it cannot be applied.

Supermesh



CIRCUIT THEOREMS - In order to design

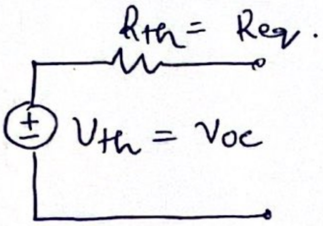
I. Superposition theorem

> In a linear system, the overall response is equal to the sum of the responses of individual inputs.

> Applies to independant sources (inputs).

II Thevenin theorem - (Linear circuits)

> A complex circuit with two output nodes could be electrically replaced by a source voltage and a series resistor.

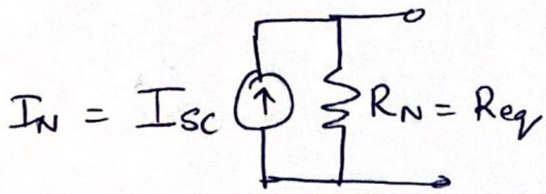


> To find R_{eq}, null all independant sources & use V_{test} & i_{test} to find $R_{eq} = \frac{V_{test}}{i_{test}}$

> V_{oc} is open circuit voltage across those two nodes.

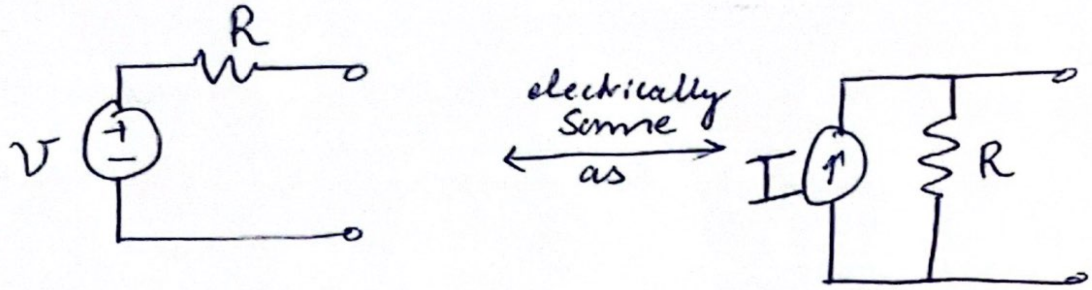
III Norton theorem - (Linear circuits)

> Replace circuit with a current source and a resistor in parallel.



> Short circuit the two nodes & find I_{sc} through it.

IV Source Transformation

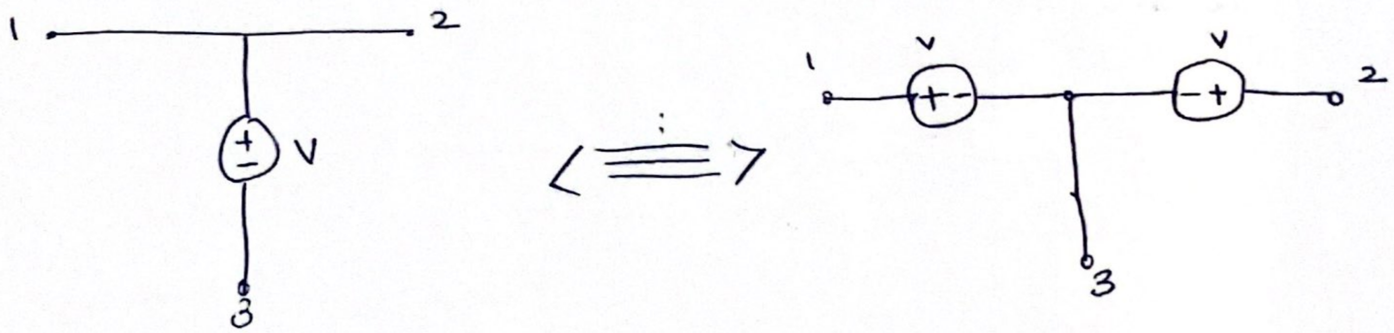


↳ However this dissipates P when no load → heat & magnetic fields are produced

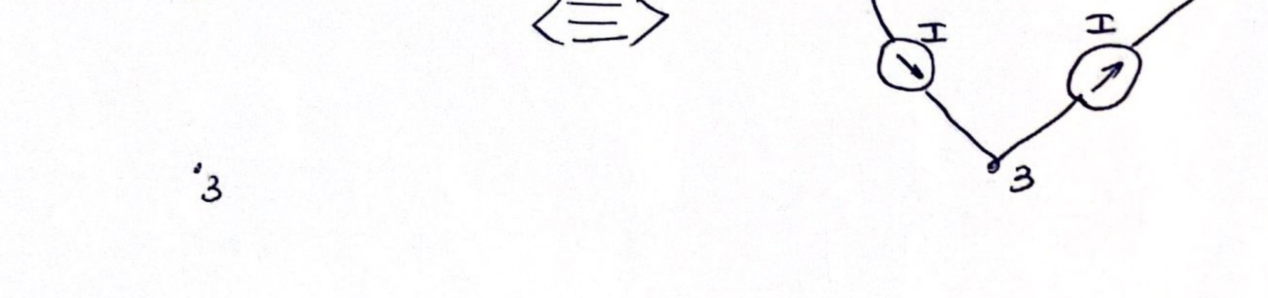
V Source Transpostation

↳ (converting π - T models in transistors)

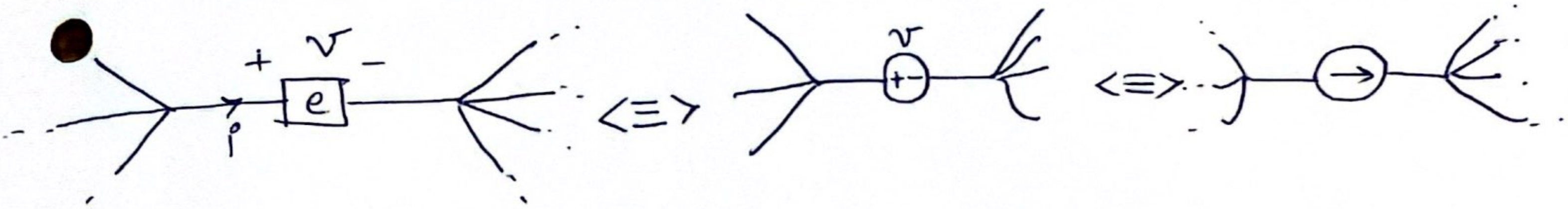
Voltage sources.



Current sources

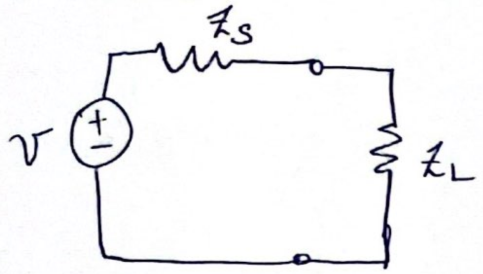


VI Substitution theorem

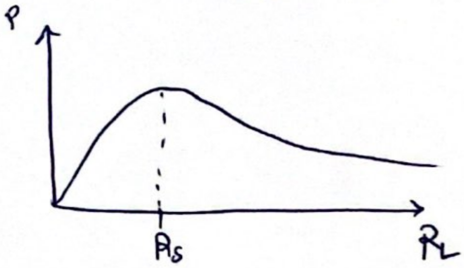


An element 'e' with a voltage 'v' across it and current 'i' through it can be replaced by a voltage source or current source

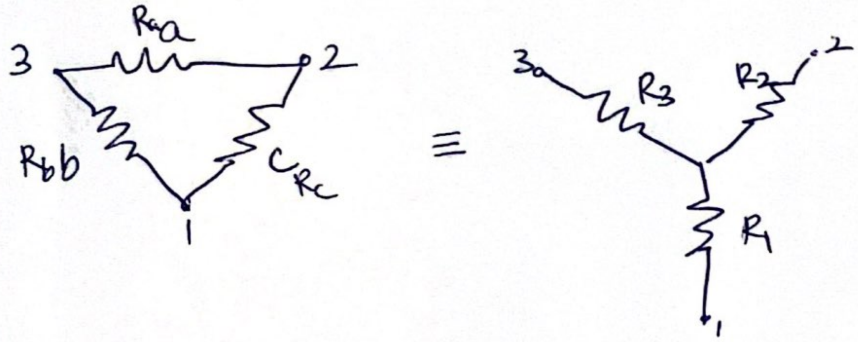
VII Maximum Power Transfer theorem.



> Maximum power transfer occurs when $Z_L = Z_s^*$



VIII Y - Δ conversion (Not a fundamental theorem).



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c}$$

$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c}$$

$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c}$$

$$R_a = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_1}$$

$$R_b = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_2}$$

$$R_c = \frac{R_1 R_2 + R_2 R_3 + R_1 R_3}{R_3}$$

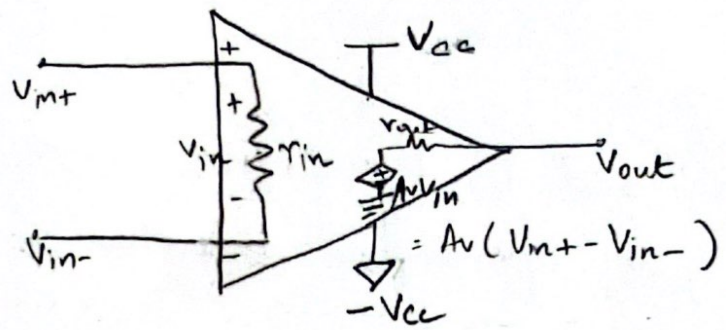
→ Active Circuits

- → Transformers are bilateral devices whereas amplifiers are unilateral devices.

→ Amplifiers have power gain whereas transformers do not have power gain.

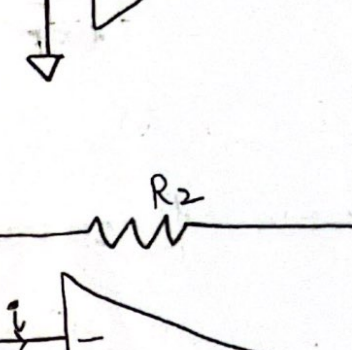
Operational Amplifier

→

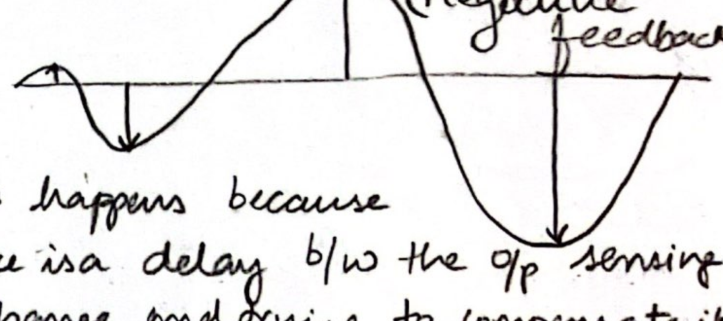


	Ideal	Practical
A_v	∞	$10^6 - 10^8$
r_{in}	∞	$10^5 - 10^3 \Omega$
r_{out}	0	$10 - 100\Omega$
V_{cc}	Large enough for o/p swing.	$1.5V - 30V$

Qualitative

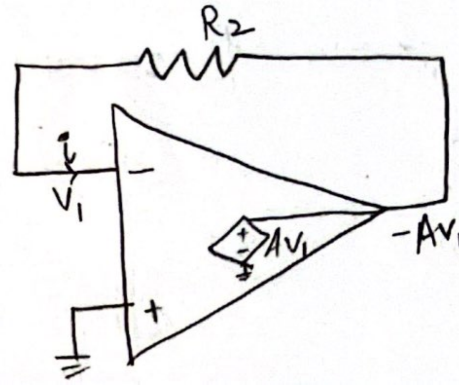


Unstable ⇒ overcompensation.
⇒ too much feedback (negative feedback)



This happens because there is a delay b/w the o/p sensing a change and trying to compensate it.

Quantitative



$-AV_1$ → Feedback forces V_1 to be 0.
More accurately it forces $V_- = V_+$ when $A \to \infty$

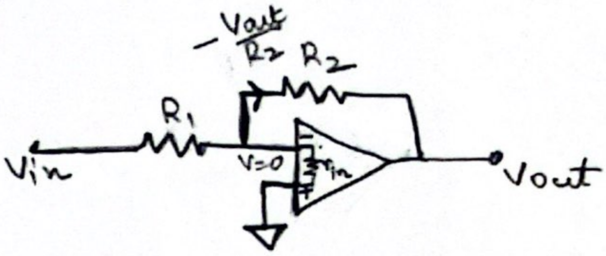
if $i = 0 \Rightarrow V_1 = -AV_1$
$\Rightarrow V_1 = 0$

→ If $A \to \infty$; and we have negative feedback. the two inputs are forced to be equal.

<u>Principle of Asymptotic equality</u>

→ Output is finite, A is infinite ⇒ V_{in} must go to 0.
(Where $V_{in} = V_+ - V_-$) only when feedback is negative

→



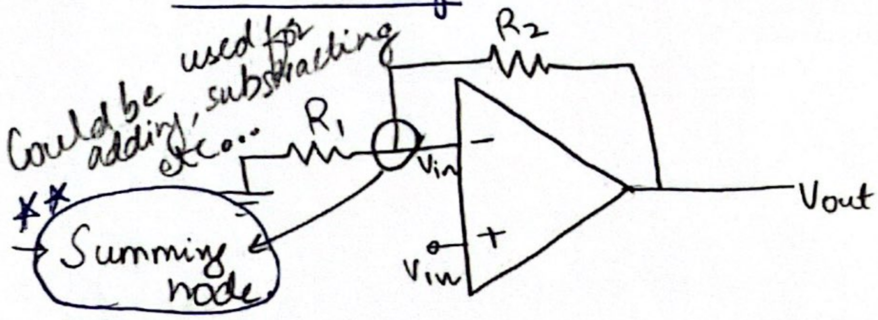
$$\boxed{\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}}$$

Regardless of V_{in} if $A \to \infty$, V_- would be 0.

→ Gain becomes desensitized to variations. Notice it is a ratio! & also just depends on resistors only.

→

<u>Non inverting.</u>

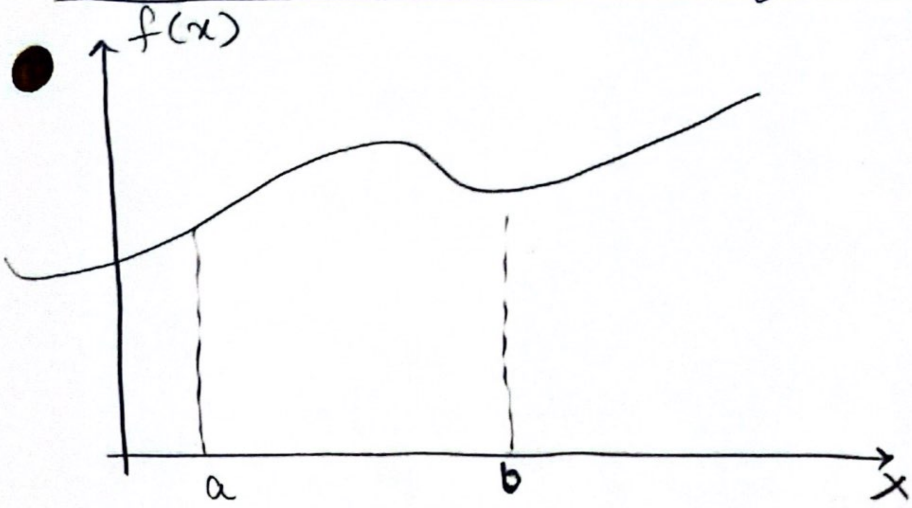


$$\boxed{\frac{V_{out}}{V_{in}} = 1 + \frac{R_2}{R_1}}$$

→ Acts as a perfect market. Everything adds to zero.

→ Op amps are not very fast ⇒ low frequency device.

Fundamental theorem of Calculus



→ $\int_a^b f(x) = F(b) - F(a)$ $\qquad F(x) = \overset{\text{Indefinite}}{\int} f(x)\,dx + c$

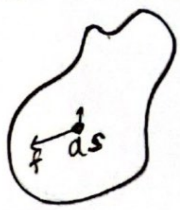
$f(x) = \frac{dF}{dx}$

→ $\int_M dw = \int_{dM} w$

M ↓ Manifold
⇒ m^{th} dimension

→ Integral of the derivative over the volume of a manifold = integral of the function on the boundaries.

→ 2D :- Stoke's theorem :-



$$\iint_S \vec{\nabla} \times \vec{f} \cdot d\vec{s} = \oint \vec{f} \cdot d\vec{l}$$

Curl of f on surface = line integral of f along boundary.

Divergence theorem :-

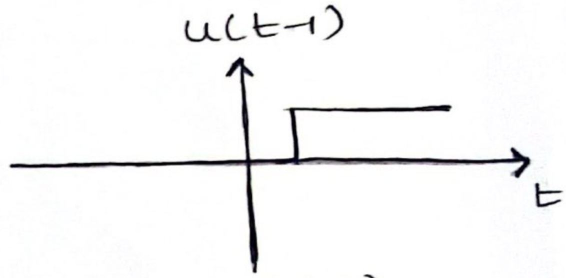
$$\iiint_V \vec{\nabla} \cdot \vec{f}\, dV = \iint_S \vec{f} \cdot d\vec{s}$$

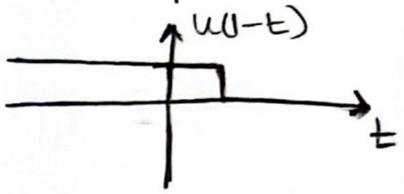
Divergence of f through a volume = surface integral of f through its surface.

Singularity Functions

① Unit step function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

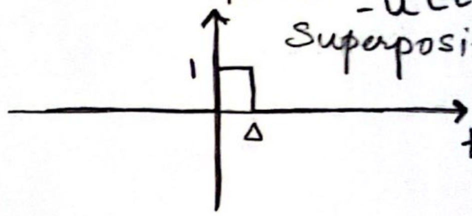




② Pulse function

$$P_\Delta(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \leq t < \Delta \\ 0 & \Delta < t \end{cases}$$

$P_\Delta(t) = u(t) - u(t-\Delta)$
Superposition!



③ Dirac Delta function

$$\left.\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{Undefined} & t = 0 \end{cases}\right\} \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

$$\delta(t) = \lim_{\Delta \to 0} \left(\frac{1}{\Delta} \cdot P_\Delta(t) \right)$$

This has area of 1 regardless of Δ
As $\Delta \to 0$ it becomes longer & thinner

Also $\boxed{\int_{0^-}^{0^+} \delta(t)\, dt = 1}$

Notice, $\int_{-\infty}^{t} \delta(t)\, dt = u(t)$

$\Rightarrow \delta(t) = \frac{d}{dt} u(t) \sim P(u(t))$

P → operator differential

$\frac{1}{P} \to$ integral operator

Sifting property

$$\to \int_{-\infty}^{+\infty} f(t')\, \delta(t'-t)\, dt' = f(t)$$

only matters at t ⇒ can be taken out of integral.

The fn takes the value of the fn at t & sifts it out!!

Operator :- i/p & o/p are functions

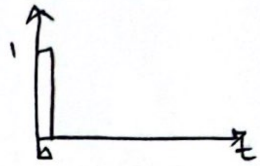
→ $P : \frac{d}{dt}$ $\qquad$ $\frac{1}{P} : \int_{-\infty}^{t} \cdot \, dt'$

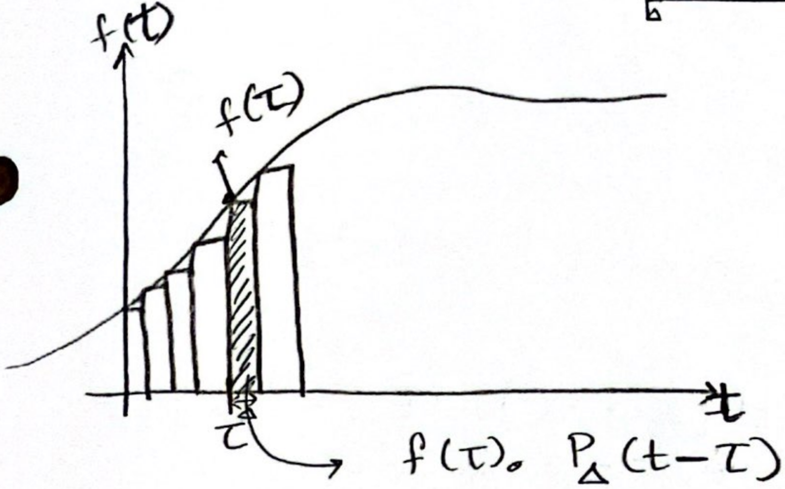
→ $\frac{1}{P} u(t) = r(t)$ 'unit ramp function'. $\left\{ r(t) = \begin{cases} 0 & t<0 \\ t & t>0 \end{cases} \right.$

→ $\frac{1}{P} [r(t)] = \frac{t^2}{2} u(t)$.

→ How can we approximate any function f(t)?

We have $P_\Delta(t) =$





Let $\tau = n\Delta$

$$\Rightarrow f(t) \approx \sum_{n=-\infty}^{+\infty} f(n\Delta) P_\Delta(t-n\Delta) = \sum_{n=-\infty}^{\infty} f(n\Delta) \frac{1}{\Delta} P_\Delta(t-n\Delta)\Delta.$$

Let $\lim_{\Delta\to 0}$ $\quad f(t) = \lim_{\Delta\to 0} \sum_{n=-\infty}^{\infty} \underbrace{f(n\Delta)}_{\tau} \underbrace{\frac{1}{\Delta} P_\Delta(t-n\Delta)}_{\delta(t-\tau)} \Delta \to d\tau$

$$\boxed{f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau}$$

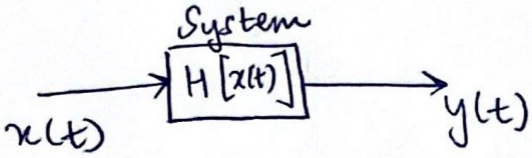
Any function can be written as a sum of δ values scaled at each 't'.

[The functionality of time is only held in the impulse function.]

One-sided functions

$f(t) = 0 \quad \forall t < 0$

System



$y(t) = H[x(t)]$

Now we know $x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

$$y(t) = H\left\{ \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \right\}$$

→ Applying superposition & assuming linear system.

Superposition Integral.

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,\underbrace{H[\delta(t-\tau)]}_{\text{Impulse Response}}\,d\tau$$

Impulse Response **WoW**

$$= \int_{-\infty}^{+\infty} x(\tau)\,h_t(\tau)\,d\tau$$

$h_t(\tau) \longleftrightarrow h(t,\tau)$

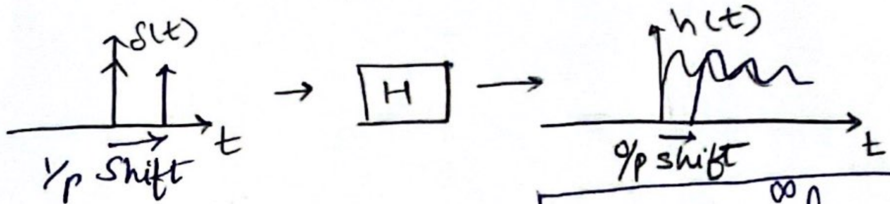
→ Remember here $h(t,\tau)$ is a fn. of both t & τ.

If at $t = \tau$ I apply an impulse at input I get some behaviour at the output that need not be an impulse. So, sum of these output behaviours gives me the output.

Next we assume it is time invariant **L.T.I**

→ Properties do not change with time.

Then the impulse response depends only on $h(t-\tau)$



Observe that now :- $\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) d\tau}$

Impulse decomposition

Convolution WOW

→ Time invariant systems assume causality ⇒ If input shifts by τ, the output also must shift by τ.

aka $y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$ ⇒ Commutative.

$\boxed{y = x * h = h * x}$

"All we need to know to fully characterize a system is it's impulse response."

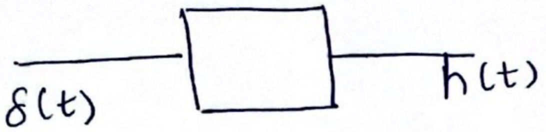
→ Beauty of impulse response

→ Since it is such a sharp, rude, harsh signal it brings out the actual behaviour of the system!

→ All the natural frequencies of a system come out when you hit it with an impluse → Piano.

It is like multiplying the Laplace transforms & Laplace transform of δ is 1 ⇒ it brings out the natural frequencies.

→ A system with impulse response of step function is an ideal integrator.



$$y(t) = h(t) * u(t)$$
$$= u(t) * h(t)$$

Time domain response

Capacitor:- An element whose charge is a function of it's voltage. $q \rightarrow q(v)$

For LTI system :- $\underline{\underline{q = Cv}} \Rightarrow i = C\frac{dv}{dt}$; $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$

$$E = \frac{1}{2} \cdot CV^2 = \frac{1}{2} \cdot \frac{q^2}{C}$$

parallel :- $C_{tot} = C_1 + C_2 + \dots$

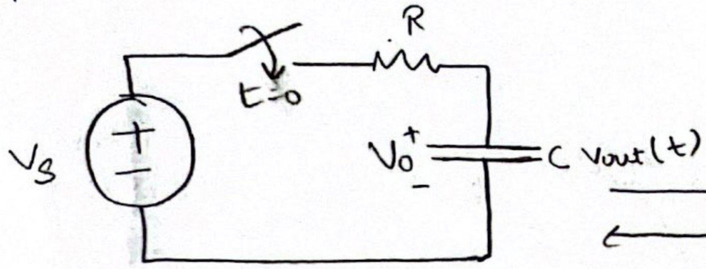
series :- $C_{tot}^{-1} = C_1^{-1} + C_2^{-1} + \dots$

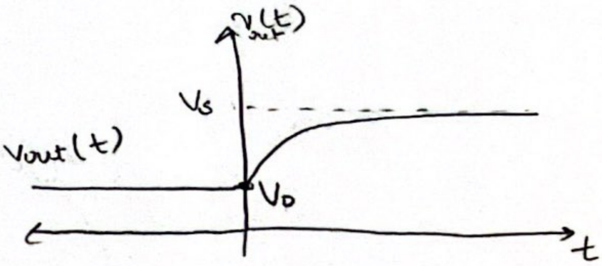
Inductor :- An element whose magnetic flux is a function of it's current.

LTI :- $\underline{\underline{\phi = Li}} \Rightarrow v = \frac{d\phi}{dt} \Rightarrow V = \frac{L\,di}{dt}$

$$i = \frac{1}{L}\int_{-\infty}^{t} v(t')dt'.$$

→





$$v(t) = \begin{cases} V_0 & ; t<0 \\ V_s + (V_0 - V_s)e^{-t/\tau} & ; t>0 \end{cases}$$

$\xrightarrow[\text{natural decomposition}]{\text{forced \& }}$

$$V_f(t) = \begin{cases} 0 & ; t<0 \\ V_s(1-e^{-t/\tau}) & ; t>0 \end{cases}$$

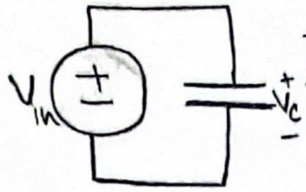
$$V_n(t) = \begin{cases} V_0 & ; t<0 \\ V_0 e^{-t/\tau} & ; t>0 \end{cases}$$

if V_s were zero.

$\xrightarrow[\text{steady state}]{\text{transient}}$

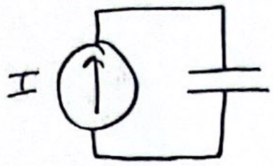
$$V_{tr}(t) = (V_0 - V_s)e^{-t/\tau} \quad t>0$$

$$V_{ss}(t) = V_s \quad t>0$$



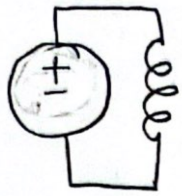
→ ∞ current at $t=0^+$ & then no current after stedystat

→ Voltage $V_c = V_{in}$ almost instantaneously.



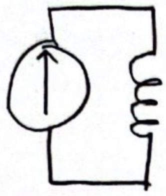
→ Voltage rises linearly to ∞. $V = \frac{1}{C} I \cdot t$

→ Current is constant.



→ current rises linearly to ∞ $i = \frac{1}{L} V \cdot t$

→ voltage is constant.

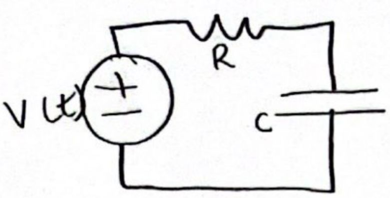


→ Infinite voltage at $t=0^+$ & then no voltage after.

→ Constant current almost instantaneously.

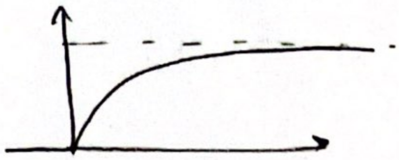
For any system with negative exponential growth.

$$V(t) = \left[V(\infty) + [V(0) - V(\infty)] e^{t/\tau} \right] u(t).$$



Step response $s(t)$ $\quad V(t) = V_s [1 - e^{-t/\tau}] u(t)$

$V_{in} = V_s u(t)$

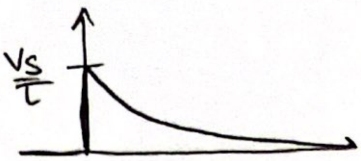


$\frac{d}{dx}$!

Impulse response $h(t)$ $\quad V(t) = \frac{V_s}{RC} e^{-t/\tau} u(t)$

$V_{in}(t) = V_s \delta(t)$

$\frac{V_s}{\tau}$



Heaviside Operator

$$p[f(t)] = \frac{df}{dt} \qquad \frac{1}{p}[f(t)] = \int_{-\infty}^{t} f(t')\,dt'$$

Cap: $i(t) = C\frac{dv}{dt} = Cp[v] = Cp\,v(t)$

Ind: $v(t) = L\frac{di}{dt} = Lp[i] = Lp\,i(t)$

In general assume Cp & Lp themselves are operators

$\Rightarrow v(t) = Z(p)\,[i(t)]$

$\hookrightarrow$ impedance operator. & $Y(p)$ = admittance operator.

→ This operator is done in the time domain! Unlike Laplace.

→ Is $p\left[\frac{1}{p}[f]\right] = \frac{1}{p}\left[p[f]\right] = f$? This is true only when $f(-\infty) = 0$

→ Eg: $p[\cos 4t\; u(t)] = -4\sin 4t\; u(t) + \cos(4t)\,\delta(t).$

$= -4\sin 4t\; u(t) + \delta(t)$

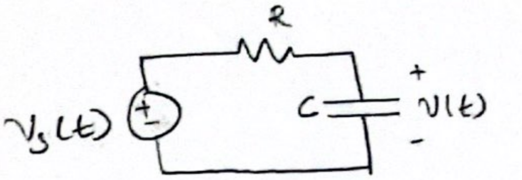
$\frac{1}{p}[\sin 5t\; u(t)] = \int_{-\infty}^{t} \sin 5t'\, u(t')\,dt'$

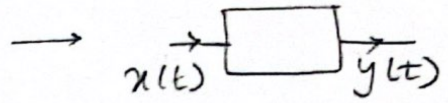
$= u(t)\int_{0}^{t} \sin 5t'\,dt' = \frac{-1}{5}\cos(5t)\Big|_{0}^{t} u(t)$

needs to be here since $\int$ can generate two sided fns.

$= \frac{1}{5}\left[1 - \cos(5t)\right] u(t).$

Low pass operator





$$\frac{dv}{dt} + \frac{1}{RC} v(t) = \frac{1}{RC} V_s(t)$$

$$\Rightarrow \quad y' + ay = x$$

$$\Rightarrow \quad py + ay = x \quad \Rightarrow \quad \underbrace{(p+a)}_{\text{assume it is a single operator.}} y = x. \quad \text{— ①}$$

$$\Rightarrow \quad y = \underbrace{\frac{1}{p+a}}_{\text{low pass operator.}} x$$

$$> \quad p[e^{at} y(t)] = a e^{at} y(t) + e^{at} p y$$

$$= e^{at}[py + ay] = e^{at}[p+a]y.$$

Sub. in ①

$$e^{at}[p+a]y = e^{at} x(t) = p[e^{at} y(t)]$$

$$\Rightarrow \quad e^{at} y(t) = \frac{1}{p} e^{at} x(t).$$

$$\Rightarrow \quad y(t) = e^{-at} \cdot \frac{1}{p} [e^{at} x(t)]$$

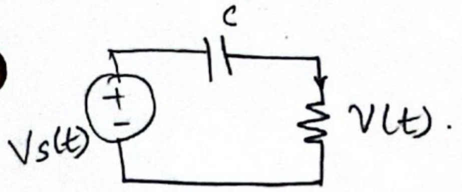
$$\boxed{\Rightarrow \quad \frac{1}{p+a} x(t) = e^{-at} \frac{1}{p} [e^{at} x(t)]}$$

1st order low pass operator.

Going back to $y(t) = \frac{1}{p+a} x(t)$. if $x(t) = \delta(t)$

$$y(t) = \frac{1}{p+a} \delta(t) = e^{-at} \int^{t} \delta(t') dt' = e^{-at} u(t)$$

High pass operator



$$y = \frac{p}{p+a} x = \left[1 - \frac{a}{p+a}\right] x$$ is the general type.

> Given: $\underbrace{\frac{p+a}{p+b}}_{\text{general response}} = \underbrace{1}_{\text{unity response}} + \underbrace{\frac{a-b}{p+b}}_{\text{low pass}}$

2nd order

$$y'' + Ay' + By = x$$

$$\Rightarrow y = \frac{1}{p+a}\left[\frac{1}{p+b}(x)\right]$$ where $a+b = A$; $ab = B$.

In general

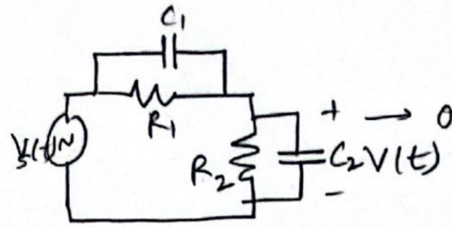
$$b_n y^n + b_{n-1} y^{n-1} + \dots + b_1 y + b_0 = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$$

$$\Rightarrow y(t) = \underbrace{\frac{a_m + \dots + a_1 p + a_0}{b_n + \dots + b_1 p + b_0}}_{\text{System operator} \to H(p)} \cdot x(t).$$

Unlike laplace this operator generates a time domain output.

> No. of natural frequencies = no. of independantly definable initial conditions = no. of degrees of freedom = order (degree) of denominator of T.F. = no. of independant energy storage elements.

Eg :-



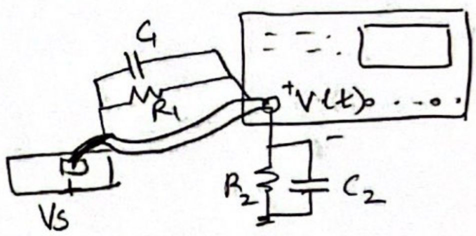
→ only one degree of freedom since $V_{C_1} + V_{C_2}$ is constrained by V

$$V(t) = \frac{R_2}{R_1+R_2} \cdot \frac{1+R_1C_1 p}{1+(R_1 \| R_2)(C_1+C_2)p} v_s(t).$$

If $R_1C_1 = (R_1 \| R_2)(C_1+C_2)$ → Dynamics cancel o/p is a purely scaled version of i/p.

$\Rightarrow \boxed{R_1C_1 = R_2C_2}$

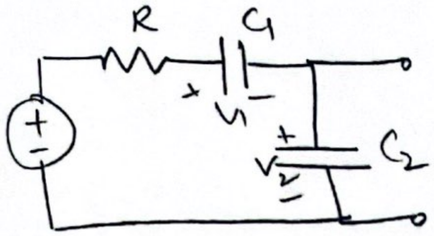
> This is how calibration of oscilloscopes & network analysers is done. To remove the dynamic effects of the probe connection



$$H(p) \cdot x(t) = \frac{1}{(p+a)^m} x(t) = e^{-at} \cdot \frac{1}{p^m} \cdot e^{at} x(t)$$

$$\boxed{H(p+r)\, x(t) = e^{-rt} H(p) e^{rt} x(t)}.$$

> Observability of a system



Order of this system is 2 since V_1 & V_2 can be independantly defined.

> But what if we combine C_1 & C_2?

> Then $C_{eq} = \dfrac{C_1 C_2}{C_1 + C_2}$. But now order = 1?

> In the former case the T.F is of the form $\dfrac{p+a}{(p+a)(p+b)}$ & therefore some of the dynamics in the system are hidden.

> Also the O/p is no longer the same when the caps are combined.

Derivative of $\delta(t) = \delta'(t) \rightarrow$ Doblet function.

Heavyside operator → subsumes laplace & fourier transforms

- $p[f(t)] = \frac{df}{dt}$

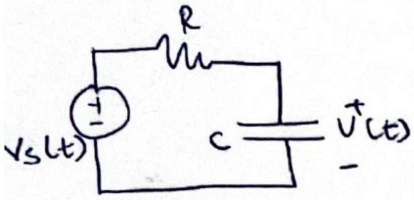
$$\frac{1}{p}[f(t)] = \int_{-\infty}^{t} f(t')\,dt'$$

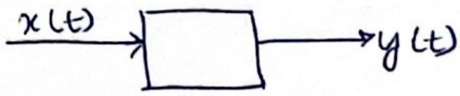
Cap:- $i(t) = \underbrace{Cp}_{\text{operator}}V(t)$

ind:- $v(t) = \underbrace{Lp}_{\text{operator}}i(t)$

→ $p\left[\frac{1}{p}[f]\right] = \frac{1}{p}\left[p[f]\right] = f$ if $\underline{\underline{f(-\infty) = 0}}$

- **Low pass operator**



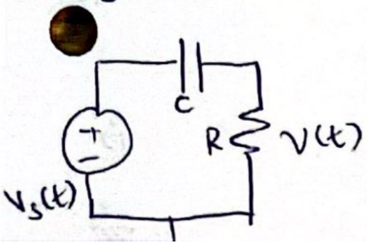


$$\hookrightarrow \frac{dv}{dt} + \frac{1}{RC}v(t) = \frac{1}{RC}v_s(t)$$

$$\underset{v(t)}{y(t)} = \frac{1}{p+a}\underset{V_s(t)}{x(t)} = e^{-at}\frac{1}{p}\left[e^{at}x(t)\right] \quad \text{where } a = \frac{1}{RC}$$

High pass operator

$$\underset{\frac{p}{p+a}(x(t))}{y(t)} = x(t) - \frac{a}{p+a}x(t)$$



2nd order system

Complex numbers :- Essentially an ordered pair.
Could define a vector.

> This vector can be in cartesian :- $x+jy$
polar :- $re^{j\theta}$
$\Big\}$ $x = r\cos\theta$. $y = r\sin\theta$. $\Big\}$ $\theta = \tan^{-1}\frac{y}{x}$

> $e^{x} = \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots$

$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$

$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!}$

$\Rightarrow \boxed{e^{j\theta} = \cos\theta + j\sin\theta.}$

Euler's relationship.

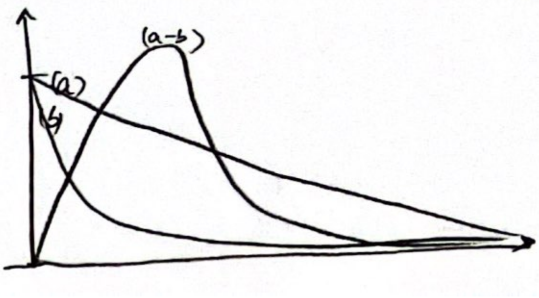
$\boxed{e^{j\pi} + 1 = 0}$

> $\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \; ; \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$

> A second order system $\frac{1}{p^2 + Ap + B}$ can be written as

$\cancel{\frac{1}{p^2 + Q_n}}$ $\quad \frac{1}{p^2 + \frac{\omega_n}{Q}p + \omega_n^2}$ where Q is quality factor.
ω_n is natural frequency.

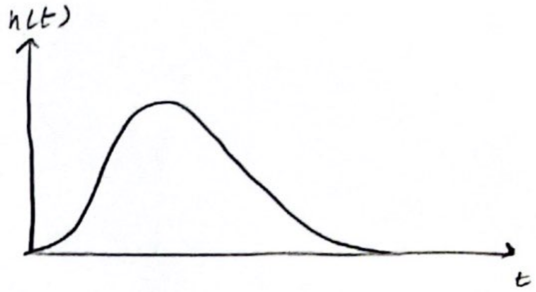
Impulse response $h(t) = \frac{1}{b-a}\left(e^{-at} - e^{-bt}\right)u(t) \Rightarrow$ Case (i) a & b are real.



If both a, b are positive

→ If either are negative
⇒ grow exponentially
⇒ unstable.

Case II $a = b \in \mathbb{R}$

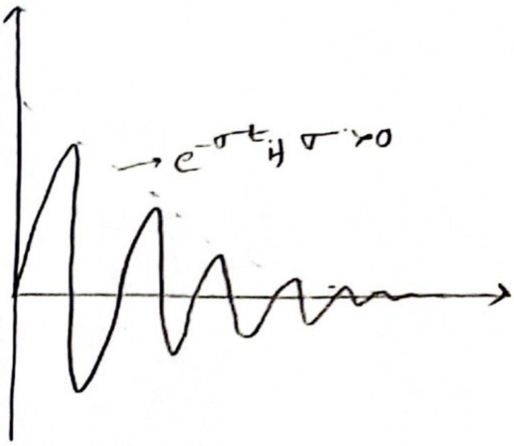


$$h(t) = e^{-bt} \cdot t \cdot u(t)$$

Case III $a, b \in \mathbb{C}$ & $a^* = b$.

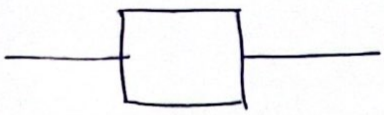
$a = \sigma + j\omega$
$b = \sigma - j\omega$

$$\boxed{h(t) = \frac{e^{-\sigma t}}{\omega} \cdot \sin(\omega t)\, u(t)}$$



Here $\sigma = 0 \Rightarrow$ marginally stable.
$\sigma > 0 \Rightarrow$ stable
$\sigma < 0 \Rightarrow$ unstable.

System Function / Transfer Function.



$x(t) = e^{st} u(t)$ where s is the complex frequency $\sigma + j\omega$.

$$H(s) = \frac{N(s)}{D(s)} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}.$$

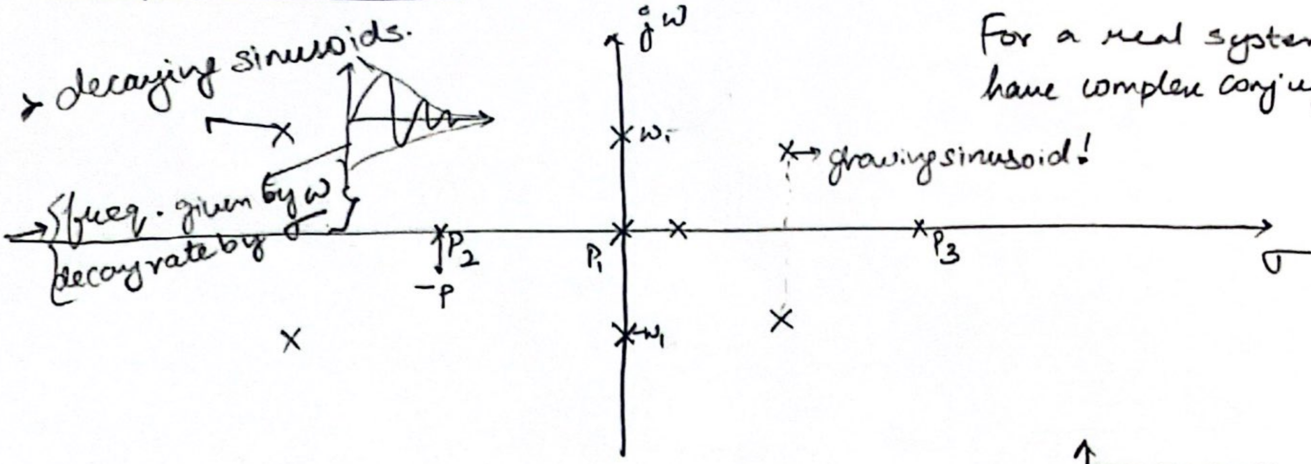
$$= H_0 \frac{\left(1 - \frac{s}{z_1}\right)\left(1 - \frac{s}{z_2}\right) \dots \left(1 - \frac{s}{z_m}\right)}{\left(1 - \frac{s}{p_1}\right)\left(1 - \frac{s}{p_2}\right) \dots \left(1 - \frac{s}{p_n}\right)}$$

$\nearrow$ DC

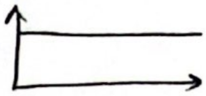
Roots of $N(s)$: zeroes. $\longrightarrow$ $s = z_i \Rightarrow H(s) = 0$
Roots of $D(s)$: poles. $\longrightarrow$ $s = p_i \Rightarrow H(s) = \infty$

Complex S plane

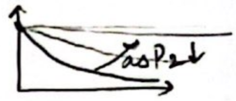
For a real system need to have complex conjugates



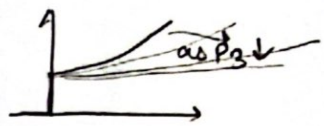
Impulse response with P_1 pole is $\frac{1}{s}$



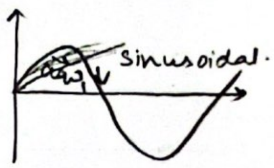
with P_2 pole is $e^{-rt}u(t)$.



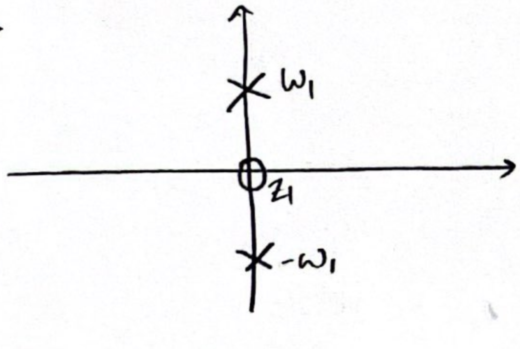
with P_3 pole is $e^{rt}u(t)$



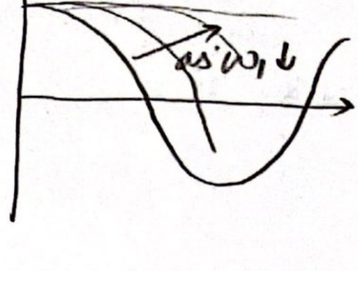
with ω_1 & $-\omega_1$



- As P_2 becomes smaller it approximates P_1 asymptotically.
- As P_3 becomes smaller it approximates P_1 asymptotically.
- As $\omega_1, -\omega_1$ become smaller they do not approximate P_1 because they actually resemble a system with $H(s) = \frac{1}{s^2}$ which is a ramp.
-

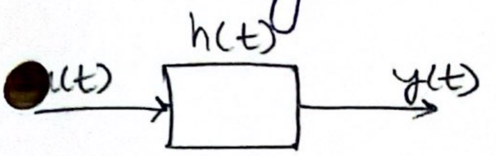


This system has a cosine response



Since at $\omega_1 \downarrow$ one pole & one zero get cancelled it approximates P_1 asymptotically.

Stability.



> If the impulse response converges to zero at $t=\infty$ it is stable.

$$\boxed{\lim_{t\to\infty} h(t) = 0} \Rightarrow \text{Stable.} \quad e^{-t}u(t) \quad \rightarrow \quad \underline{\text{BIBO Stability.}}$$

bounded input bounded output

$$\lim_{t\to\infty} h(t) = \infty \Rightarrow \text{unstable.} \quad e^{t}u(t)$$

$$\lim_{t\to\infty} h(t) \leq \text{constant} \Rightarrow \text{marginally stable.} \quad \sin(\omega t),\ u(t).$$

> $H(p) = \dfrac{1}{p-s_0} \rightarrow H(s) = \dfrac{1}{s-s_0} \rightarrow h(t) = e^{s_0 t}u(t).$

$|h(t)| = |e^{\sigma t}|\, u(t) \Rightarrow$ Stability is captured by σ alone.

$\sigma < 0 \Rightarrow$ Stable.

$\sigma > 0 \Rightarrow$ unstable.

* $\sigma = 0 \Rightarrow$ marginally stable. only for single pole system. or single pair of complex conjugate poles.

> $H(s) = \dfrac{1}{(s-s_0)^m} \Rightarrow h(t) = \dfrac{t^{m-1}}{(m-1)!} e^{s_0 t} u(t)$

$|h(t)| = \dfrac{t^{m-1}}{(m-1)!} \cdot |e^{\sigma t}|\, u(t).$

$\sigma < 0 \Rightarrow$ Stable.

$\sigma > 0 \Rightarrow$ unstable.

$\sigma = 0 \Rightarrow \begin{cases} \infty & m > 1 \rightarrow \text{unstable} \\ 1 & m = 1 \rightarrow \text{marginally stable} \end{cases}$

→ Zeroes change when input & output change whereas poles don't. since poles represent the natural frequencies.

→ ## Routh Hurwitz Criterion.

If you have a polynomial.

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

$$\begin{array}{cccc}
a_n & a_{n-2} & a_{n-4} & \cdots \\
a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
b_n & b_{n-1} & b_{n-2}. & \\
c_n & c_{n-1} & c_{n-2} & \cdots \\
\vdots & \vdots & \vdots & \\
0 & & &
\end{array}$$

$$b_n = \frac{a_{n-1}\,a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

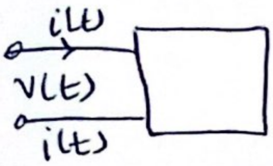
$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

→ The number of sign changes in the first column is equal to the number of RHP poles.

→ If there is a zero in first column it means a pole on the $j\omega$ axis.

Network Synthesis

Complex impedance.



$$v(t) = Z(p)\,i(t)$$

$$V(p)\,\delta(t) = Z(p)\,I(p)(\delta(t))$$

Laplace

$$V(s) = Z(s)\,I(s).$$

$$Z(s) = \frac{V(s)}{I(s)} = \left.\frac{v(t)}{i(t)}\right|_{i(t) = e^{st}u(t)}.$$

Admittance

$$Y(s) = \frac{I(s)}{V(s)}$$

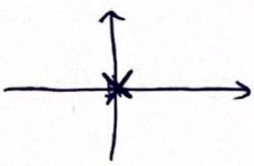
→

$$Z(s) = R$$

$$\boxed{Z(s) = R.}$$

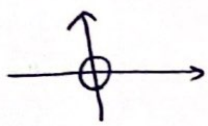
$$Z(s) = (SC)^{-1}$$

$$\boxed{Z(s) = \frac{1}{SC}}$$



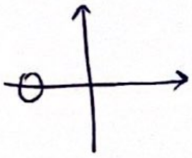
$$Z(s) = \frac{1}{(SL)^{-1}}$$

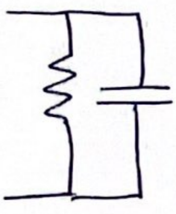
$$\boxed{Z(s) = SL}$$



→

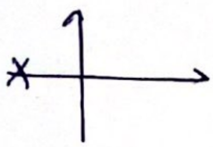
$$Z(s) = R + Ls$$

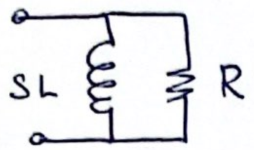




$$Y(s) = G + SC$$

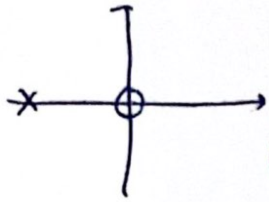
$$Z(s) = \frac{1}{\frac{1}{R} + SC}.$$

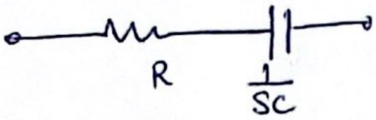


→ 

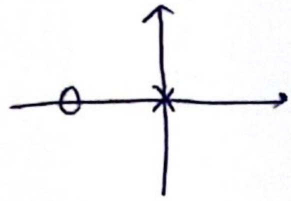
$$Y(s) = G + \frac{1}{Ls}$$

$$Z(s) = \frac{Ls}{1+GLs} = \frac{Ls}{1+\frac{Ls}{R}}$$



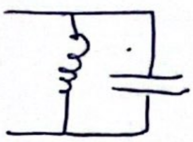
→ 

$$Z(s) = \frac{1+RsC}{sC}$$



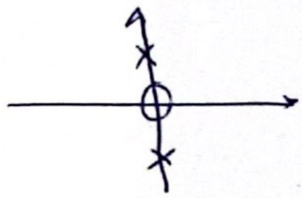
→ 

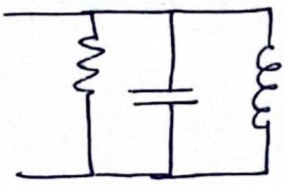
$$Z(s) = sL + \frac{1}{sC} = \frac{1+LCs^2}{Cs}$$

→ 

$$Y(s) = sC + \frac{1}{Ls} = \frac{1+s^2LC}{Ls}$$

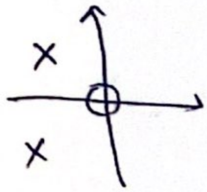
$$Z(s) = \frac{sL}{1+s^2LC}$$



→ 

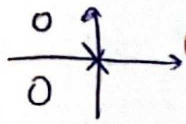
$$Z(s) = \frac{1}{Y(s)}$$

$$Y(s) = G + sC + \frac{1}{sL}$$



→ 

$$Y(s) = \frac{Cs}{1+RCs+LCs^2}$$



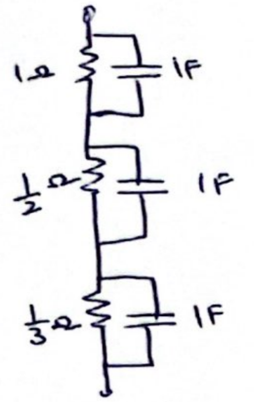
→ Build a network whose impedance is

- $Z(s) = \dfrac{1+2s+s^2}{s}$ → maybe RLC in series combination.

→ What if it is harder?

$$Z(s) = \frac{3s^2+12s+11}{s^3+6s^2+11s+6} = \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}$$

$\Rightarrow Z_1(s) + Z_2(s) + Z_3(s)$. →



→ What is the problem with this?

- If $Z(s) = \dfrac{1}{s+1} - \dfrac{1}{s+2} + \dfrac{1}{s+3}$

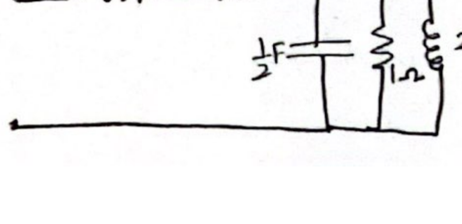
→ this is a problem!

→ Let $Z(s) = \dfrac{2s^3+5s^2+6s+1}{s^2+2s+1}$ → long division!

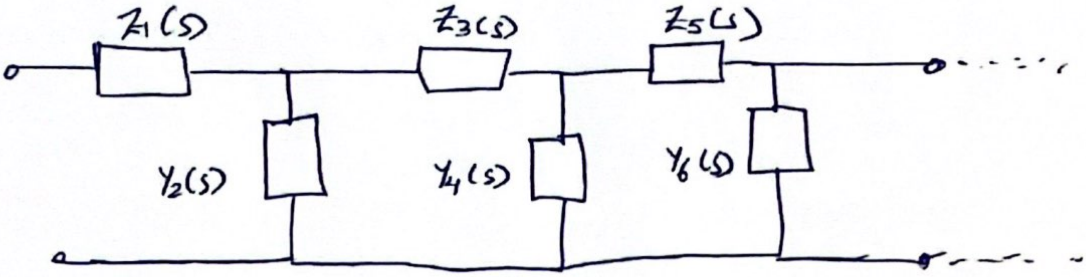
$= 2s+1 + \dfrac{2s}{s^2+2s+1}$ → flip this & continue.

$= 2s+1 + \dfrac{1}{\dfrac{s^2+2s+1}{2s}}$

$= 2s+1 + \dfrac{1}{\frac{s}{2} + + \frac{1}{2s}}$ → $Y(s) =$



→ Ladder synthesis



$$Z(s) = Z_1(s) + \cfrac{1}{Y_2(s) + \cfrac{1}{Z_3(s) + \cfrac{1}{Y_4(s) + \cfrac{1}{Z_5(s) + \cfrac{1}{Y_6(s)}}}}}$$

Bode plots

> For basics look at control systems notes.

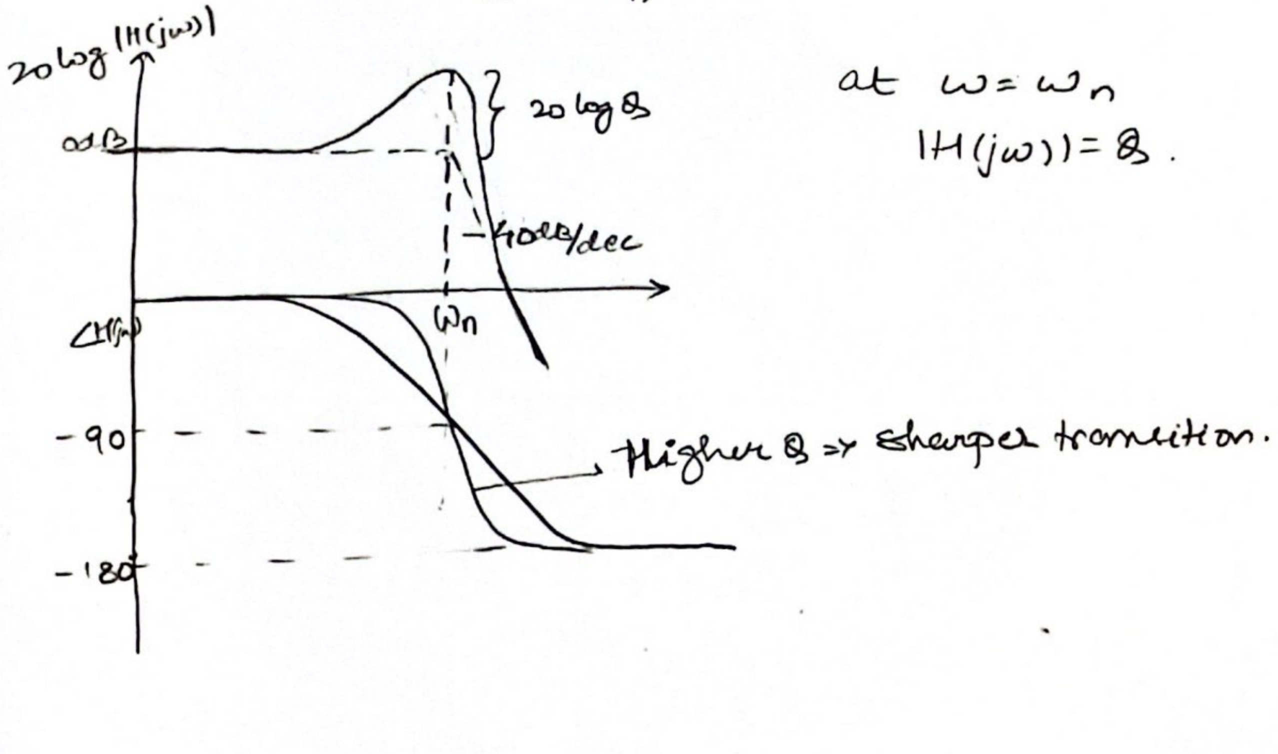
> Contd...

For a second order system

$$H(s) = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_n} + \frac{s^2}{\omega_n^2}}$$

where Q is quality factor
ω_n is natural freq.

$$H(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + \frac{j}{Q} \cdot \frac{\omega}{\omega_n}}$$



at $\omega = \omega_n$
$|H(j\omega)| = Q$.

Fourier Series and Fourier Transform

Fourier series

> A periodic series can be written as a weighted sum of sines and cosines.

> If $x(t) = x(t+nT)$

$$\Rightarrow x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right]$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(n\omega_0 t)\,dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)\,dt$$

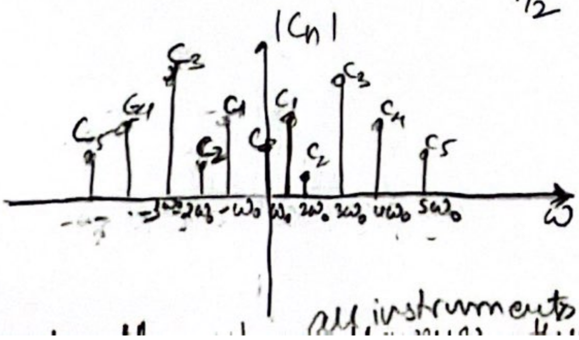
$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

$$\Rightarrow x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \frac{e^{jn\omega_0 t} + e^{-j\omega_0 t}}{2} - j b_n \frac{e^{jn\omega_0 t} - e^{-j\omega_0 t}}{2}\right]$$

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}}$$

$$\begin{cases} c_n^* = c_n \\ c_n = \frac{a_n - jb_n}{2} \\ c_0 = \frac{a_0}{2} \end{cases}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\omega_0 nt}\,dt$$



> Shows how strong each periodic component is

> 3rd harmonic here is strong.

> The reason a given note sounds the same on all instruments is because of the same fundamental frequency. &

> What about an aperiodic waveform?

→ Take the integral from $-\infty$ to ∞ since we can think of it as having a period $\simeq \infty$.

→ As $T \to \infty$ $\quad \omega_0 = \frac{2\pi}{T} \to 0 \Rightarrow$ The discrete frequency components become a continuous function.

→ $$x(t) = \sum_{n=-\infty}^{+\infty} C_n \, e^{jn2\pi f_0 t}$$

$$e^{+jn2\pi f_0 t} = \int_{-\infty}^{\infty} e^{j2\pi f t} \, \delta(f - nf_0) \, df$$ using sifting property

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{+\infty} e^{+j2\pi f t} \, \delta(f - nf_0) \, df$$

$$= \int_{-\infty}^{\infty} df \; e^{+2\pi f t} \underbrace{\sum_{n=-\infty}^{\infty} C_n \, \delta(f - nf_0)}_{\text{Let this be } X(f)}$$

↳ bunch of shifted weighted impulses.

IFT

$$\Rightarrow x(t) = \int_{-\infty}^{\infty} X(f) \, e^{j2\pi f t} \, df = F^{-1}[X(f)]$$

FT

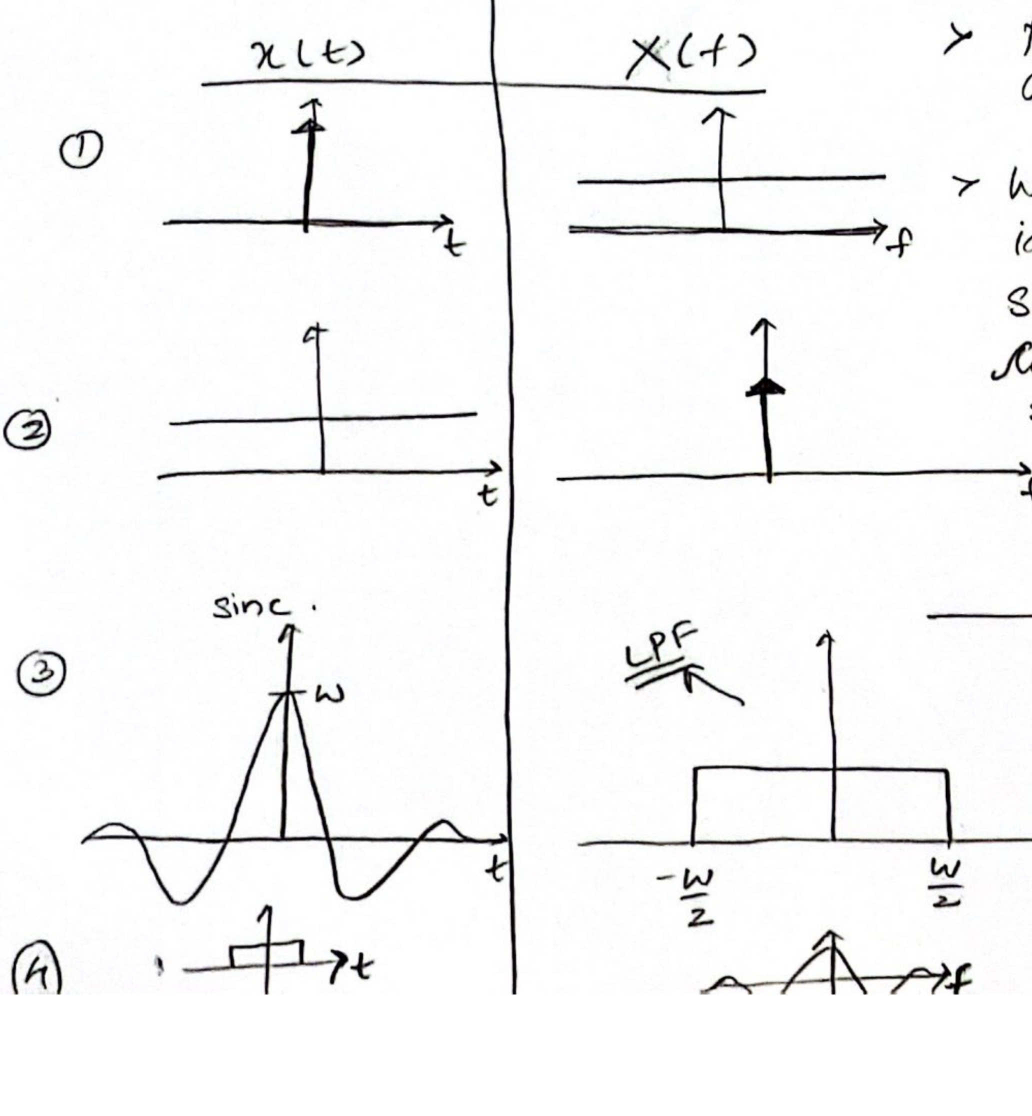
$$X(f) = \int_{-\infty}^{\infty} x(t) \, e^{-j2\pi f t} \, dt = F[x(t)]$$

> Fourier transform is a special case of the laplace transform when $S = j\omega$. The adv. of using F.T is that it has a simple inverse, "it is it's own inverse".

> $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(\omega)\, e^{+j\omega t}\, d\omega \simeq \sum_{n=-\infty}^{\infty} e^{+jn\omega_0 t} \cdot \omega_0 \cdot \frac{X(n\omega_0)}{2\pi}$$

$$\simeq \sum_{n=-\infty}^{\infty} \frac{\omega_0 X(n\omega_0)}{2\pi} e^{+jn\omega_0 t}$$

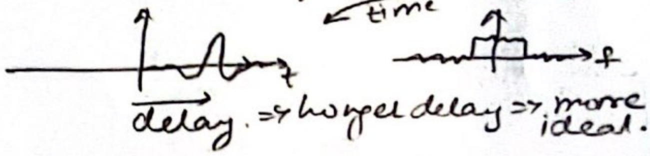
$$= \sum_{n=-\infty}^{\infty} C_n e^{+jn\omega_0 t}$$ → same as fourier series when we remove the infinitesimallness of ω.



> Notice that ① & ② are the 2 limits of ③

> We cannot have an ideal low pass filter since it would violate causality. Notice that signal exists before t = 0.

> A non ideal LPF would be as follows.



⇒ longer delay ⇒ more ideal.

→ Tradeoff b/w BW & delay.

Fourier Transform: Modulation & Sampling

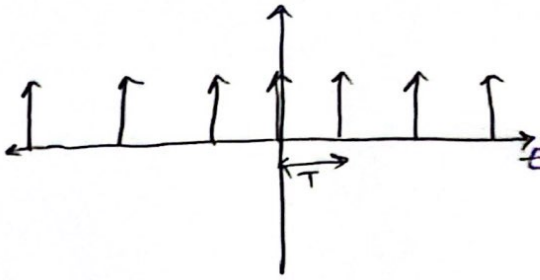
- FT is used for mixing / modulation in transmitters and receivers.

Sampling.

$$Ш_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

"shaw?"

↳ Dirac comb aka impulse train



› Since it is periodic :- $\sum c_n e^{j2\pi f_0 t}$

where, $c_n = \frac{1}{T}\int_{-T/2}^{T/2} Ш_T(t) e^{j2\pi f_0 t} dt$

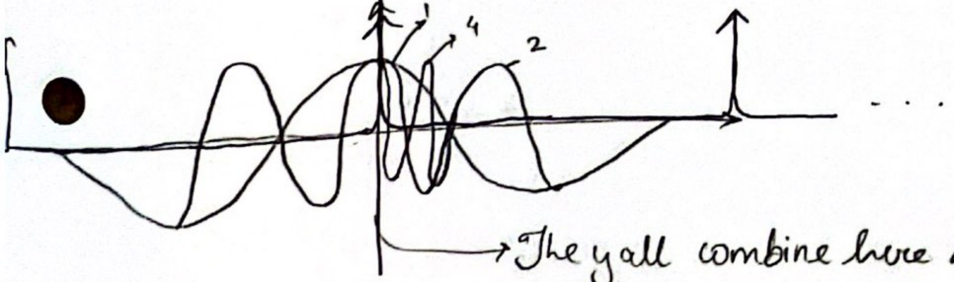
↳ only one impulse falls in this range

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{j2\pi f_0 t} dt$$

1

$$= \frac{1}{T}$$

∴ $Ш_T(t) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} e^{j2\pi n f_0 t}$ ⟹ an infinite sum of cosines are producing this impulse train.



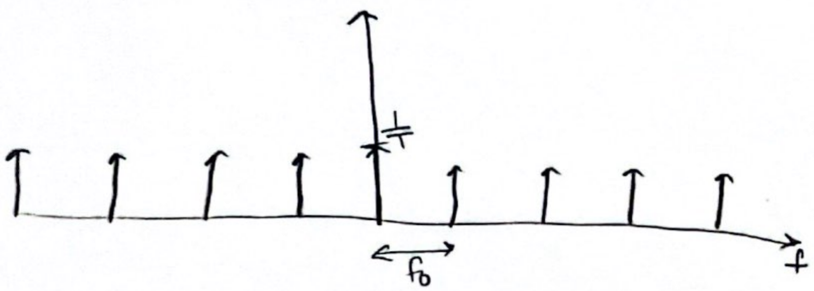
→ They all combine here & cancel everywhere else until the next T, 2T ...

> Taking F.T

$$\frac{1}{T} \cdot Ш_{\frac{1}{T}}(f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

↳ another impulse train $f_0 = \frac{1}{T}$

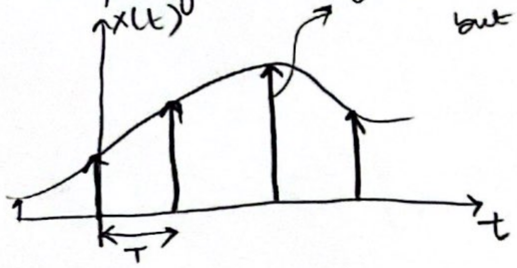
$$\Rightarrow \frac{1}{T} \cdot Ш_{f_0}(f) = \frac{1}{T} \sum_{n=-\infty} \delta\left(f - \frac{n}{T}\right)$$



Think about how?

The impulse train is periodic with all energy concentrated at the natural frequencies. Therefore it can be reconstructed by cosines at those natural frequencies.

Sampling

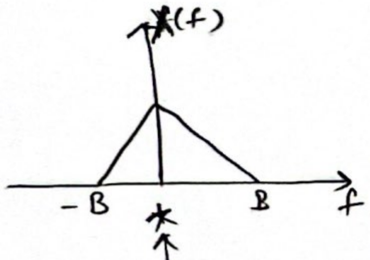


These are still infinite in height but have different areas.

$$y(t) = x(t) \cdot Ш_T(t)$$

$$\Rightarrow Y(f) = X(f) * Ш_{\frac{1}{T}}(f)$$

Let $x(f)$



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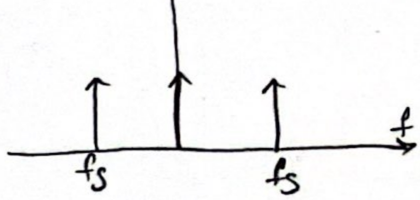
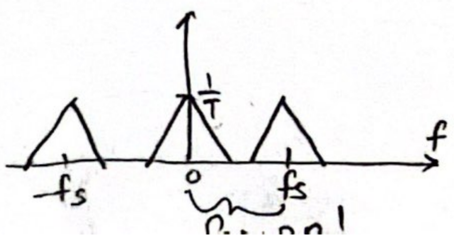




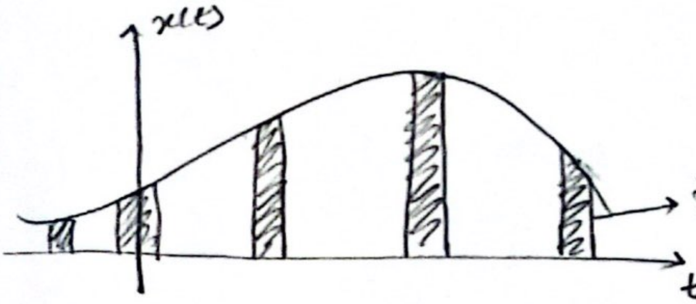
→ Here, $f_s > 2B$ (B-W) to ensure no overlap. This is the Nyquist rate.

→ If $f_s < 2B \Rightarrow$ Aliasing $\Rightarrow$ cannot reconstruct

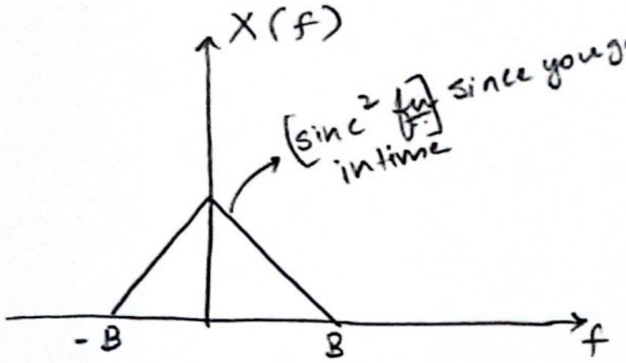
→ To reconstruct just use a LPF @ half the sampling rate.

→ Audio CD sampling is @ 44K samples to cover up to 20KHz. & leave some margin

In reality we have a pulse train

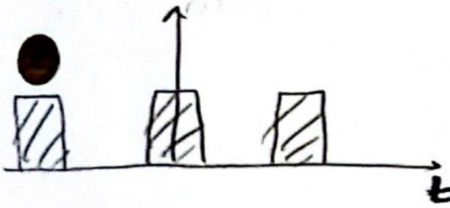


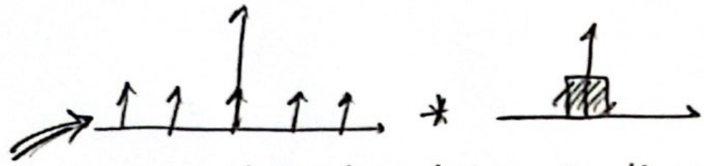
→ track and reset sampling.



[sinc^2 $\frac{f\tau}{F}$] in time since you get it by convolving two pulses [in frequency]

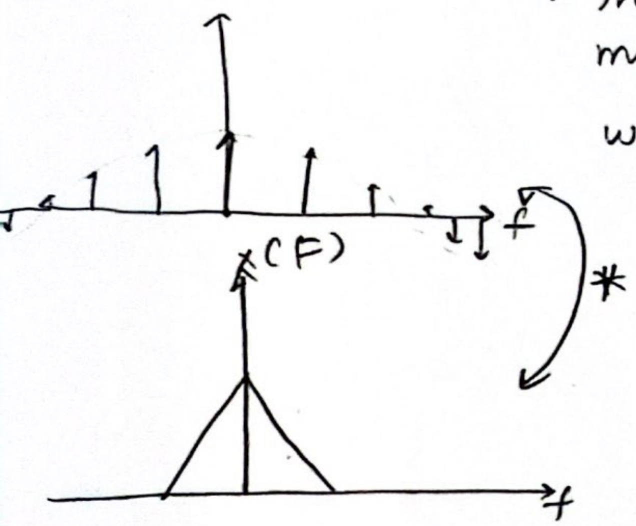
pulse train

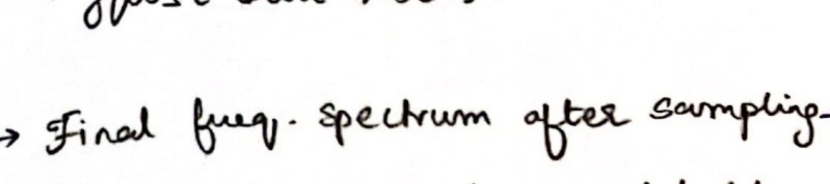




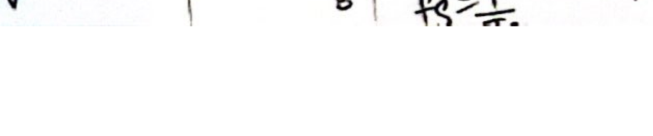
→ convolving an impulse train in time domain with a single pulse gives us a pulse train.

→ In frequency domain it is the same as multiplying the impulse train (it is same in freq domain) with sinc function.





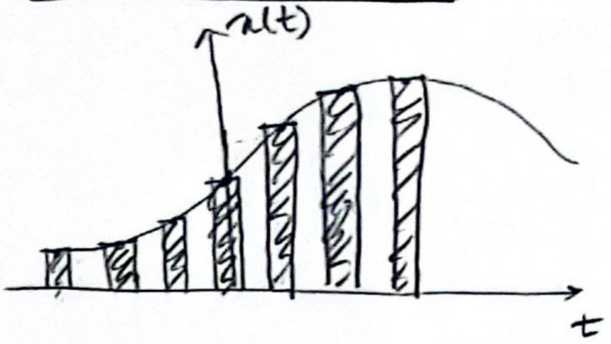
→ Nyquist still holds.



→ Final freq. spectrum after sampling.

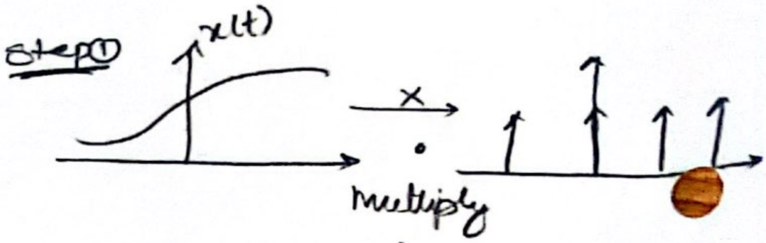
→ In reality we use track and hold & not track and reset since it is better while implementing. Else might lose info.

→ Track and hold.



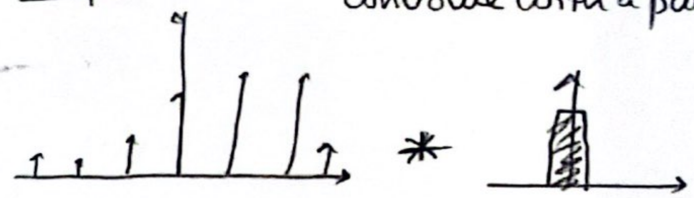
⟸

Step①



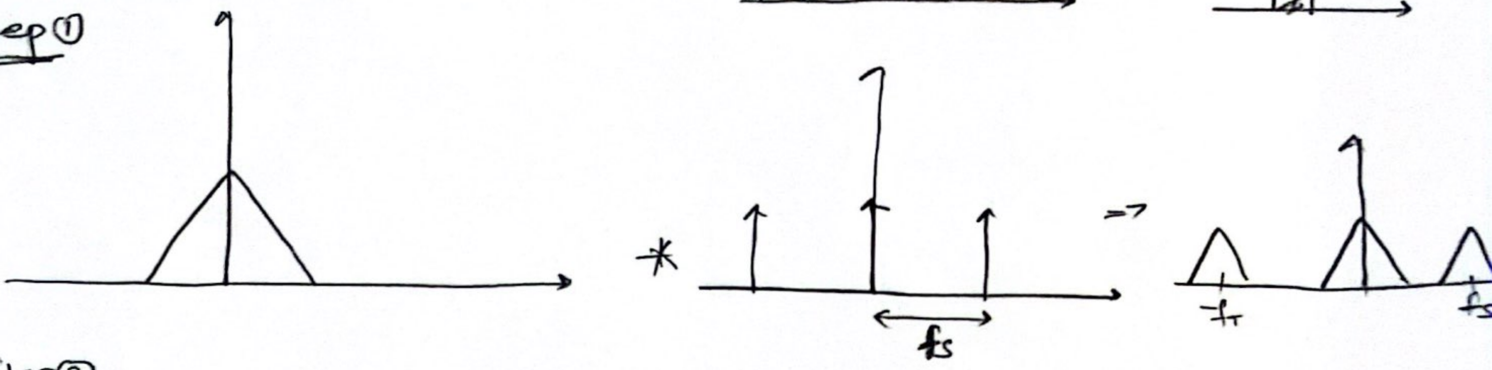
Multiply & convolve with a pulse.

Step②

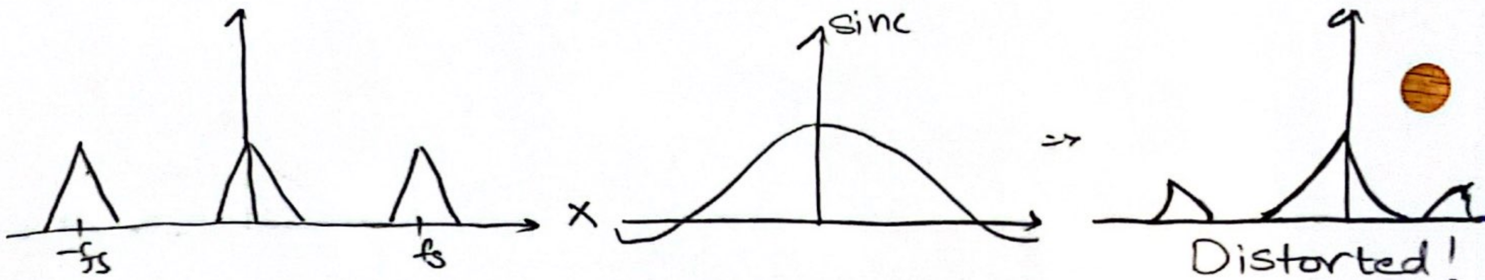


In frequency

Step①



Step②

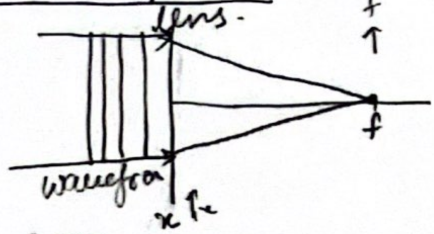


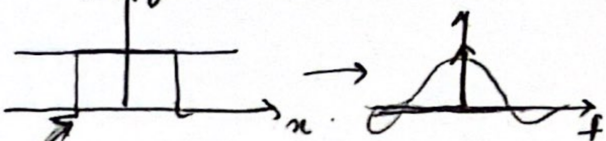
Distorted!

→ Since this distortion is deterministic, it is not a big deal!

→ Track and hold is superior due to info not being lost.

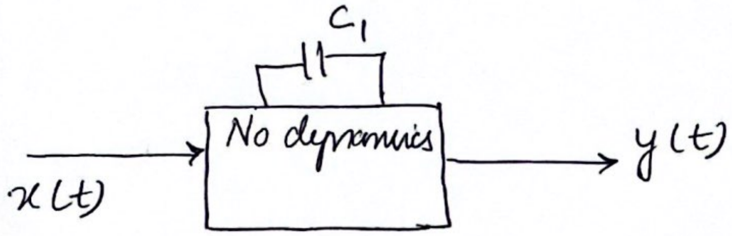
→ Fourier Optics.





window is applied ⇒ sinc fn at f ⟶ Diffraction!

Time and Transfer Constants (TTC)



> For a first order system you could reduce the high frequency TF to 3 low frequency calculation

> First order ⇒ only one energy storing element such as C, L.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H^0 + \tau_1^0 H^1 s}{1 + \tau_1^0 s}$$

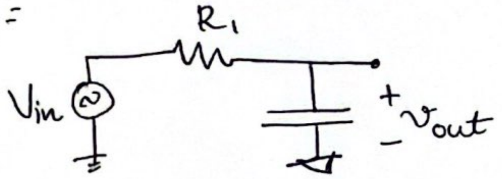
Where the transfer constants

$$H^0 = H\Big|_{C_1 \to 0} \Rightarrow \text{open circuit}$$

$$H^1 = H\Big|_{C_1 \to \infty} \Rightarrow \text{short circuit}$$

$\tau_1^0 = C_1 R_1^0$; $R_1^0 \to$ Resistance seen by the Cap. when we null all independent sources.

Example:-



$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$H^0 = 1$ since C is open

$H^1 = 0$ since C is short

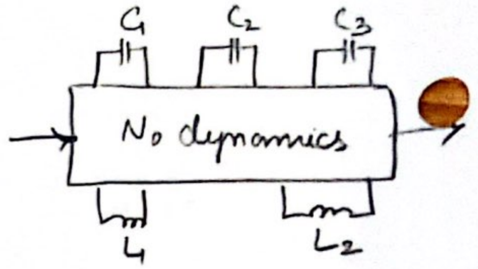
$R_1^0 = R$ since v is short

$\Rightarrow \tau_1^0 = R_1 C_1$

$$\Rightarrow H(s) = \frac{1}{1 + R_1 C_1 s}$$

> For an nth order system.

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \dots + a_m s^m}{1 + b_1 s + b_2 s^2 + \dots + b_n s^n}$$



$a_0 = H^0$

$a_1 = \sum_{i=1}^{n} H^i \tau_i^0$

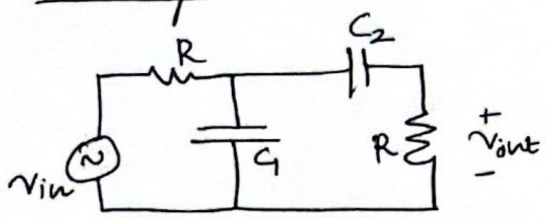
$b_1 = \sum_{i=1}^{n} \tau_i^0$ ↳ sum of zero valued time constants

$= \sum_{i=1}^{n} R_i^0 C_i$ ↳ Resistance seen by R_i when all other L/C are zeroed. or ∞ zeroed.

$a_2 = \sum_{i}\sum_{j}^{i<j} H^{ij} \tau_i^0 \tau_j^i$

$b_2 = \sum_{j=1}^{n}\sum_{i=j}^{n} \tau_i^0 \tau_j^i = \sum_{i}\sum_{j}^{i<j} \tau_i^0 \tau_j^i$ ↳ Time constant of j when i is ∞ Value.

Example.



$H^0 = 0$ both caps are open

$H^1 = 0$ C_1 short C_2 open

$H^2 = \frac{1}{2}$ resistive divider

$H^{12} = 0$ both are shorted

$\tau_1^0 = C_1 R_1^0 = C_1 R$ C_2 open V_{in} nulled.

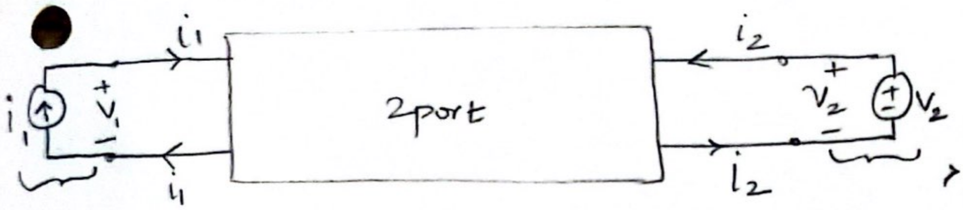
$\tau_2^0 = 2RC_2$ Vin short C_1 open

$\tau_2^1 = RC_2$ C_1 shorted

$$H(s) = \frac{H^2 \tau_2^0}{1 + R(C_1 + 2C_2)s + \tau_1^0 \tau_2^1 s}$$

$$\boxed{H(s) = \frac{RC_2 s}{1 + R(C_1 + 2C_2)s + R^2 C_1 C_2 s^2}}$$

Two port networks



We need $i_1 = i_1$ and $i_2 = i_2$

→ This is done by using a current/ voltage source

LTI

$$\left.\begin{aligned} a_1 v_1 + a_2 v_2 + b_1 i_1 + b_2 i_2 + c_1 = 0 \\ a_3 v_1 + a_4 v_2 + b_3 i_1 + b_4 i_2 + c_2 = 0 \end{aligned}\right\}$$ Two equations to describe a two port network.

c_1, c_2 are governed by independant sources inside.

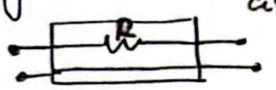
→ No independent sources ⇒ $c_1, c_2 = 0$ ⇒ No offset.

$$\Rightarrow \begin{aligned} a_1 v_1 + a_2 v_2 + b_1 i_1 + b_2 i_2 = 0 \\ a_3 v_1 + a_4 v_2 + b_3 i_1 + b_4 i_2 = 0 \end{aligned}$$

$\Rightarrow \tilde{A}\vec{V} = \tilde{B}\vec{I}$ where $\tilde{A} = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$

$\tilde{B} = \begin{pmatrix} -b_1 & -b_2 \\ -b_3 & -b_4 \end{pmatrix}$

→ Sometimes $\det|A| = 0$ & we cannot do this analysis. ↓Eg is this circuit.



$\vec{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\vec{I} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$

$$\boxed{\vec{V} = \tilde{Z}\vec{I}}$$

$$\boxed{\tilde{Z} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}}$$

$\det|A| \neq 0$ for inverse

$\Rightarrow \vec{V} = A^{-1}B\vec{I} = \tilde{Z}\vec{I}$

where, $\tilde{Z} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$

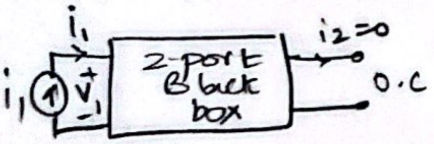
- $v_1 = Z_{11} i_1 + Z_{12} i_2$

  $v_2 = Z_{21} i_1 + Z_{22} i_2$

$Z_{11} = \left.\dfrac{v_1}{i_1}\right|_{i_2=0}$

For LTI systems with no independent sources:

→ Z_{11} find V_1, when i_1 is applied and port 2 is open

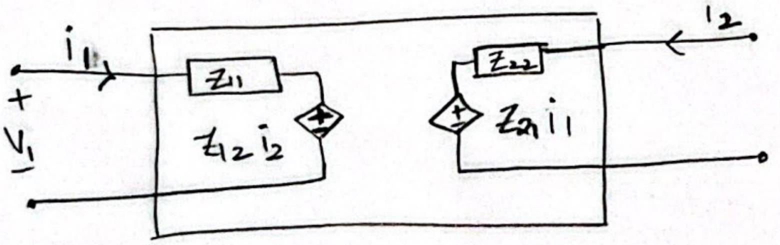


apply i_1 & measure V_2

→ $Z_{12} = \frac{V_1}{i_2}\Big|_{i_1=0}$; $Z_{21} = \frac{V_2}{i_1}\Big|_{i_2=0}$; $Z_{22} = \frac{V_2}{i_2}\Big|_{i_1=0}$

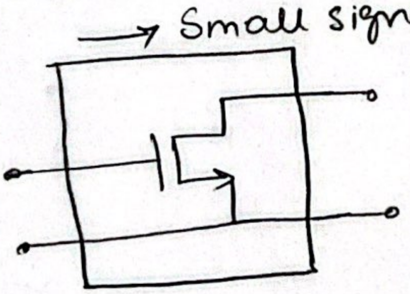
→ Z parameters are to completely characterize an LTI 2 port network. (Often used in network analysers)

→ Can we come up with an equivalent circuit?

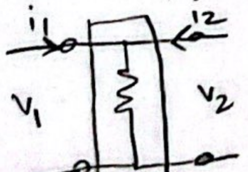


$V_1 = Z_{11} i_1 + Z_{12} i_2$

→ Small signal model is done this way since it is in fact a linearization.



→ Example:-



$Z_{11} = \frac{V_1}{i_1}\Big|_{i_2=0} = R = Z_{22}$

$Z_{21} = \frac{V_2}{i_1}\Big|_{i_2=0} = R = Z_{21}$

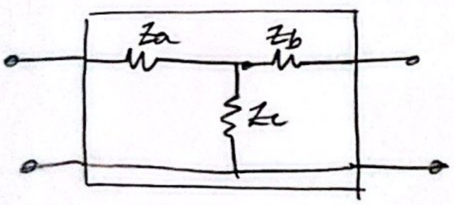
$\Rightarrow Z = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$

Reciprocity of 2 port networks.

- Reciprocal : $Z_{12} = Z_{21} \rightarrow$ Z matrix is symmetrical.

- Let's say $Z_{12} = Z_{21}$

- T or Y matrix



$Z_{11} = Z_a + Z_c$

$Z_{21} = Z_c$

$Z_{22} = Z_b + Z_c$

$Z_{12} = Z_c$

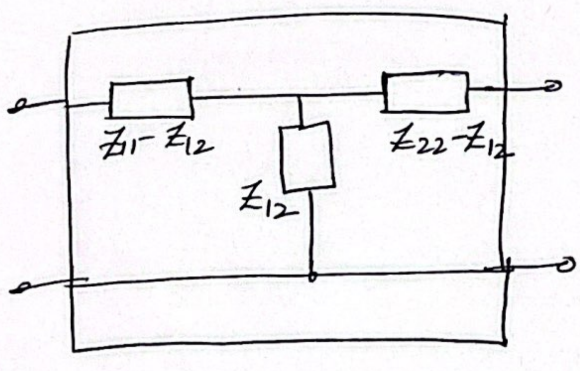
- $Z_{12} = Z_{21} = Z_c$

$\Rightarrow$ $Z_a = Z_{11} - Z_{12}$

$Z_b = Z_{22} - Z_{12}$

$Z_c = Z_{12}$

$\Rightarrow$ Any reciprocal 2port network can be modeled as a T network with Z_a, Z_b, Z_c.



- Could be used for ∇ - Y conversion.
- Generally speaking a network without any sources is reciprocal

> Going back to 2 port:- If $\tilde{A}\vec{V} = \tilde{B}\vec{I}$ & $\det|A| = 0$ we could find $\vec{I} = \tilde{B}^{-1}\tilde{A}^{-1}\vec{V}$ if $\det|B| \neq 0$.

> Called Y parameters:- $\vec{I} = \tilde{Y}\vec{V}$

$$i_1 = y_{11}V_1 + y_{12}V_2$$
$$i_2 = y_{21}V_1 + y_{22}V_2$$

$$Y_{11} = \left.\frac{i_1}{V_1}\right|_{V_2=0 \text{ short!}} \qquad Y_{21} = \left.\frac{i_2}{V_1}\right|_{V_2=0}$$

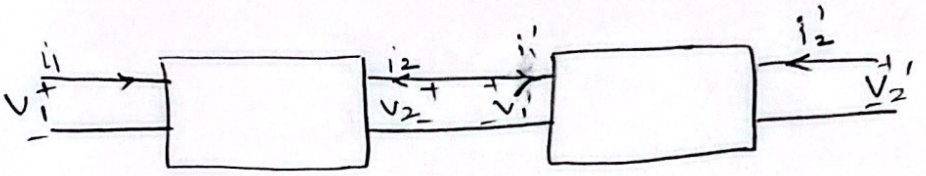
$$Y_{12} = \left.\frac{i_1}{V_2}\right|_{V_1=0} \qquad Y_{22} = \left.\frac{i_2}{V_2}\right|_{V_1=0}$$

> $Y_{11} \neq \frac{1}{Z_{11}}$ since we are nulling different quantities.

> However from $\vec{A}\vec{V} = \vec{B}\vec{I}$ we see that the matrices Z & Y are inverses.

$$\Rightarrow \boxed{\tilde{Z} = \tilde{Y}^{-1}} \rightarrow \frac{\text{cofactor}}{\text{determinant}} \Rightarrow \boxed{Z_{11} = \frac{y_{22}}{y_{11}y_{22} - y_{12}y_{21}}}$$

› How do we characterize multiple 2 port networks.



› Chain (Transmission) parameters (ABCD parameters)

› $\begin{pmatrix} V_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} V_2 \\ i_2 \end{pmatrix}$

$V_2 = V_1'$

But, $i_1' = -i_2$

are they equal?

$\begin{pmatrix} V_1' \\ i_1' \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} V_2' \\ i_2' \end{pmatrix}$

› So, we define $\begin{pmatrix} V_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ -i_2 \end{pmatrix}$

$A = \left. \frac{V_1}{V_2} \right|_{i_2 = 0}$

Apply V_1
& measure V_2
& take the inverse ratio

$B = \left. \frac{-V_1}{i_2} \right|_{V_2 = 0}$

Short V_2 & measure the current through it

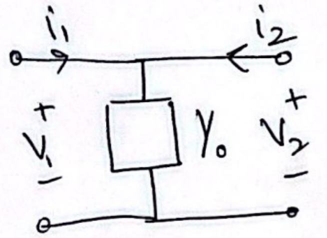
$C = \left. \frac{i_1}{V_2} \right|_{i_2 = 0}$

Apply i_1 & find V_2

$D = \left. -\frac{i_1}{i_2} \right|_{V_2 = 0}$

Apply i_1 & find i_2 through the short.

Eg:-



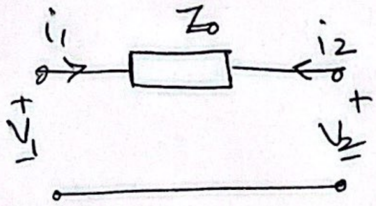
$A = \left.\dfrac{V_1}{V_2}\right|_{i_2=0} = 1$ $\quad C = Y_0$

$B = \left.-\dfrac{V_1}{i_2}\right|_{V_2=0} = 0$ $\quad D = 1$

$i_2 \to \infty$

$\Rightarrow$ ABCD $\begin{pmatrix} 1 & 0 \\ Y_0 & 1 \end{pmatrix}$

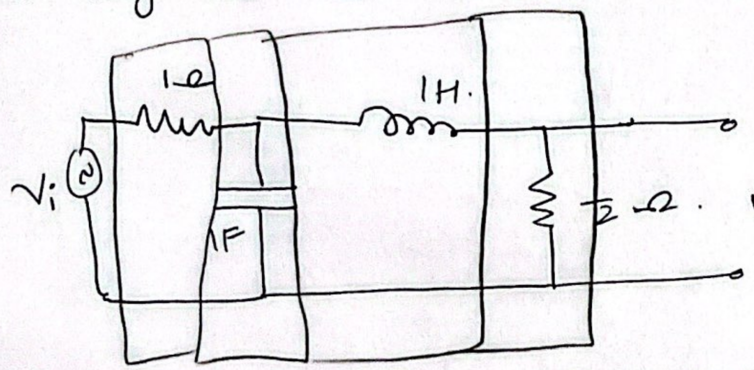
Eg:-



$A = 1$
$B = Z_0$
$C = 0$
$D = 1$

ABCD $\begin{pmatrix} 1 & Z_0 \\ 0 & 1 \end{pmatrix}$

Let's say we have.



$$\begin{pmatrix} V_i \\ i_i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} V_o \\ i_o \end{pmatrix}$$

$$= \begin{pmatrix} s+1 & 1 \\ s & 1 \end{pmatrix}\begin{pmatrix} 2s+1 & s \\ 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} V_i \\ i_i \end{pmatrix} = \begin{pmatrix} 2s^2+3s+3 & s^2+s+1 \\ 2s^2+s+2 & s^2+1 \end{pmatrix}\begin{pmatrix} V_o \\ i_o \end{pmatrix} \Rightarrow H(s) = \frac{V_o}{V_{in}} = \frac{1}{2s^2+3s+3}$$

since $i_o = 0$

- The short coming of ABCD parameters are that we cannot simulate a short/open at the terminals at very high frequencies since an open will have some capacitance and ~~an open~~ short will have some inductance.

  What is the solution?

  → S parameters!

- S parameters are used when the source has a fixed well defined impedance (hopefully R at 50Ω).

- We look at ration of reflected to incident power at the port (2port) and that is S_{ij}.

OP AMPS

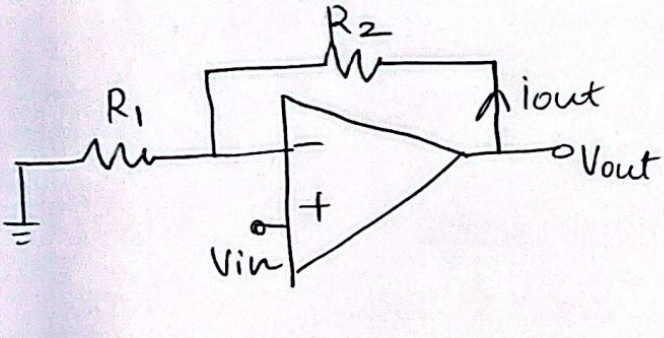
Rule #1 : Both inputs must be at same voltage when negative feedback is used. The output adjusts itself to make sure of this.

Rule #2 : Input impedance is ∞. => No current flows into the inputs. Output impedance is 0. => Can draw as much current as we need.

Rule #3 : With no feedback, the output jumps to the rail if $V_{in+} > V_{in-}$ and vice versa. Since open loop gain is ∞.

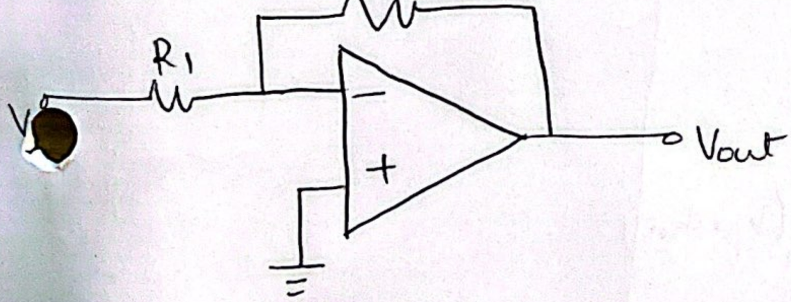
Configurations

① Non inverting amplifier. / Voltage to current converter



$$\frac{V_{out}}{V_{in}} = 1 + \frac{R_2}{R_1} \quad \Bigg| \quad \frac{i_{out}}{V_{in}} = \frac{1}{R_1}$$

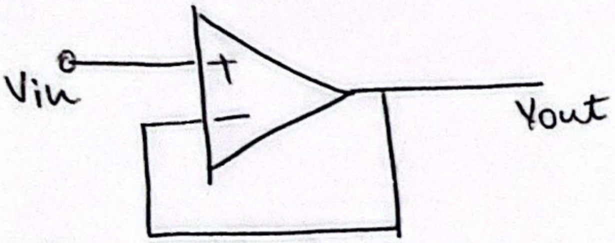
② Inverting amplifier.



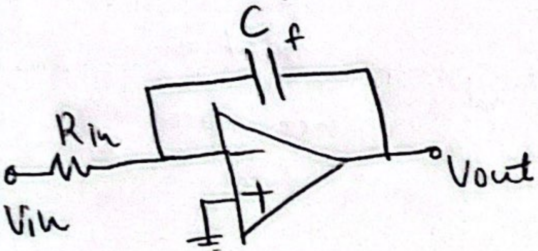
$$\frac{V_{out}}{V_{in}} = -\frac{R_2}{R_1}$$

~~[illegible]~~

Voltage follower / Buffer

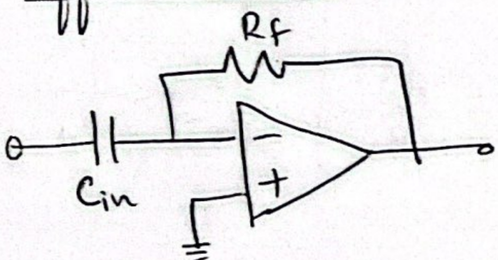


Integrator



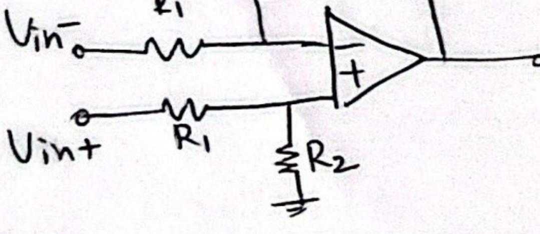
$$V_{out}(t) = -\frac{1}{R_{in} C_f} \int V_{in}(t)\, dt$$

Differentiator



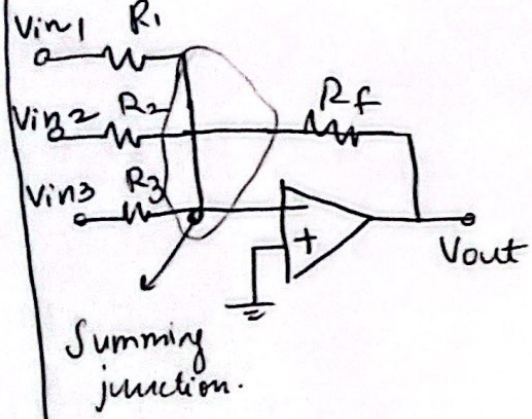
$$V_{out}(t) = -R_f C_{in} \frac{d}{dt} V_{in}(t)$$

Difference / Differential



$$V_{out} = \frac{R_f}{R_{in}} (V_{in+} - V_{in-})$$

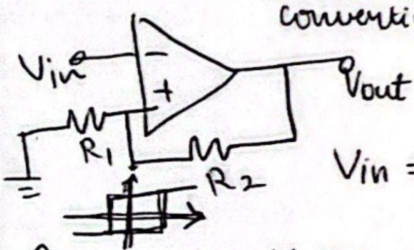
Summing amplifier.



$$V_{out} = -\frac{R_f}{R_1} V_{in1} - \frac{R_f}{R_2} V_{in2} - \frac{R_f}{R_3} V_{in3}$$

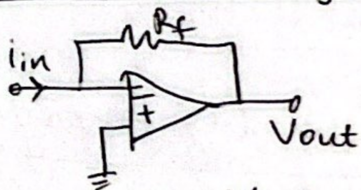
Schmidt trigger.

→ 2 thresholds for converting noisy signals into logic signals.



$$V_{in} = -\frac{R_1}{R_2} V_{out}$$

Current → Voltage (TIA)



$$\frac{V_{out}}{I_{in}} = -R_f$$

Control systems - Poles and zeroes

- Let $G(s) = \dfrac{Y(s)}{X(s)}$ where $g(s) = \mathcal{L}\{g(t)\}$

$$g(t) = y(t) * x(t)$$
$$G(s) = Y(s) \cdot X(s)$$

<u>Poles</u> :- S values at which $s \Rightarrow \infty$

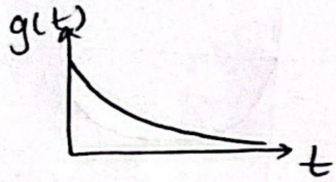
<u>Zeroes</u> :- S values at which $s \Rightarrow 0$

//

Stability from poles and time domain equivalence

- → Let $G(s) = \dfrac{1}{s+2}$ at $S = -2$ we have a pole.

$\Rightarrow g(t) = e^{-2t}u(t) \longrightarrow$



STABLE

→ Let $G(s) = \dfrac{s+2}{s^2+5s} \Rightarrow$ at $s = 0, -5$ where poles exist

$g(t) =$ decaying sinusoidal exponential. [Recall Laplace transforms].

→ ∴ poles $< 0 \Rightarrow$ stable.

poles $= 0 \Rightarrow$ marginally stable.

poles $> 0 \Rightarrow$ unstable.

Control systems - Frequency response

→ When discussing frequency response we only care about $s = j\omega$ term.

→ We can represent the transfer function frequency response on the Re-Im plane of $G(s)$.

Let $G(s) = \dfrac{1}{s+1}$

$G(j\omega) = \dfrac{1}{1+j\omega}$

① At $\omega = 0 \Rightarrow s = 0 \Rightarrow G = \frac{1}{1}$

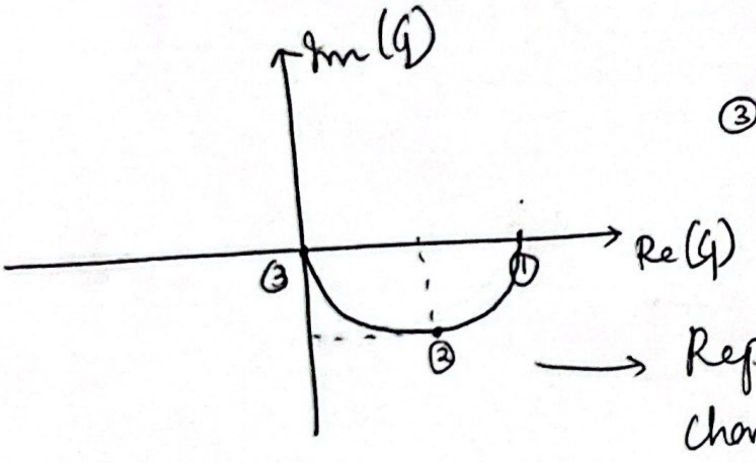
$\therefore \angle G = 0^\circ$ & $|G| = 1$

② At $\omega = 1 \Rightarrow s = j \Rightarrow G = \dfrac{1}{1+j} = \frac{1}{2} - j\frac{1}{2}$

$\angle G = -45^\circ$ & $|G| = \dfrac{1}{\sqrt{2}}$

③ At $\omega = \infty \Rightarrow s = j\infty \Rightarrow G = \dfrac{1}{1+j\infty}$

$\angle G = -90^\circ$ $|G| = 0$



→ Represents how transfer function changes in both Mag & Phase over all frequencies but here we can't visualise the frequency so we use Bode plots.

Control systems - Bode plots

Linear time invariant systems

→ Frequency of input and output is ALWAYS same.

→ For sinusoidal functions only amplitude and phase can change.

$$B \sin \omega t \longrightarrow \boxed{LTI} \longrightarrow A \sin(\omega t + \phi)$$

magnitude & phase change

→ We want to see how magnitude & phase change over the whole frequency spectrum.

∴ Bode plots.

Gain (dB) vs log scale f.

power gain

Phase vs. log f.

(In degrees) ⇒ $2\pi = 360°$

→ Why use $20 \log \frac{V_o}{V_i}$?

Bell telephone labs was trying to quantify loss in audio levels in early 20th century & used log scales since the ear perceives sound on a log scale.

$10 \log_{10} \Delta power$ was approx. the smallest power attenuation detectable.

To honour Alexander Graham bell they called it deci-Bell (since Bell was too large a unit we use deciBell)

→ Bode plot is for all frequencies. So sweep across all the frequencies or use Laplace transform.

→ When multiple systems are cascaded, body plots provide an additive property for both mag. & phase.

→ $X(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$

Solve for $H(s) = \dfrac{Y(s)}{X(s)}$

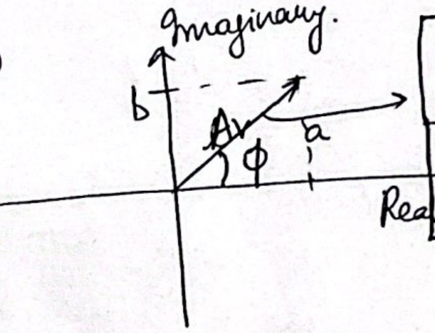
Here $s = \sigma + j\omega$

→ When talking about frequency response of a system we only care about steady state response where the exponential term 'σ' is 0.

→ <u>Steady state $s = j\omega$</u>

$\therefore H(j\omega) = a + jb.$

→ Plotting $H(j\omega)$



$A_v = \sqrt{a^2 + b^2}$

$\phi = \arctan \dfrac{b}{a}$

→ Only Imaginary part changes with ω.

→ We use a system of poles and zero to draw bode plots by hand.

→ Transfer function = $H(s)$

→ S.S freq. response = $H(j\omega)$

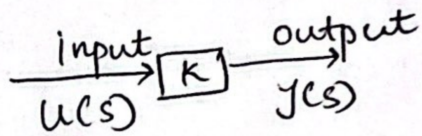
→ $H(j\omega)$ = real + imaginary.

→ gain = $|H(j\omega)|$

→ Phase = $\angle H(j\omega)$

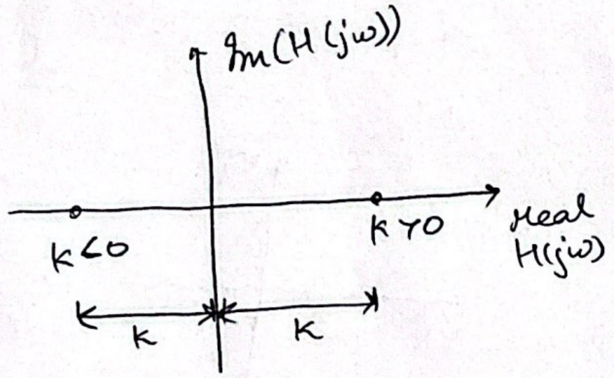
Example

$H(s) = K$ → constant



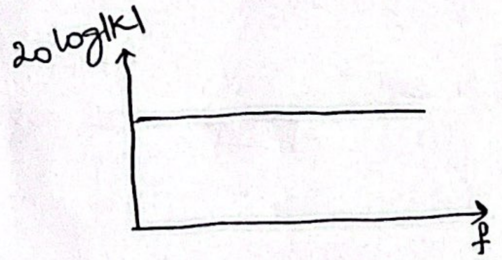
$|H(s)| = K$

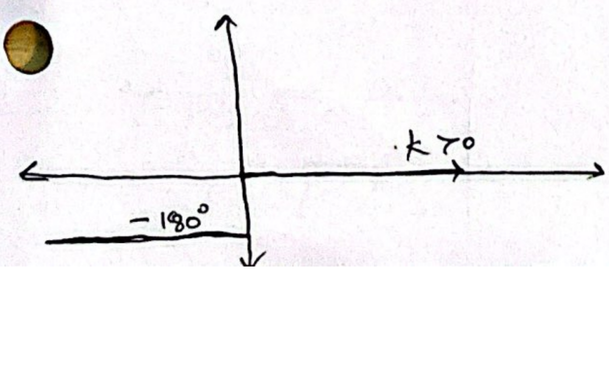
$\arg(H(s)) = \tan^{-1}\left(\frac{0}{K}\right)$



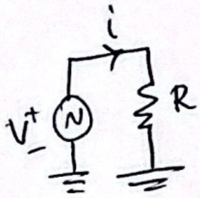
gain is always K

Phase can be 0 or $-180°$
(K>0) (K<0)





Electrical example



$v(t) = i(t)\,R.$

$V(s) = i(s)\,R.$

$\therefore \dfrac{i(s)}{V(s)} = H(s) = \dfrac{1}{R} \rightarrow |H(j\omega)| = \dfrac{1}{R}$

$\angle H(j\omega) = 0^\circ$

Poles and zeroes

→ A pole is a value of 's' for which $H(s) \rightarrow \infty$

→ A zero is a value of 'S' for which $H(s) \rightarrow 0$

→ A pole at origin $\Rightarrow \dfrac{1}{s}$

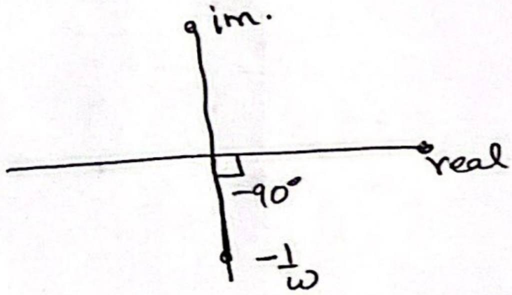
→ A zero at the origin $\Rightarrow s.$

Pole at origin

→ Notice it is laplace transform of $\int$.

$H(s) = \dfrac{1}{s}$

$H(j\omega) = \dfrac{1}{j\omega} = -\dfrac{j}{\omega}$



$\Rightarrow$ real $= 0$

im $= -\dfrac{1}{\omega}$

gain $= \dfrac{1}{\omega}$

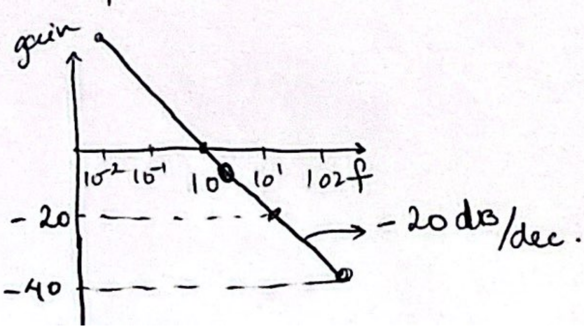
phase $= -90^\circ$

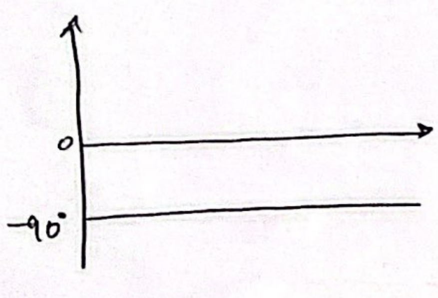
Notice

$\sin\omega t \longrightarrow \boxed{\int dt} \longrightarrow -\dfrac{1}{\omega}\cos\omega t$

phase.

gain





Real non zero pole

$$G(s) = \frac{1}{s+1} \quad \text{pole at } s=-1$$

Let $G(j\omega) = \dfrac{1}{j\omega+1}$

at low frequencies $\quad G(j\omega) \simeq \dfrac{1}{0+1} = 1$

$\Rightarrow \angle G = 0^\circ \; ; \; |G| = 1 \rightarrow$ 0dB

at high frequencies $\quad G(j\omega) \equiv \dfrac{1}{j\omega} = \dfrac{-j}{\omega}$

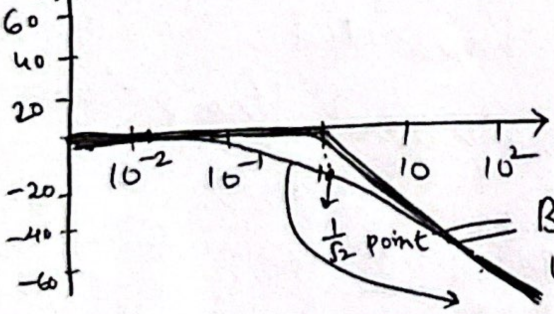
$\Rightarrow \angle G = -90^\circ ; \; |G| = \dfrac{1}{\omega}$

Notice this is -20 dB/dec like a zero pole.

at $\omega = 1 \quad G(j\omega) = \dfrac{1}{1+j}$

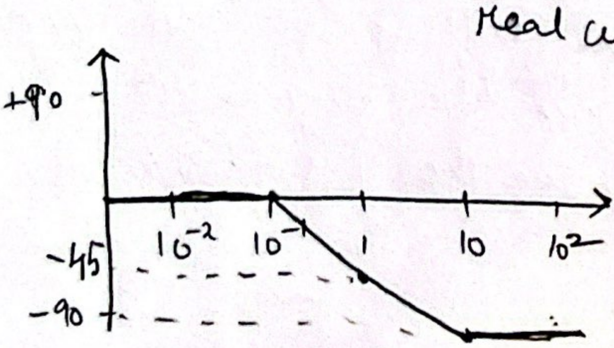
$|G| = \dfrac{1}{\sqrt{2}} . \quad \angle G = -45^\circ$

Approximating Bode plot we see asymptotes at $\omega = 0, \infty$

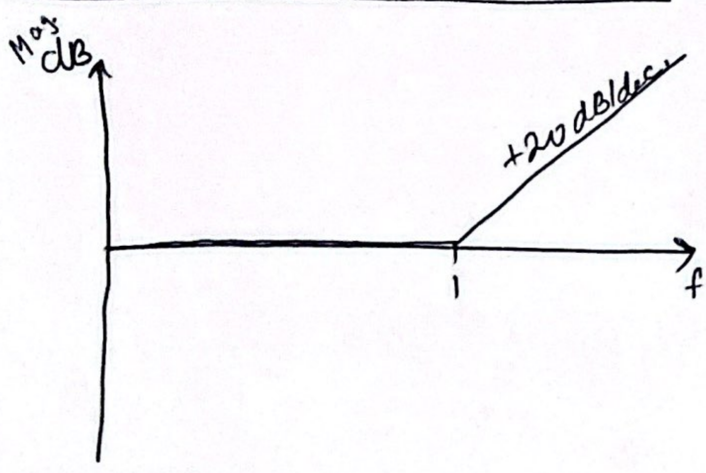


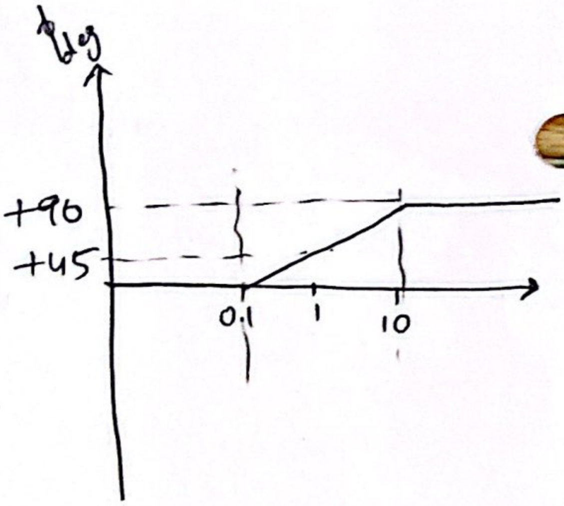
Real curve.

Because $\frac{1}{\sqrt{2}}$ pt. coincides with pole $|s_p|$ we can call it our ω_{-3dB} $\rightarrow$ freq when $20\log|A_V| \rightarrow \frac{1}{\sqrt{2}}$



Real non zero - zero





NOTE

→ Therefore for finding the point at which magnitude plot turns we use $|S_z|$ & $|S_p|$
To find 0.1× pt. at which phase plot turns
and 10× pt. at which it turns we use $|S_z|$, $|S_p|$.
This is just the consequence of how poles and zeroes affect the log plots and does not represent how transfer function goes to ∞ because these points are NOT the poles. We just found these points from the poles.

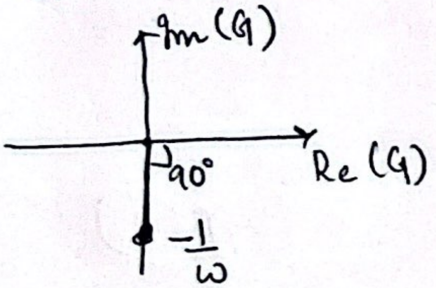
→ To visualize the T.F going to ∞ we also need to consider the σ term. Here we are ignoring it and just looking at ω. Not ignoring σ & looking in time domain we would see the T.F → ∞/0 at poles & zeroes.

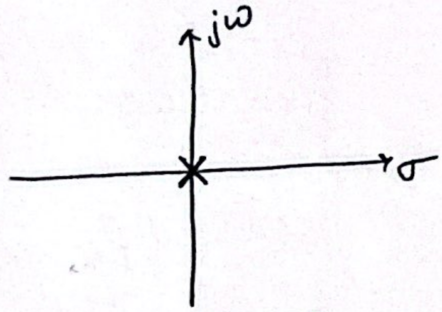
Pole Zero Plot $\longleftrightarrow$ Bode plot

Case ① :- Pole at origin $\rightarrow$ Integrator

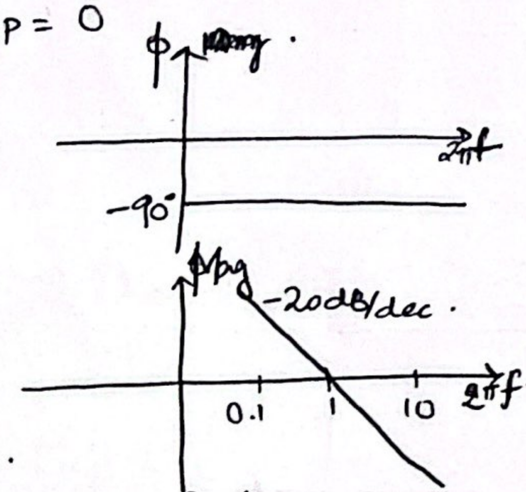
$$G(s) = \frac{1}{s} \Rightarrow G(j\omega) = \frac{1}{j\omega}$$

$$s_p = 0$$





pole zero plot

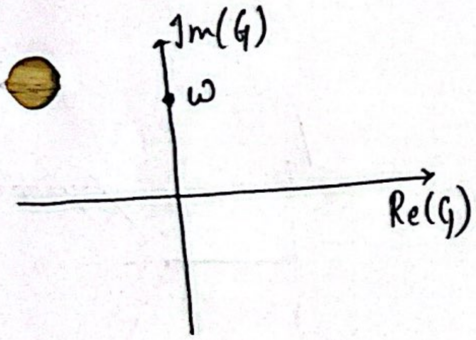


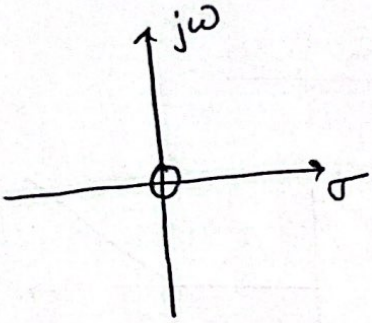
Bode

Case ② Zero at origin $\rightarrow$ Differentiator.

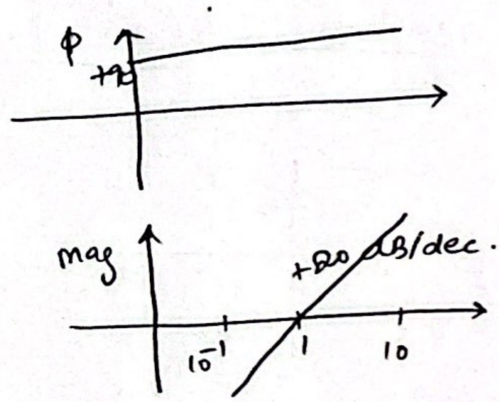
$$G(s) = s \Rightarrow G(j\omega) = j\omega$$

$$s_z = 0$$



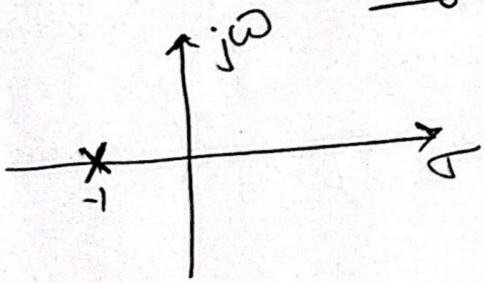


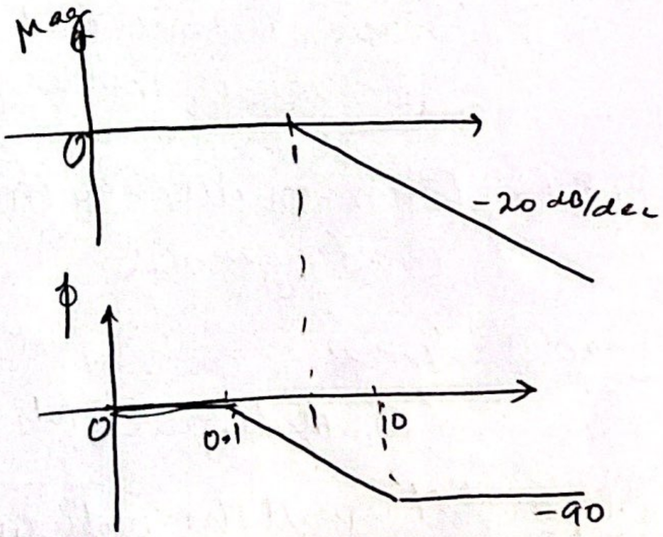
pole zero plot



Case ③ Real pole negative

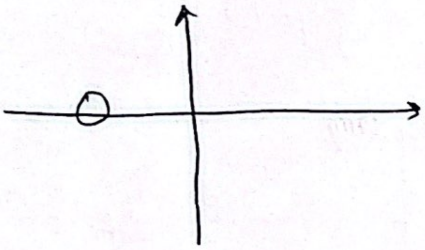
$$G(s) = \frac{1}{s+1} \Rightarrow G(j\omega) = \frac{1}{1+j\omega}$$

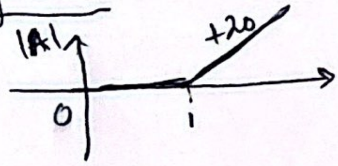


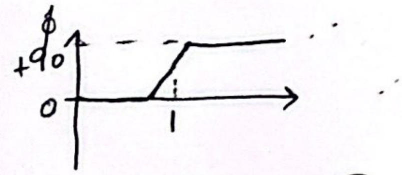


Case ④ Real zero negative

LHP

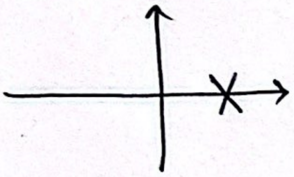


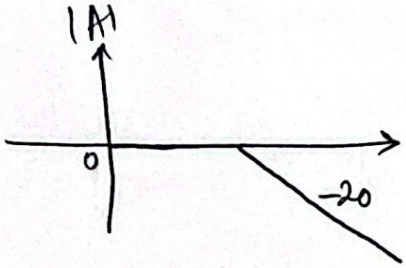


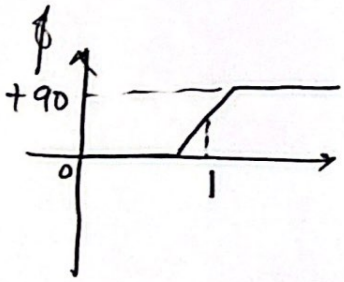


Exactly opposite of a real negative pole.

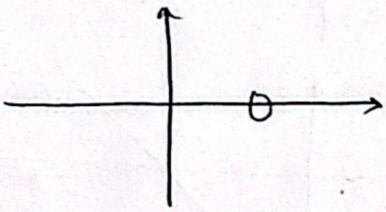
Case ⑤ Real positive pole (Phase is flipped)

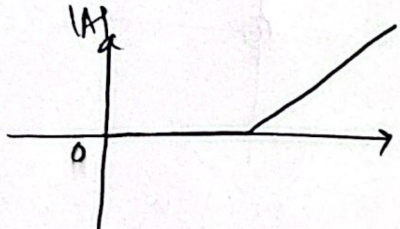


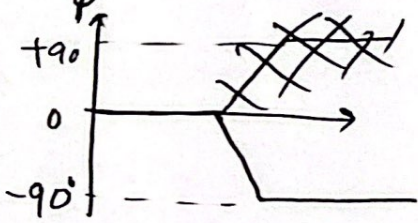




Case ⑥ Real positive zero







Rules for drawing Bode Plot

Step ① Rewrite T.F such that numerator & denominator have 0 constant value. Even $H(s) = M \cdot \dfrac{"Ax^2+Bx+1"}{"\alpha x^2+\beta x+1"}$ for values inside factors. of terms. $\longrightarrow M \cdot \dfrac{(Ax+1)(Bx+1)}{(Cx+1)(Dx+1)}$

Step ② Find multiplier ('m') & (S_p & S_z) poles and zeroes.

Step ③ Draw Bode plots for each pole & zero. and also for multiplier M. Mark the pointers from S_z, S_p

Hint:- This has 0 phase or 180° phase (+ve) (-ve)

Step ④ Add up all the plots since in log scale multiplication is addition.

Let's look at an example

$$G(s) = \frac{s+1}{(s+0.1)(s+10)}$$

$s_z = -1$
$s_{p1} = -0.1$
$s_{p2} = -10$

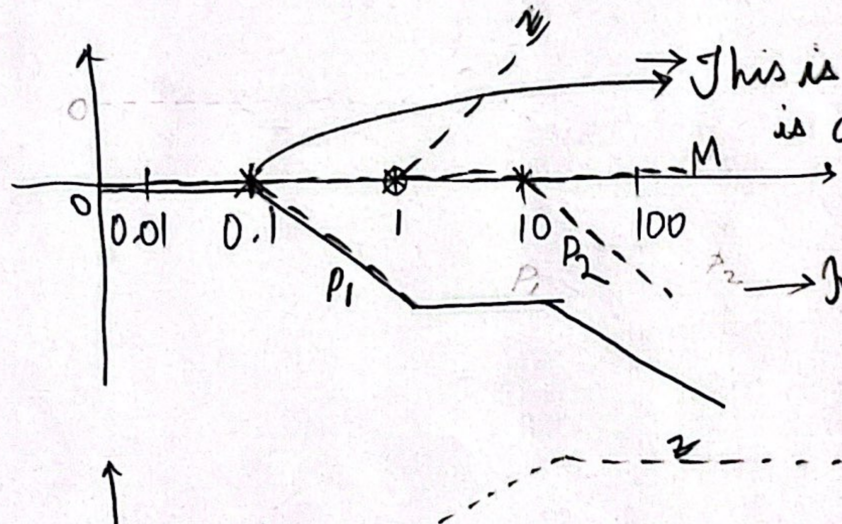
Step ①

$$G(s) = \frac{s+1}{\left(s+\frac{1}{10}\right)(s+10)} = \frac{s+1}{\frac{10}{10}(s\times 10+1)\left(\frac{s}{10}+1\right)}$$

$$= \frac{s+1}{(10s+1)\left(\frac{s}{10}+1\right)}$$

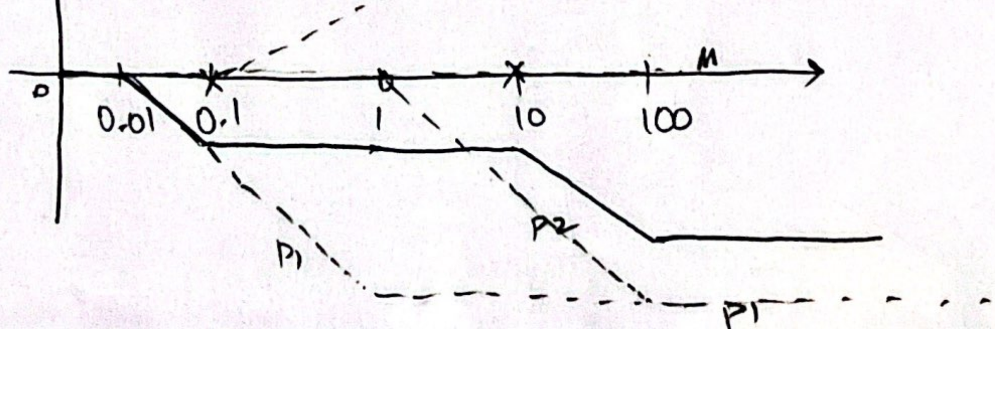
$$= \frac{\frac{s}{s_z}+1}{\left(\frac{s}{s_{p1}}+1\right)\left(\frac{s}{s_{p2}}+1\right)}$$

Step ② $M=1$, $s_z=-1$, $s_{p1}=-0.1$, $s_{p2}=-10$



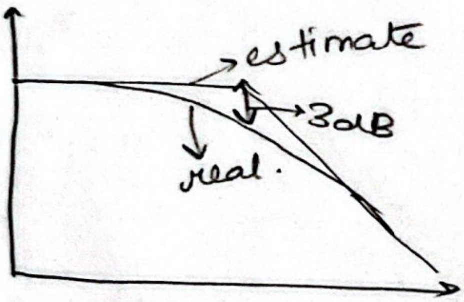
→ This is the small pole and ∴ is dominating the freq response.

→ In case of rising edge, the larger pole would dominate.

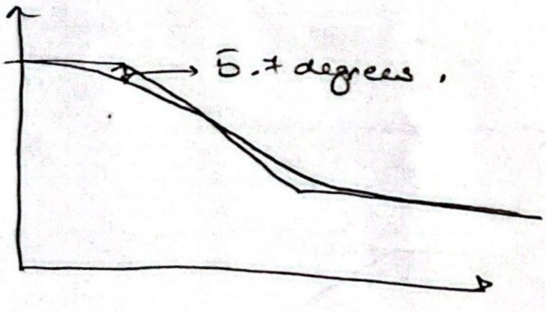


Trivia

- decade is a factor of 10. Whereas, octave is a factor of 2. It comes from western music where freq. of each octave (do re mi fa so la ti do) is double that of the previous octave (From do to do)
- Division into 7 notes is arbitrary to western music.
- 20 dB per decade = 6 dB per octave.
- Magnitude plot max error is 3dB

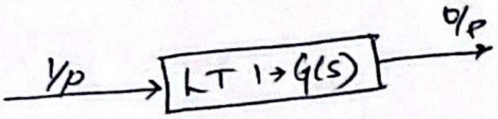


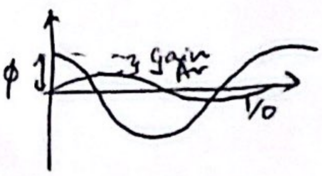
- Phase plot max error is 5.7 degrees.



Gain and Phase Margins

Review :-





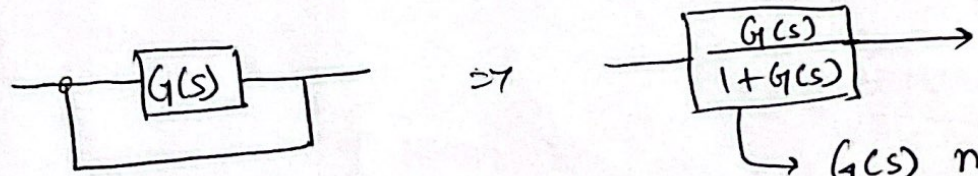
→ Gain & phase depend on f.
∴ Bode plots.

→ Gain margin & phase margin

The extra gain we can use before the system oscillates and becomes unstable.

→ Less margin = less stable since small variants in system can cause instability.

→ But what makes a system unstable?

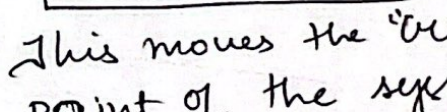


→ G(s) must become −1 for gain of ∞.

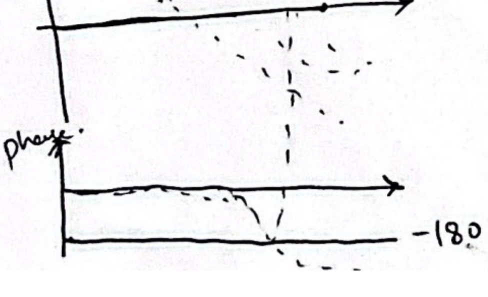
If any one frequency hits −1, the whole system becomes unstable.

=> Gain = 1 → (0dB)
phase = −180°

→ Margin tells us how far we really are from the −1 point.



If we add a gain K to the current system



→ This moves the "cross over" point of the system closer to the −1 point in Gain plot.
=> RISKY. Becomes less stable.

[Margin is how much we can move around the gain & phase before we hit that −1 point.]

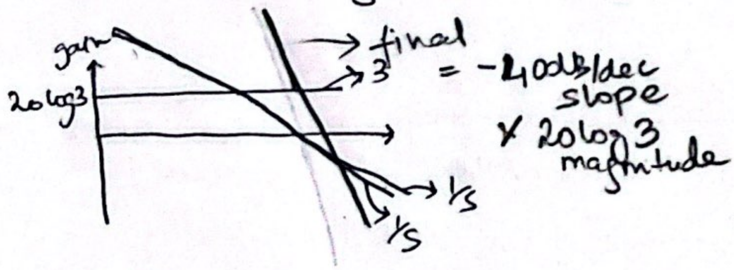
→ Since we use a log scale we can just add. seperate plots drawn for each of the multiplicand:

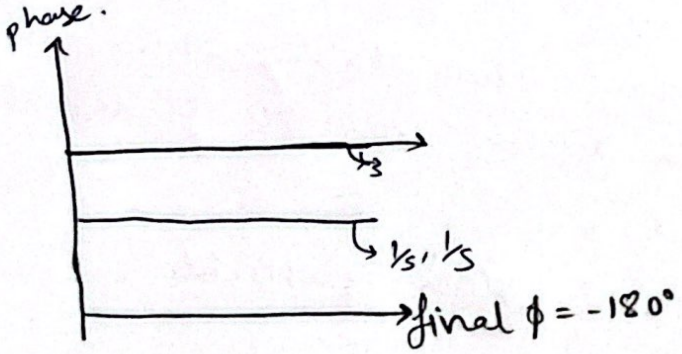
<u>Example</u>

$H(s) = \frac{5}{s} \Rightarrow 5 \cdot \frac{1}{s}$; $H(s) = \frac{3}{s^2} = 3 \cdot \frac{1}{s} \cdot \frac{1}{s}$

↓

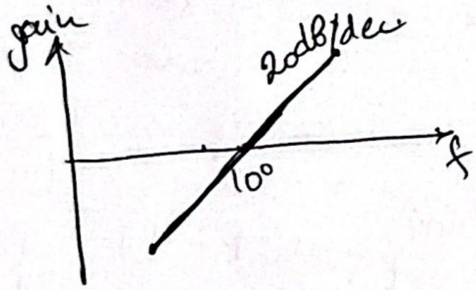
response of 3
+
response of $\frac{1}{s}$
+
response of $\frac{1}{s}$

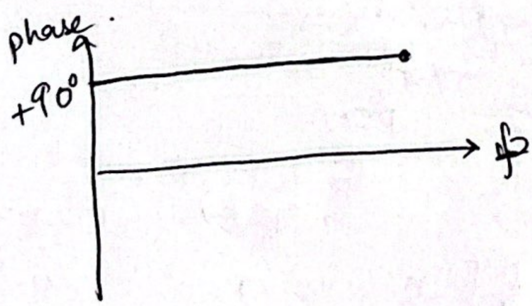




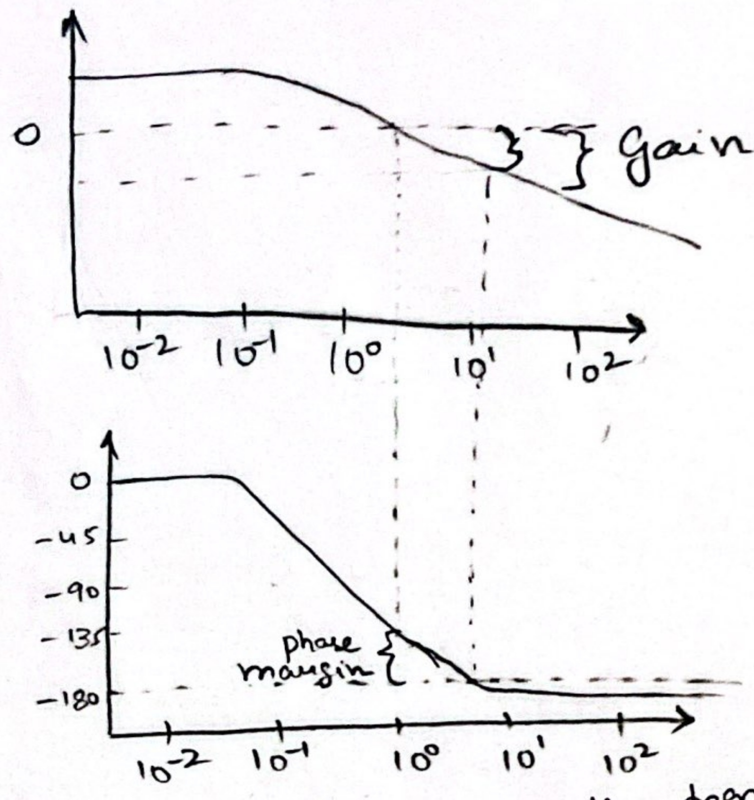
→ <u>Zero at the origin.</u>

$S = \frac{1}{\left(1/s\right)} \rightarrow$ pole $\Rightarrow$ response of 1 − response of 1/s.





Gain & Phase margins on the Bode plot



One parameter being changed in the transfer function can affect both margins since both plots change.