MATH 555 - Complex Analysis

Lec 1

Complex Numbers

> A complex number is an "ordered pair" (a,b) of $a,b \in \mathbb{R}$
means $(a_1,b_1) = (a_2,b_2)$ only if $a_1 = a_2$, $b_1 = b_2$

The set of complex number is denoted $\mathbb{C}$.

Say $\alpha = (a,b)$ and $\beta = (c,d)$ are in $\mathbb{C}$

Define binary operations "+" and "·" by

$\alpha + \beta = (a+c,\ b+d)$ $\quad \alpha \cdot \beta = (ac-bd,\ ad+bc)$

Easy to check: provided we assume knowledge of $\mathbb{R}$.

1) $+, \cdot$ are commutative $\Rightarrow \alpha+\beta = \beta+\alpha$; $\alpha\beta = \beta\alpha$

2) $+, \cdot$ are associative
$\Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ & $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

3) Distributive law:
$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$

> Usual convention in $\mathbb{C}$ is to write $\alpha = (a,b)$ as $\alpha = a+ib$
where $i = (0,1)$.

Note, $i^2 = (0,1)\cdot(0,1) = (-1,0)$

<u>Def</u>: If $\alpha = a+ib$, then $\left.\begin{array}{l} a = Re\{\alpha\} \\ b = Im\{\alpha\} \end{array}\right\} \Rightarrow \alpha = \beta \Rightarrow \begin{array}{l} Re\,\alpha = Re\,\beta \\ Im\,\alpha = Im\,\beta \end{array}$

$(a,b) + (c,d)$

$= (a+ib) + (c+id) = (a+ib+c)+id = (a+c+ib)+id = (a+c)+(ib+id)$

$= (a+c) + i(b+d) = (a+c,\ b+d)$

<u>HW exercise</u>: $(a+ib)(c+id)$

Additive identity is $(0,0)$ or 0 since $\alpha+0 = 0+\alpha = \alpha$

Multiplicative identity is $(1,0)$ or 1 since $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$

additive inverse of $\alpha = a+ib$ is $-\alpha$ since $\alpha + (-\alpha) = a+ib-a-ib$

<u>Def:</u> The conjugate of $\alpha = a+ib$ is $\bar{\alpha} = a-ib$.

$$\alpha\bar{\alpha} = a^2+b^2$$

So if $\alpha \neq 0$ then either $a \neq 0$ or $b \neq 0$ so $a^2+b^2 \neq 0$

Hence $\dfrac{1}{a^2+b^2} \in \mathbb{R}$ and from $\alpha\bar{\alpha} = a^2+b^2$ we have,

$$\alpha\left(\frac{\bar{\alpha}}{a^2+b^2}\right) = 1$$

So any $\alpha \in \mathbb{C}$ such that $\alpha \neq 0$ has a multiplicative inverse. which is $\dfrac{\bar{\alpha}}{a^2+b^2}$.

<u>Theorem:</u> The complex numbers with "+" and "·" are a <u>field</u>.

<u>Note:</u> A <u>field</u> is a set F with two binary operations eg. "+" & "·" that satisfy all of the properties described above and such that if $\alpha, \beta \in F$ then $\alpha+\beta \in F$ and $\alpha \cdot \beta \in F$

<u>Facts:</u>

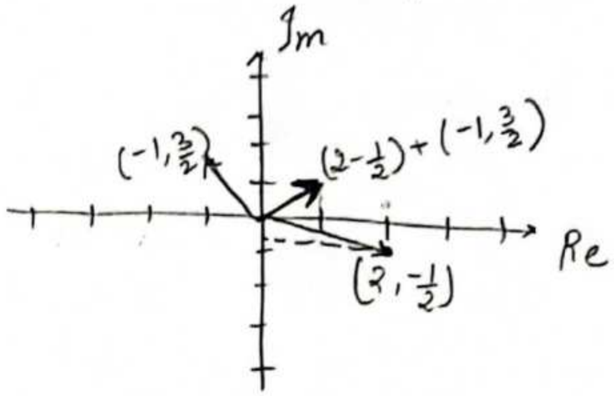
$$\overline{\alpha+\beta} = \bar{\alpha}+\bar{\beta}$$

$$\overline{\alpha\beta} = \bar{\alpha}\,\bar{\beta}$$

$$\overline{\left(\frac{\alpha}{\beta}\right)} = \frac{\bar{\alpha}}{\bar{\beta}}$$

The corresponding $(a,b) \longleftrightarrow a+ib$ indicates that $\mathbb{C}$ can be represented as the "complex plane"

Eg: $2-\frac{1}{2}i$ is



n this sense

$$|2-\tfrac{1}{2}i| = \sqrt{2^2+\left(\tfrac{1}{2}\right)^2} = \frac{\sqrt{17}}{2}$$

Def: The "modulus" or absolute value of $\alpha = a+ib$ is

$$|\alpha| = \sqrt{a^2+b^2} \Rightarrow |\alpha| = \sqrt{\alpha\bar{\alpha}}$$

In polar coordinates, (a,b) is (r,θ) where $r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$a = r\cos\theta;\ b = r\sin$

Analogously, $a+ib = |\alpha|(\cos\theta + i\sin\theta) = |\alpha| e^{i\theta}$

Def: θ is called the argument of $a+ib$, denoted $\theta = \arg\alpha$.

$\alpha = r(\cos\theta + i\sin\theta)$ is a very useful representation.

Let's

$$\left.\begin{aligned} \alpha &= r(\cos\theta + i\sin\theta) \\ \beta &= \rho(\cos\phi + i\sin\phi) \end{aligned}\right\} \alpha\beta = r\rho\left(\cos(\theta+\phi) + i\sin(\theta+\phi)\right)$$

Thm: $|\alpha\beta| = |\alpha||\beta| \qquad \arg(\alpha\cdot\beta) = \arg(\alpha) + \arg(\beta)$

Similarly, $\left|\frac{\alpha}{\beta}\right| = \frac{|\alpha|}{|\beta|}$ and $\arg\left(\frac{\alpha}{\beta}\right) = \arg(\alpha) - \arg(\beta)$

Corollary: De-Moivre's Thm

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

$\star$

$\rightarrow$ What about $\sqrt[n]{r(\cos\theta + i\sin\theta)}$?

Say $\beta = \sqrt[n]{r(\cos\theta + i\sin\theta)} = \rho(\cos\phi + i\sin\phi)$

$\Rightarrow \beta^n = \rho^n(\cos n\phi + i\sin n\phi)$

$\Rightarrow \rho^n = r$ & $\cos(n\phi) = \cos\theta$ & $\sin n\phi = \sin\theta$

$\Rightarrow \rho = r^{\frac{1}{n}} \in [0, \infty)$ and $n\phi = \theta + 2\pi k$.

$\Rightarrow \phi = \dfrac{\theta + 2\pi k}{n}$ for some $k \in \mathbb{Z}$

$\Rightarrow \beta = r^{\frac{1}{n}}\left(\cos\left(\dfrac{\theta + 2\pi k}{n}\right) + i\sin\left(\dfrac{\theta + 2\pi k}{n}\right)\right)$ for $k \in \mathbb{Z}$

But $\cos\left(\dfrac{\theta + 2\pi k}{n}\right) = \cos\left(\dfrac{\theta + 2\pi j}{n}\right)$ iff $k - j$ is a multiple of n.

$\Rightarrow$ <u>There are n possible solutions</u> for $\beta = r^{\frac{1}{n}}\left(\cos\left(\dfrac{\theta + 2\pi k}{n}\right) + i\sin\left(\dfrac{\theta + 2\pi k}{n}\right)\right)$

for $k = 0, 1, 2, \ldots, n-2, n-1$.

Once k reaches n it is the same as $k = 0$. $k = n+1$ is same as $k = 1$ & so on.

Triangle Inequality.

$|\alpha+\beta| \leq |\alpha| + |\beta|$

Proof: Say $\alpha = a+ib$, $\beta = c+id$. $\Rightarrow \alpha+\beta = (a+c)+i(b+d)$

If $x, y \in \mathbb{R} \Rightarrow$ Youngs inequality $2xy \leq x^2+y^2$ Proof $0 \leq (x-y)^2$
$\Rightarrow 0 \leq x^2+y^2-2xy.$

So $2acbd \leq a^2d^2+b^2c^2$

$\Rightarrow a^2c^2 + 2acbd + b^2d^2 \leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$

$\Rightarrow (ac+bd)^2 \leq (a^2+b^2)(c^2+d^2)$

If $ac+bd \leq 0$ then $ac+bd \leq \underbrace{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}_{\because \text{ non negative}}$

If $ac+bd > 0$ then also $ac+bd \leq \sqrt{a^2+b^2}\sqrt{c^2+d^2}$ } check claim below

$\Rightarrow 2ac+2bd \leq 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}$

$\Rightarrow a^2+2ac+c^2+b^2+2bd+d^2 \leq a^2+b^2+2\sqrt{a^2+b^2}\sqrt{c^2+d^2}+c^2+d^2$

$\Rightarrow (a+c)^2+(b+d)^2 \leq \left(\sqrt{a^2+b^2}+\sqrt{c^2+d^2}\right)^2$

$\Rightarrow \sqrt{(a+c)^2+(b+d)^2} \leq \sqrt{a^2+b^2}+\sqrt{c^2+d^2}$

$= |\alpha+\beta| \leq |\alpha|+|\beta|$

Claim: $x^2 \leq y^2$ & $x, y \geq 0 \Rightarrow x \leq y$

Proof: In $\mathbb{R}$: If $a \leq b$ and $c \geq 0$ then $ac \leq bc$ → axiom.

Suppose $x > y \Rightarrow x^2 > xy$ & $xy > y^2 \Rightarrow x^2 > xy > y^2 \Rightarrow x^2 > y^2$ → contradiction

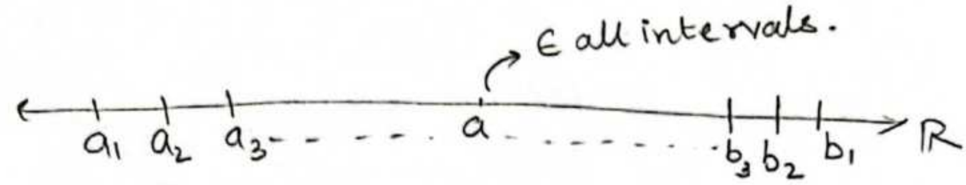
Lec2 Limits : Giving a Topology to $\mathbb{C}$

Principles of Nested Intervals:

Let $I_n = [a_n, b_n] \subseteq \mathbb{R}$ satisfy : 1) $I_{n+1} \subseteq I_n \;\forall n \in \mathbb{N}$ ("nested")

2) $\lim_{n\to\infty} b_n - a_n = 0$

Then $\exists$ one and only one point $a \in \mathbb{R}$ s.t $a \in I_n \;\forall n \in \mathbb{N}$



note: $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a \leq \cdots \leq b_3 \leq b_2 \leq b_1$

$\Rightarrow b_k > a_j \;\forall j,k.$

Note: The above principle relies on the axiomatic defn. of $\mathbb{R}$.

$\mathbb{R}$ is a complete ordered field.

$\mathbb{R}$ always has a least upper bound if an upper bound exists

Suppose $S \subseteq \mathbb{R}$ and $\exists\; L \in \mathbb{R}$ s.t $s \leq L \;\forall s \in S$ (upper bound)

Then $\exists\; \mu \in \mathbb{R}$ s.t $s \leq \mu \;\forall s \in S$ and

$\mu \leq L \;\forall L$ upper bounds of S

μ is called the least upper bound, $\mu = \sup S$

eg: if $S = [-5, 2)$ then $\sup S = 2$

Proof :- Say $k \in \mathbb{N}$ fixed since $[a_j, b_j] \subseteq [a_k, b_k]$ $\forall j \geq k$
we know that $a_k \leq a_j \leq b_j \leq b_k$ $\forall j \geq k$

In particular $a_k \leq b_j$ $\forall j \geq k$.

If $j < k$ then $[a_k, b_k] \subseteq [a_j, b_j]$ so
similarly $a_j \leq a_k \leq b_k \leq b_j$ and so $a_k \leq b_j$ for $j < k$

Hence $\forall j \in \mathbb{N}$, $a_k \leq b_j$

$\Rightarrow \forall j, k \in \mathbb{N}$ $a_k \leq b_j$

So if $S = \{a_n \mid n \in \mathbb{N}\}$ then b_j is an upper bound of S $\forall j \in \mathbb{N}$

So if $a = \sup S$ (which exists since S is bounded and $\mathbb{R}$ has completeness axiom) then

$a_j \leq a \leq b_j$ $\forall j \in \mathbb{N}$ So $\underbrace{a \in I_j \ \forall j \in \mathbb{N}}$

Proved there exists a point in all I_n intervals. Now why is it unique?

Suppose there was another point $\tilde{a}$ s.t $\tilde{a} \in I_n$ $\forall n \in \mathbb{N}$ and $\tilde{a} \neq a$ (Let's assume $\tilde{a} < a$)

Then $a_n \leq \tilde{a} < a \leq b_n$ $\forall n \in \mathbb{N}$

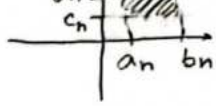
Hence $b_n - a_n \geq a - \tilde{a} > 0$

$$\Rightarrow \underbrace{\lim_{n\to\infty} b_n - a_n}_{a - \tilde{a} \leq 0} \geq \lim_{n\to\infty} a - \tilde{a} = a - \tilde{a} > 0$$

contradiction $\Rightarrow$ $\tilde{a}$ doesn't exist.

Corollary Principles of Nested Rectangles.

Say $R_n = \{ x + iy \mid x \in [a_n, b_n] \text{ and } y \in [c_n, d_n] \} \in \mathbb{C}$



(1) $R_{n+1} \subseteq R_n \ \forall\ n \in \mathbb{N}$

(2) $\lim\limits_{n\to\infty} \sqrt{(b_n - a_n)^2 + (d_n - c_n)^2} = 0 \Rightarrow \text{diag} \to 0$

Then $\exists$ exactly one $\alpha \in \mathbb{C}$ s.t. $\alpha \in R_n \ \forall\ n \in \mathbb{N}$

Proof: Claim :- $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ & $[c_{n+1}, d_{n+1}] \subseteq [c_n, d_n] \ \forall\ n \in \mathbb{N}$

Proof:- Suppose $\exists\ x_0 \in [a_{n+1}, b_{n+1}]$ s.t. $x_0 \notin [a_n, b_n]$

or $\exists\ y_0 \in [c_{n+1}, d_{n+1}]$ s.t. $y_0 \notin [c_n, d_n]$

Then $x_0 + iy_0 \in R_{n+1}$ but $x_0 + iy_0 \notin R_n$

contradiction of (1)

Since $\lim\limits_{n\to\infty} b_n - a_n = 0$ and $\lim\limits_{n\to\infty} d_n - c_n = 0$.

$\exists$ exactly one $a \in \mathbb{R}$ and one $c \in \mathbb{R}$ s.t

$a \in [a_n, b_n]$ & $c \in [c_n, d_n] \ \forall\ n \in \mathbb{N}$

$\Rightarrow \alpha = a + ic \in R_n \ \forall\ n \in \mathbb{N}$. Try to prove uniqueness as an exercise.

Proof by contradiction for A & B should start with not A or not B is true.

Lec 3: Limits in $\mathbb{C}$

Def: A sequence $\{z_n\}_{n\in\mathbb{N}} \subseteq \mathbb{C}$ of terms $z_1, z_2, z_3 \ldots$ Converges to $\alpha \in \mathbb{C}$ if for each $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t $|z_n - \alpha| < \epsilon$ For every $n \geq N$

Ex: Say $z_n = \beta^n$ for some $\beta \in \mathbb{C}$ s.t $|\beta| < 1$. Then z_n converges to 0.

→ For convergence proofs start with ϵ & then find N.

Claim: $r \in \mathbb{R}$, $0 \leq r < 1$. Say $\epsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t

$r^n < \epsilon \ \forall\, n \geq N$

Proof:- Suppose $\epsilon > 0$, then since $|\beta| < 1$, $\exists\, N \in \mathbb{N}$ s-t

$|\beta|^n < \epsilon$ for all $n \geq N$

So $|z_n - 0| = |\beta^n| = |\beta|^n < \epsilon$ for $n \geq N$

Def:- α is a limit point of $\{z_n\}_{n\in\mathbb{N}}$ if for $\epsilon > 0$, $\exists$ infinitely many $n \in \mathbb{N}$ s.t $|z_n - \alpha| < \epsilon$

Ex: $z_n = i^n$. Then z_1, z_2, z_3 is $\underbrace{i, -1, -i, 1}, i, -1, -i, 1 \ldots$

All four values are limit points.

<u>Note</u>: If $z_n \to \alpha$ (converges to α) then α is a limit point.

<u>Def</u>:- $\{z_n\}$ is <u>bounded</u> if $\exists\ M \in \mathbb{R}, M > 0$ s.t $|z_n| < M$ $\forall n \in \mathbb{N}$.

<u>Theorem</u>: <u>Bolzano Weierstrauss</u>.

Every bounded sequence $\{z_n\}$ has at least one limit point $\alpha \in \mathbb{C}$

<u>Proof</u>: $\{z_n\}$ bounded $\Rightarrow \exists\ M > 0$ S.T $|z_n| < M\ \forall n \in \mathbb{N}$

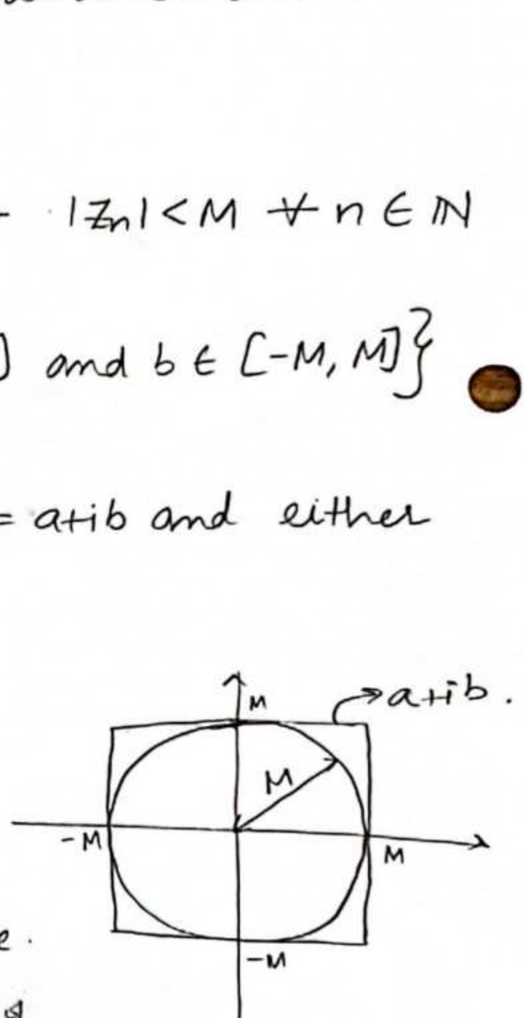
So, $\{z_n\} \subseteq \{a+ib \mid a \in [-M, M] \text{ and } b \in [-M, M]\}$

[Why? If not, then $\exists\ n \in \mathbb{N}$ S.T $z_n = a+ib$ and either

$|a| > M$ or $|b| > M$.

Then $a^2 > M^2$ or $b^2 > M^2 \Rightarrow a^2+b^2 > M^2$

$\Rightarrow |z| > M$. Condradiction]

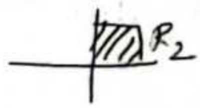


> Infinitely many z_n must be in one of the 4 squares of side length M. $\Rightarrow$ Atleast one of these 4 squares must contain an infinite sequence of z_ns.

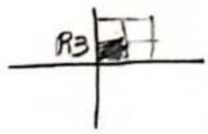
> WLOG Say it is in top right quadrant. Again divide that into 4 pieces of side $\frac{M}{2}$. $\Rightarrow$ Continue this process.

$$R_1 = \underbrace{[-M, M]}_{Re.} \times \underbrace{[-M, M]}_{Im} \; ; \; diag(R_1) = \sqrt{(2M)^2 + (2M)^2}$$
$$= 2\sqrt{2}\, M.$$

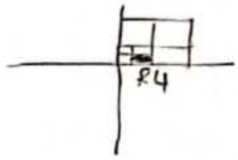
Say, $R_2 = [0, M] \times [0; M]$; $diag(R_2) = \sqrt{2}\, M$



Say, $R_3 = \left[0, \frac{M}{2}\right] \times \left[0, \frac{M}{2}\right]$; $diag(R_3) = \frac{\sqrt{2}\, M}{2}$



Similarly, Say $R_4 = \left[\frac{M}{4}, \frac{M}{2}\right] \times \left[0, \frac{M}{4}\right]$; $diag(R_4) = \frac{\sqrt{2}\, M}{4}$



→ Note that $R_1, R_2, R_3 \ldots$ are nested rectangles since,

i) $R_{i+1} \subseteq R_i \;\; \forall\, i \in \mathbb{N}$

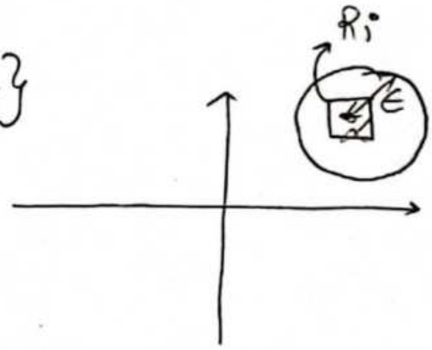
ii) $\lim_{i \to \infty} diag\, R_i = 0$ $\left[diag\, R_i = \frac{\sqrt{2}\, M}{2^{i-2}} \right]$

Therefore $\exists$ exactly one $\alpha \in \mathbb{C}$ s.t $\alpha \in R_i \;\; \forall\, i \in \mathbb{N}$.

Now, suppose $\epsilon > 0$. Then, $\exists\, R_i$ s.t

$$R_i \subseteq B_\epsilon(\alpha) = \{ z \in \mathbb{C} \mid |z - \alpha| < \epsilon \}$$

↳ Ball of radius ϵ around α.



Pick i s.t $\frac{\sqrt{2}\, M}{2^{i-2}} < \frac{\epsilon}{2}$

⇒ z must lie inside circle.

⇒ Infinitely many z_n inside R_i ⇒ Infinitely many z_n inside $B_\epsilon(\alpha)$ which is the defn. of a limit point.

Ex: $\{ z_n = 1 + (i)^n \}$ is bounded since $|z_n| \le 2$ ⇒ $\{z_n\}$ has a limit pt.

Theorem: Cauchy Convergence Criteria.

$\{Z_n\}$ converges $\overset{\text{iff}}{\Longleftrightarrow}$ for each $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t

$|Z_n - Z_m| < \epsilon \quad \forall\, n, m \geq N$ [Any 2 numbers chosen greater than N must give Z_n, Z_m such that they are arbitrarily close.]

Need to prove i) convergent sequence $\Rightarrow$ Cauchy sequence

& ii) Cauchy sequence $\Rightarrow$ convergent sequence

Proof: $\Rightarrow$: Say $Z_n \to \alpha$ as $n \to \infty$

Given $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t

$$|Z_n - \alpha| < \frac{\epsilon}{2} \quad \forall\, n \geq N$$

So if $m \geq N$ then also $|Z_m - \alpha| < \frac{\epsilon}{2}$

$$\Rightarrow |Z_n - Z_m| = |Z_n - \alpha + \alpha - Z_m|$$

$$\leq |Z_n - \alpha| + |\alpha - Z_m|$$

$$= |Z_n - \alpha| + |Z_m - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$$\Rightarrow |Z_n - Z_m| < \epsilon$$

$\Leftarrow$: First we show $\{Z_n\}$ is bounded:

Claim: $\exists\, M > 0$ s.t $|Z_n| \leq M \quad \forall\, n \in \mathbb{N}$

Proof: Let $\epsilon = 1$ Then $\exists\, N_1 \in \mathbb{N}$ s.t

$$|Z_n - Z_m| < 1 \quad \forall\, n, m \geq N_1$$

Let $m = N_1$ so $|Z_n - Z_{N_1}| < 1 \quad \forall\, n \geq N_1$

↳ Looks bounded.

Hence $|z_n| = |z_n - z_{N_1} + z_{N_1}|$

$$\leq |z_n - z_{N_1}| + |z_{N_1}|$$

$$< 1 + |z_{N_1}| \quad \forall n \geq N_1$$ This takes care of all ∞ points $\geq N_1$.

Let $M = \max\{|z_1|, |z_2|, \ldots, |z_{N_1 - 1}|, |z_{N_1}| + 1\}$

Then $|z_n| \leq M \quad \forall n \in \mathbb{N} \Rightarrow$ Bounded! $\Rightarrow$ limit point exists.

> If the max term is $|z_3|$ then $|z_n| \leq M$ since $M = |z_3|$

This takes care of all finite points till N_1

<u>Now we apply Bolzano-Weierstrass</u>

$\exists\ \alpha \in \mathbb{C}$ limit point of $\{z_n\}$

Given $\epsilon > 0$, $\exists\ N > 0$ s.t $|z_n - z_m| < \frac{\epsilon}{2} \quad \forall n, m \geq N$

Also since α is a limit point, $\exists$ infinitely many z_n in

$B_{\epsilon/2}(\alpha) = \{z \in \mathbb{C} \mid |z - \alpha| < \frac{\epsilon}{2}\}$

So $\exists\ \tilde{N} \geq N \quad |z_{\tilde{N}} - \alpha| < \frac{\epsilon}{2}$

Such an $\tilde{N}$ must exist since any neighbourhood of α must contain ∞ no. of points.

Hence for $n \geq N$, $|z_n - \alpha| = |z_n - z_{\tilde{N}} + z_{\tilde{N}} - \alpha|$

$$\leq |z_n - z_{\tilde{N}}| + |z_{\tilde{N}} - \alpha|$$

$$\underset{\text{Cauchy}}{<} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

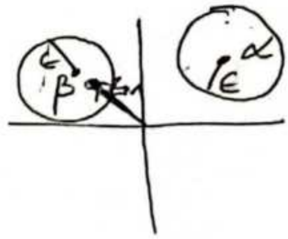
$$\Rightarrow |z_n - \alpha| < \epsilon \quad \forall n \geq N.$$

→ Cauchy sequences converge $\Rightarrow$ can only have one limit point.

Claim: If $\lim_{n\to\infty} z_n = \alpha$ then α is the only limit point of $\{z_n\}$

Proof:- Suppose $\exists\ \beta \neq \alpha$ and β is a limit point of $\{z_n\}$

$\forall\ \epsilon > 0,\ \exists\ \infty$ many z_n s.t $|z_n - \beta| < \epsilon$



Let $\epsilon = \dfrac{|\alpha - \beta|}{4}$

$\exists\ N$ s.t $|z_n - \alpha| < \epsilon \quad \forall\ n \geq N$

Since β is a limit point, $\exists\ z_m$ s.t

$|z_m - \beta| < \epsilon$ for infinitely many m. so pick an $m \geq N$. Such an m must always exist.

$|z_m - \alpha|$ can be shown to be $> \epsilon$ which is a condradiction.

$$|z_m - \alpha| + |\beta - z_m| \geq |z_m - \alpha + \beta - z_m|$$

$\Rightarrow \quad |z_m - \alpha| + |\beta - z_m| \geq |\beta - \alpha|$, where we have chosen $|\beta - \alpha|$ s.t $|\beta - \alpha| \geq 2\epsilon$

$\Rightarrow$ circles do not overlap.

$$\Rightarrow \quad |z_m - \alpha| \geq \underbrace{|\beta - \alpha|}_{\geq 2\epsilon} - \underbrace{|\beta - z_m|}_{< \epsilon}$$

$\Rightarrow \quad |z_m - \alpha| > \epsilon \longrightarrow$ condradiction!

$\Rightarrow\ \beta$ is not a limit pt!

Ch 2 #13

Claim: If $z_n \to \alpha$ then $\frac{z_1 + \cdots + z_n}{n} \to \alpha$ as $n \to \infty$

Proof: Suppose $\epsilon > 0$. Then $\exists\ \tilde{N} \in \mathbb{N}$ s.t

$$|z_n - \alpha| < \frac{\epsilon}{2} \quad \forall\ n > \tilde{N}$$

Consider $b = |z_1 - \alpha| + |z_2 - \alpha| + \cdots + |z_{\tilde{N}} - \alpha|$

Then $\exists\ N > \tilde{N}$ s.t $\frac{b}{N} < \frac{\epsilon}{2}$ $\to$ Since b is fixed, pick a big enough no. N s.t $\frac{b}{N} < \frac{\epsilon}{2}$

Now if $n \geq N$ then

$$\left| \frac{z_1 + \cdots + z_n}{n} - \alpha \right| = \frac{1}{n} |z_1 + \cdots + z_n - n\alpha|$$

$$= \frac{1}{n} |z_1 - \alpha + \cdots + z_n - \alpha|$$

$$\leq \frac{1}{n} (|z_1 - \alpha| + \cdots + |z_n - \alpha|)$$

$$< \frac{1}{n} \left(b + (n - \tilde{N}) \frac{\epsilon}{2}\right)$$ All terms after $\tilde{N}$ upto n are less than $\epsilon/2$.

$$= \frac{b}{n} + \left(\frac{n - \tilde{N}}{n}\right)\left(\frac{\epsilon}{2}\right)$$

$$< \frac{b}{N} + \frac{n}{n}\left(\frac{\epsilon}{2}\right) \quad (\because n > N)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\blacksquare$

Ch3 Complex Functions.

Def: A (single valued) function $f: \mathbb{C} \to \mathbb{C}$ assigns to each $z \in \mathbb{C}$ exactly one output $f(z) \in \mathbb{C}$

Def: $f: \mathbb{C} \to \mathbb{C}$ is one-to-one or injective if $f(z_1) = f(z_2)$ if and only if $z_1 = z_2$.

Ex: 1) $f(z) = \bar{z}$ is one to one

If $f(a+bi) = f(c+di)$ then $a-bi = c-di$

So $a=c$, $-b=-d \Rightarrow b=d \Rightarrow a+bi = c+di$

2) $f(z) = |z|$ is not injective

$f(-1) = 1 = f(1)$ but $1 \neq -1$

Def: $f: \mathbb{C} \to \mathbb{C}$ is onto or surjective if

$$\mathbb{C} = \{ f(z) \mid z \in \mathbb{C} \} \quad \text{ie. } f(\mathbb{C}) = \mathbb{C}$$

Ex: 1) $f(z) = z^2$ is surjective. (not true for real numbers)

If $w \in \mathbb{C}$ then $\exists\ r \geq 0$, $\theta \in [0, 2\pi)$ s.t $w = re^{i\theta}$

Then $f(z) = w$

$$z = \sqrt{r}\left(\cos\left(\tfrac{\theta}{2}\right) + i\sin\left(\tfrac{\theta}{2}\right)\right)$$

$$z^2 = w$$

2) $f(z) = \frac{1}{z}$ is not onto since there is not $z \in \mathbb{C}$ s.t $f(z) = 0$

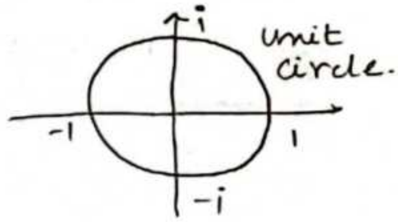
Def :- A <u>curve</u> in $\mathbb{C}$ is a one parameter set of points

$\{x(t)+iy(t) \mid a \le t \le b\}$.

Such that $x(t)$ and $y(t)$ are continuous functions of $t \in [a,b]$

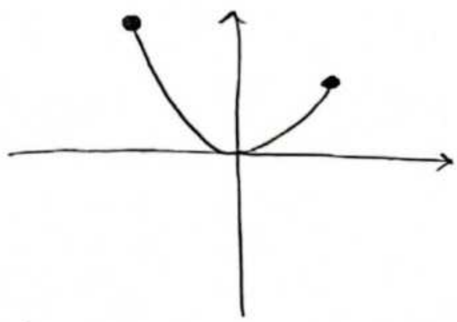
A curve is closed if $x(a)+iy(a) = x(b)+iy(b)$ & $a \ne b$.

Ex: 1) $\{\underbrace{\cos(2\pi t)}_{x(t)} + i\underbrace{\sin(2\pi t)}_{y(t).} \mid 0 \le t \le 1\}$ is a closed curve.



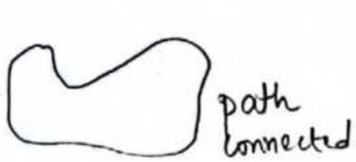
2) $\{t+iy(t) \mid -3 \le t \le 2\}$ is a graph of the function $y(t)$ on $[-3,2]$

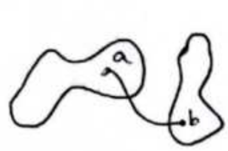
eg. if $y(t) = t^2$



Def: A set $E \subseteq \mathbb{C}$ is <u>path connected</u> if for each pair of points $z_1, z_2 \in E$, $\exists$ a curve $z(t) = \{x(t)+iy(t) \mid a \le t \le b\}$ s.t. $z(a) = z_1$, $z(b) = z_2$ and $z(t) \in E$ $\forall$ $t \in [a,b]$

Ex:





Def: $z_0 \in E$ is an <u>interior point</u> if $\exists\ \delta > 0$ s.t

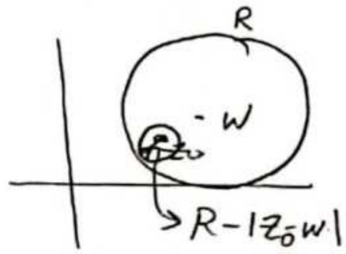
$$B_\delta(z_0) = \{ z \in \mathbb{C} \mid |z - z_0| < \delta \} \subseteq E$$

E is open if every $z \in E$ is an interior point.

Ex: 1) $B_R(w) = \{ z \in \mathbb{C} \mid |z - w| < R \}$ is open $\forall\, w, R$.

Why? Say $z_0 \in B_R(w)$ Then $|z_0 - w| < R$.

Let $\delta = R - |z_0 - w| > 0$



If $z \in B_\delta(z_0)$ then

Choose z in $B_\delta(z_0)$ and show it lies in $B_R(w)$

$$|z - w| = |z - z_0 + z_0 - w| \leq \underbrace{|z - z_0|}_{<\delta} + |z_0 - w| < \delta + |z_0 - w| = R.$$

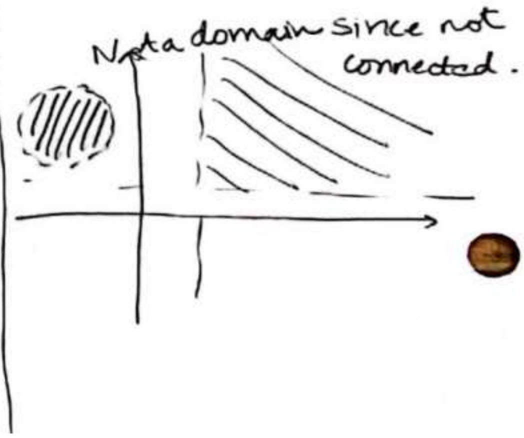
$\Rightarrow z \in B_R(w) \Rightarrow B_\delta(z_0) \subseteq B_R(w)$

2) $H = \{ x + iy \mid x > a, y > b \}$ is open and unbounded.

Def: $G \subseteq \mathbb{C}$ is a <u>domain</u> if it is <u>open</u> and <u>connected</u>.

Note: This is different from usual notion of domain
It is defined this way for convenience.

Both $B_R(w)$ and H above are domains.



Def: $G^c = \{z \in \mathbb{C} \mid z \notin G\}$

For a domain G, the set G^c has 2 types of points.

1) exterior point: $z_0 \in G^c$ and $\exists\ \delta > 0$ s.t $B_\delta(z_0) \subseteq G^c$

2) boundary point: $z_0 \in G^c$ and $\forall\ \delta > 0$, $B_\delta(z_0) \cap G \neq 0$ and $B_\delta(z_0) \cap G^c \neq 0$

Ex: $B_R(w)^c = \{z \in \mathbb{C} \mid |z - w| \geq R\}$

boundary is $\{z \in \mathbb{C} \mid |z - w| = R\}$

Def: $f : G \to \mathbb{C}$ domain G. We say $\lim_{z \to z_0} f(z) = c$ if for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t $|f(z) - c| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and $B_\delta(z_0) \subseteq G$. Here δ is a fn of both ϵ & z_0 *

** If $\lim_{z \to z_0} f(z) = f(z_0)$, then f is continuous at z_0.

If $f(z)$ is continuous at every z_0, then f is continuous on G.

Ex: $f(z) = \bar{z}$ is continuous on $\mathbb{C}$

Proof: Say $z_0 \in \mathbb{C}$ and $\epsilon > 0$. Let $\delta = \epsilon$. Then if $|z - z_0| < \delta$,

$$|f(z) - f(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon$$

Ex: $f(z) = \frac{1}{z}$ is continuous at every $z \neq 0$.

Proof: Say $z_0 \neq 0$ and $\epsilon > 0$.

$$|f(z) - f(z_0)| = \left|\frac{1}{z} - \frac{1}{z_0}\right| = \left|\frac{z_0 - z}{z z_0}\right| = \frac{1}{|z||z_0|}|z_0 - z|$$

$z \neq 0$

Say $|z - z_0| < \delta$ for $\delta > 0$ to be determined.

Then $|z_0| = |z_0 + z - z| \leq |z_0 - z| + |z| < \delta + |z|$

$\Rightarrow |z| > |z_0| - \delta \Rightarrow \frac{1}{|z|} < \frac{1}{|z_0| - \delta} \Rightarrow \frac{1}{|z|} < \frac{2}{|z_0|}$

(Since $z \neq 0$, we want $\delta < |z_0|$. Pick $\delta < \frac{|z_0|}{2}$)

$$|f(z) - f(z_0)| = \frac{1}{|z||z_0|}|z - z_0| < \frac{2\delta}{|z_0|^2}$$

Let $\delta < \min\left\{\frac{|z_0|}{2}, \frac{\epsilon |z_0|^2}{2}\right\}$ since z_0 is fixed δ is well defined.

$\Rightarrow |f(z) - f(z_0)| < \epsilon$

Note:- If $f : G \to \mathbb{C}$ is a domain, then $\forall z_0 \in G$, $\exists r > 0$ s.t

$B_\delta(z_0) \subseteq G \ \forall \delta \leq r$.

Def: $\overline{G} = G \cup \partial G$ where ∂G are the boundary points on G.
Closure of G.

→ What if: $f: \tilde{G} \to \mathbb{C}$ for $G \subsetneq \tilde{G} \subseteq \bar{G}$?

Then for $z_0 \in \tilde{G} \setminus G$ ie($z \in \tilde{G}$ but $z_0 \notin G$)

How do we define continuity?

For each $\epsilon > 0$ $\exists\, \delta > 0$ s.t if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \epsilon$
and $z \in \tilde{G}$

And we can say $\lim_{\substack{z \to z_0 \\ z \in \tilde{G}}} f(z) = f(z_0)$

Real no. analogy

$g: [a,b) \to \mathbb{R}$ continuous at $x = a$?

$\Rightarrow \lim_{x \to a^+} g(x) = g(a)$

$\Rightarrow \left.\begin{array}{l} x \to a \\ x \geq a \end{array}\right\} \Longleftrightarrow \begin{array}{l} |x - a| < \delta \\ x \in [a, b) \end{array}$

→ A stronger condition is (for G domain and $G \subseteq \tilde{G} \subseteq \bar{G}$)

(contain some boundary. maybe all or none too)

Def: $f: \tilde{G} \to \mathbb{C}$ is uniformly continuous on $\tilde{G}$ if

for each $\epsilon > 0$, $\exists\ \delta > 0$ s.t

if $|z_1 - z_2| < \delta$ for $z_1, z_2 \in \tilde{G}$ then $|f(z_1) - f(z_2)| < \epsilon$

→ Uniformly continuous is for a domain whereas continuity is for points.

(A set E is closed if it also contain all its limit points)

<u>Def</u>: A set E is <u>closed</u> if and only if it contains all its Boundary points..

<u>Ex</u>: $\overline{B_R}(w) = \{z \in \mathbb{C} \mid |z-w| \leq R\}$
$\hookrightarrow$ Boundary points are $\{z \in \mathbb{C} \mid |z-w| = R\}$

<u>Def</u>: $E \subseteq \mathbb{C}$ is <u>bounded</u> if $\exists\, M > 0$ s.t $|z| < M \;\forall\, z \in E$.

<u>Thm*</u>: If $f(z)$ is continuous on a closed path connected and bounded set E, then $f(z)$ is uniformly continuous on E.

Big statement!

Proof of this theorem relies on

<u>Thm (Heine-Borel)</u>

$(\delta, z) \in S \times E$
$\Rightarrow \delta \in S$ & $z \in E$

set of positive real nos.

If $G \subseteq \mathbb{C}$ is a closed and bounded set and $G \subseteq \bigcup_{(\delta, z) \in S \times E} B_\delta(z)$ for a subset $S \subseteq (0, \infty)$ and $E \subseteq G$ then $\exists$ finite subset $\{(\delta_1, z_1), \dots, (\delta_n, z_n)\} \subseteq S \times E$ such that $G \subseteq B_{\delta_1}(z_1) \cup B_{\delta_2}(z_2) \cup \dots \cup B_{\delta_n}(z_n)$

<u>Proof of Thm*</u>:

Suppose $\epsilon > 0$ For each $w \in E$, $\exists\, \delta_w > 0$ s.t if $|z-w| < \delta_w$ and $z \in E$ then $|f(z) - f(w)| < \frac{\epsilon}{2}$ (by continuity of f in E)

Recall $B_{\frac{\delta_w}{2}}(w) = \{z \in \mathbb{C} \mid |z-w| < \frac{\delta_w}{2}\}$

Now, $E \subseteq \bigcup_{w \in E} B_{\frac{\delta_w}{2}}(w)$ since if $w \in E$ then $w \in B_{\frac{\delta_w}{2}}(w)$ and so

$w \in \bigcup_{w \in E} B_{\frac{\delta_w}{2}}(w)$

By Heine Borel, $\exists$ finitely many $w_1, \dots, w_n$ s.t
$E \subseteq B_{\delta_1}(w_1) \cup \dots \cup B_{\delta_n}(w_n)$ where $\delta_i = \delta_{w_i}/2$

Let $\delta = \min\{\delta_1, \dots, \delta_n\}$ Say $z_1, z_2 \in E$ s.t $\longrightarrow$ Key step key step!

$|z_1 - z_2| < \delta$. Since $z_1 \in E \subseteq B_{\delta_1}(w_1) \cup \ldots \cup B_{\delta_n}(w_n)$, $\delta_n = \frac{\delta_{w_n}}{2}$

$\exists\, k \in \{1, \ldots, n\}$ s.t $z_1 \in B_{\delta_k}(w_k)$, ie. $|z_1 - w_k| < \frac{\delta_{w_k}}{2} < \delta_{w_k}$ ①

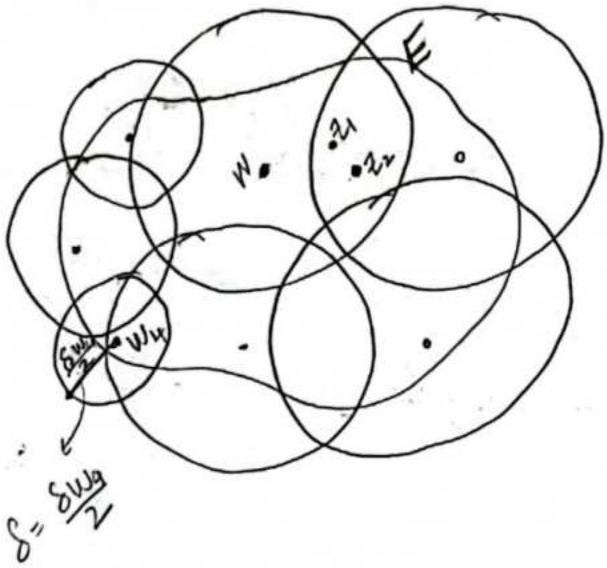
Hence, $|z_2 - w_k| = |z_2 - z_1 + z_1 - w_k| \leq |z_2 - z_1| + |z_1 - w_k|$

$$< \delta + \frac{\delta_{w_k}}{2} \leq \frac{\delta_{w_k}}{2} + \frac{\delta_{w_k}}{2} = \delta_{w_k} \quad ②$$

Thus, $|f(z_1) - f(z_2)| \leq |f(z_1) - f(w_k)| + |f(z_2) - f(w_k)|$

from ① & ②

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$



By picking $|z_1 - z_2|$ to be smaller than the smallest ball we force z_2 to lie in a ball that has a radius of the ball around z_1 + the smallest ball. In the worst case $\Rightarrow z_1$ is on the edge of the smallest ball, z_2 would lie with a ball twice its size. Therefore, both z_1 & z_2 lie in this bigger ball & therefore $|f(z_1) - f(z_2)|$ is bounded!

EX: Say $f: \mathbb{C} \to \mathbb{C}$ is defined by

$$f(r(\cos\theta + i\sin\theta)) = r\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) \text{ for } r \geq 0, \theta \in [-\pi, \pi]$$

Show f is not continuous at some $z \in \mathbb{C}$

Proof: Notice that the problem region is when $\theta \simeq -\pi$ or π

since $\cos(-\pi) + i\sin(-\pi) = -1 = \cos(\pi) + i\sin(\pi)$

but, $\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}) = -i$

and $\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$

(Since $\forall \epsilon > 0 \ \exists \delta$ we choose $\epsilon = 1/2$)

Suppose $\epsilon = \frac{1}{2}$. If f is continuous at $z_0 = -1$

then $\exists \ \delta > 0$ s.t if $|z - (-1)| = |z+1| < \delta$

then $|f(z) - f(-1)| = |f(z) - i| < \frac{1}{2}$

Consider $z_n = \cos(-\pi + \frac{1}{n}) + i\sin(-\pi + \frac{1}{n})$

So $|z_n + 1| = |1 + \cos(-\pi + \frac{1}{n}) + i\sin(-\pi + \frac{1}{n})|$

$\sin(-\pi + \frac{1}{n}) = \sin(-\pi)\cos(\frac{1}{n}) + \sin(\frac{1}{n})\cos(-\pi) = -\sin(\frac{1}{n})$

$1 + \cos(-\pi + \frac{1}{n}) = 1 + \cos(\frac{1}{n})\cos(\pi) + \sin(\frac{1}{n})\sin(\pi) = 1 - \cos(\frac{1}{n})$

From Calculus, as $x \to 0$, $\sin(x) \to 0$ and $1 - \cos(x) \to 0$

Hence, $\exists N$ large enough s.t $|\sin(\frac{1}{n})| < \frac{\delta}{2} \ \forall n \geq N$

and $|1 - \cos(\frac{1}{n})| < \frac{\delta}{2}$. Hence $|z_n + 1| < \frac{\delta}{\sqrt{2}} < \delta$. $\forall n \geq N$

Continuity implies

$$|f(z_n)+i| < \frac{1}{2} \quad \forall\ n \geq N$$

But $\quad |f(z_n)+i| = \left|i + \cos\left(-\frac{\pi}{2}+\frac{1}{2n}\right) + i\sin\left(-\frac{\pi}{2}+\frac{1}{2n}\right)\right|$

$$= \left|-i - \sin\left(\tfrac{1}{2n}\right) - i\cos\left(\tfrac{1}{2n}\right)\right|$$

$$= \sqrt{\left(1+\cos\left(\tfrac{1}{2n}\right)\right)^2 + \sin^2\left(\tfrac{1}{2n}\right)} > 1$$

Facts: If $f(z)$ and $g(z)$ are continuous at z_0, then

① $(f+g)(z)$ is continuous at z_0

② $(fg)(z)$ is continuous at z_0

③ If $g(z_0) \neq 0$ then $(f/g)(z)$ is continuous at z_0.

Ex: Show $f(z) = \frac{1}{3+z^3}$ is uniformly continuous on

$$B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\}$$

Any poly. is continuous. (Thm)

Proof: $g(z) = z^3+3$ can be checked to be continuous on $\mathbb{C}$.

$\Rightarrow$ By Fact ③, if $g(z_0) \neq 0$ then $\frac{1}{g(z)}$ is continuous at z_0

If $g(z_0) = z_0^3+3 = 0$, then $z_0^3 = -3$. So $|z_0|^3 = 3$

$\Rightarrow |z_0| = \sqrt[3]{3} > 1$. Hence $f(z) = \frac{1}{g(z)}$ is continuous on $\overline{B_1(0)} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. By Thm $f(z)$ is uniformly cont. on $\overline{B_1(0)}$

So if $\epsilon > 0$, then $\exists\, \delta > 0$ S.T if $|z_1 - z_2| < \delta$

and $z_1, z_2 \in \overline{B_1(0)}$ then $|f(z_1) - f(z_2)| < \epsilon$

In particular, the same is true if $z_1, z_2 \in B_1(0) \subseteq \overline{B_1(0)}$

<u>Ex</u>: $f(z) = \frac{1}{z}$ is continuous but not uniformly continuous on

$\mathbb{C} \setminus \{0\}$

<u>Proof</u>: Suppose $\epsilon = \frac{1}{3}$. If $f(z)$ is uniformly continuous on $\mathbb{C} \setminus \{0\}$

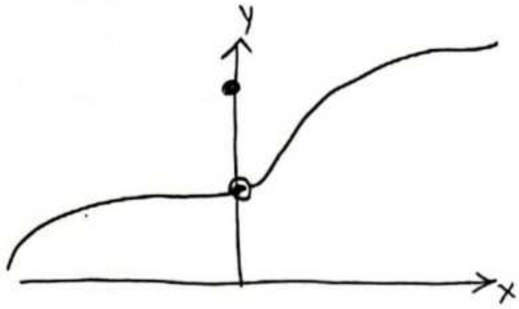
then $\exists\, \delta > 0$ S.T if $\underset{(z_1, z_2 \neq 0)}{|z_1 - z_2| < \delta}$ then $\left|\frac{1}{z_1} - \frac{1}{z_2}\right| < \frac{1}{3}$

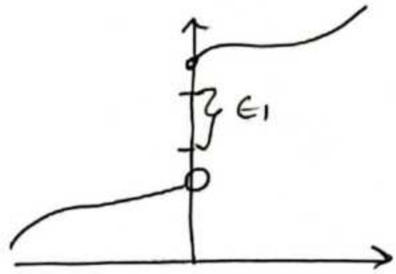
Well, $\exists\, n \in \mathbb{N}$ S.T $\frac{1}{2n} < \delta$. Let $z_1 = \frac{1}{n}$ and $z_2 = \frac{1}{2n}$

$\Rightarrow |z_1 - z_2| = \frac{1}{2n} < \delta$

But $|f(z_1) - f(z_2)| = |n - 2n| = n \geq 1 > \frac{1}{3}$ $\divideontimes$ $\blacksquare$

Limits, Continuity and Uniform Continuity intuition.





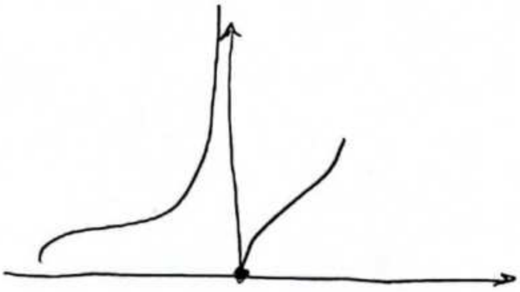
Limit from left exits & from right exists but they are not equal

Limit exists at $x=0$ but function is not continuous.

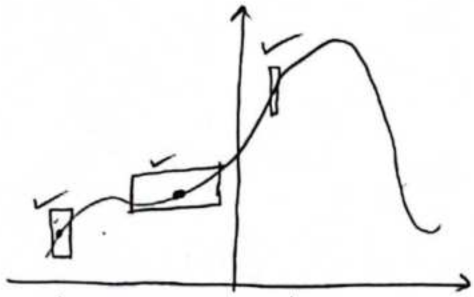
Limit does not exist at $x=0$ since for $\epsilon = \epsilon_1 > 0$ there is no $\delta > 0$ s.t $|f(x_1) - f(x_2)| < \epsilon$

> In continuity the $\delta \longrightarrow \delta(z_0, \epsilon)$. Therefore at different pts. the size of the "rectangle" that $f(z)$ passes through can be different.

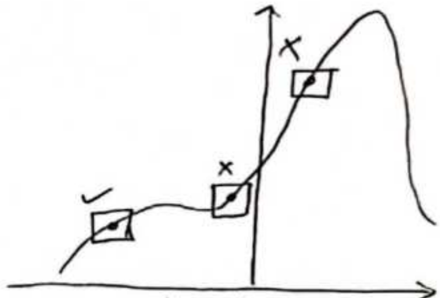
> In uniform continuity, $\delta \longrightarrow \delta(\epsilon)$ so the size of the rectangle must be same everywhere, and at all points it must satisfy. Stricter condition and smoother graph.



Limit does not exist. (from the left too)



Continuous!



Not uniformly continuous

> Uniform continuity is a stricter condition!

> Limit exists & equal to fn value ⇒ continuous!

Proofs of uniform continuity

> Statement: $\forall \epsilon > 0, \exists \delta > 0$ S.T $\forall z_1, z_2 \in G$ with $|z_1 - z_2| < \delta$ we have $|f(z_1) - f(z_2)| < \epsilon$.

Direct proof

Step① Start with $|z_1 - z_2| < \delta$.

Express/expand $|f(z_1) - f(z_2)|$ and find a value for ϵ in terms of δ S.T $|f(z_1) - f(z_2)| < \epsilon$.

Step② Write δ as $\delta(\epsilon)$ and construct the proof. as.

Let $\epsilon > 0$. Choose $\delta = \delta(\epsilon)$, then $\forall z_1, z_2 \in G$ with $|z_1 - z_2| < \delta$ we have $|f(z_1) - f(z_2)| = \ldots < \epsilon$.

> Proofs on continuity are the same except that δ can be a function of both ϵ & z_0.

Proof of not uniformly continuous.

> Assume that it is uniformly continuous,

> Pick an ϵ (say $\epsilon = 1$) and try to find a δ such that $|z_1 - z_2| < \delta$. For this you can choose functions for z_1 & z_2 in terms of n S.T $|z_1 - z_2| < \delta$ for some n (say $n \to \infty$). Use this $z_1(n)$ & $z_2(n)$ in $f(z_1)$ & $f(z_2)$ & show a contradiction for $|f(z_1) - f(z_2)| < \epsilon (= 1)$.

> Since not uniform can be shown at one pt. we can pick specific values for z_1 & z_2 & ϵ.

Chap 3 Summary

$\exists \Rightarrow$ we can pick a value.
$\forall \Rightarrow$ pick a fn that ensures ≤ 0 values are all accounted for

X: Sequence $\{z_n\} \longrightarrow \alpha$

For each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $|z_n - \alpha| < \epsilon \ \forall \ n \geq N$

$\bar{X}$: Sequence $\{z_n\} \not\longrightarrow \alpha$

$\exists \ \epsilon > 0$ s.t $\forall N \in \mathbb{N} \ \exists \ n \geq N$ s.t $|z_n - \alpha| > \epsilon$

Y: $\{z_n\}$ is bounded

$\exists M \in \mathbb{R}, M > 0$ s.t $|z_n| < M \ \forall n \in \mathbb{N}$.

$\bar{Y}$: $\{z_n\}$ is unbounded.

$\forall M \in \mathbb{R}, M > 0 \ \exists n \in \mathbb{N}$ s.t $|z_n| > M$.

Bolzano Weierstrass : Every bounded sequence has atleast one limit point $\alpha \in \mathbb{C}$.

*Cauchy convergence criteria:
$\{z_n\}$ converges if and only if for each $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t
$|z_n - z_m| < \epsilon \ \forall \ n, m \geq N$.

Thm: If $\lim_{n \to \infty} z_n = \alpha$ then α is the only limit point of $\{z_n\}$

Z: Limit!
$\lim_{z \to z_0} f(z) = c$ if for each $\epsilon > 0$, $\exists \ \delta > 0$ s.t, $|f(z) - c| < \epsilon$
wh ($\cancel{\forall}$ $0 < |z - z_0| < \delta$ and $B_\delta(z_0) \subseteq G$)

$\bar{Z}$: $\lim_{z \to z_0} f(z) \neq c$ if $\exists \ \epsilon > 0$ s.t $\forall \ \delta > 0$ s.t $|f(z) - c| > \epsilon$
w. ($\exists$ $0 < |z - z_0| < \delta$ and $B_\delta(z_0) \subseteq G$.)

Continuity $\lim_{\Delta z \to 0} f(z_0 + \Delta z) = f(z_0)$

> If $\lim_{z \to z_0} f(z) = f(z_0)$, then f is continuous at z_0.

> If $f(z)$ is continuous at every z_0, then f is continuous on G. $\Rightarrow \forall \epsilon > 0 \ \exists \ \delta > 0$ s.t $|f(z) - f(z_0)| < \epsilon$ when $|z - z_0| < \delta \ \forall z_0 \in G$.

Uniform continuity

$f : \tilde{G} \to \mathbb{C}$ is uniformly continuous on $\tilde{G}$ if

for each $\epsilon > 0$, $\exists \ \delta > 0$ s.t $\forall \ |z_1 - z_2| < \delta$ for $z_1, z_2 \in \tilde{G}$

then $|f(z_1) - f(z_2)| < \epsilon$.

Negation: $\exists \ \epsilon > 0$, s.t $\forall \ \delta > 0 \ \nexists \ |z_1 - z_2| < \delta$ for $z_1, z_2 \in \tilde{G}$

then $|f(z_1) - f(z_2)| > \epsilon$.

Thm*: If $f(z)$ is continuous on a closed, path connected and bounded set E, then $f(z)$ is uniformly continuous on E.

Thm: Any polynomial of z is continuous.

Closed: A set G is closed if it contains all its limit points.

Reverse triangle inequality

$$|z_1 - z_1| \geq |z_1| - |z_2|$$

Bounded: $\exists \ M > 0$ s.t $|f(z)| \leq M \ \forall z \in E$

Unbounded: $\forall \ M > 0 \ \exists \ z \in E$ s.t $|f(z)| > M$

Ch 4

Differentiation in $\mathbb{C}$

Def: $f: G \longrightarrow \mathbb{C}$ (G is a domain, ie open/path connected)

f is differentiable at $z \in G$ if

$$f'(z) = \lim_{\substack{\Delta z \to 0 \\ \Delta z \neq 0 \\ z+\Delta z \in G}} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists}$$

*) If $f'(z)$ exists $\forall z \in G$, then f is analytic in G

If $z_0 \in G$ and $\exists \ \delta > 0$ s.t $f'(z)$ exists $\forall z \in B_\delta(z_0)$, then f is analytic at z_0

Ex: 1) $f(z) = z^n$ $(n \in \mathbb{N})$ is analytic on $\mathbb{C}$

Proof: $\lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{\cancel{z^n} + \binom{n}{1} z^{n-1} \Delta z + \dots + (\Delta z)^n - \cancel{z^n}}{\Delta z}$$

Here we use that polynomials are continuous

$$= \binom{n}{1} z^{n-1}$$

$= n z^{n-1}$ which is finite $\forall z \in \mathbb{C}$

2) $f(z) = \bar{z}$ is not differentiable at any $z \in \mathbb{C}$

Proof: $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{(z+\Delta z)} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$

Does this limit exist?

[Recall HW Ch 3 #5:

$f(z) \to \alpha$ as $z \to z_0 \Rightarrow f(z_n) \to \alpha \ \forall z_n \to z_0$ as $n \to \infty$]

In other words, given 2 sequences $\{z_n\}, \{w_n\}$ converging to 0,

if $\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \alpha$ exists, then $\lim_{n \to \infty} \frac{\overline{z_n}}{z_n} = \alpha = \lim_{n \to \infty} \frac{\overline{w_n}}{w_n}$

Let $z_n = \frac{1}{n}$ and $w_n = \frac{i}{n}$

$\lim_{n \to \infty} \frac{\overline{z_n}}{z_n} = \lim_{n \to \infty} \frac{(1/n)}{(1/n)} = 1$

$\lim_{n \to \infty} \frac{\overline{w_n}}{w_n} = \lim_{n \to \infty} \frac{(-i/n)}{i/n} = -1$ $\Rightarrow$ limit does not exist!

Facts: Suppose f & g diff. at z, then

① $(f+g)'(z) = f'(z) + g'(z)$

② $(fg)'(z) = f'(z)g(z) + g'(z)f(z)$

③ If $g(z) \neq 0$ then $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

④ If ϕ is diff at $w = f(z)$ then for $\phi \circ f(z) = \phi(f(z))$

$$(\phi \circ f)'(z) = \phi'(f(z))f'(z) = \phi'(w)f'(z)$$

Note: As seen in example # on page 29, the limit must be the same from all directions, this motivates the "Cauchy Riemann Equations" relating partial derivates when considering $\mathbb{C} \sim \mathbb{R}^2$ with complex derivatives.

Complex Differentials

Say $f'(z)$ exists. Define $\Delta f = f(z+\Delta z) - f(z)$

Then since $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z)$ exists,

Claim: $\Delta f = f'(z)\Delta z + \epsilon(z, \Delta z)\Delta z$
where $\epsilon(z, \Delta z) \to 0$ as $\Delta z \to 0$

Proof: $\lim_{\Delta z \to 0} \Big[\underbrace{\frac{f(z+\Delta z) - f(z)}{\Delta z} - f'(z)}_{\epsilon(z, \Delta z)} \Big] = 0$ $\;\rightarrow$ Here z is fixed.

$\epsilon(z, \Delta z)$ can be made continuous by defining it to be 0 explicitly at $\Delta z = 0$

Def: The differential of f is denoted $df = f'(z)\,dz$, so $f'(z) = \frac{df(z)}{dz}$

Def: $u: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x, y) if

$\exists\; \epsilon_1(x, y, \Delta x, \Delta y)$ and $\epsilon_2(x, y, \Delta x, \Delta y)$ s.t

$\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ and $\exists\; A(x,y), B(x,y)$

s.t $\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y) = A\Delta x + B\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$

Fact: If $u(x,y)$ is diff. at (x,y) then

$\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ exist at (x,y)

Proof: $$\frac{\partial u}{\partial x} \triangleq \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$ $\leftarrow \Delta u$ with $\Delta y = 0$

$$= \lim_{\Delta x \to 0} \frac{A\Delta x + \epsilon_1 \Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} A + \epsilon_1$$

$$= A$$

If $f: G \to \mathbb{C}$ for domain $G \subseteq \mathbb{C}$ we can write

$$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$$

where $u(x,y) = u(x+iy) = \mathrm{Re}\, f(z)$
$v(x,y) = v(x+iy) = \mathrm{Im}\, f(z)$

Thm: $f = u+iv$ is differentiable at $z_0 = x_0 + iy_0$ if and only if
① $u(x,y)$ and $v(x,y)$ are diff. at (x_0, y_0)

and ② $$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{Cauchy Riemann Eqs.}} \text{ at } (x_0, y_0)$$

Proof: $\Rightarrow$ Suppose $f(z)$ diff at $z_0 = x_0 + iy_0$

Then $\exists$ $\epsilon(z_0, \Delta z)$ S.T $\epsilon(z_0, \Delta z) \to 0$ as $\Delta z \to 0$

and $\Delta f = f'(z_0)\Delta z + \epsilon \Delta z$

We can write $f'(z_0) = \alpha + i\beta$ and $\epsilon = \epsilon_1 + i\epsilon_2$

$$\underbrace{\Delta f = \Delta u + i\Delta v}_{f = u+iv} = (\alpha + i\beta)\underbrace{(\Delta x + i\Delta y)}_{\Delta z} + (\epsilon_1 + i\epsilon_2)(\Delta x + i\Delta y)$$

$$\Delta u + i\Delta v = (\alpha\Delta x - \beta\Delta y + \epsilon_1\Delta x - \epsilon_2\Delta y) + i(\beta\Delta x + \alpha\Delta y + \epsilon_1\Delta y + \epsilon_2\Delta x$$

$$\Rightarrow \Delta u = \alpha\Delta x - \beta\Delta y + \epsilon_1\Delta x - \epsilon_2\Delta y$$
$$\Delta v = \beta\Delta x + \alpha\Delta y + \epsilon_2\Delta x + \epsilon_1\Delta y$$

Now for u and v to be diff, we need $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

Since $\epsilon \to 0$ as $\Delta z \to 0$ we know $|\epsilon| \to 0$ as $|\Delta z| \to 0$

But $|\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. So as $\Delta x, \Delta y \to 0$ we know $|\Delta z| \to 0$. Also, $0 \leq |\epsilon_1| = \sqrt{\epsilon_1^2} \leq \sqrt{\epsilon_1^2 + \epsilon_2^2} = |\epsilon|$

Hence, by squeeze thm from calculus as $\Delta x, \Delta y \to 0$

Since $|\epsilon| \to 0$ as $0 \leq |\epsilon_1| \leq |\epsilon|$, we have $|\epsilon_1| \to 0$

Claim: If $\lim_{z \to z_0} |g(z)| = 0$ then $\lim_{z \to z_0} g(z) = 0$

Proof: Suppose $\epsilon > 0$ Then $\exists$ $\delta > 0$ S.T if $0 < |z - z_0| < \delta$ then $||g(z)| - 0| < \epsilon$. But $||g(z)| - 0| = |g(z)|$

So $|g(z)| < \epsilon$ for $0 < |z - z_0| < \delta$

Hence, $\epsilon_1 \to 0$ as $\Delta x, \Delta y \to 0$

(Notice $\epsilon_1 : \mathbb{C} \to \mathbb{R}$, but claim proof still holds)

Similarly, $\epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ and $-\epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

Hence u & v are differentiable.

Thus, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are

$$\frac{\partial u}{\partial x} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta u}{\Delta x} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\alpha \Delta x + \epsilon_1 \Delta x}{\Delta x} = \alpha.$$

Similarly, $$\frac{\partial u}{\partial y} = \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{\Delta u}{\Delta y} = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{-\beta \Delta y - \epsilon_2 \Delta y}{\Delta y} = -\beta$$

and $\frac{\partial v}{\partial x} = \beta$ and $\frac{\partial v}{\partial y} = \alpha$

Hence, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Note

> $\Delta u = A\,\Delta x + B\,\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ should be true for any direction along which Δx and Δy go to zero.

This is when u is differentiable.

$\Leftarrow$: Say u, v differentiable and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Then
$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \overset{-\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y}}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$
$$\Delta v = \frac{\partial v}{\partial x}\Delta x + \underset{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial y}}\Delta y + \tilde{\epsilon}_1 \Delta x + \tilde{\epsilon}_2 \Delta y$$

for $\epsilon_1, \epsilon_2, \tilde{\epsilon}_1, \tilde{\epsilon}_2 \to 0$ as $\Delta x, \Delta y \to 0$

We compute Δf

$$\Delta f = \Delta u + i\,\Delta v$$

$$= \frac{\partial u}{\partial x}\Delta x + i\frac{\partial v}{\partial x}\Delta x - \frac{\partial v}{\partial x}\Delta y + i\frac{\partial u}{\partial x}\Delta y + \epsilon_1 \Delta x + i\tilde{\epsilon}_1 \Delta x + \epsilon_2 \Delta y + i\tilde{\epsilon}_2 \Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) i\,\Delta y + (\epsilon_1 + i\tilde{\epsilon}_1)\Delta x + (\tilde{\epsilon}_2 - i\epsilon_2)\, i\,\Delta y$$

$$= \underbrace{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)}_{f'(z)}\underbrace{(\Delta x + i\Delta y)}_{\Delta z} + \underbrace{\left((\epsilon_1 + i\tilde{\epsilon}_1)\frac{\Delta x}{\Delta z} + (\tilde{\epsilon}_2 - i\epsilon_2)\frac{\Delta y}{\Delta z}\right)}_{\epsilon}\Delta z$$

Now, remains to show $\epsilon \longrightarrow 0$ as $\Delta z \longrightarrow 0$

Well,

$$|\epsilon| = \left| (\epsilon_1 + i\tilde{\epsilon}_1)\frac{\Delta x}{\Delta z} + (\tilde{\epsilon}_2 - i\epsilon_2)\frac{\Delta y}{\Delta z} \right|$$

$$\leq \left|(\epsilon_1 + i\tilde{\epsilon}_1)\frac{\Delta x}{\Delta z}\right| + \left|(\tilde{\epsilon}_2 - i\epsilon_2)\frac{\Delta y}{\Delta z}\right|$$

$$= |\epsilon_1 + i\tilde{\epsilon}_1| \frac{|\Delta x|}{|\Delta z|} + |\tilde{\epsilon}_2 - i\epsilon_2| \frac{|\Delta y|}{|\Delta z|}$$

Now, $\dfrac{|\Delta x|}{|\Delta z|} = \dfrac{|\Delta x|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \leq 1$. Also, $\dfrac{|\Delta y|}{|\Delta z|} \leq 1$

$$|\epsilon| \leq \underbrace{|\epsilon_1| + |\tilde{\epsilon}_1| + |\tilde{\epsilon}_2| + |\epsilon_2|}_{\text{go to 0 as } \Delta x, \Delta y \to 0}$$

Hence $\epsilon \to 0$ as $\Delta z \longrightarrow 0$

■

<u>Note:</u>

$\Delta f = c\Delta z + \epsilon \Delta z$ $\qquad\qquad \Delta f = f(z + \Delta z) - f(z)$.

$\dfrac{\Delta f}{\Delta z} = c + \epsilon$.

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} (c + \epsilon) = c$$

independant of Δz

$\lim_{\Delta z \to 0} \dfrac{\Delta f}{\Delta z} = f'(z) \quad \because \Delta f = f(z + \Delta z) - f(z)$

Also note from the proof: $\boxed{f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}}$

Note: We just showed that if $f'(z)$ exists then.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Useful fact: If $g : \mathbb{R}^2 \to \mathbb{R}$ and $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are continuous in $B_\delta((x_0, y_0)) = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\}$ for some $\delta > 0$, then g is diff. at (x_0, y_0).

Use this to check condition ① of CR Thm.

Note: Suppose $f = u + iv$ is differentiable at $z_0 = x_0 + iy_0$

Then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at (x_0, y_0)

So $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y}$$

Assuming 2^{nd} derivatives exist

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = -\frac{\partial^2 v}{\partial y \partial x}$$

Thm (Clairaut-Schwarz)

Suppose $\frac{\partial^2 g}{\partial y \partial x}$ and $\frac{\partial^2 g}{\partial x \partial y}$ continuous in $B_\delta((x_0, y_0))$ for some $\delta > 0$

Then $\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2 g}{\partial y \partial x}$ at (x_0, y_0)

So if $\frac{\partial^2 v}{\partial x \partial y}$ and $-\frac{\partial^2 v}{\partial y \partial x}$ continuous around (x_0, y_0) then $\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}_{\text{Laplace Eq (Harmonic fn)}}$

Ex: Say f is analytic on $\mathbb{C}$ and $f = u + iv$.
If $u(x,y) = x^2 - y^2$, what can $v(x,y)$ be?

Note, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$ ✓

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \Rightarrow 2x = \dfrac{\partial v}{\partial y} \Rightarrow V = 2xy + C_1(x).$

$-\dfrac{\partial u}{\partial y} = \dfrac{\partial V}{\partial x} \Rightarrow 2y = \dfrac{\partial V}{\partial x} \Rightarrow V = 2xy + C_2(x)$

So $2xy + C_1(x) = 2xy + C_2(y) \Rightarrow C_1(x) = C_2(y) = C$

So $f(x+iy) = x^2 - y^2 + i(2xy + c)$

Note:

If u is a harmonic fn $\exists$ v that is also harmonic & $f = u + iv$ is continuous & analytic.

Ex: (Ch 4 # 3) If $f'(z) = 0$ $\forall z \in G$ then show $f(z) = \alpha$ constant on G

Proof: $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} + i\dfrac{\partial v}{\partial x} = 0$

$\Rightarrow \dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = 0$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} = 0 &\Rightarrow u(x,y) = C_1(y) \\ \frac{\partial u}{\partial y} = 0 &\Rightarrow u(x,y) = C_2(x) \end{aligned}\right\} \Rightarrow C_1(y) = C_2(x)$$

$\Rightarrow C_1(y) = C_2(x) = C$ constant $\Rightarrow u(x,y) = c$

Similarly $v(x,y) = d$ Let $\alpha = c + id$.

Proofs of differentiability

I > Differentiability of $u(x,y)$ at (x_0,y_0)

i) Compute $\left.\frac{\partial u}{\partial x}\right|_{x_0,y_0} = \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0,y_0)}{h}$

ii) compute $\left.\frac{\partial u}{\partial y}\right|_{x_0,y_0} = \lim_{h\to 0} \frac{u(x_0,y_0+h) - u(x_0,y_0)}{h}$

iii) Show that,
$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y) - f(x_0,y_0) - \left(\left.\frac{\partial u}{\partial x}\right|_{x_0,y_0}\right)x - \left(\left.\frac{\partial u}{\partial y}\right|_{x_0,y_0}\right)y}{\sqrt{x^2+y^2}} = 0$$

use fn. deriv

II If $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous in $B_\delta(x_0,y_0)$ for some δ

$B_\delta(x_0,y_0) = \{(x,y)\in\mathbb{R}^2 \mid \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\}$ then $u(x,y)$ is differentiable at (x_0,y_0). (This is true for any open set & not just for a small ball ofcourse)

Note that the converse is NOT true for the partial derivatives case.

> NOT Differentiability of $f(z)$.

$\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \tilde{f}(z,\Delta z) \rightsquigarrow$ Replace this limit with

$\lim_{n\to 0} \tilde{f}(z_n)$ and choose $z_n = \frac{1}{n}, \frac{-1}{n}, \frac{1+i}{n}, \frac{1-i}{n}, \ldots$ & show that the limit is different along different directions

Differentiability of $f(z)$

(I) > Prove that $\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists and is finite.

(II) Use Cauchy theorem: Show both conditions are true!

i) u, v are differentiable

ii) $u_x = v_y$; $u_y = -v_x$.

Continuity of $f(z)$

(I) > $\lim_{\Delta z \to 0} f(z+\Delta z) = f(z)$ Show this

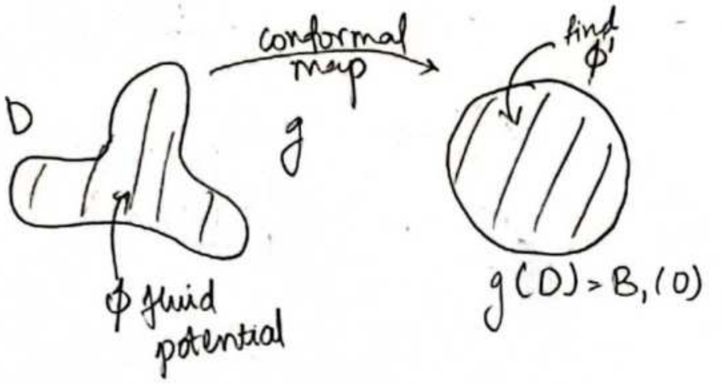
(II) Use ϵ, δ definition.

Conformal Maps

<u>Idea</u>: A conformal map "preserves" angles.

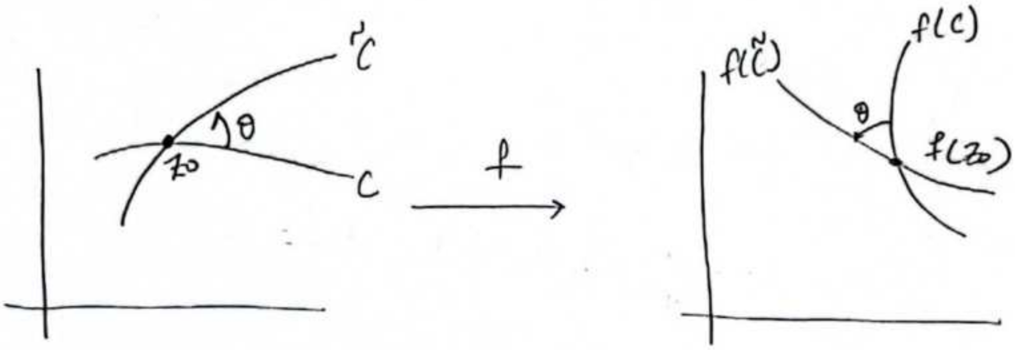
<u>Motivation</u>: Mobius transformations, solving Laplace's eq. on domains.

Want to know when a function f is conformal at $z_0 \in \mathbb{C}$.



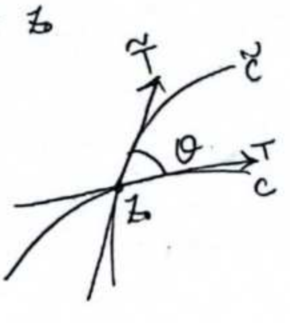
$g'(z) \neq 0 \Rightarrow$ find ϕ' and do a change of variable since $g'(z)$ is nonzero

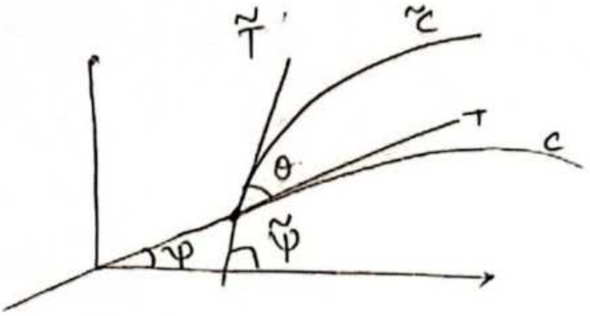
In $\mathbb{R}$ $\int f(u(x))\,dx = \int f(u) \frac{du}{u'(x)}$ $\Rightarrow$ we need $u'(x) \neq 0$



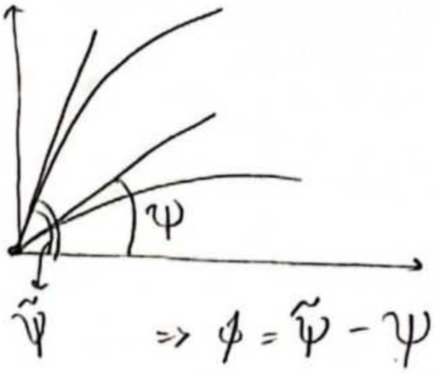
Now how to define θ?

Given 2 curves $C, \tilde{C}$ define θ by the angle between tangent vectors $T, \tilde{T}$ at z_0





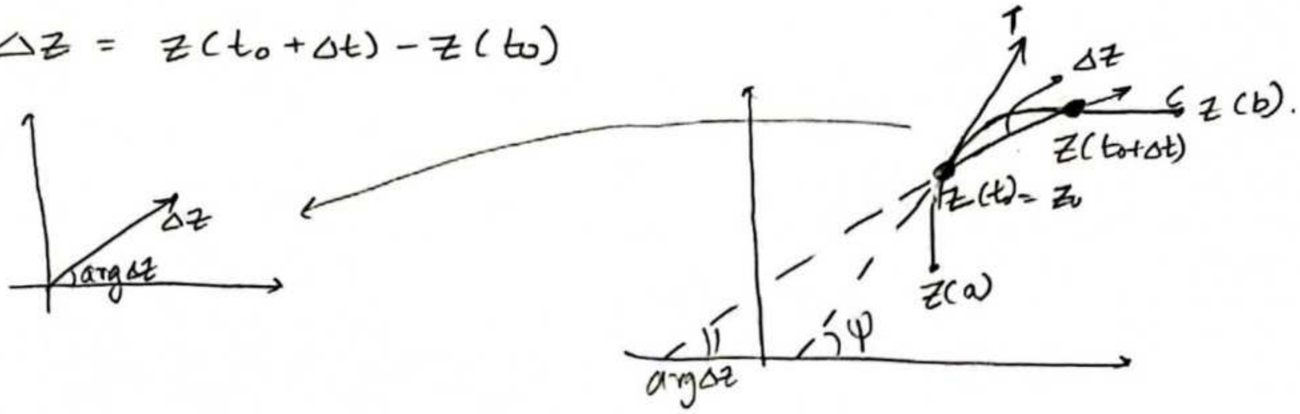
translate → to origin.



$\Rightarrow \phi = \tilde{\psi} - \psi$

$C = \{ z(t) \mid z : [a,b] \to \mathbb{C} \}$ → continuous.

$\Delta z = z(t_0 + \Delta t) - z(t_0)$



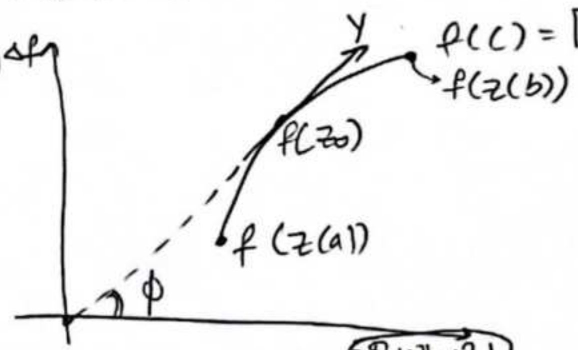
$$\psi = \lim_{\Delta t \to 0} \arg(\Delta z). = \lim_{\Delta z \to 0} \arg \Delta z$$

since $\Delta t \to 0 \Rightarrow \Delta z \to 0$

Now, under f we have $\Delta f = f(z(t_0 + \Delta t)) - f(z(t_0))$
$= f(z_0 + \Delta z) - f(z_0)$.

What is the difference between ψ and ϕ?

$\Rightarrow \phi = \lim_{\Delta t \to 0} \arg \Delta f$



$$\phi - \psi = \lim_{\Delta z \to 0} \arg \Delta f - \lim_{\Delta z \to 0} \arg \Delta z$$

$$= \lim_{\Delta z \to 0} (\arg \Delta f - \arg \Delta z) = \lim_{\Delta z \to 0} \arg\left(\frac{\Delta f}{\Delta z}\right) \overset{\text{why?}}{=} \arg\left(\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z}\right)$$

why? → (Why? 1)

Only allowed if $\underline{\underline{f'(z_0) \neq 0}}$

why? → (Why? 2)

$= \arg f'(z_0)$.

Assuming f is diff. no matter what path we take, the derivative [illegible]

Q Why? 1: Chp 2 #7

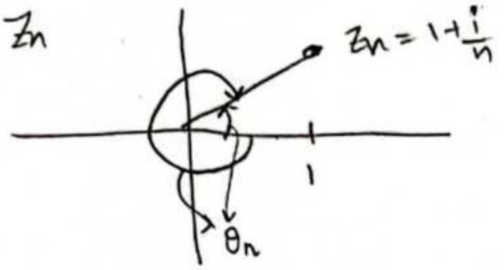
- If $z_n \to \alpha$, $\alpha \neq 0$. then "$\arg z_n \to \arg \alpha$" ($\arg(0)$ undefined.)

[This means, if θ is any argument of α, then $\exists$ choices θ_n for $\arg z_n$. S.T $\theta_n \to \theta$.]

Eg: $z_n = 1 + \frac{i}{n}$; $\alpha = 1$ $\Rightarrow \arg \alpha = 0, 2\pi, -8\pi, \ldots$

Say $\theta = 0$;

z_n



- Proof: $z_n \to \alpha \Rightarrow |z_n| \to |\alpha| \neq 0$

$$\Rightarrow \frac{z_n}{|z_n|} \to \frac{\alpha}{|\alpha_n|} \qquad \left[a_n \to a \ \& \ b_n \to b \Rightarrow \frac{a_n}{b_n} \to \frac{a}{b} \text{ if } b \neq 0 \right]$$

$\frac{z_n}{|z_n|} = \cos\theta_n + i \sin\theta_n$ for some $\theta_n = \arg z_n$.

Also, $\frac{\alpha}{|\alpha|} = \cos\theta + i\sin\theta$

$\Rightarrow \cos\theta_n + i\sin\theta_n \to \cos\theta + i\sin\theta$.

So $\lim_{n\to\infty} |\cos\theta_n + i\sin\theta_n - (\cos\theta + i\sin\theta)| = 0$

$\Rightarrow \lim_{n\to\infty} |\cos\theta_n - \cos\theta + i(\sin\theta_n - \sin\theta)| = 0$

- Facts: $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

$$\lim_{n\to\infty} \underbrace{\left| 2\sin\left(\frac{\theta_n-\theta}{2}\right)\left(-\sin\left(\frac{\theta_n+\theta}{2}\right)+i\cos\left(\frac{\theta_n+\theta}{2}\right)\right)\right|}_{2\left|\sin\left(\frac{\theta_n-\theta}{2}\right)\right| \underbrace{\left|-\sin(\gamma)+i\cos(\gamma)\right|}_{1}} = 0$$

$$\Rightarrow \boxed{\lim_{n\to\infty}\left|\sin\left(\frac{\theta_n-\theta}{2}\right)\right| = 0}$$

Fact if $|a_n| \to 0 \Rightarrow a_n \to 0$
only for 0

$$\Rightarrow \lim_{n\to\infty} \sin\left(\frac{\theta_n-\theta}{2}\right) = 0 \Longrightarrow \sin\left(\lim_{n\to\infty}\frac{\theta_n-\theta}{2}\right) = 0$$

$\sin x$ is continuous.
$\Rightarrow \lim_{x\to c} f(x) = f(c)$

$$\Rightarrow \lim_{n\to\infty} \frac{\theta_n-\theta}{2} \in \{\pi k \mid k \in \mathbb{Z}\}$$

$$\Rightarrow \lim_{n\to\infty} \theta_n - \theta = 2\pi k \text{ for som } k \in \mathbb{Z} \quad \blacksquare$$

To ensure $k=0$ we can appropriately choose the right θ_n.

So $\phi - \psi = \arg f'(z_0)$ only defined if $f'(z_0) \neq 0$

Now do the same to $\tilde{C}$ and $\tilde{T}$.

$\tilde{\phi} - \tilde{\psi} = \arg(f'(z_0))$ also $\Rightarrow$ No matter what curve is chosen, the rotation is exactly the same!

angle for tangent to $f(\tilde{C})$ at $f(z_0)$

Hence, $\underbrace{\tilde{\phi} - \phi} = \tilde{\psi} + \underline{f'(z_0)} - (\psi + \underline{f'(z_0)}) = \underbrace{\tilde{\psi} - \psi}$

angle is preserved!

Thus <u>Fact</u>: If $f: G \to \mathbb{C}$ is continuous on G and $f'(z_0)$ exists for $z_0 \in G$ and $f'(z_0) \neq 0$, then f is conformal at z_0

Integrals in $\mathbb{C}$

<u>Def</u>: A curve C is <u>smooth</u> if $C = \{z(t) \mid a \le t \le b\}$ and $z'(t)$ is continuous and $z'(t) \neq 0 \ \forall\, a \le t \le b$.

$\Rightarrow$ as t changes we are always moving along z (See note below)

Now, let $G \subseteq \mathbb{C}$ a domain and $C \subseteq G$ s.t $C = \{z(t) \mid a \le t \le b\}$ and $z(a) = \alpha$, $z(b) = \beta$.

Let $\{\underbrace{z_0}_{\alpha}, z_1, \ldots, \underbrace{z_n}_{\beta}\} \subseteq C$ s.t $z_k = z(t_k)$ for $t_k \in [a,b]$ and $a = t_0 < t_1 < t_2 < \ldots < t_n = b$.

Define $\Delta z_k = z_k - z_{k-1}$ for $k = 1, \ldots, n$.

l_k = length along C from z_{k-1} to z_k

$\lambda = \max\{l_1, l_2, \ldots, l_n\}$ $\quad w_k \in [z_{k-1}, z_k]$ arbitrary.

$\rightarrow$ points arbitrarily chosen b/w z_{k-1}, z_k (See note)

<u>Def</u>: f is <u>integrable</u> along C if

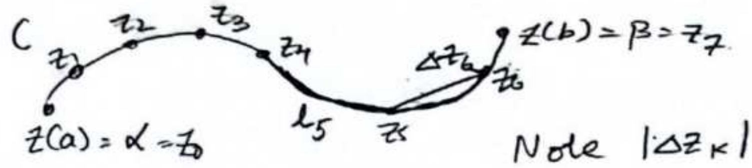
$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(w_k) \Delta z_k \text{ exists.}$$

regardless of the choice of w_k, z_k.

We write $$\boxed{\int_C f(z)\,dz = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(w_k)\Delta z_k}$$

Note,
$z(t) = x(t) + iy(t)$
$\Rightarrow z'(t) = x'(t) + iy'(t)$

<u>Note</u>:



$z'(t) \neq 0$
$\Rightarrow t_1 \neq t_2 \Rightarrow z_1 \neq z_2$

Note $|\Delta z_k| \neq l_k$

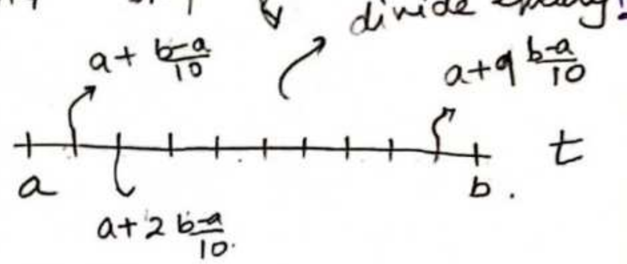
Remark: This means if $\int_C f(z)\,dz$ exists, then

if $C = \{z(t) \mid a \le t \le b\}$ and

$S_n \subseteq C$ s.t $S_n = \{ z(a + k(\frac{b-a}{n})) \mid k = 0, 1, \dots, n\}$ → breaks interval [into equal parts.

and let $w_k = z(a + k(\frac{b-a}{n}))$ $k = 0, 1, \dots, n-1$ → picks w = left end pts. of interval

divide equally!

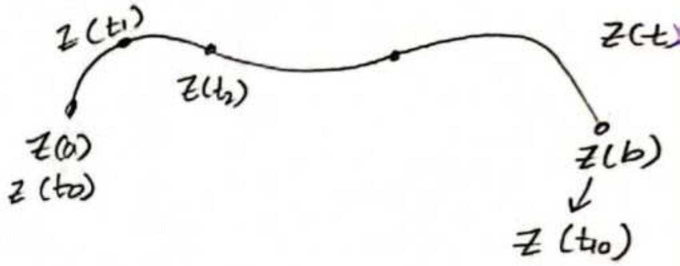


Then $$\boxed{\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(w_k)\,\Delta z_k = L}$$

But if I pick $\tilde{S}_n = C$ s.t

$\tilde{S}_n = \{ z\left(a\left(\frac{n-k}{n}\right)^2 + \left(\frac{k}{n}\right)^2 b \right) \mid k = 0, 1, 2 \dots n\}$

$= \{\tilde{z}_0, \dots, \tilde{z}_n\}$



and

$$\tilde{w}_k = z\left(\tfrac{1}{2}\left(a\left(\tfrac{n-k}{n}\right)^2 + b\left(\tfrac{k}{n}\right)^2 + a\left(\tfrac{n-k+1}{n}\right)^2 + b\left(\tfrac{k-1}{n}\right)^2\right)\right)$$ ↳ midway.

Then also

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(\tilde{w}_k)\,\Delta\tilde{z}_k = L$$

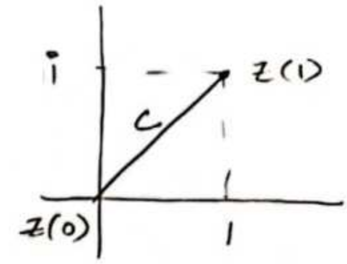
↳ $\Delta \tilde{z}_k = \tilde{z}_k - \tilde{z}_{k-1}$

Ex: Say $C = \{\underbrace{t + it}_{z(t)} \mid 0 \le t \le 1\}$, $f(z) = z^2$

For example if we take

$$\{z_0, \dots, z_n\} = \{0, \tfrac{1}{n} + \tfrac{i}{n}, \tfrac{2}{n} + \tfrac{2i}{n}, \dots, \tfrac{n-1}{n} + \tfrac{n-1}{n} i, 1+i\}$$

and $\underset{\text{midpt}}{w_k} = \dfrac{z_{k-1} + z_k}{2} = \dfrac{2k-1}{2n} + \left(\dfrac{2k-1}{2n}\right) i$

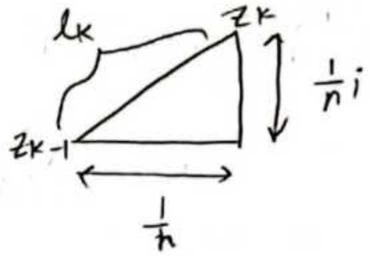


Then $\Delta z_k = z_k - z_{k-1}$

$$= \frac{k}{n} + \left(\frac{k}{n}\right) i - \left(\frac{k-1}{n}\right) - \left(\frac{k-1}{n}\right) i$$

$$= \frac{1}{n} + \frac{1}{n} i$$

$$l_k = \sqrt{\left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^2} = \frac{\sqrt{2}}{n}$$



So $\lambda = \max\{l_1, \dots, l_n\} = \dfrac{\sqrt{2}}{n}$.

Hence $\lambda \to 0 \leftrightarrow n \to \infty$

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(w_k) \Delta z_k = \lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{2k-1}{2n} + \frac{(2k-1)}{2n} i\right)^2 \left(\frac{1}{n} + \frac{1}{n} i\right)$$

$$= \lim_{n \to \infty} \sum_{k=1}^{n} \left(2i \frac{(2k-1)^2}{4n^2}\right)\left(\frac{1}{n} + \frac{1}{n} i\right) = \frac{(-2+2i)}{4} \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$= \frac{-2+2i}{4} \lim_{n \to \infty} \frac{1}{n^3}\left[4\left(\frac{1}{6} n(n+1)(2n+1)\right) - 4\left(\frac{1}{n} n(n+1)\right) + n\right]$$

$$= \frac{-2+2i}{\cancel{4}} \cdot \frac{\cancel{8}^{\cancel{4}}}{\cancel{6}\,3} = \boxed{-\frac{2}{3}(1-i)}$$

Note: ☆☆☆

$$\int_{z(t)} f(z)\,dz = \int_{t} f(z(t))\, z'(t)\,dt$$

<u>Theorem:</u> If $f(z)$ is continuous in domain G and $C \subseteq G$ smooth curve, then $\int_C f(z)\,dz$ exists.

<u>Proof:</u> Write $f(x+iy) = u(x,y) + i\,v(x,y)$. $C = \{z(t) \mid a \le t \le b\}$, $z(t) = x(t)+iy(t)$

Then u and v are continuous.

Let $z_k = z(t_k)$ s.t. $t_0 = a$, $t_n = b$, $t_0 < t_1 < \dots < t_n$.

and w_k on the arc $\overset{\frown}{z_{k-1} z_k} \subseteq C$, say $w_k = z(s_k)$; $s_k \in [t_{k-1}, t_k]$

Then write $w_k = \alpha_k + i\beta_k$ and $f(w_k) = \underbrace{u_k}_{u(\alpha_k,\beta_k)} + i\,\underbrace{v_k}_{v(\alpha_k,\beta_k)}$

and $\Delta z_k = \underbrace{z_k}_{x_k+iy_k} - \underbrace{z_{k-1}}_{x_{k-1}+iy_{k-1}} = \Delta x_k + i\,\Delta y_k$

Then $$\sum_{k=1}^{n} f(w_k)\Delta z_k = \sum_{k=1}^{n} (u_k + i v_k)(\Delta x_k + i\Delta y_k)$$

$$= \sum_{k=1}^{n} (u_k \Delta x_k - v_k \Delta y_k) + i(u_k \Delta y_k + v_k \Delta x_k).$$

↳ looks like a dot product! ⇒ line integral in $\mathbb{R}^2$

Now, recall defn. of line integral on $\mathbb{R}^2$.
Say $F = (u,v) = (u(x,y), v(x,y)) \in \mathbb{R}^2$ a vector field.
The line integral along a parameterized curve $C = \{\gamma(t) \mid a \le t \le b\}$

is $$\int_C F(\gamma)\,d\gamma = \int_a^b F(\gamma(t)) \cdot \gamma'(t)\,dt.$$

↳ dot product

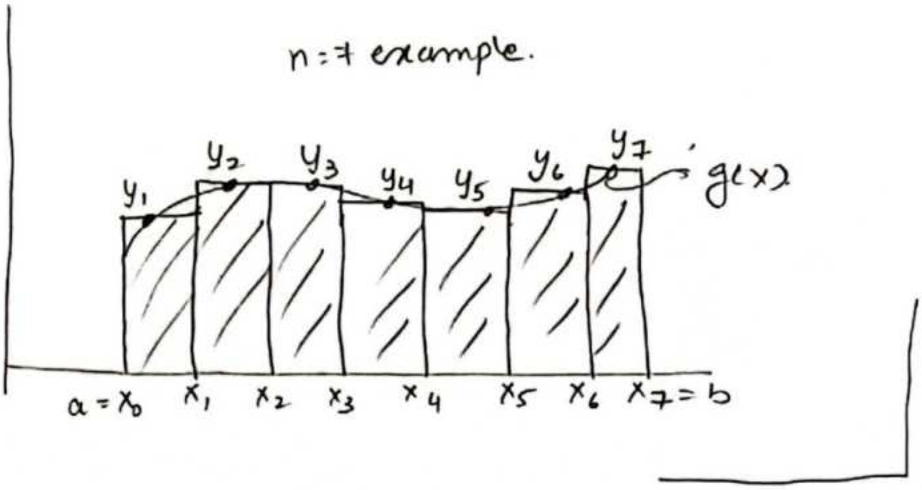
If F is cont. & C is a smooth curve then $\int_C$ exists ($\Rightarrow$ it is finite)

~~sufficient cond.~~

F is cont. $\Rightarrow u$ & v are

Now, by Riemann Sums,

$$\int_a^b g(x)\,dx = \lim_{\lambda \to 0} \sum_{K=1}^{n} g(y_k)(x_K - x_{K-1})$$



$\Delta x_K = x_k - x_{k-1} = x(t_K) - x(t_{k-1})$

Mean value thm $\rightarrow$ $= \underbrace{x'(c_K)}_{\text{some } c_K \in [t_{k-1}, t_K]}(t_K - t_{K-1})$

Since $z'(t)$ continuous, so is $x'(t)$

Thus, $\lim_{c_K \to t} x'(c_K) = x'(t)$.

$\Delta x_K = x'(c_K)\,\Delta t_k$

$\Rightarrow \Delta x_K \cong x'(s_K)\Delta t_k$.

M.V.T

f' is continuous on the interval $[a, b]$

Then $\exists\ c \in (a,b)$ S.T

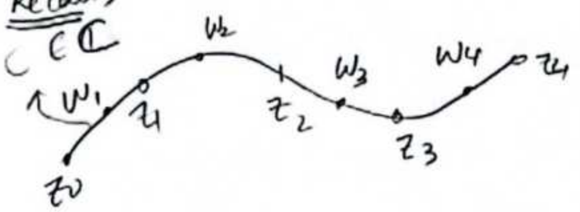
$f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$\Rightarrow f'(c)(b-a) = f(b) - f(a)$

Such a pt. always exists.

So in summary... imagining $\Delta x_K \approx x'(\underset{\text{some pt. b/w}}{s_K})\Delta t_K$

Recall,



$W_K = z(s_K)$ where

$t_{K-1} \leq s_K \leq t_K$

$z(t) = x(t) + i\,y(t)$

$z_K = z(t_K) = x(t_K) + i\,y(t_K)$

$\Delta x_K = x(t_K) - x(t_{K-1})$

$= x'(c_K)\underbrace{(t_K - t_{K-1})}$ for some c_K.

$\approx x'(s_K)\quad \Delta t_K.$ $\rightarrow$ Since s_K & c_K are

Real part

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} u_k \Delta x_k - v_k \Delta y_k \simeq \lim_{\lambda \to 0} \sum_{k=1}^{n} \left(u_k x'(s_k) \Delta t_k - v_k y'(s_k) \Delta t_k \right)$$ 45

$$\simeq \lim_{\lambda \to 0} \sum_{k=1}^{n} \Big(\underbrace{u(x(s_k), y(s_k))}_{\text{Re } f(w_k)}, -v(x(s_k), y(s_k)) \Big) \cdot (x'(s_k), y'(s_k)) \Delta t_k.$$

$\Delta t_k = t_k - t_{k-1}$

This is where the approximation comes in.

Remember $w_k = z(s_k)$..

If $F = (u, -v)$, $r(t) = (x(t), y(t))$

then $g(t) = F(r(t)) \cdot r'(t) \leftarrow \int_a^b g(t)\, dt \rightarrow$ then Riemann sums are

$a = t_0 < t_1 \cdots < t_n = b$; $s_k \in [t_{k-1}, t_k]$

$$\therefore \quad = \int_a^b \left(u(x(t), y(t)), -v(x(t), y(t)) \right) \cdot (x'(t), y'(t))\, dt.$$

Which is finite for continuous vector field (u, v) on C. → From multi variable calc.

Similarly,

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} u_k \Delta y_k + v_k \Delta x_k = \int_a^b \left(u(x(t), y(t)), v(x(t), y(t)) \right) \cdot (y'(t), x'(t))\, dt$$

Hence, $\lim_{\lambda \to 0} \sum_{k=1}^{n} f(w_k) \Delta z_k$ exists ■

This proof gives us the concise form.

$$\boxed{\begin{aligned} \int_C f\, dz &= \int_C (u + iv)(dx + i\,dy) \\ &= \int_a^b R(t)\, dt + i \int_a^b I(t)\, dt \end{aligned}}$$

$R(t) = \text{Re } f(z(t))\, z'(t)$

$I(t) = \text{Im } f(z(t))\, z'(t)$

> Suppose $C = C_1 \cup C_2 \cup \dots \cup C_n$ in the sense

$$C_k = \{ z_k(t) \mid a_k \le t \le b_k \}$$

and $\quad C = \{ z(t) \mid a \le t \le b \}$

where $b_k = a_{k+1}$ for $k \in \{1, \dots, n-1\}$, $a = a_1$, $b = b_n$

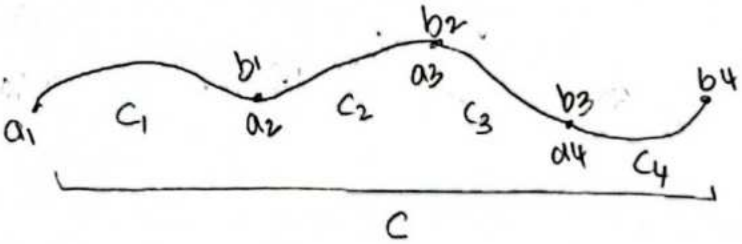
and so $[a, b] = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_n, b_n]$

equal.

and $z_k(b_k) = z_{k+1}(a_{k+1})$ for $k \in \{1, 2, \dots, n-1\}$

and $z(t) = z_k(t)$ for $t \in [a_k, b_k]$ and $z'(t)$ continuous.

This requires $\lim\limits_{t \to a_{k+1}^+} z_{k+1}'(t) = \lim\limits_{t \to b_k^-} z_k'(t)$. derivates from both sides equal.



Then if $f(z)$ continuous on C,

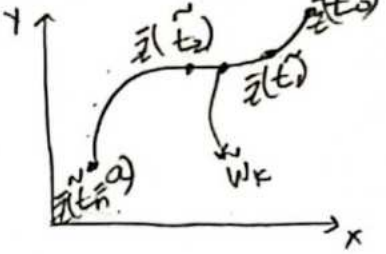
$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \dots + \int_{C_n} f(z)\,dz$$

If C_k are smooth but $C = C_1 \cup \dots \cup C_n$ is not then C is piecewise smooth and we define $\int_C f(z)dz = \int_{C_1} f(z)dz + \dots + \int_{C_n} f(z)dz$

Def: If $C = \{z(t) \mid a \le t \le b\}$ is a piecewise smooth curve then C^- denote the curve traversed in the opposite direction

eg: $C^- = \{z^-(t) \mid 0 \le t \le 1\}$ where $z^-(t) = z(ta + (1-t)b)$

Thm: Suppose C piecewise smooth and $\int_C f(z)dz$ exists then $\int_{C^-} f(z)\,dz = -\int_C f(z)\,dz$.



Proof:

Suppose C, and so C^-, is smooth.

Let $C^- = \{z^-(t) \mid 0 \le t \le 1\}$

$C = \{z(t) \mid a \le t \le b\}$ $\qquad z^-(t) = z(ta + (1-t)b)$.

Let $0 = \tilde{t}_0 < \tilde{t}_1 < \dots < \tilde{t}_n = 1$ be a partition of $[0,1]$

Let $S = \sum_{k=1}^{n} f(\tilde{w}_k)(z^-(\tilde{t}_k) - z^-(\tilde{t}_{k-1}))$ for $\tilde{w}_k \in \overset{\frown}{z^-(t_{k-1})\, z^-(t_k)}$ are in C^-

So $S = \sum_{k=1}^{n} f(\tilde{w}_k)(z(\tilde{t}_k a + (1-\tilde{t}_k)b) - z(\tilde{t}_{k-1} a + (1-\tilde{t}_{k-1})b))$

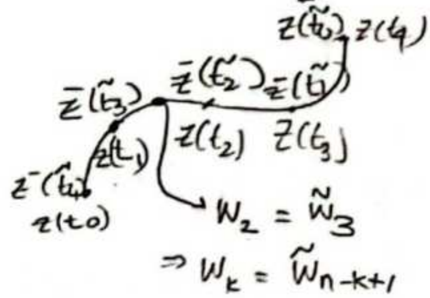
Since $a < b$ and $\tilde{t}_{k-1} < \tilde{t}_k$ we get

$$\tilde{t}_k a + (1-\tilde{t}_k) b < \tilde{t}_{k-1} a + (1-\tilde{t}_{k-1}) b$$

iff $$\underbrace{(\tilde{t}_k - \tilde{t}_{k-1})}_{>0} a < \underbrace{(\tilde{t}_k - \tilde{t}_{k-1})}_{>0} b \quad \text{since } a<b$$

So $$a = \underset{t_0}{\underset{\parallel}{\tilde{t}_n a + (1-\tilde{t}_n)}} < \underset{t_1}{\underset{\parallel}{\tilde{t}_{n-1} a + (1-\tilde{t}_{n-1}) b}} < \dots < \underset{t_n}{\underset{\parallel}{\tilde{t}_0 a + (1-\tilde{t}_0) b}} =$$

is a partition of $[a,b]$ for C



Also, $w_k = \tilde{w}_{n-k+1} \in \overset{\frown}{z(t_{k-1})\, z(t_k)} \subseteq C$

$$f(\tilde{w}_{n-k+1}) \qquad \tilde{z}(\tilde{t}_{n-k+1}) \qquad \tilde{z}(\tilde{t}_{n-k})$$

So $$S = \sum_{k=1}^{n} f(w_k)\underbrace{(z(t_{k-1}) - z(t_k))}_{-\Delta z_k}$$

$$= -\sum_{k=1}^{n} f(w_k)\Delta z_k$$

Let $\lambda \to 0$ to get the thm ∎

<u>thm:</u> Suppose $\int_C f(z)\,dz$ and $\int_C g(z)\,dz$ exist then for any $\alpha, \beta \in \mathbb{C}$

$$\int_C \alpha f(z) + \beta g(z)\, dz = \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz.$$

<u>Thm</u>: Suppose C piecewise smooth $\int_C f(z)dz$ exists and $|f(z)| \le M \ \forall z \in C$. Then

$$\left|\int_C f(z)dz\right| \le Ml \quad \text{where } l \text{ is length of } C.$$

<u>Proof</u>: Consider partition $\{z_k\}$ of C and choice $w_k \in \widetilde{z_{k-1}z_k} \subseteq C$

$$\left|\sum_{k=1}^{n} f(w_k)\Delta z_k\right| \le \sum_{k=1}^{n} |f(w_k)||\Delta z_k| \le M \sum_{k=1}^{n} |\Delta z_k|$$
$$\le Ml.$$

<u>Approximating by Polygon Curves.</u>

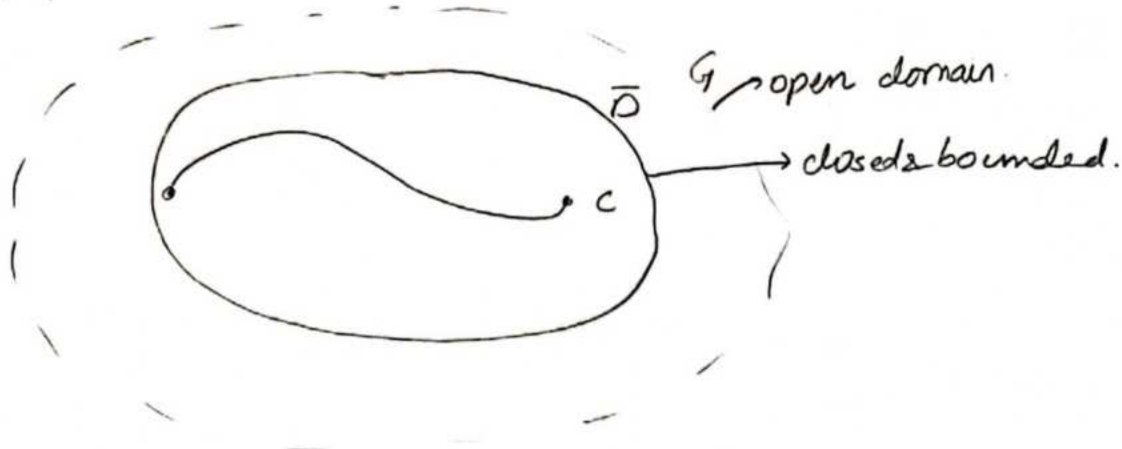
<u>Def</u>: Polygon curve is piecewise smooth $L = L_1 \cup L_2 \cup \dots \cup L_n$ where each L_k is a line segment.

Can $\int_C f(z)dz$ be approximated by $\int_L f(z)\,dz$?

<u>Lemma</u>: Let $C \subseteq G$ (domain) a piecewise smooth curve and $f(z)$ continuous on G. then $\forall \epsilon > 0, \exists L$ polygon s.t

$$\left|\int_C f(z)\,dz - \int_L f(z)\,dz\right| < \epsilon.$$

Idea of proof.



$C \subseteq \bar{D} \subseteq G$ — f continuous on G ↪ f uniformly continuous on $\bar{D}$.

$C \subseteq G$ ↪ open ⇒ for each $z \in C$, $\exists\ \delta_z > 0$ s.t. $B_\delta(z) \subseteq G$. (Since G open)

closed & bounded. ⇒ Since $C \subseteq \bigcup_{z \in C} B_{\frac{\delta_z}{2}}(z)$, we get (Heine Borel) $\exists\ z_1, \dots z_n$ and

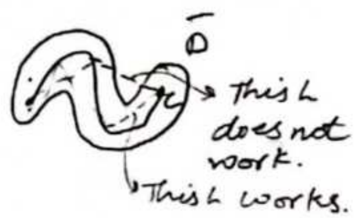
$$\delta_k = \frac{\delta_{z_k}}{2} \text{ s.t. } C \subseteq \underbrace{B_{\delta_1}(z_1) \cup \dots \cup B_{\delta_n}(z_n)}_{\text{closed \& bounded}}$$ → closed balls

$$\subseteq \underbrace{B_{\delta_{z_1}}(z_1) \cup \dots \cup B_{\delta_{z_n}}(z_n)}$$ → open balls.

$$\subseteq G$$

⇒ Let $\bar{D} = B_{\delta_{z_1}}(z_1) \cup \dots \cup B_{\delta_{z_n}}(z_n)$

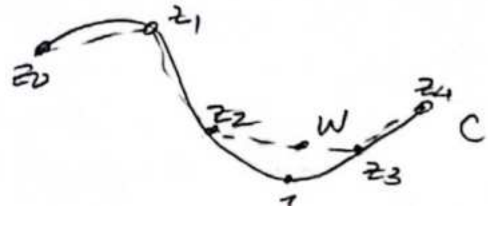
⇒ $C \subseteq \bar{D} \subseteq \underline{\underline{G}}$.



f is uniformly continuous on $\bar{D}$.

Pick L & C close enough so $|f(z) - f(w)| < \delta$

$$\tilde{f}(z) = \begin{cases} f(w_1) & z \in \widehat{z_0 z_1} \text{ arc} \\ f(w_2) & z \subset \widehat{z_1 z_2} \\ \vdots & \end{cases}$$



Compare $\underbrace{\int_C f \longleftrightarrow \int_C \tilde{f}}_{\epsilon/2}$

line segment

$$g = \begin{cases} f(w_1) & z \in \overline{z_0 z_1} \\ f(w_2) & z \in \overline{z_1 z_2} \end{cases}$$

Compare $\int_L f \longleftrightarrow \int_L g$

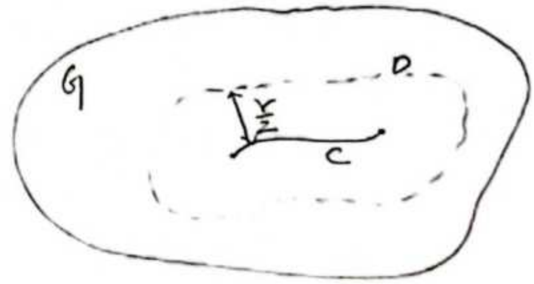
$$\int_C \tilde{f} = \int_L g$$

Proof: Say $\epsilon > 0$

Step 1: $\exists\, r > 0$ s.t. $B_r(z) \subseteq G \quad \forall\, z \in C$.

Let $D = \{ z \in G \mid |z - w| < r/2 \text{ for some } w \in C \}$,

$\bar{D}$ = closure of D.



Proof: Suppose not. Then

$\forall\, r > 0,\ \exists\, z \in C$ s.t. $\underbrace{B_r(z) \not\subseteq G}_{B_r(z) \cap G^c \neq \emptyset}$

$\mathbb{C} \setminus G$

Let $r = \frac{1}{n}$, $\exists\, z_n \in C \subseteq G$ s.t. $B_{\frac{1}{n}}(z_n) \cap G^c \neq \emptyset$

C is bounded and $\{z_n\} \subseteq C$ so $\{z_n\}$ bounded.

So $\exists\, \alpha \in \mathbb{C}$ s.t. α is a limit point of $\{z_n\}$

Fact: C is a closed set (since interval $[a,b]$ is closed & bounded & $z(t)$ is continuous).

Fact: Closed sets contain all of their limit points.

$\Rightarrow \alpha \in C$.

> We will show $B_r(\alpha) \cap G^c \neq \phi \quad \forall\, r > 0$.

> Suppose $\exists\, r > 0$ s.t $B_r(\alpha) \subseteq G$

> α limit pt of $\{z_n \mid n \in \mathbb{N}\}$ $\Rightarrow$ $\exists$ infinitely many z_n in the ball $B_{\frac{r}{2}}(\alpha)$.

> Thus, $\exists\, z_m \in B_{\frac{r}{2}}(\alpha)$ s.t $\frac{1}{m} < \frac{r}{2}$

> Recall, $B_{\frac{1}{m}}(z_m) \cap G^c \neq \phi$. So $\exists\, w \in B_{\frac{1}{m}}(z_m) \cap G^c$.

> $|w-\alpha| \leq \underbrace{|w-z_m|}_{<\frac{1}{m}} + \underbrace{|z_m-\alpha|}_{<\frac{r}{2}} < \underbrace{\frac{1}{m}}_{<\frac{1}{r}} + \frac{r}{2} < \frac{r}{2} + \frac{r}{2} = r$.

$\Rightarrow w \in B_r(\alpha) \Rightarrow w \in G$ $\cancel{*}$

$\Rightarrow B_r(\alpha) \cap G^c \neq \phi \quad \forall\, r > 0$

Also $B_r(\alpha) \cap G \neq \phi \quad \forall\, r > 0$ since $\alpha \in G$

α is a boundary pt. of G & G^c $\cancel{*}$

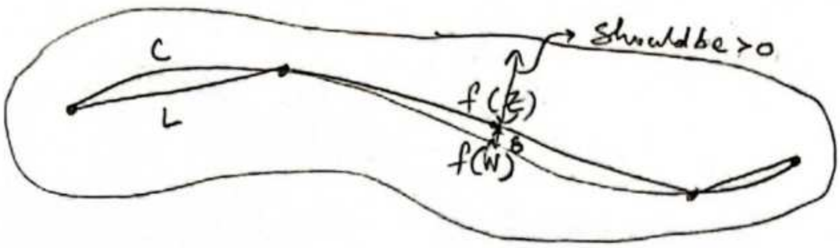
Since G is open, G^c is closed so $\alpha \in G^c$

<u>Step 2</u> For $\epsilon > 0$, $\exists\ \delta > 0$ s.t $|f(z) - f(w)| < \frac{\epsilon}{2l}$ whenever $|z - w| < \delta$ and $z, w \in D$. Here l is the length of C.

<u>Proof</u>: $\bar{D}$ is closed and bounded. f is continuous in G.

closure of D

$\Rightarrow f$ is uniformly continuous in $\bar{D}$.



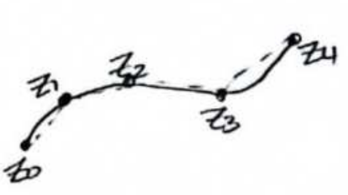
Uniform continuity guarantees that $f(z)$ & $f(c)$ are close to each other.

<u>Step 3</u> Construct L s.t $L \subseteq D$ and L is close to C. → step 4.

Let $\delta' = \min\{\delta, \frac{r}{2}\}$ and $C = \{z(t) \mid a \le t \le b\}$

Pick a partition $z_0 = z(a),\ z_1 = z(t_1), \ldots, z_n = z(b)$ s.t

$l_k = \text{length}\,(\underset{\text{on } C}{\widehat{z_k z_{k+1}}}) < \delta'$. $\quad k = 0, 1, \ldots, n-1$



$L = L_0 \cup L_1 \cup \ldots \cup L_{n-1}$

$L_k = \underbrace{\{(1-t) z_k + t z_{k+1} \mid 0 \le t \le 1\}}$ line connecting z_k, z_{k+1}

Say $w \in L_k$ for some $k \in \{0, 1, \ldots, n-1\}$

$|w - z_k| = |(1-t) z_k + t z_{k+1} - z_k| = |-t z_k + t z_{k+1}|$

$= t\,|z_{k+1} - z_k| \le 1 \cdot l_k < \delta' \le r/2$

$w \in B_{\frac{r}{2}}(z_k),\ z_k \in C \Rightarrow w \in D \Rightarrow L \subseteq D$ //

Let $S = \sum_{k=1}^{n} f(z_k)\Delta z_k$ for $\{z_0, z_1, \dots, z_n\}$ s.t

$l_k = \text{length}(\widehat{z_k z_{k+1}}) < \delta' \ \forall k.$

<u>Step 4</u> Compare S with $\int_C f(z)dz$ and $\int_L f(z)dz$.

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \dots + \int_{\gamma_n} f(z)dz \quad \text{where } \gamma_k = \widehat{z_{k-1} z_k} \subseteq C$$

<u>Claim</u>: $f(z_k)\Delta z_k = f(z_k)(z_k - z_{k-1}) = \int_{\gamma_k} f(z_k)dz$

<u>Proof</u>: $\int_{\gamma_k} f(z_k)dz = \lim_{\lambda\to 0} \sum_{m=0}^{N} f(z_k)\Delta w_m$ where $w_0, \dots, w_N$ partition of γ_k.

constant value

$$= \lim_{\lambda\to 0} f(z_k) \sum_{m=0}^{N} \Delta w_m$$

$$= \lim_{\lambda\to 0} f(z_k)(w_N - w_0)$$

end pts. of γ_k

$$= \lim_{\lambda\to 0} f(z_k)(z_k - z_{k-1})$$

$$= f(z_k)\Delta z_k \ //$$

$$S = \sum_{k=1}^{n} f(z_k)\Delta z_k = \sum_{k=1}^{n} \int_{\gamma_k} f(z_k)dz$$

$$\left| S - \int_C f(z)dz \right| = \left| \sum_{k=1}^{n} \int_{\gamma_k} f(z_k)dz - \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz \right|$$

$$= \left| \sum_{k=1}^{n} \int_{\gamma_k} (f(z_k) - f(z))dz \right| \le \sum_{k=1}^{n} \left| \int_{\gamma_k} (f(z_k) - f(z))dz \right|$$

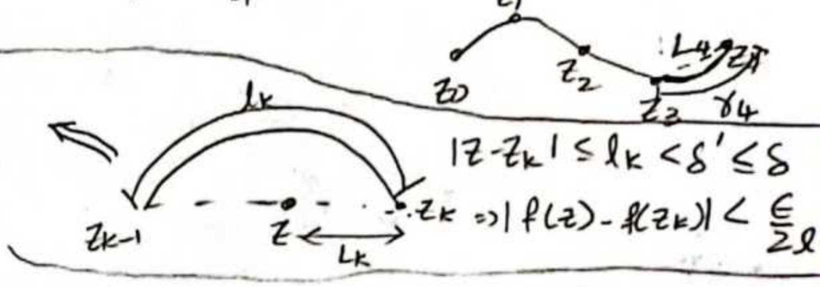
Since $\left| f(z_k) - f(z) \right| < \frac{\epsilon}{2l}$ because $z, z_k \in \gamma_k$

$\Rightarrow |z - z_k| < l_k < \delta' \leq \delta$. {uniform continuity}

$$\Rightarrow \left| S - \int_{C} f(z)\,dz \right| \leq \sum_{k=1}^{n} \frac{\epsilon}{2l} \cdot l_k \leq \frac{\epsilon}{2l} \sum_{k=1}^{n} l_k = \frac{\epsilon}{2l} \cdot l = \frac{\epsilon}{2}.$$

Similarly, $S = \sum_{k=1}^{n} f(z_k)\,\Delta z_k = \sum_{k=1}^{n} \int_{L_k} f(z_k)\,dz$

and $\left| S - \int_{L} f(z)\,dz \right| < \frac{\epsilon}{2}$



Thus,

$$\left| \int_{C} f\,dz - \int_{L} f\,dz \right| \leq \left| \int_{C} f\,dz - S \right| + \left| S - \int_{L} f\,dz \right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \blacksquare$$

Def: A closed polygonal curve L is a polygonal curve

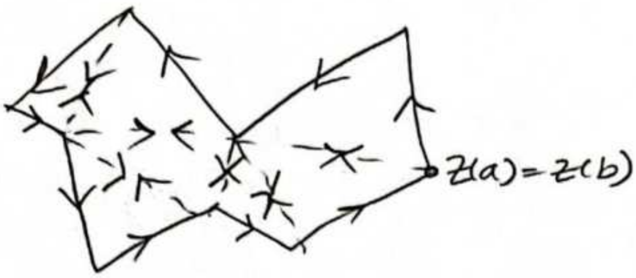
$L = \{ z(t) \mid a \leq t \leq b \}$ s.t $z(a) = z(b)$

A triangle is a closed polygonal curve with 3 vertices.

Lemma: Say $f(z)$ is continuous in G and L is a closed polygonal curve in G. Then,

$$\int_L f(z)\,dz = \int_{\Delta_1} f(z)\,dz + \int_{\Delta_2} f(z)\,dz + \dots + \int_{\Delta_n} f(z)\,dz$$

where Δ_K are triangles in G s.t $\{\Delta_1, \dots, \Delta_n\}$ is a triangulation of the region enclosed by L, eg:

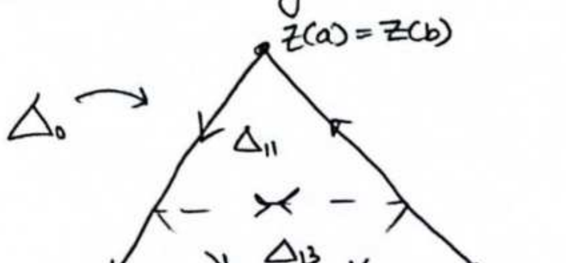


Theorem: Cauchy Integral Theorem

Let $f(z)$ be analytic in a simply connected domain G, → no holes or separations.

Then $\int_C f(z)\,dz = 0$ for every piecewise smooth closed curve C.

Proof: We just have to prove this for a triangle Δ_0.



$$\int_{\Delta_0} f(z)\,dz = \sum_{k=1}^{4} \int_{\Delta_{1,k}} f(z)\,dz$$

Say, $\left|\int_{\Delta_0} f(z)\,dz\right| = M$ Want to show $M=0$

Suppose $M>0$

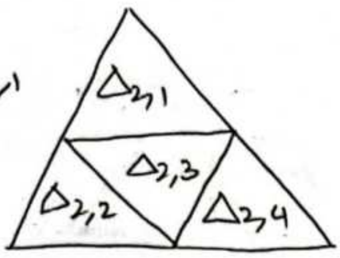
<u>Claim</u>: At least one of $\left|\int_{\Delta_{1,k}} f(z)dz\right| \geq \frac{M}{4}$

<u>Proof</u>: Suppose $\left|\int_{\Delta_{1,k}} f(z)dz\right| < \frac{M}{4}$ $\forall$ $k=1,2,3,4$

Then $\left|\int_{\Delta_0} f(z)\,dz\right| = \left|\sum_{k=1}^{4}\int_{\Delta_{1,k}} f(z)dz\right| \leq \sum_{k=1}^{4}\left|\int_{\Delta_{1,k}} f(z)dz\right| < M$

WLOG Say, $\Delta_{1,1}$ is the triangle with $\left|\int_{\Delta_{1,1}} f(z)dz\right| \geq \frac{M}{4}$

Repeat $\Delta_{1,1}$



So $\exists$ k s.t $\left|\int_{\Delta_{2,k}} f(z)\,dz\right| \geq \frac{M}{16} = \frac{M}{4^2}$

Let it be $\Delta_{2,1}$. continuing this we get

$\left|\int_{\Delta_{n,1}} f(z)dz\right| \geq \frac{M}{4^n}$ for each $n \in \mathbb{N}$.

Now, $\Delta_0 \supseteq \Delta_1 \supseteq \Delta_2 \cdots$ is a nested sequence of triangles whose area goes to zero.

By analogy to nested rectangle theorem, $\exists$ unique $z_0 \in \mathbb{C}$ s.t $z_0 \in \Delta_{n,1}$ $\forall$ $n \in \mathbb{N}$

Since $f(z)$ analytic on G $f'(z_0)$ exits/is finite.

This means, $\lim_{z \to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| = 0$

Hence for each $\epsilon > 0$, $\exists$ $\delta > 0$ s.t if $0 < |z-z_0| < \delta$ then,

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$$

<u>Claim</u>: Say L is a line segment from z_1 to z_2

Then $\int_L z\,dz = \frac{1}{2}(z_2^2 - z_1^2)$

<u>Proof</u>: For a curve $C = \{z(t) \mid a \le t \le b\}$; Parameterize $L = \{(1-t)z_1 + t z_2 \mid 0 \le t \le 1\}$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

So, $$\int_L z\,dz = \int_0^1 ((1-t)z_1 + t z_2) \cdot (-z_1 + z_2)\,dt$$

$$= (-z_1 + z_2)\left[z_1 \int_0^1 (1-t)\,dt + z_2 \int_0^1 t\,dt\right]$$

$$= \frac{1}{2}(-z_1 + z_2)(z_1 + z_2) = \frac{1}{2}(z_2^2 - z_1^2)$$

Hence for a triangle $\int_\Delta z\,dz = \int_{L_1} z\,dz + \int_{L_2} z\,dz + \int_{L_3} z\,dz = 0$.

Similarly $\int_\Delta 1\,dz = 0$.

So, $$\int_{\Delta_{n,1}} (f(z) - f(z_0) - (z - z_0) f'(z_0))\,dz = \int_{\Delta_{n,1}} f(z)\,dz - f(z_0) \underbrace{\cancel{\int_{\Delta_{n,1}} 1\,dz}}_{0}$$

$$- f'(z_0) \underbrace{\cancel{\int_{\Delta_{n,1}} z\,dz}}_{0} + f'(z_0) z_0 \underbrace{\cancel{\int_{\Delta_{n,1}} 1\,dz}}_{0}$$

$$= \int_{\Delta_{n,1}} f(z)\,dz.$$

Now pick $\epsilon < \frac{M}{l^2}$ where l is the perimeter of Δ_0.

Then the perimeter of $\Delta_{1,n}$ is $\frac{l}{2^n}$.

Now choose n large enough so that $\Delta_{1,n} \subseteq B_\delta(z_0)$.

where $|z - z_0| < \delta$ implies $|f(z) - f(z_0) - (z-z_0)f'(z_0)| < \epsilon |z - z_0|$

Then, for z on $\Delta_{1,n}$, $|f(z) - f(z_0) - (z - z_0)f'(z_0)| < \epsilon |z - z_0|$

$$\Rightarrow |f(z) - f(z_0) - (z - z_0) f'(z_0)| < \frac{M}{l^2} l_n = \frac{M}{l 2^n}$$

Hence,

$$\left| \int_{\Delta_{n,1}} f(z)dz \right| = \left| \int_{\Delta_{n,1}} f(z) - f(z_0) - f'(z_0)(z - z_0) dz \right| < \underbrace{\frac{M}{l 2^n}}_{\text{Max value.}} \cdot \underbrace{l_n}_{\text{perimeter of curve}} = \frac{M}{4^n}$$

$$\Rightarrow \left| \int_{\Delta_{n,1}} f(z)dz \right| < \frac{M}{4^n} \quad \divideontimes$$

Summary of proof idea.

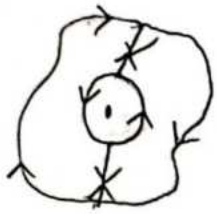
> Use nested triangles to find z_0 that lies inside some n^{th} Δ^{le}. $\int_{\Delta_{nth}} f(z)dz \geq \frac{M}{4^n}$. Pick n^{th} Δ^{le} inside $B_\delta(z_0)$ where δ guarantees that $\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon$ for $\epsilon < \frac{M}{l^2}$. Use this to find an upper bound on $|f(z) - f(z_0) - (z - z_0)f'(z_0)|$, since $\epsilon < \frac{M}{l^2}$ & $|z - z_0| < l_n$

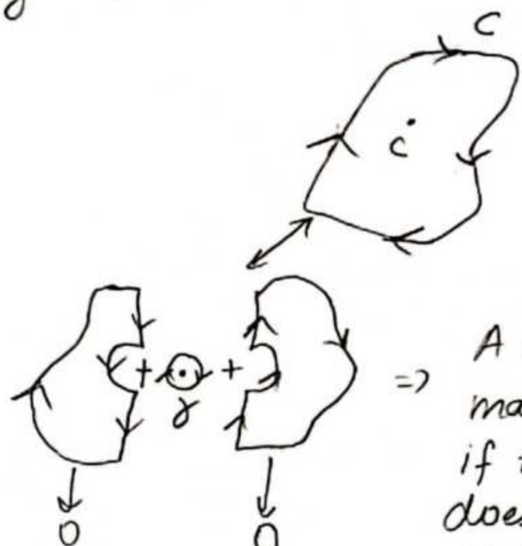
Show that $\int_{\Delta_n} f(z)dz = \int f(z) - f(z_0) - (z - z_0)f'(z_0)$ ✱ key step!. Use $\int < M \times C$ to show that $\int_{\Delta_{nth}} f(z)dz < \frac{M}{4^n} \Rightarrow \divideontimes$.

Ex: Let C be a piecewise smooth curve that is closed & simple. (simple ⇒ not self intersecting) containing 0.

Compute $\int_C \frac{dz}{z}$ assuming C is

Solution:





⇒ A curve can be made into a circle if the transition does not go over a pole.

$$\Rightarrow \int_C \frac{dz}{z} = \int_\gamma \frac{dz}{z}$$

Let $\gamma = \{\cos t \overset{\text{clockwise}}{-} i\sin t \mid 0 \le t \in 2\pi\}$

Then
$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{1}{z(t)} z'(t)\,dt = \int_0^{2\pi} \frac{1}{\cos t - i\sin t}(-\sin t - i\cos t)\,dt$$

$$= \int_0^{2\pi} \frac{(\cos t + i\sin t)(-\sin t - i\cos t)\,dt}{\cos^2 t + \sin^2 t}$$

$$= \int_0^{2\pi} \cancel{-\sin t\cos t}^{\,0} + \cancel{\sin t\cos t}^{\,0} - i(\cancel{\sin^2 t + \cos^2 t}^{\,1})\,dt$$

$$= \int_0^{2\pi} -i\,dt = -2\pi i$$

In general,
$$\int_C \frac{dz}{z - z_0} = -2\pi i \text{ if } C \text{ is clockwise}$$
$$\int_\gamma \frac{dz}{z - z_0} = 2\pi i \text{ if } C \text{ is counterclockwise}$$
& C containing z_0.

Theorem : Cauchy Integral Formula.

Let $f(z)$ be analytic in G and $C \subseteq G$ piecewise smooth closed counterclockwise simple curve and the interior region of C is in G*.

int(C)

Then for any $z_0 \in \text{int}(C)$

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} \, dz$$

Note to self.

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + f'(z_0) + f''(z_0)(z-z_0) \dots$$

Therefore residue is $f(z_0)$ & $\int_C = 2\pi i \, f(z_0)$!

Proof: $g(z) = \dfrac{f(z)}{z-z_0}$ is analytic in G except at z_0

Let γ_R be any circle of radius R around z_0 s.t $\overline{B_R(z_0)} \subseteq G$.

Then,
$$\int_C g(z)\,dz = \int_{\gamma_R} g(z)\,dz \quad \forall \; R > 0.$$

Hence
$$\int_C g(z)\,dz = \lim_{R \to 0} \int_{\gamma_R} g(z)\,dz$$

and we just have to show $\lim_{R\to 0} \int_{\gamma_R} g(z)\,dz = 2\pi i \, f(z_0)$.

In other words, given $\epsilon > 0$, we need to find $\delta > 0$ s.t $\forall R < \delta$, we have $\left| \int_{\gamma_R} g(z)dz - 2\pi i f(z_0) \right| < \epsilon$.

Notice,

$$\left| \int_{\gamma_R} g(z)dz - 2\pi i f(z_0) \right| = \left| \int_{\gamma_R} g(z)dz - f(z_0) \int_{\gamma_R} \frac{dz}{z-z_0} \right|$$

$$= \left| \int_{\gamma_R} \frac{f(z)}{z-z_0} dz - f(z_0) \int \frac{dz}{z-z_0} \right|$$

$$= \left| \int_{\gamma_R} \frac{f(z)-f(z_0)}{z-z_0} dz \right|$$

For $z \in \gamma_R$, $|z-z_0| = R$.

Also f is continuous & hence uniformly continuous in $\overline{\text{int}(C)}$

Thus for any given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t if $|z-z_0| < \delta$ then

$|f(z) - f(z_0)| < \frac{\epsilon}{2\pi}$. Thus $\left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \frac{\epsilon}{2\pi R}$ on γ_R.
for $R < \delta$

So $\left| \int \frac{f(z)-f(z_0)}{z-z_0} dz \right| < \frac{\epsilon}{2\pi R} \cdot \text{length}(\gamma_R) = \epsilon$ for $R < \delta$.

<u>Idea of proof</u>: Make circle small enough so that $R < \delta \Rightarrow |z-z_0| < R < \delta$

$\Rightarrow \left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \frac{\epsilon}{2\pi} \cdot \frac{1}{R}$ & use $\int |f| < f_{max} \cdot C$. to show $\left| \int \frac{f(z)}{z-z_0} - f(z_0) . 2\pi i \right| <$

<u>Theorem</u>: Say $f(z)$ is analytic in G. Then $f(z)$ is infinitely differentiable in G and

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz} \quad \forall z_0 \in G,\ n \in \mathbb{N}\ \&\ z_0 \in \operatorname{int}(C)$$

C is a piecewise smooth closed simple counterclockwise curve.

<u>Proof</u> (by induction)

For $n=0$, we showed $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$

Say $f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^n}\,dz$

Then

$$f^{(n)}(z_0) = \lim_{z\to z_0} \frac{f^{(n-1)}(z) - f^{(n-1)}(z_0)}{z-z_0} = \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} \quad (1)$$

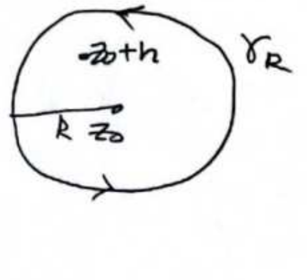
Now for $R>0$ small enough, $\overline{B_R(z_0)} \subseteq \operatorname{int}(C)$.

and so

$$\int_C \frac{f(z)}{(z-z_0)^k}\,dz = \int_{\gamma_R} \frac{f(z)}{(z-z_0)^k}\,dz \quad \forall \quad k \in \mathbb{N}$$

Let $|h|<R$ so

$z_0+h \subseteq \operatorname{int}(\gamma_R) \subseteq \operatorname{int}(C)$



Hence by induction hypothesis $f^{(n-1)}(z_0+h) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(z)}{(z-(z_0+h))^n} dz$

and similar to before, $\int_C \frac{f(z)}{(z-z_0-h)^n} dz = \int_{\gamma_R} \frac{f(z)}{(z-z_0-h)^n} dz$

$$\Rightarrow f^{(n-1)}(z_0+h) = \frac{(n-1)!}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-z_0-h)^n} dz$$

$$f^{(n)}(z_0) = \lim_{h\to 0} \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h}$$ Substitute

$$f^{(n)}(z_0) = \lim_{h\to 0} \frac{(n-1)!}{2\pi i h} \left[\int_{\gamma_R} \left(\frac{f(z)}{(z-z_0-h)^n} - \frac{f(z)}{(z-z_0)^n} \right) dz \right]$$

$$= \lim_{h\to 0} \frac{(n-1)!}{2\pi i h} \left[\int_{\gamma_R} f(z) \left(\frac{(z-z_0)^n - (z-z_0-h)^n}{(z-z_0)^n (z-z_0-h)^n} \right) dz \right] \quad (2)$$

Let $a = z - z_0$; $b = z - z_0 - h$

$$\Rightarrow a^n - b^n = \underbrace{(a-b)}_{h} \left(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1} \right) = h \left(\sum_{k=0}^{n-1} a^{n-1-k} b^k \right)$$

Now, $f(z)$ continuous in $\overline{\text{int}(\gamma_R)}$, so $M = \max_{z \in \overline{\text{int}(\gamma_R)}} |f(z)|$ exists

Also, $|a-b| = h$ and $|z-z_0| = R$ so $|z-z_0-h| \le R+h < 2R$

Also, $|z-z_0-h| \geq ||z-z_0| - |h|| = R-h.$

From ① & ②

$$\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} = \frac{(n-1)!}{2\pi i h}\left[\int_{\gamma_R} f(z)\left(\frac{(z-z_0)^n-(z-z_0-h)^n}{(z-z_0)^n(z-z_0-h)^n}\right)dz\right]$$

Hence,

$$\left|\frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} - \frac{n!}{2\pi i}\int_{\gamma_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$

$$= \left|\frac{(n-1)!}{2\pi i h}\int_{\gamma_R} f(z)\left(\frac{(z-z_0)^n-(z-z_0-h)^n}{(z-z_0)^n(z-z_0-h)^n}\right)dz - \frac{n!}{2\pi i}\int_{\gamma_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$

$$= \frac{(n-1)!}{2\pi}\left|\int_{\gamma_R} f(z)\left[\frac{(z-z_0)\left[(z-z_0)^n-(z-z_0-h)^n\right] - hn(z-z_0-h)^n}{h(z-z_0)^{n+1}(z-z_0-h)^n}\right]dz\right|$$

$$= \frac{(n-1)!}{2\pi}\left|\int_{\gamma_R} f(z)\left[\frac{(z-z_0)\left[h\cdot\sum_{k=0}^{n-1}(z-z_0)^{n-1-k}(z-z_0-h)^k\right] - hn(z-z_0-h)^n}{h(z-z_0)^{n+1}(z-z_0-h)^n}\right]dz\right|$$

$$= \frac{(n-1)!}{2\pi}\left|\int_{\gamma_R} f(z)\left[\frac{(z-z_0)^n + (z-z_0)^{n-1}(z-z_0-h)+\dots+(z-z_0)(z-z_0-h)^{n-1} - n(z-z_0-h)^n}{(z-z_0)^{n+1}(z-z_0-h)^n}\right.\right.$$

Now, $\left[\sum_{k=0}^{n-1}(z-z_0)^{n-k}(z-z_0-h)^k\right] - n(z-z_0-h)^n$

$$= \sum^{n-1}\left[(z-z_0)^{n-k}(z-z_0-h)^k - (z-z_0-h)^n\right]$$

$$= \sum_{k=0}^{n-1} (z-z_0-h)^k \left[(z-z_0)^{n-k} - (z-z_0-h)^{n-k} \right] \leftarrow a^{n-k} - b^{n-k}$$

use formula for $a^n - b^n$

$$= \sum_{k=0}^{n-1} (z-z_0-h)^k \, h \left(\sum_{j=0}^{n-k-1} (z-z_0)^{n-k-1-j} (z-z_0-h)^j \right)$$

$$\Rightarrow RHS = \frac{(n-1)!}{2\pi} \left| \int_{\gamma_R} \underbrace{f(z)}_{<M} \left[\frac{\sum_{k=0}^{n-1} \underbrace{(z-z_0-h)^k}_{<2R} h \left(\sum_{j=0}^{n-k-1} \underbrace{(z-z_0)}_{R}^{n-k-1-j} \underbrace{(z-z_0-h)}_{<2R}^j \right)}{\underbrace{(z-z_0)}_{R}^{n+1} \underbrace{(z-z_0-h)^n}_{\geq R-|h|}} \right] \right|$$

$$\Rightarrow RHS \leq \frac{(n-1)!}{2\pi} Mh \frac{\sum_{k=0}^{n-1} (2R)^k \sum_{j=0}^{n-k-1} R^{n-k-1-j} (2R)^j}{R^{n+1} (R-|h|)^n}$$

LHS = RHS

$$\Rightarrow \left| \frac{f^{(n-1)}(z_0+h) - f^{(n-1)}(z_0)}{h} - \frac{n!}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}} dz \right|$$

$$\leq \frac{(n-1)!}{2\pi} Mh \frac{\sum_{k=0}^{n-1} (2R)^k \sum_{j=0}^{n-k-1} R^{n-k-1-j} (2R)^j}{R^{n+1} (R-|h|)^n}$$

$$\longrightarrow 0 \quad \text{as } h \to 0$$

Indefinite Integrals.

$g : \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ s.t $G'(x) = g(x)$

$$G(x) = \int g(x)\,dx + C$$

$$G(x) = \int_a^x g(x)\,dx + \underbrace{C}_{= G(a)} \rightarrow \text{Fund Thm of Calc.}$$

theorem*: Let $f(z)$ analytic in G. $\rightarrow$ Simply connected.

Let $F(z) = \int_{z_0}^{z} f(w)\,dw$ denotes integral along any curve from z_0 to z (p.w smooth) then $F(z)$ is a single valued function s.t $F'(z) = f(z)$.

Proof: **Claim①** $F(z)$ is a function.

Proof: Say C_1 and C_2 are p.w. smooth curves from z_0 to z. then $C_1 \cup C_2^-$ is a p.w. smooth closed curve.

Hence $\int_{C_1 \cup C_2^-} f(w)\,dw = 0 \Rightarrow \int_{C_1} f(w)\,dw = \int_{C_2} f(w)\,dw = \int_{z_0}^{z} f(w)\,dw$ //

Claim② $F'(z) = f(z)$.

Proof: $F(z+h) - F(z) = \int_{z_0}^{z_0+h} f(w)\,dw - \int_{z_0}^{z} f(w)\,dw$

Let C be from z_0 to z, C_h be from z to $z+h$.

Let $C \cup C_h$ is pw smooth from z_0 to $z+h \to \tilde{C} = C \cup C_h$.

So,
$$\int_{z_0}^{z+h} f(w)\,dw = \int_{\tilde{C}} f(w)\,dw = \int_{C_h} f(w)\,dw + \int_C f(w)\,dw$$

$$\Rightarrow F(z+h) - F(z) = \int_{C_h} f(w)\,dw + \cancel{\int_C f(w)\,dw} - \cancel{\int_C f(w)\,dw}.$$
$$= \int_z^{z+h} f(w)\,dw$$

Now, also $\displaystyle\int_z^{z+h} f(z)\,dw = f(z)\int_z^{z+h} dw = f(z)\cdot h.$

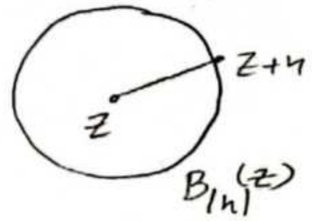
Hence, $$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \frac{1}{|h|}\left|\int_z^{z+h} f(w)\,dw - \int_z^{z+h} f(z)\,dw\right|$$
$$= \frac{1}{|h|}\left|\int_z^{z+h} (f(w) - f(z))\,dw\right|$$

f is analytic in G so for h small enough, $\overline{B_{|h|}}(z) \subseteq G$ and so f uniformly continuous on $\overline{B_{|h|}}(z)$.

So for $\epsilon > 0$, $\exists\ \delta > 0$ s.t. if $|w - z| < \delta$ then $|f(w) - f(z)| < \epsilon$

Pick any $|h| < \delta$.

Since $\overline{B_h(z)} \subseteq G$, the straight segment from z to $z+h$ is in G and has length $|h|$.



$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \leq \frac{1}{|h|} \epsilon |h| = \epsilon$$

$\blacksquare$

Theorem** If $F'(z) = f(z)$ and $G'(z) = f(z)$ then $F(z) = G(z) + C$ for a constant $C \in \mathbb{C}$.

Proof: Say $F(z) - G(z) = u(x,y) + i\, v(x,y)$

So, $\frac{d}{dz}\big(F(z) - G(z)\big) = f(z) - f(z) = 0$ and so

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial v} = 0 = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} = 0 \Rightarrow u, v \text{ are constants}$$ $\blacksquare$

Notation: For any $F(z)$ s.t. $F'(z) = f(z)$ and $f(z)$ analytic we have $\int_{z_0}^{z} f(w)\, dw = F(z) - F(z_0)$

Since $F(z) = \int_{z_0}^{z} f(w)\, dw + C$ so $F(z_0) = \int_{z_0}^{z_0} f(w)\, dw + C$ with $\int_{z_0}^{z_0} f(w)\, dw = 0$

Note: The above 2 theorems. * & ** can be generalized to continuous $f(z)$

Theorem (Morera): $f(z)$ is continuous in G and $\int_C f(z)\,dz = 0$ $\forall$ p.w smooth closed C in G. Then f is analytic.

Proof: Let $F(z) = \int_{z_0}^{z} f(z)\,dz$. Then $F'(z) = f(z)$ (from thm*)

So $F(z)$ analytic (since $F'(z)$ exists). But the $F^{(n)}(z)$ exists $\forall n \in \mathbb{N}$ $\Rightarrow$ $f'(z) = F''(z)$ $\Rightarrow$ f analytic $\blacksquare$

→ This is "almost" the converse of Cauchy's Thm. Here G need not be simply connected but to go from analytic $\rightarrow \oint = 0$ we need the domain to be simply connected.

Complex Series

We define $\sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} \sum_{k=1}^{n} z_k$

$\uparrow$ infinite series $\qquad$ $\uparrow$ n^{th} partialsum (call it S_n)

<u>Def</u>: We say $\sum_{k=1}^{\infty} z_k$ <u>converges</u> if the limit $\lim_{n\to\infty} \sum_{k=1}^{n} z_k$ exists. If not, $\sum_{k=1}^{\infty} z_k$ <u>diverges</u> $\hookrightarrow$ also includes oscillating sequences

<u>Ex</u>: $\sum_{k=0}^{\infty} w^k$ converges if $|w|<1$ and diverges if $|w|\geq 1$

<u>Proof</u>: $S_n = \sum_{k=0}^{n} w^k = 1+w+w^2+\dots+w^n$

$$(1-w)\sum_{k=0}^{n} w^k = \sum_{k=0}^{n} w^k - w^{k+1} = 1-\cancel{w}+\cancel{w}-\cancel{w^2}+\cancel{w^2}-\dots-\cancel{w^n}+\cancel{w^n}-w^{n+1}$$
$$= 1-w^{n+1}$$

$S_n = \frac{1-w^{n+1}}{1-w}$ So $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-w^{n+1}}{1-w}$ $\qquad$ $1-w\neq 0$ since $|w|<1$

If $|w|<1$ then $\lim_{n\to\infty} |w|^{n+1}=0$ so $\lim_{n\to\infty} w^{n+1}=0$ $\rightarrow$ only true if lim$\to 0$

Thus $\boxed{\lim_{n\to\infty} S_n = \frac{1}{1-w}}$

Theorem: If $\sum_{k=1}^{\infty} z_k$ converges then $\lim_{k\to\infty} z_k = 0$
↳ terms shrinking.

Proof: $z_n = \sum_{k=1}^{n} z_k - \sum_{k=1}^{n-1} z_k = S_n - S_{n-1}$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \sum_{k=1}^{\infty} z_k - \sum_{k=1}^{\infty} z_k = 0 \quad \blacksquare$$

This step works since both limits exist.

Ex:- Say $|w| \geq 1$. Then $\lim_{n\to\infty} |w|^n = \begin{cases} 1 & |w| = 1 \\ \infty & |w| > 1 \end{cases} \neq 0$

$\Rightarrow \sum_{k=0}^{\infty} w^k$ diverges → This is the NOT statement of the above thm & not the converse.

Def: If $\sum_{k=1}^{\infty} |z_k|$ converges, then we say $\sum_{k=1}^{\infty} z_k$ is absolutely convergent. If $\sum_{k=1}^{\infty} z_k$ converges but $\sum_{k=1}^{\infty} |z_k|$ does not, then $\sum_{k=1}^{\infty} z_k$ is conditionally convergent.

Remark: We can see if $\sum_{k=1}^{\infty} |z_k|$ converges then $\sum_{k=1}^{\infty} z_k$ converges.

Let $\tilde{S}_n = \sum_{k=1}^{n} |z_k|$. Since $\{\tilde{S}_n\}$ converges, it is Cauchy so given $\epsilon > 0$, $\exists N > 0$ s.t if $n, m \geq N$ then $|S_n - S_m| = \left| \sum_{k=1}^{n} z_k - \sum_{k=1}^{m} z_k \right|$
$= \left| \sum_{k=m+1}^{n} z_k \right| \leq \sum_{k=m+1}^{n} |z_k| = \tilde{S}_n - \tilde{S}_m = |\tilde{S}_n - \tilde{S}_m| < \epsilon$. Hence $\{S_n\}$ converges
$\Rightarrow \sum^{\infty} z_k$ converges //

Ex: $z_k = (-1)^{k+1}\frac{1}{k}$ is conditionally convergent as a series

$$\sum_{k=1}^{n} z_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{(-1)^{n+1}}{n}$$

Now, say $\epsilon > 0$, let $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \epsilon$. If $n, m \geq N$ then

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} (-1)^{k+1}\frac{1}{k} \right| = \left| \frac{(-1)^{m+2}}{m+1} + \frac{(-1)^{m+3}}{m+2} + \dots + \frac{(-1)^{n+1}}{n} \right|$$

$$\leq \frac{1}{m+1} \text{ (why?)}$$

$$< \frac{1}{N} < \epsilon \quad //$$

Say $m = 9$, $n = 15$

$$\left| -\frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \frac{1}{14} + \frac{1}{15} \right|$$

$$= \left| \underbrace{\frac{1}{10} - \frac{1}{11}}_{>0} + \underbrace{\frac{1}{12} - \frac{1}{13}}_{>0} + \underbrace{\frac{1}{14} - \frac{1}{15}}_{>0} \right|$$

$$= \frac{1}{10} + \underbrace{\left(\underbrace{-\frac{1}{11} + \frac{1}{12}}_{<0} \underbrace{- \frac{1}{13} + \frac{1}{14}}_{<0} \underbrace{- \frac{1}{15}}_{<0} \right)}_{<0}$$

$$< \frac{1}{10}$$

However, $\sum_{k=1}^{\infty} |z_k| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges

Proof: Suppose $\tilde{S}_n = \sum_{k=1}^{n} \frac{1}{k}$ converges.

then any subsequence should converge to the same value.

So $\lim_{n\to\infty} \tilde{S}_{2^n} = \lim_{n\to\infty} \tilde{S}_n$ where $\tilde{S}_{2^n} = \sum_{k=1}^{2^n} \frac{1}{k}$

$$= \underbrace{(1)}_{\text{upto } 2^0} + \underbrace{\left(\frac{1}{2}\right)}_{\text{upto } 2^1} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\text{upto } 2^2} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\text{upto } 2^3} + \dots$$

$$+ \left(\frac{1}{2^{n-1}+1} + \frac{1}{2^{n-1}+2} + \dots + \frac{1}{2^n} \right)$$

The last term is $\left(\underbrace{\frac{1}{2^{n-1}+1}}_{\geq \frac{1}{2^n}} + \underbrace{\frac{1}{2^{n-1}+2}}_{\geq \frac{1}{2^n}} + \cdots + \underbrace{\frac{1}{2^n}}_{\geq \frac{1}{2^n}}\right)$ $\quad 2^n \to 2^{n-1}+2^{n-1}$

$$\Rightarrow \geq \underbrace{\frac{1}{2^n} + \frac{1}{2^n} + \cdots + \frac{1}{2^n}}_{2^{n-1} \text{ times}} = \frac{1}{2}$$

So $a_n \geq \frac{n}{2}$ since $\tilde{S}_{2^n} \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = 1 + \frac{n}{2}$

$\Rightarrow \lim_{n \to \infty} \tilde{S}_{2^n} = +\infty$ diverges $\divideontimes$ ■

<u>Fact:</u> If $\sum z_k$ and $\sum w_k$ converge then

$$\sum (z_k \pm w_k) = \sum z_k \pm \sum w_k \text{ converges.}$$

<u>Def:</u> A <u>rearrangement</u> of a series $\sum_{k=1}^{\infty} z_k$ is a series

$\sum_{k=1}^{\infty} z_{\phi(k)}$ where $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection.

<u>theorem:</u> If $\sum z_k$ is absolutely convergent, then $\sum z_{\phi(k)}$ converges for any rearrangement and $\sum z_{\phi(k)} = \sum z_k$

Series of Functions

<u>Def</u>:- By $\sum_{k=1}^{\infty} f_k(z)$, we mean for each $z_0 \in \mathbb{C}$,

$\sum_{k=1}^{\infty} f_k(z_0) = f_1(z_0) + f_2(z_0) + \dots$ is the series where

each term is f_k evaluated at $z_0 \in \mathbb{C}$.

If for each $z \in G \subseteq \mathbb{C}$ $\sum_{k=1}^{\infty} f_k(z)$ converges as a series in $\mathbb{C}$

then $f(z) = \sum_{k=1}^{\infty} f_k(z)$ is a well defined function on G

with $f(z_0) = f_1(z_0) + f_2(z_0) + \dots$ for each $z_0 \in G$.

<u>Ex</u>:- Let $f_k(z) = z^k$ for $k = 0, 1, \dots$ on $G = B_1(0)$ $(|z| < 1)$.

So for $z \in B_1(0)$,

$$\sum_{k=0}^{\infty} f_k(z) = \lim_{n\to\infty} \sum_{k=0}^{n} f_k(z) = \lim_{n\to\infty} \sum_{k=0}^{n} z^k = \lim_{n\to\infty} \frac{1 - z^{n+1}}{1-z} = \frac{1}{1-z}$$

So $\sum_{k=0}^{\infty} f_k(z) = \frac{1}{1-z}$ for $z \in B_1(0)$ //

> Let's call $S_n(z) = \sum_{k=1}^{n} f_k(z)$ n^{th} partial sum.

Def: $\sum_{k=1}^{\infty} f_k(z)$ <u>converges uniformly</u> on a set G if given any $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t $|S_n(z) - S(z)| < \epsilon$ $\forall\, n \geq N$ and $\forall\, z \in G$ where $S(z) = \sum_{k=1}^{\infty} f_k(z)$.

So for every $\epsilon > 0$, eventually, $\sum_{k=1}^{N} f_k(z)$ is within ϵ of $S(z)$ independant of $z \in G$.

<u>Intuition</u>: > We have for each $f_k(z)$ a mapping from $z \to f_k(z)$. Adding up all these ∞ mappings gives us $f(z)$. If $f(z)$ is finite at all z then the sum of $f_k(z)$ is convergent.

> <u>Uniform convergence</u>: If you pick a large enough value of k then every point z in $f_k(z)$ should be within ϵ of the corresponding point on the final value $f(z) \Rightarrow \forall\, z$ on G the values of $f_k(z)$ should be ϵ-close to $f(z)$. It is much stricter!

Ex:- Consider $S(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} f_k(z)$ for $f_k(z) = z^k$, $z \in B_1(0)$

<u>Claim</u>: $\sum_{k=0}^{\infty} f_k(z)$ does not converge uniformly on $B_1(0)$

<u>Proof</u>:- $|S(z) - S_n(z)| = \left| \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} \right| = \left| \frac{z^{n+1}}{1-z} \right| = \frac{|z|^{n+1}}{|1-z|}$

Suppose for $\epsilon > 0$ & $\epsilon < 1$ $\exists N \in \mathbb{N}$ s.t. $\frac{|z|^{n+1}}{|1-z|} < \epsilon$ $\forall n \geq N$ $\forall z \in B_1(0)$

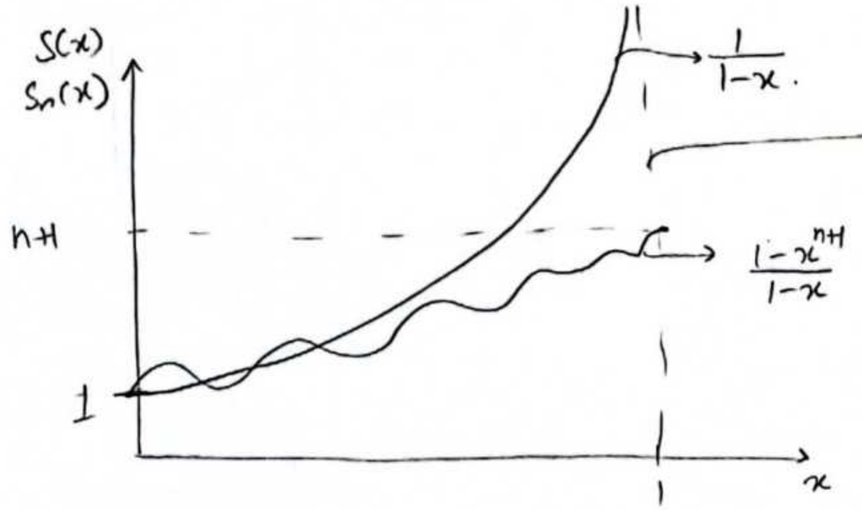
Then let $z_n = 1 - \frac{1}{n} \in B_1(0)$. So it should be that

$$\frac{|z_n|^{n+1}}{|1-z_n|} = n\left(1-\frac{1}{n}\right)^{n+1} < \epsilon \quad \forall \ n \geq N$$

But, $\lim_{n\to\infty} n\left(1-\frac{1}{n}\right)^{n+1} = +\infty$ so $\exists n \geq N$ s.t.

$$|S_n(z_n) - S(z_n)| > \epsilon \quad \text{※}$$

<u>Intuition</u> Say $S(x) = \frac{1}{1-x} \Rightarrow S_n(x) = \frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n$



As $x \to 1$ when n is finite the diff. is ∞ & therefore there is no N such that $|S_n(x) - S(x)| < \epsilon$.

<u>Theorem</u>: If $S(z) = \sum_{k=1}^{\infty} f_k(z)$ is uniformly convergent in G (open domain) and $f_k(z)$ is continuous at $z_0 \in G$ $\forall\, k \in \mathbb{N}$, then $S(z)$ is also continuous in G.

<u>Proof</u>: Say $z_0 \in G$. So $\exists$ $r > 0$ s.t $B_r(z_0) \subseteq G$

Let $\delta < r$ in this proof

By uniform convergence

Say $\epsilon > 0$. So $\exists$ $N \in \mathbb{N}$ s.t if

$n \geq N$ then $|S(z) - S_n(z)| < \epsilon/3$ $\forall$ $n \geq N$, $\forall$ $z \in G$.

In particular $|S(z) - S_N(z)| < \epsilon/3$ $\forall$ $z \in G$

since each is continuous (here it is a finite sum)

Now, $S_N(z) = f_1(z) + \dots + f_N(z)$ is continuous so

$\exists$ $\delta > 0$ s.t $\delta < r$ and $|S_N(z) - S_N(z_0)| < \frac{\epsilon}{3}$ $\forall$ $z \in B_\delta(z_0)$

Hence if $|z - z_0| < \delta$, then

$$|S(z) - S(z_0)| \leq \underbrace{|S(z) - S_N(z)|}_{< \frac{\epsilon}{3}} + \underbrace{|S_N(z) - S_N(z_0)|}_{< \frac{\epsilon}{3}} + \underbrace{|S_N(z_0) - S(z_0)|}_{< \frac{\epsilon}{3}}$$

continuity of finite sum.

$< \epsilon$ $\blacksquare$

uniform convergence

Theorem: Suppose $|f_k(z)| \le a_k$ for $k \in \mathbb{N}, \forall z \in G$ and $\sum_{k=1}^{\infty} a_k$ converges. Then $\sum_{k=1}^{\infty} f_k(z)$ and $\sum_{k=1}^{\infty} |f_k(z)|$ converge uniformly in G..

EX:- $f_k(z) = z^k \Rightarrow$ in $B_1(0)$ $\quad |f_k(z)| \le 1 \quad \Rightarrow a_k = 1$

& $\sum_{k=1}^{\infty} a_k$ diverges.

Intuition:- Each $f_k(z)$ is bounded & if these sequence of bounds converges then $f_k(z)$ & $|f_k(z)|$ are uniformly convergent

Proof:- Step 1: Show $\sum_{k=1}^{\infty} |f_k(z)|$ converges if $z \in G$.

(comparison test)

Proof 1:- Let $\tilde{S}_n = \sum_{k=1}^{n} |f_k(z)|$. Say $\epsilon > 0$. $\exists N \in \mathbb{N}$

s.t $\left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| < \epsilon \quad \forall n,m \ge N$. ← Cauchy criteria.

$$\left| \tilde{S}_n - \tilde{S}_m \right| = \left| \sum_{k=1}^{n} |f_k(z)| - \sum_{k=1}^{m} |f_k(z)| \right| = \sum_{k=m+1}^{n} |f_k(z)|$$ → Finite series comparison test applied here.

$$\le \sum_{k=m+1}^{n} a_k = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| < \epsilon \quad ; \quad n,m \ge N$$

So $\{\tilde{S}_n\}$ is Cauchy $\Rightarrow \{\tilde{S}_n\}$ converges $\Rightarrow \sum_{k=1}^{\infty} |f_k(z)|$ converges

$\Rightarrow \tilde{S}(z) = \sum_{k=1}^{\infty} |f_k(z)|$ for $z \in G$ is well defined.

Step 2 : Uniform convergence

Given $\epsilon > 0$. let $\sum_{k=1}^{\infty} a_k = A < \infty$., $A_n = \sum_{k=1}^{n} a_k$.

$\exists N \in \mathbb{N}$ s.t $|A - A_n| < \epsilon \;\; \forall\, n \geq N$. since $\lim_{n \to \infty} A_n = A$.

$$A - A_n = \left[\lim_{j \to \infty} \sum_{k=1}^{j} a_k\right] - \sum_{k=1}^{n} a_k = \lim_{j \to \infty}\left[\sum_{k=1}^{j} a_k - \sum_{k=1}^{n} a_k\right]$$

$$= \lim_{j \to \infty} \sum_{k=n+1}^{j} a_k = \sum_{k=n+1}^{\infty} a_k.$$

$$|\tilde{S}(z) - \tilde{S}_n(z)| = \left|\lim_{m \to \infty} \sum_{k=n+1}^{m} |f_k(z)|\right| = \lim_{m \to \infty} \sum_{k=n+1}^{m} |f_k(z)|$$

We know $\sum_{k=n+1}^{m} |f_k(z)| \leq \sum_{k=n+1}^{m} a_k \leq \sum_{k=n+1}^{\infty} a_k < \epsilon$

$|\tilde{S}(z) - \tilde{S}_n(z)| < \epsilon$ independant of z // $\blacksquare$

<u>Theorem</u>:- $S(z) = \sum_{k=1}^{\infty} f_k(z)$ converges uniformly on a p.w. smooth curve C. Say $f_k(z)$ continuous on the curve C. $\forall\ k \in \mathbb{N}$. Then $\int_C S(z)\,dz = \sum_{k=1}^{\infty} \int_C f_k(z)\,dz$.

<u>Proof</u>:- $S(z)$ is continuous on C. so $\int_C S(z)\,dz$ exists.

By uniform convergence, given $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t

$$|S(z) - S_n(z)| < \frac{\epsilon}{l} \quad \forall\ n \geq N. \ \forall\ z \in C.$$

l ← length of curve C.

$$\left|\int_C S(z)\,dz - \int_C S_n(z)\,dz\right| = \left|\int_C (S(z) - S_n(z))\,dz\right|$$

Since both integrals are continuous ↳ earlier Thm.

$$< \frac{\epsilon}{l} \cdot l \quad \forall n \geq N.$$

$$< \epsilon.$$

$$\Rightarrow \underbrace{\int_C S(z)\,dz = \lim_{n\to\infty} \int_C S_n(z)\,dz}_{\text{Since we have now shown this the rest follows trivially from finite sum properties.}} = \lim_{n\to\infty} \int_C \sum_{k=1}^{n} f_k(z)\,dz$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \int_C f_k(z)\,dz = \sum_{k=1}^{\infty} \int_C f_k(z)\,dz$$

■

Ch 6 #8

Ratio test: Suppose $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ for $a_n \in \mathbb{C}$.

If $L < 1$ then $\sum_{k=1}^{\infty} a_k$ converges absolutely.

$L > 1$ then $\sum_{k=1}^{\infty} a_k$ diverges.

$L = 1 \Rightarrow$ no information

Alternate: $L = \limsup \left|\frac{a_{n+1}}{a_n}\right|$ & if $L < 1 \to$ converges

$L_m = \sup \left\{ \left|\frac{a_{n+1}}{a_n}\right| \;\middle|\; n > m \right\}$

$\hookrightarrow$ least upper bound, kind of max value.

$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \lim_{m\to\infty} L_m$.

Ex:- $a_n = 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{8}, 0, \frac{1}{16} \ldots$ $\Rightarrow a_n = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2^{n/2}} & n \text{ even} \end{cases}$

$\sup \{ a_n \mid n > m \} = \frac{1}{2^{\left(\frac{m+1}{2}\right)}}$ if m is odd $\Rightarrow \limsup = 0$.

$b_n = 0, 1, 0, 1, 0, 1 \ldots$

$\sup \{ b_n \mid n > m \} = 1 \Rightarrow \limsup = 1$ {no limit but yes limsup}

Proof: $L<1$:

Given $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t $\left| \left|\frac{a_{n+1}}{a_n}\right| - L \right| < \epsilon \quad \forall n \geq N$.

Pick ϵ s.t $L + \epsilon < 1$.

$$\left|\frac{a_{n+1}}{a_n}\right| \leq L + \left| \left|\frac{a_{n+1}}{a_n}\right| - L \right| < L + \epsilon < 1 \qquad \forall n \geq N$$

Consider $\sum_{k=n}^{\infty} |a_k|$. Let $A_m = \sum_{k=n}^{m} |a_k|$. Then for $k \geq n$

$$|a_k| = \left| \frac{a_k}{a_{k-1}} \cdot \frac{a_{k-1}}{a_{k-2}} \cdot \frac{a_{k-2}}{a_{k-3}} \cdots \frac{a_{n+1}}{a_n} \cdot a_n \right|$$

$$= |a_n| \prod_{j=n+1}^{k} \left| \frac{a_j}{a_{j-1}} \right| \longrightarrow L+\epsilon$$

$$< |a_n| \, (L+\epsilon)^{k-n}$$

change of variable $k-n \to k$.

$$A_m < \sum_{k=n}^{m} |a_n| (L+\epsilon)^{k-n} = |a_n| \sum_{k=0}^{m-n} (L+\epsilon)^k$$

$$\Rightarrow \quad A_m < |a_n| \frac{1 - (L+\epsilon)^{m-n+1}}{1 - (L+\epsilon)}$$

Since $L + \epsilon < 1$, $\sum_{k=0}^{\infty} (L+\epsilon)^k$ converges. Apply comparison test to $\sum_{k=n}^{\infty} |a_k|$ & $|a_n| \sum_{k=0}^{m-n} (L+\epsilon)^k$ to show convergence.

Theorem (Weierstrauss)

$S(z) = \sum_{k=1}^{\infty} f_k(z)$ converges in domain G, f_k analytic in G.

and $\sum_{k=1}^{\infty} f_k(z)$ converges uniformly on every $\bar{D} \subseteq G$ that is closed/bounded. then ① $S(z)$ analytic in G.

② $S^{(n)}(z) = \sum_{k=1}^{\infty} f_k^{(n)}(z) \quad \forall n \in \mathbb{N}, z \in G$

and converges uniformly in each $\bar{D} \subseteq G$.

<u>Proof</u>: ① Say $z_0 \in G$. $\exists R > 0$ s.t $\overline{B_R(z_0)} \subseteq G$.

Let γ_R = boundary of $\overline{B_R(z_0)} \leftarrow$ circle.

Since $S(z) = \lim_{j \to \infty} \sum_{k=1}^{j} f_k(z)$ converges uniformly on γ_R. and $\overline{B_R(z_0)}$.

Then $\frac{n!}{2\pi i} \frac{S(z)}{(z - z_0)^{n+1}} = \lim_{j \to \infty} \sum_{k=1}^{j} \frac{n!}{2\pi i} \frac{f_k(z)}{(z-z_0)^{n+1}}$ $\forall n \in \mathbb{N}$, $n \geq 0$

is uniformly convergent on γ_R.

Why? Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\left|S(z) - \sum_{k=1}^{j} f_k(z)\right| < \frac{2\pi R^{n+1}}{n!}\epsilon$

$\forall j \geq N$, $\forall z \in \gamma_R$. Hence $\left| \frac{n!}{2\pi i} \frac{S(z)}{(z-z_0)^{n+1}} - \sum_{k=1}^{j} \frac{n!\, f_k(z)}{2\pi i (z-z_0)^{n+1}} \right|$

$$\Rightarrow \frac{n!}{2\pi \underbrace{|z-z_0|}_{R}^{n+1}} \underbrace{\left| s(z) - \sum_{k=1}^{j} f_k(z) \right|}_{\leq \frac{2\pi R^{n+1}\epsilon}{n!}} < \epsilon. \ \forall j \geq N \ \& \ z \in \gamma_R.$$

Since f_k is analytic, then f_k continuous. $\rightarrow$ continuous on γ_R

By past theorem, $\displaystyle\int_{\gamma_R} \frac{n!}{2\pi i} \frac{s(z)}{(z-z_0)^{n+1}} dz = \sum_{k=1}^{\infty} \int_{\gamma_R} \frac{n!}{2\pi i} \frac{f_k(z)}{(z-z_0)^{n+1}} dz$

$$n=0 \Rightarrow \frac{1}{2\pi i}\int \frac{s(z)}{z-z_0} dz = \sum_{k=1}^{\infty} \int_{\gamma_R} \frac{1}{2\pi i}\left(\frac{f_k(z)}{z-z_0}\right) dz$$

$$= \sum_{k=1}^{\infty} f_k(z_0) \leftarrow$$ f_k analytic so we use Cauchy formula.

$$= s(z_0)$$

We have shown $\displaystyle s(z_0) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{s(z)}{z-z_0} dz$

Why does this imply $s(z)$ analytic?

[Recall the induction proof to show $\displaystyle f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}} dz$

The only use of $f(z)$ analytic was to get the $n=0$ step

For $n \geq 1$ we just needed the $n=0$ step and $f(z)$ continuous (and hence bounded) on γ_R]

In earlier proof for $n=0$ case alone we used $f(z)$ as analytic & here we already have the $n=0$ case so we don't need analyticity. We can use the $k-1, k$ induction proof just as before.

We have 1) $n=0$ step

2) f_K continuous and $S(z) = \sum_{K=1}^{\infty} f_K(z)$ converges uniformly on $\bar{B}_R(z_0)$. So $S(z)$ continuous on $\bar{B}_R(z_0)$.

So by Induction argument,

$$S^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma_R} \frac{S(z)}{(z-z_0)^{n+1}} dz .$$

$n=1 \Rightarrow S'(z_0)$ exists $\Rightarrow S(z)$ analytic at z_0. //

By proof above, for $z_0 \in G$

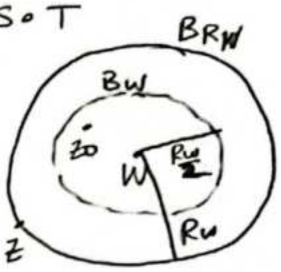
$$S^{(n)}(z_0) = \sum_{K=1}^{\infty} \frac{n!}{2\pi i} \int_{\gamma_R} \frac{f_K(z)}{(z-z_0)^{n+1}} dz$$

$$= \sum_{K=1}^{\infty} f_K^{(n)}(z_0)$$

② It remains to show $\sum_{K=1}^{\infty} f_K^{(n)}(z) = S^{(n)}(z)$ converges uniformly in each $\bar{D} \in G$. closed/bounded.

<u>Proof of ②</u>

Since $\sum_{K=1}^{j} f_K(z)$ converges uniformly to $S(z)$ as $j \to \infty$ in any $\bar{D} \subseteq G$, then it <u>converges uniformly on γ_{R_W}</u>, where γ_{R_W} = boundary of $\overline{B_{R_W}(w)}$ with $R_W > 0$ s.t $\overline{B_{R_W}(w)} \subseteq G$. Let $B_W = B_{\frac{R_W}{2}}(w)$



Then any $z_0 \in B_W$ is in $\mathrm{int}(\gamma_{R_W})$. Hence,

$$\frac{n!}{2\pi i} \int_{\gamma_{R_W}} \frac{f_K(z)}{(z-z_0)^{n+1}}\, dz = f_K^{(n)}(z_0) \quad \text{and} \quad \frac{n!}{2\pi i} \int_{\gamma_{R_W}} \frac{S(z)}{(z-z_0)^{n+1}} = S^{(n)}(z_0)$$

<u>Claim:</u> $\sum_{K=1}^{\infty} f_K^{(n)}(z)$ converges uniformly to $S^{(n)}(z)$ in B_W.

<u>Proof</u>: Given $\epsilon > 0$, since γ_{R_W} closed/bounded, $\exists\, N_W \in \mathbb{N}$ s.t

$$\left| \sum_{K=1}^{j} f_K(z) - S(z) \right| < \frac{R_W^n \epsilon}{2^{n+1} n!} \quad \forall\ j \geq N_W,\quad z \in \gamma_{R_W}$$

Hence,
$$\left| \sum_{K=1}^{j} \frac{n!}{2\pi i} \int_{\gamma_{R_W}} \frac{f_K(z)}{(z-z_0)^{n+1}}\, dz - \frac{n!}{2\pi i} \int_{\gamma_{R_W}} \frac{S(z)}{(z-z_0)^{n+1}}\, dz \right|$$

$$= \frac{n!}{2\pi} \left| \int_{\gamma_{R_W}} \frac{1}{(z-z_0)^{n+1}} \left(\sum_{k=1}^{j} f_k(z) - S(z) \right) dz \right|$$

Now, if $z \in R_W$ and $z_0 \in B_W$ then $|z - z_0| \geq \frac{R_W}{2}$

since $R_W = |z - W| \leq |z - z_0| + |z_0 - W| \leq |z - z_0| + R/_2$

$$\leq \frac{n! \text{ length}(\gamma_{R_W})}{2\pi (R_W/_2)^{n+1}} \cdot \frac{R_W^n \epsilon}{2^{n+1} n!} = \epsilon \quad \forall z_0 \in B_W \quad \forall j \geq N_W$$

Now, $\bar{D} \subseteq \bigcup_{W \in \bar{D}} B_W$ hence, $\exists$ finite subset s.t

$\bar{D} \subseteq B_{W_1} \cup B_{W_2} \cup \ldots B_{W_n}$

Now let $N = \max \{N_{W_1}, N_{W_2}, \ldots, N_{W_m}\}$

Given $\epsilon > 0$, if $j \geq N$ and $z_0 \in \bar{D}$, then

$\exists$ $W_\ell \in \{W_1, \ldots, W_m\}$ s.t $z \in B_{W_\ell}$ and also then

$N \geq N_{W_\ell}$ so

$$\left| \sum_{k=1}^{j} f_k^{(n)}(z_0) - S^{(n)}(z_0) \right| = \left| \sum_{k=1}^{j} \frac{n!}{2\pi i} \int_{\gamma_{R_{W_\ell}}} \frac{f_k(z)}{(z-z_0)^{n+1}} dz - \frac{n!}{2\pi i} \int_{\gamma_{R_{W_\ell}}} \frac{S(z)}{(z-z_0)^{n+1}} dz \right|$$

$$< \epsilon$$

So $\sum_{k=1}^{j} f_k^{(n)}(z)$ converges uniformly to $S^{(n)}(z_0)$ $\blacksquare$

Ex: Let $f_K(z) = z^K$; $S(z) = \frac{1}{1-z}$ → for $|z|<1$

Claim: $\forall\ \overline{D} \subseteq B_1(0)$ closed and bounded $\sum_{K=0}^{\infty} f_K(z)$ converges uniformly. → (This is not closed)

Proof: $\overline{D}$ closed & bounded. Hence $\exists\ r>0$ s.t $\overline{D} \subseteq \overline{B_r(0)}$ and $r<1$. (Why?) → watch lec 10/27/20 (57:22)

Say $\epsilon>0$ For any $z \in \overline{D}$,

$$\left| S(z) - \sum_{K=1}^{n} f_K(z) \right| = \left| \frac{1}{1-z} - \frac{1-z^{n+1}}{1-z} \right| = \left| \frac{z^{n+1}}{1-z} \right|$$

$$\leq \frac{r^{n+1}}{1-r} \quad \text{(Why?)}$$

Pick N s.t $\frac{r^{N+1}}{1-r} < \epsilon$. So $\left| S(z) - \sum_{K=0}^{n} f_K(z) \right| < \epsilon \quad \forall\ z \in \overline{D}$

Hence, since z^K analytic in $B_1(0)$ from Weierstrauss Thm.

$$S'(z) = \sum_{K=0}^{\infty} f_K'(z) \Rightarrow \frac{1}{(1-z)^2} = \sum_{K=1}^{\infty} K z^{K-1} = \sum_{K=0}^{\infty} (K+1) z^K$$

$$S^{(n)}(z) = \sum_{K=0}^{\infty} (K+n)(K+n-1)\dots(K+1) z^K.$$

$$\boxed{\frac{n!}{(1-z)^{n+1}} = \sum_{K=0}^{\infty} \frac{(K+n)!}{K!} z^K}$$

Power Series.

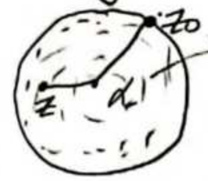
Def: $\sum_{k=0}^{\infty} c_k (z-\alpha)^k \quad c_k, \alpha \in \mathbb{C}$ is a power series.

Note: A power series always converges when $z = \alpha$.

Lemma: Suppose $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges at $z = z_0 \neq \alpha$

Then $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges absolutely for all

z s.t. $|z-\alpha| < |z_0 - \alpha|$



→ converges absolutely inside circle

Proof: Since $\sum_{k=0}^{\infty} c_k (z_0-\alpha)^k$ converges, $\lim_{k\to\infty} c_k (z_0-\alpha)^k = 0$.

Hence $\exists\, M > 0$ s.t. $|c_k (z_0 - \alpha)^k| \le M \quad \forall\, k \in \mathbb{N}$

(Every sequence that converges is bounded)

Thus,

$$\left| c_k (z-\alpha)^k \right| = \left| c_k (z_0-\alpha)^k \right| \left| \frac{z-\alpha}{z_0-\alpha} \right|^k \le M r^k$$

where $r = \dfrac{|z-\alpha|}{|z_0-\alpha|} < 1$.

Since $\sum_{k=0}^{\infty} M r^k = \dfrac{M}{1-r}$ converges, so does $\sum_{k=1}^{\infty} |c_k (z-\alpha)^k|$ $\blacksquare$

Radius of Convergence Theorem.

Suppose $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges for a subset $S \subseteq \mathbb{C}$ but $S \neq \mathbb{C}$

Then $\exists\, R>0$ s.t it converges (absolutely) if $|z-\alpha|<R$ and

diverges if $|z-\alpha|>R$.

<u>Proof</u>: Suppose $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ diverges for $z_0 \in \mathbb{C}$, $z_0 \notin S$

If $|z-\alpha| > |z_0-\alpha|$ and $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges then by previous lemma, $\sum_{k=0}^{\infty} c_k (z_0-\alpha)^k$ converges. Contradiction.

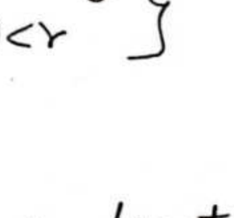
Hence $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ diverges $\forall\, z$ s.t $|z-\alpha| > |z_0-\alpha|$

Thus if $E = \{ r \in \mathbb{R} \mid r \geq 0$ and $\sum_{k=1}^{\infty} c_k (z-\alpha)^k$ converges $\forall\, |z-\alpha|<r \}$

then E has an upper bound, namely $|z_0-\alpha|$

Thus, by least upper bound property of $\mathbb{R}$, E has a least upper bound. Call it R.

So if $|z-\alpha|<R$ the $|z-\alpha|<r_0<R$ for some $r_0<R$



<u>Claim</u>: $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$ converges when $|z-\alpha|<R$

<u>Proof</u>: Suppose not. Then, as above, if $\tilde{z} \in \mathbb{C}$ s.t

$|z-\alpha| < r_0 < |\tilde{z}-\alpha| < R$ then $\sum c_k (\tilde{z}-\alpha)^k$ diverges.

Hence any $\tilde{\gamma} > r_0$ is not in E. So r_0 is an upper bound for E. But $r_0 < R \Rightarrow$ $\divideontimes$

Now, suppose $z_0 \in \mathbb{C}$ and $|z-\alpha| > R$

Claim: $\sum_{k=0}^{\infty} c_k (z_0-\alpha)^k$ diverges when $|z_0 - \alpha| > R$

Proof: Suppose not, let $r_0 = |z_0 - \alpha|$ and let $\tilde{\gamma} = \frac{r_0+R}{2} < r_0$

Say $|z-\alpha| \leq \tilde{\gamma}$. By previous lemma, since $\sum_{K=0}^{\infty} c_k (z_0-\alpha)^k$ converges, so does $\sum_{K=0}^{\infty} c_k (z-\alpha)^k$ $\forall$ $z \in \mathbb{C}$

S.T $|z-\alpha| \leq \tilde{\gamma} < r_0$. But $R < \tilde{\gamma}$ and $\tilde{\gamma} \in E$.

So R is not an upper bound $\divideontimes$

Def: The number R above is the radius of convergence of $\sum_{K=0}^{\infty} c_k (z-\alpha)^k$

Thm: If $S(z) = \sum_{K=0}^{\infty} c_k (z-\alpha)^k$ has radius of convergence R, then it converges uniformly on $\overline{B_r(\alpha)}$ $\forall$ $r < R$ and $S(z)$ is continuous on $B_R(\alpha)$.

Proof: Say $|z-\alpha| \le r$ $(z \in \overline{B_r(\alpha)})$ $r < R$

Consider $\alpha + r \in \overline{B_r(\alpha)}$ so $z - \alpha = r$

Since $\alpha + r \in B_R(\alpha)$ so $\sum_{k=0}^{\infty} c_k(\alpha + r - \alpha)^k$ converges abs.

So $\sum_{k=0}^{\infty} |c_k| r^k$ converges

So given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t if $n \ge N$, then

$$\left| \sum_{k=0}^{\infty} |c_k| r^k - \sum_{k=0}^{n} |c_k| r^k \right| = \sum_{k=n+1}^{\infty} |c_k| r^k < \epsilon$$

So if $|z-\alpha| \le r$, then

$$\left| \sum_{k=0}^{\infty} c_k(z-\alpha)^k - \sum_{k=0}^{n} c_k(z-\alpha)^k \right| \le \sum_{k=n+1}^{\infty} |c_k| |z-\alpha|^k$$

$$\le \sum_{k=n+1}^{\infty} |c_k| r^k < \epsilon \quad \forall n \ge N$$

$\Rightarrow \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ converges uniformly on $\overline{B_r(\alpha)}$

Since $c_k(z-\alpha)^k$ are continuous, then $s(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ is continuous on $\overline{B_r(\alpha)}$ $\forall r < R$ $\Rightarrow s(z)$ is continuous on $B_R(\alpha)$. Since every point $z \in B_R(\alpha)$ is in some $\overline{B_r(\alpha)}$.

■

<u>Thm</u>: Say $S(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$ has radius of convergence R. Then $S(z)$ is analytic in $B_R(\alpha)$ and

$$S^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\dots(k-n+1) c_k (z-\alpha)^{k-n} = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} c_{n+k} (z-\alpha)^k$$

has radius of convergence R.

<u>Proof</u>: <u>By</u> previous theorem, $S(z)$ converges uniformly in each $\overline{B_r(\alpha)}$ for $r<R$. Hence if $\overline{D} \subseteq B_R(\alpha)$ closed and bounded, since $\exists\, r<R$ s.t. $\overline{D} \subseteq \overline{B_r(\alpha)}$, $S(z)$ converges uniformly on each $\overline{D} \subseteq B_R(\alpha)$ closed/bounded.

Thus by Weierstrauss thm, $S(z)$ is analytic and

$$S^{(n)}(z) = \sum_{k=0}^{\infty} \frac{d^n}{dz^n}\left(c_k (z-\alpha)^k\right) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} c_{n+k} (z-\alpha)^k \quad \forall z \in B_R(\alpha)$$

converges uniformly in each $\overline{D} \subseteq B_R(\alpha)$

Now, since $S^{(n)}(z)$ converges $\forall\, z \in B_R(\alpha)$, if R_n is radius of convergence of $S^{(n)}(z)$, then $R_n \geq R$.

We now show $R_n > R$ is not possible by induction.

<u>Base case</u>: $R_0 = R$.

<u>Induction</u>: Suppose $R_{n-1} = R$

<u>Claim</u>: $R_n = R$

Proof: Suppose $R_n > R$. Let C be a p.w. smooth curve from α to a point z_0 in the annulus

$$\{z \in \mathbb{C} \mid R < |z| < R_n\}$$

Since C is closed/bounded, $S^{(n)}(z)$ converges uniformly on C

Hence since each $\frac{(n+k)}{k!} C_{n+k} (z-\alpha)^k$ is continuous,

$$\int_C S^{(n)}(z)\,dz = \sum_{k=0}^{\infty} \int_C \frac{(n+k)!}{k!} C_{n+k} (z-\alpha)^k\,dz \text{ converges}$$

But,

$$\int_C \frac{(n+k)!}{k!} C_{n+k} (z-\alpha)^k\,dz = \frac{(n+k)!}{(k+1)!} C_{n+k} (z_0-\alpha)^{k+1} - \frac{(n+k)!}{(k+1)!} C_{n+k} (\alpha-\alpha)^{k+1}$$

by "antiderivation" thm.

$$= \frac{(n+k)!}{(k+1)!} C_{n+k} (z_0-\alpha)^{k+1}$$

$$\Rightarrow \int_C S^{(n)}(z)\,dz = \sum_{k=0}^{\infty} \frac{(n+k)!}{(k+1)!} C_{n+k} (z_0-\alpha)^{k+1} = \sum_{k=1}^{\infty} \frac{(n-1+k)!}{k!} C_{n-1+k} (z_0-\alpha)^{k}$$

converges.

But this is the series $S^{(n-1)}(z) - (n-1)!\,C_{n-1}$ at z_0

which diverges for $|z_0| > R$. $\divideontimes$ $\blacksquare$

Methods to Determine Radius of convergence.

Def: $\limsup_{n\to\infty} a_n$ for $a_n \in \mathbb{R}$ is defined by $\lim_{n\to\infty} A_n$

where $A_n = \sup \{ a_k \mid k \geq n \} = \sup \{ a_n, a_{n+1}, a_{n+2}, \dots \}$

Ex: 1) $a_n = 1, \frac{1}{2}, 1, \frac{1}{4}, 1, \frac{1}{8}, \dots$

$A_n = 1 \quad \forall \; n \in \mathbb{N} \Rightarrow \limsup_{n\to\infty} a_n = 1$

2) If $\lim_{n\to\infty} a_n$ exists, then $\lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$

Say $\lim a_n = a \in \mathbb{R}$

Given $\epsilon > 0$, $\exists \; N$ s.t $|a_n - a| < \epsilon \;\; \forall \; n \geq N$

So for $n \geq N$, $a_n < a + \epsilon$

$A_N = \sup \{ a_N, a_{N+1}, a_{N+2}, \dots \} \leq a + \epsilon$

$\lim A_N = a$

Rigorous proof is from Real Analysis.

3) $1, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, 16, \dots$

$\sup \{ a_n, a_{n+1}, \dots \} = +\infty \Rightarrow A_n = \infty$

Thm - Cauchy Hadamard

The series $\sum_{n=0}^{\infty} c_n z^n$ has radius of convergence

$R = \frac{1}{l}$ where $l = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $l = +\infty \longleftrightarrow R = 0$

and $l = 0 \longleftrightarrow R = +\infty$.

Ex: 1) Consider $\sum_{n=0}^{\infty} \frac{z^n}{n^s}$ for $s \geq 0$

Then $\sqrt[n]{|c_n|} = \frac{1}{n^{s/n}} = \frac{1}{e^{(s\ln n)/n}}$

$\ulcorner\; n^{s/n} = (n^s)^{1/n} = (e^{s\ln n})^{1/n} = e^{\frac{s\ln n}{n}} \;\lrcorner$

Now as $n \to \infty$, $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$\Rightarrow \lim_{n\to\infty} \frac{1}{e^{\frac{s\ln n}{n}}} = \frac{1}{e^0} = 1 \Rightarrow \limsup \sqrt[n]{|c_n|} = 1 \Rightarrow R = 1$

2) $\sum_{n=0}^{\infty} n!\, z^n$: $\lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} (n!)^{1/n}$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\ln(n!)}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n}(\ln(n) + \dots + \ln(1))}$$

$$\geq \lim_{n\to\infty} e^{\left[\frac{1}{n}\int_1^{n-1} \ln(x)\,dx\right]}$$

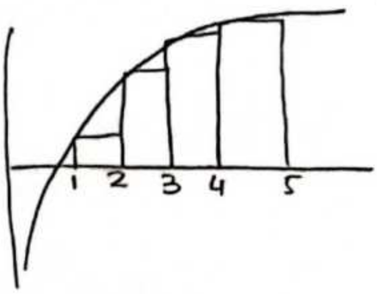
$\hookrightarrow$ e is increasing.

$\ln(1) + \ln(2) + \ldots + \ln(n-1) + \ln(n)$

$= \ln(1)(2-1) + \ln(2)(3-2) + \ldots + \ln(n-1)(n-(n-1))$

$+ \ln(n)(n+1-n)$

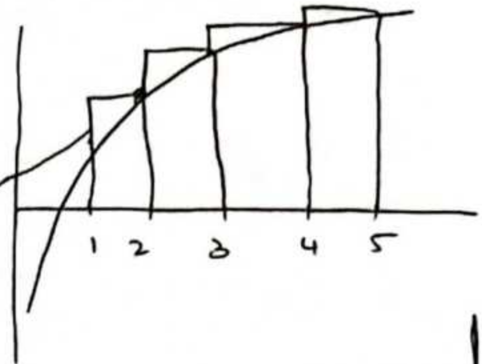
Riemann rectangles



$\int_1^{n+1} \ln(x)\,dx$ is greater

$\Rightarrow \int_1^{n+1} \ln(x)\,dx$ is smaller.

Here this is greater



$= \lim\limits_{n\to\infty} e^{\frac{1}{n}\left[x\ln x - x\Big|_1^{n-1}\right]}$

$= \lim\limits_{n\to\infty} e^{\frac{1}{n}\left(\underbrace{(n-1)}_{1}\underbrace{\ln(n-1)}_{\to\infty} - \underbrace{(n-1)}_{1} + \underbrace{1}_{0}\right)} = \infty$

$\Rightarrow R = 0 \Rightarrow \sum n!\, z^n$ converges only at $z = 0$

3) $\sum c_n z^n$ where $c_n = \begin{cases} 3^{\frac{n}{2}+\frac{1}{2}} & \text{if } n \text{ odd} \\ 2^{n/2} & \text{if } n \text{ even} \end{cases}$

$A_n = \sup\left\{ \sqrt[k]{|c_k|} \;\middle|\; k \geq n \right\} = \sup \left\{ \begin{array}{c|l} 3^{1/2 + 1/2k} & k \text{ odd}, k \geq n \\ 2^{1/2} & k \text{ even}, k \geq n \end{array} \right\}$

$= 3^{\frac{1}{2}+\frac{1}{2n}}$

Series looks

↳ 1, 3, 2, 9, 4, 27, 8, ...

$\Rightarrow \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} A_n = \sqrt{3} \Rightarrow R = \frac{1}{\sqrt{3}}$

Special Functions.

Def Consider $\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Since $\lim_{n\to\infty} (n!)^{1/n} = \infty$,

$\lim_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0$ So $R = \infty$.

Define the exponential function.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

cosine function

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Sine function

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

Since $R = +\infty$ for e^z, $\cos(z)$, $\sin(z)$, these functions are analytic on all of $\mathbb{C}$, ie. entire

Facts: 1) $e^{z_1+z_2} = e^{z_1}e^{z_2}$. Since e^z absolutely convergent we can rearrange order of terms and multiply.

$$e^{z_1}\cdot e^{z_2} = \left(1+z_1+\frac{z_1^2}{2!}+\cdots\right)\left(1+z_2+\frac{z_2^2}{2!}+\cdots\right)$$

$$= \left(1+z_1+\frac{z_1^2}{2!}+\cdots\right)+\left(z_2+z_1z_2+\frac{z_1^2z_2}{2!}+\cdots\right)$$

$$+\left(\frac{z_2^2}{2!}+\frac{z_2^2z_1}{2!}+\cdots\right)$$

$$= 1+(z_1+z_2)+\left(\frac{z_1^2}{2!}+z_1z_2+\frac{z_2^2}{2!}\right)+\cdots$$

$$= 1+(z_1+z_2)+\frac{(z_1+z_2)^2}{2!}+\cdots$$

$$= e^{(z_1+z_2)}$$

2) $$e^z e^{-z} = e^0 = \sum_{n=0}^{\infty}\frac{0^n}{n!} = 1 \Rightarrow e^{-z} = \frac{1}{e^z}$$

$$\Rightarrow \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

3) $$e^{iz} = \sum_{n=0}^{\infty}\frac{(iz)^n}{n!} = 1+iz-\frac{z^2}{2!}-\frac{iz^3}{3!}+\cdots$$

$$= \left(1-\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)+i\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)$$

$$\Rightarrow \boxed{e^{iz} = \cos(z)+i\sin(z).}$$ Euler's Formula.

4) $\cos(-z) = \sum_{n=0}^{\infty} (-1)^n \frac{(-z)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos(z)$

Similarly $\sin(-z) = -\sin(z)$

5) $e^{iz} + e^{-iz} = \cos z + i\sin z + \cos z - i\sin z$

$= 2\cos z$

$\Rightarrow \boxed{\cos z = \frac{e^{iz}+e^{-iz}}{2}}$ & $\boxed{\sin z = \frac{e^{iz}-e^{-iz}}{2i}}$

6) If $z \in \mathbb{C}$. Then $z = r(\cos\theta + i\sin\theta)$

Hence $\boxed{z = re^{i\theta}}$

7) $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin y)$

$\Rightarrow |e^z| = e^x$

8) $e^{i(z_1+z_2)} = e^{iz_1} e^{iz_2} \Rightarrow \cos(z_1+z_2) + i\sin(z_1+z_2)$

$= (\cos z_1 + i\sin z_1)(\cos z_2 + i\sin z_2)$

$\vdots$

$\Rightarrow \cos(z_1+z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$

$\sin(z_1+z_2) = \cos(z_1)\sin(z_2) + \cos(z_2)\sin(z_1)$

9) $\cos(z+2\pi) = \cos z$, $\sin(z+2\pi) = \sin z$

10) $1 = \cos(z-z) = \cos^2 z + \sin^2 z$.

11) $\sin z = 0 \Rightarrow \frac{e^{iz} - e^{-iz}}{2i} = 0 \Rightarrow$ i) $e^{iz} = 0 = e^{-iz}$ or
ii) $e^{iz} = e^{-iz}$

(i) not possible since $|e^{iz}| = |e^{ix-y}| = e^{-y} \neq 0$

(ii) $\Rightarrow e^{2iz} = 1 \Rightarrow e^{2ix-2y} = 1 \Rightarrow e^{-2y} = 1$ & $e^{2ix} = 1$

$e^{-2y} = 1 \Rightarrow y = 0$; $e^{2ix} = \cos 2x + i\sin 2x = 1 \Rightarrow \begin{matrix} \cos 2x = 1 \\ \sin 2x = 0 \end{matrix}$

$\Rightarrow x = \pi k \quad k \in \mathbb{Z}$

$\Rightarrow \boxed{z = \pi k}$ for $\sin(z) = 0$

12) Similarly $\cos(z) = 0 \Leftrightarrow z = \frac{\pi}{2} + \pi k \quad k \in \mathbb{Z}$.

Def: Hyperbolic cosine $\cosh(z) = \frac{e^z + e^{-z}}{2}$
Hyperbolic sine $\sinh(z) = \frac{e^z - e^{-z}}{2}$

13) $\cosh(z) = \cos(iz)$ & $\sinh(z) = -i\sin(iz)$

14) $\Rightarrow \cosh^2 z - \sinh^2 z = 1$

15) Since $R = +\infty$ and each term is analytic

$$\frac{d}{dz} e^z = \frac{d}{dz}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dz}\left(\frac{z^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$$

$$\frac{d}{dz}\cos z = \sum_{n=0}^{\infty} \frac{d}{dz}\left((-1)^n \frac{z^{2n}}{(2n)!}\right) = \sum_{n=1}^{\infty}(-1)^n \frac{z^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!}$$

$$= -\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} = -\sin z$$

Similarly, $\frac{d}{dz}\sin z = \cos z$, $\frac{d}{dz}\cosh z = \sinh z$,

$$\frac{d}{dz}\sinh z = \cosh z$$

<u>Def</u>: $f(z)$ is <u>univalent</u> in domain G (open/connected) if

① $f(z)$ is analytic in G

② $f(z)$ is one-to-one in G (injective)

$$f(z_1) = f(z_2) \iff z_1 = z_2$$

G is called the <u>domain of univalence</u>.

<u>Fact</u>: $f(z)$ univalent in $G \Rightarrow f'(z) \neq 0 \quad \forall z \in G$.

<u>Thm</u>: If $f(z)$ is univalent in domain G (open/connected) then $E = f(G) = \{f(z) \mid z \in G\}$ is open/connected. (domain)

<u>Proof</u>: ① <u>Connected</u>: Say $w_1, w_2 \in E$. So $\exists\, z_1 z_2 \in G$

S.T $w_1 = f(z_1)$ and $w_2 = f(z_2)$

G connected $\Rightarrow \exists$ continuous curve $C = \{z(t) \mid a \le t \le b\}$

S.T $z(t)$ continuous and $z(a) = z_1$; $z(b) = z_2$.

$\Rightarrow$ Since f is analytic, f is also continuous.

$\Rightarrow f(z(t))$ is continuous and $f(z(a)) = f(z_1) = w_1$.

and $f(z(b)) = f(z_2) = w_2$.

$\Rightarrow E$ is connected.

② <u>Open</u>: Say $w_0 \in E$, $w_0 = u_0 + i v_0$. Let $z_0 \in G$

S.T $z_0 = x_0 + i y_0$, $f(z_0) = w_0$.

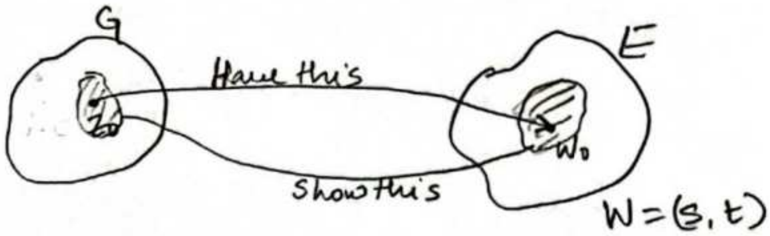
Let $f(x+iy) = u(x,y) + i\,v(x,y)$

Consider $F : \mathbb{R}^4 \to \mathbb{R}^2$, $F(s,t,x,y) = (\underbrace{s - u(x,y)}_{F_1}, \underbrace{t - v(x,y)}_{F_2})$

Then $F(u_0, v_0, x_0, y_0) = (u_0 - u(x_0,y_0), v_0 - v(x_0,y_0)) = (0,0)$
↳ need to solve this.

Ex: $y^2 x - \sin(y) + 5x^2 e^y = 2$

↳ Wish $y = f(x)$ ↳ can solve for a small region on x.
↳ Implicit fcn thm



<u>Implicit Function theorem</u>: Say $F: \mathbb{R}^{n+m} \to \mathbb{R}^m$ is continuously differentiable and $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ s.t.

$F(a,b) = \underset{(0,0,\ldots m\text{ times})}{0} \in \mathbb{R}^m$.

If the matrix
$$\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & & & \\ \frac{\partial F_m}{\partial y_1} & \frac{\partial F_m}{\partial y_2} & \cdots & \frac{\partial F_m}{\partial y_m} \end{bmatrix} = \left[\frac{\partial F_i}{\partial y_j}\right]_{i,j \in \{1,\ldots,m\}}$$

is invertible (where we consider $\mathbb{R}^{n+m}$ to consist of points (x,y) where $(x,y) = (x_1, \ldots x_n, y_1, \ldots y_m)$) then $\exists$ open ball $B \subset \mathbb{R}^n$ s.t. $a \in B$ ... unique continuously differentiable function $g: B \to \mathbb{R}^n$ with $g(a) = b$ and $F(x, g(x)) = 0 \quad \forall\, x \in B$.

$F: \mathbb{R}^{2+2} \to \mathbb{R}^2 \quad F(s,t,x,y) = (s - u(x,y),\ t - v(x,y))$

$$\begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial u}{\partial x} & -\frac{\partial u}{\partial y} \\ -\frac{\partial v}{\partial x} & -\frac{\partial v}{\partial y} \end{bmatrix}$$ invertible if $\det \neq 0$

$$\det [\ ] = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

CR eqs $\Rightarrow = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = |f'(z)|^2 \neq 0 \quad \forall z \in G$

$f'(z) \to \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$

from fact shown earlier.

By the Implicit function theorem, $\exists$ continuously diff function $g: B \to \mathbb{R}^2$ s.t. $(u_0, v_0) \in B$ and $F(s,t,g(s,t)) = (0,0)$ $\forall\, (s,t) \in B$. Write $g(s,t) = (\tilde{x}(s,t), \tilde{y}(s,t))$, so

$s = u(\tilde{x}(s,t), \tilde{y}(s,t)) \qquad t = v(\tilde{x}, \tilde{y})$

g is cont. diff on B.

$\Rightarrow \tilde{x}, \tilde{y}$ cont. diff. on B, $\quad \tilde{x}, \tilde{y}: B \to \mathbb{R}$

Since G is open, $\exists\ \epsilon > 0$ s.t. $B_\epsilon(x_0 + iy_0) \subseteq G$.

Since g is continuous, $\exists\ \delta > 0$ s.t. if $|(s,t) - (u_0, v_0)| < \delta$ then $|g(s,t) - g(u_0, v_0)| < \epsilon$

So $g(B_\delta(u_0, v_0)) \subseteq B_\epsilon(x_0, y_0) = B_\epsilon(x_0 + iy_0) \subseteq G$

Note if $(s,t) \in B_\delta(u_0, v_0)$ then $(s,t) \in B$. So $F(s, t, g(s,t)) = 0$.

$\Rightarrow B_\delta(u_0, v_0) \subseteq E \Rightarrow E$ is open $\blacksquare$

<u>thm:</u> Let $f(z)$ univalent on G and $E = f(G)$.
Then $\exists$ inverse $f^{-1}: E \to G$ s.t. f^{-1} is univalent

and $$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} \quad \forall\ w \in E.$$

<u>Proof:</u> $f: G \to E$ is onto (surjective) and one-to-one (injective)

So $f^{-1}: E \to G$ exists and is one to one (bijective).

[$w \in E$, $\exists\ z \in G$ s.t. $f(z) = w$ (onto)
$\exists$ only one $z \in G$ s.t. $f(z) = w$ so $f^{-1}(w)$ has only z as the output]

$w_0 \in E$. Show $(f^{-1})'(w_0)$ exists.

$$(f^{-1})'(w_0) = \lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0}$$

Let $\{W_n\} \subseteq E$ s.t $W_n \to W_0$ and $W_n \neq W_0$. $\forall$ n.

Then $\exists$ $\{z_n\} \subseteq G$ s.t $f^{-1}(W_n) = z_n$ $\forall$ n.

f^{-1} is continuous in E. since f is continuous.

So $\lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} f^{-1}(W_n) = f^{-1}(\lim\limits_{n\to\infty} W_n) = f^{-1}(W_0) = z_0 \in E$.

$$\lim_{n\to\infty} \frac{f^{-1}(W_n) - f^{-1}(W_0)}{W_n - W_0} = \lim_{n\to\infty} \frac{z_n - z_0}{f(z_n) - f(z_0)}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{f(z_n)-f(z_0)}{z_n - z_0}\right)} = \frac{1}{f'(z_0)} = \frac{1}{f'(f^{-1}(W_0))} \text{ exists.}$$

exists since f analytic, $f'(z) \neq 0$ $\forall z \in G$. ∎

Ex:

① $f(z) = z^n$. Say $w = z^n$, $W = re^{i\theta}$

So $z = r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) = r^{\frac{1}{n}} e^{i\frac{\theta + 2\pi k}{n}}$

$k = 0, 1, 2, \ldots, n-1$

Hence $f(z)$ is univalent on $G_\alpha = \{z \in \mathbb{C} \mid \alpha < \arg z < \alpha + \frac{2\pi}{n}\}$

open G_α.

If $G \subseteq \mathbb{C}$ open, $G_\alpha \subseteq G$, and $G_\alpha \neq G$ then G can't be a domain of univalence.

Pick eg. $G_0 = \{ z \in \mathbb{C} \mid 0 < \arg z < \frac{2\pi}{n} \}$

$\Rightarrow f^{-1}(w) = r^{1/n} e^{i\theta/n} = \sqrt[n]{w} \qquad f^{-1} : E_0 \to G_0$

$E_0 = \{ w \in \mathbb{C} \mid w \notin (0, \infty) \}$ $\longrightarrow$ since $\arg z > 0$

For $f^{-1} : E_0 \to G_0$, $(f^{-1})'(w) = \dfrac{1}{f'(f^{-1}(w))} = \dfrac{1}{n f^{-1}(w)^{n-1}} = \dfrac{\sqrt{w}}{nw}$

Ex: 2 $\quad f(z) = e^z$. f is univalent on the sets

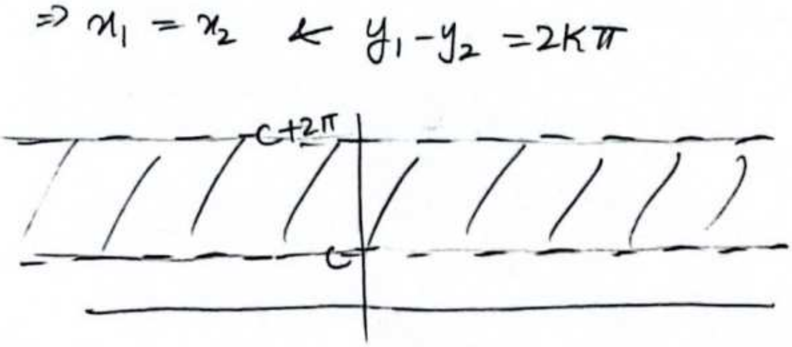
$G_c = \{ z \in \mathbb{C} \mid c < \operatorname{Im}(z) < c + 2\pi \}$

Why?

$e^{z_1} = e^{z_2}$ where $z_j = x_j + i y_j$

$\Rightarrow e^{x_1} e^{iy_1} = e^{x_2} e^{iy_2} \quad \Rightarrow e^{x_1} = e^{x_2}$ and $y_1 - y_2 = 2k\pi$

$\Rightarrow x_1 = x_2$ & $y_1 - y_2 = 2k\pi$



Consider $G_0 = \{ x+iy \mid 0 < y < 2\pi \}$

$f(G_0) = ?$

$f(x+iy) = \underset{\substack{\uparrow \\ \text{any radius (positive)}}}{e^x}\, e^{iy}$ $\longrightarrow$ any nonzero angle.

$\Rightarrow f(G_0) = E_0 = \{ x+iy \mid x+iy \notin [0, \infty) \}$

$$> (f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))} = \frac{1}{e^{f^{-1}(w)}} = \frac{1}{f(f^{-1}(w))} = \frac{1}{w}.$$

$f^{-1}: E_0 \longrightarrow G_0 \qquad f^{-1}(w) = \ln w$

$$\boxed{\ln w = \ln|w| + i \arg w}$$

Since $w = e^{x+iy} = e^x e^{iy} \Rightarrow |w| = e^x \quad \arg w = y$

Ex: $\ln(-2) = \ln|2| + i\arg(-2)$

$= \ln|2| + i\,((2k+1)\pi)$

k depends on the choice of domain of univalence for e^z.

Branches / Branch Points

- Consider $f(z) = z^n$. Domains of univalence

$G_k = \{z \in \mathbb{C} \mid \frac{2k\pi}{n} < \arg z < \frac{2(k+1)\pi}{n}\} \qquad k = 0, 1, \ldots, n-1$

$G_k \cap G_j = \phi \quad \forall\ j, k \in \{0, 1, \ldots n-1\}$ and

$$\bigcup_{k=0}^{n-1} \overline{G_k} = \bigcup_{k=0}^{n-1} \{z \in \mathbb{C} \mid \frac{2k\pi}{n} \le \arg z \le \frac{2(k+1)\pi}{n} \text{ or } z = 0\}$$

$= \mathbb{C}.$

On each G_k, inverse is defined. Call it $(\sqrt[n]{w})_k$

$w \in \mathbb{C} \setminus \{x \in \mathbb{R} \mid x \geq 0\} = E$

$\underbrace{(\sqrt[n]{w})_k}^{\text{singlevalued.}} : E \to G_k$ are single valued and are called branches of $\underbrace{\sqrt[n]{w}}_{\text{multivalued}}$

- Ex: $f(z) = e^z$. Let $G_k = \{z \in \mathbb{C} \mid 2k\pi < \operatorname{Im} z < 2(k+1)\pi\}$ $k \in \mathbb{Z}$

Branches are: $(\ln w)_0$, $(\ln w)_{+1}$, $(\ln w)_{-1}$, $(\ln w)_2$, ... are branches of $\ln w$.

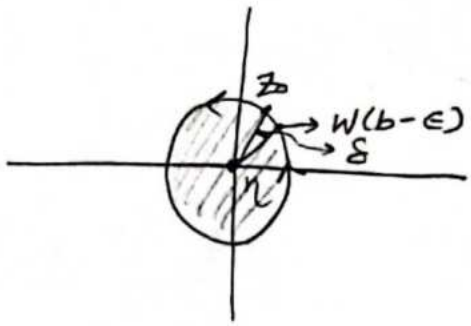
Def: A branch point is a point $\eta \in \mathbb{C}$ s.t a single loop around any closed curve C s.t $\eta \in \operatorname{int} C$ takes every branch of a multivalued function to another branch.

Ex: Consider $\sqrt[n]{w}$

① $\eta \neq 0$ $C = \{w(t) \mid a \leq t \leq b\}$

$\sqrt[n]{w(t)}$ stays in the same branch since $\arg(w(t))$ never increases by 2π when you complete the loop around C.

② $\eta = 0$



$C = \{W(t) \mid a \le t \le b\}$

$W(a) = W(b) = z_0$. $\arg(W(b-\epsilon))$

$$= \arg z_0 + (2\pi - \delta)$$

Say $\sqrt[n]{z_0} \in G_k$ so $\arg \sqrt[n]{z_0} \in \left(\frac{2k\pi}{n}, \frac{2(k+1)\pi}{n}\right)$

$$\arg(\sqrt[n]{W(b-\epsilon)}) \in \Big(\underbrace{\frac{2k\pi}{n} + \frac{2\pi-\delta}{n}}_{\text{close to } \frac{2(k+1)\pi}{n}}, \underbrace{\frac{2(k+1)\pi}{n}, \frac{2\pi\delta}{n}}_{\text{close to } \frac{2(k+2)\pi}{n}}\Big)$$

$\Rightarrow$ for ϵ small enough, $\arg(\sqrt[n]{W(b-\epsilon)}) \in \left(\frac{2(k+1)\pi}{n}, \frac{2(k+2)\pi}{n}\right)$

$\Rightarrow$ moved from G_k to G_{k+1}

Taylor Series

Let $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$, $\alpha \in \mathbb{C}$ has radius of convergence R. Then for $\{z \in \mathbb{C} \mid |z-\alpha| < R\}$, we can differentiate term by term.

$$f'(z) = \sum_{n=0}^{\infty} \frac{d}{dz}\left(c_n (z-\alpha)^n\right) = \sum_{n=1}^{\infty} n\, c_n (z-\alpha)^{n-1}$$

$$= \sum_{n=0}^{\infty} (n+1)\, c_{n+1} (z-\alpha)^n$$

$$f''(z) = \sum_{n=1}^{\infty} (n+1)\, n\, c_{n+1} (z-\alpha)^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1)\, c_{n+2} (z-\alpha)^n$$

$$\vdots$$

$$f^k(z) = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)\, c_{n+k} (z-\alpha)^n$$

Let $z = \alpha$ So $f^k(\alpha) = k \cdot (k-1) \cdot \ldots 2 \cdot 1 \cdot c_k$

$$\Rightarrow \boxed{c_k = \frac{f^k(\alpha)}{k!}}$$

If $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$, then $c_n = \dfrac{f^n(\alpha)}{n!}$

Def: A power series of the form $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$ for an analytic function $f(z)$ is a Taylor Series for $f(z)$ around $\alpha \in \mathbb{C}$.

If $C = \{z \in \mathbb{C} \mid |z-\alpha| = \rho\}$ $(\rho < R)$, then Cauchy formula gives $f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}} dz$

Thus

$$C_k = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{k+1}} dz$$

Thm: Suppose $f(z)$ is analytic on the $B_R(\alpha)$. Then

$$f(z) = \sum_{n=0}^{\infty} C_n (z-\alpha)^n \quad \forall \ z \in B_R(\alpha) \text{ where } C_n = \frac{f^{(n)}(\alpha)}{n!}$$

Proof: Say $w \in B_R(\alpha)$ and $\gamma = \{z \in \mathbb{C} \mid |z-\alpha| = r\}$ with $r < R$, and $w \in \text{int}(\gamma) = B_r(\alpha)$.

Then $f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz$

$$\frac{1}{z-w} = \frac{1}{(z-\alpha)-(w-\alpha)} = \frac{1}{z-\alpha} \cdot \frac{1}{1-\left(\frac{w-\alpha}{z-\alpha}\right)}$$

Since $\left|\frac{w-\alpha}{z-\alpha}\right| = \frac{|w-\alpha|}{r} < 1$

So $\sum_{n=0}^{\infty}\left(\frac{W-\alpha}{Z-\alpha}\right)^n$ converges. to $\frac{1}{1-\left(\frac{W-\alpha}{Z-\alpha}\right)}$

So $\frac{1}{Z-W} = \frac{1}{Z-\alpha}\sum_{n=0}^{\infty}\left(\frac{W-\alpha}{Z-\alpha}\right)^n = \sum_{n=0}^{\infty}\frac{(W-\alpha)^n}{(Z-\alpha)^{n+1}}$

$\Rightarrow f(W) = \frac{1}{2\pi i}\int_{\gamma}\sum_{n=0}^{\infty}\frac{f(Z)}{(Z-\alpha)^{n+1}}(W-\alpha)^n\,dZ$

$f(Z)$ analytic $\Rightarrow$ $f(Z)$ continuous on γ & γ is closed & bounded set (10 closed curve)

So $\exists\ M \in \mathbb{R}$ s.t. $M = \max_{Z\in\gamma}|f(Z)|$

$\underbrace{\left|\frac{f(Z)}{(Z-\alpha)^{n+1}}\cdot(W-\alpha)^n\right|}_{f_n(Z)} \le \underbrace{\frac{M}{\gamma}\left(\frac{|W-\alpha|}{\gamma}\right)^n}_{a_n}$ where $\frac{|W-\alpha|}{\gamma} < 1$

Recall Thm: If $|f_n(z)| \le a_n\ \forall\ n\in\mathbb{N}$ & $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} f_n(z)$ converges uniformly.

Recall Proof: $\left|\sum_{n=0}^{\infty} f_n(z) - \sum_{n=0}^{N} f_n(z)\right| \le \sum_{n=N+1}^{\infty} a_n < \epsilon$

independant of Z.

Since $\frac{|W-\alpha|}{\gamma} < 1$, the series $\sum_{n=0}^{\infty}\frac{M}{\gamma}\left(\frac{|W-\alpha|}{\gamma}\right)^n$ converges

Hence $\sum_{n=0}^{\infty}\frac{f(Z)}{(Z-\alpha)^{n+1}}(W-\alpha)^n$ converges uniformly on γ.

So, $\displaystyle \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{f(z)}{(z-\alpha)^{n+1}}\cdot (w-\alpha)^n \, dz$

$$= \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz\right]}_{\frac{1}{n!} f^{(n)}(\alpha)} (w-\alpha)^n$$

$\displaystyle \Rightarrow f(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(w-\alpha)^n$ which also converges since we earlier had a uniformly converging sequence

$\blacksquare$

Def: If $f(z)$ is analytic on $B_\delta(z_0)$ for some $\delta > 0$ then z_0 is a <u>regular point</u>. Otherwise we call z_0 a <u>singular point</u>.

<u>Thm</u>: Say $\displaystyle f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ Taylor Series, $\displaystyle c_n = \frac{f^{(n)}(\alpha)}{n!}$ with ROC $= R$. Then $f(z)$ has atleast one singular point on circle $\{z \in \mathbb{C} \,/\, |z-\alpha| = R\} = C$.

<u>Proof</u>: Suppose not. Then every $z \in C$ is regular.

$\Rightarrow \exists\ \delta_z > 0$ s.t $f(z)$ analytic on $B_{\delta_z}(z)$.

Since C is closed and bounded and $\displaystyle C \subseteq \bigcup_{z\in C} B_{\delta_z}(z)$

So $\exists\ z_1, \ldots, z_n$ s.t $\displaystyle C \subseteq \bigcup_{k=1}^{n} B_{\delta_k}(z_k)$

It can be shown that $\exists\ r>0$ s.T

$$B_{R+r}(\alpha) \subseteq B_R(\alpha) \cup B_{\delta_{z_1}}(z_1) \cup \ldots \cup B_{\delta_{z_n}}(z_n)$$

$f(z)$ analytic on $B_{R+r}(\alpha) \Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n$

on $B_{R+r}(\alpha)$. So ROC is atleast $R+r > R$. $\divideontimes$ ∎

<u>Thm:</u> (Liouville)

If $f(z)$ is entire and $|f(z)| \le M$ (bounded) $\forall\ z \in \mathbb{C}$, then $f(z) = \text{constant}$

<u>Proof:</u> $f(z)$ analytic on all of $\mathbb{C} \Rightarrow f(z)$ analytic on $B_R(0)\ \forall\ R>0$

So on each $B_R(0)$, $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \ \forall\ z \in \mathbb{C}.$$

Now, $\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z^{n+1}}\,dz \quad \forall\ n \in \mathbb{N},\ \forall\ R>0$
where $\gamma_R = \{|z| = R\}$

$$\Rightarrow \left|\frac{f^{(n)}(0)}{n!}\right| \le \frac{1}{2\pi}\cdot\frac{M}{R^{n+1}}\cdot 2\pi R = \frac{M}{R^n} \quad \forall\ R>0,\ n \in \mathbb{N}$$

Send $R \to \infty$ to get $f^{(n)}(0) = 0 \quad \forall\ n \ge 1$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0)$$

∎

Thm: Uniqueness

Suppose $\sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} b_n (z-z_0)^n \ \forall z \in E$ s.t z_0 is a limit point of E. Then $a_n = b_n \ \forall n \in \mathbb{N}$.

(Assume both converge in E)

→ Say $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$

for $x = x_1, x_2, \ldots, x_{n+1}$ ⇒ we get a system of equations & can solve for

$0 = (a_n - b_n) x^n + (a_{n-1} - b_{n-1}) x^{n-1} + \ldots + (a_1 - b_1) x + a_0 - b_0$.

In case of a power series we need ∞ly many points ⇒ a series with a limit point.

Proof: Let $z_n \in E$ s.t $z_n \neq z_0$ (limit point) and $z_n \neq z_m$ if $n \neq m$ and $z_n \neq 0$ and $|z_n - z_0| \leq \frac{|z_1 - z_0|}{n} \ \forall n \in \mathbb{N}$. and $\lim_{n\to\infty} z_n = z_0$.

Since $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ converges at $z = z_1$, the series is analytic and hence continuous in $\{z \in \mathbb{C} \mid |z-z_0| < |z_1 - z_0|\}$. ← contains $\{z_n\}$

By continuity, $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_k (z_n - z_0)^k = \sum_{k=0}^{\infty} a_k (z_0 - z_0)^k = a_0$

Similarly $\lim_{n\to\infty} \sum_{k=0}^{\infty} b_k (z_n - z_0)^k = b_0$. But $\sum_{k=0}^{\infty} a_k (z_n - z_0)^k = \sum_{k=0}^{\infty} b_k (z_n - z_0)^k$

$\forall z_n \in E$ ⇒ $a_0 = b_0$. (base case of induction)

<u>Induction:</u> Assume $a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2 \ldots a_j = b_j$

$$\sum_{k=0}^{\infty} a_k (z_n - z_0)^k = \sum_{k=0}^{\infty} b_k (z_n - z_0)^k \qquad \forall\, n \in \mathbb{N}$$

$$\Rightarrow \sum_{k=j+1}^{\infty} a_k (z_n - z_0)^k = \sum_{k=j+1}^{\infty} b_k (z_n - z_0)^k$$

$(z_n - z_0)^{j+1} \neq 0$. Divide both sides.

$$\sum_{k=j+1}^{\infty} a_k (z_n - z_0)^{k-(j+1)} = \sum_{k=j+1}^{\infty} b_k (z_n - z_0)^{k-(j+1)}$$

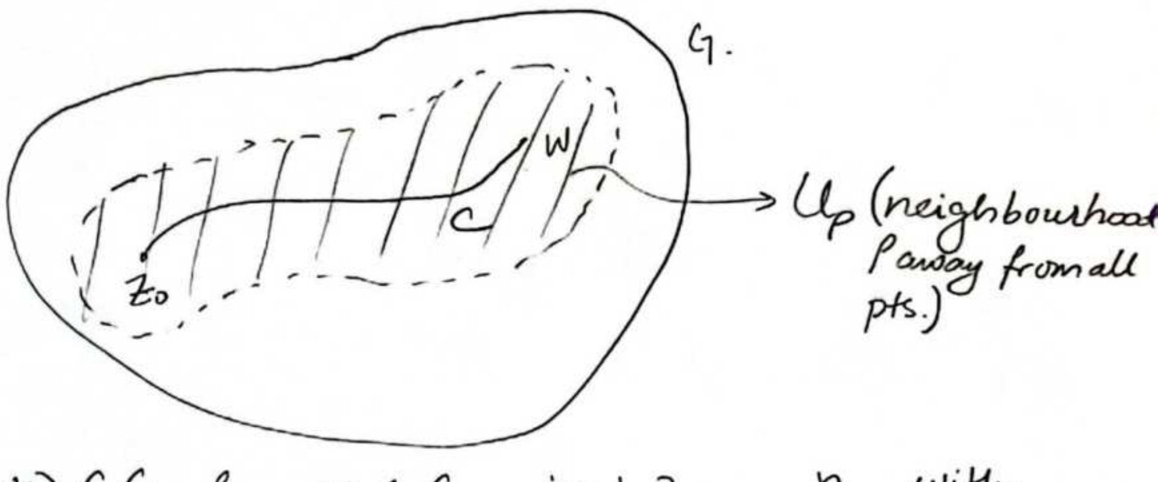
Take $n \to \infty$ (similar to earlier) $\Rightarrow a_{j+1} = b_{j+1}$ ■

Thm (Uniqueness of Analytic Functions)

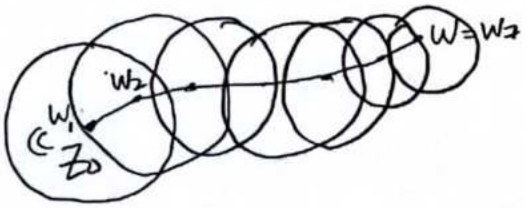
Say $f(z)$ and $g(z)$ analytic in domain G and $f(z) = g(z)$ $\forall z \in E$ s.t $E \subseteq G$ and E has a limit point $z_0 \in G$.
Then $f(z) = g(z)$ $\forall z \in G$.

Proof: Say $w \in G$, $w \neq z_0$. Let $C \subseteq G$ a curve connecting w and z_0. Then $\exists\ \rho > 0$ s.t.

$$U_\rho = \{ z \in \mathbb{C} \mid |z - \alpha| < \rho,\ \alpha \in C \} \subseteq G$$



Now, let $B_\rho(w_j) \subseteq G$ for $w_j \in C$, $j = 1, 2, \dots, n$. with $w_1 = z_0$, $w_n = w$ and $B_\rho(w_j) \cap B_\rho(w_{j+1}) \neq \phi$. and $C \subseteq \bigcup_{j=1}^{n} B_\rho(w_j)$ and $w_{j+1} \in B_\rho(w_j)$



Now, $f(z) = g(z) \ \forall \ z \in E \cap B_\rho(w_1)$. Also since $f(z)$ and $g(z)$ analytic on $B_\rho(w_1)$. $\exists$ power series $f(z) = \sum_{n=0}^{\infty} a_n (z-w_1)^n = \sum_{n=0}^{\infty} b_n (z-w_1)^n$. on a subset $E \cap B_\rho(w_1)$ with limit point $w_1 = z_0 \Rightarrow$ by previous uniqueness theorem, $a_n = b_n \ \forall n \in \mathbb{N}$.

So $f(z) = g(z)$ in $B_\rho(w_1)$. Hence $f(w_2) = g(w_2)$ and $f(z) = g(z) \ \forall \ z \in B_\rho(w_1) \cap B_\rho(w_2) \subseteq B_\rho(w_2)$

w_2 is a limit pt. of $B_\rho(w_1) \cap B_\rho(w_2)$. So repeat the above argument to get $f(z) = g(z) \ \forall z \in B_\rho(w_2)$

And repeat for all $B_\rho(w_3), B_\rho(w_4) \ldots B_\rho(w_n)$

Hence $f(w_n) = g(w_n) \Rightarrow f(w) = g(w)$ $\blacksquare$

Hence, if $f(z) = g(z)$ on any open ball or curve then $f(z) = g(z)$ on the whole domain G. So for example, if $f(z) = c \ \forall z \in C$, a curve, the $f(z)$ is the constant function in G.

<u>Def</u>: Say $f(z)$ is a non zero analytic function in G

s.t $f(z) = \sum_{n=m}^{\infty} C_n (z-z_0)^n$ for $z_0 \in G$ <u>and $C_m \neq 0$</u>

then z_0 is a <u>zero of order m</u>. If $m=1$, z_0 is a <u>simple zero</u>

<u>Thm</u>: Every zero of a non zero analytic function is <u>isolated</u>

i.e. $\exists\ \delta > 0$ s.t $f(z) \neq 0\ \forall\ z \in B_\delta(z_0) \setminus \{z_0\}$ and $f(z_0) = 0$.

<u>Proof</u>: If not, then $\exists$ sequence z_n s.t $z_n \neq z_0$,

$z_n \in B_{1/n}(z_0)$ and $f(z_n) = 0$. So z_0 is a limit pt. of

$E = \{z_n\}$. and $f(z) = 0$ on E. By uniqueness thm,

$f(z) = 0$ everywhere in the domain ∎

<u>Thm:</u> If $f(z)$ analytic in G and $|f(z)|$ is constant in G

then $f(z)$ is constant in G.

<u>Proof</u>: $|f(z)|^2 = u(x,y)^2 + v(x,y)^2$ if $f = u + iv$

Say $M = |f(z)|\ \forall\ z \in G$. then

$$M^2 = u(x,y)^2 + v(x,y)^2$$

$$\Rightarrow \frac{\partial}{\partial x}(M^2) = 0 = \frac{\partial}{\partial x}\left(u(x,y)^2 + v(x,y)^2\right) = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial y}(M^2) = 0 = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial y}$$

Case i) $M \neq 0$. Then $u(x,y) \neq 0$ or $v(x,y) \neq 0$ or

$u(x,y)$ and $v(x,y) \neq 0$ at each $z \in G$.

Since $|f(z)| \neq 0$.

So $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $2\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence,

$$\det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{bmatrix} = 0 \Rightarrow \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0 \quad \forall z \in G$$

$\Rightarrow$ by C-R eqs $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

So $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0 \quad \forall z \in G$.

Thus, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad \forall z \in G$. By CR, $V_x = V_y = 0 \quad \forall z \in G$

$\Rightarrow u(x,y) = C_1$, $v(x,y) = C_2$ constant $\forall z \in G$. $\Rightarrow f(z) = C_1 + iC_2$ $\forall z \in G$.

Case ii) $M = 0$, then $|f(z)| = 0 \ \forall z \in G \Rightarrow f(z) = 0 \quad \forall z \in G$

$\blacksquare$

Thm (Maximum Principle)

If $f(z)$ analytic and non constant in domain G, then $|f(z)|$ has no maximum in G, ie $\nexists\ z_0 \in G$ s.t $|f(z)| \leq |f(z_0)|\ \forall\ z \in G$.

Proof: Suppose $\exists\ z_0 \in G$ s.t $|f(z)| \leq |f(z_0)| = M\ \forall\ z \in G$.

Then $\forall\ B_R(z_0) \subseteq G$, we have for γ_R = boundary of $B_R(z_0)$

$f(z_0) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z - z_0}\, dz$. Parameterize $\gamma_R = \{z_0 + Re^{it} \mid 0 \leq t \leq 2\pi\}$

$$\text{So } f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z(t))}{z(t) - z_0}\, z'(t)\, dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{Re^{it}}\, i Re^{it} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

$$\Rightarrow |f(z_0)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(z_0 + Re^{it})\, dt \right|$$

Now, suppose $\exists\ t_0$ s.t $|f(z_0 + Re^{it_0})| < M - 2\epsilon$ for some $\epsilon > 0$

By continuity, $\exists\ \delta > 0$ s.t for $t \in (t_0 - \delta, t_0 + \delta)$. we have

$|f(z_0 + Re^{it})| < M - \epsilon$.

Hence

$$M = |f(z_0)| = \left| \int_0^{2\pi} f(z_0 + Re^{it})\, dt \right|$$

$$= \left| \int_{t_0+\delta}^{2\pi} f(z_0+Re^{it})\,dt + \int_{t_0-\delta}^{t_0+\delta} f(z_0+Re^{it})\,dt + \int_0^{t_0-\delta} f(z_0+Re^{it})\,dt \right|$$

$$\leq \left| \int_{t_0+\delta}^{2\pi} f(z_0+Re^{it})\,dt \right| + \left| \int_{t_0-\delta}^{t_0+\delta} f(z_0+Re^{it})\,dt \right| + \left| \int_0^{t_0-\delta} f(z_0+Re^{it})\,dt \right|$$

$$\leq M(2\pi - (t_0+\delta)) + (M-\epsilon)((t_0+\delta)-(t_0-\delta)) + M(t_0-\delta)$$

$$< M \quad \divideontimes$$

$\Rightarrow |f(z)| = M \quad \forall \quad z \in \gamma_R \Rightarrow |f(z)| = M$ on $B_R(0)$ $\forall B_R(0) \subseteq G$.

$\forall\ B_R(z_0) \subseteq G$

By last theorem, $f(z) =$ constant on $B_R(0)$

$\Rightarrow$ By uniqueness, $f(z) =$ constant on G contradiction

since f non constant

$\blacksquare$

<u>Application</u>: Any harmonic <u>fn</u> $u: G \to \mathbb{R}$ that attains its maximum in G is a constant function.

Laurent series.

<u>Thm</u>: Consider series $\sum_{n=0}^{\infty} c_n (z-\alpha)^{-n}$ and $l = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$

Then ① If $l=0$ series converges absolutely $\forall\, z \neq \alpha$

② If $0<l<\infty$ then series converges absolutely for $\{z \in \mathbb{C} \mid |z-\alpha| > l\}$ and diverges for $\{z\in\mathbb{C} \mid |z-\alpha|<l\}$

③ If $l=\infty$, series diverges $\forall\, z\in\mathbb{C}$.

<u>Def</u>: The <u>Laurent series</u> $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ converges if and only if

$$\underbrace{\sum_{n=0}^{\infty} c_n (z-\alpha)^n}_{\text{regular part}} \text{ and } \underbrace{\sum_{n=1}^{\infty} c_{-n} (z-\alpha)^{-n}}_{\text{principal part.}} \text{ both converge.}$$

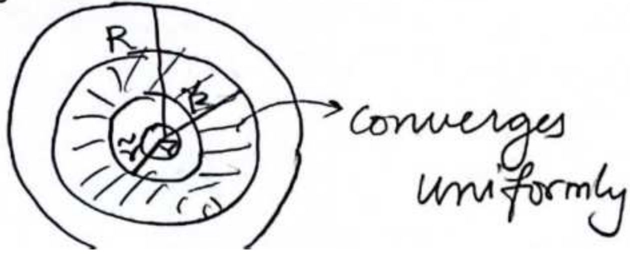
Say $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has radius of convergence R and $r = \limsup_{n\to\infty} \sqrt[n]{|c_{-n}|}$.

Then $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ converges absolutely and uniformly on

$\{z\in\mathbb{C} \mid \tilde{r} \le |z-\alpha| \le \tilde{R}\}$ provided $r<\tilde{r}\le\tilde{R}<R$, and

annulus.

$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ is analytic on that annulus.



Thm: If $C = \{z \in \mathbb{C} \mid |z-\alpha| = \rho\}$ and $r < \rho < R$ and

$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ converges for $r < |z-\alpha| < R$ then

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}} \, dz \quad \text{for} \quad n \in \mathbb{Z}.$$

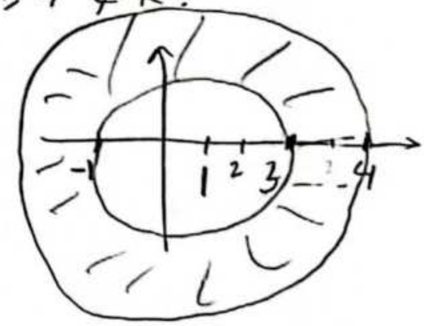
Also similar to Taylor Series,

Thm: If $f(z)$ is analytic in $K = \{z \in \mathbb{C} \mid r < |z-\alpha| < R\}$ then $f(z)$ has a Laurent expansion, $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$, for $z \in K$ with coefficients $c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-\alpha)^{n+1}} \, dz$ as above.

EX: ① $f(z) = \frac{1}{z+1} + \frac{2}{4-z}$ is analytic in $\mathbb{C} \setminus \{-1, 4\}$

Then for example if $\alpha = 1$, we have

$$K = \{z \in \mathbb{C} \mid \underset{\substack{\uparrow \\ \text{dist for} \\ \alpha=1 \text{ to } z=-1}}{2} < |z-1| < \underset{\substack{\uparrow \\ \text{dist} \\ \alpha=1 \text{ to } z=4}}{3}\} \longleftarrow \text{ so } -1, 4 \notin K.$$



So on K, a Laurent expansion exists

$$\frac{1}{z+1} = \frac{1}{2-(1-z)} = \frac{-1}{1-z} \cdot \frac{1}{1 - \frac{2}{1-z}}$$

if

$\left|\frac{2}{1-z}\right|<1 \Rightarrow \frac{1}{z+1} = -\frac{1}{1-z} \cdot \sum_{n=0}^{\infty} \left(\frac{2}{1-z}\right)^n = \sum_{n=0}^{\infty} (-2)^n (z-1)^{-n-1}$

$$= \sum_{n=-1}^{-\infty} (-2)^{+n+1} (z-1)^n \quad \text{if } |z-1|>2$$

$$\frac{2}{4-z} = \frac{2}{3-(z-1)} = \frac{2}{3} \cdot \frac{1}{1-\left(\frac{z-1}{3}\right)} = \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left(\frac{z-1}{3}\right)^n$$

if $\left|\frac{z-1}{3}\right|<1$

$$= \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} \cdot (z-1)^n$$

So, $\frac{1}{z+1} + \frac{2}{4-z} = \sum_{n=-1}^{-\infty} (-2)^{+n+1} (z-1)^n + \sum_{n=0}^{\infty} \frac{2}{3^{n+1}} (z-1)^n$

if $\underbrace{\frac{|z-1|}{3}<1 \text{ and } \left|\frac{2}{1-z}\right|<1}$

$\Rightarrow |z-1|<3$ and $|1-z|>2$

$\Rightarrow z \in K.$

② $f(z) = \frac{1}{z(z-1)(z-2)}$ analytic except at $z=0,1,2$

Consider annulus $0<|z|<1$

$f(z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$. Solve for A, B, C to get

$$f(z) = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$= \underbrace{\frac{1}{2z}}_{\text{principle.}} + \underbrace{\frac{1}{1-z} + \frac{(-1/4)}{1-\frac{z}{2}}}_{\text{regular}}$$

$$= \frac{1}{2z} + \sum_{n=0}^{\infty} z^n - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$$= \frac{1}{2z} + \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+2}}\right) z^n$$

<u>Def</u>:- If $f(z)$ (analytic) is defined on $B_\delta(z_0) \setminus \{z_0\}$ but not at z_0, then z_0 is an <u>isolated singular point</u> of f

By theorem of Laurent expansion, $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ on $B_\delta(z_0) \setminus \{z_0\}$.

<u>Def</u>: 1) If $c_n = 0 \ \forall \ n<0$, z_0 is a <u>removable singularity</u>.

2) If $c_n = 0 \ \forall \ n < m < 0$ for some $m \in \mathbb{Z} \cap [-\infty, -1]$ then z_0 is a <u>pole</u> of order m with $c_m \neq 0$.

3) If $c_n \neq 0$ for infinitely many $n<0$, z_0 is an <u>essential singularity</u>.

If (1) $\leftrightarrow$ removable, then $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ on $B_\delta(z_0)\setminus\{z_0\}$ for some $\delta>0$. But $\tilde{f}(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$ is analytic on $B_\delta(z_0)$, with $f(z) = \tilde{f}(z)$ except at $z=z_0$.

So, $\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \tilde{f}(z) = \tilde{f}(z_0) = c_0$.

Ex: $f(z) = \frac{z^2-1}{z-1}$ has a removable singularity at $z=1$

$f(z) = \frac{\sin z}{z}$ " " " " " $z=0$.

If (2) $\leftrightarrow$ pole, then for some $m\in\mathbb{N}$, on $B_\delta(z_0)\setminus\{z_0\}$

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{c_{-1}}{z-z_0} + \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

If $m=1$, z_0 is a simple pole.
If $m>1$, z_0 is a multiple pole.

So $(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \dots + c_{-1}(z-z_0)^{m-1} + \sum_{n=0}^{\infty} c_n (z-z_0)^{n+m}$ is a power series and z_0 is a removable singularity of $(z-z_0)^m f(z)$ and so $\lim_{z\to z_0} (z-z_0)^m f(z) = c_{-m} \neq 0$.

Thus $\lim_{z\to z_0} f(z) = \infty$. $\Rightarrow$ pole still exists.

Thm: Say z_0 is a zero of order m of $f(z)$ analytic in $B_\delta(z_0)$.
Then z_0 is a pole of order m of $\frac{1}{f(z)}$.

Def: For isolated singularity z_0 of $f(z)$, define the residue of f at z_0 as $\operatorname*{Res}_{z=z_0} f(z) = c_{-1}$ where

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n.$$

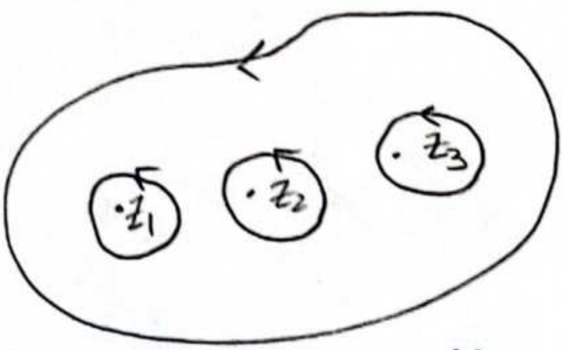
Fact: If z_0 is a removable singularity, then $\operatorname*{Res}_{z=z_0} f(z) = 0$.

Thm (Residue)

If $f(z)$ is analytic on pw smooth C and in $int(C)$ except at $z = z_1, z_2, \ldots, z_m$ (isolated singular points) in $int(C)$, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{m} \underset{z=z_k}{\operatorname{Res}} f(z)}$$

Proof: $\exists$ circles $\gamma_1, \gamma_2 \ldots \gamma_m$ s.t. $z_k \in int(\gamma_k)$ and $\overline{int(\gamma_k)} \cap \overline{int(\gamma_j)} = \phi \quad \forall j \neq k$ and $\overline{int(\gamma_k)} \subseteq int(C)$ $\forall k$



As we have done before,

$$\int_C f(z)\,dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z)\,dz$$

Say at z_k, $f(z) = \sum_{n=-\infty}^{\infty} c_{n,k}(z-z_k)^n$

Then since series converges uniformly (and absolutely) on γ_k.

$$\int_{\gamma_k} f(z)\,dz = \sum_{n=-\infty}^{\infty} \int_{\gamma_k} c_{n,k}(z-z_k)^n\,dz = \sum_{n=-\infty}^{\infty} c_{n,k} \int_{\gamma_k} (z-z_k)^n\,dz$$

Now, $\displaystyle\int_{\gamma_k} (z-z_k)^n\,dz = \int_0^{2\pi} r_k^n e^{itn} \cdot i r_k e^{it}\,dt.$

$$\gamma_k = \{ z_k + r_k e^{it} \mid 0 \le t \le 2\pi \}$$

$$= i r_k^{n+1} \int_0^{2\pi} e^{it(n+1)} dt = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{if } n \neq -1 \end{cases}$$

So $$\int_{\gamma_k} f(z)\,dz = C_{-1,k} \cdot 2\pi i = 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$

$$\Rightarrow \int_C f(z)\,dz = \sum_{k=1}^{m} 2\pi i \operatorname*{Res}_{z=z_k} f(z) \quad \blacksquare$$

Calculating Residues (without calculating Laurent Series)

> Say z_0 is a simple pole of $f(z)$.

$$\Rightarrow f(z) = \frac{C_{-1}}{z - z_0} + \sum_{n=0}^{\infty} C_n (z - z_0)^n \quad \text{for } z \in B_\delta(z_0) \text{ some } \delta > 0$$
& some $C_{-1}, C_0, C_1 \ldots \in \mathbb{C}$

Then $$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} C_{-1} + \sum_{n=0}^{\infty} C_n (z - z_0)^{n+1} = C_{-1} = \operatorname*{Res}_{z=z_0} f(z)$$

> Say z_0 is a pole of order $m > 1$. So

$$f(z) = \frac{C_{-m}}{(z - z_0)^m} + \frac{C_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{C_{-1}}{z - z_0} + \sum_{n=0}^{\infty} C_n (z - z_0)^n$$

$$\Rightarrow (z - z_0)^m f(z) = C_{-m} + C_{-m+1}(z - z_0) + \cdots + C_{-1}(z - z_0)^{m-1} + \sum_{n=0}^{\infty} C_n (z - z_0)^{m+n}$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) = (m-1)!\, C_{-1} + \sum_{n=0}^{\infty} C_n \frac{(m+n)!}{(n+1)!}(z-z_0)^{n+1}$$

$$\text{So } \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right) = (m-1)!\; C_{-1}$$

$$\Rightarrow \operatorname*{Res}_{z=z_0} f(z) = C_{-1} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)$$

Ex: #21 (a) : $f(z) = \dfrac{1}{z^3 - z^5} = \dfrac{1}{\underbrace{z^3(1-z^2)}_{\to \text{ order } 3.}}$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(3-1)!} \lim_{z\to 0} \frac{d^2}{dz^2}\left(z^3 f(z)\right)$$

$$= \frac{1}{2} \lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{1}{1-z^2}\right) = \frac{1}{2} \lim_{z\to 0} \frac{d}{dz}\left(\frac{2z}{(1-z^2)^2}\right)$$

$$= \frac{1}{2} \lim_{z\to 0}\left(\frac{2}{(1-z^2)^2} + \frac{8z}{(1-z^2)^3}\right) = 1$$

#21(g) $f(z) = \sin z \, \sin\left(\frac{1}{z}\right)$ → $z=0$ pole.

For $z \neq 0$, $\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{z}\right)^{2n+1} = \frac{1}{z} - \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$

For any $z \in \mathbb{C}$, $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

So, $\sin\left(\frac{1}{z}\right)\sin z$ has no odd powers of z.

$\Rightarrow$ For $\mathbb{C} \setminus \{0\}$, if we write $\sin(z)\sin\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$

then $C_{-1} = 0 \Rightarrow \operatorname{Res}_{z=0} f(z) = 0$

<u>Ex:</u> <u>#26</u> Let $C = \left\{ |z-2| = \frac{1}{2} \right\}$

$$\int_C \frac{dz}{(z-1)(z-2)^2} = 2\pi i \operatorname{Res}_{z=2} \underbrace{\frac{1}{(z-1)(z-2)^2}}_{\text{order } 2} \qquad \text{since } 1 \notin \operatorname{int}(C)$$

$$\operatorname{Res}_{z=0} \frac{1}{(z-1)(z-2)^2} = \frac{1}{(2-1)!} \lim_{z\to 2} \frac{d}{dz}\left(\frac{1}{z-1}\right) = \lim_{z\to 2} \frac{-1}{(z-1)^2} = -1$$

$$\Rightarrow \boxed{\int_C \frac{dz}{(z-1)(z-2)^2} = -2\pi i}$$

Or

$$\frac{1}{(z-1)(z-2)^2} = \frac{1}{(z-2)^2}\left(\frac{1}{1+(z-2)}\right) = \frac{1}{(z-2)^2} \cdot \sum_{n=0}^{\infty} (-1)^n (z-2)^n$$

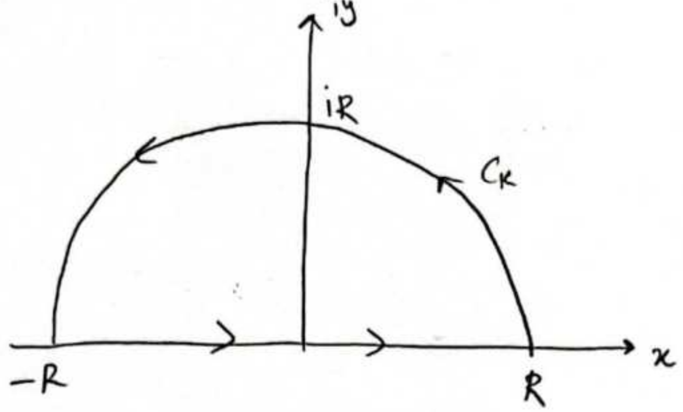
$$= \frac{1}{(z-2)^2} - \underbrace{\frac{1}{z-2}}_{C_{-1} = -1} + 1 - (z-2) + \cdots$$

Applications : Improper Integrals

Ex: Compute $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} dx$ on $\mathbb{R}$ for $a>0$

Let $f(z) = \frac{1}{(z^2+a^2)^3}$. So for $x \in \mathbb{R}$, $f(x) = \frac{1}{(x^2+a^2)^3}$

Consider curve C_R:



$\exists\ r>0$ s.t if $R>r$ then $ia \in \text{int}(C)$

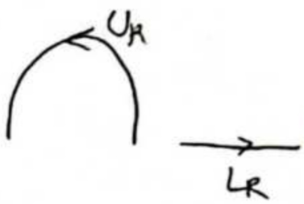
Now, $\underset{z=ia}{\text{Res}}\ f(z) = \frac{1}{(3-1)!} \lim_{z\to ia} \frac{d^2}{dz^2}\left((z-ia)^3 f(z)\right)$

$$= \frac{1}{2} \lim_{z\to ia} \frac{d^2}{dz^2}\left(\frac{1}{(z+ia)^3}\right)$$

$$= \frac{1}{2} \lim_{z\to ia} \frac{12}{(z+ia)^5} = \frac{6}{(2ia)^5} = \frac{3}{16a^5 i}$$

So $\int_{C_R} f(z)dz = 2\pi i \cdot \frac{3}{16a^5 i} = \frac{3\pi}{8a^5}$

Now, $\int_{C_R} f(z)\,dz = \int_{L_R} f(z)\,dz + \int_{U_R} f(z)\,dz$



$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx.$$ Next on U_R.

$$|f(z)| = \frac{1}{|z^2+a^2|^3} \le \frac{1}{||z|^2-|a^2||^3} \le \frac{1}{(R^2-a^2)^3}$$

$$\Rightarrow \left|\int_{U_R} f(z)\,dz\right| \le \text{length}(U_R)\,\frac{1}{(R^2-a^2)^3} = \frac{\pi R}{(R^2-a^2)^3} \to 0 \text{ as } R\to\infty$$

Hence $\lim_{R\to\infty} \int_{U_R} f(z)\,dz = 0$ So,

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty}\left[\int_{U_R} f(z)\,dz + \int_{L_R} f(z)\,dz\right]$$

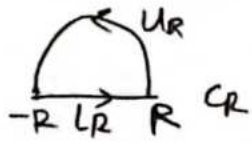
$$\Rightarrow \lim_{R\to\infty}\left(\frac{3\pi}{8a^5}\right) = 0 + \int_{-\infty}^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{8a^5} \quad \text{for } a>0.$$

Ex: $\displaystyle\int_0^\infty \frac{\cos(x)}{x^2+a^2}\,dx = ?$ $a>0$

We can choose $\dfrac{\cos(z)}{z^2+a^2}$ or $\dfrac{e^{iz}}{z^2+a^2}$.

Pick $f(z) = \dfrac{e^{iz}}{z^2+a^2}$. Let C_R, U_R, L_R as before



On U_R, $\displaystyle |f(z)| = \frac{|e^{iz}|}{|z^2+a^2|} \le \frac{e^{-y}}{R^2-a^2} \le \frac{1}{R^2-a^2}$ ← $y \ge 0$

So $\displaystyle \left| \int_{U_R} f(z)\,dz \right| \le \pi R \cdot \frac{1}{R^2-a^2} \longrightarrow 0$ as $R \to \infty$.

$\displaystyle \operatorname*{Res}_{z=ia} f(z) = \lim_{z\to ia} (z-ia) f(z) = \lim_{z\to ia} \frac{e^{iz}}{z+ia} = \frac{e^{-a}}{2ia}$

$\displaystyle \Rightarrow \int_{C_R} f(z)\,dz = 2\pi i \left(\frac{e^{-a}}{2ia} \right) = \frac{\pi e^{-a}}{a}$

Let $R\to\infty$ so $\displaystyle \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}$

$\displaystyle \Rightarrow \int_{-\infty}^{\infty} \left(\frac{\cos x}{x^2+a^2} + i\,\frac{\sin x}{x^2+a^2} \right) dx = \frac{\pi e^{-a}}{a}$

Take real part $\Rightarrow \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{a}.$

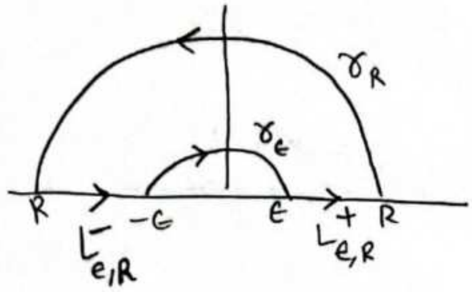
Now, $\dfrac{\cos x}{x^2+a^2}$ is even so $\displaystyle\int_{0}^{\infty} \frac{\cos x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2a}$

<u>Ex:</u> $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx$ $\longrightarrow$ undefined at $x=0$,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_{\epsilon}^{R} \frac{\sin x}{x}\,dx = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \frac{1}{2}\left(\int_{\epsilon}^{R} \frac{\sin x}{x}\,dx + \int_{-R}^{-\epsilon} \frac{\sin x}{x}\,dx\right)$$

$\longrightarrow$ even fn.

Let $C_{R,\epsilon}$ be



Consider, $f(z) = \dfrac{e^{iz}}{z}$

On $f(z)$ analytic on int$(C_{R,\epsilon})$. So $\displaystyle\int_{C_{R,\epsilon}} f(z)\,dz = 0$

On γ_R, we use integration by parts.

<u>Fact</u>: $\int_C f'(z)g(z)\,dz = f(z_2)g(z_2) - f(z_1)g(z_1) - \int_C f(z)g'(z)\,dz$

for f, g analytic on C from z_1 to z_2.

<u>Proof</u>: Let $H(z) = f(z)\,g(z)$ analytic.

Then $h(z) = H'(z) = f'(z)g(z) + f(z)g'(z)$

By thm* page 69 of notes, $\int_C h(w)\,dw = H(z_2) - H(z_1)$ ■

So, $$\int_{\gamma_R} \frac{e^{iz}}{z}\,dz = \int_{\gamma_R} \frac{1}{iz}\frac{d}{dz}(e^{iz})\,dz$$

$$= \left.\frac{e^{iz}}{iz}\right|_{z=-R} - \left.\frac{e^{iz}}{iz}\right|_{z=R} + \int_{\gamma_R} \frac{e^{iz}}{iz^2}\,dz$$

$$= -\frac{e^{iR} + e^{-iR}}{iR} + \int_{\gamma_R} \frac{e^{iz}}{iz^2}\,dz$$

Now, $$\left|\frac{e^{iR} + e^{-iR}}{iR}\right| \le \frac{|e^{iR}| + |e^{-iR}|}{R} = \frac{2}{R} \to 0 \quad \text{as } R \to \infty$$

$$\left|\int_{\gamma_R} \frac{e^{iz}}{iz^2}\,dz\right| \le \pi R \cdot \frac{1}{R^2} \to 0 \quad \text{as } R \to \infty$$

$|e^{iz}| = e^{-y} \le 1$ for $y \ge 0$

So, $\int_{\gamma_R} \frac{e^{iz}}{z} dz \to 0$ as $R \to \infty$.

Next, for γ_ϵ, consider $\frac{e^{iz}}{z} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n z^{n-1}}{n!}$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{i^{n+1} z^n}{(n+1)!}$$

Now, $P(z) = \sum_{n=0}^{\infty} \frac{i^{n+1} z^n}{(n+1)!}$ converges on all $z \in \mathbb{C}$.

So $P(z)$ analytic $\Rightarrow$ On $B_1(0)$, $P(z)$ is bounded.

So $\exists$ $M > 0$ s.t $|P(z)| \leq M$ $\forall$ $z \in \gamma_\epsilon$ for $\epsilon < 1$.

So $\left| \int_{\gamma_\epsilon} P(z) dz \right| \leq M \cdot 2\pi\epsilon \to 0$ as $\epsilon \to 0$.

Next, $\int_{\gamma_\epsilon} \frac{dz}{z} = \int_{\pi}^{0} \frac{i\epsilon e^{it}}{\epsilon e^{it}} dt = -i\pi \longrightarrow -\pi i$ as $\epsilon \to 0$

$\hookrightarrow z(t) = \epsilon e^{it}$

$$\Rightarrow \int_{\gamma_\epsilon} \frac{e^{iz}}{z} dz \longrightarrow -\pi i + 0 = -\pi i \text{ as } \epsilon \to 0$$

thus, $\lim_{\substack{\epsilon \to 0 \\ R \to \infty}} \left[\int_{\epsilon}^{R} \frac{e^{iz}}{z} dx + \int_{-R}^{-\epsilon} \frac{e^{iz}}{z} dx \right] = 0 - (-\pi i + 0) = \pi i$

Im parts $\Rightarrow$ $\int_0^{\infty} \frac{\sin x}{x} dx + \int_{-\infty}^{0} \frac{\sin x}{x} dx = \pi$ $\Rightarrow$ $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$.

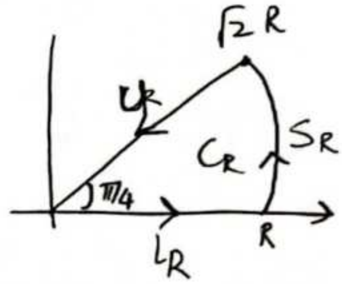
Ex: $\int_0^\infty \cos(x^2)\,dx = ?$; $\int_0^\infty \sin(x^2)\,dx = ?$

Let $f(z) = e^{iz^2} \Rightarrow f(x) = \cos(x^2) + i\sin(x^2)$

$f(z)$ has no singular pts. But we know $\int_0^\infty e^{-x^2}dx$

ie. when $iz^2 = -x^2 \Rightarrow z^2 = ix^2$

Consider segment $z = \sqrt{i}\,x$ for $x=0$ to $x=R$.



Note $\sqrt{i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ So, $\int_{C_R} f(z)\,dz = 0$

$$\int_{L_R} f(z)dz = \int_0^R \cos(x^2)dx + i\int_0^R \sin(x^2)dx$$

On S_R,

$$\int_{\gamma_R} e^{iz^2}dz = \int_{\gamma_R} \frac{1}{2iz}\frac{d}{dz}\left(e^{iz^2}\right)dz$$

$$= \frac{1}{2iz}e^{iz^2}\Big|_{z=\sqrt{i}R} - \frac{1}{2iz}e^{iz^2}\Big|_{z=R} + \int_{\gamma_R}\frac{e^{iz^2}}{2iz^2}dz$$

$$= \frac{1}{2i}\left[\frac{e^{-R^2}}{R} - \frac{e^{iR^2}}{R} + \int_{\gamma_R}\frac{e^{iz^2}}{z^2}dz\right]$$

$\frac{e^{-R^2}}{R}, \frac{e^{iR^2}}{R} \to 0$ as $R \to \infty$.

On γ_R, $\left|\dfrac{e^{iz^2}}{z^2}\right| = \dfrac{|e^{-2xy}|}{R^2} \le \dfrac{1}{R^2}$ So $\left|\int_{\gamma_R} f(z)\,dz\right| \le \dfrac{\pi R}{4}\cdot\dfrac{1}{R^2}$

$z^2 = x^2 - y^2 + 2ixy$

$\longrightarrow 0$ as $R \to \infty$

Finally, $\displaystyle\int_{U_R} f(z)\,dz = \int_R^0 e^{-x^2}\sqrt{i}\,dx = -\sqrt{i}\int_0^R e^{-x^2}dx \to -\sqrt{i}\int_0^\infty e^{-x^2}dx$ as $R \to \infty$

Claim: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

Proof: (Euler) $\displaystyle\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}r\,dr\,d\theta = \pi$

So $\displaystyle\int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2} \Rightarrow \int_{U_R} f(z)\,dz = -\frac{\sqrt{\pi}}{2}\sqrt{i}$

So,

$$0 = \int_{C_R} f(z)\,dz = \int_{U_R} f(z)\,dz + \int_{\gamma_R} f(z)\,dz + \int_{L_R} f(z)\,dz$$

As $R \to \infty$

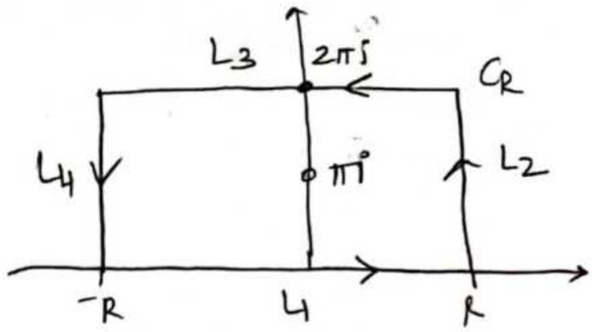
$$0 = -\frac{\sqrt{\pi}}{2}\sqrt{i} + 0 + \int_0^\infty \cos(x^2) + i\sin(x^2)\,dx.$$

$\displaystyle\Rightarrow \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$ & $\displaystyle\int_0^\infty \sin(x^2)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}}$ $\because \sqrt{i} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

<u>Ex:</u> $\displaystyle\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = ?$ $\quad (0<a<1)$

$\hookrightarrow$ why? So $\dfrac{e^{ax}}{1+e^x} \to 0$ fast enough as $x \to \infty$.

$$f(z) = \frac{e^{az}}{1+e^z}, \quad f(x) = \frac{e^{ax}}{1+e^x} \text{ on } \mathbb{R}.$$



Singular when $z = i(2k+1)\pi$

Note on L_3,

$$f(z) = \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}} = \frac{e^{ax}}{1+e^x}\cdot e^{a2\pi i}$$

So, $$\int_{L_3} f(z)\,dz = \int_R^{-R} f(x+2\pi i)\,dx = -\int_{-R}^{R} \frac{e^{ax}e^{a2\pi i}}{1+e^x}\,dx$$

$$\int_{L_1} f(z)\,dz = \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx$$

On L_2, $$|f(z)| = \left|\frac{e^{a(R+iy)}}{1+e^{R+iy}}\right| = \frac{e^{aR}}{|1+e^R e^{iy}|} \le \frac{e^{aR}}{|1-e^R|} = \frac{e^{aR}}{e^R-1}$$

for R large.

So, $$\left|\int_{L_2} f(z)\,dz\right| \le \frac{2\pi e^{aR}}{e^R-1} \to 0 \text{ as } R\to\infty \quad \text{since } a<1$$

On L_4, $$|f(z)| \le \frac{e^{-aR}}{1-e^{-R}} \text{ for } R \text{ large.}$$

So, $\left| \int_{L_4} f(z)\,dz \right| \leq \dfrac{2\pi e^{-aR}}{1-e^{-R}} \to 0$ as $R \to \infty$.

Finally, $\displaystyle\int_{C_R} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=\pi i} f(z)$

$$\frac{1}{1+e^z} = \frac{1}{1+e^{\pi i} e^{z-\pi i}} = \frac{1}{1-e^{z-\pi i}} = \frac{1}{(-z+\pi i)-\left(\sum_{n=2}^{\infty} \frac{(z-\pi i)^n}{n!}\right)}$$

$$= \frac{-1}{z-\pi i} \cdot \frac{1}{1+\sum_{n=1}^{\infty} \frac{(z-\pi i)^n}{(n+1)!}}$$

Now, $\displaystyle\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ for $|z|<1$

$g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(z-\pi i)^n}{(n+1)!}$ is analytic on $\mathbb{C}$ and so for z close enough to πi, $|g(z)| < 1$. Since $g(\pi i) = 0$

So, $$\frac{1}{1+e^z} = \frac{-1}{z-\pi i} \cdot \left(\sum_{n=0}^{\infty} (-1)^n \left(\sum_{k=1}^{\infty} \frac{(z-\pi i)^k}{(k+1)!} \right)^n \right)$$

$$= \frac{-1}{z+\pi i} + \text{higher powers of } z-\pi i$$

Also, $e^{az} = e^{a(z-\pi i)} e^{a\pi i} = e^{a\pi i} + e^{a\pi i}(z-\pi i) + \dots$

$$\Rightarrow \operatorname*{Res}_{z=\pi i} \frac{e^{az}}{1+e^{z}} = (-1)\, e^{a\pi i} \Rightarrow \int_{C_R} f(z)\,dz = -2\pi i\, e^{a\pi i}$$

Thus,

$$-2\pi i\, e^{a\pi i} = -\int_{-\infty}^{\infty} \frac{e^{ax} e^{2\pi a i}}{1+e^{x}}\,dx + \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{-2\pi i\, e^{a\pi i}}{1-e^{2\pi a i}} = \pi \frac{2i}{e^{\pi a i} - e^{-\pi a i}} = \underline{\underline{\frac{\pi}{\sin(\pi a)}}}$$

Argument Principle.

Def: The logarithmic residue of $f(z)$ at $z=a$ is

$$\operatorname*{Res}_{z=a} \frac{f'(z)}{f(z)}$$

Say $z=a$ is a zero of order r. Then

$$f(z) = C_r(z-a)^r + C_{r+1}(z-a)^{r+1} + \cdots$$

$$f'(z) = rC_r(z-a)^{r-1} + (r+1)C_{r+1}(z-a)^r + \cdots$$

$$\frac{f'(z)}{f(z)} = \frac{rC_r(z-a)^{r-1} + (r+1)C_{r+1}(z-a)^r + \cdots}{C_r(z-a)^r + C_{r+1}(z-a)^{r+1} + \cdots}$$

$$= \frac{r}{z-a} \cdot \frac{1 + \frac{r+1}{r}\frac{C_{r+1}}{C_r}(z-a) + \frac{r+2}{r}\frac{C_{r+2}}{C_r}(z-a)^2 + \cdots}{1 + \frac{C_{r+1}}{C_r}(z-a) + \frac{C_{r+2}}{C_r}(z-a)^2 + \cdots}$$

analytic & for z close to a, $\longleftarrow$

$$= \frac{r}{z-a}\left(1+\frac{r+1}{r}\,\frac{c_{r+1}}{c_r}(z-a)+\dots\right)\left(\sum_{n=0}^{\infty}(-1)^n\left(1+\frac{c_{r+1}}{c_r}(z-a)+\dots\right)^n\right)$$

$$= \frac{r}{z-a} + \text{higher powers of } (z-a)$$

$$\Rightarrow \operatorname*{Res}_{z=a} \frac{f'(z)}{f(z)} = r$$

Say $z=b$ is a pole of order s.

$$f(z) = \frac{c_{-s}}{(z-b)^s} + \frac{c_{-s+1}}{(z-b)^{s-1}} + \dots$$

$$f'(z) = \frac{-s\,c_{-s}}{(z-b)^{s+1}} + \frac{(1-s)\,c_{-s+1}}{(z-b)^s} + \dots$$

Calculate as before $\dfrac{f'(z)}{f(z)} = \dfrac{-s}{z-b}$ + higher powers of $z-b$.

$$\Rightarrow \operatorname*{Res}_{z=b} \frac{f'(z)}{f(z)} = -s.$$

By Residue Theorem, if $f(z)$ has zeroes at $a_1, a_2, \dots, a_m$ and poles at $b_1, \dots, b_n$ and analytic elsewhere on $\operatorname{Int}(C)$ for a simple closed smooth curve C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{m} \operatorname*{Res}_{z=a_k} \frac{f'(z)}{f(z)} + \sum_{k=1}^{n} \operatorname*{Res}_{z=b_k} \frac{f'(z)}{f(z)}$$

$$= \sum_{k=1}^{m} r_k - \sum_{k=1}^{n} s_k$$

s_k → order of pole b_k

r_k → order of zero a_k.

Ex: $f(z) = \dfrac{z^2}{(1+z)^3}$ $\quad C = \{|z| < 3\}$

$N = 2 \leftarrow z = 0$ is order 2 zero

$P = 3 \leftarrow z = -1$ is order 3 pole

$f(z) = \dfrac{(z+1)\sin z}{z^2(z-1)^3}$

$N = 1 \leftarrow z = -1$ order 1 zero

$P = 1+3 \leftarrow z = 1$ pole order 3

$\dfrac{\sin z}{z^2} = \dfrac{1}{z} + \cdots$

> $N - P = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f'(z)}{f(z)} dz = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{d}{dz} (\ln f(z))\, dz$

$$= \frac{1}{2\pi i} \underbrace{\Delta_C \ln(f(z))}$$

defined as change in $\ln(f(z))$ when z makes one loop around C.

$\ln z = \ln|z| + i\arg(z)$

$\Rightarrow \ln f(z) = \ln|f(z)| + i\arg f(z)$

$\Rightarrow \Delta_C \ln f(z) = i\,\Delta_C \arg f(z).$

$\Rightarrow N - P = \dfrac{1}{2\pi i}\Delta_C \ln f(z)$

$\Rightarrow \boxed{N - P = \dfrac{1}{2\pi}\Delta_C \arg f(z)}$ argument principle!

Thm (Rouché)

$f(z), g(z)$ analytic on $\overline{\text{int}(C)}$, C is pw smooth simple closed curve and $|f(z)| > |g(z)|$ $\forall z \in C$.

Then $f(z)$ and $f(z)+g(z)$ have the same number of zeroes in int(C), counting order.

Proof: $|f(z)| > |g(z)| \geq 0 \Rightarrow f(z) \neq 0$ $\forall z \in C$.

$$\Delta_C \arg[f(z)+g(z)] = \Delta_C \arg\left[f(z)\left(1+\frac{g(z)}{f(z)}\right)\right]$$

$$= \Delta_C \arg f(z) + \Delta_C \arg\underbrace{\left\{1+\frac{g(z)}{f(z)}\right\}}_{\text{analytic on } C}$$

Since, $\left|\dfrac{g(z)}{f(z)}\right| < 1$ so $1+\dfrac{g(z)}{f(z)} \in B_1(1) \Rightarrow \Delta_C \arg\left\{1+\dfrac{g(z)}{f(z)}\right\} = 0.$

$\Rightarrow \Delta_C \arg(f(z)+g(z)) = \Delta_C \arg f(z)$

$\Rightarrow N_{f+g} = N_f$ ∎

Ex: Find # of zeroes of $h(z) = z^9 + z^5 - 8z^3 + 2z + 1$

in $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$

First consider $B_1(0)$.

Usual trick : $h(z) = f(z) + g(z)$. Pick $|f(z)| > |g(z)|$ s.t

we know zeroes of $f(z)$.

Let $f(z) = -8z^3$; $g(z) = z^9 + z^5 + 2z + 1$ since on $|z| = 1$, $8z^3$ dominates.

$|-8z^3| = 8|z|^3 = 8$ on C_1 ie $|z| = 1$

$|g(z)| \leq |z^9| + |z^5| + |2z| + 1 = 5.$

$\Rightarrow$ on C_1 $|f(z)| > |g(z)|$

In $B_1(0)$, f & $f+g = h$ have same no. of zeroes.

Hence, $h(z)$ has 3 zeroes in $B_1(0)$

Consider, $C_2 = \{|z| = 2\}$, $B_2(0)$

$h = f + g = z^9 + z^5 - 8z^3 + 2z + 1$

Let $f = z^9$; $g = z^5 - 8z^3 + 2z + 1$

$|f(z)| = 2^9$; $|g(z)| \leq 2^5 + 2^6 + 2^2 + 1 < 2^9 = |f(z)|$

$\Rightarrow h(z)$ has 9 zeroes in $B_2(0)$

$\Rightarrow$ 6 zeroes in $1 \leq |z| \leq 2$.