Classical Optics - Light matter interactions.

Reading: Siegman Lasers Ch: 2

Lorentz: Model atomic response as harmonic oscillator.

- Classical model works well with fitting parameters which come from quantum theory: resonance freq ω_0, oscillator strength f, and damping γ.

> Transitions b/w atomic or molecular energy levels corresponds to an oscillation in space of charge.

Lorentz model.

$\vec{p} = -e\vec{r}$ → dipole moment

↳ "strength of a dipole"

> How much torque it would experience in an electric field

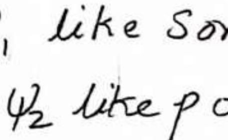
> How much force it would experience in a gradient field.

Qualitative QM approach to atomic dipoles.

> E_1 → ground state with symmetric wavefn. ψ_1 like S orbital.

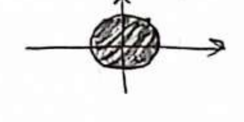
> E_2 → excited state with antisymmetric wavefn. ψ_2 like p orbital.

In 1D: $\psi_1(-x) = \psi_1(x)$; $\psi_2(-x) = -\psi_2(x)$.



→ ψ_1 in 3D

In 3D: $\psi_1(-\bar{r}) = \psi_1(\bar{r})$; $\psi_2(-\bar{r}) = -\psi_2(\bar{r})$



→ ψ_2 in 3D

> Probability density fn. is given by $\psi^*\psi$

> Electronic charge density $\rho(\bar{r}) = -e\,\psi^*(\bar{r})\psi(\bar{r})$ by definition. (since $|\psi|^2$ is prob per unit volume)

⇒ Average position of the electron is $\langle \bar{r} \rangle = \int \psi^*(\bar{r})\,\psi(\bar{r})\, d^3\bar{r}$

⇒ Dipole moment of the system $\vec{p} = -e\langle \bar{r} \rangle$

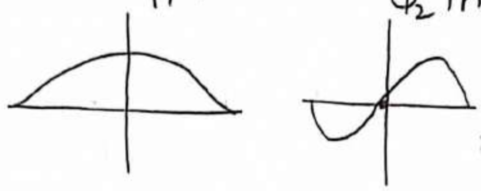
$$\Rightarrow \vec{P} = \int \vec{r}\,(-e\psi^*\psi)\,d^3\vec{r}$$

$$\boxed{\vec{P} = \int \vec{r}\rho(\vec{r})\,d^3\vec{r}}$$ where $\rho(\bar{r}) = -e\psi^*\psi$

> If system is in ψ_1 or ψ_2, $\rho(\bar{r})$ is symmetric and $\bar{r}$ is antisymmetric so $\vec{P} = 0$ which just says the centers of the +ve & -ve charges cancel.

> If the atom is in a superposition state, assume the amplitudes are equal

ψ_1 in 1D $\qquad$ ψ_2 in 1D



$$\Rightarrow \psi_{tot} = \psi_1 + \psi_2$$

Recall, $\psi_1(\bar{r},t) = \psi_1(\bar{r})\,e^{-iE_1t/\hbar}$

$\psi_2(\bar{r},t) = \psi_2(\bar{r})\,e^{-iE_2t/\hbar}$

$\Rightarrow |\psi_1|^2$ & $|\psi_2|^2$ are symmetric but $\psi_1\psi_2^*$ is antisymmetric $\Rightarrow \bar{r}\psi_1\psi_2^*$ is symmetric.

$$\Rightarrow \vec{P} = e\underbrace{\int \bar{r}\,|\psi_1(\bar{r})|^2 d^3\bar{r}}_{0} + e\underbrace{\int \bar{r}\,|\psi_2(\bar{r})|^2 d^3\bar{r}}_{0} + e\int \bar{r}\,\psi_2^*\psi_1\,e^{-i(E_1-E_2)t/\hbar}\,d^3\bar{r}$$

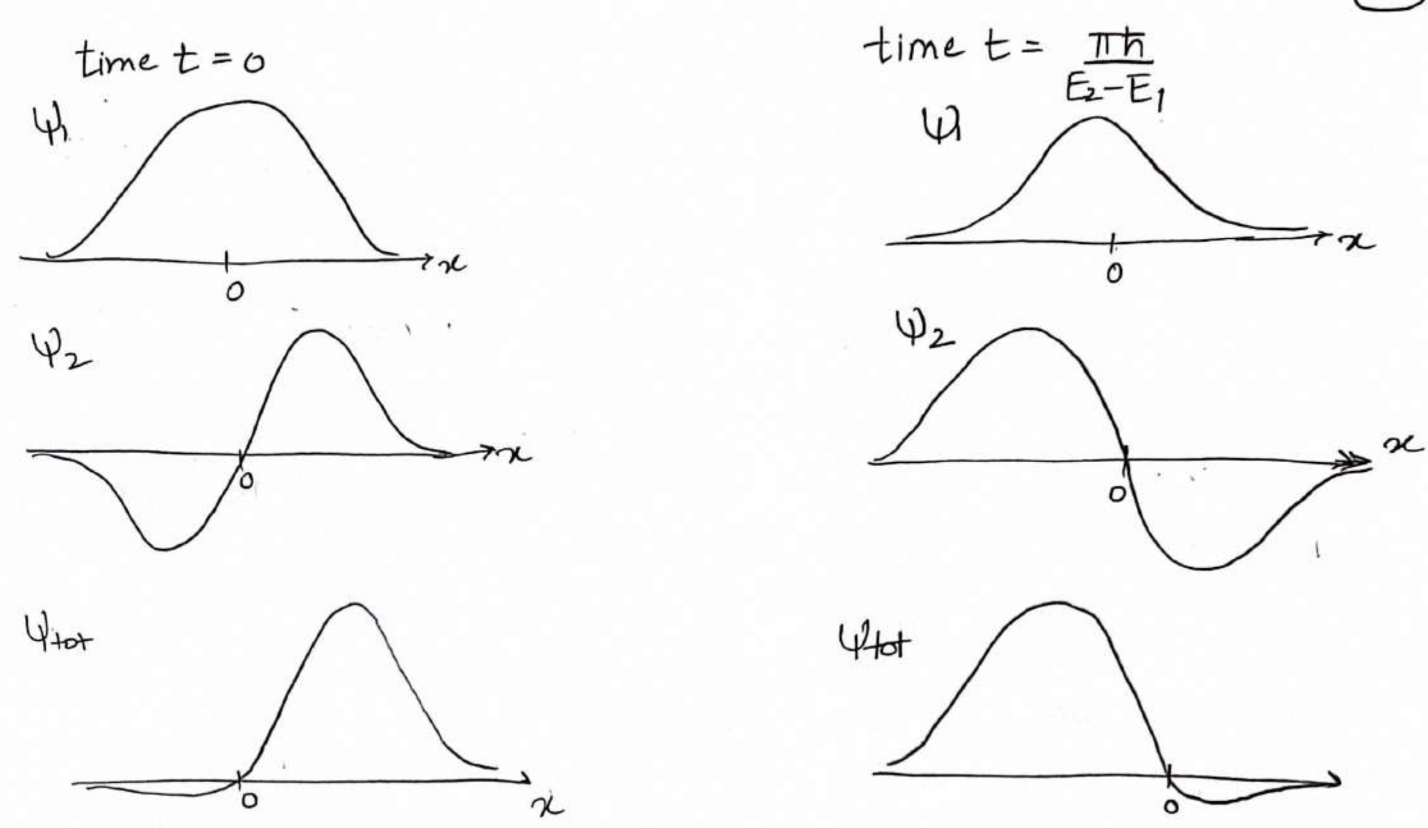
$+$ c.c. of ↑

$\Rightarrow \vec{P}$ oscillates with a frequency given by $E_2 - E_1$! → Hertzian dipole

> Assume Energy origin to be the ground state $\Rightarrow E_1 = 0$

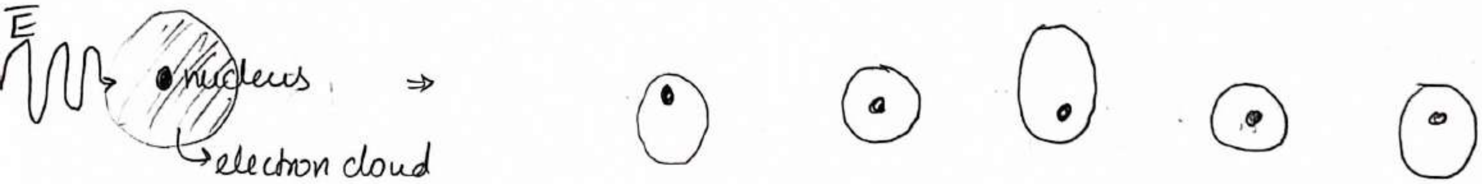
$\Rightarrow \psi_1(\bar{r},t) = \psi_1(\bar{r}) =$ constant in time.

> Let us look at a 1D slice for pictoral insight.



→ Recall charge density $\rho(\vec{r}) = -e\,|\psi_{tot}(x,t)|^2$ → oscillates periodically.

→ How do we put the atom in a superposition state? By illuminating it with light.



This displacement of the electron from S orbital can mathematically be written as "mixing" in some amount of p-orbital. More generally we can perform a mode expansion of the displaced electron cloud as,

$$\psi_{disp} = \sum_{n,l,m} a_{nlm}\,\psi_{nlm}$$

$$= a_{1s}\psi_{1s} + a_{2s}\psi_{2s} + a_{2p}\psi_{2p} + \dots$$

Where n, l, m are the quantum numbers of the valence electron.

where $a_{nlm} = \int \psi_{nlm}^{*}\,\psi_{disp}\,d^3\vec{r}$

Eg:- For hydrogen atom,

> ψ_{1s} is the ground state,

> ψ_{2p}, ψ_{2s} are the first excited states.

> By symmetry a_{2s} will be zero $\rightarrow$ why? and the largest coefficients will be $a_{1s} = \int \psi_{1s}^* \psi_{disp} \, d^3\vec{r}$ and $a_{2p} = \int \psi_{2p}^* \psi_{disp} \, d^3\vec{r}$

> If $\hbar\omega = E_2 - E_1$ for the optical field, these will be the only 2 states.

Review of Electric Dipole Radiation.

Given sources $\bar{J}(\bar{r},t)$ and $\rho(\bar{r},t)$, $\bar{A}$ satisfies-

$$\nabla^2 \bar{A} - \mu_0 \epsilon_0 \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu_0 \bar{J}$$

General soln:

$$\vec{A}(\bar{r},t) = \frac{\mu_0}{4\pi} \int_{\bar{r}'} d^3\bar{r}' \int dt' \, \frac{\bar{J}(\bar{r}',t')}{|\bar{r}-\bar{r}'|} \, \delta\left(t' + \frac{|\bar{r}-\bar{r}'|}{c} - t\right)$$

When $t' = t - \frac{|\bar{r}-\bar{r}'|}{c}$ $\Rightarrow$ when $t = \frac{|\bar{r}-\bar{r}'|}{c}$ we have $t' = 0$

$\Rightarrow$ t' clock starts when then & follows what happened at $t=0$ from then on.

$\Rightarrow$ Basically you remove $\frac{|\bar{r}-\bar{r}'|}{c}$ from t & that time is t' when the δ function fires & extracts the value of current accordingly. It extracts a "retarded" value of the current density. Each pt. in space of the current distribution provides a contribution at $\vec{r}$ which are added up!

If the current is harmonic,

$$\bar{J}(r,t) = \bar{J}(\bar{r})\, e^{i\omega t}$$

$$\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int_{V'} \bar{J}(\bar{r}')\, \frac{e^{-ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\, d^3\bar{r}'.$$

In optics $|\bar{r}'| \ll |\bar{r}|$ $\Rightarrow$ $|\bar{r}-\bar{r}'| = r - \underbrace{\bar{r}'\cdot\hat{r}}_{\cong 0} = r$

$\Rightarrow$

$$\bar{A}(\bar{r}) \cong \frac{\mu_0}{4\pi} \frac{e^{-ikr}}{r} \int_{V'} \bar{J}(\bar{r}')\, d^3r'$$

Note $\bar{J} = 0$ on the surface since V' bounds the charges completely.

$$\int_{V'} \bar{J}(\bar{r}')\, d^3\bar{r}' = -\int_{V'} \bar{r}'\, (\nabla\cdot J)\, d^3\bar{r}' \longrightarrow \text{integration by parts in 3D.}$$

<u>Digression</u>

This expression & the necessary condition $\bar{J} = 0$ are not a trivial formulation. <u>Proof</u>:

Consider,

$$\nabla\cdot(x\bar{J}) = \nabla x\cdot\bar{J} + x\,\nabla\cdot\bar{J} = J_x + x\,\nabla\cdot\bar{J}$$

Similarly $\nabla\cdot(y\bar{J}) = J_y + y\,\nabla\cdot\bar{J}$; $\nabla\cdot(z\bar{J}) = J_z + z\,\nabla\cdot\bar{J}$

$$\Rightarrow \sum_i \hat{x}_i\, (\nabla\cdot(x_i\bar{J})) = \bar{J} + \bar{r}(\nabla\cdot\bar{J})$$

$$\Rightarrow \int_{R^3} \sum_i \hat{x}_i (\nabla\cdot(x_i\bar{J}))\, d^3R = \int_{R^3} \bar{J}\, d^3R + \int_{R^3} \bar{r}\,(\nabla\cdot\bar{J})\, d^3R$$

$\Rightarrow \sum_i \hat{x}_i \oint_S (\hat{n}\cdot(x_i\bar{J}))\, d^3R$ = RHS. But $\hat{n}\cdot\bar{J} = 0$ on the surface

$\Rightarrow \oint = 0$ $\Rightarrow$ RHS $= 0$

This is analogous to the integration by parts in 1D.

$$\int_a^b d(uv) = \int_a^b u\,dv + \int_a^b v\,du$$

$$\Rightarrow [uv]_a^b = \int_a^b u\,dv + \int_a^b v\,du$$

Therefore when uv is 0 at the "boundaries" $uv\big|_a^b = 0$

$$\Rightarrow \int_a^b u\,dv = -\int_a^b v\,du$$

> In 3D instead of $d(uv) = u\,dv + v\,du$ we used $\nabla.(x\bar{J}) = \nabla x . \bar{J} + x\,\nabla.\bar{J}$ & used the div. thm to form a surface integral that vanished.

> However it is not always a trivial matter to get the right form of the 3D diff eqn.

<u>Coming back.</u>

$$\int_{V'} \bar{J}(\bar{r}')d^3\bar{r}' = -\int_{V'} \bar{r}'(\nabla.\bar{J})\,d^3\bar{r}' = i\omega\int \bar{r}'\rho(\bar{r}')\,d^3\bar{r}'$$

But we know that $\vec{P} = \int_{V'} \bar{r}'\underbrace{\rho}_{\to -e|\psi|^2 \to \text{prob. density} \Rightarrow \text{prob. per unit volume.}}(\bar{r}')\,d^3\bar{r}'$ from earlier derivation.

$$\Rightarrow \boxed{\vec{A} = i\omega\frac{\mu_0}{4\pi}\frac{e^{-ikr}}{r}\vec{P}}$$

where $\vec{P}$ is the dipole moment of the atomic system with the charges in their "cloud" distribution

Assume $\vec{P} = p_0\,\hat{z}\cos\omega t$ we can find the fields.

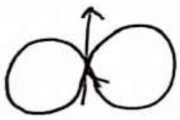
$$\vec{E} = (E_r, E_\theta, E_\phi) = \left(0, \frac{-\omega^2 p_0}{4\pi\epsilon_0 c^2} \sin\theta \frac{\cos\omega(t - \frac{r}{c})}{r}, 0\right)$$

$$\vec{H} = (H_r, H_\theta, H_\phi) = \left(0, 0, -\frac{\omega^2 p_0}{4\pi c} \sin\theta \frac{\cos\omega(t-\frac{r}{c})}{r}\right)$$

$$\vec{S} = (S_r, S_\theta, S_\phi) = \left(\frac{\omega^4 p_0^2}{16\pi^2\epsilon_0 c^3} \sin^2\theta \frac{\cos^2\omega(t-\frac{r}{c})}{r^2}, 0, 0\right)$$

Note: Radiated power $\propto$ square of dipole moment magnitude

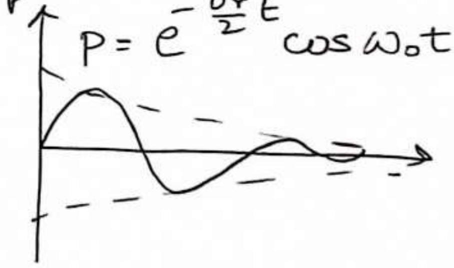
" " $\propto \omega^4$

Power pattern is that of a Hertzian dipole 

Average power radiated

$$\langle W \rangle = \frac{p_0^2 \omega^4}{12\pi\epsilon_0 c^3}$$

Radiative decay



Energy of an oscillator,

$$U(t) = \frac{1}{2} m\omega_0^2 z(t)^2 + \frac{1}{2} m\left(\frac{dz}{dt}\right)^2$$

Where $z(t) = -\frac{1}{e} P = -\frac{1}{e} e^{-\frac{\gamma_r}{2}t} \cos\omega_0 t$

We can show $U(t) = -\frac{1}{\gamma_r} \frac{dU(t)}{dt}$

Equating $\frac{dU(t)}{dt}$ to $\langle W \rangle$ we get

replaced in QM by atomic dipole matrix element

spatial avg.

$$\gamma_r = \frac{1}{\tau_r} = \frac{1}{4\pi\epsilon_0}\left(\frac{2\pi^2 e^2}{m}\right) \cdot \frac{1}{3} \cdot \left(\frac{\omega_0^2}{\pi^2 c^3}\right)$$

DOS in 3D of free space.

> τ_r is the classic radiative lifetime.

& energy decays exponentially as $U(t) = U(0)\, e^{-t/\tau_r}$.

In general $\gamma_r = \frac{1}{4\pi\epsilon_0} \rho_0(\omega)$ ⟶ DOS seen by the dipole (including an angular average)

⇒ Decay rate can be engineered by modifying the density of EM modes with which the dipole interacts.

> The modification of radiative decay via control of the DOS is known as the Purcell effect in Quantum theory of radiation.

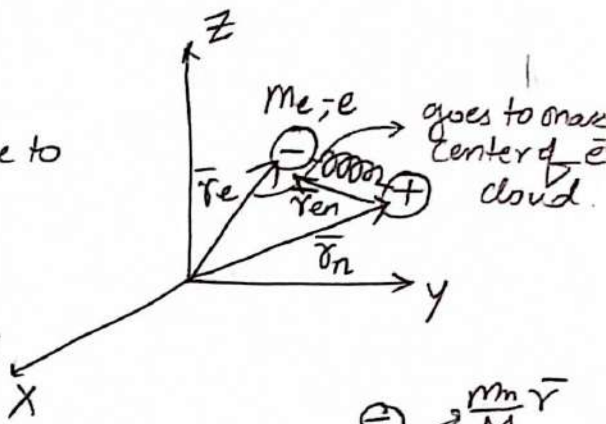
> Modern photonics research focuses a lot on controlling the τ_r through DOS using waveguides, crystals, metamaterials etc.

> $\tau_r \simeq 10ns$ but for lasers we can get $\simeq \mu s/ms$ since they can operate in "forbidden regions".

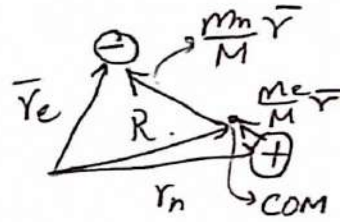
Driven Lorentz Oscillator

- $\overline{r}_{en} = \overline{r}_e - \overline{r}_n$

  $\vec{F}_{en} = -\vec{F}_{ne}$ is the force on the electron due to the ion/nucleus.



This is not a coulomb force but rather a harmonic restoring force when slightly displaced. This can be justified for small displacements in QM.

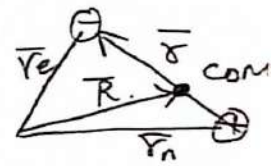


$$\vec{F} = q(\overline{E} + \overline{v} \times \overline{B}) \cong q\overline{E}$$

$$\Rightarrow \quad m_e \frac{d^2\vec{r}_e}{dt^2} = -e\,\overline{E}(\overline{r}_e, t) + \overline{F}_{en}(\overline{r}_{en})$$

Newton's Law $F = ma$.

- $$m_n \frac{d^2\overline{r}_n}{dt^2} = e\,\overline{E}(\overline{r}_n, t) + \overline{F}_{ne}(\overline{r}_{en})$$



Introducing a center of mass coordinate $\vec{R} = \dfrac{m_e\overline{r}_e + m_n\overline{r}_n}{m_e + m_n}$

we have $\overline{r}_e = \overline{R} + \frac{m_n}{M}\overline{r}$; $\overline{r}_n = \overline{R} - \frac{m_e}{M}\overline{r}$ where $\overline{r} = \overline{r}_{en}$ & $M = m_e + m_n$

Subs. into Newton's Eqs.

$$m_e \frac{d^2\overline{R}}{dt^2} + m\frac{d^2\overline{r}}{dt^2} = -e\,\overline{E}\left(\overline{R} + \frac{m_n}{M}\overline{r}, t\right) + \overline{F}_{en}(\overline{r})$$

- $$m_n \frac{d^2\overline{R}}{dt^2} - m\frac{d^2\overline{r}}{dt^2} = e\overline{E}\left(\overline{R} - \frac{m_e}{M}\overline{r}, t\right) - \overline{F}_{en}(\overline{r})$$

Where, $m = \dfrac{m_e m_n}{M}$ (close to m_e). We want to decouple $\overline{r}$ & $\overline{R}$

Center of mass motion

Adding the 2 eqns $\Rightarrow$

$$M\frac{d^2\bar{R}}{dt^2} = e\left[\bar{E}\left(\bar{R}-\frac{m_e}{M}\bar{r},t\right) - \bar{E}\left(\bar{R}+\frac{m_n}{M}\bar{r},t\right)\right]$$

Taylor expansion, (It is valid since $\frac{\bar{r}\cdot\nabla E}{E}$ is small due to dipole approx)

$$E_j(x+\Delta x) \cong E_j(x) + \frac{\partial E_j}{\partial x}\Delta x.$$

In 3D, $\frac{\partial E_j}{\partial x} = \hat{x}\cdot\nabla E_j$

$$\Rightarrow E_j(\bar{R}+\Delta R) \cong E_j(\bar{R}) + \left(\sum_{j=1}^{3}\Delta R_j\,\hat{x}_j\right)\cdot\nabla E_j \qquad j = x,y,z$$

In vector form

$$\bar{E}(\bar{R}+\Delta\bar{R}) \cong \bar{E}(\bar{R}) + \overline{\Delta R}\cdot\nabla\bar{E}$$

$$\Rightarrow \bar{E}\left(R \pm \frac{m_l}{M}\bar{r},t\right) \simeq \bar{E}(\bar{R},t) \pm \frac{m_l}{M}\bar{r}\cdot\nabla\bar{E}(\bar{R},t) \quad ; \quad \text{where } l = n, e$$

$$\Rightarrow M\frac{d^2\bar{R}}{dt^2} = e\left[-\frac{m_e}{M}\bar{r}\cdot\nabla\bar{E}(\bar{R},t) - \frac{m_n}{M}\bar{r}\cdot\nabla\bar{E}(\bar{R},t)\right]$$

$$\Rightarrow \boxed{M\frac{d^2\bar{R}}{dt^2} = -e\bar{r}\cdot\nabla\bar{E}(\bar{R},t)}$$

The COM is accelerated by a gradient in the incident field.

> The dipole moment of a Lorentz oscillator depends on $\bar{E}$ by a factor α called polarizability.

$$\Rightarrow \vec{P} = -e\bar{r} = \alpha\vec{E} \rightarrow \text{linear assumption.}$$

$$\Rightarrow M\frac{d^2\bar{R}}{dt^2} = \alpha \vec{E}\cdot\nabla\vec{E}$$

Note, $\nabla(\bar{E}\cdot\bar{E}) = \bar{E}\cdot\nabla\bar{E} + (\nabla\bar{E})\cdot\bar{E} = 2\bar{E}\cdot\nabla\bar{E}$

$$\Rightarrow \boxed{M\frac{d^2\bar{R}}{dt^2} = \frac{\alpha}{2}\nabla(\bar{E}\cdot\bar{E})}$$

$\bar{E}$ is a real field & $\therefore$ we have $\bar{E}\cdot\bar{E}$ & not $E^*\cdot\bar{E} = |\bar{E}|^2$

$\therefore$ the force oscillates at $\cos^2\omega t$ but this is too high so let's find time average over one cycle.

$$\vec{E} = \hat{e}\,E_0(\bar{R})\cos\omega t$$

$$\Rightarrow \boxed{\langle\bar{F}\rangle = \frac{\alpha}{4}\nabla E_0^2(\bar{R})}$$

E_0^2 → Force along gradient of Intensity
⇒ pushes atom/molecule towards higher Intensity.

→ α for a small dielectric sphere is $\alpha = n_m^2\left(\frac{n_r^2-1}{n_r^2+2}\right)$ where $n_r = n/n_m$ where n is ior of particle & n_m is i.o.r of medium.

Lorentz oscillator - internal coordinate

Subtracting (ii) from (i),

$$2m\frac{d^2\bar{r}}{dt^2} = -e\left[2\bar{E}(\bar{R},t) + \underbrace{\left(\frac{m_e-m_n}{M}\right)\bar{r}\cdot\nabla\bar{E}(\bar{R},t)}_{=0}\right] + 2\bar{F}_{en}(\bar{r}) + \underbrace{(m_e-m_n)\frac{d^2\bar{R}}{dt^2}}_{=0}$$

$=0$ → dipole approx ← $=0$

$$\Rightarrow \frac{d^2\bar{r}}{dt^2} = -\frac{e}{m}\bar{E}(\bar{R},t) + \frac{1}{m}\bar{F}_{en}(\bar{r})$$

Lorentz argued that $\bar{F}_{en}(\bar{r}) = -m\omega_0^2\bar{r}$ → harmonic restoring force.

$$\Rightarrow \frac{d^2\bar{r}}{dt^2} + \omega_0^2 \bar{r} = -\frac{e}{m} \bar{E}(\bar{R},t) \longrightarrow$$ undamped harmonic oscillator

> Even a single atom decays due to radiation.

$$\Rightarrow \boxed{\frac{d^2\bar{r}}{dt^2} + \gamma_r \frac{dr}{dt} + \omega_0^2 \bar{r} = -\frac{e}{m} \bar{E}(R,t)}$$

Let $\bar{r}(t=0) = \bar{r}_0$ is the initial amplitude.

Trial solution $\bar{r}(t) = \bar{r}_0 \, e^{(-\Gamma + i\omega)t}$. Solve for Γ & ω.

$$\Rightarrow (-\Gamma + i\omega)^2 + \gamma_r(-\Gamma + i\omega) + \omega_0^2 = 0$$

$$\Rightarrow (\Gamma^2 - \omega^2 - \gamma_r\Gamma + \omega_0^2) + i(-2\omega\Gamma + \omega\gamma_r) = 0$$

Imag part $\Rightarrow$ $\omega(-2\Gamma + \gamma_r) = 0$

$$\Rightarrow \boxed{\Gamma = +\frac{\gamma_r}{2}} \longrightarrow \text{dipole decay rate.}$$

Real part $\Rightarrow$ $\boxed{\omega = \sqrt{\omega_0^2 - \left(\frac{\gamma_r}{2}\right)^2}} \longrightarrow$ shifted or "renormalized" (red shifted) from ω_0 (almost negligible).

$$\Rightarrow \bar{r}(t) = \bar{r}_0 \, e^{i\omega t - \frac{\gamma_r t}{2}} \quad \text{where } \omega \simeq \omega_0$$

$$U(t) = \frac{1}{2} m\omega_0^2 |\bar{r}(t)|^2 + \frac{1}{2} m \left|\frac{d\bar{r}}{dt}\right|^2$$

$\boxed{U(t) = U(t_0)\, e^{-\gamma_r t}}$ Energy decays at γ_r. $\rightarrow$ twice as fast as polarization.

Polarizability of a single isolated oscillator

Since $\bar{p} = -e\bar{r}$

we can rewrite the Lorentz equation as,

$$\frac{d^2\bar{p}}{dt^2} + \gamma_r \frac{d\bar{p}}{dt} + \omega_0^2 \bar{p} = \frac{e^2}{m} \bar{E}(\bar{R},t).$$

Let $\bar{R} = 0$ and $\hat{e}$ be the incident polarization $\bar{E}(t) = E_0 e^{i\omega t}\hat{e}$

> we expect a solution of the form $\bar{p}(t) = \hat{\epsilon} p_0 e^{i\omega t}$
Where $\bar{p}_0$ is the complex dipole amplitude allowing for phase shifts w.r.t $\bar{E}$.

$$\Rightarrow -\omega^2 p_0 + i\omega\gamma_r p_0 + \omega_0^2 p_0 = \frac{e^2}{m} E_0$$

$$\Rightarrow p_0 = \left[\frac{e^2}{m} \cdot \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma_r}\right] E_0 = \alpha E_0$$

where $\boxed{\alpha = \frac{e^2}{m} \cdot \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma_r}}$ → Polarizability!
(note linear response)

- At resonance, the dipole moment lags the driving field by 90°.
- Far from resonance we can ignore $i\omega\gamma_r$ ⇒ no damping & the $\bar{p}$ is in phase with

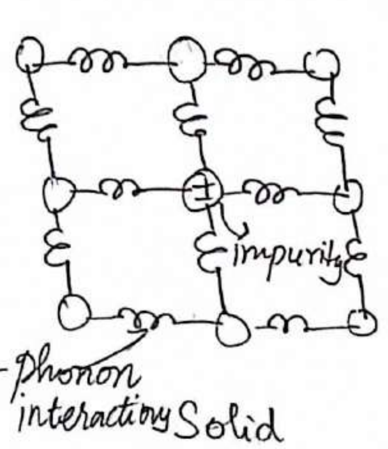
Dipole - environment interactions and dephasing

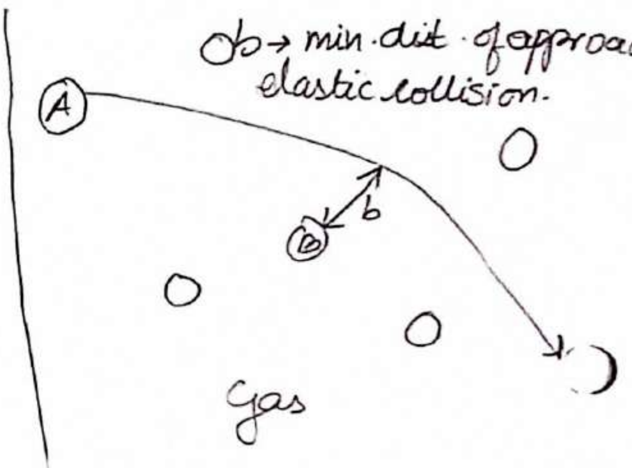
> Environmental interactions dominate the radiative response of the atoms.

> Collisions happen very frequently. Elastic collisions merely change the trajectory but do not take energy away from the dipole oscillation.. Inelastic collisions provide a nonradiative channel for energy decay of the oscillator, in which case

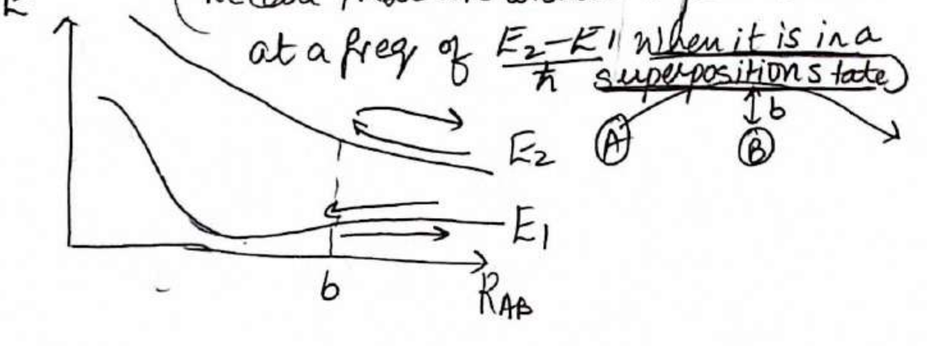
$$\gamma = \gamma_r + \gamma_{nr}$$

> Elastic & inelastic interactions can also occur in solids.





> The phonons modulate the amplitude & phase of the dipole.



> R_{AB} goes from $\infty \to b \to \infty$

⇒ phase advance of the dipole becomes time dependant.

$$\Rightarrow e^{-i(E_2-E_1)t/\hbar} \longrightarrow e^{-i\Delta E(t)\,t/\hbar} = e^{-i\phi(t)}.$$

This is a phenomenolo-gical argument to extend the Lorentz model.

where $\Delta E(t) = E_2(t) - E_1(t)$

Therefore the total phase change from the collision $\phi_{collision} = \int_{\tau_{coll}} \underbrace{\frac{\Delta E(t)}{\hbar}}\,dt$ $\quad \tau_{coll} \sim \frac{\text{atomic si-}}{V_{th}} \simeq 100\,fs.$

> The argument is that in a medium each atom undergoes these phase shifts completely at random and the ϕ_{coll} is thus a random variable equally distributed from $[0, 2\pi]$.

> The effect of the collision is thus to randomize the phase of dipole oscillations. The collision duration is much longer than the time period of oscillation. So the interaction is more of a random (AM, PM) modulation.

> In the Lorentz model we can add this effect as a random force apart from the harmonic restoring force.

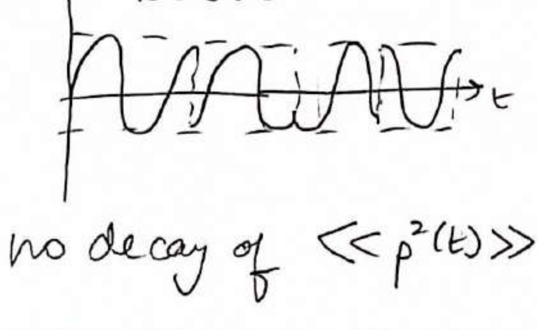
$$\Rightarrow \boxed{\frac{d^2\bar{p}}{dt^2} + \gamma \frac{d\bar{p}}{dt} + \omega_0^2 \bar{p} = \frac{e^2}{m} \bar{E} + \underline{\underline{\bar{F}_\gamma(t)}}}$$

γ → decay rate due to γ_r, γ_{nr} & collision interactions.

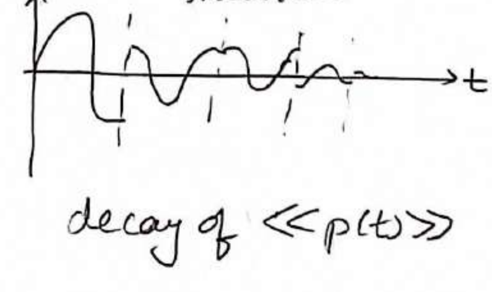
> Only the statistical properties of $\bar{F}_\gamma(t)$ are known.

The equation describes the $\bar{p}$ for a single atom/oscillator that interacts with a surrounding "bath". → described by $\bar{F}_\gamma(t)$

→ A <u>wrong</u> picture where interactions are instant looks as follows.



no decay of $\langle\langle p^2(t) \rangle\rangle$



decay of $\langle\langle p(t) \rangle\rangle$

Macroscopic Polarization.

- Maxwell's Equations for macroscopic fields are the Spatial averages over atomic scale fluctuations.
- Polarization enters through $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

where $\vec{P} = \underbrace{\frac{1}{V}}_{\text{volume}} \sum_{j=1}^{\overbrace{N}^{\text{total no. of "oscillators"}}} \vec{p}_j$ → Lorentz dipole moment of each atom.

Let $N = \frac{N}{V}$ where N is the number density ⇒ no. of oscillators per unit vol.

$$\vec{P} = N \left[\frac{1}{N} \sum_{j=1}^{N} \vec{p}_j \right]$$

$$\Rightarrow \boxed{\vec{P} = N \cdot \langle\langle \vec{p} \rangle\rangle}$$ → ensemble average dipole moment of all the oscillators.

- In the language of stat mech, we can consider each atom to be different "realisations" of identical oscillators to justify the ensemble average interpretation.

→ We can use the eqn. $\vec{p}(t) = \vec{p}_0 e^{i\omega t}$ and extend it when $F_r(t)$ is present as $\vec{p}(t) = \vec{p}_0(t) e^{i\omega t}$ since $F_r(t)$ adds a "modulation."

→ Real experiments have long averaging times so we really only observe over one averaging time period,

$$\langle \vec{p}_0(t) \rangle = \frac{1}{T} \int_t^{t+T} \vec{p}_0(t')\,dt'.$$ For an ergodic process $\langle \vec{p}_0(t) \rangle = \langle\langle \vec{p}_0(t) \rangle\rangle$

Dephasing: Decay of the Macroscopic Polarization.

> For a system with no collisions if an impulse incident field is applied, the macroscopic polarization is maximum since all atoms/dipoles line up $\vec{P} = N p_0 \hat{e}$.

> Introducing collisions causes an immediate dissipation due to the random phase & amplitude modulations. The macroscopic polarization decays (even without energy loss in the elastic case).

> Assume collision rate $\gamma_c = \frac{1}{T_2'}$. T_2' is the avg time b/w randomizing collisions.

→ Big assumption made of this being a rate process → (heuristic argument)

> By the definition of a rate process the fractional change in the ensemble average $\langle\langle p \rangle\rangle$ in a time Δt will be,

$$\frac{\Delta \langle\langle p(t) \rangle\rangle}{\langle\langle p(t) \rangle\rangle} = -\frac{\Delta t}{T_2'}$$

→ decays!

→ this ensure exponential decay.

$$\Rightarrow \frac{d}{dt} \langle\langle p(t) \rangle\rangle = -\frac{1}{T_2'} \langle\langle p(t) \rangle\rangle$$

$$\Rightarrow \langle\langle p(t) \rangle\rangle = P_0 e^{-t/T_2'}$$

$$\boxed{\Rightarrow \bar{P}(t) = \hat{e} N p_0 e^{-t/T_2'}}$$

This decay of macroscopic polarization due to interactions of oscillators at random is called dephasing.

→ Phase coherence is lost. T_2' is dephasing time & $\frac{1}{T_2'}$ is the dephasing rate.

→ So far we have not considered decay in macroscopic polarization due to radiative & nonradiative (inelastic) loss. Adding them in we get,

$$\boxed{\overline{P}(t) = \hat{\varepsilon} N p_0 e^{-t/T_2}}$$

where, $\frac{1}{T_2} = \frac{\gamma_r + \gamma_{nr}}{2} + \frac{1}{T_2'}$ is the total decay rate.

use, $\frac{1}{T_1} = \gamma_r + \gamma_{nr} \rightarrow T_1$ is the energy decay time. since only these process cause energy decay.

$$\Rightarrow \frac{1}{T_2} = \frac{1}{2T_1} + \frac{1}{T_2'}$$

> Generally $T_2' \ll 2T_1 \Rightarrow$ dephasing dominates macroscopic pol-decay.

<u>Equation of motion for average polarization.</u>

$\gamma = \gamma_r + \gamma_{nr} + \frac{2}{T_2'} = \frac{2}{T_2}$ & note $\langle\langle \overline{F}_\gamma(t) \rangle\rangle = 0. \rightarrow$ must!

In most case a long time average $\langle \overline{F}_\gamma(t) \rangle = 0$. → ergodic

$$\Rightarrow \frac{d^2 \langle\langle \bar{p}(t) \rangle\rangle}{dt^2} + \gamma \frac{d}{dt} \langle\langle \bar{p}(t) \rangle\rangle + \omega_0^2 \langle\langle p(t) \rangle\rangle = \frac{e^2}{m} \vec{E}(\overline{R}, t).$$

use $\langle\langle \bar{p}(t) \rangle\rangle . N = \overrightarrow{P}$

$$\Rightarrow \boxed{\frac{d^2 \vec{P}}{dt^2} + \gamma \frac{d\vec{P}}{dt} + \omega_0^2 \vec{P} = \frac{Ne^2}{m} \vec{E}(\vec{R}, t)}$$

Solve this for index of refraction.

Lorentz eqn. for Macroscopic polarization.

Optical Susceptibility

Recall,

$$\frac{d^2\bar{P}}{dt^2} + \gamma\frac{d\bar{P}}{dt} + \omega_0^2\bar{P} = \frac{Ne^2}{m}\bar{E}(t).$$

Consider the steady state response to a time harmonic incident field,

$$\bar{E}(t) = \hat{e}\, E_0\, e^{i\omega t}$$

We expect a soln. of the form.

$$\bar{P}(t) = \hat{e}\, P_0\, e^{i\omega t} \qquad \text{where } P_0 \text{ is complex.}$$

Substituting -

$$-\omega^2 P_0 + i\omega\gamma P_0 + \omega_0^2 P_0 = \frac{Ne^2}{m}E_0$$

So,

$$\boxed{P_0 = \frac{Ne^2}{m}\left[\frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}\right]E_0}$$

Write $\bar{P} = \epsilon_0 \chi \vec{E}$

$$\Rightarrow \boxed{\chi(\omega) = \frac{Ne^2}{\epsilon_0 m}\cdot\frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}} \longrightarrow \text{electric susceptibility.}$$

Here the field $\bar{E}(t)$ is not the incident field, it is the resultant field which must be derived self consistently with $\bar{P}(t)$ so we need an equation for $\bar{E}$ & $\bar{P}$ that describes propagation of $\bar{E}$.

Wave equation with sources.

Fact: Any vector field $\bar{A}$ can be split into a longitudinal field $\bar{A}^L$ and a transverse field $\bar{A}^T$ such that

$$\nabla \cdot \bar{A}^T = 0 \qquad \nabla \times \bar{A}^L = 0.$$

$$\Rightarrow \quad \bar{E} = \bar{E}^L + \bar{E}^T$$

Since $\nabla \times \bar{E}^L = 0 \quad \Rightarrow \bar{E}^L = -\nabla \Phi$

$$\Rightarrow \quad \nabla \cdot \bar{E} = \nabla \cdot \bar{E}^L = -\nabla^2 \Phi$$

Gauss' Law $\Rightarrow \quad \nabla^2 \Phi = -\frac{\rho}{\epsilon_0}$ $\Rightarrow$ Longitudinal field arise from response to charges in electrostatics problems.

> Waves that propagate into the far field and that can be described by the wave eqn. are transverse.

$$\nabla \times (\nabla \times E^T) = -\mu_0 \frac{\partial}{\partial t} (\nabla \times H^T)$$

$$= \nabla(\nabla \cdot \bar{E}^T) - \nabla^2 \bar{E}^T = -\nabla^2 \bar{E}^T$$

$$\nabla \times \bar{H}^T = \bar{J}^T + \frac{\partial \bar{D}^T}{\partial t} = \bar{J}^T + \frac{\partial}{\partial t} (\epsilon_0 \bar{E}^T + \bar{P})$$

Therefore, the sources of radiated fields are $\frac{\partial}{\partial t} J^T$ and $\frac{\partial^2 \bar{P}}{\partial t^2}$

$\bar{J}$ is usually used for free charges & $\bar{P}$ for bound charges but it can be hard to distinguish them in optics.

In fact, $\frac{\partial \bar{P}}{\partial t} = \rho \bar{v} = \bar{J}$ from earlier $\Rightarrow \bar{J} = i\omega \bar{P}$ for time harmonic charges.

$\rightarrow 90°$ phase shift (physically obvious)

> We will lump $\bar{J}$ & $\bar{P}$ into $\bar{P}$. (Don't double count)

$$\Rightarrow \nabla^2 \bar{E}^T = \mu_0 \frac{\partial^2}{\partial t^2} (\epsilon_0 \bar{E}^T + \bar{P})$$

Dropping the transverse superscript since we only care about travelling waves,

$$\boxed{\nabla^2 \bar{E} - \mu_0 \epsilon_0 \frac{\partial^2 \bar{E}}{\partial t} = \mu_0 \frac{\partial^2 \bar{P}}{\partial t^2}}$$

This equation must be solved simultaneously with the Lorentz equation for $\bar{E}(t)$ & $\bar{P}(t)$.

- Time Harmonic Fields - Helmholtz Equation.

Let $\bar{E} = \bar{E}(\bar{r}) e^{i\omega t}$ $\Rightarrow \bar{P} = \epsilon_0 \chi \bar{E}$ where $\chi = \chi(\omega) = \frac{Ne^2}{\epsilon_0 m} \left[\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} \right]$

Inserting into above equation.

$$\nabla^2 \bar{E}(\bar{r}) + \frac{\omega^2}{c^2} (1+\chi) \bar{E}(\bar{r}) = 0$$

Plane wave solutions

$$\Rightarrow E(\bar{r}) = \hat{e} E_0 e^{-i \bar{k} \cdot \bar{r}} \Rightarrow \boxed{k^2 = \frac{\omega^2}{c^2} [1 + \chi(\omega)]}$$

→ Dispersion relation.

→ Note that k is complex due to $\chi(\omega)$.

- → Define, $\boxed{\tilde{n}^2(\omega) = 1 + \chi(\omega)}$

$$\Rightarrow k^2 = \frac{\tilde{n}^2 \omega^2}{c^2}$$

→ We can write $\tilde{n} = n(1 - i\chi)$

where n is the familiar refractive index

and $-n\chi$ is the imaginary part that accounts for loss.

$$e^{-ikz} = e^{-\frac{in\omega z}{c}} e^{-\frac{n\chi\omega z}{c}}$$

$$= e^{-\frac{in\omega z}{c}} e^{-z/d}$$

where $d = \frac{c}{n\omega\chi}$ is the <u>skin depth</u>. That is the length over which amplitude decays by $1/e$.

<u>Index of refraction in transparent media.</u>

⇒ no loss ⇒ $\gamma_r \to 0$
& $\omega \ll \omega_0$

$\omega_0^2 - \omega^2 \gg \omega\gamma$

then,

$$\chi(\omega) \cong \frac{Ne^2}{\epsilon_0 m} \cdot \frac{1}{\omega_0^2 - \omega^2} \quad \text{which is real}$$

$$n^2(\omega) = 1 + \chi(\omega) = 1 + \frac{Ne^2}{\epsilon_0 m} \cdot \frac{1}{\omega_0^2 - \omega^2} \implies \text{Dispersion since } n \text{ is a fn. of } \omega.$$

i) Visible frequencies are s.t $\omega_0 > \omega \Rightarrow n > 1$.

ii) $n\uparrow$ as $\omega\uparrow$ or as $\lambda\downarrow$ ⇒ "Normal dispersion" ⇒ "red is faster than blue"

> In real materials several resonances may exist so an empirical model which assumes $\simeq 3$ resonances is used which fits the data well.

$$n^2 = 1 + \sum_j \frac{A_j}{\omega_{0j}^2 - \omega^2}$$

$A_j \propto N_j \Rightarrow$ number density of the j^{th} resonator.

Alternative form: Sellmeier's formula.
$$\boxed{n^2 = 1 + \sum_j \frac{B_j \lambda^2}{\lambda^2 - \lambda_j^2}}$$
$\lambda_j = \frac{2\pi c}{\omega_{0j}}$

B_j & λ_j tables exist

→ Note that $n<0$ or $n<1$ are possible. $n \simeq 0$ aka ENZ epsilon near zero materials.

Resonant absorption

What happens as $\omega \to \omega_0$?

$$\chi(\omega) = \frac{Ne^2}{\epsilon_0 m} \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

$$\omega \simeq \omega_0 \Rightarrow \omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \simeq 2\omega_0(\omega_0 - \omega)$$

$$\Rightarrow \boxed{\chi(\omega) \simeq \frac{Ne^2}{2\epsilon_0 m \omega_0} \cdot \frac{1}{\omega_0 - \omega + i\frac{\gamma}{2}}}$$

→ "Complex Lorentzian" only valid for $\omega \simeq \omega_0$

$$\chi(\omega) = \chi'(\omega) + i\chi''(\omega) = \frac{Ne^2}{\epsilon_0 m \omega_0 \gamma}\left[\frac{(\omega_0 - \omega)\gamma/2}{(\omega_0 - \omega)^2 + \left(\frac{\gamma}{2}\right)^2} - i \frac{(\gamma/2)^2}{(\omega_0 - \omega)^2 + \left(\frac{\gamma}{2}\right)^2}\right]$$

Let $\chi_0'' = \frac{Ne^2}{\epsilon_0 m \omega_0 \gamma}$ → strength of the light matter interactions.

Using γ_r from earlier,

$$\chi_0'' = \frac{3}{4\pi^2}(N\lambda^3)\frac{\gamma_r}{\gamma}$$

Recall $\gamma = \gamma_r + \gamma_{nr} + \frac{2}{T_2'}$

$$\Rightarrow \frac{\gamma_r}{\gamma} = \frac{\gamma_r}{\underbrace{\gamma_r}_{\text{photons}} + \underbrace{\gamma_{nr}}_{\text{phonons.}} + \underbrace{\frac{2}{T_2'}}_{\text{dephasing.}}} \leq 1$$

The equality occurs if polarization decays purely through radiation.

- In reality dephasing dominates ⇒ $\frac{\gamma_r}{\gamma} \ll 1$ & the strength of the interactions is greatly reduced.
- In heroic experiments you can have $\frac{\gamma_r}{\gamma} \to 1$ & $\chi_0'' = \text{constant} \times N\lambda^3$ and the max. strength of light matter interactions is therefore given by the number of oscillators in a volume of cubic wavelength. Later we will see that this strength can be expressed as a "cross section" of an atom (similar to RCS) that the light interacts with.

Real atomic dipole transitions.

- Recall, $\gamma_r = \dfrac{e^2 \omega_0^2}{6\pi \epsilon_0 m c^3}$

Real atoms have γ_r that vary over many orders of magnitude, and this is accounted for phenomenologically as

$$\gamma_r = \frac{e^2 f \omega_0^2}{6\pi \epsilon_0 m c^3}$$ where f is the "oscillator strength" of the transition.

→ In the classical model f is a fitting parameter. In QM it can be derived from first principles.

Optical Susceptibility

$$\chi(\omega) = -\chi_0'' \left[\frac{(\omega-\omega_0)\,\gamma/2}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2} + i\, \frac{(\gamma/2)^2}{(\omega-\omega_0)^2 + \left(\frac{\gamma}{2}\right)^2} \right]$$

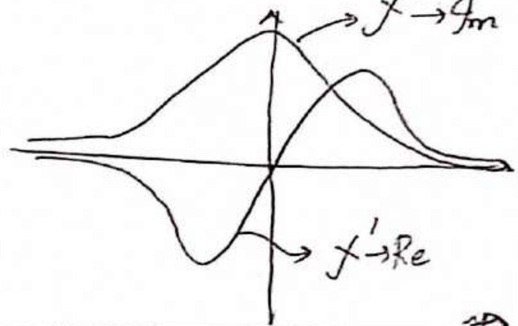
$\omega - \omega_0$ is called "detuning". Also the dephasing rate γ will turn out to be the width of the absorption spectra.

⇒ Let us define $\Delta = \dfrac{\omega - \omega_0}{\gamma/2}$ as the "normalised detuning" ie. the freq relative to the resonance freq, normalised to the spectral width.

$$\chi(\Delta) = -\chi_0''\left[\frac{\Delta}{1+\Delta^2} + i\frac{1}{1+\Delta^2}\right]$$

> The real part of χ is responsible for dispersion and the function $\frac{\Delta}{1+\Delta^2}$ is called a "dispersive line shape".

> The imaginary part of χ is responsible for optical absorption and the function $\frac{1}{1+\Delta^2}$ is called the "Lorentzian line shape".



Absorption and Dispersion.

$\chi(\omega)$ is complex $\Rightarrow$ $n(\omega)$ is complex.

$$\Rightarrow \tilde{n}(\omega) = \sqrt{1+\chi(\omega)} = n(\omega)\left[1 - i\kappa(\omega)\right]$$

Consider,

$$E(\bar{r}, t) = \underset{pol}{\hat{e}}\ \underbrace{\mathcal{E}(z)}_{envelope}\ \underbrace{e^{-i\beta z}}_{carrier}\ e^{i\omega t}$$

$\mathcal{E}(z)$ is complex and hence accounts for both amplitude change & phase change <u>w.r.t carrier</u> $e^{-i\beta z}$

Helmholtz $\Rightarrow$

$$\frac{d^2\mathcal{E}(z)}{dz^2} - 2i\beta\frac{d\mathcal{E}(z)}{dz} - \beta^2\mathcal{E}(z) + \frac{\omega^2}{c^2}(1+\chi)\mathcal{E}(z) = 0$$

> Assume envelope varies slowly along z w.r.t wavelength.

$$\left|\frac{d^2\mathcal{E}}{dz^2}\right| \ll 2\beta \left|\frac{d\mathcal{E}}{dz}\right| \qquad \text{where } \beta = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \quad -2i\beta \frac{d\mathcal{E}}{dz} + \beta^2 \chi \mathcal{E} = 0$$

$$\Rightarrow \boxed{\frac{d\mathcal{E}}{dz} = -i\frac{\beta}{2}\chi\mathcal{E}}$$

Here by assigning $\beta = \frac{\omega}{c}$ the damping appears in the envelope & not as the imaginary part of k.

Let $\chi = \chi' + i\chi''$

decay of field amp.

$$\Rightarrow \quad \mathcal{E}(z) = \mathcal{E}_0 e^{\alpha_m z - i\Delta\beta_m z} \qquad \text{where } \alpha_m = \frac{\beta}{2}\chi''(\omega)$$

$$\& \quad \Delta\beta_m = \frac{\beta}{2}\chi'(\omega)$$

phase shift w.r.t vaccum phase shi[ft]

$\alpha_m \rightarrow$ absorption coefficient

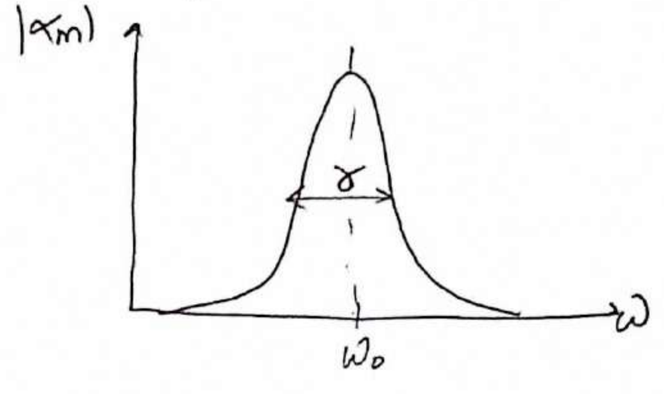
$\Delta\beta_m \rightarrow$ phase shift factor.

Beer's Law of absorption

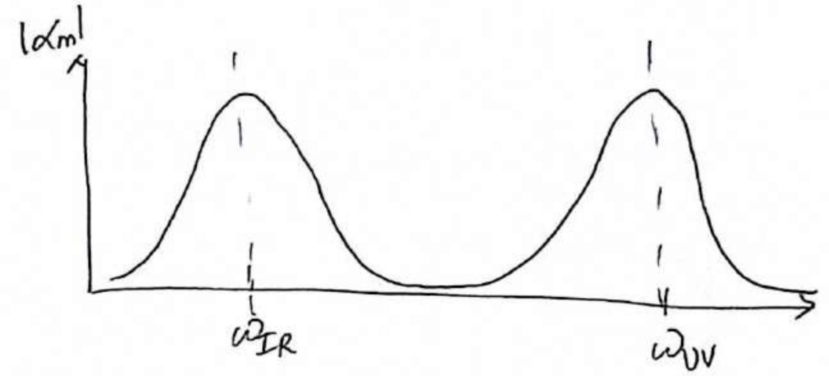
$$\chi'' \sim -\chi_0'' < 0 \quad \text{and } \alpha_m < 0 \quad \Rightarrow |\mathcal{E}(z)| = |\mathcal{E}_0| e^{\alpha_m z}$$

$$\Rightarrow |\mathcal{E}(z)| = |\mathcal{E}_0| e^{-|\alpha_m| z}$$ $\Rightarrow$ exp. decay with propagation.

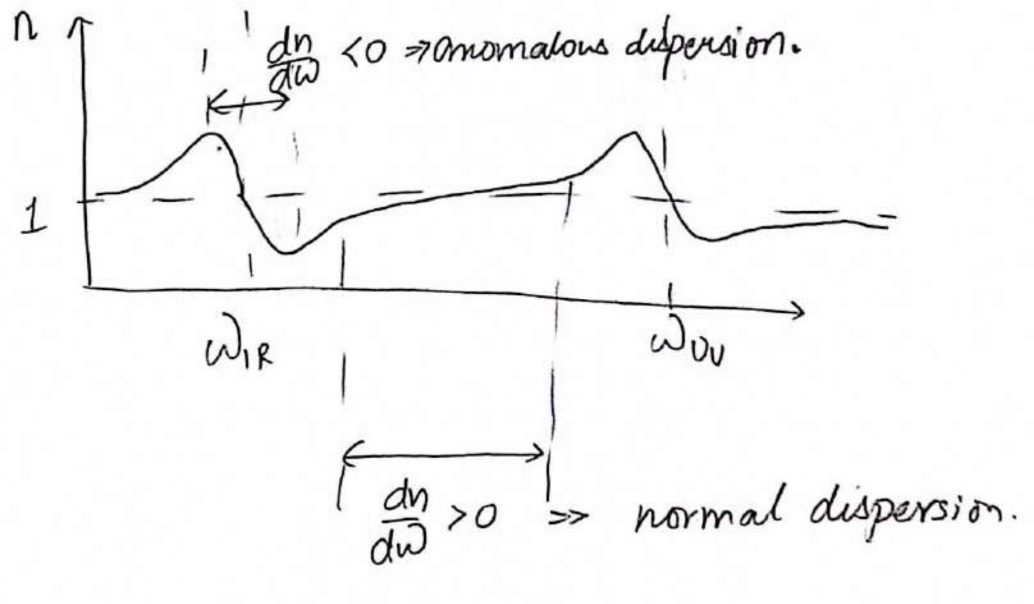
> The aborption spectra is a "Lorentzian" with FWHM of γ.



Ex:



Different materials have diff. & multiple resonances from diff. modes of vibration of the electrons, diff atomic transitions & so on.

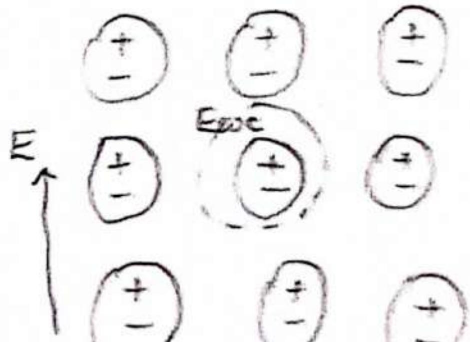


Dense Dielectrics

> In solids if atoms can interact via dipole-dipole interactions (most solids) we refer to them as dense dielectrics

> The dipole induced on a single atom is the sum of the dipole induced by the external field plus the polarization induced by the other dipoles surrounding it.

$\Rightarrow \quad \vec{p} = \alpha \vec{E}_{loc}$ ↳ external field + field due to neighbouring dipoles



> This is modeled as a small spherical cavity surrounded by a uniform dielectric polarized by an external electric field.

$\Rightarrow \quad \vec{E}_{loc} = \vec{E} + \frac{1}{3\epsilon_0}\vec{P}$ ↳ macroscopic polarization of medium.

Recall $\vec{P} = N \langle\langle \vec{p} \rangle\rangle$

$$\Rightarrow \quad \frac{d^2\vec{p}}{dt^2} + \gamma\frac{d\vec{p}}{dt} + \omega_0^2\vec{p} = \frac{Ne^2}{m}\vec{E}_{loc} = \frac{Ne^2}{m}\left[\vec{E} + \frac{1}{3\epsilon_0}\vec{P}\right]$$

or

$$\frac{d^2\vec{p}}{dt^2} + \gamma\frac{d\vec{p}}{dt} + \left(\omega_0^2 - \frac{Ne^2}{3\epsilon_0 m}\right)\vec{p} = \frac{Ne^2}{m}\vec{E}$$

So,

$$\frac{d^2\vec{p}}{dt^2} + \gamma\frac{d\vec{p}}{dt} + \omega_0'^2\vec{p} = \frac{Ne^2}{m}\vec{E}$$

Here we have defined a new resonance frequency ω_0' by

$$\omega_0' = \sqrt{\omega_0^2 - \frac{Ne^2}{3\epsilon_0 m}}$$

dipole dipole interactions cause a red-shift. AKA Lorentz-Lorenz red-shift.

* In classical theory ω_0 is anyway a measured/fitted value so there isn't a distinction between ω_0' & ω_0.

Metals and Plasmons.

> Since it is difficult to distinguish bound & free charges, the formalism of $\frac{d^2\bar{P}}{dt^2}$ and $\frac{d\bar{J}}{dt}$ are equivalent.

$$\chi(\omega) = \frac{Ne^2}{\epsilon_0 m} \cdot \frac{1}{\omega_0^2 - \omega^2 + i\omega\gamma}$$

> In a metal there are no restoring forces $\Rightarrow \omega_0 \longrightarrow 0$

> γ is due to electron- electron and electron - phonon interactions.

$$\Rightarrow \chi(\omega) = \frac{-Ne^2}{\epsilon_0 m} \cdot \frac{1}{\omega^2 - i\omega\gamma}$$

In metals it is convenient to seperate the contributions from free electrons & from the ionic lattice to the permittivity.

$$\Rightarrow \epsilon_r(\omega) = \epsilon_L \left[1 + \chi(\omega) \right]$$

$\chi(\omega)$ → free electrons.

ϵ_L → lattice → constant.

$$\Rightarrow \epsilon_r(\omega) = \epsilon_L \left[1 - \frac{\omega_p^2}{\omega^2 - i\omega\gamma} \right]$$

where, $\boxed{\omega_p^2 = \frac{Ne^2}{\epsilon_0 m}}$ is the plasma frequency.

m → use "effective mass" m^* for a specific metal.

$$\epsilon_r(\omega) = \epsilon'(\omega) + i\epsilon''(\omega)$$

$$\Rightarrow \epsilon'(\omega) = \epsilon_L\left(1 - \frac{\omega_p^2}{\omega^2+\gamma^2}\right)$$

$$\epsilon''(\omega) = -\epsilon_L \frac{\gamma}{\omega}\left(\frac{\omega_p^2}{\omega^2+\gamma^2}\right)$$

Drude model

Valid when contributions from bound electrons is very small (Al is a good drude metal)

→ Plasma freq. of metals is typically in UV.

Conductivity $\bar{J} = \sigma \bar{E}$

$$\epsilon_r(\omega) = \epsilon_L - \frac{i\sigma(\omega)}{\omega\epsilon_0}$$

(i) Low freq. limit

$$\omega \ll \gamma \ll \omega_p$$

10^9 10^{13} 10^{16}

$$\epsilon' = \epsilon_L\left(1 - \frac{\omega_p^2}{\gamma^2}\right) \quad \& \quad \frac{\omega_p^2}{\gamma^2} \gg 1$$

$$\Rightarrow \epsilon' \simeq -\epsilon_L\frac{\omega_p^2}{\gamma^2} \quad \& \quad \epsilon'' = -\epsilon_L\frac{\gamma}{\omega}\cdot\frac{\omega_p^2}{\gamma^2} = -\epsilon_L\frac{\omega_p^2}{\omega\gamma}$$

$$\Rightarrow \frac{|\epsilon'|}{|\epsilon''|} = \frac{\epsilon_L\,\omega_p^2/\gamma^2}{\epsilon_L\,\omega_p^2/\omega\gamma} = \frac{\omega}{\gamma} \ll 1$$

$\Rightarrow \epsilon_r \simeq i\epsilon''$ almost pure imaginary.

$$\epsilon_r \simeq -i\,\epsilon_L\,\frac{\omega_p^2}{}$$

- Conductivity, $\epsilon_r - \epsilon_L = -i\dfrac{\sigma}{\omega\epsilon_0}$

$\epsilon_r \gg \epsilon_L$

$$\Rightarrow -i\,\epsilon_L \frac{\omega_p^2}{\omega\gamma} = -\frac{i\sigma}{\omega\epsilon_0}$$

$\sigma = \sigma' + i\sigma''$

$\sigma \simeq \sigma'$ almost purely real.

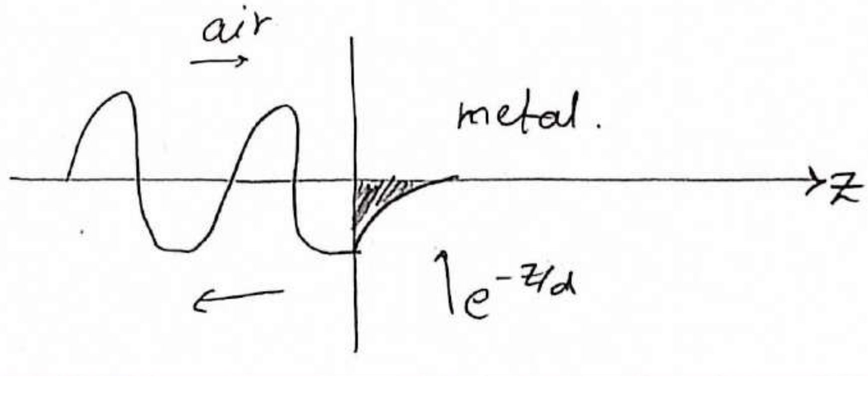
$\sigma \simeq \epsilon_0 \epsilon_L \dfrac{\omega_p^2}{\gamma}$ & $\omega_p \gg \gamma \Rightarrow \sigma$ large.

- (ii) $\underline{\gamma \ll \omega \ll \omega_p}$ (optical freq)

$$\epsilon' \simeq \epsilon_L\left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad \& \quad \epsilon'' \simeq -\epsilon_L \frac{\omega_p^2\gamma}{\omega^3}$$

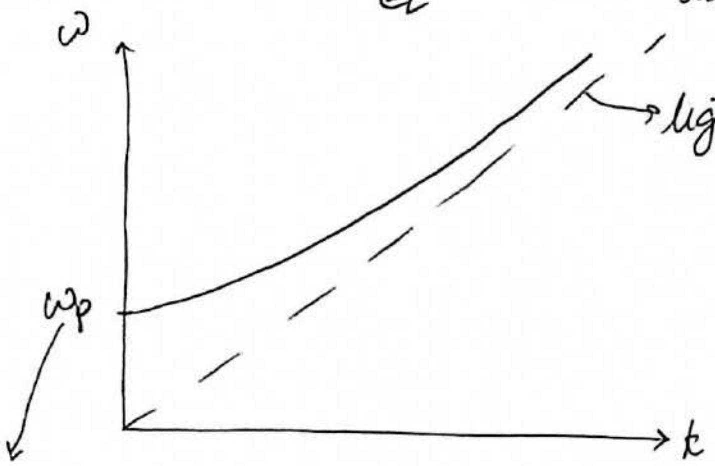
$\dfrac{|\epsilon'|}{|\epsilon''|} = \dfrac{\omega}{\gamma} \gg 1 \Rightarrow \epsilon_r(\omega)$ mostly real & negative.

with small imaginary part. $\longrightarrow R \simeq 90-95\%$ & $5-10\%$ absorption loss in metal.



Recall, $k^2 = \frac{\omega^2}{c^2}\epsilon_r(\omega) = \frac{\epsilon_L}{c^2}(\omega^2 - \omega_p^2)$ 1)

or $\omega^2 = \omega_p^2 + \frac{c^2}{\epsilon_L}k^2$ → Dispersion relation. if we neglect damping.



At this frequency the oscillations have ∞ wavelength ⟹ oscillations occur over the entire length. Here $k = \epsilon_r = 0$

2)

> Plasmons

> Charge oscillations in a bulk metal are known as plasmons.

$$\epsilon' = \epsilon_L\left(1 - \frac{\omega_p^2}{\omega^2}\right)\bigg|_{\omega=\omega_p} = 0$$

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_r\epsilon_0\vec{E} = \underline{\underline{0}} \quad \text{so } \vec{P} = -\epsilon_0\vec{E}$$

$$\& \ \nabla\times\vec{H} = -\frac{\partial\vec{D}}{\partial t} = 0.$$

The zeroes of the dielectric fn correspond to longitudinal excitations / oscillations of the electron plasma, which do not radiate or couple to propagating transverse EM fields.

> Using $\bar{J}$ instead of $\bar{P}$ we have $\bar{J} = \sigma \bar{E}$

$$\nabla^2 \bar{E} = \mu_0 \epsilon_0 \epsilon_L \frac{\partial^2 \bar{E}}{\partial t^2} + \mu_0 \sigma \frac{\partial \bar{E}}{\partial t}$$

Time harmonic $\bar{E} = \bar{E}(\bar{r}) e^{i\omega t}$

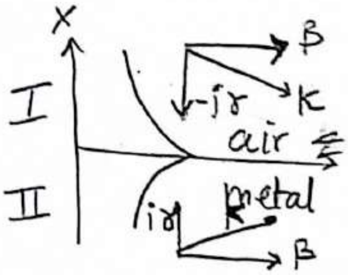
$$\Rightarrow \nabla^2 \bar{E}(\bar{r}) + \frac{\omega^2}{c^2}\left(\epsilon_L - \frac{i\sigma}{\omega \epsilon_0}\right) \bar{E}(\bar{r}) = 0$$

$$\Rightarrow \boxed{\epsilon_r = \epsilon_L - \frac{i\sigma}{\omega \epsilon_0}}$$

Surface Plasmons.

> An EM wave propagating at the interface b/w a metal & a dielectric.

> Only TM exists



$$\bar{H}(x, z) = \hat{y}\, H(x)\, e^{-i\beta z}$$

$$\bar{k} = -i\gamma \hat{x} + \beta \hat{z}$$

$$\Rightarrow \bar{k}\cdot\bar{r} = -i\gamma x + \beta z$$

$$\Rightarrow e^{-i\bar{k}\cdot\bar{r}} = e^{-\gamma x} e^{-i\beta z}$$

$$\frac{d^2 H(x)}{dx^2} + \left[\epsilon_r(\omega)\frac{\omega^2}{c^2} - \beta^2\right] H(x) = 0$$

$$H_I(x) = H_0 e^{-\gamma_I x} \qquad H_{II}(x) = H_0 e^{\gamma_{II} x}$$

Note here that β must be the same in both regions.

$$\Rightarrow \gamma_I^2 + \epsilon_I \frac{\omega^2}{c^2} - \beta^2 = 0 \quad \& \quad \gamma_{II}^2 + \epsilon_{II}\frac{\omega^2}{c^2} - \beta^2 = 0$$

$$\Rightarrow \gamma_I = \sqrt{\beta^2 - \epsilon_I \frac{\omega^2}{c^2}} \quad \& \quad \gamma_{II} = \sqrt{\beta^2 - \epsilon_{II}\frac{\omega^2}{c^2}}$$

Applying B.C

$\rightarrow$ Note $H_I(0) = H_{II}(0) = H_0$

$\rightarrow$ Continuity of D_x

$$\nabla \times \bar{H} = \frac{\partial \bar{D}}{\partial t} = i\omega \bar{D} = i\omega(D_x \hat{x} + D_z \hat{z})$$

$$\Rightarrow \nabla \times \vec{H}\Big|_x = \frac{\partial H_z}{\partial y}^{\rightarrow 0} - \frac{\partial H_y}{\partial z} = -\frac{\partial H_y}{\partial z} \text{ must be continuous.}$$

$$\frac{\partial H_i}{\partial z} = -i\beta H_i \quad (i = I, II)$$ $\Rightarrow$ β must be continuous which we saw from phase matching.

$$\beta^2 = \gamma_I^2 + \epsilon_I \frac{\omega^2}{c^2} = \gamma_{II}^2 + \epsilon_{II}\frac{\omega^2}{c^2} \quad \text{—①}$$

For $\epsilon_I = 1$ (air) & $\epsilon_{II} = \epsilon_r(\omega)$ for metal we have

$$\gamma_I^2 + \frac{\omega^2}{c^2} = \gamma_{II}^2 + \epsilon_r(\omega)\frac{\omega^2}{c^2} \qquad \text{where } \epsilon_r(\omega) = \epsilon_L\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

From continuity of E_z, use $\nabla \times \bar{H} = i\omega \bar{D} = i\omega \epsilon_r \epsilon_0 \vec{E}$ to see,

$$\frac{1}{\epsilon_I}\frac{\partial H_I}{\partial x} = \frac{1}{\epsilon_{II}}\frac{\partial H_{II}}{\partial x}$$

$$\Rightarrow \boxed{-\frac{\gamma_I}{\epsilon_I} = \frac{\gamma_{II}}{\epsilon_{II}}} \qquad \Rightarrow \boxed{\epsilon_r(\omega) = -\frac{\gamma_{II}}{\gamma_I}} \quad \text{②}$$

$\rightarrow$ must be negative!!

Dispersion relations

- From ① & ②

$$\epsilon_r(\omega) = \frac{-\sqrt{\beta^2 - \epsilon_r(\omega)\frac{\omega^2}{c^2}}}{\sqrt{\beta^2 - \frac{\omega^2}{c^2}}}$$

$$\Rightarrow \omega = c\beta\sqrt{\frac{\epsilon_r(\omega)+1}{\epsilon_r(\omega)}} \Leftrightarrow \boxed{\beta = \frac{\omega}{c}\sqrt{\frac{\epsilon_r(\omega)}{1+\epsilon_r(\omega)}}}$$

→ Need to match to this β.

In an ideal metal we can write $(\gamma = 0)$

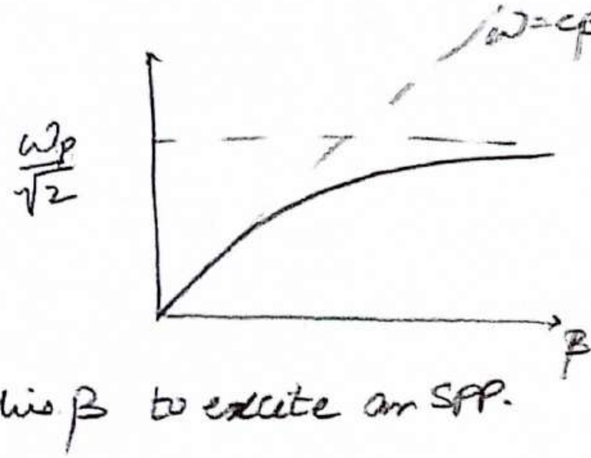
$$\epsilon_r(\omega) = \epsilon_L\left(1 - \omega_p^2/\omega^2\right)$$

- $\Rightarrow$

$$\beta = \frac{\omega}{c}\sqrt{\frac{\epsilon_L\left(1-\frac{\omega_p^2}{\omega^2}\right)}{1+\epsilon_L\left(1-\frac{\omega_p^2}{\omega^2}\right)}}$$

> When $\omega \ll \omega_p$; $\beta \sim \frac{\omega}{c}$.

> When $\omega = \frac{\omega_p}{\sqrt{1+\frac{1}{\epsilon_L}}}$; β has a pole & if $\epsilon_L \simeq 1$ which is true in many cases we get $\boxed{\omega = \frac{\omega_p}{\sqrt{2}}}$



- $$\boxed{\beta = \frac{\omega}{c}\sqrt{\frac{\epsilon_d\,\epsilon_r(\omega)}{\epsilon_d + \epsilon_r(\omega)}}}$$

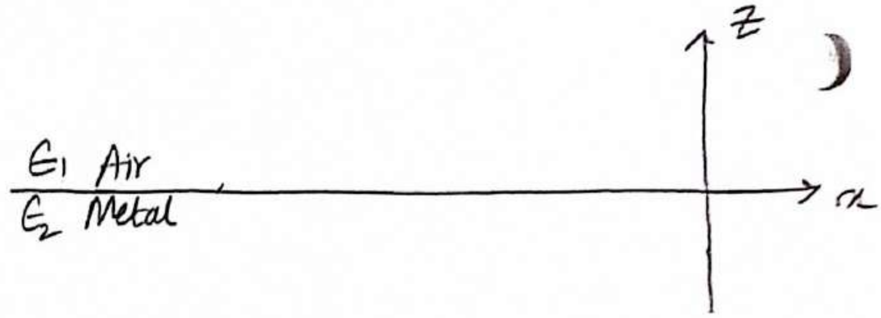
dielectric

Match to this β to excite an SPP.

Surface Plasmon Polaritons (youtube - diff. notation of ikr)

TE: $z>0$

$$\bar{E}_1 = E\hat{y}\, e^{i(k_x x - \omega t)} e^{-\alpha_1 z}$$



$z<0$

$$\bar{E}_2 = E\hat{y}\, e^{i(k_x x - \omega t)} e^{\alpha_2 z} \qquad (\alpha_1, \alpha_2 > 0)$$

$$\nabla \times \bar{E} = -\mu_0 \frac{\partial \bar{H}}{\partial t} \longrightarrow \mu_0 i\omega \bar{H}$$

$$\Rightarrow \quad H_x = \frac{-1}{\mu_0 i\omega}\frac{\partial}{\partial z}(E_y) \quad \& \quad H_z = \frac{+1}{\mu_0 i\omega}\frac{\partial}{\partial x}(E_y)$$

$$\Rightarrow \underline{z>0}: \quad \bar{H}_1 = \frac{1}{\mu_0 i\omega}\left[\alpha_1 \hat{x} + ik_x \hat{z}\right] E_1 e^{i(kx-\omega t)} e^{-\alpha_1 z}$$

$$\underline{z<0}:- \quad \bar{H}_2 = \frac{1}{\mu_0 i\omega}\left[-\alpha_2 \hat{x} + ik_x \hat{z}\right] E_2 e^{i(kx-\omega t)} e^{\alpha_2 z}$$

@ $z=0$ $H_{\parallel}$ continuous. X $\Rightarrow$ cannot exist as TE since $\alpha_1 \neq \alpha_2$

TM:

$$\nabla \times \bar{H} = \epsilon \frac{\partial \bar{E}}{\partial t} \quad \& \quad \bar{H}_1 = H_1 \hat{y}\, e^{i(k_x x - \omega t)} e^{-\alpha_1 z}$$

$$\bar{H}_2 = H_2 \hat{y}\, e^{i(k_x x - \omega t)} e^{-\alpha_1 z}$$

$$\Rightarrow E_x = \frac{1}{i\omega\epsilon}\frac{\partial H_y}{\partial z} \qquad E_z = -\frac{1}{i\omega\epsilon}\frac{\partial H_y}{\partial x}$$

$\underline{z>0}$: $\quad \bar{E}_1 = \frac{1}{\epsilon_1 i\omega}\left[-\alpha_1 \hat{x} - i k_x \hat{z}\right] H_1 e^{i(k_x x - \omega t)} e^{-\alpha_1 z}$

$\underline{z<0}$: $\quad \bar{E}_2 = \frac{1}{\epsilon_2 i\omega}\left[\alpha_2 \hat{x} - i k_x \hat{z}\right] H_1 e^{i(kx - \omega t)} e^{\alpha_2 z}$

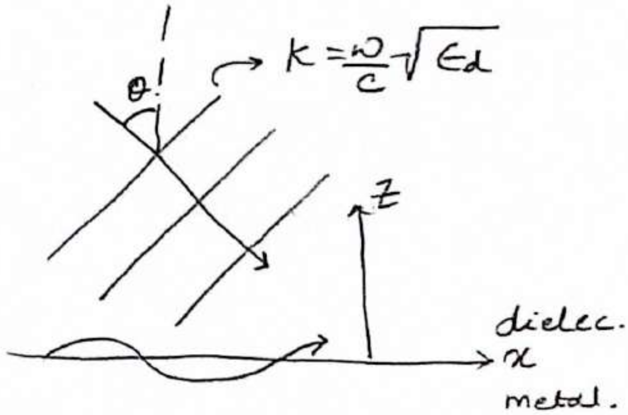
$H_{\parallel}$ @ $z=0 \Rightarrow H_1 = H_2$

$E_{\parallel}$ @ $z=0 \Rightarrow \boxed{-\frac{\alpha_1}{\epsilon_1} = \frac{\alpha_2}{\epsilon_2}} \Rightarrow -\frac{\alpha_1}{\alpha_2} = \frac{\epsilon_1}{\epsilon_2}$

Here αs are the damping factor.

Similar to earlier we have

$$\boxed{k_x = \frac{\omega}{c}\sqrt{\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}}}$$



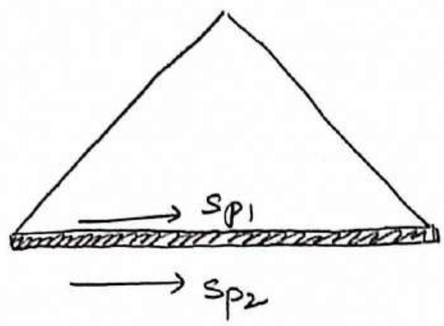
<u>Plane wave interaction</u>

$k_x = \frac{\omega}{c}\sqrt{\epsilon_d}\ \sin\theta$

$k_x = \frac{\omega}{c}\sqrt{\frac{\epsilon_d \epsilon_m}{\epsilon_d + \epsilon_m}}$

$\Rightarrow \boxed{\underbrace{\sin\theta}_{[-1,1]} = \underbrace{\sqrt{\frac{\epsilon_m}{\epsilon_d + \epsilon_m}}}_{>1}}$ $\Rightarrow$ Won't work.

Kretschmann Configuration



$$K_{x_{pw}} = K_{x_{sp}}$$

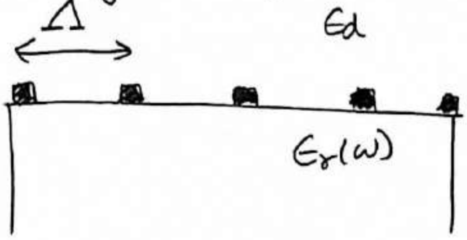
$$\frac{\omega}{c}\sqrt{\epsilon_d}\,\sin\theta = \frac{\omega}{c}\sqrt{\frac{\epsilon_m \epsilon_{air}}{\epsilon_m + \epsilon_{air}}}$$

$\hookrightarrow$ since it is excited in air

$$\Rightarrow \sin\theta = \sqrt{\frac{\epsilon_m}{\epsilon_d(\epsilon_m + 1)}} = \sqrt{\frac{+8^2}{4\times 7}} = \sqrt{\frac{2}{7}} \quad \left[\begin{matrix}\epsilon_d \simeq 4 \\ \epsilon_m = -8\end{matrix}\right.$$

$\Rightarrow$ Possible to excite on the outside of the metal strip.

Grating configuration



$$\beta = k_{z_{incident}} + \frac{2\pi}{\Lambda} \rightarrow \mathbb{K}$$

$$\Rightarrow \frac{\omega}{c}\sqrt{\frac{\epsilon_d \epsilon_r(\omega)}{\epsilon_d + \epsilon_r(\omega)}} = \sin\theta\,\frac{\omega}{c}\sqrt{\epsilon_d} + \mathbb{K}$$

$$\Rightarrow \frac{\omega}{c}\sqrt{\epsilon_d}\left[\sqrt{\frac{\epsilon_r(\omega)}{\epsilon_d + \epsilon_r(\omega)}} - \sin\theta\right] = \mathbb{K}.$$

$$\Rightarrow \omega = \frac{\mathbb{K}\,c}{\sqrt{\epsilon_d}}\left[\sqrt{\frac{\epsilon_r(\omega)}{\epsilon_d + \epsilon_r(\omega)}} - \sin\theta\right]^{-1}$$

$$\epsilon_r(\omega) \simeq -\frac{i\,\sigma(\omega)}{\omega\,\epsilon_0} \qquad \sigma(\omega) \rightarrow 55500000 \text{ im } MS \qquad \simeq -i(62.7\times 10^6)$$

$\epsilon_d \simeq 4.35$ for oxide U.

$\Rightarrow$ Cannot use a real metal! If $\epsilon_r(\omega) \cong -8$ & $\epsilon_d \simeq 4$ and $\theta = 45°$

Linear Response Theory - Kramers Kronig's Relations

Recall,

$$\chi(\Delta) = -\chi_0'' \left[\frac{\Delta}{1+\Delta^2} + i \frac{1}{1+\Delta^2} \right] \quad \text{where } \Delta = \underbrace{\frac{\omega-\omega_0}{\gamma/2}}_{\text{normalized detuning.}}$$

Features of a harmonic model

i) Linearity $\bar{P} \propto \bar{E}$

ii) Causality

iii) Time shift invariance

$$\bar{P}(t) = \epsilon_0 \int_{-\infty}^{\infty} \chi(t,t')\, \bar{E}(t')\, dt' \longrightarrow \text{most general form.}$$

$\chi(t,t')$ → response function. (Impulse Response)

Time shift invariance & causality

$$\Rightarrow \quad \bar{P}(t) = \epsilon_0 \int_{-\infty}^{\infty} \chi(\cancel{t-t'})\, \bar{E}(t')\, dt' \qquad \& \ \chi(t-t') = 0 \text{ for } t'>t$$

(*)

$$\Rightarrow \bar{P}(t) = \epsilon_0 \int_{-\infty}^{t} \chi(t-t')\, \bar{E}(t')\, dt'$$

→ For a real function $f(t)$ we have $f(t) = f^*(t)$ since no Imaginary part exists.

$$\Rightarrow F^*(\omega) = \int_{-\infty}^{\infty} f^*(t)\, e^{i\omega t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt = F(-\omega)$$

Applying Fourier Transforms to Eq ⊛

$$\boxed{\bar{P}(\omega) = \epsilon_0 \chi(\omega) \bar{E}(\omega)}$$

Impulse Response of the Harmonic Oscillator

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 + i\omega\gamma}$$ where $\omega_p^2 = \frac{Ne^2}{\epsilon_0 m}$

$\gamma = \gamma_r + \gamma_{nr} + \frac{2}{T_2'}$

$$\chi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega) e^{i\omega t} d\omega$$

$$= \frac{\omega_p^2}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega\gamma} d\omega \longrightarrow \text{use complex integration}$$

Look at notes of 530/537 for complex integral evaluation.

$$\boxed{\chi(t) = \frac{\omega_p^2}{\omega_a} \sin \omega_a t \; e^{-\frac{\gamma t}{2}} \; \theta(t)}$$

$\omega_a = \sqrt{\omega_0^2 - \gamma^2/4}$

$\gamma = \gamma_r + \gamma_{nr} + \frac{2}{T_2'}$

$\omega_p = \frac{Ne^2}{\epsilon_0 m}$

$\theta(t) \rightarrow$ unit step fn.
↳ causality.

$\omega_a \rightarrow$ renormalized oscillation frequency

$\frac{\gamma}{2} \rightarrow$ damping rate

$\chi(t)$ is real. & is a damped sinusoid for $t \geq 0$.

Kramers Kronig's Relations

i) Causality $\Rightarrow \chi(t) = 0$ for $t<0$

ii) Boundedness $\Rightarrow \chi(t)$ bounded for all t

iii) Damping $\Rightarrow \chi(t) \to 0$ as $t \to \infty$

iv) Time shift invariance $\Rightarrow \chi(t,t') = \chi(t-t')$.

guarantee that $\chi(\omega)$ is analytic in the LHP $\rightarrow$ by this convention.

Complex Analysis gives (Check 530/537 notes)

$$\chi'(\omega) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi''(\omega')}{\omega' - \omega} d\omega'$$

$$\chi''(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\chi'(\omega')}{\omega' - \omega} d\omega'$$

Hilbert Transforms $\Rightarrow \chi'$ & χ'' form a Hilbert transform pair.

Also $\chi(-\omega) = \chi^*(\omega)$

$\rightarrow$ $Re\{\chi(\omega)\} = \chi'(\omega)$ is even in ω

$Im\{\chi(\omega)\} = \chi''(\omega)$ is odd in ω.

$$\Rightarrow \boxed{\begin{aligned} \chi'(\omega) &= -\frac{2}{\pi} \int_0^{\infty} \frac{\omega' \chi''(\omega')}{\omega'^2 - \omega^2} d\omega' \\ \chi''(\omega) &= \frac{2\omega}{\pi} \int_0^{\infty} \frac{\chi'(\omega')}{\omega'^2 - \omega^2} d\omega' \end{aligned}}$$

$\rightarrow$ More useful since $\omega \geq 0$.

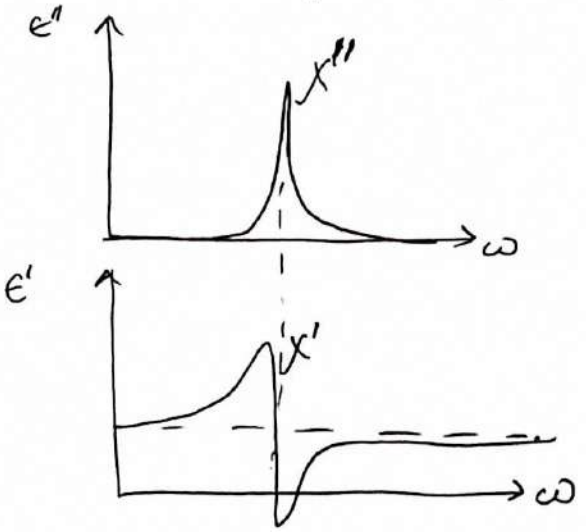
Also $\epsilon(\omega) = 1 + \chi(\omega)$

Physical Significance

i> Absorption $\Leftrightarrow$ Dispersion.

ii> Measure either Absorption or Dispersion is enough.

iii> Sharp absorption peak $\Rightarrow$ Sharp change in ϵ or n.



Linear Response - Nonmonochromatic light

→ **Phase velocity**

$$v_p = \frac{\omega}{k(\omega)} = \frac{c}{n(\omega)} \longrightarrow \text{velocity of a wavefront.}$$

→ A pure harmonic wave extends to $\pm\infty$ in space & time & has no information

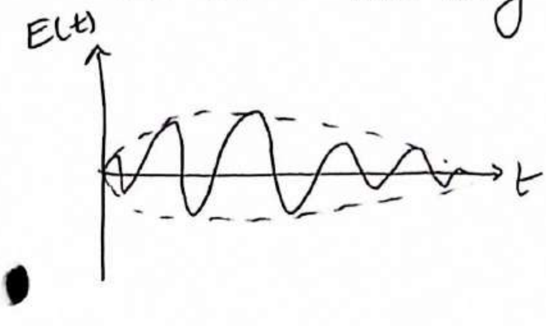
Wave packet

> Construct a wave packet around $\bar{\omega}$ with $BW = \Delta\omega$

$$\Rightarrow E(z,t) = \int_{\bar{\omega}-\frac{\Delta\omega}{2}}^{\bar{\omega}+\frac{\Delta\omega}{2}} E(\omega)\, e^{i(\omega t - \beta z)}\, d\omega$$

$$= \int_{-\infty}^{\infty} \mathcal{E}(\omega)\, e^{i(\omega t - \beta z)}\, d\omega$$ where $\mathcal{E}(\omega)$ is defined only over $\bar{\omega}-\frac{\Delta\omega}{2}$ to $\bar{\omega}+\frac{\Delta\omega}{2}$.

Since $\Delta\omega \ll \bar{\omega}$ we say the wave is quasimonochromatic

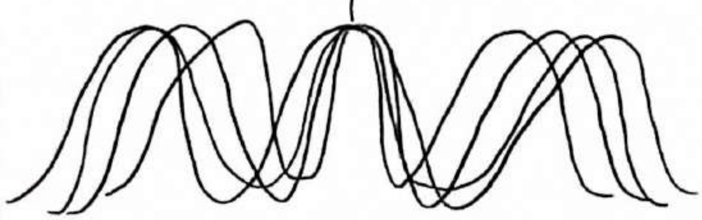


v_p is different for different frequencies.

Group Velocity.

> Velocity of the wave packet/envelope.

Construction of envelope



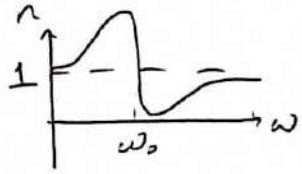
At the peak of packet the waves are all in phase. ⇒ phase variation is zero w.r.t frequency.

$$\Rightarrow \phi = \omega t - \beta z \Rightarrow \frac{d\phi}{d\omega} = 0 \Rightarrow t = \frac{d\beta}{d\omega} z$$

Group delay $\quad t_g = \frac{d\beta}{d\omega} z = \frac{z}{V_g}$

⇒ Group velocity $\boxed{V_g = \left.\frac{d\omega}{d\beta}\right|_{\bar{\omega}}}$

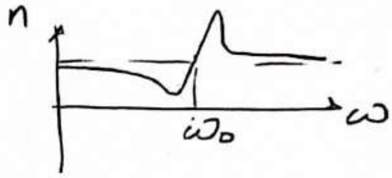
$$\boxed{V_g = \frac{c}{n + \omega \frac{dn}{d\omega}}} \qquad \text{since } \frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{n\omega}{c}\right) = \frac{n}{c} + \frac{\omega}{c}\frac{dn}{d\omega}$$



> Close to resonance $\frac{dn}{d\omega} < 0$ (When $\omega \simeq \omega_0$) and $V_g > c$ but this does not violate relativity since absorption is very strong and the pulse does not propagate beyond one pulse width or so.

> If $\Delta\omega_p \gg \gamma$, the pulse shape is severely distorted and V_g can

> When light propagates through a gain medium (population inversion) we can have phase, group & energy velocities > 0. $\chi \propto N \propto$ & $N = N_1 - N_2$



$\Rightarrow$ If $N_2 > N_1$ (pop. inv) χ flips sign &

> This was resolved by R. Chiao (1993)

> Causality is not violated. This happens due to a pulse reshaping and there is no information at the peak that wasn't already in the tail of the pulse. Relativity is not violated

---

Pulse Propagation.

1) Given $E_i(z_i, t) \rightarrow$ find $E(z_i, \omega) = F\{E_i(z_i, t)\}$

2) $E(z, \omega) = H(\omega)\, E(z_i, \omega)$
   $H(\omega) \rightarrow$ Transfer function.

$$H(\omega) = |H(\omega)|\, e^{-i\phi(\omega)}$$

in a medium :- $H(\omega) = e^{-i\tilde{k}(\omega)z}$ with $\tilde{k}(\omega) = \frac{\tilde{n}(\omega)\,\omega}{c}$

or $H(\omega) = e^{-\alpha_m(\omega)z - i\Delta\beta_m(\omega)z}$ with $\alpha_m = \frac{\beta}{2}\chi''$ &

$\Delta\beta_m = \frac{\beta}{2}\chi'$.

3) $E(z, t) = F^{-1}\{E(z, \omega)\}$

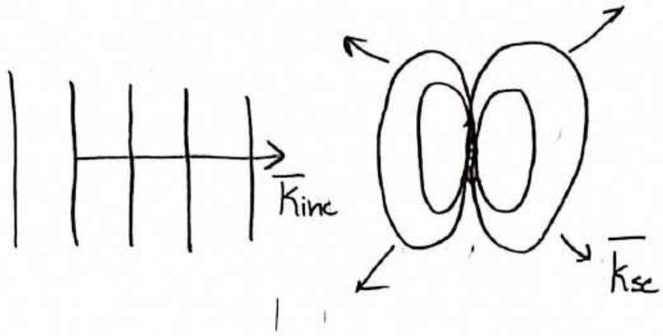
Or directly use time domain convolution.

$$E_{out}(t) = h(t) \circledast E_{in}(t).$$

$h(t) \rightarrow$ real impulse response.

Light Scattering

① Single dipole oscillator.



Induced dipole moment

$$\vec{p}(\omega) = \alpha(\omega)\,\hat{e}\,E_{inc}\cos\omega t$$

Lorentz:- $\alpha(\omega) = \dfrac{e^2/m}{\omega_0^2 - \omega^2 + i\omega\gamma}$ polarizability

Power radiated by dipole

$$\langle W \rangle = \frac{p^2\,\omega^4}{6\pi\epsilon_0 c^3}$$

$$= \frac{\omega^4 \alpha^2 E_{inc}^2}{6\pi\epsilon_0 c^3}\langle \cos^2\omega t \rangle$$

where $\frac{1}{\epsilon_0 c}\langle|\bar{S}|\rangle = E_{inc}^2\langle\cos^2\omega t\rangle$ is the avg Poynting vector.

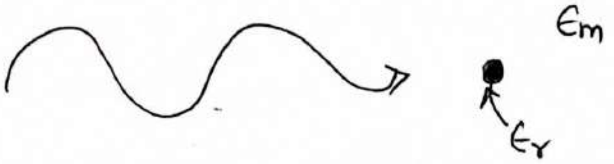
$$\Rightarrow \boxed{\langle W \rangle = \left[\frac{\omega^4 \alpha^2(\omega)}{12\pi\epsilon_0^2 c^4}\right]\underbrace{\langle|\bar{S}|\rangle}_{I_{inc}}}$$

I_{inc} → Intensity incident

$$\Rightarrow \boxed{\langle W \rangle = \underbrace{\sigma(\omega)}_{\text{scattering cross section.}}\, I_{inc}(\omega)}$$

2 Small Spherical Particle. $(a \ll \lambda)$

↳ (Rayleigh scattering.)



Electrostatic approximation.

$$\Rightarrow \alpha = a^3 \frac{\epsilon_r - \epsilon_m}{\epsilon_r + 2\epsilon_m}$$

$$\Rightarrow \sigma = \frac{\omega^4 a^6}{12\pi \epsilon_0^2 c^4} \left(\frac{\epsilon_r - \epsilon_m}{\epsilon_r + 2\epsilon_m} \right)^2$$

- For frequency ω far away from resonances σ goes as ω^4 or $\sigma \propto \frac{1}{\lambda^4}$

> For metal particles when $\epsilon_r < 0$ we can find $\epsilon_r = -2\epsilon_m$

$\Rightarrow$ "Fröhlich condition".

Which gives rise to strong colors in visible light from metal nanoparticles. (Stained glass in Churches that don't fade).

When $a \sim \lambda$, multiple resonance modes (quadrupoles, octupoles etc.) may be excited

- Which are analysed using Mie theory.

> $a \gg \lambda \Rightarrow \sigma$ is basically constant and approaches the physical dimensions. White light scatters into white light. (clouds, milk etc) $\to$ most white stuff.

$$\sigma \sim \pi a^2$$

---

③ N molecules in a confined region $\ll \lambda$

$$\vec{P}_{blob} = N \vec{P}_1$$

$$\sigma_N = \frac{\omega^4 N^2 \alpha^2(\omega)}{12\pi \epsilon_0^2 c^4}$$ dependance on $N^2 \Rightarrow$ "Coherent scattering".

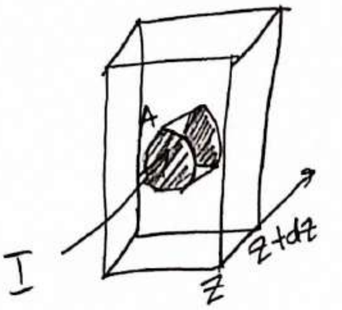
Coherent since dipoles are all inphase.

$$\boxed{\sigma_N = N^2 \sigma_{1\,dipole}}$$

When "blob" size $\geq \lambda$ we get $\sigma_N = N \sigma_{1\,dipole}$.

(4) Scattering by a collection of N independant scatterers

→ Suppose there are N scatterers per unit volume.



$P(z) = A\,I(z)$

$P(z+dz) = A\,I(z+dz)$

Power scattered out of the beam = # atoms × scattered power per atom

$$P_{scatt} = (N \cdot A \cdot dz)\,\sigma I(z)$$

Conservation of energy ⇒ $P(z) - P(z+dz) = NA\sigma I\,dz$

$P(z) - P(z+dz) \longrightarrow -A\,dI(z)$ in $\lim_{dz \to 0}$

or $$\frac{dI}{dz} = -\sigma N I$$

$$\Rightarrow \boxed{I(z) = I_{inc}\, e^{-\sigma N z}}$$ Exponential attenuation from pure scattering (ignoring absorption)

More density ⇒ more scattering

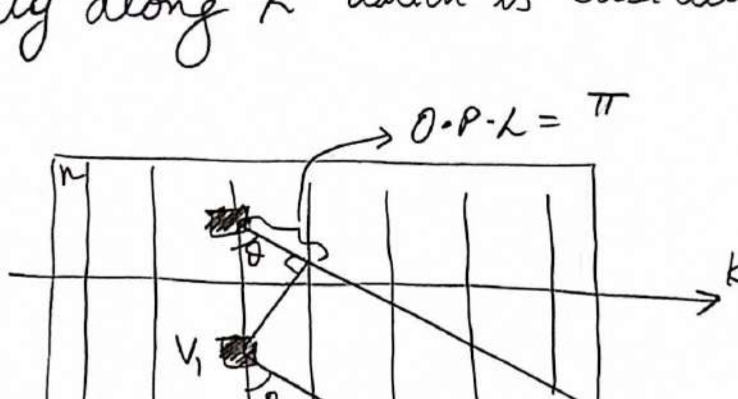
> Here we have made a big assumption that each atom scatters independantly of others. ⇒ each atom scatters power out of beam direction.

> Accounting for absorption ⇒ $\sigma_{extinction} = \sigma_{absorption} + \sigma_{scattering}$.

#5 Scattering by a perfectly uniform distribution of scatterers

> There is a seeming contradiction b/w previous models (where a uniform dielectric that does not absorb light) that allow non attenuating propagation through a dielectric and #4 (which suggests atoms always cause attenuation by scattering).

Consider a plane wave inside a dielectric. For any direction θ, given a volume ΔV_1 that scatters along θ, there is always a volume ΔV_2 from where scattering along θ is out of phase with original scatterer. Therefore scattering occurs only along $\bar{k}$ which is basically the wave propagation itself.

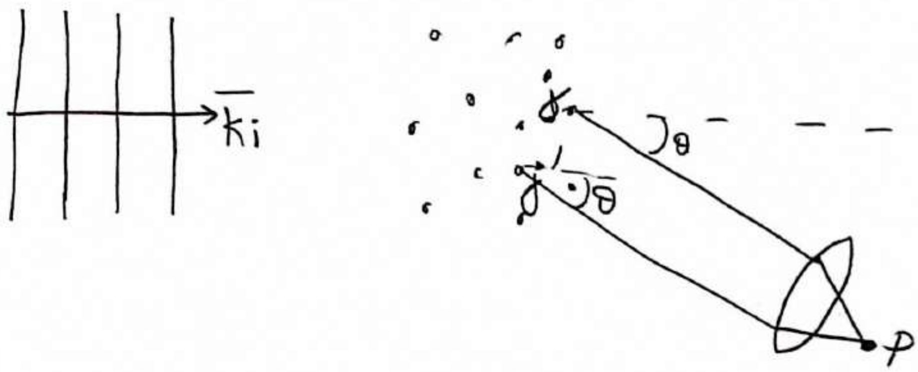


If ΔV_1 & $\Delta V_2 \ll \lambda^3$ they are identical scatterers.

→ Fluctuations in n with space cause attenuation (ie side scattering)

#6 N point scatterers over region $\gg \lambda$

- Consider plane wave incident on a collection of scatterers. Use a lens to focus all light scattered at an angle θ to a point P.



- Field at P due to j^{th} dipole is

$$E_j = E_0 e^{-i\bar{k}\cdot\bar{r}_j + i\theta_j}$$ where E_0 is field amplitude from one dipole.

$\Rightarrow$ Intensity at P is proportional to

$$I_p = \left(\sum_{j=1}^{N} E_j\right)\left(\sum_{j'=1}^{N} E_{j'}^{*}\right) = |E_p|^2$$

$$= \sum_{jj'} |E_0|^2 e^{-i\bar{k}\cdot(\bar{r}_j - \bar{r}_{j'}) + i(\phi_j - \phi_{j'})}$$

- The incident wave sets up the relative phases as

$$\phi_j - \phi_{j'} = \bar{k}_i \cdot (\bar{r}_j - \bar{r}_{j'})$$

Thus

$$I_p = I_0 \sum_{jj'} e^{-i(\bar{k}-\bar{k}_i)\cdot(\bar{r}_j - \bar{r}_{j'})}$$

i) Forward scattering. : $\bar{k} = \bar{k}_i$

$$I_p = I_0 \left(\sum_{j=1}^{N} 1\right)\left(\sum_{j'=1}^{N} 1\right) = N^2 I_0$$

↳ goes as N^2 like coherent scattering.

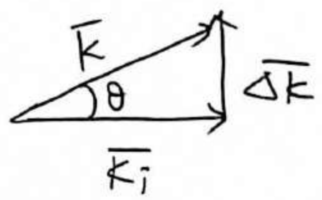
ii) Side scattering $\bar{k} \neq \bar{k}_i$

Define $\Delta \bar{k} = \bar{k} - \bar{k}_i$ as the "scattering wave vector"

We've been assuming elastic scattering. (i.e. no absorption)

so that,

$$|\bar{k}| = |\bar{k}_j| = \frac{\omega}{c}$$



$$I_p = I_0 \sum_{j=1}^{N}\sum_{j'=1}^{N} e^{-i\Delta\bar{k}\cdot(\bar{r}_j - \bar{r}_{j'})}$$

We'll consider 2 cases - dipole scatterers arranged randomly or in a perfect crystal.

a) <u>Random Scatterers</u>

⇒ for any atom j, the sum

$$\sum_{j'} e^{-i\Delta\bar{k}\cdot(\bar{r}_j-\bar{r}_{j'})} = e^{-i\Delta\bar{k}\cdot\bar{r}_j}\underbrace{\sum_{j'\neq j} e^{i\Delta\bar{k}\cdot\bar{r}_{j'}}} + 1$$

This term oscillates wildly over j' & averages to 0.

Thus $I_p = I_0 \sum_{j=1}^{N} 1 = N I_0$

Thus we recover the case of N "independant" scatterers when they are <u>randomly</u> <u>positioned</u>.

⇒ Incoherent scattering $\propto N$. ie $I = N I_0$.

Side scattering occurs & $I(z) = I_{inc}\, e^{-\sigma N z}$

b) <u>Scattering centers arranged on a perfect lattice.</u>

> It is equivalent to the homogeneous case when $\lambda \gg a$.
⇒ only coherent forward scattering occurs.

<u>Real life:</u> Thermal fluctuations & finite medium size along with defects always cause density fluctuations ⇒ exponential attenuation.

Wavelength dependance of scattered Light

Why is the Sky blue?

Recall $\sigma(\omega) = \dfrac{\omega^4 \alpha^2(\omega)}{12\pi \epsilon_0 c^4}$

> When $\alpha(\omega)$ shows strong resonances we may see colourful scattering at those frequencies (metal nanoparticles)

> Rayleigh Scattering

> Nonresonant scattering in transparent media when $a \ll \lambda$.

> $\omega \ll \omega_0$, $\omega\gamma \ll \omega_0^2 \Rightarrow \alpha_{CEO}(\omega) \cong \dfrac{e^2}{m\omega_0^2} \Rightarrow$ freq independant

↳ classical electric oscillator

$\Rightarrow \sigma \propto \omega^4 \Rightarrow$ blue light (400nm) scatters 16 times more than red light (800nm). $\Rightarrow$ Blue sky.

> Red sunset $\Rightarrow$ more atmosphere $\Rightarrow$ blue light scatters away + red light scatters towards observer.

> Clouds are large water molecules & scatter equally at all frequencies $(a \geq \lambda)$

> Gray clouds are due to attenuation of white light as it passes through the clouds.

> Green flash light after sunset: Blue is scattered away + green bends more than red. Yellow and orange are absorbed by water vapour + O_4 molecules.

Geometrical Optics

From Waves to Rays

$\nabla^2 E + k^2 \bar{E} = 0$ where $k^2 = \frac{n^2\omega^2}{c^2}$, let $k_0 = \frac{\omega}{c}$

Consider the solution (scalar soln.)

$E(\bar{r},t) = A(\bar{r})\, e^{-ik_0 S(\bar{r})} e^{i\omega t}$ → (Wentzel, Kramers, Brillouin) WKB Solution.

1) It is a scalar solution
2) Assume A varies slowly with $\bar{r}$.
3) $S(\bar{r})$ is called the Eikonal function.

Such a solution is useful when $n \to n(\bar{r})$.

Ex: $n = n(z)$

$\Rightarrow E(z,t) = A(z)\, e^{i[(\omega t - k_0 S(z))]}$

or

$E(z) = A(z)\, e^{-ik_0 S(z)}$

Substituting into HH.

$$\Rightarrow \left[\frac{d^2A}{dz^2} - A\left(k_0\frac{dS}{dz}\right)^2\right] - i\left[2k_0\frac{dS}{dz}\frac{dA}{dz} + Ak_0\frac{d^2S}{dz^2}\right] + k^2A = 0$$

$$\Rightarrow \frac{d^2A}{dz^2} + \left[\frac{n(z)^2\omega^2}{c^2} - \left(k_0\frac{dS}{dz}\right)^2\right]A = 0 \quad \& \quad 2k_0\frac{dS}{dz}\frac{dA}{dz} + Ak_0\frac{d^2S}{dz^2} = 0$$

If $n(z)$ varies slowly, then A should be slowly varying and $\frac{d^2A}{dz^2} \to 0$ (aka WKB approximation)

$$\Rightarrow \frac{n(z)^2\omega^2}{c^2} - k_0^2\left(\frac{dS}{dz}\right)^2 = 0$$

$$\Rightarrow \frac{dS(z)}{dz} = n(z)$$

$$\Rightarrow \boxed{S(z) = \int_0^z n(z')\,dz'} \longrightarrow \text{WKB solution for phase.}$$

$$\Rightarrow E(z) = \underbrace{A(z)}_{\text{slowly varying}}\, e^{-ik_0\int_0^z n(z')\,dz'}$$

WKB Solution looks locally like a plane wave

Physical meaning: Assume $n(z) = n \Rightarrow S(z) = nz$

$\Rightarrow$ optical path length = index $\times$ physical length.

$\Rightarrow S(z)$ is the optical path length & $k_0 S = 2\pi\left(\frac{S}{\lambda}\right)$

$\Rightarrow \frac{S}{\lambda}$ is the "number of waves" in a given OPL.

WKB in 3D

$n = n(\bar{r})$

$\Rightarrow E(\bar{r}) = A(\bar{r})\, e^{-i k_0 S(\bar{r})}$

$\underline{\underline{HH}} \Rightarrow \nabla^2 A + [n^2 - (\nabla S).(\nabla S)] k_0^2 A = 0$

$\& \quad 2(\nabla S).(\nabla A) + (\nabla^2 S) A = 0$

Neglect $\nabla^2 A$ assuming $n(\bar{r})$ varies slowly.

$\Rightarrow \quad \nabla S . \nabla S = \boxed{|\nabla S|^2 = n^2(\bar{r})} \longrightarrow$ "Eikonal equation"

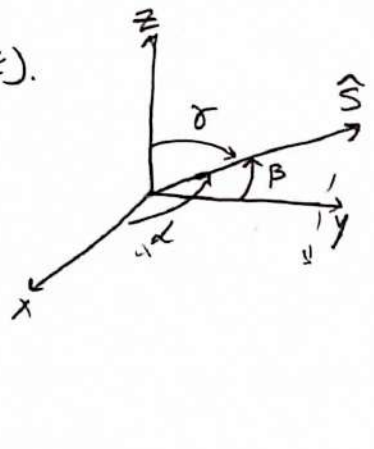
Note that when $S(\bar{r}) =$ constant, this describes a surface of constant phase $\Rightarrow$ the phase is $\phi = k_0 S$ everywhere on that surface.

Examples.

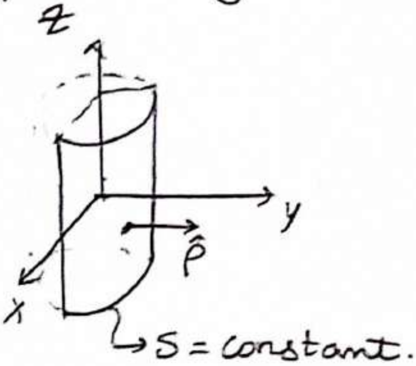
i) $n(\bar{r}) = n =$ constant (uniform medium)

Suppose initial solution is a plane wave with directional cosines α, β and γ. $\Rightarrow S = n(\alpha x + \beta y + \gamma z)$.

$\& \; \nabla S = n\hat{s}$ where $\hat{s} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$



ii) $n =$ constant, initial wave surface is a cylinder



$\Rightarrow$ $S = n\rho$ & $\hat{S} = \hat{\rho}$

$\nabla S = n\hat{\rho} \rightarrow \nabla S = n\hat{S}$

iii) $n =$ constant, initial wave = sphere

$S = nr$; $\nabla S = n\hat{r}$; $\nabla S = n\hat{S}$

In general

$$\boxed{\nabla S = n(\bar{r})\,\hat{S}(\bar{r})}$$

Where $\hat{S}(\bar{r})$ is a unit vector normal to the surface at $\bar{r}$.

∇S is a vector perpendicular to the surface $S =$ constant.

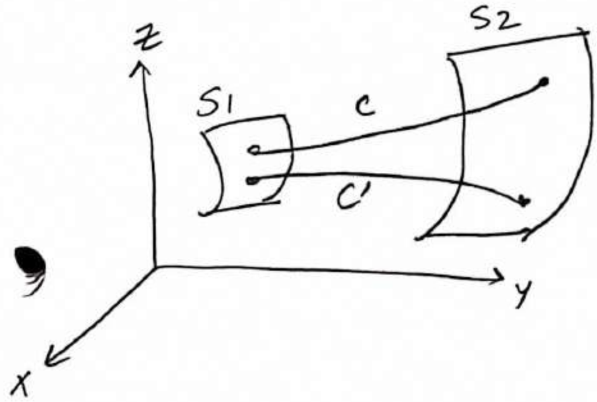
Def: A ray is the path taken by $\hat{S}$ as the corresponding wave propagates through space.

When is this GO formalism valid? When $\lambda \to 0$ since then A is basically constant over a wavelength and $\nabla^2 A \to 0$.

- Note, $\int_A^B |\nabla S|\, ds = S(\bar{r}_B) - S(\bar{r}_A) = \underbrace{\int_A^B n\, ds}_{\text{optical path length from } A \to B}$.

$\Rightarrow$

Two rays going from S_1 (one surface of constant phase) to S_2 (another surface of constant phase), the optical path length is the same.



$$\int_c n\,ds = \int_{c'} n\,ds = S_2 - S_1$$

Consider the rate of change of $n\hat{s}$ along a ray:

$$\frac{d}{ds}(n\hat{s}) = \frac{d}{ds}(\nabla S)$$

$$= \hat{s} \cdot \nabla (\nabla S)$$

$$= \frac{\nabla S}{n} \cdot \nabla(\nabla S)$$

$$= \frac{1}{2n} \nabla (\nabla S)^2$$

$$= \frac{1}{2n} \nabla n^2 = \nabla n$$

$$\Rightarrow \boxed{\frac{d}{ds} n\hat{s} = \nabla n}$$ "ray equation"

Given $n(\bar{r})$ & an initial ray direction $\hat{s}_0$ we can find the path taken by the ray.

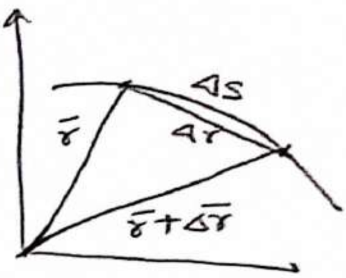
→ In words, the change in the optical path along the path is given by the gradient of the index of refraction.

→ s here is the parameter describing the ray.

Note:

$$\hat{S} = \frac{d\bar{r}}{ds} = \hat{x}\frac{dx}{ds} + \hat{y}\frac{dy}{ds} + \hat{z}\frac{dz}{ds}$$

ds = length element along curve



Examples.

i) $n =$ constant ; $\nabla n = 0$

$$\Rightarrow \frac{d}{ds}(n\hat{s}) = 0 \Rightarrow \quad \hat{s} = \text{constant} \quad \Rightarrow \text{straight line.}$$

ii) $n = n(y)$; $\nabla n = \frac{dn}{dy}\hat{y}$ & ray travelling along $\hat{y}$.

$$\Rightarrow \frac{d}{ds} = \frac{d}{dy} \quad ; \quad \hat{s}_0 = \hat{y}$$

$$\Rightarrow \frac{d}{dy}(n\hat{s}) = \frac{dn}{dy}\hat{y} \Rightarrow \hat{s} = \hat{y} \quad \Rightarrow \text{no change in direction.}$$

iii) $n = n(y)$ and $\hat{s}_0 = \hat{x}$, the ray bends towards higher n.

iv) In general, $n = n(y)$ $\hat{s}_0 = \hat{x}\sin\theta_0 + \hat{y}\cos\theta_0$

Vector ray equation can be split into 3 components.

- x: $\frac{d}{ds}(n \sin\theta) = \frac{dn}{dx} = 0$ $\qquad \hat{s} = \hat{x}\sin\theta + \hat{y}\cos\theta + \hat{z}\gamma$.

y: $\frac{d}{ds}(n\cos\theta) = \frac{dn}{dy} \neq 0$

z: $\frac{d}{ds}(n\gamma) = 0$ where γ is direction cosine along z.

$\rightarrow n\gamma = \text{constant}$ but $\gamma_0 = 0 \Rightarrow \gamma = 0 \Rightarrow$ ray stays in xy plane.

x: $n\sin\theta = \text{constant} = n_0 \sin\theta_0 \Rightarrow$ Snells Law is locally obeyed.

- y: If $\frac{dn}{dy} > 0$, the ray gradually turns towards y axis.

In general, the ray bends towards higher refractive index.

Fermat's Principle

The loosest form of the statement is:

"Of all the paths light might take b/w 2 points, the actual path taken is the one that requires the least time".

- Relationship b/w differential path element & time element is

$c\,dt = n\,ds \Rightarrow c\int_A^B dt = \int_A^B n\,ds$.

$(v = \frac{c}{n})$

Eikonal equation gives,

$$\nabla S = n\hat{s}$$

$$\underbrace{\nabla \times (\nabla S)}_{0} = \nabla \times n\hat{s} = 0$$

$$\Rightarrow \iint_A \nabla \times (n\hat{s}) \cdot d\bar{a} = 0$$ where A is an open surface

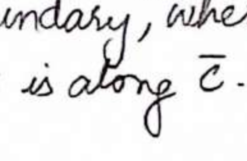
$$\Rightarrow \boxed{\oint_C n\hat{s} \cdot d\bar{r} = 0.} \longrightarrow$$ Lagrange's integral invariant.

This gives an easy proof of Fermat's Principle.

> Consider a ray passing through points P_1 and P_2 along a curve $\bar{C}$.

> Consider another curve $\bar{C}'$ also passing through P_1 & P_2.

Apply Lagrange's invariant to the loop,



$$\int_{\bar{C}} n\hat{s} \cdot d\bar{r} - \int_{\bar{C}'} n\hat{s} \cdot d\bar{r} = 0$$

Along $\bar{C}$ $\hat{s} \parallel d\bar{r}$ $\Rightarrow \int_{\bar{C}} n\,ds = \int_{\bar{C}'} n\hat{s} \cdot d\bar{r}$

dr is along the boundary, whereas ds is along $\bar{C}$.

Triangle inequality $\Rightarrow \hat{s} \cdot d\bar{r} \leq ds \Rightarrow \int_{\bar{C}} n\,ds \leq \int_{\bar{C}'} n\,ds$

"Path of wavefront normals is also the path of least time."

Generalized Fermat's Principle.

"The optical path traversed from P_1 to P_2 is that for which the integral $\int_{P_1}^{P_2} n\,ds$ is stationary."

$\Rightarrow$ The path integral is a local optimum.

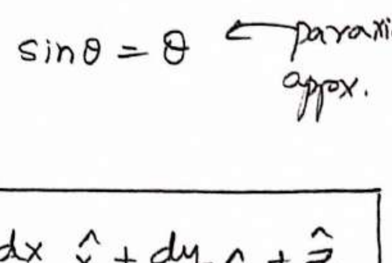
Mathematically $\delta \int_{P_1}^{P_2} n\,ds = 0$ $\Rightarrow$ small change in path does not change the integral value.

We can use the calculus of variations to derive the ray equation from this statement.

Graded Index (GRIN) optics

$n = n(\rho) = n(\sqrt{x^2+y^2})$ $\Rightarrow$ Cylindrical symmetry.

Use paraxial rays approximation $\Rightarrow$ rays propagating at small angles with respect to z axis.



$$ds = dz\sqrt{1+\left(\frac{dx}{dz}\right)^2+\left(\frac{dy}{dz}\right)^2}$$

$$\overset{\text{Taylor}}{\simeq} dz\left[1+\frac{1}{2}\left(\frac{dx}{dz}\right)^2+\frac{1}{2}\left(\frac{dy}{dz}\right)^2\right] \quad \& \quad \frac{dx}{dz} = \tan\theta = \sin\theta = \theta \leftarrow \text{paraxial approx.}$$

$$\Rightarrow \frac{dx}{dz}, \frac{dy}{dz} \ll 1 \text{ and } \boxed{ds \simeq dz} \Rightarrow \boxed{\hat{s} = \frac{dx}{dz}\hat{x} + \frac{dy}{dz}\hat{y} + \hat{z}}$$

Ray equation,

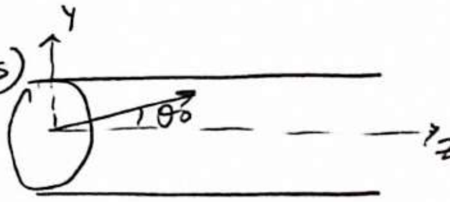
$$\frac{d}{dz}(n\hat{s}) = \frac{d}{dz}\left(n\frac{dx}{dz}\hat{x} + n\frac{dy}{dz}\hat{y} + n\hat{z}\right) = \nabla n$$

n does not vary with z so we have 2 equations,

$$\frac{d}{dz}\left(n\frac{dx}{dz}\right) = \frac{\partial n}{\partial x}$$

$$\frac{d}{dz}\left(n\frac{dy}{dz}\right) = \frac{\partial n}{\partial y}.$$

Meridional Rays



- Rays that intersect the optical axis (z axis)
- Choose ray in YZ plane WLOG.
- Rays that do not intersect the optical axis are called skew rays.
- Ray always remains in YZ plane due to symmetry.

$\Rightarrow$ Solve $\frac{d}{dz}\left(n\frac{dy}{dz}\right) = \frac{\partial n}{\partial y}$. subject to initial conditions

$$y(0) = y_0; \quad \left.\frac{dy}{dz}\right|_0 = \tan\theta_0 \cong \theta_0$$

Assume $n^2(y) = n_0^2(1-\alpha^2 y^2)$

Approximately,

$$n(y) \cong n_0 \left(1 - \frac{1}{2}\alpha^2 y^2\right)$$

$$\frac{\partial n}{\partial y} = -n_0 \alpha^2 y$$

Ray equation :

$$\frac{d}{dz}\left(n \frac{dy}{dz}\right) = n \frac{d^2 y}{dz^2} = -n_0 \alpha^2 y$$

$$\frac{d^2 y}{dz^2} = -\frac{n_0}{n}\alpha^2 y \cong -\alpha^2 y$$

Since $n \cong n_0$ if $\alpha^2 y^2 \ll 1$

$$\frac{d^2 y}{dz^2} + \alpha^2 y = 0$$

$\Rightarrow$ Solution,

$$y(z) = A \cos\alpha z + B \sin\alpha z$$

$$y(0) = y_0 \Rightarrow A = y_0$$

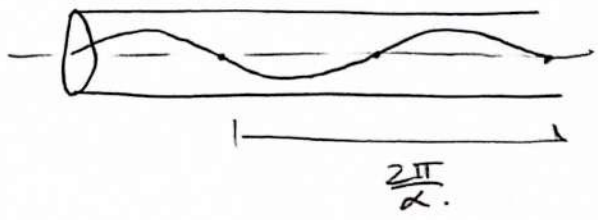
$$\left.\frac{dy}{dz}\right|_0 = \theta_0 \Rightarrow \alpha B = \theta_0 \Rightarrow B = \frac{\theta_0}{\alpha}$$

$$\boxed{y(z) = y_0 \cos\alpha z + \frac{\theta_0}{\alpha}\sin\alpha z}$$

$$\Rightarrow \boxed{\frac{dy}{dz} = \theta(z) = -y_0 \alpha \sin\alpha z + \theta_0 \cos\alpha z}$$

$\Rightarrow$ Meridional rays are periodic with period or "pitch" $= \frac{2\pi}{\alpha}$

$$y_{max} = \sqrt{y_0^2 + \left(\frac{\theta_0}{\alpha}\right)^2}$$



Skew rays

$$n^2 = n_0^2(1-\alpha^2\rho^2) = n_0^2(1-\alpha^2(x^2+y^2))$$

$$\Rightarrow n \simeq n_0\left[1-\frac{1}{2}\alpha^2(x^2+y^2)\right]$$

$$\frac{d^2x}{dz^2} = -\alpha^2 x \quad ; \quad \frac{d^2y}{dz^2} = -\alpha^2 y$$

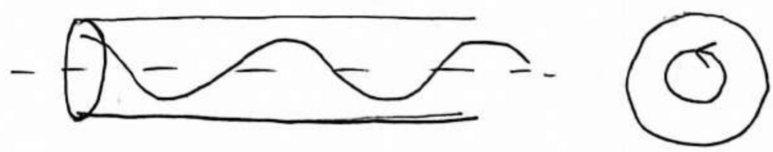
$\Rightarrow$ $x(z)$, $y(z)$ are harmonic functions with period $\frac{2\pi}{\alpha}$

Initial conditions : (x_0, y_0) & $(\theta_{x0}, \theta_{y0})$. WLOG choose $x_0 = 0$

$$x(z) = \frac{\theta_{x0}}{\alpha}\sin\alpha z \qquad y(z) = y_0\cos\alpha z + \frac{\theta_{y0}}{\alpha}\sin\alpha z.$$

Special case : $\theta_{y0} = 0$; $\theta_{x0} = \alpha y_0$

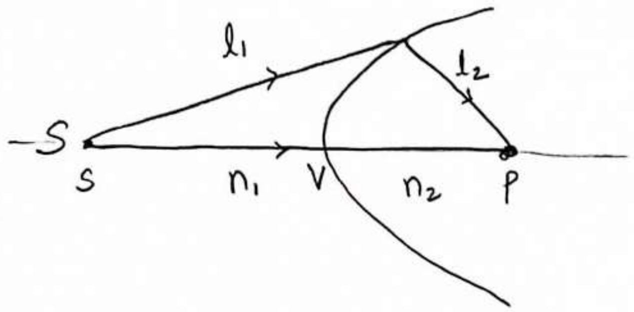
$$\Rightarrow \left.\begin{aligned} x(z) &= y_0\sin\alpha z \\ y(z) &= y_0\cos\alpha z \end{aligned}\right\}$$ helical ray along $\rho = y_0 = $ Constant.



Lenses and Imaging Systems.

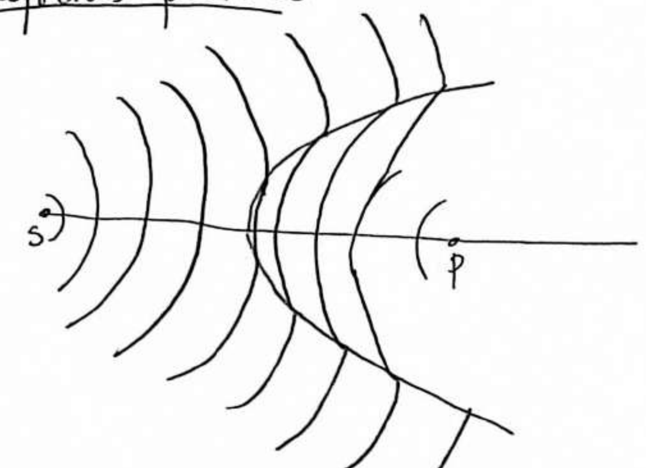
Review of basic GO

Image formation by a refracting surface.



Fermat's Principle $\Rightarrow n_1 l_1 + n_2 l_2 = \text{constant}$
This is the equation of a "cartesian oval" of revolution.

Wavefronts picture



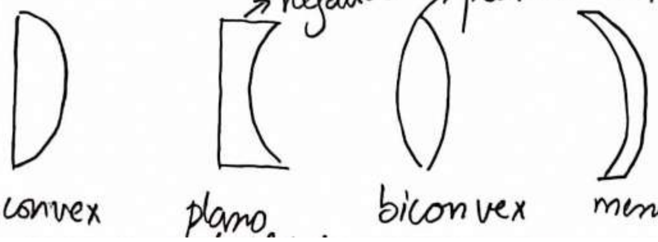
> When s is at infinity it can be shown that the Cartesian oval reduces to all ellipse -

> Cartesian ovals are aspherical and cannot be efficiently manufactured so we use spherical lenses instead.

Practical Lenses

> For paraxial rays the cartesial oval can be approximated by a sphere. The imperfections caused due to this approximation are called spherical aberrations.

$\Rightarrow$ negative $\Rightarrow$ positive $\Rightarrow$ thicker in middle.



planoconvex plano. biconvex meniscus.

Sign convention.

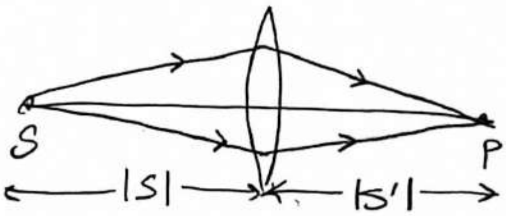
i) optical axis = z axis

ii) vertex of surface (V) is positioned at the origin.

iii) left ⇒ −ve ; right ⇒ positive.

Thin lens equation.

$$\boxed{\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}}$$

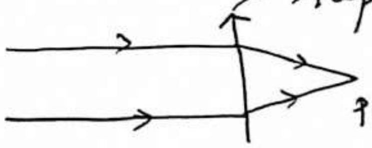


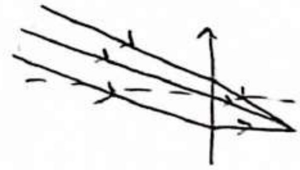
Special cases

⇒ represents the thin lens.

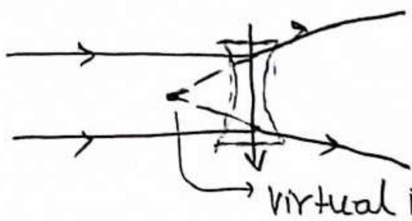
i) $s = -\infty \Rightarrow s' = f$

similarly $s = -f \Rightarrow s' = \infty$





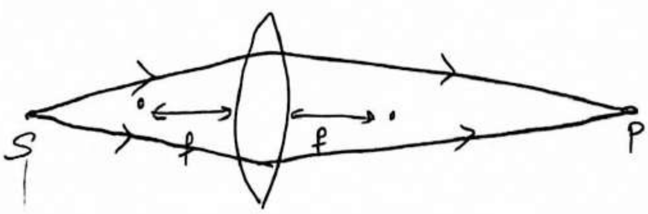
ii) $f < 0 \Rightarrow s' = f < 0$



⇒ virtual image.
⇒ cannot capture on screen.

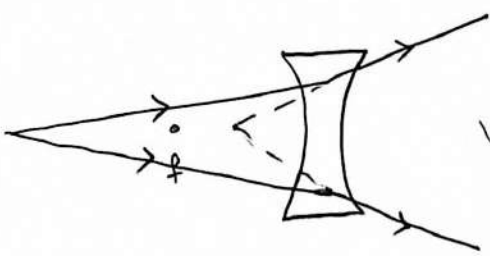
iii) $s < -f$

a) $f > 0$



real image

b) $f < 0$



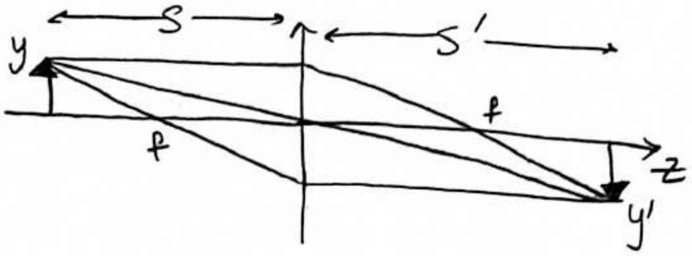
virtual image

Image formation by ray tracing.

Rules

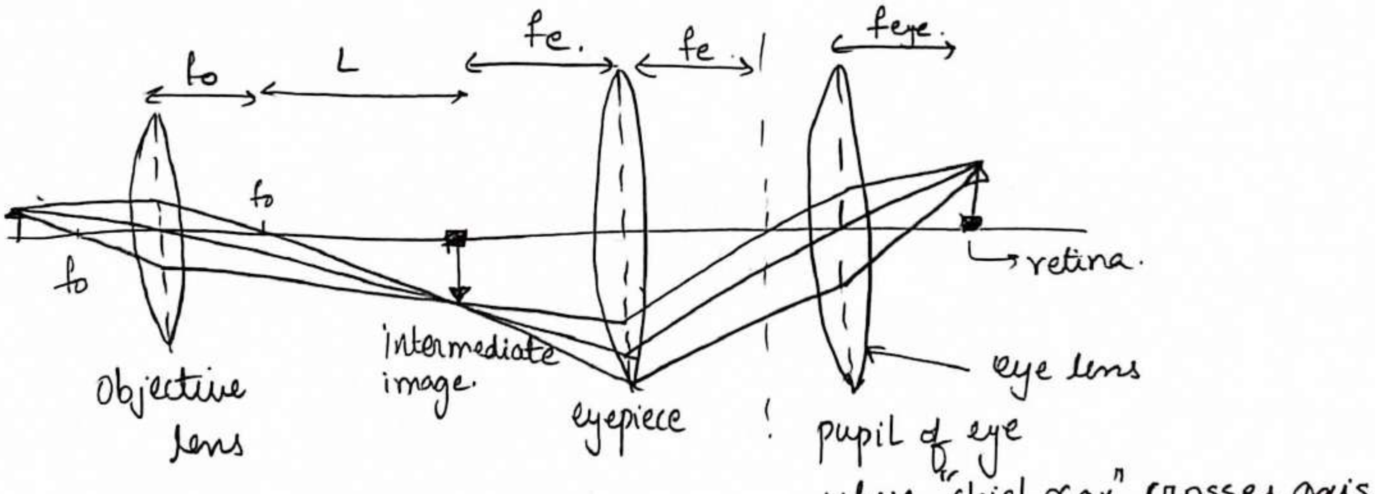
i) Ray through centre of lens is <u>undeviated</u>.

ii) Collimated rays pass through focus.



Magnification $= \frac{y'}{y} < 0 \Rightarrow$ image inverted & real.

Compound Microscope.

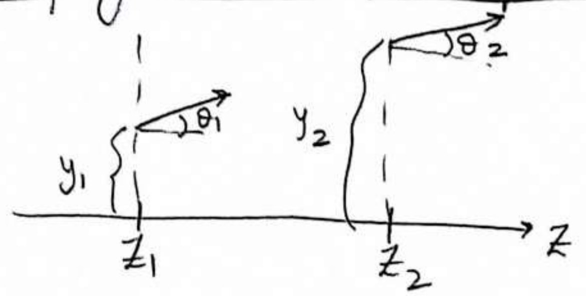


$L =$ "tube length" $\simeq 160\,mm$ usually

$\frac{1}{f_{eye}} = 60$ diopters ($1\,\text{diopter} = 1\,m^{-1}$)

Paraxial ray tracing and Matrix (ABCD) optics.

1) Propagation in free space.



$$y_2 = y_1 + (z_2 - z_1)\tan\theta$$

$$\Rightarrow \boxed{\begin{aligned} y_2 &\simeq y_1 + d\theta \\ \theta_2 &= \theta_1 \end{aligned}}$$ where d = distance b/w reference planes.

$$\Rightarrow \begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}}} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \overleftrightarrow{T}$$

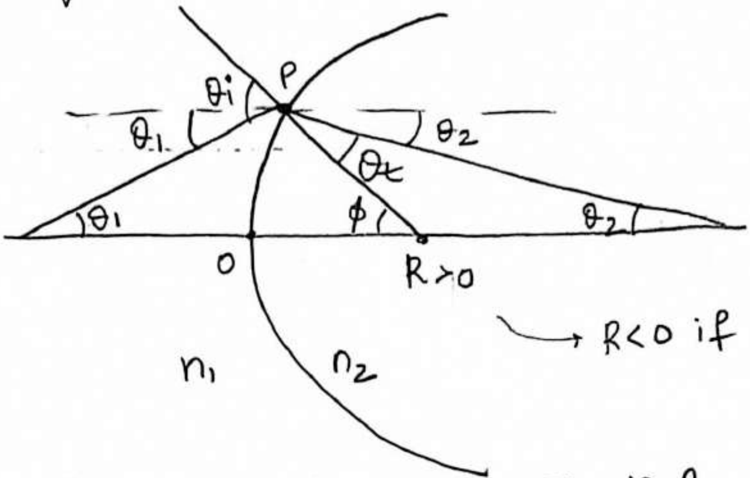
> $\begin{bmatrix} y \\ \theta \end{bmatrix}$ is a "ray vector" $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ is the ray transfer matrix of free space.

2) Refraction at planar interface

$y_2 = y_1$ $\quad n_1 \sin\theta_1 = n_2 \sin\theta_2 \Rightarrow \theta_2 \simeq \frac{n_1}{n_2}\theta_1$

$$\Rightarrow \begin{bmatrix} y_2 \\ \theta_2 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}} \begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix}$$

3. Refraction at spherical surface.



→ $R<0$ if center is to the left of vertex.

$n_1 \sin\theta_i = n_2 \sin\theta_t \Rightarrow n_1\theta_i = n_2\theta_t$

Geometry ...

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{(n_2-n_1)}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$$

$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \frac{n_1}{n_2}$ always! Some books use convention $\begin{bmatrix} y \\ n\theta \end{bmatrix}$ to make det $= 1$

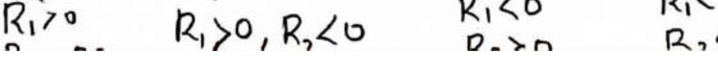
for a paraxial system.

4. Thin lens.

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{(1-n)}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{(n-1)}{nR_1} & \frac{1}{n} \end{bmatrix}$$

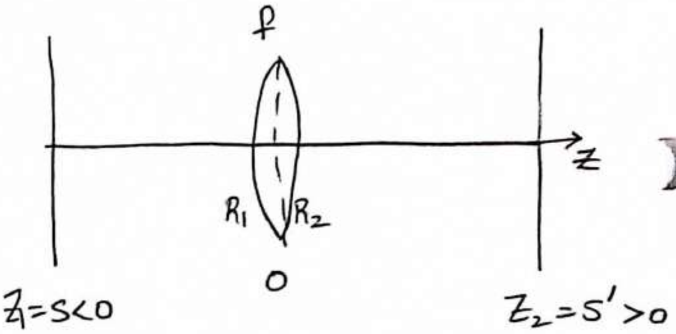
R_2 R_1

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} = \overleftrightarrow{S}$$



$R_1 > 0$ $R_1 > 0, R_2 < 0$ $R_1 < 0$, $R_2 > 0$ $R_1 < 0$, $R_2 < 0$

Imaging.



$$\sigma = \overleftrightarrow{T_2} \cdot \overleftrightarrow{S} \cdot \overleftrightarrow{T_1}$$

$$= \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & 1 \end{bmatrix} \begin{bmatrix} 1 & -s \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - s'(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & -s + s'\left\{ 1 + s(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right\} \\ -(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) & 1 + s(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

> For image formation all rays coming from a point y_1 must end up at y_2 independant of θ_1 $\Rightarrow B = 0$.

$$B = 0 \Rightarrow \frac{1}{s} - \frac{1}{s'} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Recall $\quad \frac{1}{s'} - \frac{1}{s} = \frac{1}{f}$

$$\Rightarrow \boxed{\frac{1}{f} = (n-1)\left[\frac{1}{R_1} - \frac{1}{R_2}\right]} \longrightarrow \text{"Lens makers equation"}$$

$\therefore$ Thin lens

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

Imaging matrix is now,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1-\frac{s'}{f} & 0 \\ -\frac{1}{f} & 1+\frac{s}{f} \end{bmatrix}$$

$$y_2 = \left(1-\frac{s'}{f}\right) y_1 \Rightarrow M = \frac{y_2}{y_1} = 1-\frac{s'}{f}$$

$$\Rightarrow M = 1 - s'\left(\frac{1}{s'} - \frac{1}{s}\right) = \frac{s'}{s}$$

$$\Rightarrow \boxed{M = \frac{s'}{s}} \qquad M<0 \Rightarrow \text{inverted.}$$

$$D = 1+\frac{s}{f} = \frac{s}{s'} = \frac{1}{M}$$

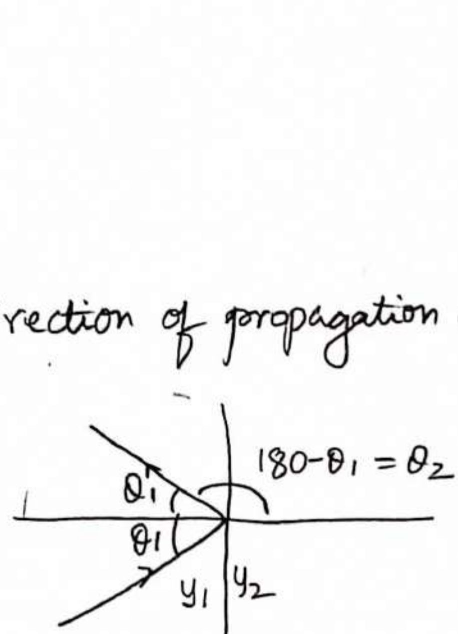
on axis $\Rightarrow y=0 \Rightarrow \theta_2 = \frac{1}{M}\theta_1 \rightsquigarrow$ angular demagnification

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} M & 0 \\ -\frac{1}{f} & \frac{1}{M} \end{bmatrix}$$

Planar mirror

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ unchanged. Only direction of propagation changes.

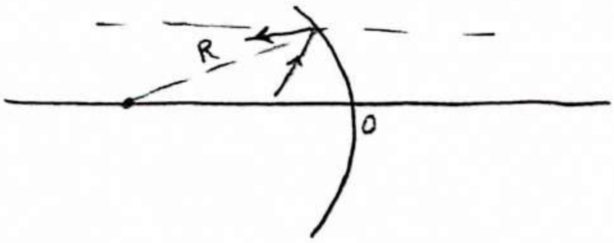
Not sure if this makes sense.



$\theta_2 = \theta_1$ or $\theta_2 = 180-\theta_1$?

→ If $z>0$ is defined by direction of propagation, then this is correct!

Spherical Mirror



$$\overleftrightarrow{M} = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 0 \end{bmatrix}$$

$R<0 \Rightarrow$ concave mirror

$R>0 \Rightarrow$ convex mirror.

Quadratic GRIN medium.

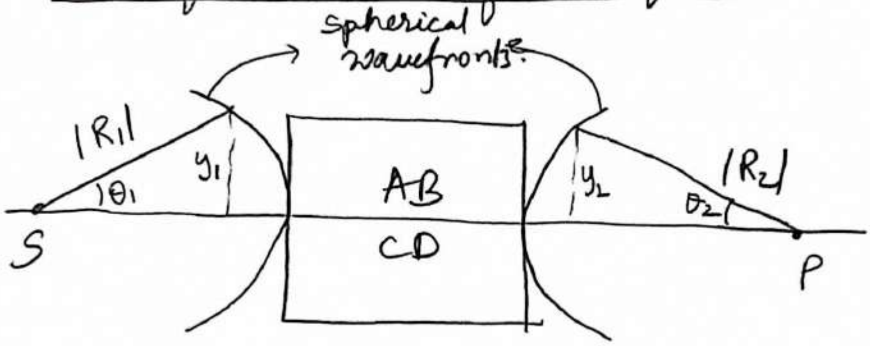
$$\frac{d^2y}{dz^2} + \alpha^2 y = 0$$

$$y(z) = y_0 \cos\alpha z + \frac{\theta_0}{\alpha} \sin\alpha z$$

$$\theta(z) = -y_0 \alpha \sin\alpha z + \theta_0 \cos\alpha z$$

$$\Rightarrow \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\alpha z & \frac{1}{\alpha}\sin\alpha z \\ -\alpha \sin\alpha z & \cos\alpha z \end{bmatrix}$$

Transformation of wavefronts

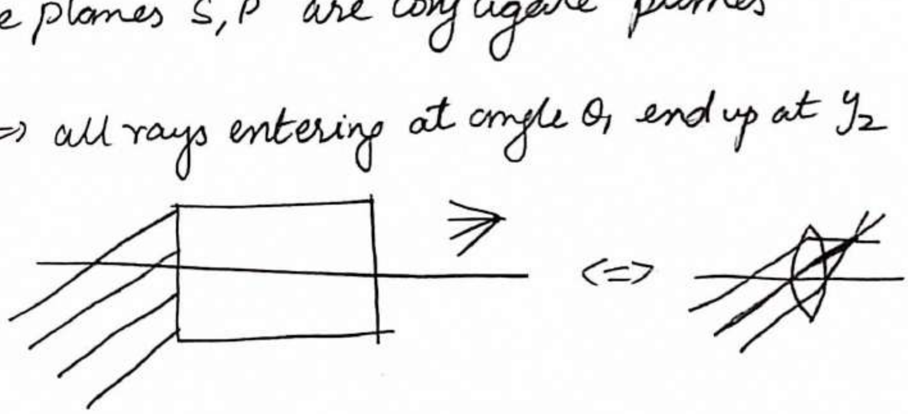


$$R_1 = \frac{y_1}{\theta_1} \quad ; \quad R_2 = \frac{y_2}{\theta_2} = \frac{Ay_1 + B\theta_1}{Cy_1 + D\theta_1}$$

$$\Rightarrow \boxed{R_2 = \frac{AR_1 + B}{CR_1 + D}} \quad \text{ABCD law!}$$

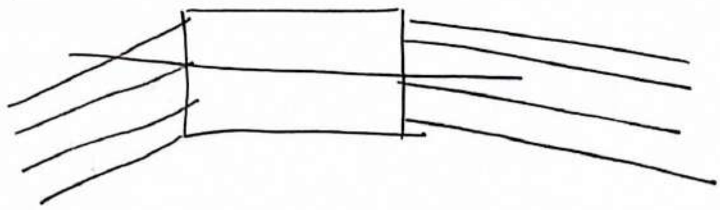
Other properties

- $B = 0 \Rightarrow$ reference planes S, P are conjugate planes
- $A = 0 \Rightarrow y_2 = B\theta_1 \Rightarrow$ all rays entering at angle θ_1 end up at y_2



- $C = 0 \Rightarrow \theta_2 = D\theta_1$

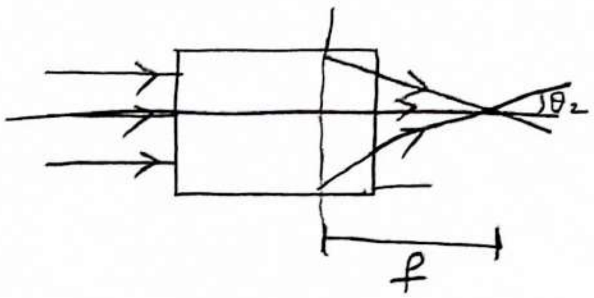
Telescopic system. with angular magnification $\frac{\theta_2}{\theta_1}$



- $D = 0$ is reciprocal of $A = 0$ case since $\theta_2 = Cy_1$.

General meaning of C

$\boxed{C = -\frac{1}{f}}$ always.

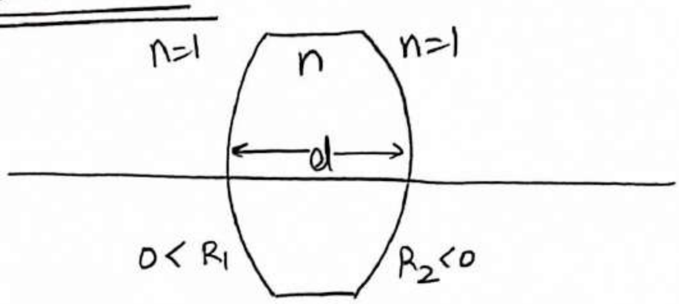


$\theta_1 = 0$

$\theta_2 = Cy_1 \Rightarrow \theta_2 = -\frac{1}{f} y_1$.

$\Rightarrow$ For a complex system find ABCD matrix & $f = \frac{-1}{C}$

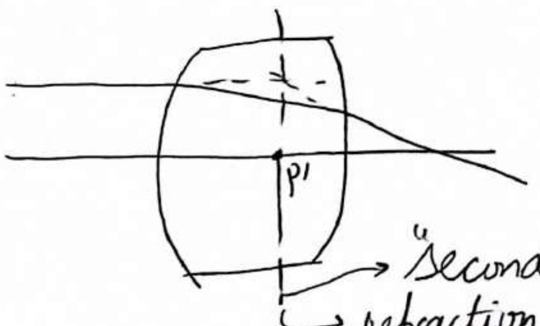
Thick lens



$$\begin{bmatrix} 1 & 0 \\ -\frac{1-n}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 + d\frac{1-n}{nR_1} & \frac{d}{n} \\ -\frac{1-n}{R_2}\left(1 + d\frac{1-n}{nR_1}\right) + \frac{1-n}{R_1} & 1 - d\frac{1-n}{nR_2} \end{bmatrix}$$

$$C = \frac{-1}{f} = \frac{1-n}{R_1} - \frac{1-n}{R_2} - \frac{(1-n)^2 d}{n R_1 R_2}$$

$$\Rightarrow \boxed{\frac{1}{f} = (n-1)\left[\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)d}{n R_1 R_2}\right]}$$ Lens Makers Eqn.

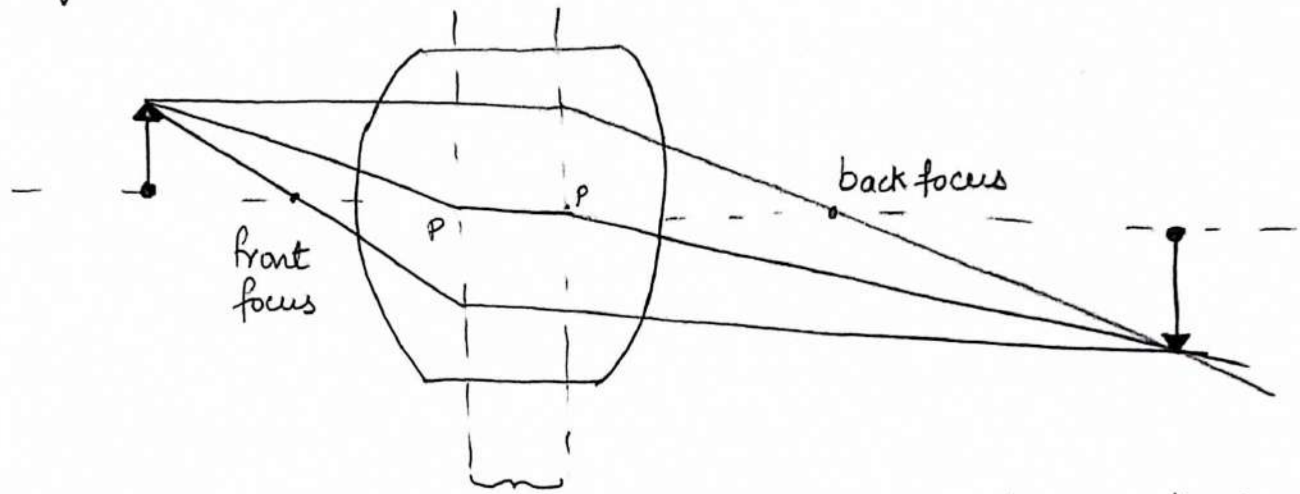


p' → "secondary principle point"

→ "Secondary principle plane"

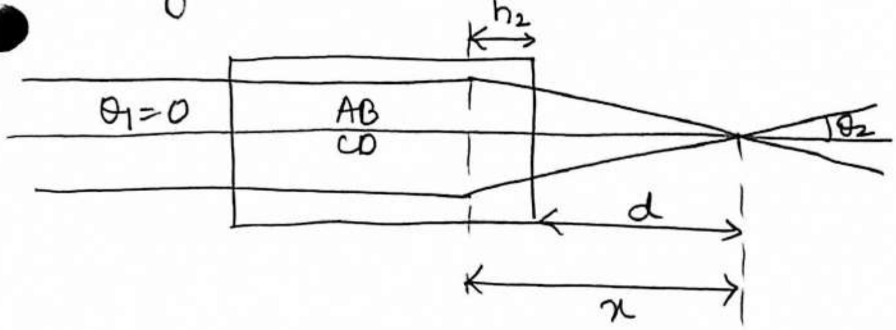
↳ refraction appears to occur here.

> Similarly a primary principle point and plane can be defined.



rays are always parallel between the 2 principle planes.

Finding the principle planes.



$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A+dC & B+dD \\ C & D \end{bmatrix}$$

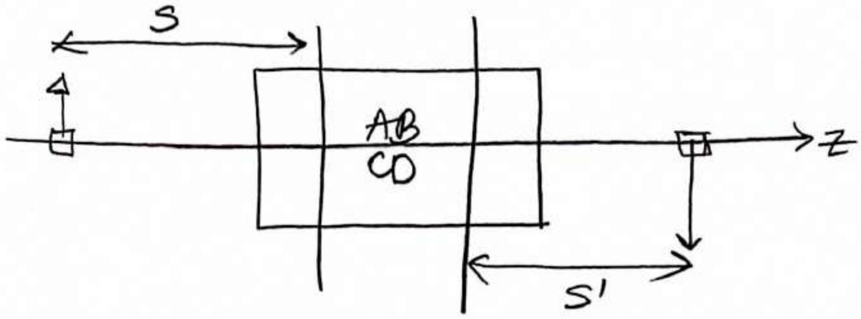
$\theta_1 = 0 \qquad y_2 = (A+dC)\, y_1$

at focus $y_2 = 0 \qquad A + dC = 0 \qquad d = -\frac{A}{C}$

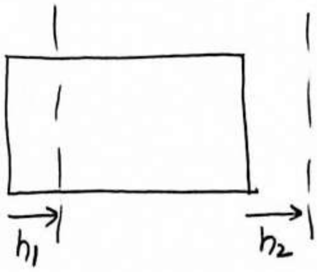
$\theta_1 = 0 \quad \theta_2 = Cy_1 \quad ; \tan\theta_2 = \theta_2 = -\frac{y_1}{x}$

$x = \frac{-y_1}{\theta_2} = \frac{-y_1}{Cy_1} = -\frac{1}{C} = f \quad ; \quad h_2 = d - x;$

$\boxed{h_2 = \frac{1-A}{C}}$ $\boxed{h_1 = \frac{D-1}{C}}$ once we have h_1 & h_2, $\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}$ applies

> 

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f}$$

> 

Here $h_1, h_2 > 0$ as drawn.

Propagation between the 2 PPs is given by.

$$\begin{bmatrix} 1 & h_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 1 & -h_1 \\ 0 & 1 \end{bmatrix}$$

Lens Aberrations

> Deviations from perfect point to point imaging are called aberrations

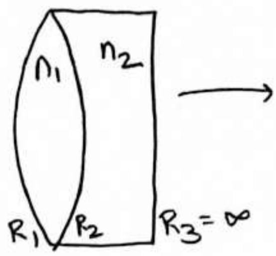
> Chromatic & monochromatic aberrations.

Chromatic aberrations

Recall $f(\lambda)^{-1} = [n(\lambda)-1]\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$ $\Rightarrow$ f is a fn. of frequency.

Blue bends more than red.

> An achromat is a combination of two lenses that can achieve equal focal lengths at two wavelengths.



$\longrightarrow$ <u>achromatic doublet.</u>

$$ABCD = \begin{bmatrix} 1 & 0 \\ -\frac{1-n_2}{R_3} & n_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n_2-n_1}{n_2 R_2} & \frac{n_1}{n_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n_1-1}{n_1 R_1} & \frac{1}{n_1} \end{bmatrix}$$

Calculating C,

$$\frac{1}{f} = \frac{n_1-1}{R_1} + \frac{n_2-n_1}{R_2} - \frac{n_2-1}{R_3}$$

Condition for achromaticity,

$$\frac{d}{d\lambda}\left(\frac{1}{f}\right) = 0 \Rightarrow \left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{dn_1}{d\lambda} + \left(\frac{1}{R_2} - \frac{1}{R_3}\right)\frac{dn_2}{d\lambda} = 0$$

> An "apochromat" uses 3 elements for equal focus at 3 wavelengths.

> Typically $R_3 \to \infty$ to reduce cost. R_1 & R_2 provide enough freedom to satisfy the above equation.

Monochromatic Aberrations.

> Generally a result from breakdown of the paraxial approximation ie $\sin\theta \cong \theta$.

> Use $\sin\theta \simeq \theta - \frac{\theta^3}{3}$ aka. 3rd order aberrations, primary aberrations or Seidal aberrations.

Interferometry

> Coherent addition of waves!

$$\bar{E}_1 = \hat{\epsilon}_1 E_{01} e^{i(\omega_1 t - \bar{k}_1 \cdot \bar{r} + \phi_1)}$$

$$\bar{E}_2 = \hat{\epsilon}_2 E_{02} e^{i(\omega_2 t - \bar{k}_2 \cdot \bar{r} + \phi_2)}$$

$$\bar{E} = \bar{E}_1 + \bar{E}_2$$

Assume it is incident on a square law detector,

$$\langle |E(t)|^2 \rangle = \frac{1}{T} \int_t^{t+T} |E(t)|^2 dt$$; $T \to$ response time of detector.

$$\Rightarrow \langle |E(t)|^2 \rangle = \langle (\bar{E}_1 + \bar{E}_2) \cdot (\bar{E}_1^* + \bar{E}_2^*) \rangle$$

$$= \langle |E_1(t)|^2 \rangle + \langle |E_2(t)|^2 \rangle$$

$$+ \langle \hat{\epsilon}_1 \cdot \hat{\epsilon}_2^* E_{01} E_{02}^* e^{i(\omega_1 - \omega_2)t} e^{-i(\bar{k}_1 - \bar{k}_2) \cdot \bar{r} + i(\phi_1 - \phi_2)} \rangle$$

$$+ c.c.$$

> Note that interference requires the 2 polarizations to be non orthogonal. i.e. $\hat{\epsilon}_1 \cdot \hat{\epsilon}_2^* \neq 0$.

> If $T < \frac{2\pi}{|\omega_1 - \omega_2|}$ $\Rightarrow \langle I(t) \rangle$ will exhibit temporal beats at $\omega_1 - \omega_2$

> If $T > \frac{2\pi}{|\omega_1 - \omega_2|}$ $\Rightarrow \langle e^{i(\omega_1 - \omega_2)t} \rangle = 0$ unless $\omega_1 = \omega_2$

Therefore interference only occurs when $\omega_1 = \omega_2$ (if T is large). It is in this sense that we say that waves need to have the same frequency to interfere.

Coherence: Frequency & waveform are identical.
$\hookrightarrow$ (constant relative phase)

From here on, $\omega_1 = \omega_2 = \omega$.

$$\Rightarrow \langle |E(t)|^2 \rangle = \langle |E_{01}|^2 \rangle + \langle |E_{02}|^2 \rangle + \langle E_{01} E_{02}^* e^{-i(\bar{k}_1 - \bar{k}_2)\cdot \bar{r} + i(\phi_1 - \phi_2)} + cc \rangle$$

or

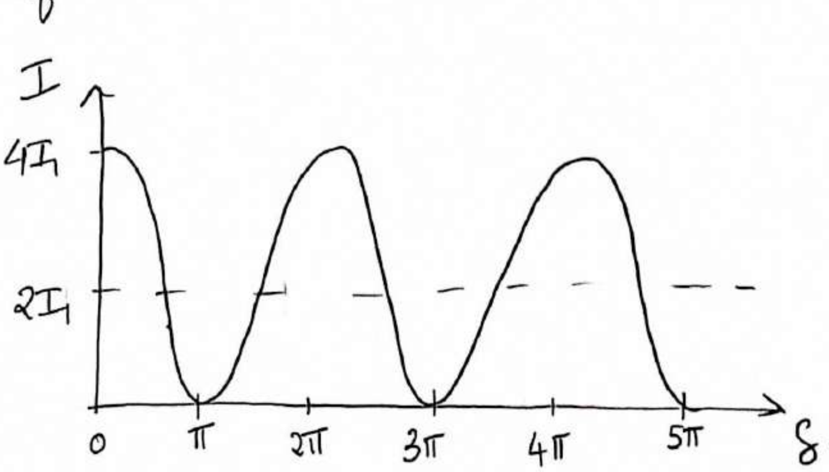
$$I(\bar{r}) = I_1 + I_2 + \sqrt{I_1 I_2} \left[e^{i(\bar{k}_1 - \bar{k}_2)\cdot\bar{r} + i(\phi_1 - \phi_2)} + cc \right]$$

Let $\delta = (\bar{k}_1 - \bar{k}_2)\cdot \bar{r} + (\phi_1 - \phi_2)$ $\rightarrow$ phase diff. b/w the 2 waves (independent of t)

$$\Rightarrow \boxed{I(\bar{r}) = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\delta.}$$

$\delta = 2n\pi \Rightarrow$ constructive.
$\delta = (2n+1)\pi \Rightarrow$ destructive.

If $I_1 = I_2 \Rightarrow I = 4 I_1 \cos^2 \frac{\delta}{2}$



Usually $\phi_1 = \phi_2 \Rightarrow$ same initial phase.

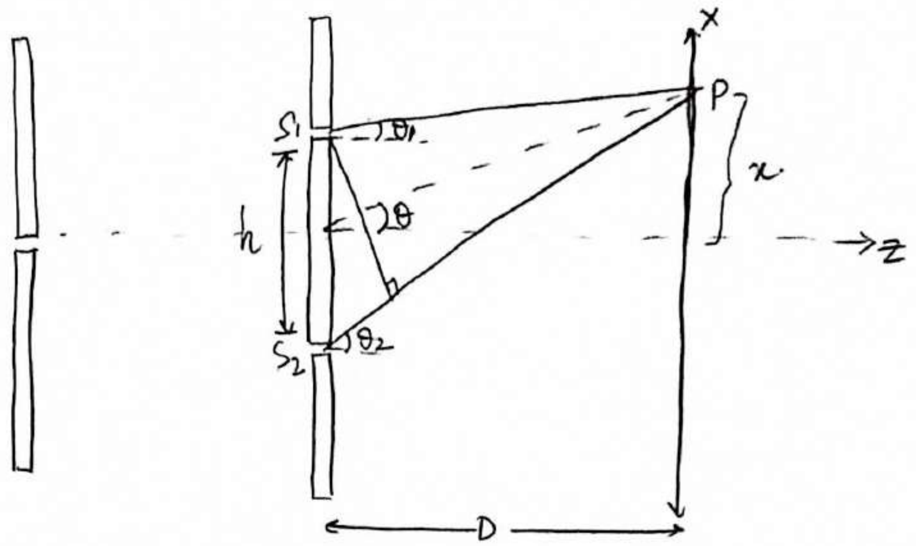
$\Rightarrow \delta = \bar{k}_1 \cdot \bar{r} - \bar{k}_2 \cdot \bar{r}$

In general if the 2 rays follow paths C_1 & C_2 we have.

$$\delta = \int_{C_1} \overline{k_1(\bar{r})} \cdot d\bar{s} - \int_{C_2} \bar{k}_2(\bar{r}) \cdot d\bar{s}$$

Young's Interference

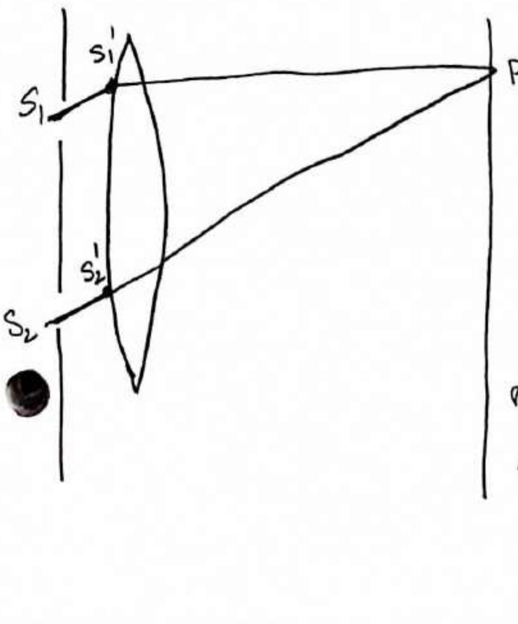
Double Slit Interference



- If $D \gg h$, $\theta_1 \simeq \theta_2 \simeq \theta$

$$OPD = \overline{S_2P} - \overline{S_1P} = h\sin\theta \simeq h\theta. \quad , \quad \theta = \tan\theta = \frac{x}{D}$$

Instead of assuming $D \gg h$, place a lens after the double slit & make $D = f$. Now $OPD = h\sin\theta$ & $\theta_1 = \theta_2 = \theta$



> $S_1'P = S_2'P$ from Fermat's Principle.

$$\underbrace{\delta}_{k(OPD)} = \frac{2\pi}{\lambda} \cdot \frac{hx}{D} = 2\pi m \Rightarrow \text{constructive.}$$

or $OPD = \underbrace{m}_{\text{"order"}}\lambda \Rightarrow$ Constructive.

"The lens effectively puts the screen at infinity".

Intensity Pattern.

› Same intensity and phase at both slits implies

$$I = 4I_1 \cos^2\frac{\delta}{2} = 4I_1 \cos^2\left(\pi \frac{Lx}{\lambda D}\right)$$

The maxima are spread out by an amount,

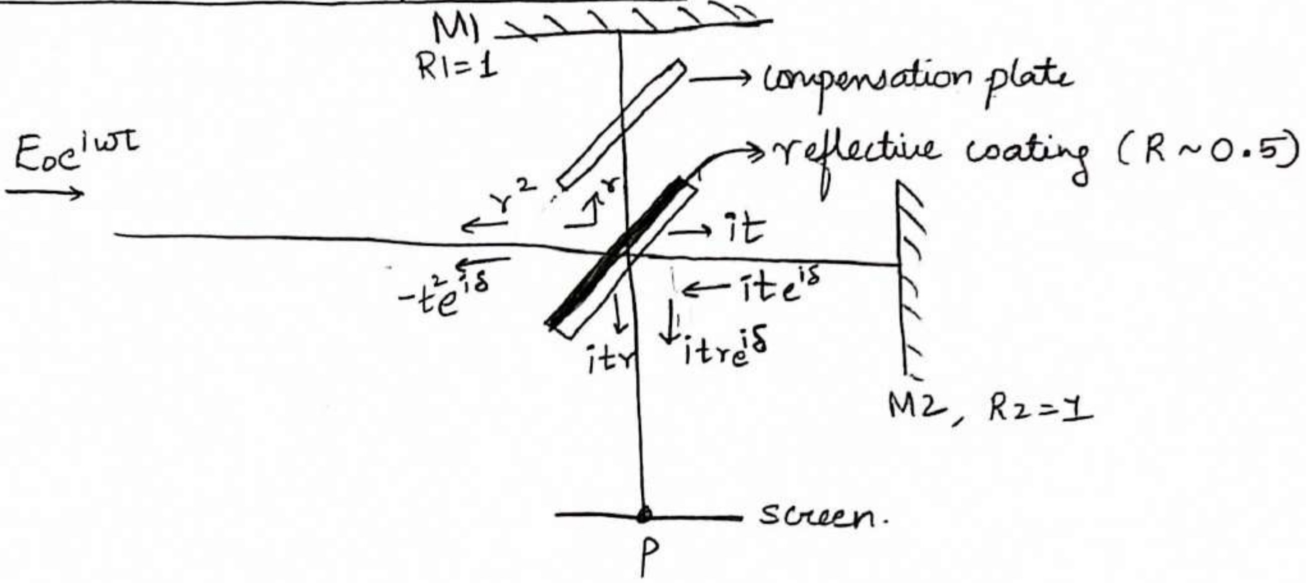
$$|\Delta x| = |m| \frac{D}{h} \Delta\lambda_0.$$

mean spacing b/w fringes $|\delta x| = \frac{D}{h}\lambda_0$.

⇒ If $|m| < \frac{\lambda_0}{\Delta\lambda}$, the spread is negligible.

→ This experiment is an example of "interference by division of wavefronts".

Michelson Interferometer



> For a lossless dielectric mirror, power conservation and time reversal symmetry give a scattering matrix as follows,

$$S = \begin{bmatrix} r & it \\ it & r \end{bmatrix} \qquad t = \sqrt{1-r^2}$$

> The reflected & transmitted waves are at 90° w.r.t each other.

> $$\delta = \frac{2\pi}{\lambda} 2d = \frac{2\omega}{c} d = \omega \tau^{\frac{2d}{c}}$$; d is excess distance of M2 w.r.t M1.

> $\tau = \frac{2d}{c}$ is the optical delay

$$E_R = r(rE_0) + it(ite^{i\delta}E_0)$$

$$= [r^2 - (1-r^2)e^{i\delta}]E_0$$

$$= \underline{0} \text{ (if } \delta = 0 \text{ and } r^2 = 0.5)$$

> $E_T = r(itE_0) + it\, e^{i\delta}(rE_0)$

$= itr(1+e^{i\delta})E_0$

If $r = t = \frac{1}{\sqrt{2}}$ & $\delta = 0$ we have $E_T = iE_0$.

> $|E_T|^2 = |itr(1+e^{i\delta})|^2 |E_0|^2$

$\Rightarrow I_t = |r|^2 |t|^2 |1+e^{i\delta}|^2 I_0$

$= RT(1+e^{i\delta})(1+e^{-i\delta}) I_0$

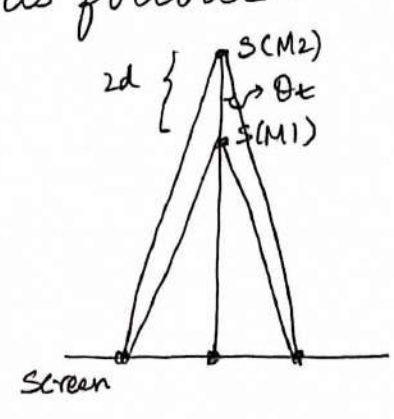
$$\boxed{I_t = 2RT(1+\cos\omega\tau) I_0}$$

$R = |r|^2 ; T = |t|^2$

If $R = T = 0.5$;

$$\boxed{I_t = \frac{1}{2}(1+\cos\omega\tau) I_0}$$

> Intensity pattern on screen is of concentric rings since source gives spherical waves. The image of S from M1 & M2 is as follows.

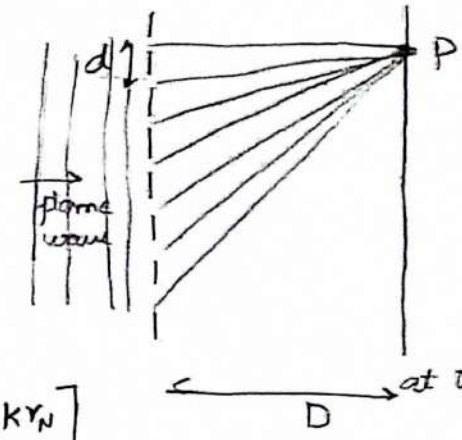


$\Rightarrow \delta = \frac{2\pi}{\lambda} \cdot 2d\cos\theta_t = 2\pi m \Rightarrow$ constructive.

Interference of N-waves.

Division of wavefront $\Leftrightarrow$ N slits.

Total field at P,



$$E = E_0(r_1)e^{i(\omega t - kr_1)} + E_0(r_2)e^{i(\omega t - kr_2)} + \ldots + E_0(r_N)e^{i(\omega t - kr_N)}$$

$$\simeq E_0(D)\, e^{i\omega t}\left[e^{-ikr_1} + e^{-ikr_2} + \ldots + e^{-ikr_N}\right]$$

(Assume $E_0(r_1) = E_2(r_2) = \ldots = E_0(D)$)

$$\Rightarrow E = E_0(D)\, e^{i(\omega t - kr_1)}\left[1 + e^{-ik(r_2 - r_1)} + \ldots + e^{-ik(r_N - r_1)}\right]$$

$r_3 - r_1 = 2(r_2 - r_1)$; $r_4 - r_1 = 3(r_2 - r_1)$; ... ; $r_N - r_1 = (N-1)(r_2 - r_1)$

We Know $r_2 - r_1 = d\sin\theta = \frac{\delta}{k}$ → phase diff. b/w adjacent slits

$$\Rightarrow E = E_0\, e^{i(\omega t - kr_1)}\left[1 + e^{-i\delta} + (e^{-i\delta})^2 + \ldots + (e^{-i\delta})^{N-1}\right]$$

$$= E_0\, e^{i(\omega t - kr_1)}\left[\frac{e^{-iN\delta} - 1}{e^{-i\delta} - 1}\right] = E_0 e^{i(\omega t - kr_1)}\left[\frac{e^{-i\frac{N\delta}{2}}\left(e^{-i\frac{N\delta}{2}} - e^{i\frac{N\delta}{2}}\right)}{e^{-i\delta/2}\left(e^{-i\frac{\delta}{2}} - e^{i\frac{\delta}{2}}\right)}\right]$$

$$= E_0\, e^{i[\omega t - kr_1 - (N-1)\delta/2]}\left[\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}\right]$$

Note, $D = r_1 + \frac{1}{2}(N-1)d\sin\theta$.

$$\Rightarrow \boxed{E = E_0(D)\, e^{i(\omega t - kD)} \left[\frac{\sin\frac{N\delta}{2}}{\sin\frac{\delta}{2}}\right]}$$

plane wave modulated by [ ] term.

(Similar to array factor from Antennas)

$$I = I_0 \frac{\sin^2\left[\frac{Nkd\sin\theta}{2}\right]}{\sin^2\left[\frac{kd\sin\theta}{2}\right]}$$

or

$$\boxed{I = I_0 \left[\frac{\sin^2\left(N\pi\frac{d\sin\theta}{\lambda}\right)}{\sin^2\left(\pi\frac{d\sin\theta}{\lambda}\right)}\right]}$$

oscillates much faster that the denominator.

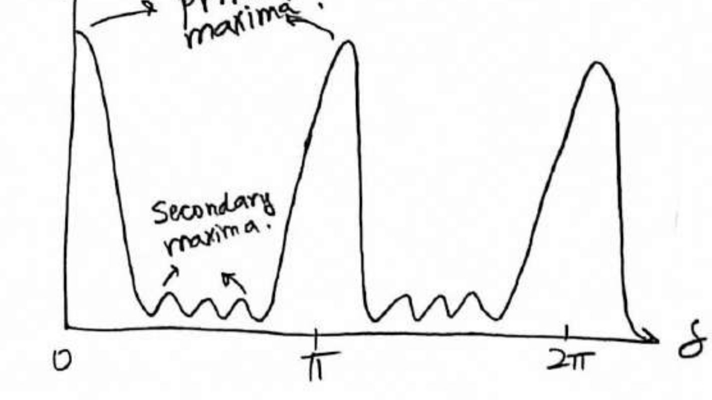
$\delta = kd\sin\theta$

As $\theta \to 0$ (ie $m=0$) $\Rightarrow \sin\theta = \theta$,

$$\Rightarrow \boxed{I = I_0 N^2}$$

$\Rightarrow$ When $\underbrace{d\sin\theta = m\lambda}_{\text{grating equation with order } m.}$ we have principle maxima.

Secondary maxima occur at $\frac{N\delta}{2} = \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$



$N \gg 1 \Rightarrow$ secondary maxima are small & principle maxima are very narrow.

$$\boxed{\Delta\theta = \frac{\lambda}{Nd\cos\theta}}$$ angular width.

Fabry Perot Etalons and Interferometers.

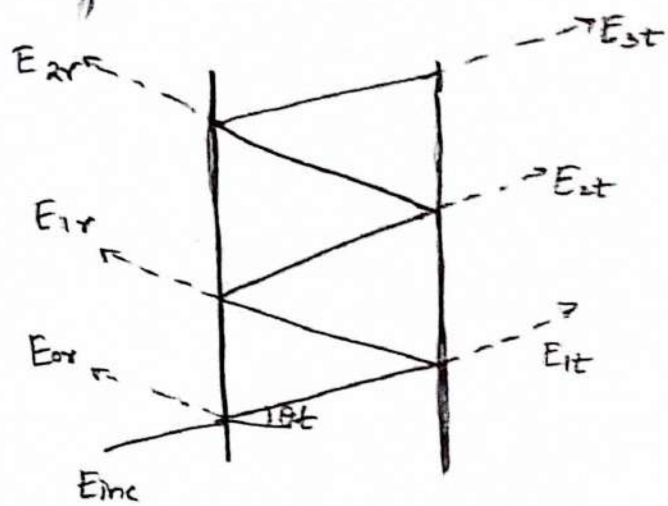
Division of Amplitude

FP Etalon: A dielectric slab with parallel faces (may be coated)

FP Interferometer: Two highly reflecting parallel mirrors.

Approach I: Follow the partial reflections.

$t_{1,2} = \sqrt{1 - r_{1,2}^2}$



> Even with 99.99% reflectivity, at some "resonant frequencies" the reflections inside build up in amplitude to a large steady state field & then $E_{1r}, E_{2r} \ldots$ add up to cancel E_{0r}. Which gives perfect transmission.

1) incident wave $\rightarrow E_{inc}$
2) one trasmission $\rightarrow i t_1 E_{inc}$.
3) propagation b/w mirrors $\rightarrow i t_1 e^{-i\delta/2} E_{inc}$.
4) transmission through second mirror $\rightarrow (it_1)(it_2)\, e^{-i\delta/2} E_{inc}$.

$$= -(t_1 t_2)\, e^{-i\delta/2} E_{inc}.$$

where $\dfrac{\delta}{2} = \dfrac{n\omega d}{c} \cos\theta_t$

$\therefore\; E_{2t} = (r_1 r_2 e^{-i\delta})(-t_1 t_2 e^{-i\delta/2}) E_{inc}$.

$E_{3t} = (r_1 r_2 e^{-i\delta})^2 (-t_1 t_2 e^{-i\delta/2}) E_{inc}$. & so on.

$$\Rightarrow E_{trans} = \left[\sum_{l=0}^{\infty} \left(r_1 r_2 e^{i\delta}\right)^l\right]\left(-t_1 t_2 e^{-i\delta_1}\right) E_{inc}.$$

$r_1 < 1$ & $r_2 < 1$ so we set $x = r_1 r_2 e^{-i\delta}$ & note $|x| < 1$

$$\Rightarrow \sum_{l=0}^{\infty} x^l = \frac{1}{1-x}$$

$$\Rightarrow \boxed{E_{trans} = \frac{-t_1 t_2 e^{-i\delta_1}}{1 - r_1 r_2 e^{i\delta}} E_{inc}.}$$

Similarly,

$$E_{0r} = r_1 E_{inc}$$

$$E_{1r} = (it_1) r_2 (it_1) e^{-i\delta} E_{inc}.$$

$$E_{2r} = r_1 r_2 e^{-i\delta} E_{1r}$$

$$\vdots$$

$$\Rightarrow E_{refl} = \left[\sum_{l=0}^{\infty} \left(r_1 r_2 e^{-i\delta}\right)^l\right] \left(-t_1^2 r_2 e^{-i\delta}\right) E_{inc} + r_1 E_{inc}.$$

$$\Rightarrow \boxed{E_{refl} = \left[r_1 - \frac{t_1^2 r_2 e^{-i\delta}}{1 - r_1 r_2 e^{-i\delta}}\right] E_{inc}.}$$

Approach II :- Scattering & Transfer matrices.

- $$S_{1,2} = \begin{bmatrix} r_{1,2} & it_{1,2} \\ it_{1,2} & r_{1,2} \end{bmatrix} \quad ; \quad M_{1,2} = \frac{1}{it_{1,2}} \begin{bmatrix} -1 & r_{1,2} \\ -r_{1,2} & 1 \end{bmatrix}$$

$$\Rightarrow M = \left\{ \left(\frac{1}{it_2} \right) \begin{bmatrix} -1 & r_2 \\ -r_2 & 1 \end{bmatrix} \right\} \begin{bmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{bmatrix} \left\{ \left(\frac{1}{it_2} \right) \begin{bmatrix} -1 & r_1 \\ -r_1 & 1 \end{bmatrix} \right\}$$

$$\Rightarrow M_{22} = \frac{-1}{t_1 t_2} \left[-r_1 r_2 e^{-i\delta/2} + e^{i\delta/2} \right]$$

$S_{21} = S_{12} = t_{21} = \dfrac{1}{M_{22}}$ which gives the same results as Approach I

- ### Intensity

$$I_{trans} = \left| \frac{-t_1 t_2 e^{-i\delta/2}}{1 - r_1 r_2 e^{-i\delta}} \right|^2 I_{inc.}$$

Consider $r_1 = r_2 = r$; $t_1 = t_2 = t = \sqrt{1-r^2}$

$$\Rightarrow I_{trans} = \frac{t^4}{|1 - r^2 e^{-i\delta}|^2} I_{inc.}$$

$$|1 - r^2 e^{-i\delta}|^2 = (1 - Re^{-i\delta})(1 - Re^{i\delta}) = 1 - 2R\cos\delta + R^2$$

- $$= 1 - 2R\left(1 - 2\sin^2\frac{\delta}{2}\right) + R^2$$

$$= (1-R)^2 + 4R\sin^2\frac{\delta}{2} \qquad \text{where } R = r^2$$

$$\Rightarrow I_{trans} = \frac{T^2}{(1-R)^2\left[1+\frac{4R}{(1-R)^2}\sin^2\frac{\delta}{2}\right]} I_{inc}.$$

Define "contrast" as $\boxed{F = \frac{4R}{(1-R)^2}}$; note $\frac{T^2}{1-R^2} = 1$.

& $R+T = 1$

$$\Rightarrow \boxed{\frac{I_{trans}}{I_{inc}} = \frac{1}{1+F\sin^2\frac{\delta}{2}}} \longrightarrow \boxed{\text{Airy function.}}$$

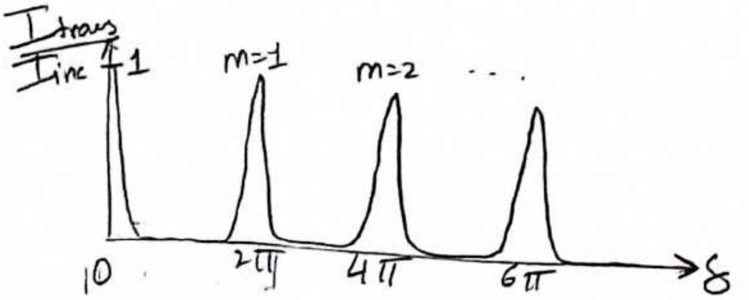
F is also called "coefficient of finesse."

> When $\delta = \frac{2nd\omega}{c}\cos\theta_t = 2n\pi$ we have $\sin^2\frac{\delta}{2} = 0$

$\Rightarrow I_{trans} = I_{inc}$!

> When $\delta = (2n+1)\pi$; $\frac{I_{trans}}{I_{inc}} = \frac{1}{1+F}$

> If R is very close to 1, F is very large. $\Rightarrow I_{trans} \ll I_{inc}$.



Define $\boxed{L = nd\cos\theta_t}$

$\Rightarrow$ Seperation b/w 2 orders ($\Delta m = 1$) is called the "free spectral range".

$$\frac{2\Delta\omega_{FSR}}{c} L = 2\pi$$

aka axial mode spacing.

$$\Rightarrow \boxed{\Delta\omega_{FSR} = 2\pi\frac{c}{2L}} \text{ or } \boxed{\Delta\nu_{FSR} = \frac{c}{2L}}$$ ✲

> Full width at half maximum,

- ie. $$\frac{1}{1+F\sin^2\frac{\delta}{2}} = \frac{1}{2}$$

$$\Rightarrow \boxed{\delta_{1/2} = \pm 2\sin^{-1}\left(\frac{1}{\sqrt{F}}\right)}$$

When $F \gg 1$ $\quad \sin\frac{\delta}{2} \simeq \frac{\delta}{2} = \pm\frac{1}{\sqrt{F}}$

$$\Rightarrow \boxed{\Delta\delta_{1/2} \cong \frac{4}{\sqrt{F}}}$$

Recall $\delta = \frac{2\omega}{c} nd\cos\theta_t = \frac{2\omega}{c}L = 2\pi\nu\frac{2L}{c}$

- $$\Delta\delta = 2\pi\Delta\nu\cdot\frac{2L}{c}$$

$$\Rightarrow \Delta\nu_{1/2} = \frac{\Delta\delta_{1/2}}{2\pi}\cdot\frac{c}{2L} = \frac{2}{\pi\sqrt{F}}\Delta\nu_{FSR}.$$

Define Finesse $\mathcal{F}$ by $\boxed{\mathcal{F} = \frac{\pi}{2}\sqrt{F}}$

$$\Rightarrow \boxed{\frac{\Delta\nu_{1/2}}{\Delta\nu_{FSR}} = \mathcal{F}^{-1}}$$

→ With large F (& $\mathcal{F}$) we have $\Delta\nu_{FSR} \gg \Delta\nu_{1/2}$ & this is useful in doing spectroscopy.

Diffraction

> Confined light (or any wave) eventually spreads.
> We consider only scalar diffraction theory ⇒ scalar fields.

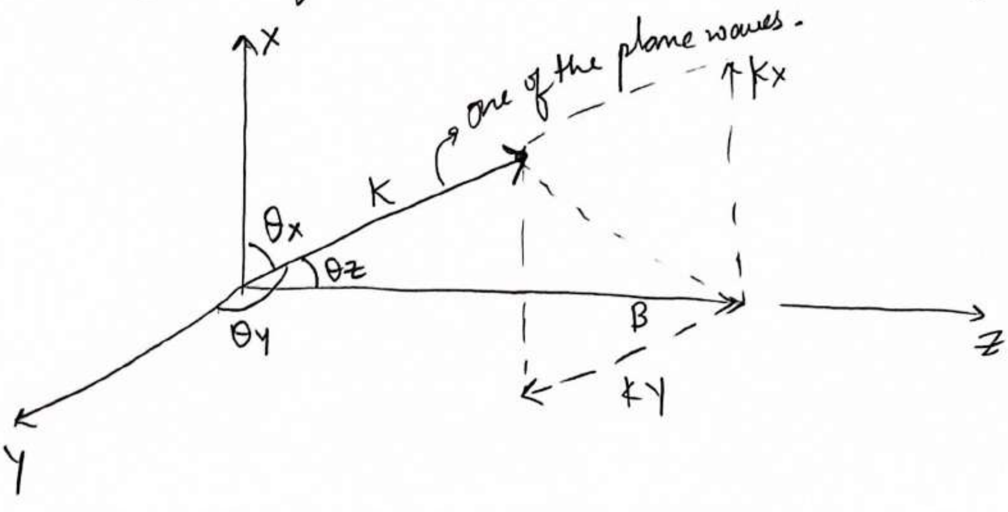
Recall Angular spectrum

$$E_A(\bar{r}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(k_x, k_y)\, e^{-i(k_x x + k_y y + \beta z)}\, dk_x\, dk_y$$

→ We break down the field into plane waves propagating at different angles. Each plane wave has a value for a given k_x, k_y called the angular spectrum (ie. $A(k_x, k_y)$)

→ $\bar{K} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z}$.

→ Therefore for a given z, the value of $\bar{E}$ can be written as a sum of plane waves in different directions (with same z). The "weight" of each direction is given by $A(k_x, k_y)$



$$\beta = \sqrt{K^2 - K_x^2 - k_y^2}$$

$K_x \rightarrow$ spatial frequency along x direction.

$\Rightarrow \quad \theta_x = \cos^{-1}\left(\frac{K_x}{K}\right)$ gives prop. angle wrt x axis

If $K_x^2 + K_y^2 > \left(\frac{2\pi}{\lambda}\right)^2 \Rightarrow \beta$ is imaginary $\Rightarrow$ evanescent modes.

> Therefore there is an upper limit on the spatial frequency that can propagate. Higher frequencies (spatial) that capture variations that are smaller are evanescent!

> Such an angular spectrum decomposition directly hints at diffraction since we have plane waves going in all directions. (Unless we started with a plane wave).

> Once we know $A(K_x, K_y)$ at z_1 ie $A(K_x, K_y; z_1)$ it is trivial to find $A(K_x, K_y; z_2)$. Let $z_1 = 0$, $z_2 = z$. Then we have

$$A(K_x, K_y; z) = A(K_x, k_y; 0)\, e^{-i\beta z}$$

> Let $H(K_x, K_y; z) = e^{-i\beta z}$ be the transfer fn of the angular spectrum.

> Therefore, propagation in free space amounts to multiplication of angular spectrum by the transfer function

$$> A(k_x, k_y; 0) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} E(x', y'; 0)\, e^{i(k_x x' + k_y y')}\, dx'\, dy'$$

$$\Rightarrow E(x, y; z) = \iint_{-\infty}^{\infty} A(k_x, k_y; 0)\, H(k_x, k_y; z)\, e^{-i(k_x x + k_y y)}\, dk_x\, dk_y$$

$$= \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} E(x', y'; 0) \times \left[\iint_{-\infty}^{\infty} H(k_x\, k_y; z)\, e^{-i[k_x(x-x') + k_y(y-y')]}\, dk_x\, dk_y \right] dx'\, dy'$$

→ The term in brackets looks like an inverse Fourier Transform & indeed it is the "spatial impulse response"

$$h(x, y; z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} H(k_x, k_y; z)\, e^{-i(k_x x + k_y y)}\, dk_x\, dk_y$$

→ The fields then are the convolution integral.

$$E(x, y; z) = \iint_{-\infty}^{\infty} E(x', y'; 0)\, h(x - x'; y - y'; z)\, dx'\, dy'$$

$$\boxed{E(x, y; z) = E(x', y'; 0) \otimes h(x, y; z)}$$

Fresnel Approximation : Propagation of Paraxial Waves.

> $H(k_x, k_y; z) = e^{-i\beta z}$. where $\beta = \sqrt{k_x^2 + k_y^2 - k^2}$

Assuming waves propagate close to z axis we have,

$$\beta = k\sqrt{1 - \frac{k_x^2}{k^2} - \frac{k_y^2}{k^2}} \simeq k\left[1 - \frac{1}{2}\left(\frac{k_x^2 + k_y^2}{k^2}\right)\right]$$

So,

$$\beta \cong k - \frac{k_x^2 + k_y^2}{2k}$$

$\sin\theta \simeq \theta$ comes from the fact that $\frac{k_x}{k} \ll 1 \Rightarrow \frac{k_x}{k} = \sin\theta \ll 1$!

$$\Rightarrow \boxed{H(k_x, k_y; z) = e^{-i\beta z} \simeq e^{-ikz}\, e^{i(k_x^2 + k_y^2)z/2k}}$$

Impulse response : $\boxed{h(x, y; z) \simeq \frac{e^{-ikz}}{i\lambda z}\, e^{-ik(x^2+y^2)/2z}}$

Conclusion

f i) Each spatial frequency component acquires a quadratic phase $(k_x^2 + k_y^2)z/2k$ on propagating over a distance z.

t ii) The initial field is convolved by a quadratic phase $k(x^2+y^2)/2z$.

In either case, the phase accumulation is quadratic

Gaussian Beams.

Assume $E(x,y;0) = E_0 \, e^{\frac{-(x^2+y^2)}{W_0^2}}$ where $W_0 \to$ beam waist

ie the radius where amplitude drops by $1/e$.

$$A(k_x, k_y; 0) = -E_0 \iint_{-\infty}^{\infty} e^{-(x^2+y^2)/W_0^2} \, e^{i(k_x x + k_y y)} \, dx\,dy$$

$$\boxed{A(k_x, k_y; 0) = E_0 \pi W_0^2 e^{-W_0^2 (k_x^2 + k_y^2)/4}}$$

From Fresnel approximation, at a distance z we have

$$A(k_x, k_y; z) = E_0 \pi W_0^2 e^{-W_0^2 (k_x^2+k_y^2)/4} \, e^{-ikz} \, e^{i(k_x^2+k_y^2)\frac{z}{2k}}$$

$$\Rightarrow E(x,y;z) = \frac{E_0}{4\pi} W_0^2 e^{-ikz} \iint_{-\infty}^{\infty} e^{-(k_x^2+k_y^2)\left(\frac{W_0^2}{4} - \frac{iz}{2k}\right)} \times e^{-i(k_x x + k_y y)} \, dk_x dk_y$$

$$= \frac{E_0}{1 - i\frac{2z}{kW_0^2}} \, e^{-(x^2+y^2)/W_0^2\left(1 - i\frac{2z}{kW_0^2}\right)}$$

Define, the Rayleigh range $z_R = \frac{kW_0^2}{2} = \frac{2\pi}{\lambda} \frac{W_0^2}{2}$

$$\Rightarrow \boxed{z_R = \frac{\pi W_0^2}{\lambda}}$$

> The factor $1 - i\frac{2z}{kW_0^2} = 1 - i\frac{z}{z_R} = \underbrace{\sqrt{1+\left(\frac{z}{z_R}\right)^2}\, e^{-i\tan^{-1}\left(\frac{z}{z_R}\right)}}_{\text{polar}}$

$$= \sqrt{1+\left(\frac{z}{z_R}\right)^2}\, e^{-i\phi(z)}$$

$$= \frac{W(z)}{W_0}\, e^{-i\phi(z)}$$

> Alternatively,

$$\frac{1}{1-i\left(\frac{z}{z_R}\right)} = \frac{z_R}{z_R - iz} = \frac{iz_R}{z + iz_R} = \frac{iz_R}{q(z)}$$

Where $\boxed{q(z) = z + iz_R}$ is the 'q-parameter'

Define : Wavefront radius as

$$\boxed{R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right]}$$

Define : Beam radius as

$$W^2(z) = W_0^2\left[1+\left(\frac{z}{z_R}\right)^2\right] \quad ; \; W_0 \text{ is the beam waist.}$$

> The Gaussian beam envelope can then be written as,

$$\mathcal{E}(x,y,z) = E_0 \frac{W_0}{W(z)}\, e^{-i\left(-\phi(z) + k(x^2+y^2)/2R(z)\right)}\, e^{-(x^2+y^2)/W^2(z)}.$$

Also,

$$\boxed{\mathcal{E}(x,y,z) = E_0\left[\frac{iz_R}{q(z)}\right]e^{-ik(x^2+y^2)/2q(z)}}$$

Note,

$$\frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)}$$

The full field $E(x,y,z) = \mathcal{E}(x,y,z)\,e^{-j\beta z}$.

Standard Form of the Gaussian Beam.

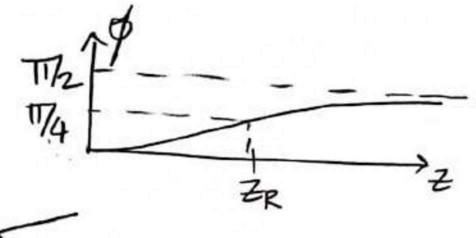
> Define origin where beam radius is minimum $\Rightarrow W(z=0) = W_0$, and the wavefront radius $R(z=0) = \infty$. Then,

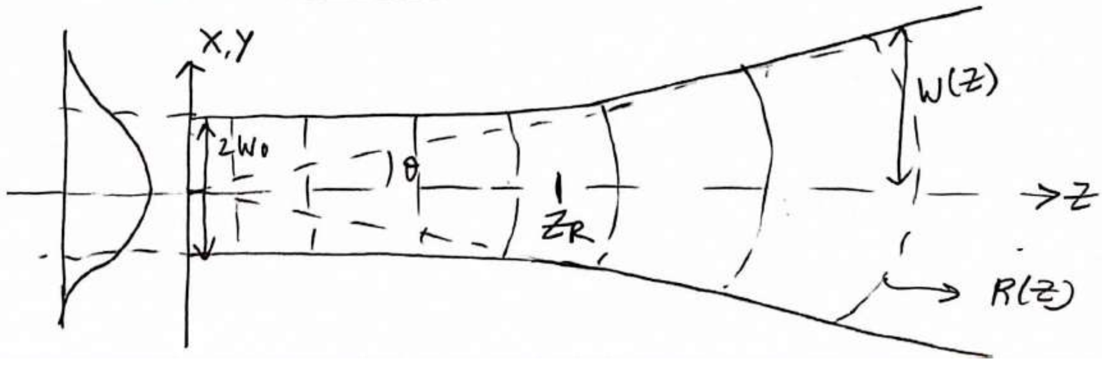
$$\frac{1}{q(z)} = \frac{1}{q_0} = \frac{1}{R(z)} - i\frac{\lambda}{\pi W^2(z)} = 0 - i\frac{\lambda}{\pi W_0^2}$$

$$\Rightarrow q_0 = i\frac{\pi W_0^2}{\lambda} = iz_R$$

$$\Rightarrow \quad W(z) = W_0\sqrt{1+\left(\frac{z}{z_R}\right)^2}\ ;\ R(z) = z\left[1+\left(\frac{z_R}{z}\right)^2\right];$$

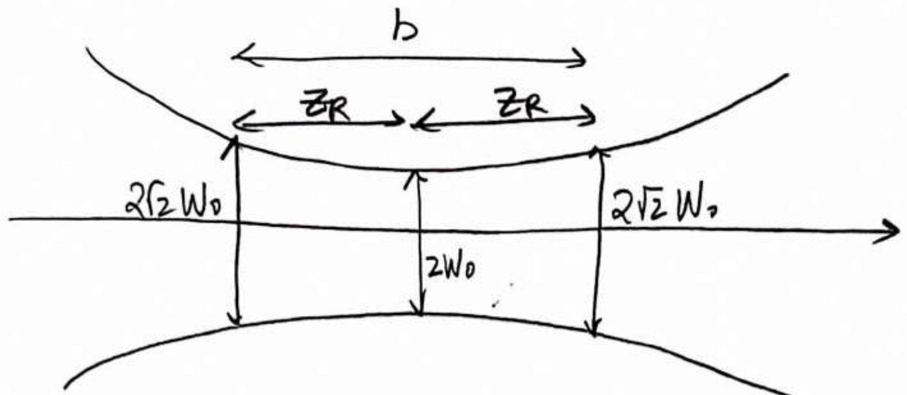
$$\boxed{\phi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)} \longrightarrow \text{Guoy phase}$$





At $z = z_R$; $W(z) = \sqrt{2}\, W_0$. Beyond z_R, the beam is not collimated and the divergence due to diffraction becomes significant.

Define $\boxed{b = 2 z_R}$ → confocal parameter.



→ A small beam waist corresponds to a small confocal parameter. Since $b = 2z_R = \frac{2\pi W_0^2}{\lambda}$

$$\theta = \tan\theta = \frac{W(z)}{z} \simeq \frac{W_0}{z_R} = \frac{\lambda}{\pi W_0}$$

$$\Rightarrow \boxed{\theta = \frac{\lambda}{\pi W_0}}$$ Angular spread of Gaussian beam.

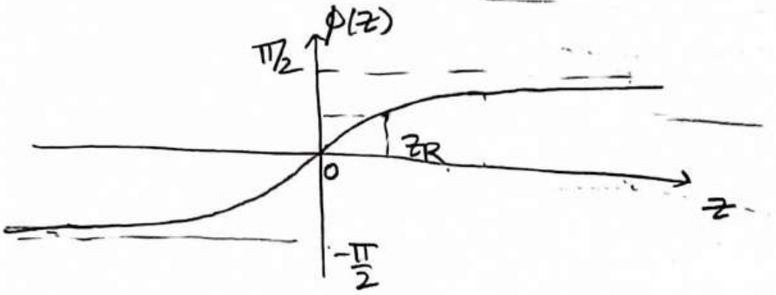
> Narrow beam ⇒ more divergence.

Guoy Phase Shift:

$e^{-ikz+i\phi(z)}$ is the phase shift of a Gaussian beam.

Where $\phi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$

> This is an additional (axial) phase shift beyond the normal phase advance, called the Guoy phase shift



From focus to ∞ $\rightarrow \frac{\pi}{2}$

From $-\infty$ to ∞ $\rightarrow \pi$.

From $-z_R$ to z_R $\rightarrow \frac{\pi}{2}$

Beam Power.

In cylindrical coordinates,

$$I(\rho) = \frac{2P}{\pi W^2} e^{-2\rho^2/W^2}$$ where P the total power in the beam.

The power transmitted through a circular aperture of diameter D is given by

$$P_{trans} = \frac{2P}{\pi W^2}\int_0^{D/2} e^{-2\rho^2/W^2} 2\pi\rho\, d\rho = P\left(1 - e^{-D^2/2W^2}\right).$$

> In reality diffraction from the edges will introduce a "ripple" that can only be reduced by increasing D.

Gaussian Beam Propagation & ABCD matrices.

- $R(z)$ and $w(z)$ are connected by

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi w^2(z)} \quad \text{—} ①$$

or $$q(z) = z - i z_R \quad \text{—} ②$$

If $q(z_1) = q_1 \Rightarrow q(z_2) = q_2 = q_1 + d$ from ②

$$\Rightarrow \quad q_2 = \frac{Aq_1 + B}{Cq_1 + D} \text{ where } \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

- A thin lens changes R but not w.

From ①, $$\frac{1}{q_2} - \frac{1}{q_1} = \frac{1}{R_2} - \frac{1}{R_1}$$

From GO we know how a thin lens changes R, ie

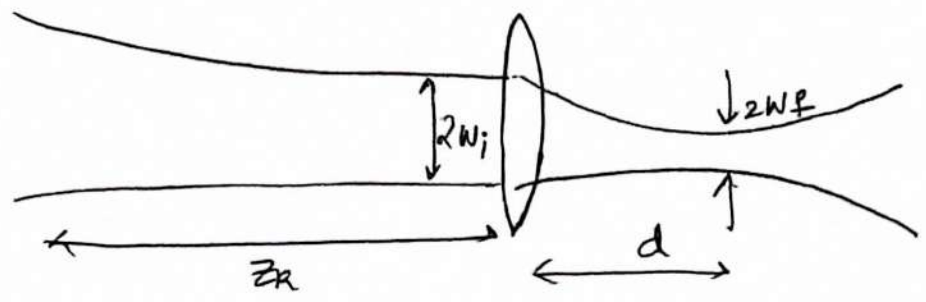
$$R_2 = \frac{AR_1 + B}{CR_1 + D} \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Combining these results gives,

$$\boxed{q_2 = \frac{Aq_1 + B}{Cq_1 + D}}$$

→ ABCD law. This law is more general than Gaussian beams. It says that the complex radius of curvature q is changed by a system exactly as R is changed.

Focussing a Gaussian Beam.



Assume
$Z_R \gg f$

$$\frac{1}{q_1} = \frac{1}{\infty} - i\frac{\lambda}{\pi W_i^2} \Rightarrow q_1 = i\frac{\pi W_i^2}{\lambda} \quad \text{where } Z_{R_i} = \frac{\pi W_i^2}{\lambda}$$

The ABCD matrix from lens to focus is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix} = \begin{bmatrix} 1-\frac{d}{f} & d \\ \frac{-1}{f} & 1 \end{bmatrix}$$

So the ABCD law gives

$$q_2 = \frac{\left(1-\frac{d}{f}\right)\left(i\frac{\pi W_i^2}{\lambda}\right) + d}{\left(\frac{-1}{f}\right)\left(i\frac{\pi W_i^2}{\lambda}\right) + 1} = \left[\frac{1}{R_2} - i\frac{\lambda}{\pi W_2^2}\right]^{-1}$$

Equating real & imaginary parts gives,

$$R_2(d) = \frac{\left(\frac{d}{Z_{R_i}}\right)^2 + \left(1-\frac{d}{f}\right)^2}{\left(\frac{d}{Z_{R_i}}\right)^2 - \frac{1}{f}\left(1-\frac{d}{f}\right)} \quad \& \quad W_2^2(d) = W_i^2\left(1-\frac{d}{f}\right)^2 + W_i^2\left(\frac{d}{Z_{R_i}}\right)^2$$

The waist of the focussed beam occurs when $R_2(d) = \infty$

Therefore,

$$\frac{d}{z_{R_i}^2} - \frac{1}{f}\left(1-\frac{d}{f}\right) = 0 \Rightarrow d\left(\frac{1}{z_{R_i}^2} + \frac{1}{f^2}\right) = \frac{1}{f},$$

$$\Rightarrow \boxed{d = \frac{f}{1+\frac{f^2}{z_{R_i}^2}}}$$

When the incident beam is well collimated ie $z_{R_i} \gg f$ then

$d = f$ as in GO

Insert d into expression for $W_2^2(d)$,

$$\boxed{W_f = \frac{\lambda f}{\pi W_i} \frac{1}{\sqrt{1+f^2/z_{R_i}^2}}} \rightarrow \text{Beam waist at focus.}$$

Again when $z_{R_i} \gg f$,

$$\boxed{W_f = \frac{\lambda f}{\pi W_i}}$$

Define beam diameters as $D_i = 2W_i$, $D_f = 2W_f$, then

$$\boxed{D_f = \frac{4}{\pi}\lambda\left(\frac{f}{D_i}\right)}$$

$\frac{f}{D_i}$ is the f-number ie $f'^{\#}$ & $\frac{4}{\pi} \simeq 1.27$

$$\Rightarrow \boxed{D_f = 1.27\lambda f'^{\#}}$$

$f'^{\#} \uparrow \Rightarrow$ fast lens since it focuses at a very close range

The fastest practical lenses have $f'^{\#} \simeq 1 \Rightarrow \boxed{D_{f_{min}} \simeq \lambda} \rightarrow$ Diffraction limit!!

Fraunhofer Diffraction

We start with the Fresnel Diffraction integral.

$$E(x,y,z) = \frac{i}{\lambda} \cdot \frac{e^{-ikz}}{z} \iint_{-\infty}^{\infty} E(x',y';0)\, e^{-ik[(x-x')^2+(y-y')^2]/2z}\, dx'\, dy'$$

Expanding the phase term,

$$k\,\frac{(x-x')^2+(y-y')^2}{2z} = k\left[\frac{x^2+y^2}{2z} - \frac{xx'+yy'}{z} + \underbrace{\frac{x'^2+y'^2}{2z}}_{\text{ignore this}}\right]$$

Fraunhofer ("far field") approximation $\Rightarrow$ $\frac{z}{k} \gg \frac{x'^2+y'^2}{2}$

$\Rightarrow$ Let $W = \max(x', y')$, we need $z \gg \frac{kW^2}{2} = \frac{\pi W^2}{\lambda}$ → looks like Rayleigh range! Hence we need to be many Rayleigh ranges away.

$$\Rightarrow \boxed{E(x',y';z) = \frac{i e^{-ikz}}{\lambda z} \cdot e^{-ik(x^2+y^2)/2z} \iint_A E(x',y';0)\, e^{ik(xx'+yy')/z}\, dx'dy'}$$

→ Fraunhoffer Diffraction integral; $k_x = k\sin\theta_x = k\tan\theta_x = \frac{kx}{z}$

$$\Rightarrow \boxed{E(x',y';z) = \frac{i e^{-ikz}}{\lambda z} \cdot e^{-ik(x^2+y^2)/2z} \iint_A E(x',y';0)\, e^{i(k_x x' + k_y y')}\, dx'dy'}$$

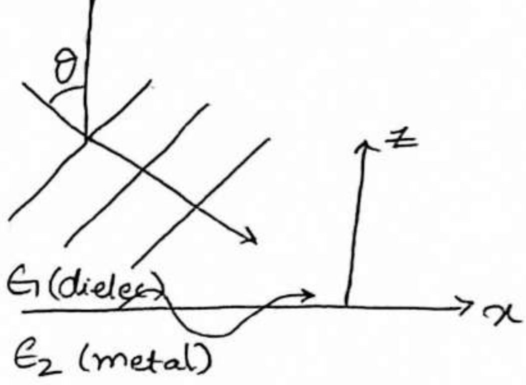
High Q tunable filters @ high frequencies.

(I) Surface Plasmon Polaritons excited by a grating.

SPP basics - Plane wave interaction.

$K_{x_1} = K_{x_2}$

$$\beta = K_{x_2} = \frac{\omega}{c}\sqrt{\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}}$$ → dispersion relation.



$$k_{x_1} = \frac{\omega}{c}\sqrt{\epsilon_1}\ \sin\theta$$

$$\Rightarrow \boxed{\sin\theta = \underbrace{\sqrt{\frac{\epsilon_m}{\epsilon_d+\epsilon_m}}}_{>1 \text{ since } \epsilon_m < 0.}}$$

Also need $\frac{\epsilon_1}{\epsilon_2} = -\frac{\alpha_1}{\alpha_2}$ → damping factor.

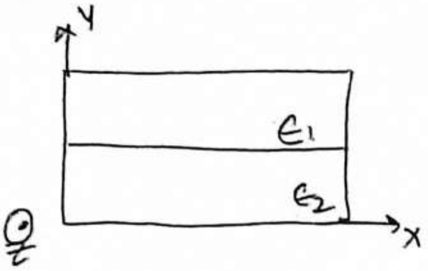
To match the momentum along x use grating!

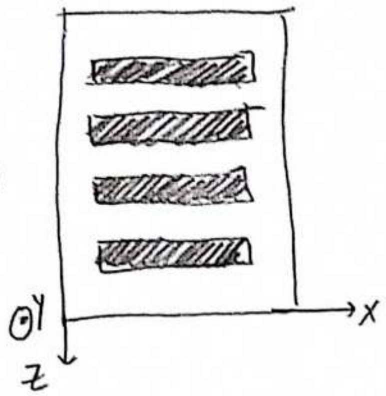
$$\beta = K_{x_1} + \underbrace{\frac{2\pi}{\Lambda}}_{K}$$

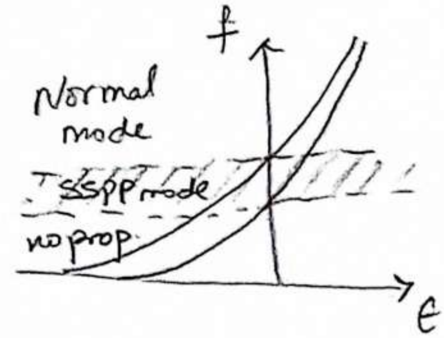
$$\Rightarrow \boxed{\frac{\omega}{c}\sqrt{\epsilon_d}\left[\sqrt{\frac{\epsilon_r(\omega)}{\epsilon_d+\epsilon_r(\omega)}} - \sin\theta\right] = K.}$$

> Design ϵ_d & $\epsilon_r(\omega)$ to get a realisable K value at THz!
> Cannot use a real metal since $\epsilon_r(\omega)$ is purely imaginary at subTHz
> How can a grating be implemented if ϵ_1 & ϵ_2 are 2D lattices
> What would the dispersion relation look like & can plasmons even be excited in a 2D lattice?
> Are the E & H fields in a lattice confined to the lumped elements?

(II) Spoof SPP in SIW for band pass response.







→ plasmon propagation band.

> TE_{10} in 1 & TE_{10} in 2 must be excited
  & ϵ_1 & ϵ_2 must have opposite signs.

> ϵ_1 & ϵ_2 in reality are positive by due to structural dispersion } this occurs.

> Can we achieve such a structural dispersion to make ϵ_1 & ϵ_2 of opposite signs in a 2D lattice? This would probably allow plasmon propagation at the interface.

> PPW : structural dispersion → $\epsilon_{eff} = \epsilon_b - \dfrac{\lambda_0^2}{4a^2}$ —(1)

↳ comes from $\lambda g_{mn} = \left[\sqrt{\frac{1}{\lambda^2} - \left(\frac{m}{2a}\right)^2 - \left(\frac{n}{2b}\right)^2}\right]^{-1}$

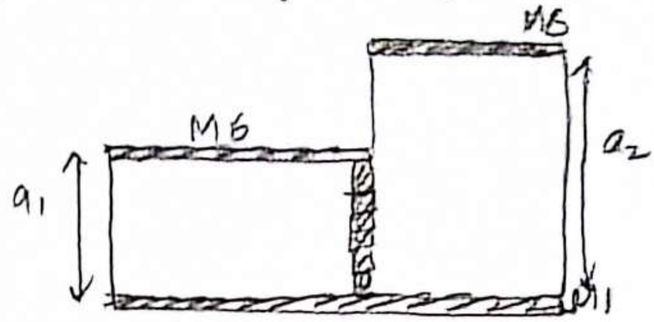
⇒ ϵ_{eff} can be negative!

> Mode coupling should be negligible.
The rods are basically killing the TM_{10} mode since it has an E_x component.

> The waveguide structural dispersion then allows ϵ_1 & ϵ_2 to be of opposite signs as per Eq(1) & a dipole along E_y can be used to excite the SPP.

> 2D lattice to control ϵ_1 & ϵ_2 possible?

* → Combine ideas Ⓘ & Ⓘⅈ using Lattice dispersion to design tunable filters?



$$\epsilon_{eff} = 4 - \frac{\lambda_0^2}{4a^2}$$