Lecture 1   MATH 420   (Hoffman & Kunze)   1

Def: A group is a set G with a binary operation $G \times G \to G$ "product" which satisfies

(1) associative law: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(2) $\exists$ element $1 \in G$ so $1x = x1$ for all $x \in G$

(3) Inverses: given $x \in G$ $\exists$ $x' \in G$ so $xx' = 1 = x'x$.

> An abelian group is a group which satisfies the commutative law:
$xy = yx$ for all $x, y \in G$.

> Sometimes, group operation is written as '+' (eg $(\mathbb{Z}, +)$ form an abelian group).

> A ring is a set R with two binary operations '$+, \cdot$' so $(R, +)$ is an abelian group.

Also:

(1) mult is associative $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(2) there is $1 \in R$ so $1 \cdot x = x \cdot 1 = x$ for all $x \in R$

(3) distributive laws: $x(y+z) = xy + xz$ & $(x+y)z = xz + yz$ for $x, y, z \in R$

R is commutative if $xy = yx$

R is a field if R is commutative & if $x \in R$, $x \neq 0$, then $\exists\ x' \in R$ so $xx' = 1 = x'x$ AND: $0 \neq 1$

Algebra

Solving algebraic equations.

eg: $x_1^2 + x_2^2 - 1 = 0$

Consequences of axioms for a ring

> If x has an inverse, there is just one:

Say x', x'' satisfy $xx' = x'x = 1$ &
$xx'' = x''x = 1$

Then $x' = x''$.

$$x' = x'1 = x'(xx'') = (x'x)x'' = 1x'' = x''.$$

> Also $0x = 0 \ \forall x$

$$0x = (0+0)x = 0x + 0x$$

Add $-0x$ to both sides

$$\Rightarrow (0x - 0x) = 0x + 0x - 0x$$

$$0 = 0x + 0$$

$$\Rightarrow 0x = 0$$

In R, we have -1. If $x \in R$, claim $(-1)x = \underset{\sim}{-x}$ additive inverse of x.

Check: $x + (-1)x \overset{?}{=} 0$

$\underline{LHS} = 1x + (-1)x = (1+(-1))x = 0x = 0$

Solving linear equations:

$$(*) \quad \text{System} \begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \dots + a_{m,n}x_n = b_m \end{cases}$$

Coefficients are in a field F.

> ERO = elementary row operation on a system. It is one of the following operations

1) multiply one equation by a non zero scalar,
2) Add a multiple of one row to a different row.
3) Interchange two rows.

System $(*)$ is transformed to a new system $(**)$

Claim: Solution set of $x_1 \dots x_n$ which satisfy $(*)$ is same for $(**)$.

Observe each elementary row operation can be undone by another ER

If R is a ring,

An $m \times n$ matrix over R is a function

$f : \{1, 2, \dots, m\} \times \{1, 2, \dots, n\} \to R.$

$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots \\ \vdots & & \\ \vdots & & \end{pmatrix}_{m \times n}$ is a picture of a matrix. (loosely called a matrix)

We can represent the system (*) by matrix equation "$Ax = b$". We can do EROs by use of matrix mult.

$Ax = b \rightsquigarrow EAx = Eb.$

Define sum, Scalar mult. & product of matrices.

Sum

$\underset{m \times n}{A} + \underset{p \times q}{B}$ defined only when $(m,n) = (p,q)$

$a_{ij} + b_{ij}$ is the i,j entry of $A+B$.

Scalar mult.

$cA_{m \times n} = (c\, a_{ij})_{m \times n}$

Product

$\underset{m \times n}{A} \underset{p \times q}{B}$ only when $n = p$

The i,j entry of AB (if $n = p$) is $\sum_{k=1}^{n} a_{ik} b_{kj}$.

Back to (*)

Define $A = \begin{pmatrix} a_{1,1} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}_{m\times n}$

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}_{n\times 1}$$

An n-tuple $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ satisfies (*) $\Longleftrightarrow A\vec{x} = \vec{b}$

Summary

Group: associative, identity element & inverse.

Abelian group: Group with commutative property.

Ring: $(R, +)$ abelian group; $(R, \cdot)$ associative, mult identity & distributive under $+$.

Commutative ring: commutative under $\cdot$.

Field: A commutative ring with mult inverse.

Lecture ②

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

$A = (a_{ij})_{\substack{i=1\dots m \\ j=1\dots n}}$

$(A|b) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \vdots & & & & \vdots \\ a_{m,1} & \dots & \dots & a_{mn} & b_m \end{array}\right)_{m \times (n+1)}$

Augmented matrix.

We shall see that EROs on matrices can be achieved by left multiplication ~~with an~~ EROM (matrix).

(1,2) → this row + row this become row this's?

Eg $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$ ERO2

$\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_m \end{pmatrix}_{m \times m}$ diagonal matrix. multiplies each row with $c_1, c_2 \dots c_m$ resp. ERO1

$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ rows 3 & 4 interchanged in the I matrix ⇒ interchanges rows 3 & 4 in A when left multiplied. ERO3

<u>Remark:</u> $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\dots$ $e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$A_{m\times n} \Rightarrow \underset{m\times n}{A}\,\underset{n\times 1}{e_k} = m\times 1$ it is column k of mtx A.

> Interchange of 2 columns can be achieved by other 2 EROs.

Eg:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

picks out 2nd column

picks out 1st column.

$$\underbrace{\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\text{EROs}} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{column interchange.}}$$

Eg:

$$\begin{pmatrix} 2 & & \\ & 3 & \\ & & 4 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \\ 4g & 4h & 4i \end{pmatrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 & & \\ & 3 & \\ & & 4 \end{pmatrix} = \begin{pmatrix} 2a & 3b & 4c \\ 2d & 3e & 4f \\ 2g & 3h & 4i \end{pmatrix}$$

Def: A, B $n \times n$ matrices ; call (A, B) an inverse pair if $AB = BA = I_n$.

> EROs form inverse pairs. $\Rightarrow$ (inverse of ERO is also ERO)

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t+u \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \text{if } u = -t \text{ we get inverse pair.}$$

$$\begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_m \end{pmatrix}\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_m \end{pmatrix} = \begin{pmatrix} c_1 d_1 & & & \\ & c_2 d_2 & & \\ & & \ddots & \\ & & & c_m d_m \end{pmatrix} \quad \Rightarrow d = \frac{1}{c} \Rightarrow \text{inverse pair.}$$

Def: $\underset{m\times n}{A}x = b$ linear system.

<u>Homogeneous</u> if $\underset{m\times 1}{b} = \underset{m\times 1}{0}$

Otherwise inhomogeneous.

<u>Remark</u>: Homogeneous systems always have $\vec{x} = \vec{0}$ as the trivial sol.

Inhomogeneous systems may have no soln.

Procedure involves rowreducing $(A|b)$ to the RREF.

<u>Def</u>: A matrix is in RRF (row reduced form)

if every nonzero row has leading entry <u>1</u> & all other entries in that column are zeros.

<u>RREF</u>: Every zero row is below every nonzero row. Also if r is the number of nonzero rows and row1 has leading column index k_1, & row2 has leading col. index k_2, so on. then $k_1 < k_2 < \dots < k_r$.

<u>Thm:</u> $A_{m\times n}$ may be transformed to matrix in RREF by finite no. of EROs.

<u>Proof</u>: (book)

If $A=0$, done. Suppose $A\neq 0$.

Let k_1 be smallest index of a nonzero column.

Take row index i_1 so entry $a_{i_1,k_1}\neq 0$.

Divide row i_1 by a_{i_1,k_1} to make it 1. Then add multiples of row i_1 to other rows to create 0s above & below.

If there are no nonzero entries off of row i_1, done.

Suppose there are, say position $(i,j)_{j>k_1}$. Let k_2 be the least such j. $\exists\, i_2\neq i_1$ so that $a_{i_2,k_2}\neq 0$. Now do EROs to make $(i_2,k_2)=1$ & 0s everywhere else on that column.

Continue on to get a RRF. Interchange rows to get RREF. ∎

<u>Note</u>: RREF is unique. (proved later)

> $Ax=b$ has same soln. set as $Rx=z$ where R is the RREF.

Let $x_{k_1},\dots,x_{k_r}$ variables, $k_1\dots k_r$ leading indices, $u_1\dots u_{n-r}$ other variables.

Eqns. look like

$$x_{k_1} + \sum_{i=1}^{r} c_{ik_1} u_i = z_1$$

$$\vdots$$

$$x_{k_r} + \sum_{i=1}^{r} c_{i,k_r} u_i = z_r$$

$$0 = z_{r+1}$$

$$\vdots$$

$$0 = z_m$$

$x_{k_1} \dots x_{k_r}$ are dependant variables & $u_1, \dots u_{n-r}$ are free variables

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Lecture 3 (OH - M & W 11-12:30) (3062 EH)

Ex: $Ax = b$

augmented matrix
$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 2 & a \\ 0 & 1 & 1 & -1 & b \\ 0 & 3 & 1 & 3 & c \end{array}\right)$$

Sol:

$$\to \left(\begin{array}{cccc|c} 0 & 1 & 0 & 2 & a \\ 0 & 0 & 1 & -3 & b-a \\ 0 & 0 & 1 & -3 & c-3a \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 0 & 1 & 0 & 2 & a \\ 0 & 0 & 1 & -3 & b-a \\ 0 & 0 & 0 & 0 & c-b-2a \end{array}\right)$$

Case i) $c-b-2a \neq 0$

divide 3 by $c-b-2a \Rightarrow$ "$0=1$" $\divideontimes$ no soln.

Case ii) $c-b-2a = 0$

$$x_2 + 2x_4 = a$$
$$x_3 - 3x_4 = b-a$$

leading variables are x_2, x_3; free variables x_1, x_4

general soln. $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ a-2x_4 \\ b-a+3x_4 \\ x_4 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} 0 \\ -2 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ a \\ b-a \\ c \end{pmatrix}$$

Each leading variable occurs only once so the row eqns. do not contradict each other.

<u>Observe</u> $Ax = 0$ homogeneous always has the trivial soln.
(Sometimes more solns.)

<u>General Gauss Jordan procedure:</u>

$Ax = b$ convert by ERO to

$Rx = z$, R is RREF.

r = no. of nonzero rows.

Equations indexed by $r+1, r+2 \dots$ look like $0 = z_k$; $k = r+1, \dots$

If $\exists$ such $z_k = 1 \Rightarrow$ "no sol."

If all are 0, system has nontrivial equations.

$$x_{k_i} + \sum_{j=1}^{n-r} c_{ij} u_j = z_i$$

$u_1 \dots u_{n-r}$ are free variables
c_{ij} Scalars.

<u>Ex:</u>

$$\underset{P}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} \overset{\tilde{P}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p & q \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \tilde{P}$ is one sided inverse of P. Not two sided inverse.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ p & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & q & 0 \end{pmatrix}$$

Def: A, B $n \times n$ matrices are inverse pairs: $AB = BA = I_n$.

Thm: Equivalent are, for $A_{n\times n}$: (nxn square matrix)

(i) A invertible (i.e. has 2 sided inverse)

(2) A is row equivalent to I_n

(3) A is a product of EROMs.

(4) $\underset{n\times n}{A}\ \underset{n\times 1}{X} = \underset{n\times 1}{0}$ has only trivial solution.

(5) $AX = \underset{n\times 1}{Y}$ solvable for all Y.

Def: $A_{m\times n}$, $B_{m\times n}$ are row equivalent if $\exists$ EROMs $E_1, \dots, E_k$ so $B = E_k E_{k-1} \cdots E_1 A$.

Observe: An EROM has a 2 sided inverse.
So if $A =$ product of EROMs, A is invertible

In general if P, Q $n \times n$ & invertible, then PQ is invertible.

$\Big(\exists\ P_1$ so $PP_1 = I = P_1 P$; $\exists\ Q_1$ so $QQ_1 = I = Q_1 Q$,
so $(Q_1 P_1)(PQ) = I = (PQ)(Q_1 P_1) \Big)$

Proof

1) A inv. $R =$ RREF of A. $\left(A^{-1}Ax = Ix = x.\right.$

$\Rightarrow Ax = 0$ has only 0 as sol.)

$Ax = 0$ has only $x = 0$ as sol.

Since

$Ax = 0 \Rightarrow A^{-1}Ax = 0$

$\Rightarrow Ix = 0 \Rightarrow \underline{\underline{x = 0}}$

$\Rightarrow Rx = 0$ has only trivial soln.

$\Rightarrow$ bottom row of R is non zero } otherwise non trivial solns. can exist. (In $n \times n$ case no. of eqns = no of variables so if one row $= 0 \Rightarrow$ free variables must exist)

[else $\begin{pmatrix} & \vdots & \\ 0 & 0 \cdots & 0 \end{pmatrix}$ so $\exists$ free variables.

so $\exists$ x sol. with some free variables $\neq 0$ $\divideontimes$]

$\Rightarrow$ no free variables & all rows nonzero ⇓

$R_{n \times n}$ all rows nonzero $\Rightarrow R = I \Rightarrow (2)$

So if $E_1 \cdots E_k$ are EROMs so $E_k \ldots E_1 A = I$.

$A = (E_k \ldots E_1)^{-1} = E_1^{-1} \ldots E_k^{-1} \Rightarrow (3)$ ((4) is done above)

Given Y

$A^{-1}Y$ solves $AX = Y \Rightarrow (5)$.

$\underline{\underline{(A(A^{-1}Y)) = Y}}$

Reverse implications

(2) $\Rightarrow$ (1) since A = product of EROMs.

(3) $\Rightarrow$ (1) " " .

(4) $\Rightarrow$ $RX = 0$ has only trivial sol. $R = RREF$ of A.

$\Rightarrow$ bottom row of R is nonzero.

$\Rightarrow R = I$ so $\underbrace{E_k \dots E_1}_{\text{EROMs}} A = I \Rightarrow (1)$

(5) $R = RREF$ for A

Claim: bottom row of R is nonzero.

if no

$$R = \begin{pmatrix} * \\ \vdots \\ 0\ 0\ \dots\ 0 \end{pmatrix} \qquad RX = R\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \text{col with } n\text{th coord} = 0$$

so if $Y = \begin{pmatrix} 0 \\ \vdots \\ 9 \end{pmatrix}$

$\Rightarrow RX = Y$ not solvable

$\Rightarrow R = I$ $\divideontimes$

Matrix multiplication.

$\underset{m\times n}{(A}\underset{n\times p}{B)}\underset{p\times q}{C} = A(BC)$ associative (straightforward proof).

$$A_{m\times n} = \underbrace{(A_1 \mid A_2 \mid \dots \mid A_n)}_{\text{columns}}$$

$$Ax = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{x_1A_1 + x_2A_2 + \dots + x_nA_n}_{\text{linear comb. of columns of } A}$$

Proof: compare coefficients

$$\underset{n\times 1}{e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} \to j^{th} \text{ row} \qquad \Rightarrow A e_j = A_j$$

Blocked matrices

$$\underset{m\times(n+p)}{A} = (\underset{m\times n}{B_1} \quad \underset{m\times p}{B_2})$$

$$\underset{(n+p)\times r}{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\begin{matrix} n\times r \\ p\times r \end{matrix}$$

$$\Rightarrow \underset{(m\times r)}{AC} = (\underset{m\times r}{B_1C_1} + \underset{m\times r}{B_2C_2})$$

Lecture 4

> Block matrices notes (to be uploaded).

Lemma: $A_{n\times n}$ Suppose A has a left inverse L & a right inverse R, then $L = R$.

Proof: $L = LI = L(AR) = (LA)R = IR = R.$ //

Corollary: If A has a 2 sided inverse, it is unique.

Thm: If $AB = I$, then $BA = I$ for A, B $n\times n$ matrices.

Proof: Suppose $AB = I$

Then $\forall \vec{y}$, $A(By) = (AB)y = y$.

So every eqn $Ax = y$ is solvable.

$\overset{\text{Thm}}{\Rightarrow}$ A is invertible, so A^{-1} is the two sided inverse $\overset{\text{lemma}}{\Rightarrow}$ $B = A^{-1}$

If $BA = I$ similar arg shows B is invertible.

If A^{-1} is the two sided inverse, $A^{-1}A = I \Rightarrow BA = I$!

Thm: Suppose $A_1 \dots A_k$ are $n\times n$ matrices, then $A := A_1 \cdots A_k$ is invertible iff each A_j is invertible.

Proof: $\Leftarrow$ (easy)

$\Rightarrow$ Given A is invertible, so all equations $Ax = y$ are solvable.

$\Rightarrow$ All eqns. $A_1 w = y$ are solvable (let $w = A_2 \dots A_k x$ where $Ax = y$)

$\Rightarrow A_1$ is invertible

$\Rightarrow A_2 \dots A_k = A_1^{-1} A$ is invertible.

Use induction on k to conclude each A_j is invertible. //

Procedure for finding $A^{-1}_{n\times n}$ when it exists.

Form $(A \mid I)_{n \times 2n}$

Row reduce $E_1, \ldots, E_k$

So $(\underbrace{E_k \ldots E_1 A}_{RREF = R} \mid E_k \ldots E_1)$

If $R = I$, then $\underbrace{E_k \ldots E_1}_{} = A^{-1}$
("B" in Thm $BA = I$).

<u>n=2</u>

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow AA' = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

If $ad-bc \neq 0$ then $\frac{1}{ad-bc} A' = \underline{A^{-1}}$

If $ad-bc = 0$ then we want to show that A is not invertible.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} ad-bc \\ cd-cd \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if c, d are nonzero, the homogeneous system $Ax = 0$ has a nontrivial soln.
$\Rightarrow$ A not invertible.

If c,d are zero but a,b non zero implies.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} ab - ab \\ -ad + bc \end{pmatrix} = \underline{0} \leftarrow \text{by imp}^{2}$$

If a,b,c,d are zero → Finally $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$ not invertible

Similar analysis to before.

Chapter 2 (Vector spaces)

F field

Def: A vector space over F is a set V with binary operations $V \times V \xrightarrow[\text{addition}]{+} V$, $F \times V \xrightarrow[\text{Scalar mult.}]{\cdot} V$. (scalar mult. is only from left)

(V, +) forms an abelian group with zero element = 0.

$(0 + v = v = v + 0$ for all $v \in V)$

Scalar multiplication satisfies,

(1) associative law

$c(dv) = (cd)v \quad \forall\, c, d \in F$ & $v \in V$

(2) distributive laws

$(c+d)v = cv + dv$

$c(v+v') = cv + cv'$

(3) $1v = v$ for all v $(1 \in F)$ & $1 \neq 0$

Ex1

$F^{m\times n}$ = all $m\times n$ matrices over F with componentwise arithmetic forms a vector space.

Ex2 Set S, Function (S,F) from S to F. is a vector space.

f, g $(f+g)\,a = f(a)+g(a)$.

Linear combination (LC)

of a sequence $v_1, \dots, v_r$ in V is some expression

$$c_1v_1 + \dots + c_rv_r = \sum_{j=1}^{n} c_j v_j \text{ for scalars } c_1 \dots c_r \in F.$$

Def: Subspace

V is a VS (vec space) over F

A subspace of V is a subset W so

(1) $\vec{0} \in W$

(2) if $\vec{x}, \vec{y} \in W$ then $a\vec{x}+b\vec{y} \in W$ $\forall\, a,b \in F$

"closure under LCs".

(1) & (2) $\Rightarrow$ (1') $W \neq \emptyset$
(2) closure under LCs.

Q: Is $W = \{0\}$ a subspace?

<u>Def</u> S is a subset of V, the span of S is the set of all LCs made from finite sequences in S. ($s_1, s_2 \ldots s_n \in S$ be a finite seq. Linear combinations of this seq from the span)

<u>Lemma</u> If I is a nonempty index set (indexes the as.) and W_a, $a \in I$ is a subspace of V then, $\bigcap_{a \in I} W_a$ is a subspace. (Intersection of subspaces forms a subspace)

<u>Proof</u>: $\forall a,\ \vec{0} \in W_a$

$$\Rightarrow \vec{0} \in \bigcap_{a \in I} W_a \Rightarrow (1).$$

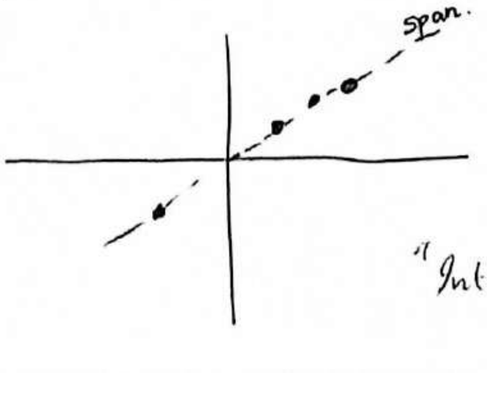
For $x, y \in \bigcap_{a \in I} W_a$ & scalars c, d, we have

$$\forall a \quad x, y \in W_a \Rightarrow cx + dy \in \underset{\underline{\underline{\forall a}}}{W_a} \text{ (subspace)}$$

$$\Rightarrow cx + dy \in \bigcap_{a \in I} W_a \Rightarrow (2). \quad //$$

<u>Def</u>: For S a subset of V, we define the subspace of V generated by S to be $\bigcap_{\substack{W \text{ subspace of } V \\ \& \ W \supseteq S}} W$.

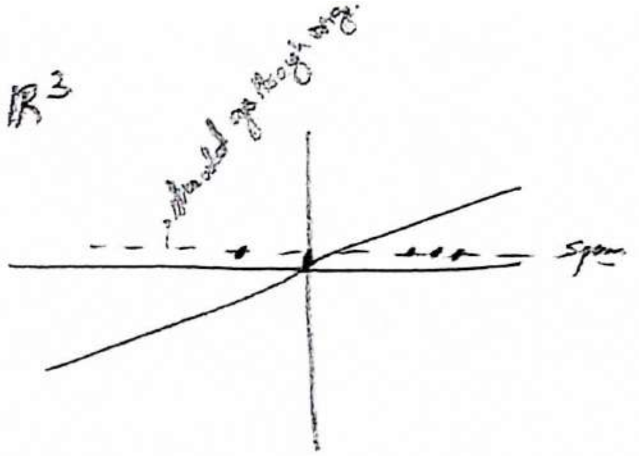
<u>EX</u>: $\mathbb{R}^2$



S is the set of 4 points.

the span & $\mathbb{R}^2$ are the 2 subspaces.

"Intersection of all subspaces of V that contain S".



Here we have span, planes containing the line & $\mathbb{R}^3$.

$\bigcap$ these sets is the span.

<u>Thm</u>: If $S \neq \emptyset$ then $\text{span}(S) =$ the subspace generated by S.

<u>Proof:</u> First we show the $\text{span}(S)$ is a subspace.

$S \neq \emptyset$ so take $v \in S$. $0 \cdot v = \vec{0}$ is a LC in $\text{span}(S)$.

$\Rightarrow \vec{0} \in \text{span}(S)$.

<u>Next</u>. Closure under LCs.

Take $x = \sum_{i=1}^{p} a_i v_i$. $y = \sum_{j=1}^{q} b_j w_j$ where

$v_1, \dots, v_p, w_1, \dots, w_q \in S$. Take scalars c, d.

We want $cx + dy \in \text{span}(S)$.

$$cx + dy = \sum_i c a_i v_i + \sum_j d b_j w_j \longrightarrow ?$$

Set $U := \bigcap_{\substack{W \text{ subspace } V \\ W \supseteq S}} W$. We want $U = \text{span}(S)$.

Observe that $\text{span}(S)$ is a subspace which contains S.

Take $x \in S$, $x = 1 \cdot x$, a LC of elements of S.

$\Rightarrow \text{span}(S) \supseteq U$

If $W \supseteq S$ then $\underset{\text{subspace}}{W} \overset{\text{since } W \text{ is a subspace}}{\supseteq} \text{span}(S) \Rightarrow U \supseteq \text{span}(S)$

$\Rightarrow U = \text{span}(S)$. //

Key idea, If $W \supseteq S$ since W is a subspace & span(S) is a subspace

↳ It must contain all linear combs of S.

$\Rightarrow W \supseteq \text{span}(S) \quad \underline{\forall W}$

$\Rightarrow U \supseteq \text{span}(S)$.

Since span(S) is a subspace & $\text{span}(S) \supseteq \text{span}(S) \Rightarrow \text{span}(S) \supseteq U$.

Lecture 5

Last time we proved that

$$\text{span}(S) = \bigcap_{\substack{W \text{ subspace of } V, \\ W \supseteq S}} W \quad \text{for a non empty subset of } V, \text{ v.s.}$$

Def: If S & T are subsets of V (v.s $\overset{\text{over}}{/F}$)

Then $S + T := \{x + y \mid x \in S, y \in T\}$.

So if W_1 and W_2 are subspaces we get the subset $W_1 + W_2$.

Lemma: $W_1 + W_2$ is a subspace.

Proof: $0 = \underset{W_1}{\underset{\sim}{0}} + \underset{W_2}{\underset{\sim}{0}} \in W_1 + W_2$

Typical elements of $W_1 + W_2$

$x = x_1 + x_2 \qquad y = y_1 + y_2$

$x_j, y_j \in W_j \quad j = 1, 2$

Scalars a, b

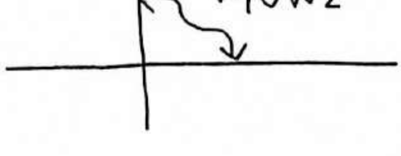
$$ax + by = a(x_1 + x_2) + b(y_1 + y_2)$$
$$= \underbrace{(ax_1 + by_1)}_{W_1} + \underbrace{(ax_2 + by_2)}_{W_2}$$
$$\in W_1 + W_2$$

Exercise: Prove that $W_1 + W_2 = \text{span}(W_1 \underset{\text{set}}{\cup} W_2)$

Eg: $V = \mathbb{R}^2$

$W_1 = x_1\text{-axis}$

$W_2 = x_2\text{-axis}$



Def V v-s

A sequence of finite lengths $v_1 \dots v_n$ is linearly dependant (LD) if there exist scalars $c_1 \dots c_n$, not all 0, so

$$\sum_{i=1}^{n} c_i v_i = 0$$

Sequence is linearly independant (LI) if not LD.

Def S subset of V, it is linearly dependant if $\exists$ a sequence of distinct vectors in S which is linearly dep. Otherwise S is LI.

Eg: the sequence 0 is LD because $\underset{\text{not } 0}{1} . 0 = 0$

A subsequence of a LI sequence is LI. i.e if $v_1 \dots v_n$ is LI. So is subseq $v_{i_1}, \dots, v_{i_k}$ where $i_1 < i_2 < \dots < i_k$.

If $\sum_{j=1}^{k} c_j v_{i_j} = 0$ then $\sum_{\substack{j=1 \\ j \neq i_1 \dots i_k}}^{n} 0 . v_j + \sum_{j=1}^{k} c_j v_{i_j} = 0$

$\Rightarrow$ all coeffs $= 0$.

Book

Def V v.s. A basis for V is a LI set which spans V.

> An ordered basis is a basis with a total ordering.

> In this course, in finite dimensional situation "basis" means ordered basis.

Eg: $F^{n\times 1}$ has a standard basis, $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

If $x \in F^{n\times 1}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} x_i e_i$.

Eg: $m\times n$ linear system $Ax = 0$. Leading variables $x_{k_1}, \dots, x_{k_r}$. $\{1, 2, \dots, n\} = K \cup J$. Where $K = \{k_1, \dots k_r\}$, $J = \{j \mid j \in \{1, \dots, n\}, j \notin K\}$ free indeces.

The solution set has basis obtained by setting one free variable $= 1$, rest $= 0$.

free $\begin{pmatrix} * \\ 1 \\ * \\ 0 \\ * \\ \vdots \\ 0 \end{pmatrix}$ $\begin{pmatrix} * \\ 0 \\ * \\ 1 \\ * \\ \vdots \\ 0 \end{pmatrix}$ ....

v_1 $\quad$ v_2

This set is LI, since $\sum n_i x_i$ has coeff n_i at position k_i.

S set $\quad V = \text{Function}(S, F)$ is a v.s

For $a \in S$, $\quad f_a \in V \quad f_a(b) = \delta_{a,b} = \begin{cases} 1 & a = b \\ 0 & a \neq b \end{cases}$

If S is finite $\{f_a \mid a \in S\}$ forms a basis.

$f \in V \quad f = \sum_{a \in S} f(a) f_a$. Not so if $|S| = \infty$.

> Because if $U = \{a_1, \ldots, a_t\}$ is a finite subset of S.
Then any LC $\sum_{i=1}^{t} c_i f_a$ is 0 on $S \setminus U$.

A constant function $g : S \to F$ where $g(b) = 1 \ \forall b$.

Def A vector space V is finite dimensional if it is spanned by a finite set.

Theorem: V v.s, $V_1 \ldots V_p$ is L.I seq. $w_1 \ldots w_q$ is a seq.
Suppose $v_1, \ldots, v_p$ is in $W = \text{span}\{w_1, \ldots, w_q\}$ then $p \leq q$

Proof: Since each $v_j \in W$, $\exists$ expressions $v_j = \sum_{i=1}^{q} a_{ij} w_i$

$A = (a_{ij})_{q \times p}$. Consider a LC $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p =$

$$\sum_{j=1}^{p} x_j v_j = \sum_{j=1}^{p} x_j \left(\sum_{i=1}^{q} a_{ij} w_i \right)$$

$$= \sum_{i=1}^{q} \left(\sum_{j=1}^{p} x_j a_{ij} \right) w_i = \sum_{i=1}^{q} \Big(\underbrace{\sum_{j=1}^{p} a_{ij} x_j}_{\text{think: } Ax} \Big) w_i \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix}$$

So the coeff. at w_i is coefficient i of column matrix $\underset{q \times p}{A} x$.

Assume $p > q$, (note $\text{rank}(A) \leq p$ & $\text{rank}(A) \leq q$) row rank = col-rank

then $A\vec{x} = \vec{0}$ has a nontrivial soln. Since $\text{rank}(A) \leq q < p$
= no. of variables.

For such x, we get $\underbrace{x_1 v_1 + x_2 v_2 + \ldots + x_p v_p}_{=0 \text{ since } A\bar{x}=0 \text{ but } \bar{x} \neq 0}$ = LC of $w_1 \ldots w_q$ with all coeffs. = 0.

But $v_1, \ldots, v_p$ is LI. $\Rightarrow$ all $x_i = 0$ $\divideontimes$

<u>Corollary</u>: If v.s V has a basis of length m, then any basis has length m. (m here is the dimension by def)

<u>Proof</u>: $v_1 \ldots v_p$ & $w_1 \ldots w_q$ are two bases of V ($V=W$).

$p \leq q$ & $q \leq p$ (by interchanging thm). // . Since $\{\vec{v}\}$ spans $\{\vec{w}\}$ & $\{\vec{w}\}$ spans $\{\vec{v}\}$.

Exercise

If S, T are subsets of V then $\mathrm{span}(S \cup T) = \mathrm{span}(S) + \mathrm{span}(T)$.

(Sum of subspaces)

Lemma (Casting out Lemma)

Let V v.s and $r \geq 2$. Suppose $v_1, \ldots, v_r$ is a linearly dependant sequence of nonzero vectors. Then $\exists$ an index j which is between 2 and r $(2 \leq j \leq r)$ so that v_j is a linear combination of $v_1, \ldots, v_{j-1}$. Consequently $\mathrm{span}\{v_1, \ldots, v_j\}$

$$= \mathrm{span}\{v_1, \ldots, v_{j-1}\}$$

$$\mathrm{span}\{v_1, \ldots, v_r\} = \mathrm{span}\{v_1, \ldots \overset{\text{omit this term}}{\hat{v}_j}, \ldots v_r\}$$

$$= \mathrm{span}\{\{v_1, \ldots, v_r\} - \{v_j\}\}.$$

Corollary: Every finite dimensional vector space has a basis.

Pf of Lemma: Let j be the least index so that the sequence

$v_1, \ldots v_j$ is LD.

There are scalars $c_1, \ldots, c_j$ s.t $\sum_{i=1}^{j} c_i v_i = 0$.

Claim: $c_j \neq 0$. Since otherwise $v_1, \ldots, v_{j-1}$ is LD $\divideontimes$.

$\Rightarrow$ span $(S) \subseteq$ span (T). If $T \subseteq$ span(S)

$\Rightarrow$ span $(T) \subseteq$ span(span(S))

$=$ span (S).

$$\Rightarrow c_j v_j = \sum_{i=1}^{j-1} (-c_i v_i)$$

$$\Rightarrow v_j = \underbrace{\sum_{i=1}^{j-1} \left(\frac{-c_i}{c_j}\right) v_i}_{\{v_1 \dots v_j\} \Rightarrow T \subseteq \text{span}(S)} \leftarrow \text{span}(v_1, \dots, v_{j-1})$$

$\Rightarrow$ span $\{v_1, \dots, v_j\} =$ span $\{v_1, \dots v_{j-1}\}$

span $\{v_1 \dots v_r\} =$ span $\{v_1 \dots v_j\} +$ span $\{v_{j+1} \dots, v_r\}$ (use Exercise)

$\Downarrow$

span $\{v_1 \dots v_r\} =$ span $\{v_1 \dots v_{j-1}, v_{j+1}, \dots v_r\}$.

<u>Corollary</u>

If $v_1, \dots v_m$ is a LI sequence in V(v s) & if $w \notin$ span $\{v_1, \dots v_m\}$ then $v_1, \dots, v_m, w$ is LI.

(Note $w \neq 0$ since it is not in a subspace)

<u>Proof</u>: If LD, $\exists$ member which is LD of previous member.

<u>Thm</u>: (1) If V is a finite dim V s. & W is a subspace then W is a f-d v s & dim $W \leq$ dim V.

(2) If $W_1 \leq W_2$ are subspaces of V (fdvs). Then dim $W_1 \leq$ dim W_2 & $W_1 = W_2$ iff dim $W_1 =$ dim W_2.

(subspace) (given $W_1 \leq W_2$)

Proof

(1) $W \leq V$ fdvs.

In W, consider LI sequences. The length of any LI sequence is at most $= \dim(V)$

Take a LI seq. of greatest length in W. say $v_1, \dots, v_p$.

$$\underbrace{\text{span}\{v_1, \dots v_p\}}_{f.d} \leq W \leq \underbrace{V}_{f.d},$$

To show $v_1, \dots v_p$ spans W.

Suppose not true: Suppose span $\{v_1 \dots v_p\}$ is a proper subspace of W.

Take some element y in W which is outside span $\{v_1 \dots v_p\}$

then $v_1, \dots v_p, y$ is LI in W $\divideontimes$ to maximality of length of $v_1, \dots, v_p$.

Conclude that W spanned by $\{v_1 \dots v_p\} \Rightarrow \dim W = p \leq \dim(V)$.

(2) Take $W_1 \leq W_2$ in V. $\dim W_1 \leq \dim W_2$ by Thm.

Suppose $\dim W_1 = \dim W_2$ but $W_1 < \underbrace{W_2}$ → proper subspace $W_1 \neq W_2$

(Same arg). Take basis $v_1 \dots v_p$ for W_1, an element $y \in W_2 \setminus W_1$

And we have a LI seq. $v_1, \dots v_p, y$ in W_2 of length $> \dim W_2$.

$\divideontimes$

Procedure Given a LI sequence $v_1 \dots v_p$ in V (fdvs). Then there are vectors $v_{p+1}, \dots, v_n$ in V so $v_1, \dots, v_p, v_{p+1}, \dots v_n$ is a basis of V.

Proof: Let $w_1, \dots, w_r$ be a finite spanning set of V.

Study $v_1, \dots v_p, w_1, \dots, w_r$.

Remove all zero vectors.

We can remove elements from the right using casting out lemma.

ok if $p \geq 2$.

if $p = 1$, $r \geq 1$ ok

$p = 1$, $r = 1$ still OK.

Lemma
$v_1, \dots v_p$ basis of subspace W of V & $v_1, \dots v_p \dots v_n$ basis for V. Then if $x \in V$, $x = \sum_{i=1}^{n} c_i v_i$ then $x \in W$
$\Leftrightarrow c_i = 0 \ \forall \ i > p$.

Def: If $x \in V_{(vs)}$ with basis $v_1, \dots v_n$ then x is written in just one way as a LC of $v_1, \dots v_n$. So if $x = \sum_{i=1}^{n} c_i v_i$
$c_1, \dots c_n$ coordinates of x w.r.t basis $\mathcal{B} = \{v_1 \dots v_n\}$
We write $[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$

Proof of Lemma. ($\Leftarrow$) trivial.

($\Rightarrow$) Suppose $x \in W$ & some $c_j \neq 0$ for $j \geq p+1$.

Since $x \in W$, x is a linear of $v_1, \dots, v_p$

$$x = \sum_{i=1}^{p} a_i v_i .$$

Then $0 = x - x = \sum_{i=1}^{n} c_i v_i - \sum_{i=1}^{p} a_i v_i$ is a LC of

$v_1 \dots v_n$ not all coeffs $=0$, $c_j \neq 0$ $\times$ ■

Thm (Dimension Formula)

V (vs), W_1, W_2 fd subspaces.

Then $W_1 + W_2$ and $W_1 \cap W_2$ are fd & $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Proof:

Take a basis $x_1 \dots x_p$ of $W_1 \cap W_2$. Extend to basis

$x_1 \dots x_p, y_1, \dots, y_q$ of W_1.

Similarly for W_2.

Extend $x_1 \dots x_p$ to basis $x_1 \dots x_p, z_1 \dots z_r$ of W_2

Then $W_1 + W_2$ is spanned by $x_1 \dots x_p, y_1 \dots y_q, z_1 \dots z_r$.

We are done if this sequence is LI. Since $\dim(W_1) = p+q$
$\dim(W_2) = p+r$
$\dim(W_1+W_2) = p+q+r$

Consider not ($\Rightarrow$LD).

Consider a LC

$$\underbrace{\sum_{i=1}^{p} a_i x_i}_{X} + \underbrace{\sum_{j=1}^{q} b_j y_j}_{Y} + \underbrace{\sum_{k=1}^{r} c_k z_k}_{Z} = 0$$

$$= 0$$

$X+Y \in (W_1 \cap W_2) + W_1 = W_1$

$X+Y = -Z \in W_2$

$\Rightarrow X+Y \in W_1 \cap W_2.$

Nice argument

By lemma, the b_j are 0. (Since $X+Y \in W_1 \cap W_2$ & $X+Y \in W_1$ we have that it can be written in terms of only $\sum a_i x_i \Rightarrow b_j = 0$)

So we have $X + Z = 0$

The coeffs a_i & c_k must be 0 since $x_1 \ldots x_p\ z_1 z_r$ is basis for W_2.

∎

Lecture 7

Thm (Dimension Formula)

V vs. W_1, W_2 f.d subspaces (V need not be f.d)

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Ex:

$\dim(V) = 10$ What are possible dimensions of $W_1 \cap W_2$?

$\dim(W_1) = 6$ $\dim(W_1 + W_2) \leq \dim(V) = 10$

$\dim(W_2) = 8$ & $\dim(W_1 + W_2) \geq 8$

$\Rightarrow \dim(W_1 \cap W_2) = 4, 5, 6.$

Coordinates

Def: V v.s basis $v_1, \ldots, v_n$.

For $x \in V$

There are unique $c_1 \ldots c_n$. So $x = \sum_{i=1}^{n} c_i v_i$.

This sequence is called the coordinate seq. for x w.r.t basis $v_1 \ldots v_n$

Lemma: $v_1 \ldots v_n$ is a basis of V iff every vector in V has a unique expression as a linear comb. of $v_1 \ldots v_n$.

Proof: $v_1 \dots v_n$ basis means it spans V so $x \in V$

has an expression.

$$x = \sum_{i=1}^{n} a_i v_i$$

Another $x = \displaystyle\sum_{i=1}^{n} b_i v_i$

$$\Rightarrow \quad 0 = x - x = \sum_{i=1}^{n} (a_i - b_i) v_i$$

Since $v_1 \dots v_n$ is LI, all $a_i - b_i$ must be 0.

So $(\Rightarrow)$ follows.

$(\Leftarrow)$ easy ∎

Notation

If $v_1 \dots v_n$ basis

If $\displaystyle\sum_{i=1}^{n} x_i v_i$ $x_i \in F$, we write $[x]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

If $\underbrace{v_1' \dots v_n'}_{\mathcal{B}'}$ is another basis. Same x, $x = \displaystyle\sum_{i=1}^{n} x_i' v_i'$

$$[x]_{\mathcal{B}'} = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}.$$

We write $v_j' = \sum_{i=1}^{n} P_{ij} v_i$

$$P_{n\times n} = (P_{ij})$$

$$x = \sum_{j=1}^{n} x_j' v_j' = \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{n} P_{ij} v_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} x_j' \right) v_i$$

So $\underset{\substack{\parallel \\ [x]_{\mathcal{B}'}}}{P X'} = \underset{\substack{\parallel \\ [x]_{\mathcal{B}}}}{X}$.

Claim: P is invertible, if not, there is a nonzero solution to $PX = \underset{(X' = 0)}{0}$. So $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. $x = \sum x_i v_i \neq 0$.

But then $X' = 0 \Rightarrow x = \sum 0 v_i' = 0$ $\divideontimes$.

<u>Subspace examples</u>

$A_{m\times n}$, then the row space is the span of the rows of A in $F^{1\times n}$.

" " col. " " " " " columns " " " $F^{m\times 1}$.

> Observe that an ERO on A preserves the row space but not the column space.

> ERO changes rows by replacing one by a L·C of the others, or by switching.

<u>Eg</u>:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \text{col. space spanned by } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to \text{spanned by } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \#$$

For subspace W of $F^{1\times n}$ a basis $v_1, \ldots, v_r$ of W is called an echelon basis if the matrix $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{pmatrix}$ is in RREF.

$A \longrightarrow R$ = a RREF, then the nonzero rows of R form an echelon basis of row space of A.

<u>Eg</u>:

$$R = \begin{array}{c} \begin{array}{cccccccc} & & k_1 & & k_2 & & k_3 & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0\,1 & \cdots & 0 & \cdots & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 1 & \cdots & 0 & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & 1 & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 & \cdots \end{pmatrix} \end{array}$$

Rows $v_1 \ldots$ etc

If $c_1\, c_2$ scalars, then $\sum c_i v_i$ has form

$$\begin{array}{c} \begin{array}{ccccccc} & \downarrow k_1 & & \downarrow k_2 & & \downarrow k_3 & \end{array} \\ (0 \cdots 0 \; c_1 \; * \cdots * \; c_2 \; * \cdots * \; c_3 \; * \quad) \end{array}$$

This is 0 $\Leftrightarrow$ all $c_i = 0$.

If $v \in W =$ row space. & $v = (a_1, a_2 \ldots a_n)$

Then $v = \sum_{i=1}^{r} a_{k_i} \cdot v_i$

Thm: W subspace of $F^{1\times n}$ has a unique echelon basis.

Corollary: $A_{m\times n}$ RREF is unique.

Proof of Thm. {Two echelon bases must be equal since $v_j = \sum c_i v_i \Rightarrow c_i = 0 \ \forall i \neq j$ since the 1's & 0's line up & cannot interfere}.

Look at leading entries for vectors in W. Define J to be the $\{ i \in \{1, \ldots, n\} \mid \exists \underset{w\neq 0}{w} \in W$ so leading entry of w is at index $i \}$.

Define $W_i := \{ v \in W \mid$ coordinates of v are 0 at index $1, 2, \ldots, i-1 \}$

W_i is a subspace of W.

contains properly.

$J = \{k_1, k_2 \ldots k_r\}$. $W = W_{k_1} \supsetneq W_{k_2} \supsetneq W_{k_3} \cdot W_{k_r} = 0$.

If $v_1 \ldots v_r$ is any echelon basis, then $W_{k_l} = \text{span}\{v_l, v_{l+1}, \ldots, v_r\}$).

If $v_1 \ldots v_r$ is an echelon basis, the form of $\sum_{i=1} c_i v_i$ shows that v_i has leading 1 at index k_i. If $v_1' \ldots v_r'$ is another echelon basis, the same observation applies.

Now write v_j' as a LC of $v_1 \ldots v_r$. It must have coefficient 1 at index k_i & coeff 0 at k_l, $l \neq i$.

$\Rightarrow \quad v_j' = \left(\sum_{l \neq j} 0\, v_l\right) + 1 \cdot v_j \quad \overset{\text{by}}{(*)}.$ $\blacksquare$

Ch 3: Linear Transformations. (LT)

Def: F field, V, W v.s $/F$.

A function $T: V \to W$ is a LT iff

$$T(ax+by) = aTx + bTy.$$

for all $x, y \in V$, scalars a, b.

Ex 1 If $A_{m\times n}$ matrix, $F^{n\times 1} \to F^{m\times 1}$ sends $x \longmapsto Ax$ is a LT.

Ex 2

$D: V \to V$

V = all smooth functions $\mathbb{R} \to \mathbb{R}$ a v.s $/\mathbb{R}$.

$D = \frac{d}{dt}$.

$J: V \to V$

$J(f) = \int_0^t f(u)\,du$ both LTs. $DJ(f) = f$.

But $JD(f) = f(t) - f(0)$.

$DJ = I_{dv} \neq JD$. (This happens in ∞ dimensions but in f.d $\underset{n\times n}{A}\,\underset{n\times n}{B} = I \Leftrightarrow BA = I$).

Thm $T . V \to W$ L.T

then kernel $(T) := \{x \in V \mid Tx = 0\}$ is a subspace of V

& $Im(T) := \{Tx \mid x \in V\}$ is a subspace of W.

Lecture 8

Thm 1: V fdvs basis $v_1 \dots v_n$, W v.s, elements $w_1, \dots w_n$ then there exists a unique LT $T: V \to W$ so $Tv_i = w_i$, for $i = 1 \dots n$.

Proof: For $x \in V$, there is a unique expression

$x = \sum_{i=1}^{n} c_i v_i$. Define $T(x) := \sum_{i=1}^{n} c_i w_i$

Claim: T is a LT. If $y \in V$, $y = \sum_{i=1}^{n} d_i v_i$

a, b scalars

$ax + by = \sum_{i=1}^{n} (ac_i + bd_i) v_i$

So, $T(ax+by) = \sum (ac_i + bd_i) w_i = aTx + bTy$. ∎

Such a LT is unique since $T(x)$ must be $\sum_{i=1}^{n} c_i T(v_i)$

Remark: If S set, W v.s, Functions (S, W) is a V.S.

V v.s. Functions (V, W) is a v.s contains $L(V,W)$ = set of LTs from V to W as a subspace

In general, if $S_1 \subseteq S_2$ sets, we have a map, Functions $(S_2, W) \xrightarrow[\text{res}]{\text{restrict domain}}$ Fns. (S_1, W)

If we take $\mathcal{B}$, a basis of V we consider Functions $(V, W) \xrightarrow{\text{res}}$ Fns. $(\mathcal{B}, W)$.

$L(V,W) \xrightarrow{\text{res to } L(V,W)}$ Fns. $(\mathcal{B}, W)$ → there is a bijection.

$\cap$ Functions (V, W)

Def: T is a LT from $V \to W$ the kernel of T $\ker(T)$ or null space is $\{x \in V \mid Tx = 0\}$ subspace of V. The image of T is $Im(T) = \{Tx \mid x \in V\}$. subspace of W. V is called domain W is called co-domain.

Def: Rank of a LT is the dimension of the image of T.

Def. Nullity of T is $\dim(\ker(T))$.

Thm: V f.d, W v.s, $T \in L(V, W)$.

Then $\dim(V) = \text{rank } T + \text{nullity } T$. { Rank + Nullity Theorem }

Proof: $\ker(T)$ subspace. Let $v_1, \dots v_m$ be basis for $\ker(T)$

Extend it to basis $v_1 \dots v_m, \dots v_n$ of V.

Then $Im(T)$ is spanned by $\underbrace{T(v_1), \dots T(v_m)}_{0}, \dots T(v_n)$.

$\Rightarrow \text{rank}(T) \leq n - m$.

Claim: $Tv_{m+1}, \dots, T(v_n)$ are LI.

Consider scalars $c_{m+1}, \dots c_n$ so that $\sum_{k=m+1}^{n} c_j T v_j = 0$.

$\Rightarrow T\left(\sum_{j=m+1}^{n} c_j v_j\right) = 0 \Rightarrow \sum_{j=m+1}^{n} c_j v_j \in \ker(T)$.

Recall Lemma.

If $v_1, \ldots, v_m \ldots v_n$ basis of V, $W = \text{span}\{v_1 \ldots v_m\}$.

then $\sum_{i=1}^{n} c_i v_i$ is in W iff $c_j = 0 \ \forall j \geq m+1$.

Lemma $\Rightarrow$ all $c_j = 0$ in proof of claim.

$\Rightarrow Tv_{m+1}, \ldots, Tv_n$ is LI $\Rightarrow \text{rank}(T) = n-m$ ∎

Note:-

$x = \sum c_j v_j \Rightarrow Tx = \sum c_j T v_j \Rightarrow$ Image of span, spans image.

Def: $A_{m\times n}$, rowrank is the dim (rowspace) ie subspace $F^{1\times 1}$ spanned by rows. Column space is dim of column space.

If we interpret A as the LT $x \to Ax$, $x \in \mathbb{R}^n$ then rank means column rank.

$$Ax = (A_1 | \ldots | A_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum x_j A_j .$$

Thm: $A_{m\times n}$, Then row rank = column rank.

Proof: R+N Thm says nullity of $A + \overset{\text{col. rank}}{\text{rank}}(A) = n$

Theory of linear systems says $\underset{\text{no. of free indeces}}{k} + \underset{\text{no. of leading Indeces}}{\text{rowrank}} = n$

But no. of free indeces is dim (soln. space $Ax = 0$). This is the nullity. $\Rightarrow k = \text{nullity} \Rightarrow$ row rank = column rank. ∎

Thm: $n = \dim V$ $m = \dim W$ then $L(V,W)$ has dim mn.

Idea: Think of $m \times n$ matrices, $e_{pq} = \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \to$ 1 in pos p,q, everywhere else 0.

Proof: We take bases $\underbrace{v_1, \dots v_n}_{\mathcal{B}}$, $\underbrace{w_1, \dots w_m}_{\mathcal{B}'}$

We define a LT E^{pq} by the condition

$$E^{p,q}(v_i) = \begin{cases} 0 & i \neq q \\ w_p & \text{if } i = q. \end{cases} = \delta_{iq} w_p$$

→ Kronecker delta. Form a basis

Take $T \in L(V,W)$. $Tv_j = \sum_{p=1}^{m} a_{pj} w_p$, for $j = 1, \dots, n$.

Form $A = (a_{pq})$, Define $U = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} E^{p,q}$.

$U \in L(V,W)$. Claim $T = U$.

Calculate $U(v_j) = \sum_p \sum_q a_{pq} \underbrace{E^{p,q}(v_j)}_{0 \text{ unless } j=q}$.

$= \sum_p a_{p,j} w_p = Tv_j$.

Since T & U agree on a spanning set for V, $T = U$.

Claim: E^{pq} are LI. Suppose the LC, $U = 0$. Then.

$0 = U(v_j) = \sum_{p=1}^{m} a_{p,j} w_p$.

Since $w_1, \ldots, w_m$ is LI all $a_{pj} = 0$.

??
Note: $e_{pq} = \begin{pmatrix} 0 & 0 & \vdots \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}_{m \times m} \to p,q \Rightarrow e_{pq} \underset{m\times n}{A} = \begin{pmatrix} 0 \\ \text{———} \\ 0 \end{pmatrix} \to$ row p is row q of A

$\underset{m\times n}{A} \cdot e_{pq} = \left(0 \;\middle|\; 0 \right) \to$ column q

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Lecture 9

- $L(V,W)$ has basis E^{pq} where $E^{pq}(v_i) = \delta_{iq} w_p$.

Suppose $U \to V \to W$

- Consider basis $\{u_1, \ldots, u_t\}$ & $\{v_1 \ldots v_n\}$ & $\{w_1 \ldots, w_m\}$
- E^{pq} as before
- $F^{rs}(u_j) = \delta_{js} v_r$

$G^{ps}(u_j) = \delta_{js} w_p$

Calculate $E^{pq} F^{rs}(u_j) = E^{pq}(\delta_{js} v_r) = \delta_{js} E^{pq}(v_r)$

$= \delta_{js} \delta_{rq} w_p = \delta_{qr} G^{ps}(u_j)$

$\Rightarrow E^{pq} F^{rs} = \delta_{qr} G^{ps}$.

<u>Def</u>: $T \in L(V,W)$ is <u>one-one</u> if $x, y \in V$, $Tx = Ty \Rightarrow x = y$

$\Leftrightarrow \ker(T) = 0$ | $Tx=0 \Rightarrow x=0$

<u>Def</u>: T is onto means $\operatorname{Im}(T) = W$.

<u>Thm</u>: T is one-to-one iff whenever $v_1 \ldots v_r$ is a LI seq. then so is $Tv_1, \ldots, Tv_r$.

<u>Proof</u>: $(\Rightarrow)$ Consider a LC,

$$\sum_{i=1}^{n} c_i T v_i = T\left(\sum_{i=1}^{n} c_i v_i\right)$$

If $=0 \Rightarrow \sum_{i=1}^{n} c_i v_i = 0$ since T is one to one.

$\Rightarrow$ All $c_i = 0$ by LI of $v_1 \ldots v_r$.

$(\Leftarrow)$ ~~Suppose $x \in \ker(T)$~~.

If $x \neq 0$, the sequence "x" is LI

$\Rightarrow \underset{\text{Since } Tx \text{ is LI}}{\underline{Tx \neq 0}}$. So $\ker(T) = 0$.

<u>Thm</u> V, W fdvs, dim V = dim W if $T \in L(V,W)$ then equivalent are

1) T is one to one
2) T is onto
3) T is an <u>iso morphism</u> (Linear Transformation L(V,W) which has a two sided inverse.)

<u>Proof</u> : $\text{rank}(T) + \text{nullity}(T) = \dim(V) = n$.

If T is 1-1, nullity = 0 $\Rightarrow$ Im(T) has dim = rank(T) = n = dim(W) $\Rightarrow$ (2).

- If T is onto, rank(T) must be n $\Rightarrow$ nullity = 0.
- In general having a 2 sided inverse map of sets $A \xrightarrow{f} B$ is equivalent to having a 2 sided inverse.
- So (1) & (2) $\Rightarrow$ (3) & (3) $\overset{\text{trivial}}{\Rightarrow}$ (1, 2). ∎

<u>Representation of LT by matrix</u>

V, W fdvs / F ; basis $\underbrace{\{v_1 \dots v_n\}}_{\mathcal{B}}$ & $\underbrace{\{w_1 \dots w_m\}}_{\mathcal{B}'}$. $T \in L(V,W)$

$T v_j = \sum_{i=1}^{m} a_{ij} w_i \ \forall j$. So T gives a matrix $A = (a_{ij})_{m \times n}$.

$Mat_{m\times n}(F)$ means the set of all matrices of size $m \times n$.

A v.s / F

We write A as $[T]_{\mathcal{B},\mathcal{B}'}$.

The function $T \longrightarrow [T]_{\mathcal{B}\mathcal{B}'}$ is a LT (exercise).

It is 1-1. So $T \longmapsto [T]_{\mathcal{B},\mathcal{B}'}$ is an isomorphism.

(since $L(V,W)$ & $\mathrm{Mat}_{m\times n}(F)$ have common dimension mn)

> We consider, $U \xrightarrow{S} V \xrightarrow{T} W$
$\{u_1, \dots u_\ell\}$ $\{v_1, \dots v_n\}$ $\{w_1, \dots w_m\}$

> If T has matrix A, S has matrix B.

Then $\overset{A}{T}\overset{B}{S}$ has matrix C.

<u>Prove :</u> $C = AB$.

$$Su_j = \sum_{i=1}^{n} b_{ij} v_i$$

$$TS\,u_j = \sum_{i=1}^{n} b_{ij}\, Tv_i = \sum_i b_{ij} \sum_{k=1}^{m} a_{ki} w_k.$$

$$= \sum_k \Big(\underbrace{\sum_i a_{ki} b_{ij}}_{\text{coeff } kj \text{ of } C} \Big) w_k$$

It equals coeff kj of AB ✓

$V = W$ f.d. $A = [T]_{\mathcal{B},\mathcal{B}}$. $\mathcal{B}'$ another basis $B = [T]_{\mathcal{B}'}$,
$\mathcal{B}$ basis

then $B = P^{-1}AP$ where $P = (p_{ij})$.

$\mathcal{B} = \{v_1, \dots, v_n\}$

$\mathcal{B}' = \{v_1', \quad , v_n'\}$

$$v_j' = \sum_{i=1}^{n} p_{ij} v_i$$

Above $\Leftrightarrow PB = AP$.

P has columns $([v_j']_{\mathcal{B}}) = \begin{pmatrix} p_{1j} \\ p_{2j} \\ \vdots \end{pmatrix}$

<u>Lecture 10</u> (When converting bases P converts the "language" in which they are expressed so it acts in the opposite manner)

$T \in L(V, W)$

V basis $\mathcal{B}\{v_1, \dots v_n\}$, $\mathcal{B}'\{v_1' \; v_n'\}$. W basis $\mathcal{C} = \{w_1, \dots, w_m\}$ &
$\mathcal{C}' = \{w_1', \dots, w_m'\}$.

$A = (a_{ij})$ = matrix of T wrt $\mathcal{B}, \mathcal{C}$. $B = (b_{ij})$ = matrix of T wrt $\mathcal{B}', \mathcal{C}'$.

$P = (p_{ij})$ $\quad v_j' = \sum_{i=1}^{n} p_{ij} v_i$. $\quad Q = (q_{ij})$ $\quad w_j' = \sum_{i=1}^{m} q_{ij} w_i$.

We have $Tv_j = \sum_{i=1}^{m} a_{ij} w_i$. $\quad Tv_j' = \sum_{i=1}^{m} b_{ij} w_i'$

$= \sum_i b_{ij} \left(\sum_k q_{ki} w_k \right)$.

Also $Tv_j' = T\left(\sum_{i=1}^{n} p_{ij} v_i\right) = \sum_{i=1}^{n} p_{ij} \cdot Tv_i = \sum_{i=1}^{n} p_{ij}\left(\sum_{k=1}^{m} a_{ki} \cdot w_k\right)$

Extract coeff of each w_k.

$\sum_{i=1}^{m} b_{ij} q_{ki} = \sum_{i=1}^{m} q_{ki} b_{ij} = kj$ entry of QB.

This equals $\sum_i p_{ij} a_{ki} = \sum_i a_{ki} p_{ij} = kj$ entry of AP.

So $QB = AP$ and $B = Q^{-1}AP$.

<u>Example</u>

$m = n = 2 \quad v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $w_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $w_2 = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

Say $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. $P = Q = \begin{pmatrix} -2 & -3 \\ 5 & 4 \end{pmatrix}$. $P^{-1} = Q^{-1} = \frac{1}{7}\begin{pmatrix} 4 & 3 \\ -5 & 2 \end{pmatrix}$

$\Rightarrow P^{-1}AP = \frac{1}{7}\begin{pmatrix} 74 & 41 \\ 68 & -39 \end{pmatrix}$

> List of standard basic matrices correspond to the E^{ij} on website

Quotient spaces (not in text)

Def: S set, an <u>equivalence relation</u> is a subset E of $S \times S$ such that

(1) $(a,a) \in E \ \forall a \in S$

(2) $(a,b) \in E \Rightarrow (b,a) \in E$

(3) (a,b) & $(b,c) \in E \Rightarrow (a,c) \in E$

We write $a \sim b \Rightarrow (a,b) \in E$.

Equivalence class of $a \in S$ is $\{ b \in S \mid (a,b) \in E \}$.

These sets <u>partition</u> S.

If $f: S \to T$ function & f is a constant on each equivalence class. ($a \sim b \Rightarrow f(a) = f(b)$)

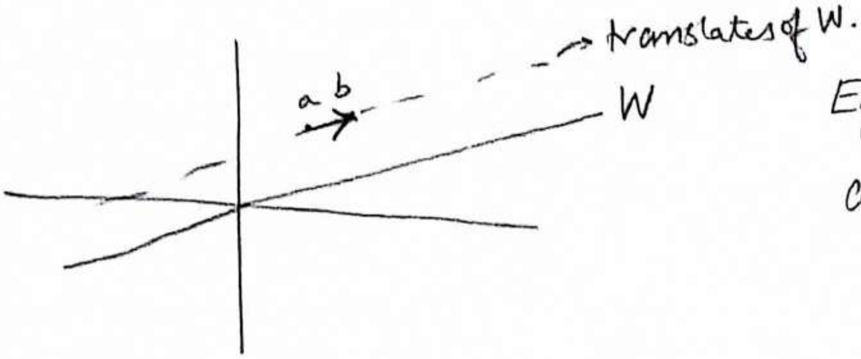
Then we get a well defined function $\underbrace{S/\sim}_{\text{set of all equivalence classes}} \to T$

<u>Notation</u>

V vs $/F$ & W subspace. <u>$\sim$ on V</u> s.t

$a \sim b \Leftrightarrow a - b \in W$ (Trivial to check $\Leftarrow$ by checking the 3 conditions)

Example : $V = \mathbb{R}^2$, $W =$ line $\{cw \mid c \in \mathbb{R}\}$ $w \neq 0$



Eq. classes $V/\sim \, = V/W$ are the translates of W

V, W as in notations.

$[a]$ for equivalence class of a in V/W.

Claim: V/W ~~has~~ inherits a structure of a vector space.

If S & T are subsets of V.

$S + T = \{x+y \mid x \in S, y \in T\}$.

$[a] + [b] = [a+b]$.

Claim

We have
$$\begin{tikzcd} V \times V \arrow[r, "+"] \arrow[rr, bend right, "f"'] & \overbrace{V \longrightarrow V/W}^{\text{map}} \\ & a \longrightarrow [a] \end{tikzcd}$$

Then f is constant on eq. classes in $V \times V / \sim$.

ie. $(v, w) \sim (v', w')$ iff $(v - v', w - w') \in W \times W$.

If true, can deduce

$[a] + [b] \subseteq [a+b]$

Take $(a,b) \in V \times V$ & $(a',b') \in V \times V$.

Suppose $a-a' \in W$ & $b-b' \in W$. Then $(a+b) - (a'+b')$

$$= \underbrace{(a-a')}_{W} + \underbrace{(b-b')}_{W}$$

$\Rightarrow \in W$.

So addition is constant on equivalence classes in set $V \times V$.

Similarly,

$F \times V \to V \to V/W$.

Take $(c,v) \longmapsto cv \longmapsto [cv]$ and $c[v] = \{ca \mid a \in [v]\}$.

$\Rightarrow c[v] \longrightarrow [cv]$. $ca \sim cv$ for all $a \in [v]$.

Check v.s axioms for V/W. The quotient map $a \longmapsto [a]$ is a LT & onto.

So from $V \times V \to V/W$ we get $V/W \times V/W \to V/W$.

$([a],[b]) \to [a+b]$ is the sum op.

If $(a,b) \sim (a',b')$ $\longrightarrow [a+b]$, $\longrightarrow [a'+b']$ These are equal $\Rightarrow$ addition is const. on eq. classes

Lemma: $q: V \to V/W$ quotient map, a LT. Then

$x \in \ker(q) \Leftrightarrow x \in W$. Also q is onto.

So if V is f.d $\dim W =$ nullity of q.

$\text{rank}(q) = \dim(V) - \dim(W)$

Proof: $x \in \ker(q) \Leftrightarrow [x] = [0]$

$\Leftrightarrow x - 0 \in W \Leftrightarrow x \in W$

$\underbrace{\text{rank}(q)}_{n - \dim(W)} + \underbrace{\text{nullity}(q)}_{\dim(W)} = n = \dim(V).$

We shall study cases where $\dim(V)$ & $\dim(W)$ are infinite but $\dim(V/W)$ is finite.

Define: Codimension of W in V is $\dim(V/W)$. If $\dim V$ is finite then $\text{codimension}(W) = \dim V - \dim W$.

<u>Def</u> A <u>linear functional</u> on V is a member of $L(V, F)$. (V vs $/F$).

<u>Ex</u>: $F[x]$ space of all polynomials.

$a \in F$ $\quad p(x) \in F[x]$ $\quad p \mapsto p(a) \in F$ is a linear functional.

<u>Ex</u>: V = all functions from $[0,1] \to \mathbb{R}$.

$a \in [0,1]$, $f \in V$, $f \to f(a)$ is a LT & L func. on V.

<u>Exam 1: next Wed 7-9, 1360EH.</u> (No quotient spaces?)

<u>Problem session 7pm Mon on Zoom.</u>

<u>Lecture 11</u>

V vs $/F$, $V^* := L(V, F)$, the dual space of V.

<u>Ex</u>: If $V = F^{n \times 1}$. Given the sequence $c_1, \dots, c_n$ of scalars, we get a linear functional which takes $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} c_i x_i$

We can think of this as $(c_1, c_2, \dots, c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (f(x))$

We know $\dim(L(V, W)) = \dim V \cdot \dim W$ for fdvs V, W.

For $\dim V = n$,

$\dim V^* = n$.

<u>Def:</u> Sequences $v_1 \ldots v_n$ in V & $f_1, \ldots, f_n$ in V^* are in <u>duality</u> if $f_i(v_j) = \delta_{ij}$.

<u>Thm:</u> If above sequences are in duality, $v_1 \ldots v_n$ is LI in V & $f_1 \ldots f_n$ is LI in V^*.

<u>Proof:</u> Consider a relation $\sum_{i=1}^{n} c_i v_i = 0$. Apply f_j:

$$\underbrace{\sum_i c_i f_j(v_i)}_{c_j \cdot 1} = f_j(0) = 0 \quad \Rightarrow \text{ each } c_j \text{ must be } 0. \Rightarrow \text{LI}.$$

$\sum b_i f_i = 0$ in V^*. Apply to v_j:

$$\sum b_i \underbrace{f_i(v_j)}_{\delta_{ij}} = 0(v_j) = 0. \; ; \quad b_j = 0 \Rightarrow \text{second seq is LI.}$$

<u>Def:</u> V fdvs with basis $v_1 \ldots v_n$, the dual basis is $f_1 \ldots f_n$, basis of V^*, so $f_i(v_j) = \delta_{ij}$ (these seqs are LI & in duality. By dimension considerations $f_1 \ldots f_n$ LI $\Rightarrow$ is a basis of V^*)

<u>Thm:</u> Above notations. Dual basis exists & for $v \in V$,

$$v = \sum_{i=1}^{n} f_i(v) v_i$$

For $g \in V^*$, $g(v) = \sum_{i=1}^{n} \underbrace{g(v_i)}_{b_i} f_i$.

Proof: Say $v = \sum c_i v_i$, Then $f_j(v) = \sum_i c_i \underbrace{f_j(v_i)}_{\delta_{ij}} = c_j$

Similarly, $g \in V^*$, $f_1 \dots f_n$ basis V^*, $\exists$ unique scalars $b_1 \dots b_n$

so $g = \sum_{i=1}^{n} b_i f_i$

Now $g(v_j) = \sum_i b_i \underbrace{f_i(v_j)}_{\delta_{ij}} = b_j$

Remarks If $f \in V^*$, $f \neq 0$, then $Im(f) = F$. So kernel

$Ker(f)$ is a proper subspace. And if $\dim(V) = n$,

$\underset{1}{\text{rank}}$ of f + $\underset{n-1}{\text{nullity}} = n$ $\Rightarrow \dim(Ker(f)) = n-1$.

Codim ($\dim(V) - \dim(W)$ if W = subspace of V.) $(Ker(f)) = 1$ (for arbitrary v.s).

Such a space is called a hyperspace. $Ker(f)$.
1 dim less than space.

Ex: Suppose $V =$ continuous functions on $(-\infty, \infty)$.

Subspace of Functions $(\mathbb{R}, \mathbb{R})$.

Suppose $a_1 < a_2 < \dots < a_n$ real. Claim $e^{a_1 x}, \dots, e^{a_n x}$ LI.

Consider a LC, $c_1 e^{a_1 x} + \dots + c_n e^{a_n x} = 0$ for all x.

Divide by $e^{a_n x}$ to assume $a_1 < \dots < a_{n-1} < a_n = 0$.

$c_1 e^{a_1 x} + \dots + c_{n-1} e^{a_{n-1} x} + c_n = 0$. Take lim as $x \to \infty \Rightarrow c_n = 0$.

Here we must remove all terms with 0 coeff before hand. $*$

This is fine since subseq is LD $\Rightarrow$ seq is LD.

$\Rightarrow$ Set $\{e^{ax} \mid a \in \mathbb{R}\}$ is LI in V.

Ex: $V = F[x] =$ all polys over F. Take $a_1, a_2, \dots, a_n$ distinct scalars in F.

For any $a \in F$, there is $\varepsilon_a \in V^*$ which is "evaluation at a"

$$\varepsilon_a(p(x)) = p(a).$$

Define: $q_i(x) \in F[x]$, $q_i(x) := \prod_{\substack{j=1 \\ j \neq i}}^{n} (x - a_j)$

Define: $p_i(x) := \dfrac{q_i(x)}{q_i(a_i)}$ $\rightarrow$ Lagrange interpolation

(Fit a polynomial of least degree through some points).

$\downarrow$

Lagrange polynomials.

Claim: The sequences $P_1, \ldots, P_n$ in V and $\epsilon_{a_1}, \ldots, \epsilon_{a_n}$ in V^* are in duality.

$q_i(a_j) = 0$ if $j \neq i$, $\quad \epsilon_{a_j}(p_i(x)) = \dfrac{q_i(a_j)}{q_i(a_i)} = 0.$ $\leftarrow 0$

$$\epsilon_{a_i}(p_i(x)) = \frac{q_i(a_i)}{q_i(a_i)} = 1. //$$

In $F[x]$, $x^0, x^1, \ldots, x^{n-1}$ is basis of {polys of deg $\leq n-1$}.

The $p_i(x)$, $i = 1 \ldots n$ form another basis for {poly of deg $\leq n-1$}.

Suppose we want a poly function $g(x)$ of deg $\leq n-1$ so that given points (a_i, b_i) $i = 1, \ldots, n$. $g(a_i) = b_i$.

Take $\boxed{g(x) = \sum_{i=1}^{n} b_i p_i(x)}$. $\quad g(a_j) = \sum_{i=1}^{n} b_i \underbrace{p_i(a_j)}_{\delta_{ij}} = b_j.$

Def: S subset of V. $S^\circ := \{f \in V^* \mid f(x) = 0 \ \forall x \in S\}$.

annihilator of S in V^*

S° is a subset

Thm: $\dim V = n$; W is a subspace of V, then

$$\underset{\substack{\cap \\ V}}{\dim(W)} + \underset{\substack{\cap \\ V^*}}{\dim(W^\circ)} = n.$$

Proof: Take basis $w_1 \dots w_m$ of W. Extend to basis

$w_1, \dots, w_m, \dots, w_n$ for V.

Take $f_1 \dots f_n$ basis of V^* dual to $w_1 \dots w_n$.

$f_i(w_j) = \delta_{ij}$

So $\underbrace{f_{m+1} \dots f_n}_{n-m}$ are in W°. (If $j > m$ then $f_j(w_i) = 0$ for $i = 1 .. m$)
($\Rightarrow f_j \in W^\circ$)

Now consider $g \in W^\circ$ Then $g \overset{\text{Thm}}{=} \sum_{i=1}^{n} g(w_i) f_i$. $\leftarrow 0$ if $i = 1, \dots, m$

$= \sum_{i=m+1}^{n} g(w_i) f_i \in \text{span}\{f_{m+1} \dots f_n\}$.

So W° is spanned by $\underbrace{f_{m+1}, \dots, f_n}_{LI}$. //

Lecture

- We discussed,

Thm. V fdvs W subspace. $W^\circ = \{f \in V^* \mid f/W = 0\}$

↳ restriction of f to W

$\dim W + \dim W^\circ = \dim V.$

$w_1 \ldots w_m$ basis W

$w_1 \ldots w_m \ldots w_n$ basis for V.

dual basis $f_1 \ldots f_n$, we showed $f_{m+1} \ldots f_n$ basis of W°.

- Using some notation:

$\ker(f_j) = \text{span}\{w_i \mid i \neq j\}$

$\ker(f_{m+1}) \cap \ker(f_{m+2}) \cap \ldots \cap \ker(f_n) = \text{span}\{w_1 \ldots w_m\} = W.$

This shows that W, dim m, is the intersection of $n-m$ subspaces of dim $n-1$.

Lemma: W subspace of V, $W \neq V$, $x \in V - W$. Then $\exists f \in V^*$ so $f/W = 0$ & $f(x) \neq 0$.

- Proof: $w_1 \ldots w_m$ basis W, $w_1 \ldots, w_m, x$ is LI, extend to a basis for V: $w_1 \ldots w_m, w_{m+1}, \ldots w_n$.

Take dual basis $f_1 .. f_n$. Then $f := f_{m+1}$ satisfies our condition.

<u>Corollary</u>: W_1, W_2 subspaces of fdvs V. Then $W_1 = W_2 \iff W_1^\circ = W_2^\circ$.

<u>Proof</u>: $(\Rightarrow)$ trivial.

$(\Leftarrow)$ suppose unequal. Then (changing indeces)

$\exists\ x \in W_1 \setminus W_2$. $\underset{\text{applied to } W_2}{\text{Lemma}} \Rightarrow \exists\ f \in W_2^\circ$ so $f(x) \neq 0$.

But $x \in W_1$ & $W_1^\circ = \underset{f}{\underset{\Psi}{W_2^\circ}} \Rightarrow f(x) = 0$ $\divideontimes$.

<u>Ex</u>: $F[x]$ polys $1, x, x^2, x^3, \ldots$

Define $f_i \in F[x]^*$ by $f_i(x_j) = \delta_{ij}$

$f_1(p(x)) = p(1)$

LC $\sum_{i=1}^{n} c_i f_i$ is 0 at all x^k, $k \geq n+1$. so dual basis doesn't really work.

Double dual

V^{**} means $(V^*)^*$

$\dim(V) = n$, $\dim(V^*) = n$, $\dim(V^{**}) = n$

For $v \in V$ define $L_v \in V^{**}$ by

$$\underset{\substack{\cap \\ V^*}}{L_v(f)} : f(v)$$

$L_v \in V^{**}$, $v \mapsto L_v$ is in $L(V, V^{**})$

$L_{cv+c'v'}(f) = f(cv+c'v') = cf(v)+c'f(v')$
$= (cL_v + c'L_{v'})(f)$.

$$L_v(af+bg) = (af+bg)(v) = a(f(v)) + bg(v)$$
$$= aL_v(f) + bL_v(g).$$

$\Rightarrow L \in L(V, V^{**})$

Observe if $L_v = 0$ then $f(v) = 0$ $\forall f \in V^*$.

<u>Claim:</u> $v=0$; if $v \neq 0$, $\exists$ a linear functional which has value $\neq 0$ on v.

Let $v_1 = v$, $v_1 \dots v_n$ basis & $f_1 \dots f_n$ dual basis.

$f_1(v_1) = 1 \neq 0$ $\divideontimes$

We conclude that L has zero kernel $L: \underbrace{V \to V^{**}}_{\text{dimn}} \Rightarrow L$ is an isomorphis

> Using L we can define E° a subspace of V to be the annihilator of subset E in V^*.

$$\begin{matrix} V & \& & V^* \\ \cup\!\mid & & \cup\!\mid \\ S & & S^\circ \end{matrix} \quad ; \quad \begin{matrix} V^* & \& & V^{**} & \overset{\text{iso}}{\underset{L}{\simeq}} & V \\ \cup\!\mid & & \cup\!\mid & & \\ E & & E^\circ & & L^{-1}(E^\circ) \end{matrix}$$

we identify V with V^{**} using L

So we get a formula like $\dim W + \dim W^\circ = \dim V$ for W subspace of V

$\dim E + \dim E^\circ = \dim V$ for subspace E of V^*.

<u>Application</u>: Given a basis $f_1 \dots f_n$ of V^*, $\exists$ a basis $v_1 \dots v_n$ of V so $f_1 \dots f_n$ is the dual bases of $v_1 \dots v_n$.

<u>Thm</u>: $S \overset{\text{subset}}{\subseteq} V_{\text{fdvs}}$. $S^{\circ\circ} = (\underset{V^*}{\underset{\cap}{S^\circ}})^\circ$ is the span of S in V.

<u>Proof</u>: Define $W := \text{span}(S)$, subspace of V. then

$W \subseteq S^{\circ\circ}$ because if $f \in V^*$ annihilates S, it annihilates W. So $\underbrace{(S^\circ)^\circ}_{\{v \in V \mid f(v)=0 \ \forall f \in W^\circ\}} \supseteq W ??$

Also $W^\circ = S^\circ \Rightarrow W \subseteq W^{\circ\circ} = S^{\circ\circ}$.

Use $\dim W + \dim W^0 = \dim V$.

$\dim W^0 + \dim W^{00} = \dim V^*$

$\Rightarrow \dim W = \dim W^{00} \Rightarrow W = W^{00}$

<u>Def</u>: V v.s. A <u>hyperspace</u> is a maximal proper subspace

i.e. subspace W of V, $W \neq V$. satisfying.

if U is a subspace $W \subseteq U \subseteq V$ then $U = W$ or $U = V$.

<u>Lemma</u>: If $f \in V^*$, $f \neq 0$ then $\ker(f)$ is a hyperspace.

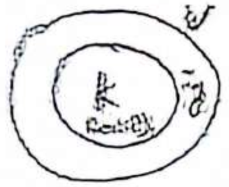
<u>Proof</u>: $K = \ker(f)$. Take $y \in V - K$. Then $f(y) \neq 0$.

<u>Claim</u>: $V = \text{span}\{K, y\} = K + \overset{\text{Field}}{F}y$

Let $x \in V$. Then $f(x) = c \in F$.

Take $x = \underbrace{\left(x - \frac{c}{f(y)} y\right)}_{\downarrow f \; 0 \;\Rightarrow\; \in K} + \frac{c}{f(y)} \cdot y$

$\Rightarrow x \in K + Fy$.



$f\left(x - \frac{c\,y}{f(y)}\right) = f(x) - \frac{f(y)}{f(y)} c = 0$

<u>key idea</u> ($f(x) = c$)

Take $x \in V$ & consider

as $x = x - \frac{c\,y}{f(y)} + \frac{c\,y}{f(y)}$

$= K + c'y$

$\ker(f)$ is a hyperspace!

$\Rightarrow$ if U is a subspace $U \supsetneq K$ the $U = V$. //

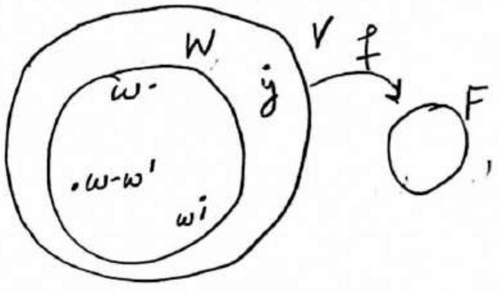
<u>Lemma:</u> If W is a hyperspace in V then $\exists f \in V^*$, $f \neq 0$

S.T $W = \ker(f)$. (Note $\ker(f) = \ker(cf)$ if $c \neq 0 \Rightarrow f$ is not unique)

<u>Proof:</u> Take $y \in V \setminus W$. Then $W + Fy = V$. $\left(\text{Since } \underbrace{W+Fy}_{\text{Span}(W \cup \{y\})} \supsetneq W\right)$

Claim $x \in V$, $\Rightarrow x$ is unique L.C. of $w + cy$ for $w \in W$

If not unique, $x = w' + c'y$ then



$$0 = x - x$$
$$= (w + cy) - (w' + c'y)$$
$$= (w - w') + (c - c')y$$

$y \notin W \Rightarrow c - c' = 0 \Rightarrow w = w' \Rightarrow$ unique

Now, define $f : V \to F$ by $\boxed{f(w + cy) = c}$

Claim: $f \in V^*$

If $x' = w' + c'y$, then $f(ax + bx') = ac + bc'$

$= af(x) + bf(x') \Rightarrow f \in V^*$

So, $\ker(f) = \{w + cy \mid c = 0\} = W$.

Lemma.

Let, $f, g \in V^*$. Then, g is a multiple of f $\iff \ker(g) \supseteq \ker(f)$.

Proof: If $g = cf$, then $\ker(f) \subseteq \ker(g)$.

$\Leftarrow$ Suppose $\ker(f) \leq \ker(g)$ If $f = 0$, then $\ker(f) = V$

$\Rightarrow \ker(g) = V \Rightarrow g = 0.$

We may assume $g \neq 0 \Rightarrow f \neq 0$ so

$\ker(f) \leq \ker(g) \subsetneq V$

$\ker(f)$ hyperspace $\Rightarrow \ker(f) = \ker(g)$

Let $K = \ker(f) = \ker(g)$. Let $y \in V \setminus K$.

then $f(y) \neq 0$. Define $c = \frac{g(y)}{f(y)}$ then $K(g - cf) \supseteq K \cup \{y$

$\Rightarrow \ker(g - cf) = V \Rightarrow g = cf$.

Thm: Let $g, f_1, \dots, f_r \in V^*$. $K = \ker(g)$, $K_j = \ker(f_j)$

Then g is a LC of $f_1 \dots f_r \iff K \supseteq \bigcap_{j=1}^{r} K_j$.

Proof $\Rightarrow$ Trivial.

$\Leftarrow$ We use induction on r, Lemma solves $r = 1$ base case. Suppose $r \geq 2$

We define $f_j' = f_j|_{k_r} \in V^*$, $g' = g|_{k_r} \in V^*$

{ Recall,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

By hyp, $K \supseteq K_1 \cap \ldots \cap K_r$.

$$\Rightarrow K \cap K_r \supseteq (K_1 \cap \ldots \cap K_r) \cap K_r.$$

$$= \underbrace{(K_1 \cap K_r)}_{\ker(f_1')} \cap \underbrace{(K_2 \cap K_r)}_{\ker(f_2')} \cap \ldots \cap \underbrace{(K_{r-1} \cap K_r)}_{\ker(f_{r-1}')}$$

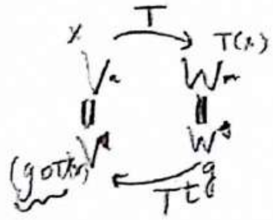
By induction, $\exists$ scalars $c_1, \ldots, c_{k-1}$ s.t $g' = \sum_{i=1}^{r-1} c_i f_i'$ in K_r

Define $h = g - \sum_{i=1}^{r-1} c_i f_i$ (no primes!) $K_r \subseteq \ker(h)$

and $K_r = \ker(f_r)$. So by lemma, $h = c_r f_r$.

$$\Rightarrow g - \sum_{i=1}^{r-1} c_i f_i = c_r f_r \Rightarrow g = \sum_{i=1}^{r} c_i f_i$$

Def: The Transpose T^t of $T \in L(V,W)$ is in $L(W^*, V^*)$ for $g \in W^*$, $T^t g = g \circ T$ so $g\circ T(v) : V \to F$



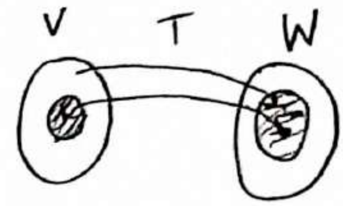
Thm: V, W & $T \in L(V,W)$

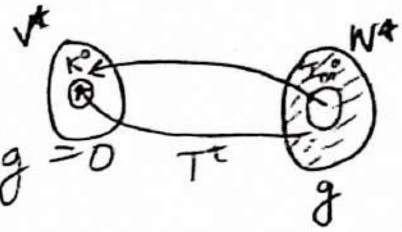
a) $\ker(T^t) = (\operatorname{Im}(T))^0$

b) If $\dim(V) = n$, $\dim(W) = m$, then

b1) $\operatorname{rank}(T) = \operatorname{rank}(T^t)$

b2) $\operatorname{Im}(T^t) = (\operatorname{Ker}(T))^0$





Proof: a) $g \in W^*$; $g \in \ker(T^t) \iff T^t g = 0$

$\iff g \circ T = 0 \Rightarrow T(x) \in \ker(g)$

$\iff \operatorname{Im} T \subseteq \ker(g)$

$\iff g \in (\operatorname{im} T)^0$

b1) If $r = \operatorname{rank}(T)$, then $\dim((\operatorname{im} T)^0) = m - r$ $\quad \dim(\operatorname{Im} T) + \dim((\operatorname{Im} T)^0)$

By (a), $\operatorname{null}(T^t) = m - r$, so,

$\operatorname{rank}(T^t) = m - (m-r) = r = \operatorname{rank}(T)$.

b2) Let $k = \ker(T)$. Claim: $\operatorname{Im}(T^t) \leq k^0$.

If $f = T^t g$ for $g \in W^*$, $v \in k$, then

$f(v) = (T^t g)(v) = g(T(v)) = g(\bar{0}) = 0 \Rightarrow f \in k^0 \Rightarrow \operatorname{Im}(T^t) \subseteq k^0$

then, $\dim(k^0) \overset{\text{lemma}}{=} n - \dim k \overset{\text{RNT}}{=} \operatorname{rank}(T) \overset{b1}{=} \operatorname{rank}(T^t)$

So we have $\operatorname{Im}(T^t) = k^0$ //

Lecture

Ch4 Polynomials

If $T \in L(V,V)$, $\dim V = n$ then T satisfies a polynomial. That is $\exists$ $p(x) = \sum_{i=0}^{m} a_i x^i$, $p \neq 0$. But $p(t) = \sum_{i=0}^{m} a_i T^i = 0$ in $L(V,V)$.

We know $\dim L(V,V) = n^2$.

Take sequence $\underset{\underset{T^0}{\|}}{1_V}, T, T^2, T^3, \dots, T^{n^2}$ cannot be LI,

So $a_0, a_1, \dots, a_{n^2}$ not all 0

So $\sum_{i=0}^{n^2} a_i T^i = 0$ So T satisfies $p(x) = \sum_{i=0}^{n^2} a_i x^i$

$T =$ scalar $\Rightarrow$ T satisfies $x - c$.

Def F field. An F-algebra, A, is a vs $/F$ which has a product $A \times A \to A$, $a, b \rightsquigarrow ab$ which commutes with scalar multiplication. If $x, y \in A$, $c \in F$ then

$c(xy) = (cx)y = x(cy)$.

Ex: $A = \mathrm{Mat}_{n \times n}(F)$: $A = (a_{ij})$, $cA = (c\, a_{ij})$

In this course, F-algebras will be associative.
May have a unit, not necessarily commutative.

Ex: $A = \mathrm{Mat}_{n\times n}(F)$ new product $A, B \in \mathrm{Mat}_{n\times n}(F)$.

$[A,B] = AB - BA$. If $n \geq 2$, non-associative F-algebra.

Def A, B F-algebras, A homomorphism of F-algebras is a function $\phi : A \to B$ so that ϕ is a LT over F and ϕ preserves products. $\phi(xy) = \phi(x)\phi(y)$.

If A & B have units we usually assume

$\phi(1_A) = \overset{\text{unit element}}{1_B}$

Def Polynomial algebra over F:

we take F^∞ = all sequences $(a_0, a_1, a_2, \dots)$. A vs.

It has a product $a = (a_0, a_1, \dots)$ $b = (b_0, b_1, \dots)$

Define $ab := (c_0, c_1, c_2, \dots)$ where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

F^∞ is an F algebra associative and commutative, with identity $(1, 0, 0, \dots)$.

Think of formal power series.

$$\left(\sum_{i=0}^{\infty} a_i x^i\right)\left(\sum_{j=0}^{\infty} b_j x^j\right)$$

The set of polynomials over F is the set of sequences in F^∞ which are almost all zero:
↳ finite nonzero entries.

ie. $(a_0, a_1, \ldots)$ for which there exist n so that

$a_i = 0$ for $i \geq n+1$.

It is a sub-algebra (F-subspace of another algebra. ie closed under products)

Both F^∞ and the algebra of polynomials have a unit & are commutative.

Denote by $F[x]$ the set of polynomials,

$x = (0, 1, 0, 0, \ldots)$

x^k = all zero sequence except for 1 at index k.

Check $x^k x^l = x^{k+l}$.

Thm F field $F[x]$ = polys in x.

B is any associative F-algebra with unit

then given an element of B ie $t \in B$,

there is a unique homomorphism of F-algebras

$F[x] \to B$ so that $x \mapsto t$.

Proof $F[x]$ has basis $1, x, x^2, x^3, \ldots$. We get a unique

LT $\phi : F[x] \longrightarrow B$ by setting $\boxed{\phi(x^k) := t^k}$.

$(t^0 = 1_B)$.

$$\text{So } \phi\left(\sum_{i=0}^{m} a_i x^i\right) = \sum_{i=0}^{m} a_i t^i$$

To <u>check that ϕ preserves products</u> it suffices to show that $\phi(yz) = \phi(y)\phi(z)$. for y, z in a basis. (by distributive law).

It is true since $x^k x^l = x^{k+l}$ & $t^k t^l = t^{k+l}$.

<u>Remarks</u>

$F^{\pm\infty}$ = all sequences indexed by integers, is a VS / F.

The product rule $c_n = \sum_{i,j} a_i b_j$ not well defined.

→ Laurent series from complex analysis.

L = all sequences $(\ldots a_{-1}, a_0, a_1, \ldots)$ in $F^{\pm\infty}$

So $a_k = 0$ for $k \ll 0$. Product rule makes sense

Distinguish between polys & poly functions.

$p \in F[x]$. Get $F \to F$, $a \to p(a)$

Consider $F :=$ integers mod 2 $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}_2$.

$p(x) = x^2 + x \neq 0$

$p(0) = 0$

$p(1) = 0$

Def: If $p(x) \in F[x]$ $\quad p = \sum_{i=0}^{n} a_i x^i$ & $a_n \neq 0$.

Define $\deg(p) = n =$ max. integer k so x^k has nonzero coefficient in $p(x)$. The degree of 0-poly is not defined OR set $= -\infty$. Observe if p & q are non zero polys, then $\deg(pq) = \deg(p) + \deg(q)$.

$$p = \sum_{i=0}^{n} a_i x^i \quad a_n \neq 0.$$

$$q = \sum_{j=0}^{m} b_j x^j \quad b_m \neq 0$$

has coeff $\underbrace{a_n b_m}_{\neq 0}$ at x^{m+n}.

nd in pq every power of x is $\leq m+n$.

$\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$

$= \max\{\deg(f), \deg(g)\}$ if $\deg f \neq \deg g$.

Thm: (Division)

Given $p(x), q(x) \in F[x]$, with $q \neq 0$. then $\exists$ $r(x)$ in $F[x]$ and $h(x) \in F[x]$ so that $p = hq + r$ where $r = 0$ or $\deg(r) < \deg(q)$.

Furthermore, such h and r are unique.

Def: R ring, S is a subring & $S \subseteq R$;
$(S,+)$ is a subgroup of $(R,+)$ and if $a, b \in S$ then $ab \in S$.

An ideal in R is I subset so $(I,+)$ is a subgroup of $(R,+)$ and $ab, ba \in I$ $\forall a \in I$, $b \in R$.

Ex: $2\mathbb{Z} \subseteq \mathbb{Z}$

<u>Def</u>: R commutative ring is an integral domain, if whenever $a, b \in R$ and $ab = 0 \Rightarrow a = 0$ or $b = 0$.

<u>Consequence</u>: If $ax = ay$ and $a \neq 0$ then $x = y$. Reason: $a(x-y) = 0$

F field (is int. domain) then $F[x]$ is an integral domain.
(Use $\deg(fg) = \deg(f) + \deg(g)$ when $f \neq 0$ & $g \neq 0$.

<u>Lemma</u>: $f, d \in F[x]$, $\deg d \leq \deg f$. Then $\exists$ poly $g(x) \in F[x]$ so $f - gd = 0$ or $\deg(f - gd) < \deg(f)$.

<u>Proof</u>: Write $f = \sum_{i=0}^{m} a_i x^i$, $a_m \neq 0$.

$d = \sum_{j=0}^{n} b_j x^j$, $b_n \neq 0$. Since $n \leq m$ we take

$g(x) := \left(\frac{a_m}{b_n}\right) x^{m-n}$.

<u>Thm 4</u>: $f, d \in F[x]$, $d \neq 0$. Then $\exists$ polys q, r so that $f = dq + r$ and $r = 0$ or $r \neq 0$ and $\deg(r) < \deg(d)$.
Furthermore, such q and r are unique.

Proof Existence.

Easy if $\deg(d) > \deg(f)$, just take $(q=0, r=f)$.

If d is a nonzero constant. Take $(q = \frac{1}{d} f, r=0)$

Assume $0 < \deg(d) \leq \deg(f)$. We prove the result by induction on $\deg(f)$.

By Lemma $\exists$ a poly g S.T $f-dg$ is 0 or $\deg(f-dg) < \deg(f)$

If $f-dg=0$ take $q=g$ and $r=0$.

If $f-dg \neq 0$ by induction, there are $q_1 r_1$ so

$$f-dg = dq_1 + r_1 .$$

$r_1 = 0$ or $\deg(r_1) < \deg(d)$.

$$f = d(\underbrace{g+q_1}_{q}) + \underbrace{r_1}_{r} .$$

Uniqueness. Suppose $f = dq_1 + r_1 = dq_2 + r_2$

$\Rightarrow d(q_1-q_2) = \underbrace{r_1 - r_2}_{0 \text{ or } \deg < \deg(d)}$. If $q_1 - q_2 \neq 0$; $\deg d(q_1-q_2) \geq \deg(d)$

$\rightarrow \divideontimes \leftarrow$

Conclude $q_1 = q_2 \Rightarrow r_1 = r_2$ //.

Def: $p(x) \in F[x]$; $c \in F$ is a root of p . if $p(c) = 0$.

Corrollary: c is a root of $p(x)$ iff $x-c$ divides $p(x)$ in $F[x]$ ie. $\exists$ poly $q(x)$ so $p(x) = (x-c)\,q(x)$.

Proof: We have polys h & r S.T $p(x) = (x-c)\,h(x) + r(x)$

Where $r = 0$ or $\deg(r) < \deg(x-c) = 1$.

$\hookrightarrow$ r is a nonzero scalar.

Taylor's Theorem.

F field of characteristic 0. Let $f(x) \in F[x]$, $f \neq 0$.

$\deg(f) = m$. Let $D = \frac{d}{dx}$, $c \in F$.

Then,

$$f(x) = \sum_{k=0}^{m} \frac{(D^k f)(c)}{k!} (x-c)^k.$$

Def: R ring with 1, we say R has characteristic $m > 0$ if

$\underbrace{1+1+1+1 \ldots +1}_{m} = 0$ and if $\forall$ integers $k > 0$, $k < m$ $\underbrace{1+1+1 \ldots 1}_{k} \neq 0$.

If R does not have positive characteristic, say it has characteristic 0.

Proof of Taylor's thm.

LHS & RHS are linear in f, so ETS (enough to show) for $f = x^m$.

We write $x^m = ((x-c)+c)^m = \sum_{j=0}^{m} \binom{m}{j} (x-c)^j c^{m-j}$

We observe that P_m = space of polys of degree $\leq m$ has dimension $m+1$. (basis $1, x, x^2, \ldots, x^m$).

Binomial Thm says that $(x-c)^j$, $j=0, \ldots, m$. spans P_m so is also a basis.

So ETS $\underbrace{\binom{m}{j}}_{\frac{m!}{j!(m-j)!}} c^{m-j} = \frac{(D^j f)(c)}{j!}$

$\Leftrightarrow \frac{m(m-1)\ldots(m-j+1)}{j!} c^{m-j} = \frac{D^j(x^m)(c)}{j!}$ ✓

Def: $c \in F$ is a root of poly $f(x)$ $\in F[x]$ of multiplicity r if $(x-c)^r$ divides $f(x)$ but $(x-c)^{r+1}$ does not.

Thm: F field char 0, $0 \neq f \in F[x]$ - $\deg(f) \leq n$. then $c \in F$ is a root of f of multiplicity r if $(D^k f)(c) = 0$ for $k = 0, 1, \ldots, r-1$. but $(D^r f)(c) \neq 0$.

<u>Proof</u> Let r be the multiplicity.

$f(x) = (x-c)^r g(x)$, $g(c) \neq 0$.

So $$f(x) = (x-c)^r \sum_{m=0}^{n-r} \frac{(D^m(g))(c)}{m!} (x-c)^m.$$

$$= \sum_{m=0}^{n-r} \frac{D^m(g)(c)}{m!} (x-c)^{m+n}$$

Also $$f(x) = \sum_{k=0}^{n} \frac{(D^k f)(c)}{k!} (x-c)^k$$

The coefficients at each $(x-c)^j$ on LHS & RHS are equal.

$(D^k f)(c) = 0$ for $0 \leq k \leq r-1$ and $(D^r f)(c) = g(c) \neq 0$.

The term at $m=0$ in Taylor. exp for g.

Also, $$\frac{(D^k f)(c)}{k!} = \frac{(D^{k-r}(g))(c)}{(k-r)!} \text{ for } k \geq r.$$

$g(c) \neq 0$

Conversely, if $(D^k f)(c) \neq 0$ for $k = 0 \ldots r-1$. &

$(D^r f)(c) \neq 0$ then Taylor expression for f is

$\sum_{k \geq r} \frac{(D^k f)(c)}{k!} (x-c)^k$ & $g(c)$ is nonzero.

To be contd & corrected!

> R ring. ideal is a subset I so $(I,+)$ is subgroup of $(R,+)$ & $a \in I$, $r \in R \Rightarrow ra \in I$ & $ar \in I$.

> If $1 \in I$, ideal then $I = R$. Such I is called the unit ideal.

> If I, J are ideals so are $I \cap J$, $I+J$ &

$IJ = \{$ LCs of ab, $a \in I$, $b \in J\}$.

> the ideal I is principal if it has the form

$I = Ra$ or $aR = \{xa \mid x \in R\}$.

In $F[x]$ every ideal is Principle.

Def: R comm. ring. We call $d \in R$ a common divisor of $a, b \in R$ if $d | a$ & $d | b$.

($d | a$ means $\exists\, a'$ so $a = da'$).

GCD: Greatest common divisor is a common divisor so that if e is any common divisor, $e | d$.

We will see that in $F[x]$, a set of polys has a g.c.d.

I in R comm ring is an ideal if $(I, +)$ is a subgroup of $(R, +)$ and if $ra \in I$ $\forall\, a \in I$ & $r \in R$.

An ideal I is principal if it has the form $Ra = \{ra \mid r \in R\}$ for some $a \in I$.

Written (a).

Remarks: I, J ideals, so are $I + J = \{x + y \mid x \in I, y \in J\}$ & $I \cap J$.

Thm: In $F[x]$, every ideal is principal. In fact, if I is a nonzero ideal there is a unique monic polynomial d s.t. $I = (d)$.

Def A polynomial is <u>monic</u> if nonzero & top coeff is 1.

Ex: In $\mathbb{Z}[x]$ the ideal $(2)+(x)$, not principle.

Remark The units in $F[x]$ (ie elements with 2 sided inverse) are just the nonzero constant polynomials.

Proof of Thm

M ideal, if $M = \{0\} = (0)$.

Assume $M \neq (0)$. In M take a poly $d(x)$ of least degree.

Claim: $M = (d)$ principal ideal. Let $f \in M$.

We may divide, $f = dq + r$ where $r = 0$ or $r < \deg(d)$.

$f \in M$, $d \in M$, $\Rightarrow dq \in M$ (ideal).

$\Rightarrow r = f - dq \in M$. (Since d is of least deg, this r cannot exist)

$\Rightarrow r = 0 \Rightarrow d \mid f \Rightarrow f \in (d) \Rightarrow M = (d)$ //

Ex: $\mathbb{R}[x_1, x_2]$. The ideal $(x_1) + (x_2) = (x_1, x_2)$ is not principal

Def: R integral domain. An element $a \neq 0$ in R is <u>reducible</u> if $\exists\, b, c \in R$. So $a = bc$ & b, c are not units.

(if u unit, $a = (au)u^{-1}$) Trivial.

Else a is irreducible. Call $p \in R$ <u>prime</u> if $p \neq 0$ and if whenever $a, b \in R$, $p \mid ab$ then $p \mid a$ or $p \mid b$.

In general, a prime is irreducible.

(p prime. If $p = ab$, a, b are non-units.
Then $p|a$ or $p|b$ by definition of prime
If $p|a$ $\exists\, c$ so $a = pc$.
$\Rightarrow p = ab = (pc)b = p(cb)$.

In integral domain, we can cancel, so $1 = bc \Rightarrow b$ is a unit $\#$
Same if $p|b$).

Remark In some rings, there are irreducible elements which are not primes.

Thm: In $F[x]$, an irreducible is prime.
Proof: Let $p \in F[x]$, $p \neq 0$, be irreducible. Take $a, b \in F[x]$

~~So~~ $p | ab \neq 0$.

to be cont'd.

Thm: In $F[x]$, if a, b nonzero polys and if $d \in F[x]$ satisfies $(d) = (a) + (b)$, then d is a gcd of a & b.

Proof: Since $\overset{1\cdot a}{a} \in (a) \subseteq (a) + (b) = (d)$ ← all multiples of d
$\Rightarrow d | a$. Similarly $d | b$. We want d to b g.c.d
$d \in (a) + (b)$

So there are polys g, b so $\underline{d = ga + hb}$

Let e be any common divisor of a & b — only gcd can be written this way //

$\Rightarrow e \mid ga$ & $e \mid hb$.

So $e \mid ga + hb = d$ //

■

Back to Thm Proof (contd...) we have $p \mid ab$.

We want $p \mid a$ or $p \mid b$.

Consider $\overset{d}{\wedge}$ gcd of p & a. It is a divisor of p, so can write

$$(d) = (1) \text{ or } (p) \quad \left(p \in (d) \;\exists \text{ poly } q \text{ so } \underset{\text{irred}}{p} = dq \Rightarrow d \text{ or } q \text{ is a unit} \right)$$

Say $d = 1$, $\exists\, g, h$ so that $ga + hp = 1$. Multiply this by b:

$$g\,ab + h\,pb = 1.b = b.$$

p divides $\Rightarrow p \mid b$.

Other case: $d = p$. But d = gcd of p & a. If so, $d \mid a$.

■

Note $1 = ga + hp \Rightarrow a, p$ are relatively prime.

Def R is integral domain, two elements a, b are associates if $\exists$ a unit u, so $b = ua \Rightarrow (b) = (a)$.

Thm: Unique Factorization in $F[x]$.

Given $f \in F[x]$, $f \neq 0$, $f \neq$ unit.

$\exists$ irreducible $p_1 \cdots p_r$ so $f = p_1 \cdots p_r$.

If $q_1 \cdots q_s$ are irreducible & $f = q_1 \cdots q_s$.

Then $r = s$ and after re-indexing, p_i & q_i are associates for $i = 1 \ldots r$.

Procedure for finding gcd of $a, b \in F[x]$.

Given $a, b \neq 0$,

Define a sequence $r_{-1}, r_0, r_1, \ldots$; $r_{-1} = a$ & $r_0 = b$.

by $\overset{a}{r_{-1}} = \overset{b}{r_0} q_1 + r_1$ as in Division algorithm.

$$r_0 = r_1 q_2 + r_2$$

$$\vdots$$

$$r_{k-2} = r_{k-1} q_k + r_k$$

$$r_{k-1} = r_k q_{k+1} + \underset{0}{\underset{\shortparallel}{r_{k+1}}}$$

$\Rightarrow r_k \neq 0$ & r_k is gcd of a, b.

We can derive an expression $r_k = g \cdot a + hb$ where $g, h \in F[x]$.

$$= (g - bf)\, a + (h + af)\, b$$

$\Rightarrow$ not unique.

Thm: Let $f \in F[x]$, $f \neq 0$, F field, f not a unit.
Then $\exists$ irreducibles $p_1 \dots p_r$ so $f = p_1 p_2 \dots p_r$. If $q_1 \dots q_s$ are irreducibles so $f = q_1 \dots q_s$ then $r = s$ after re-indexing we get that p_i & q_i are associates for $i = 1 \dots r$.

Proof: Existence of factorization is done by induction on $\deg(f) > 0$. If f is irreducible, done.

If not, there are polynomials g, h of positive degree so that $f = gh$. Since $\deg(g) < \deg(f)$. $\exists$ irreducibles $p_1 \cdots p_k$ so $g = p_1 \dots p_k$. Since $\deg(h) < \deg(f)$ $\exists$ irreducibles $p_{k+1} \dots p_r$ so $h = p_{k+1} \dots p_r$. $\Rightarrow f = p_1 \dots p_r$.

Uniqueness

Use induction on $\deg(f)$. If f is irred, then $r = s = 1$.
Suppose $r \geq 2$, then $s \geq 2$.

In $F[x]$, an irreducible is prime. So $\underset{\text{(irred)}}{p_1}$ divides $f = q_1 \dots q_s$.

So $\exists i$ so $p_1 | q_i$. Re-index so $i = 1$. $p_1 | q_1$ $\exists$ poly r_1 so

$\underset{\text{irred}}{\overset{\uparrow}{q_1}} = p_1 r_1$. Def of irred $\Rightarrow$ one of p_1, r_1 is a unit. $\Rightarrow r_1$ is a unit

$\Rightarrow p_1, q_1$ are associates. Since $p_1(p_2 \dots p_r) = \underset{\underset{p_1 r_1}{\shortparallel}}{q_1}(q_2 \dots q_s)$

$\Rightarrow p_2 \dots p_r = r_1(q_2 \dots q_s) = (r_1 q_2) q_3 \dots q_s$.

By induction (on $\deg(p_2 \dots p_r)$)

$r-1 = s-1$.

and after re-indexing

p_i & q_i are associates for $i = 2, 3, \dots, r$. ■

Determinants

We shall study matrices over K (a comm. ring).

Recall: If V fdvs, $T \in L(V,V)$, $\exists$ poly $f \neq 0$ so $f(T) = 0$ in $L(V,V)$ → motivation for studying polynomials.

We consider functions f: $Mat_{n \times n}(K) \to K$ which have certain properties.

(1) n-linear: If we consider f as a function of the n rows of the matrix. $f(V_1, \dots, V_n)$, $A = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix}$ then f is linear as a fn of row i when row j is fixed for all $j \neq i$.

$$f(V_1, \dots, V_{i-1}; aV_i + bV_i', V_{i+1}, \dots, V_n)$$

$$= a f(V_1 \dots V_{i-1}, V_i, V_{i+1}, \dots, V_n) + b f(V_1, \dots, V_{i-1}, V_i', V_{i+1}, \dots V_n).$$

Let f be an n-linear function.

(2) f is symmetric if $\forall\, i \neq j$ $f(V_1, \dots V_i \dots V_j \dots V_n)$ switch these two

$= f(V_1 \dots V_j \dots V_i \dots V_n)$.

f is alternating if f is 0 whenever 2 arguments at different indices are equal.

) f is a <u>determinant function</u> if f is alternating &

$f(I_n) = 1$

$$f\Big(\underbrace{(1,0,\dots 0)}_{e_1^t}, \underbrace{(0,1,0,\dots,0)}_{e_2^t}, \dots, \underbrace{(0,\dots,0,1)}_{e_n^t}\Big).$$

<u>Remarks</u>

A LC of n linear functions is n-linear.

A LC of Symmetric functions is symmetric.

" " " alternating " " alternating.

<u>Remarks</u>

$Mat_{1\times 1}(K) \longrightarrow K$

$(a) \longrightarrow a$ is 1-linear & symmetric.

$Mat_{2\times 2}(K) \longrightarrow K$

usual det $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ad-bc$

$\det\begin{pmatrix} a & b \\ a & b \end{pmatrix} = 0 \Rightarrow$ This is alternating & <u>2-linear</u>.

$$\det\begin{pmatrix} pa+qa' & pb+qb' \\ c & d \end{pmatrix} = (pa+qa')d - (pb+qb')c = p(ad-bc) + q(a'd-b'c)$$

$$= p\left(\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + q\left(\det\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}\right)$$

$\Rightarrow$ linearity in row 1. Similarly it is linear in row 2.

Lemma: Suppose that the n-linear function $f : \mathrm{Mat}_{n\times n}(k) \to k$ has the property that f has value 0 when 2 adjacent rows are equal. Then f is alternating.

Proof: If $f(\ldots \overset{i}{v} \; \overset{j}{v} \ldots) = 0 \;\; \forall\, v.$

Then $f(\ldots, w, \ldots, w', \ldots) = -f(\ldots w', \ldots, w \ldots) \;\; \forall$ all w, w'.

Reason: $f(\ldots, w+w', \ldots, w+w', \ldots) = 0 \quad \forall\; w, w'$

$$0 = \cancel{f(\ldots, w, \ldots, w)}^{\,0} + f(\ldots, w, \ldots, w', \ldots)$$
$$+ f(\ldots w', \ldots w, \ldots) + \cancel{f(\ldots w', \ldots w', \ldots)}^{\,0}$$

Now suppose, we have indeces.

$$i<j \qquad f(\ldots, \overset{i}{w}, \ldots, \overset{j}{w}, \ldots) = -f(\ldots, \overset{i}{w}, \ldots, \overset{j-1}{w}, \ldots)$$

$$\vdots$$

$$= \pm \underbrace{f(\ldots, w, w, \ldots)}_{\substack{\parallel \\ 0}} \text{ by hypothesis.}$$

Thm. For all $n \geq 1$, a determinant function exists.

Proof: We saw cases $n = 1, 2$.

We construct such a function for $n \geq 2$. $A_{n\times n} = (a_{ij})$.

Given indices i, j we let $A(i|j) = (n-1) \times (n-1)$ matrix obtained by striking row i & col. j.

Let D be our $(n-1)\times(n-1)$ det. function. Define D_{ij} as

$Mat_{n\times n}(K) \longrightarrow K$

$$D_{ij}(A) := D(A(i|j)).$$

Define $E_j(A)$ to be $= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} D_{ij}(A)$. $j = 1 \ldots n$.

Then E_j is a det function of degree $n \times n$.

Thm: Let D be an alternating $(n-1)$ linear fn of rows $\mathrm{Mat}_{n\times n}(k) \to k$. For $A = (a_{ij})_{n\times n}$, define

$$E_j(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} D(A(i|j))$$ Then E_j is n-linear & alternating. If in addition $D(I_{n-1}) = 1$, then $E_j(I_n) = 1$.

Proof: E_j is a LC of functions $\mathrm{Mat}_{n\times n}(k) \to k$. $A \xrightarrow[f_{ij}]{} a_{ij} D(A(i|j))$

Claim: these are n-linear whence so is E_j.

$f_{ij} = g_{ij} h_{ij}$, $g_{ij}(A) = a_{ij}$ ← is linear as function of row i but const as a function of rows $i' \neq i$

$h_{ij}(A) = D(A(i|j))$

is $(n-1)$ linear as fn of rows other than row i & const. as a fn of row i.

Conclude that f_{ij} is n-linear.

So E_j is n-linear.

Claim: E_j is alternating.

Suppose two adjacent rows k & $k+1$ are equal.

If $i \notin \{k, k+1\}$ then $D(A(i|j)) = 0$.

$E_j(A) = (-1)^{i+j} a_{ij} D(A(i|j))$ equal

opposite $+ (-1)^{i-1+j} a_{i-1,j}$ equal $D(A(i-1|j))$

$= 0$ //

If $D(I_{n-1}) = 1$, we calculate $E_j(I_n) = \sum_{i=1}^{n} (-1)^{i+j} \underset{\substack{\uparrow \\ \delta_{ij}}}{a_{ij}} \cdot D(I_n(i|j))$

$$= (-1)^{j+j} \cdot 1 \cdot D(I_{n-1}) = 1$$

<u>Uniqueness.</u>

D n-linear $Mat_{n\times n}(K) \to K$, $\epsilon_k = e_k^T = (0, \dots, 0, \underset{k}{1}, 0, \dots 0)$

$$D(A) = D\left(\sum_{j_1=1}^{n} a_{1j_1} \epsilon_{j_1}, \sum_{j_2=1}^{n} a_{2j_2} \epsilon_{j_2}, \dots, \sum_{j_n=1}^{n} a_{nj_n} \epsilon_{j_n}\right)$$

$$= \sum_{j_1 j_2 \cdots j_n = 1}^{n} a_{1j_1} a_{2j_2} \cdots a_{nj_n} D(\epsilon_{j_1}, \epsilon_{j_2}, \dots, \epsilon_{j_n}).$$

Next assume D is alternating

Then $D(\epsilon_{j_1}, \dots \epsilon_{j_n}) = 0$ if any two $j_1 \cdots j_n$ are equal.

So $D(A) = \displaystyle\sum_{j_1 \cdots j_n \text{ distinct}} a_{1j_1} a_{2j_2} \cdots a_{nj_n} D(\epsilon_{j_1} \cdots \epsilon_{j_n})$.

Let sym_n = set of all permutations of $\{1, 2, 3 \dots, n\}$.

The symmetric group of degree n.

The product in sym_n is composition of functions.

$\sigma \circ \tau$ means the fn $k \to \sigma(\tau(k))$. $|sym_n| = n!$

So

$$D(A) = \sum_{\sigma \in sym_n} a_{1\sigma_1}\, a_{2\sigma_2} \cdots a_{n\sigma_n}\, D(\epsilon_{\sigma_1}, \epsilon_{\sigma_2}, \ldots, \epsilon_{\sigma_n})$$

σ corresponds to the permutation $\begin{array}{cccccc} 1 & 2 & 3 & \cdots & n \\ \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ j_1 & j_2 & j_3 & & j_n \end{array}$

$\sigma_k \longleftrightarrow \sigma(k)$.

If we interchange indeces j_p & j_q $p \neq q$

$$D(\epsilon_{\sigma_1}, \epsilon_{\sigma_2}, \ldots \epsilon_{\sigma_p} \ldots \epsilon_{\sigma_q} \ldots \epsilon_n) = -D(\epsilon_{\sigma_1} \ldots \epsilon_{\sigma_q} \ldots \epsilon_{\sigma_p} \cdots \epsilon_{\sigma_n})$$

no change to σ_k if $k \neq p, q$

So $D(A)$ can be written as $\displaystyle\sum_{\sigma \in Sym_n} \underset{\substack{\uparrow \\ \text{some } \pm}}{sign(\sigma)}\, a_{1\sigma_1}\, a_{2\sigma_2} \ldots a_{n\sigma_n}\, D(\epsilon_1, \epsilon_2 \ldots \epsilon_n)$

Verify that every perm. on $\{1, 2, \ldots, n\}$ is a product on <u>transpositions</u> (= a perm. which fixes all points but two, these two are interchanged)

<u>Proof by induction</u>

$n = 1$ trivial

$n = 2 \Rightarrow$ ∘ ∘ ∘⟷∘

Say $n \geq 3$, take $\sigma \in Sym_n$. If $\sigma n = n$, then σ as a perm on $1, 2, \ldots, n-1$ is a product of transpositions on $1, 2, \ldots, n-1$.

Consider each of these transpositions as a perm. on $1, 2, \ldots, n$

Then σ is a product of the above transpose.

If $\sigma n = m \neq n$. use the transpose.

$\tau \in \mathrm{Sym}_n$. $\tau n = m$, $\tau m = n$, $\tau k = k$, for $k \neq m, n$.

Then $\tau\sigma$ fixes n. By previous case, there are transpositions

$\tau_1, \ldots, \tau_\ell$ so $\tau\sigma = \tau_1 \ldots \tau_\ell$. $= \underbrace{\tau\tau}_{\text{Identity}}\sigma = \tau\tau_1 \ldots \tau_\ell$ //

$$\Rightarrow D(A) = \left(\sum_\sigma \mathrm{sgn}(\sigma)\, a_{1\sigma_1} \cdots a_{n\sigma_n} \right) D(\epsilon_1, \ldots \epsilon_n).$$

If D is a det. fn, $D(\epsilon_1 \ldots \epsilon_n) = D(I_n) = 1$.

$\Rightarrow$ For a det fn $D(A) = \sum_\sigma \mathrm{sgn}(\sigma)\, a_{1\sigma_1} a_{2\sigma_2} \ldots a_{n\sigma_n}$

So for general n-linear alt fn D,

$$\boxed{D(A) = \det(A) \cdot \underbrace{D(I)}_{\text{Scalar.}}}$$

The function $\mathrm{sign}(\sigma) = (-1)^m$ where $\exists$ m transpositions $\tau_1 \ldots \tau_m$ so

$\sigma = \tau_1 \tau_2 \ldots \tau_m$.

<u>Example</u>: In sym_n (pq) means transposition $p \leftrightarrow q$.

In sym_3 $(12) = (13)(23)(13)$

1 2 3
3 2 1
3 1 2
2 1 3

Such m is unique mod 2.

If $\sigma = \tau_1 \tau_2 \ldots \tau_m$ then $\overset{\det}{D}(\epsilon_{\sigma 1}, \ldots, \epsilon_{\sigma n}) = D(\epsilon_{\tau_1\tau_2 \ldots \tau_m 1}, \ldots \epsilon_{\tau_1\tau_2 \ldots \tau_m n})$

$= -D(\epsilon_{\tau_2\tau_3 \cdots \tau_m 1}, \ldots, \epsilon_{\tau_2\tau_3 \cdots \tau_m n}) = (-1)^2 D(\epsilon_{\tau_3\tau_4 \cdots \tau_m 1}, \ldots) = (-1)^m \underbrace{D(\epsilon_1 \ldots \epsilon_n)}_{D(I) = 1.}$

Lecture

Warmup?

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

1) $\det(M) = 1$.

2) $\det(AD - BC) = \det\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$.

3) $\underset{0}{\det(A)}\,\underset{0}{\det(D)} - \underset{0}{\det(B)}\,\underset{0}{\det(C)} = 0$.

Going back to thm (uniqueness).

$E_1 = (1, 0, \dots, 0)$

$E_{\sigma_1} = (0, 0, \dots, \underset{\sigma_1}{\underset{\downarrow}{1}}, \dots 0)$

$D(E_{\sigma_1}, \dots E_{\sigma_n}) = (-1)^m$

If σ is a product of m transpositions we deduce that if σ is a product of m' transpositions then $(-1)^m = (-1)^{m'}$ so define $\mathrm{sgn}(\sigma) = (-1)^m$

Then we conclude that $D(A) = \sum_\sigma \mathrm{sign}(\sigma)\, a_{1,\sigma_1} \dots a_{n,\sigma_n}$.

This proves D is unique.

Thm: $\det(A^t) = \det(A)$

Notation: $A^t_{i,\sigma_i} = A_{\sigma_i, i}$

$$\det(A^t) = \sum_\sigma \mathrm{sgn}(\sigma) \underbrace{A^t_{1,\sigma_1} \dots A^t_{n,\sigma_n}}_{\underbrace{A_{\sigma_1,1} \dots A_{\sigma_n,n}}_{A_{1,\sigma_1^{-1}} \dots A_{n,\sigma_n^{-1}}}}$$

Also $\mathrm{sign}(\sigma\tau) = \mathrm{sign}(\sigma)\,\mathrm{sign}(\tau) \Rightarrow \mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma^{-1}) = \mathrm{sign}(\mathrm{Id}) = 1$

$\Rightarrow \mathrm{sign}(\sigma^{-1}) = \mathrm{sign}(\sigma)$.

$$\Rightarrow \det(A^t) = \sum_{\sigma \in Sym_n} \text{sign}(\sigma^{-1}) A_{1,\sigma^{-1}_1} \cdots A_{n,\sigma^{-1}_n}$$

$$= \sum_{\tau^{-1} \in Sym_n} \text{sign}(\tau) A_{1,\tau_1} \cdots, A_{n,\tau_n}.$$

$$= \sum_{\tau} \text{Sign}(\tau) A_{1,\tau_1} \cdots A_{n,\tau_n} = \det(A).$$

<u>Corollary</u>: All row properties are also column properties w.r.t determinants.

<u>Two Laplace expansions</u>

$$\forall j \quad \det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)).$$

$$\forall i \quad \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)).$$

(uses $\det(A^t) = \det(A)$).

$$> A_{n \times n} \quad \det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A(i|j)).$$

$C_{ij} = (-1)^{i+j} \det(A(i|j))$ is the i,j cofactor for A

$$\Rightarrow \det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Lemma If $j \neq k$, then $\sum_{i=1}^{n} a_{ik} C_{ij} = 0$

Proof: $B :=$ A with col j replaced by column k.

$$\Rightarrow 0 = \det(B) = \sum_{i=1}^{n} b_{ij} (-1)^{i+j} \det(B(i|j))$$

$$= \sum_{i=1}^{n} a_{ik} (-1)^{i+j} \det(A(i|j))$$

$$= \sum_{i=1}^{n} a_{ik} C_{ij}$$

Corollary $\sum_{j=1}^{n} a_{ik} C_{ij} = \delta_{kj} \det(A)$.

Def: The classical adjoint of A is $\operatorname{adj}(A) := (C_{ji})_{n \times n}$.

So $\boxed{\operatorname{adj}(A) A = \det(A) I_n.}$ (use Corollary).

Thm: $A \cdot \operatorname{adj}(A) = \det(A) I_n$ for all $A_{n\times n}$.

We observe $A^t(i|j) = A(j|i)^t$

$$(-1)^{i+j} \det(A^t(i|j)) = (-1)^{j+i} \det(A(j|i))$$

$$\Rightarrow \operatorname{adj}(A^t) = \operatorname{adj}(A)^t$$

Now use $\operatorname{adj}(A^t) A^t = \det(A^t) I_n$.

Transpose both sides.

$\Rightarrow A \underbrace{\operatorname{adj}(A^t)^t}_{\operatorname{adj}(A)} = \det(A) I_n$.

Example

$P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $Q \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $PQ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $QP = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Thm: A, B $n \times n$ then $\det(AB) = \det(A) \det(B)$.

Proof: Define $\boxed{f(A) \overset{= \det(AB)}{:=} \det(R_1 B; \dots; R_n B)}$.

rows

$A_{n\times n} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \end{pmatrix}$ f is n-linear & altesnating.

$\Rightarrow \underbrace{f(A)}_{\det(AB)} = \underbrace{f(I)}_{\det(B)} \det(A)$. from Thm.

set of elements that have 2 sided inverse under multiplication Units of K

Corollary: If A is inuertible in $\mathrm{Mat}_{n\times n}(K)$, then $\det(A) \in K^\times$

If $\det(A) \in K^\times$ then A is inuertible in $\mathrm{Mat}_{n\times n}(K)$.

Proof: If $AB = I \Rightarrow \det(A)\det(B) = \det(AB) = \det(I) = 1$.

Conversely, $\operatorname{adj}(A) A = \underbrace{\det(A)}_{\text{unit}} I$.

$\Rightarrow \frac{1}{\det(A)} \operatorname{adj}(A)$ is a matrix in $\mathrm{Mat}_{n\times n}(K)$.

$\Rightarrow \left(\frac{1}{\det(A)} \operatorname{adj}(A)\right) A = I$

$$\Rightarrow \boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).}$$

Cramer's Rule

$A_{n\times n}$ $\det(A)$ is a unit in ring K. $\Rightarrow \det(A) \in K^*$

Then a system $Ax = y_{n\times 1}$ has a unique solution

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ where $x_k = \dfrac{\det(B_k)}{\det(A)}$ & B_k is A with

col k replaced by $y := \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\underbrace{\operatorname{adj}(A)\, A}_{\det(A) I} x = \operatorname{adj}(A)\, y.$$

$$\Rightarrow \quad x = \frac{1}{\det(A)} \operatorname{adj}(A)\, y.$$

$$\det(A) x_j = (\operatorname{adj}(A)\, y)_j$$

Cramer's rule contd.

$Ax = y$, $\det(A) \in K^*$

$B_j = A$ with col. j replaced by y.

Then the unique soln. is

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{where} \quad x_j = \frac{\det(B_j)}{\det(A)}$$

$$\underbrace{\underbrace{\mathrm{adj}(A)\, A}_{\det(A) I} x}_{\det(A) x} = \mathrm{adj}(A)\, y$$

So $\det(A)\, x_j = (\mathrm{adj}(A) y)_j$

$= (\text{row } j \text{ of } \mathrm{adj}(A))\, y.$

$= \sum_{i=1}^{n} c_{ij} y_i = \det(B_j)$ ✓ Laplace expansion along col j.

$\det(B_j)$ expanded along its column j is (by Laplace), $\sum_{i=1}^{n} b_{ij} c_{ij}$

$$\det(B_j) = \det(A_1 | \dots | A_{j-1} | y | A_{j+1} | \dots | A_n)$$

$$y = \sum_{k=1}^{n} x_k A_k$$

$$= \sum_k x_k \underbrace{\det(A_1 | \dots | A_{j-1} | A_k | A_{j+1} | \dots | A_n)}_{0 \text{ unless } k = j}$$

(repeated col $\Rightarrow$ det $= 0$)

$= x_j \det(A_1 | \quad | A_n)$

$= x_j \det(A)$

$\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C)$

$A_{p\times p} \quad B_{p\times q} \quad C_{q\times q}$

Define: $\boxed{f(C) := \det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}}$

Then f is q-linear & alt. So by Thm.

$$f(C) = \underset{\det\begin{pmatrix} A & B \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}}{\underset{\parallel}{f(I)}} \det(C).$$

(use row ops).

Define $g(A) := \det\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$

It is p-linear & alt $\Rightarrow g(A) = \underset{\det(I_{pq}) = 1.}{\underset{\parallel}{g(I)}} \det(A).$

Canonical Forms.

$T \in L(V,V)$ $\dim V = n$ then T satisfies a poly $\neq 0$ of degree $\leq n^2$.
$\dim(L(V,V)) = n^2$

$$\underbrace{I_V, T, T^2, \ldots, T^{n^2}}_{1+n^2} \Rightarrow L \cdot C = 0 \text{ since } \deg = n^2+1$$

The set of polynomials in $F[x]$ satisfied by T is an <u>ideal in</u> $F[x]$.

(If f, g are satisfied by T, so are $(af+bg)$.

If $h \in F[x]$, then $f(x)h(x) \xrightarrow{T} \underbrace{f(T)\,h(T)}_{0}$.)
$f(T) = 0$

Def: $m(x)$ the monic poly. generating the ideal in $F[x]$ of polys which vanish on T is called the minimum polynomial for T.
(minimal)

We shall prove that $\deg(m_T(x)) \leq n$.

Quotient structures

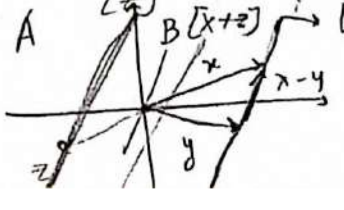
(Equivalence rel. is w.r.t to a subspace or subgroup)
(Set of eq. classes form a VS or 0 element is B)
↳ V/W → quotient space.

Abelian Groups. $(A,+)$ abelian group, B is a subgroup.

Equivalence relation $x, y \in A$ $x \sim y \Leftrightarrow x - y \in B$.

$[x] = \{z \in A \mid z \sim x\}$. Define $[x] + [y] := \{u + v \mid u \in [x], v \in [y]\}$

(This is just $[x+y]$



$[x] = x + B$
↳ eq class or coset.

Proof: Given $u \in [x]$, $v \in [y]$, $u - x \in B$, $v - y \in B$

$(u+v) - (x+y) = (u-x) + (v-y) \in B. \Rightarrow u+v \in [x+y]$.

Conversely, if $w \in [x+y]$ then $w = \underset{[x]}{\underset{\uparrow}{x}} + \underset{\text{claim in } [y]}{\underset{\uparrow}{(w-x)}}$

$(w-x) - y = w - (x+y) \in B$.

This sum makes A/B = set of eq. classes, an abelian group.

Its 0 element is $[0] = B$. $-[x] = [-x]$.

Vector spaces $/F$. $(V, +)$ W subspace is a subgroup under $+$. So can have quotient group V/W.

This also has a scalar mult. on it defined by

$$\underbrace{c[x]}_{\substack{\| \\ \{cy \mid y \in [x]\}}} := [cx].$$

$\{z \in V \mid z \sim cx\}$

ie. $z - cx \in W$

Claim: equal

LHS $\subseteq$ RHS: Need $cy \sim cx$ $\forall y$. $\Rightarrow$ $cy - cx \in W$ & $c(y-x) \in W$

$c(y-x) \in W$ subspace.

RHS $\subseteq$ LHS (exercise).

each vector here is $[x]$.

$x \sim y \overset{\text{def}}{\Longleftrightarrow} [x] = [y]$ $\quad$ $\ker(q) = W$ since $w \sim 0$

Check that $V|W$ becomes a VS with this scalar mult.

Also the quotient map. $q : V \longrightarrow V/W$ ie $x \longmapsto [x]$

is a LT with Kernel W. Image $= V/W$.

If $T \in L(V,V)$ & W is a subspace which admits T (i.e $T(W) \subseteq W$)

Also one says W is <u>T-invariant</u>,

Then we can define two new Linear Transformations (LTs):

$T_W : W \longrightarrow W.$

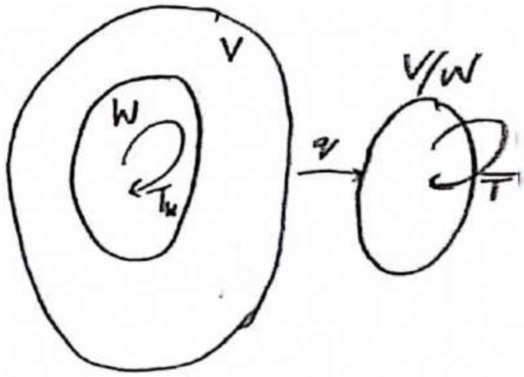
Also,

$\overline{T} : V/W \longrightarrow V/W$

$[x] \xmapsto{T} [Tx]$

$y \in [x] \Rightarrow y - x \in W \Rightarrow T(y-x) \in W.$

$\Rightarrow Ty - Tx \in W$, so $Ty \sim Tx$.



The equivalence classes just collapse W to 0 & make a new VS V/W.

<u>Thm:</u> V n-dim, $T \in L(V,V)$ Then T satisfies a poly of deg $\leq n$.

<u>Proof:</u> If $\dim V = 1$, T scalar & satisfies $x - a$ for some $a \in F$

Suppose $\dim(V) > 1$. Suppose that there is a subspace $W \neq 0, V$ so that W is T-invariant.

$\dim W = p$ $(p<n)$. $\dim(V/W) = q$ $(q<n)$. $p+q = n$. (RTN thm)

$\Rightarrow \overline{T}$ satisfies a poly $g(x)$ of degree $\leq q$. (induction hypothesis)

T_W satisfies a poly. $h(x)$ of deg $\le p$. by induction on dim.

For any vector $v \in V$. $g(\overline{T}) = 0$. i.e. $g(\overline{T})[v] = [0]$.

$$g(T)v \sim 0 \Rightarrow g(T)v \in W.$$

For any $w \in W$. $h(T)w = 0$. because $h(T_W) = 0$.

$$\Rightarrow \underbrace{h(T)\, g(T)}_{f(T)} v = 0 \quad \forall \; v \in V.$$

where $f(x) := g(x)\, h(x)$

$\deg \le p+q = n$.

If there is no such W. then can show there is a basis of V of the form $v, Tv, T^2v, \dots, T^{n-1}v$

$$T = \begin{pmatrix} & & & -a_0 \\ 1 & & 0 & -a_1 \\ & \ddots & & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix}$$

V vs $T \in L(V,V)$

$v \in V$

span $\{v, Tv, T^2v, \dots\}$ $Z(T,v)$

Claim: This is T-invariant

Typical element is $\sum_{i=0}^{p} a_i T^i v$

Apply T: $\sum_{i=0}^{p} a_i T^{i+1} v \in$ span $\{v, Tv, \dots\}$

Last time we started proof for

Thm:- If $\dim V = n$, $T \in L(V,V)$. Then $\exists$ poly $p(x)$ of degree $\leq n$ so $p(x) \neq 0$, $p(T) = 0$

Proof :- We showed if $\exists$ T-invariant subspace $W \neq 0, V$.

Then $\dim(W)$, $\dim(V/W) < n$.

So by induction,

T_W satisfies poly of deg $\leq \dim(W)$.

$\bar{T} \in L(\bar{V}, \bar{V})$, $\bar{V} = V/W$. satisfies poly $h(x)$ of deg $\leq \dim(\bar{V})$

$\Rightarrow g(T)\, h(T) = 0$

$\deg(gh) \leq n$

$g(\bar{T}) = \overline{g(T)}$ (in posted notes?)

Suppose that there is no such W. Then for $v \in V$, $v \neq 0$,

$Z(T,v) = V$. $\longrightarrow$ span $\{v, Tv, T^2v, T^3v \dots\}$ {since this span is T-invariant, the only candidate is V}

Suppose m is the smallest integer so $f(T) = 0$ for some monic poly. $f \neq 0$ & $\deg f = m$.

↳ minimal

So $\forall\ k \geq m$, $T^k v \in \text{span}\{v, Tv, T^2v, \dots, T^{m-1}v\}$

[Since $f(x) = x^m + \sum_{i=0}^{m-1} a_i x^i$

$$0 = f(T)v = T^m v + \sum_{i=0}^{m-1} a_i T^i v$$

$\Rightarrow T^{m+k} v \in \text{span}\{v, Tv, \dots, T^{m-1}v\}\ \forall\ k \geq 0$]

{Since all powers $\geq m$ can be written as a comb. of lower order terms}

$\Rightarrow m \leq n$ since $v, Tv, \dots, T^{m-1}v$ is LI.} therefore we find a spanning set Z s.t. any T^k is in it.

So $\dim(Z(T,v)) = m$.

$Z(T,v) = n \Rightarrow m = n$. (foreshadows that $m(x) = c(x)$)

So V has a basis of the form $v, Tv, \dots, T^{n-1}v$.

A = matrix for T in this basis

$$\Rightarrow A = \begin{pmatrix} 0 & & & & -c_0 \\ 1 & 0 & & 0 & \vdots \\ & 1 & 0 & & \vdots \\ & & 1 & \ddots & \vdots \\ & 0 & & 0 & \vdots \\ & & & 1 \quad 0 & -c_{n-2} \\ & & & \quad 1 & -c_{n-1} \end{pmatrix}$$

where T satisfies the polynomial

$$f(x) = x^n + \sum_{i=0}^{n-1} c_i x^i$$

$$\Rightarrow T^n + \sum_{i=0}^{n-1} c_i T^i = 0$$

$$\Rightarrow T^n v = \sum_{i=0}^{n-1} (-c_i) T^i v.$$

<u>Def</u> $f(x)$ monic polynomial of degree n

$$f(x) = x^n + \sum_{i=0}^{n-1} c_i x^i$$

The <u>companion matrix</u> for f is

$$C_f = \begin{pmatrix} 0 & & & & & -c_0 \\ 1 & 0 & & \text{\Large 0} & & \vdots \\ & 1 & \ddots & & & \vdots \\ & & \ddots & & & \vdots \\ & \text{\Large 0} & & \ddots & & \vdots \\ & & & 0 & & \\ & & & 1 & 0 & -c_{n-2} \\ & & & & 1 & -c_{n-1} \end{pmatrix}$$

<u>Claim</u>: $f(C_f) = 0$ in $\underbrace{\mathrm{Mat}_{n\times n}(F)}_{\text{ring of } n\times n \text{ matrices.}}$

$\Leftrightarrow f(T) = 0$ (T, v as before)

$f(T)v = \left(T^n + \sum_{i=0}^{n-1} c_i T^i\right)(v) = T^n v + \sum_{i=0}^{n-1} c_i T^i v$ which is 0.

Now let $k \geq 0$,

Then $f(T)T^k v = T^k f(T) v$ (because T^k, T^ℓ commute $\forall\, k, \ell$)

$$= T^k \underbrace{f(T)v}_{0} = 0$$

$\blacksquare$

Let $T = [M]_B$. We want to show $A = S^{-1}MS$ where $S = [v, Tv, \dots, T^{n-1}v]$

$\Leftrightarrow SA = MS$; $SA = S[e_2, e_3, \dots, e_n, -\vec{c}]$ where $\vec{c} = (c_0; c_1, \dots, c_{n-1})$

$Se_i =$ col i of $S = T^{i-1}v = M^{i-1}v \Rightarrow SA = [Tv, T^2v, \dots, T^{n-1}v, -S\vec{c}]$

$MS = T[v, Tv, \dots] = [Tv, T^2v, \dots, T^n v]$. Left to show that $-S\vec{c} = T^n v$

$S\vec{c} = S(c_0 e_1 + c_1 e_2 + \dots + c_{n-1}e_n) = c_0 Se_1 + c_1 Se_2 + \dots + c_{n-1} Se_n = c_0 v + c_1 Tv + \dots + c_{n-1}T^{n-1}v = -T^n v$

since $T^n + \sum^{n-1} c_i T^i v = 0$ //

Def $A_{n\times n}$, The characteristic polynomial for A is

$$C_A(x) = \det(xI - A)$$ $\rightarrow$ det. over comm ring $F[x]$.

If $T \in L(V,V)$ & is represented by a matrix A.

Def $C_T(x) = C_A(x)$.

If B = mtx of T in a new basis, then $\exists$ invertible S so

$$B = S^{-1}AS$$

$$\det(xI - B) = \det(xI - S^{-1}AS) = \det(S^{-1}(\overset{\text{commutes with } S}{xI} - A)S)$$

$$= \det(\cancel{S^{-1}})\,\cancel{\det}(S)\det(xI - A) = \det(xI - A)$$

<u>Thm (Cayley Hamilton)</u>

$A_{n\times n}$ satisfies it's characteristic polynomial i.e. $C_A(A) = 0$.

Equivalently $C_T(T) = 0 \quad \forall\, T \in L(V,V)$, V fdvs.

<u>Proof(different from book)</u>

If $n = 1$ the T is scalar & mtx $(c) \Rightarrow C_T(x) = x - c$

$\Rightarrow C_T(c) = T - cI = 0$ ✓

Suppose $n \geq 2$ & suppose there is a T invariant subspace $W \neq 0, V$.

Basis $v_1 \dots v_m$ for W

extend basis to $v_1 \dots v_n$ basis V.

mtx for T has form $\left(\begin{array}{c|c} A_{m\times m} & B_{m\times(n-m)} \\ \hline 0 & C_{(n-m)\times(n-m)} \end{array}\right) = \mathcal{B}$

$C_T(x) = \det(xI - \mathcal{B})$.

$$= \det \left(\begin{array}{c|c} xI_{m\times m} - A & -B \\ \hline 0 & xI_{n-m} - C \end{array}\right)$$

$\Rightarrow \quad = \underline{C_A(x) \cdot C_c(x)}$.

To show: $C_A(T)\, C_c(T) = 0$

$A = $ mtx T_W → restriction of T to W.

$C = $ mtx $\bar{T}$ on $\bar{V} = V/W$ → $\mathcal{L}T$ on quotient space.

using basis $[v_{m+1}], \dots, [v_n]$

So $c_C(\bar{T}) = 0$ by induction on dim.

ie $\bar{T} \in L(\bar{V}, \bar{V})$ satisfies its $C_C(x)$ → char. pol.

$\Rightarrow c_C(T)$ satisfies $\operatorname{Im}(\overset{\text{no bar}}{c_C(T)}) \subseteq W$

because $\operatorname{Im}(c_C(\bar{T})) = 0$.

$\Rightarrow \operatorname{Im}(c_A(T)\, c_C(T)) = C_A(T)\,(\underbrace{\operatorname{Im}(c_C(T))}_{\subseteq W})$

$\Rightarrow$ " $\subseteq C_A(T)(W) = 0$ since $C_A(T_W) = 0$. by induction.

$*$

Finally suppose that there is no such W.

Then V has basis of the form $v, Tv, T^2v, \dots, T^{n-1}v$.

mtx $\leftarrow$ for T on V w.r.t basis is C_f, where $f = c_T(x)$

We know $f(C_f) = 0$.

> A, B $n\times n$ matrices. Similar iff $\exists$ S invertible so that $B = S^{-1}AS$.

Given $T \in L(V,V)$.
We shall study decompositions of V into subspaces which are T invariant.

<u>Def</u>: V is a direct sum of subspaces $V_1 \dots V_r$ if every $x \in V$ has a unique expression $x \in \sum x_i$, $x_i \in V_i$.

<u>Lemma</u>: V is a direct sum of $V_1 \dots V_r$ iff $V = V_1 + V_2 + \dots + V_r$ & $\forall$ $i = 1 \dots r$, $V_i \cap \left(\sum_{j\neq i} V_j\right) = 0$.

<u>Proof</u>: If V is the direct sum, then $V_1 + \dots + V_r = V$.

If $x \in V_i \cap \left(\sum_{j \neq i} V_j\right)$ then $x = x + 0 + \dots + 0$ is an expression $x \in V_i$ and $x = 0 + \left(\sum_{j\neq i} x_j\right)$ is another $x_j \in V_j$ $\Rightarrow x = 0$ by uniqueness ...

First decomposition Theorem version 1 (not in book)

aka Primary decomposition Thm v1

<u>Thm</u> V vs/F, $T \in L(V,V)$, $0 \neq f \in F[x]$ so $f(T) = 0$.

Suppose $g, h \in F[x]$ $f = gh$, $(g,h) = 1$ $\Rightarrow$ rel prime.

Then there are T-invariant subspaces V_1, V_2 of V so that

$$V = V_1 \oplus V_2 \ , \ \underbrace{g(T)\big|_{V_1}}_{\text{restricted to}} = 0 \ , \ h(T)\big|_{V_2} = 0$$

Furthermore if $f(x) = m_T(x)$, then $g(x) = m_{T_1}(x)$ where $T_1 \in L(V_1, V_1)$ & $T_1(x) = T(x)$ for $x \in V_1$.

& $h(x) = m_{T_2}(x)$ where $T_2 \in L(V_2, V_2)$, $T_2(x) = T(x)$ for $x \in V_2$

<u>Proof</u>: Define $V_1 := \ker(g(T))$, $V_2 := \ker(h(T))$.

Since $(g,h) = 1$, there are polynomials p, q s.t. $1 = pg + qh$ in $F[x]$

$$\Rightarrow 1_V = p(T)g(T) + q(T)h(T).$$

$$x \in V \Rightarrow x = 1x = \underbrace{p(T)g(T)x}_{\text{in } V_2} + \underbrace{q(T)h(T)x}_{\text{in } V_1}.$$

(Similar arg).

$$p(T)g(T)x \xrightarrow{h(T)} h(T)p(T)g(T)x = p(T)f(T)x = 0$$

So $V = V_1 \oplus V_2$.

Suppose $x \in V_1 \cap V_2$.

$$x = 1_V x = p(T)g(T)x + q(T)h(T)x.$$

$$= 0 + 0 = 0$$ since $x \in V_1 \cap V_2$ (so both $g(T)$ & $h(T)$ kill x)

<u>Lemma</u>: If $S, T \in L(V,V)$ & they commute ($ST = TS$), the S leaves invariant $\mathrm{Ker}(T)$ & $\mathrm{Im}(T)$.

<u>Proof</u>: Take $x \in \mathrm{Ker}(T)$, we want $Sx \in \mathrm{Ker}(T)$.

Check $T(Sx) = S\,T(x) = S.0 = 0$.

Take $y \in \mathrm{Im}(T)$ so $y = Tx$ for some x.

$$\Rightarrow Sy = S(Tx) = T(Sx) \in \mathrm{Im}(T). \;\rfloor$$

The lemma proves T leaves V_1, V_2 invariant since T commutes with $g(T)$ & $h(T)$.

<u>Lemma</u>: $T \in L(V,V)$, $m_T(x)$ = minimal poly of T.

If W is a T-invariant subspace, $T_W \in L(W,W)$ ($T_W x = Tx$ for $x \in W$), then $m_{T_W}(x)$ divides $m_T(x)$.

Also, we have $\bar{T} \in L(\bar{V}, \bar{V})$, $\bar{V} = V/W$.

defined by $\bar{T}[x] = [Tx]$. Then $m_{\bar{T}}(x) \mid m_T(x)$.

<u>Proof</u>: $m_T(T_W) = 0$ because $m_T(T)x = 0 \;\; \forall x \in V$.

So $m_{T_W}(x) \mid m_T(x)$. (We use $\theta: F[x] \to L(W,W)$, $x \mapsto T_W$

$\mathrm{Ker} = (m_{T_W}(x))$ & $m\,(x)$ is ...)

Also $m_T(\bar{T})[x]$

$= [m_T(T)x] = [0] \quad \forall x \in V.$

Second statement of Thm

If $f(x) = m_T(x)$, then $g(x) = m_{T_1}(x)$, $h(x) = m_{T_2}(x)$.

$m_{T_1}(x) \mid g(x)$, $m_{T_2}(x) \mid h(x)$

$f = gh$. $(g,h) = 1 \Rightarrow m_{T_1}(x)$ & $m_{T_2}(x)$ are rel. prime.

$(m_{T_1}(x), m_{T_2}(x)) = 1$.

$\Rightarrow m_{T_1}(x)\, m_{T_2}(x) \mid gh = f$.

(if $(p,q) = 1$ & $p \mid f$, $q \mid f \Rightarrow pq \mid f$ & let $f = pf_1$ for $f_1 \in F[x]$.

Since $1 = ap + bq$, $f_1 = \underbrace{apf_1}_{f} + bqf_1 \Rightarrow q$ divides $f_1 \Rightarrow pq$ divides f)

q divides

Claim: $m_{T_1}(T)\, m_{T_2}(T) = 0$

Apply to $x \in V$, $x = \underset{V_1}{\underset{\cap}{x_1}} + \underset{V_2}{\underset{\cap}{x_2}}$, $\Rightarrow m_T(x)$ divides $m_{T_1}(x) m_{T_2}(x)$

$\Rightarrow m_T(x) \mid m_{T_1}(x)\, m_{T_2}(x)$.

This is unideal generator

T substituted here kills x so it is in the ideal generated by $m_T(x)$

So $f = m_T(x)$ equals $m_{T_1}(x)\, m_{T_2}(x)$.

We know, if V fdvs & $T \in L(V,V)$ then $\underset{\text{min pol}}{m_T(x)} \mid \underset{\text{Char pol}}{C_T(x)}$.

We can show that $m_T(x)$ & $C_T(x)$ have the same irreducible divisors.

For eg, if T has $m_T(x) = x^2(x-1)^3$, then

$$C_T(x) = x^a(x-1)^b \Rightarrow a \geq 2,\ b \geq 3.$$

(proves previous statement)

__Thm__ V fdvs, $T \in L(V,V)$ then $C_T(x)$ divides $m_T(x)^n$,

where $n = \dim(V)$.

__Proof__: Suppose $W \neq 0$, V is T-invariant & $n_1 = \dim W$,

$n_2 = \dim(V/W)$, we can choose basis for W & an extension to basis for V. T rep. by matrix $\left(\begin{array}{c|c} \underbrace{A}_{n_1 \times n_1} & B \\ \hline 0 & \underbrace{C}_{n_2 \times n_2} \end{array}\right)$

$C_T(x) = C_1(x)\, C_2(x)$ where C_1 = char pol of T_W & C_2 = char. pol $\bar{T}$.

$$C_T(x) = \det(xI - M) = \det\left(\begin{array}{c|c} xI - A & -B \\ \hline 0 & xI - C \end{array}\right) = \det(xI - A)\det(xI - C)$$

Remarks on proof of Primary decomposition theorem

(1) Could define

$V_1 = Im(h(T))$

$V_2 = Im(g(T))$

(2) We have projections.

for $V = \underset{x_1}{V_1} + \underset{x_2}{V_2}$

$P_i(x) \in V_i$

$x = P_1 x + P_2 x$

$$1_V = \underbrace{p(T)g(T)}_{P_2} + \underbrace{q(T)h(T)}_{P_1}$$

polys in T

choosing diff $p, q \Rightarrow p'(T), q'(T)$ is still the same as $p(T), q(T)$

$\Rightarrow P_1, P_2$ remain same for any p, q

Check. $x = \overbrace{p(T)\,g(T)}^{x_2}\,x + \overbrace{q(T)\,h(T)\,x}^{x_1}$.

$\underset{V_2}{\cap}$ $\underset{V_1}{\cap}$

Calculate $\underbrace{q(T)h(T)x}_{\underset{V_1}{\cap}} = \underbrace{q(T)\,\underbrace{h(T)p(T)}_{=0}\,g(T)x}_{0\text{ map}} + \underbrace{(q(T)h(T))^2 x}_{P_1(P_1(x))}$

$q(T)\,h(T)\,(p(T)\,g(T))x = 0.$

So if $S \in L(V,V)$ & $ST = TS$, then S commutes with the polys P_1, P_2 in T. $\Rightarrow S$ leaves V_1 & V_2 invariant

$\underline{\text{Ex}}$. If $T = \begin{pmatrix} c & \\ & c \end{pmatrix}$. scalar.

S = any commutes with T

but T leaves any 1-dim space invariant but S does not generally.

$$V = Fe_1 \oplus Fe_2 \qquad S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

<u>Going back to Thm from earlier.</u>

<u>Thm</u>. V fdvs $T \in L(V,V)$, $c(x)$ char poly, $m(x)$ min pol

Then $m(x) \mid c(x)$ & $c(x) \mid m(x)^n$, $n = \dim(V)$. Consequently $m(x)$ & $c(x)$ have the same set of irreducible divisors.

<u>Proof</u>. Suppose W is a T-inv subspace $\neq 0, V$, we have $T_W \in L(W,W)$ & $\overline{T} \in L(\overline{V}, \overline{V})$ where $\overline{V}$ is the quotient space V/W

We argue by induction ($n=1$, (a), $x-a = c(x) = m(x)$).

Choose basis for W, extend to basis for V to get T represented by matrix $\begin{pmatrix} A & B \\ \underset{W}{0} & D \end{pmatrix}$. If $p = \dim W$, $q = \dim \overline{V}$ then

$c_{T_W}(x) \mid m_{T_W}(x)^p$ & $c_{\overline{T}}(x) \mid m_{\overline{T}}(x)^q$. $\Rightarrow c_T(x) = c_{T_W}(x)\, c_{\overline{T}}(W)$.

(T_W rep by A & $\overline{T}$ rep by D). $m_{T_W}(x) \mid m_T(x)$ & $m_{\overline{T}}(x) \mid m_T(x)$.

$\Rightarrow c_T(x) \mid m_T(x)^{p+q} = m_T(x)^n$.

What if there is no such W?

Recall $Z(T,v) = \text{span}\{v, Tv, T^2v, \ldots\}$ for $v \in V$.

This is T-invariant.

Here, $Z(T,v) = V$ if $v \neq 0$.

So V has basis $v, Tv, \ldots, T^{d-1}v$, for some $d \leq \deg(C_T(x))$.

But $Z(T,v) = V \Rightarrow n = d$.

We have $T^d v + \sum_{i=0}^{d-1} a_i T^i v = 0$. So T satisfies $f(x) = x^n + \sum_{i=0}^{n} a_i x^i$

and does not satisfy a poly of deg $\leq d-1$ because

$v, Tv, \ldots, T^{d-1}v$ is a basis.

$\Rightarrow f(x)$ is the minimum poly of T ie $m_T(x)$ & T is rep by mtx

$C_f \Rightarrow C_T(x) = \det(xI - C_f) = f \Rightarrow m_T(x) = C_T(x)$. ✓

<u>Def</u>: $T \in L(V,V)$ V fdvs $v \in V$ then we have $\{f(x) \in F[x] \mid f(T)v$

ideal in $F[x]$ (annihilator ideal) & it has monic generator $m_{T,v}(x)$

So $m_{T,v}(x) \mid m_T(x)$

<u>Ex</u>: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a \neq b$ $\quad m_T(x) = (x-a)(x-b)$

$m_{T,e_1}(x) = x - a \quad m_{T,e_2}(x) = x - b$.

$m_{T,0}(x) = 1$. $v = \begin{pmatrix} p \\ q \end{pmatrix}$ $\begin{matrix} p \neq 0 \\ q \neq 0 \end{matrix}$ $\Rightarrow m_{T,v}(x) = (x-p)(x-q)$.

POST EXAM 2

<u>Def</u>: $T \in L(V,V)$, V fdvs.

$$\vec{v} \in V \to \{ f(x) \in \mathbb{F}[x] \mid f(T)\vec{v} = 0 \} := S.$$

monic generator $m_{T,v}(x)$

$\to m_{T,\vec{v}}(x) \mid m_T(x)$

<u>Eg</u>: $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ $a \neq b$, $m_{Te_1}(x) = (x-a)$, $m_{Te_2}(x) = x-b$

$m_{T,0}(x) = 1$

if $\vec{v} = \begin{pmatrix} p \\ q \end{pmatrix}$ $m_{T,v}(x) = (x-p)(x-q)$

R ring (associative with identity) if you have a homomorphism of rings $R \xrightarrow{f} S$ f preserves sums & products. $f(x+y) \to fx + fy$
$f(xy) \to f(x)f(y)$
and $f(1_R) = 1_S$.

$k = \ker f =$ 2 sided ideal.

R & S are abelian, K is a subgroup $(R,+)$

$r \in R$ $k \in K$; $f(rk) = f(r)\cancel{f(k)}^{0} = 0$

$f(kr) = \cancel{f(k)}^{0} f(r) = 0$

$\to K$ is an ideal.

R/K has product $[a][b] = [ab]$.

$(a+k)(b+k) \subseteq ab+k$ (not necessarily equal).

Making new Rings & Fields

$\mathbb{Z}/p\mathbb{Z}$ is a field as p & q are coprime $\exists a,b$ s.t. $1 = ap+bq$

quotient map $a \mapsto [a]$, $R \longrightarrow R/K$ is a ring homomorphism.

If F is a finite field, then $|F| = q$ where $q = p^k$ for prime p.

Let R be an $\mathbb{F}$-algebra then an ideal is an F-subspace
K ideal, $a_i \in K$, $c \in \mathbb{F}$. $\rightarrow$ $c a_1 = \underbrace{(c\,1)}_{\in R} a_1 \in K$

$\mathbb{F}[x]$, take any ideal $= (m(x)) = K$.
Say $d = \deg(m(x))$. $\mathbb{F}[x]/(m(x))$

$P_d := \{ f \in \mathbb{F}[x] \mid \deg(f) \leq d-1 \text{ or } f = 0 \}$.

> Claim: $\mathbb{F}[x] = K \oplus P_d$.

Given $f \in \mathbb{F}[x]$ $f = qm + r$ for some $r = 0$ or $\deg(r) < \underbrace{\deg(m)}_{d}$.

$\Rightarrow f \in \mathbb{F}[x] \rightarrow f \equiv r \bmod m$ and $\in P_d$.

P_d is generally not a subring (so not an ideal)
Not a subring if $d \geqslant 2$
But we can model R/K (quotient algebra) by products in
P_d mod m.

If $\pi : F[x] \rightarrow F[x]/K$ then $\mathrm{Im}(\pi) = \mathrm{span}([x^k])$ & has basis
$[1], \dots, [x^{d-1}]$.

So $f(x), g(x) \in \mathbb{F}[x]$.

$\rightarrow$ $[f][g] = [fg] = [r]$ where r = remainder of $fg \bmod (m(x))$

$\rightarrow$ $\overline{P_d'} = \text{Im } P_d$ in $\mathbb{F}[x]/k$.

$\overline{P_d} = \mathbb{F}[x]/_k$ is an F algebra.

<u>Eg:</u> Take $\mathbb{R}[x]$ and (x^2+1)

$P_d = \mathbb{R}[x]/(x^2+1)$ has basis $[1]$, $[x]$.

$[x]^2 = [x^2] = [-1] = -[1]$.

$\rightarrow$ P_d is iso to $\mathbb{C}$ by $[a+bx] \xrightarrow{\phi} a+bi$

$P_d = \mathbb{R}[x]/(x^2+x+1)$

Take $w = e^{2\pi i/3} = \frac{1}{2} - \frac{1}{2}\sqrt{3}i$

$w^2 = e^{4\pi i/3} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i = \overline{w} = w^{-1}$

So P_d can be show iso $(\mathbb{C})$ by taking $[a+bx] \rightarrow a+bw$.

Note $F[x]/_k = F[x] \bmod k$.

Let $T \in L(V,V)$ $\mathbb{F}[x] \xrightarrow{\theta} L(V,V)$ where $\theta(x) \rightarrow T$

Let $k = \ker\theta$

$\text{Im}(\theta) =$ subring $(L(V,V))$ spanned by $(1, T, T^2, \dots)$

<u>$\text{Im}(\theta)$</u> is isomorphic to $F[x]/_k$; $\mathbb{F}$-Algebra but $L(V,V)$ is <u>No</u>

comm

Eg: Take $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Mat_{2\times 2}(\mathbb{R})$.

$A^2 = -I$, minimum poly $= x^2+1$

$\text{im}(\theta) = \text{span}(I, A)$ is isomorphic to $\mathbb{C}$. by

$$[aI + bA] \to a+bi$$

Def :- Sum of rings R_1 & R_2

$$R_1 \oplus R_2 = \{(x,y) \mid x \in R_1, y \in R_2\}$$

Component wise operations

$$(a,b) + (a',b') = (a+a', b+b').$$

$$(x,y) + (x',y') = (xx', yy').$$

T satisfies a polynomial f → i.e. $f \in F[x] \to f(T) = 0 \to I, T, T^2 \ldots$ are L

Eg: $T \in L(V,V)$ where min poly$(T) = m_T(x) \to m_T(x) = (x-a)(x-b)$

$a \neq b \in \mathbb{F}$

Claim: $\text{Im}(\theta) \simeq F \oplus F$

If T is represented by a diagonal matrix.

$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ note adding / multiply diagonal matrices.

Think of PDT: T satisfies $\underbrace{(x-a)}\,\underbrace{(x-b)}$

relatively prime.

$V = V_a \oplus V_b$

$\ker(T-a)$ $\quad$ $\ker(T-b)$

T acts on V_a as multiplication by a,

T " " V_b " " by b

$\theta : \mathbb{F}[x] \to L(V,V)$

$\theta : x \to T$

$Im(\theta)$ is a sub $\mathbb{F}$ algebra of $L(V,V)$

an F-algebra homomorphism $Im(\theta) \to \mathbb{F} \oplus \mathbb{F}$

Take a poly $f \in \mathbb{F}[x]$

$f(T) \in Im(\theta)$ $\quad$ send $f(T) \to (\underbrace{f(T)_a}_{\text{Scalar } V_a}, f(T)_b)$

$f(T)_a = f(a)$

<u>Def</u> Characteristic of a ring.

R (ring). There is a ring homomorphism from $\mathbb{Z} \xrightarrow{\phi} R$

$n \to \underbrace{1+1+\cdots+1}_{n}$ if $n \geq 0$

$-n \to \underbrace{-(1+\cdots+1)}_{|n|}$ if $n < 0$

$\ker(\phi) = ?$

Ideal is some (m), $m \geq 0$

m is the characteristic of R.

A positive characteristic means $\underbrace{1+1+1\cdots+1}_{m} = 0$ by $\underbrace{1+1\cdots+1}_{k} \neq 0$ if $0 \leq k < m$

If $x, y \in R$ and x, y commute, then $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ $\forall n \geq 0$

<u>Proof by induction.</u> (use $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$) - easy.

If $xy \neq yx$ then this is not necessarily true.

If x, y commute & $p = \text{char}(R) > 0$, p is prime. then $(x+y)^p = x^p + y^p$

Reason: $\binom{p}{k} \equiv 0 \mod p$ if $0 < k < p$.

Eg: $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (int mod p). char $= p$

$\mathbb{F}_p[x] \ni x^p - 1 = (x-1)^p$

$$x^p - 1 \in \mathbb{C}[x]$$
$$\shortparallel$$
$$\prod_{k=0}^{p-1} (x - \zeta^k) \text{ where } \zeta = e^{2\pi i/p}$$

Suppose R, S comm. rings + $\phi : R \to S$ ring homo.

$\phi(1_R) = 1_S$.

Then we have a homomorphism of rings.

$$\mathrm{Mat}_{n\times n}(R) \xrightarrow{\phi} \mathrm{Mat}_{n\times n}(S)$$

$$\phi(a_{ij})_{n\times n} = (\phi(a_{ij}))_{n\times n}$$

$$\phi(A+B) = \phi(A) + \phi(B).$$

$A = (a_{ij})$, $B = (b_{kl})$

$$(AB)_{il} = \sum_{k=1}^{n} a_{ik} b_{kl} \xrightarrow{\phi} \sum_{k=1}^{n} \phi(a_{ik})\,\phi(b_{kl})$$

$$= (\phi(A) \cdot \phi(B))_{il}$$

We also have that $\det(\phi(A)) = \phi(\det(A))$.

Apply ϕ to $\sum_{\sigma} \mathrm{sgn}(\sigma)\, a_{1\sigma 1} \cdots a_{n\sigma n} = \det(A)$.

& ϕ respects sums & products so it is applied to everything $= \det(\phi(A))$.

Eg: $A = \begin{pmatrix} 137 & 14 \\ 66 & -56 \end{pmatrix}$ invertible?

$\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$ $\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ det is $[2] \in \mathbb{Z}/3\mathbb{Z}$

$\Rightarrow \det(A) \neq 0.$

$T \in L(V,V)$ V fdvs

- $Tr(T) = Tr(\underbrace{A}_{\text{represents } T})$.

  $Tr(AB) = Tr(BA)$.

  If B reps $T \Rightarrow B = S^{-1}AS$.

  $Tr(B) = Tr(S^{-1}AS) = Tr(AS^{-1}S) = Tr(A)$

In HW, there is an infinite dim. space & LTs T, S so

$$\underline{ST - TS = 1_V}$$

- But if V fdvs over F of char 0. then $I \neq AB - BA$ for any A, B.

$\rightarrow \dim(V) = Tr(I) = Tr(AB) - Tr(BA) = 0$ in infinite dim vs.

There are examples in positive char, where $Tr(I) = 0$.

$P = 2$ $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\Rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{in mod } 2}.$$

- $[A, B] := AB - BA$ <u>commutator</u> of A, B

In a group, commutator of x, y

is $x^{-1}y^{-1}xy$ or $xyx^{-1}y^{-1}$.

Terminology

$T \in L(V,V)$ linear operator.

endomorphism

↳ i.e $V \to V$

onto map → epimorphism. (surjection)

one-one map → monomorphism (injection)

Def: V fdvs $T \in L(V,V) \to$ aka $Hom_F(V,V)$ → F linear homomorphism from $V \to V$

is diagonalizable iff there is a basis so mtrx for T is diagonal.

Thm: T is diagonalizable iff $m_T(x) = (x-c_1)(x-c_2)\dots(x-c_r)$ where $c_1, c_2, \dots c_r$ are distinct scalars.

Proof: $(\Rightarrow)$ then $\exists$ $\mathcal{B}$ basis. Index basis so $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \dots \cup \mathcal{B}_r$ where the elements of $\mathcal{B}_i$ are in $\ker(T - c_i 1_V)$.

(A = mtx of T, $c_1, \dots, c_r$ are distinct scalars on diagonal of A)

Let $V_i := \text{span}(\mathcal{B}_i)$

Then claim $f(x) = \prod_{i=1}^{r} (x - c_i)$ is the minimal polynomial.

$f(T) = \prod (T - c_i) = 0$ on V because $\mathcal{B}_i \subseteq \ker(T - c_i)\ \forall\ i$.

$m_T(x) \mid f(x)$

if $m_T(x) \neq f(x)$.

Then $m_T(x)$ divides some f_K

$\left(f_K = \prod_{j \neq K} (x - c_j) \right)$

then $f_K(T)$ acts as $\prod_{j \neq K} (c_K - c_j)$ on $V_K \Rightarrow f_K(T) \neq 0$.

But $f_K(T) = 0$ ⚡

Conclude: $m_T(x) = f(x)$ //

<u>Converse</u>: Suppose $m_T(x) = \prod_{i=1}^{r} (x - c_i)$ $c_1 \dots c_r$ distinct.

Then PDT says for $m_T(x) = (x - c_1) h(x)$,

$h(x) = \prod_{j \neq 1} (x - c_j)$ that $V = \underset{\ker(T - c_1)}{\underset{\parallel}{V_1}} \oplus V_2 = \ker(h(T))$

& $m_{T_1}(x) = (x - c_1)$

$T_1 \in L(V_1, V_1)$

$T_1(x) = T(x)|_{V_1}$

$m_{T_2}(x) = h(x)$

$T_2 \in L(V_2, V_2)$

$T_2(x) = T(x)|_{V_2}$

We argue by induction on $\dim V$ that V has a basis of eigenvectors for T.

We have such a basis for V_1

by induction there is one for T_2.

So union of these bases give diag. mtx for T.

Projections

$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$.

$x \in V$ $x = \sum_{i=1}^{r} x_i$ $\qquad x_i$ unique in V_i.

$P_i(x) = x_i \qquad P_i \in L(V,V)$. projections for the given direct sum.

$W \subseteq V$

P_W not always defined.

These $P_1 \ldots P_r$ satisfy

$P_i^2 = P_i$, $\quad P_i P_j = 0$ if $i \neq j$ & $1_V = \sum_{i=1}^{r} P_i$.

Conversely, given a seq $E_1 \ldots E_r$ in $L(V,V)$.

So $1 = \sum E_i$, $E_i E_j = \delta_{ij} E_i$ then $E_1 \ldots E_r$ is the set of projections for a direct sum.

Define $V_i := \operatorname{Im}(E_i)$, $i = 1 \ldots r$.

Then $1_V = \sum_{i=1}^{r} E_i \Rightarrow V = V_1 + \cdots + V_r$, $x \in V$ $x = \sum \underset{\substack{\uparrow \\ V_i}}{E_i x}$

Suppose $x = \sum_{i=1}^{r} x_i$, $x_i \in V_i$,

We want $x_i = E_i x$ (if so, $V = V_1 \oplus \cdots \oplus V_r$)

$E_i x = E_i \left(\sum_{i=1}^{r} x_j \right) = \sum_{j} E_i x_j$

$x_j \in V_j = \operatorname{Im}(E_j)$.

So if $i \neq j$,

$E_i x_j \in E_i(V_j) = E_i E_j(V) = 0V = 0.$

$\Rightarrow E_i x = E_i x_i \qquad x_i \in V_i = E_i V_i \quad x_i = E_i v$ for some v

$\Rightarrow E_i x_i = E_i E_i v = E_i^2 v = E_i v = x_i$

Primary Decomposition Theorem (Standard version)

$T \in L(V,V)$ T satisfies $f(x) = \prod_{i=1}^{r} p_i(x)^{m_i}$, $p_1 \ldots p_r$ distinct monic irreducibles. Then $V = V_1 \oplus \cdots \oplus V_r$, where V_i is T-invariant and $p_i(T)^{m_i} V_i = 0$.

The projections E_i for this direct sum are polynomial expressions in T.

If $f(x) = m_T(x)$, then $T_i \in L(V_i, V_i)$ defined by $T_i x = Tx$, for $x \in V_i$, has min poly $p_i(x)^{m_i}$.

<u>Proof</u>: For a factorization of f into rel. prime factors we get a T-invariant decomposition for which the projections are poly expressions in T. We prove the thm by induction on r.

Write $f(x) = g(x)h(x)$, where $g(x) = p_1(x)^{m_1}$,

$h(x) = \prod_{j \neq 1} p_j(x)^{m_j}$

Primary Decomposition theorem (Proof)

- $\underline{r=1}$: Trivial.

Let $\underline{r \geq 2}$, fix index $i \in \{1, 2, \dots, r\}$.

Let $g_i(x) := p_i(x)^{m_i}$ & $h_i(x) = \prod_{j \neq i} p_j(x)^{m_j}$

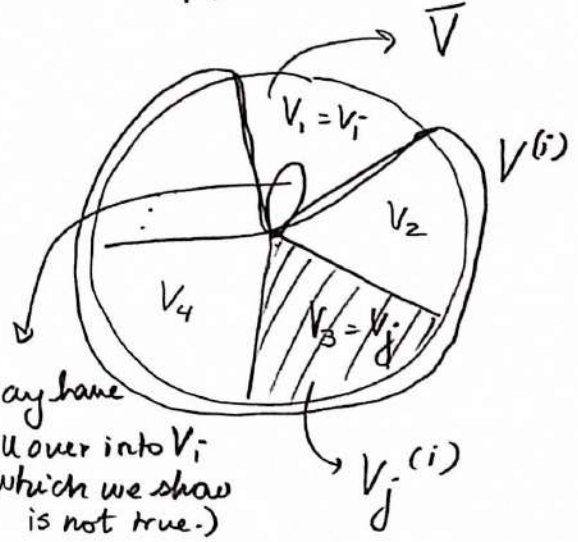
From BPDT we have $V = V_i \oplus V^{(i)}$ where $V_i = \text{null}(g_i(T))$

& $V^{(i)} = \text{null}(h_i(T))$.

Induction on r implies we can decompose

$$V^{(i)} = \bigoplus_{j \neq i} V_j^{(i)} \quad . \quad V_j^{(i)} = \text{null}\left(p_j(T)^{m_j}\big|_{V^{(i)}}\right)$$

- $\Rightarrow V_j^{(i)} = V_j \cap V^{(i)}$



We need to show $V_j^{(i)} = V_j$.

$V_j^{(i)} \subseteq V_j$ is trivial.

$\underline{\text{Claim:}}$ $V_j \subseteq V_j^{(i)}$.

Consider $\overline{V} = V/V^{(i)}$ & $\overline{T}$ induced by T on $\overline{V}$.

$\overline{T}$ satisfies $p_i(x)^{m_i}$. $p_i(x)^{m_i}$ & $p_j(x)^{m_j}$ are relatively prime

$\Rightarrow 1 = p_i(x)^{m_i} l(x) + p_j(x)^{m_j} k(x)$. Let $x = \overline{T} \Rightarrow \overline{p_j(T)^{m_j}}$ is invertible

- $\Rightarrow \ker(p_j(T)^{m_j})$ restricted to $\overline{V}$ must be $\{0\}$. $\Rightarrow V_j \subseteq V^{(i)}$

$\Rightarrow V_j = V^{(i)} \cap \ker(p_j(T)^{m_j}) = V_j^{(i)}$ //

It remains to show that $p_i(T)^{m_i}$ is the minimal polynomial of the endomorphism induced by T on V_i.

Assume not true.

$\Rightarrow$ T_i satisfies $p_i(x)^{m_i - 1}$ $\Rightarrow$ T satisfies $p_i(x)^{m_i - 1} \prod_{i \neq j} p_j(x)^{m_j}$

which contradicts the hypothesis that $f(x)$ is the min poly. $\blacksquare$

Diagonalizability & Triangulability.

Def $T \in L(V,V)$ $v \neq 0$ in V is an eigenvector for eigenvalue $c \in F$ if $Tv = cv \iff v \in \ker(T-c)$ & $v \neq 0$.

$\ker(T-c)$ is eigenspace of T for eigenvalue c

Def Eigenbasis for T is a basis for which each member is an eigenvector.

Thm: V fdvs, equivalent are:

1) T has eigenbasis
2) min poly has form $(x-c_1)\dots(x-c_r)$ for distinct scalars $c_1, \dots, c_r$
3) If $V = V_1 \oplus \dots \oplus V_r$ is decomp from POT, $E_1 \dots E_r$ projections, then $\exists$ scalars $c_1 \dots c_r$ ST $T = c_1E_1 + c_2E_2 + \dots + c_rE_r$

Proof: Assume 1)

Let $c_1 \dots c_r$ be all distinct eigenvalues, then $(T-c_1)(T-c_2)\dots(T-c_r) = 0$ because each member of the eigenbasis is in it's kernel $\Rightarrow (2)$

Assume (2)

Take $V = V_1 \oplus \cdots \oplus V_r$ as described,

$V_i = \ker(T - c_i I)$. Recall projections E_i act as I on V_i,

0 on V_j for $j \neq i$ $\Rightarrow$ $\sum_{i=1}^{r} c_i E_i$ acts as c_i on V_i. $\forall i$,

$$\Rightarrow T = \sum_{i=1}^{r} c_i E_i$$

Assume (3)

Let $\mathcal{B}_i$ be a basis for V_i, $\forall i$

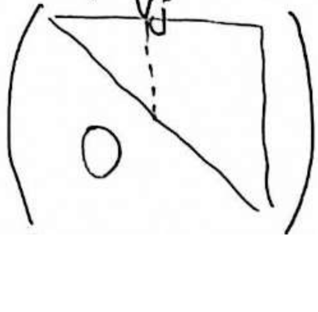
Then $\mathcal{B}_i$ consists of eigenvectors & so does the basis

$\dot{\bigcup}_{i=1}^{r} \mathcal{B}_i$ for V.

∎

Def $T \in L(V,V)$, V fdvs is triangulable if there exists a basis $v_1, \ldots, v_n$ so mtx for T is upper triangular. Rmk (Equivalent is) $\Leftrightarrow$ there is a chain of subspaces $0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$ so $\dim V_i = i$ & V_i is T-invariant for all i

($\Rightarrow$) Take $V_i := \mathrm{Span}\{v_1 \ldots v_i\}$. Form of mtx $\Rightarrow$ each V_i is T-invariant



$T v_j = LC$ of $v_1 \ldots v_j$ $\forall j$.

($\Leftarrow$) Given such a chain, take $v_i \in V_i - V_{i-1}$ for

$i = 1 \dots n$

$\Rightarrow V_i = \text{span}\{v_1 \dots v_i\}$

T-invariant $\Rightarrow$ Mtx of T is upper triangular.
for all i

<u>Lemma</u>

If $T \in L(V,W)$, W' is a subspace of W. Then

$V' := T^{-1}(W') = \{x \in V \mid Tx \in W'\}$ is a subspace of V.

Also $V' \supseteq \operatorname{Ker}(T)$ since W' is a subspace & contains $\{0\}$

<u>Proof</u> $T0 = 0 \Rightarrow 0 \in V'$.

If $x, y \in V'$ & $a, b \in F$. Then $T(ax+by) = a\underset{W'}{\underset{\cap}{Tx}} + b\underset{W'}{\underset{\cap}{Ty}} \in W$, subsp

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$\mathcal{F}$ a family of endomorphisms of V. (fd vs)

<u>Thm 1</u> If $\mathcal{F}$ is a commuting $\underset{\text{set}}{\text{family}}$ of diagonalizable edomorphisms, $\exists$ a common eigenbasis.

<u>Thm 2</u> If $\mathcal{F}$ is a commuting family of Triangulable endomorphisms, $\exists$ a basis S.T matrix for every member of $\mathcal{F}$ is upper triangular.

Proof of Thm 1

Trivial if $\mathcal{F}$ is all scalar transformations.

Assume otherwise. There exists $T \in \mathcal{F}$ so that

$m_T(x) = (x-c_1) \dots (x-c_r),\ r \geq 2.$

T-invariant decomp. $V = V_1 \oplus V_2 \oplus \dots \oplus V_r$ where

$V_i = \ker(T - c_i)$, all $\neq 0$. All V_i are invariant by every $S \in \mathcal{F}$.

because S commutes with $T - c_i$. $\left\{ \text{Commuting LTs leave Ker \& Im invariant} \right\}$

Use induction on $\dim V$. If $\dim V = 1$, trivial.

Assume $\dim V > 1$. Each $\dim(V_i) < \dim(V)$ since $r \geq 2$.

So we get a family $\mathcal{F}_i$ of commuting endomorphisms on V_i by restricting $S \in \mathcal{F}$ to V_i. S induces $S_i \in L(V_i, V_i)$

By induction, $\exists$ basis $\mathcal{B}_i$ common eigenbasis $\forall\ \mathcal{F}_i$

$\Rightarrow \mathcal{B} = \bigcup_{i=1}^{r} \mathcal{B}_i$ is a common eigenbasis for $\mathcal{F}$.

Lemma: V fdvs, $T \in L(V,V)$ is triangulable iff $m_T(x)$ factors completely into linear factors. i.e. $m_T(x) = \prod (x-c_i)^{m_i}$

$c_1 \dots c_r$ distinct, $m_i \geq 1$.

Proof Suppose T is triangulable, $\exists$ basis so $M = $ mtx for T has form $M = \begin{pmatrix} a_{11} & & X \\ & a_{22} & \\ 0 & & \ddots \end{pmatrix}$

$$\det(xI - M) = \prod_{i=1}^{n} (x - a_{ii}) \quad //$$

Conversely assume $m_T(x) = \prod_{i=1}^{r} (x - c_i)^{m_i}$

PDT : $V = V_1 \oplus \ldots \oplus V_r$, $V_i = \ker\left((T - c_i)^{m_i}\right)$

Prove by induction on $\dim V$, ($\dim = 1$ trivial)

If $r \geq 2$ we observe that every $S \in \mathcal{F}$ fixes each V_i (commuting family)

So we observe that the induced family $\mathcal{F}_i$ on V_i is pairwise commuting. By induction $\exists$ basis $\mathcal{B}_i$ of V_i which simultaneously triangulizes $\mathcal{F}_i$. $\mathcal{B} = \bigcup_{i=1}^{r} \mathcal{B}_i \rightarrow$ mtx of form $\begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_r \end{pmatrix}$

Each M_i is upper triangular.

$\mathcal{F}$ family of endos of V, fdvs, mutually commuting.

Assume $\mathcal{F}$ consists of triangulable endomorphisms.

$\Leftrightarrow$ each $T \in \mathcal{F}$ has $C_T(x)$ a product of linear factors.

> If W, subspace of V, W is $\mathcal{F}$ invariant. Then (each member of) the family $\mathcal{F}_W$ induced on W by $\mathcal{F}$ is triangulizable on W if $\mathcal{F}$ is triangulizable on V.

Proof Triangulizable $\Leftrightarrow$ char poly is a product of linear factors. Use fact that if $T \in L(V,V)$ leaves W invariant then the char poly of T_W & $\bar{T}$ divide $C_T(x)$. //

Lemma 2

A commuting family $\mathcal{F}$ of triangulizable endomorphisms of V leaves invariant a 1-dim subspace of V.

Proof: Induction on $\dim V$, $\dim = 1$ trivial, so assume $\dim V \geq 2$

If $\exists T \in \mathcal{F}$ so that T has 2 different eigenvalues c_1, c_2

Write $C_T(x) = (x - c_1)^p g(x)$

$(g, (x - c_1)) = 1$; $\deg(g) > 1$

$V = V_1 \oplus V_2$

PDT for T, $V_1 = \ker (T-c_1)^p \neq 0$

$V_2 = \ker (g(T)) \neq 0$

Both V_i are invariant by $\mathcal{F}$.

By induction $\mathcal{F}$ leaves invariant a 1-space in V_1 which is also a 1-dim subspace in V.

Now we may assume every endo in $\mathcal{F}$ has just one eigenvalue

Suppose $T \in \mathcal{F}$ & $m_T(x)$ has $\deg \geq 2$.

$\Rightarrow m_T(x) = (x-c)^p$

Then $W := \ker (T-c)^{p-1} \neq 0, V$.

It is $\mathcal{F}$ invariant. By induction, W has a 1-dim space which is $\mathcal{F}$-invariant.

Finally we suppose every $T \in \mathcal{F}$ has min poly of degree 1.

$\Rightarrow \mathcal{F}$ consists of scalar transformations. So any 1-dim subspace is $\mathcal{F}$-invariant. //

<u>Thm 2</u> $\mathcal{F}$ is simultaneously triangulizable

<u>Proof</u>: Induction on dim, dim = 1 trivial.

Suppose dim ≥ 2.

$\exists$ a 1-dim subspace W which is $\mathcal{F}$-invariant.

We consider the family $\bar{\mathcal{F}}$ induced by $\mathcal{F}$ on $\bar{V} = V/W$.

$\bar{\mathcal{F}}$ is a commuting family. (if $S, T \in \mathcal{F}$, then on $\bar{V}$,

$$\bar{S}\bar{T}[v] = \bar{S}[Tv] = [S(Tv)] = [STv] = [TSv]$$
$$= \bar{T}[Sv] = \bar{T}\bar{S}[v])$$

By induction $\bar{\mathcal{F}}$ is triangulable on $\bar{V} \Leftrightarrow \exists$ chain of subspaces

$$V_0' \subseteq V_1' \subseteq \ldots \subseteq V_{n-1}' = \bar{V}$$

$\dim(V_i') = i$ & all V_i' are $\bar{\mathcal{F}}$ invariant.

We now define $V_{i+1} := q^{-1}(V_i')$; $q : V \xrightarrow{\text{quotient map}} \bar{V} = V/W$

subspace which contains $Ker(q) = W$.

$q(v) = [v]$

Define $V_0 := \{0\}$, we have $V_0 \subseteq V_1 \subseteq V_2 \ldots \subseteq V_n = V$.

select $\rightarrow v_1$ $\quad v_2 \in V_2 - V_1$ $\quad v_n \in V_n - V_{n-1}$ $\}$ form a basis

<u>Claim:</u> $\dim(V_{i+1}) = i+1$ from RNT, think of $q|_{V_{i+1}} : V_{i+1} \xrightarrow{\text{onto}} V_i'$

& V_i' has dim i. $\underset{\substack{\| \\ Ker(q) \\ \text{1-dim}}}{W} \subseteq V_{i+1}$ $\Rightarrow$ rank of $q|_{V_{i+1}}$ is i $\Rightarrow$ $i+1$ is dim of domain of V_{i+1} //

$q : V_{i+1} \longrightarrow V_i'$

$rank(q) = \dim(V_i') = i$

$nullity(q) = \dim(W) = 1$

$\Rightarrow \dim(V_{i+1}) = i+1$ (RNT).

We want each V_{i+1} to be T-invariant for every $T \in \mathcal{F}$.

Take $x \in V_{i+1}$, want $Tx \in V_{i+1}$, Consider $[Tx] = \underbrace{T[x]}_{\in V_i'}$ since each V_i' is $\mathcal{F}$ invariant.

$\Rightarrow Tx \in q^{-1}(V_i') = V_{i+1}$ by def.

__Claim:__ $v_1, v_2, \ldots, v_n$ is a basis for V. $q(\text{span}(v_1 \ldots v_{i+1})) = V_i'$,

rank + nullity for $q : \text{span}\{v_1 \ldots v_{i+1}\} \to V_i'$
$n-1 \qquad 1$

rank $= i$, nullity $= 1$, domain is $i+1$-dimensional.

$\Rightarrow \text{span}\{v_1, \ldots v_{i+1}\} = V_{i+1}$ //

The basis $v_1, \ldots v_n$ triangulizes each endomorphism in $\mathcal{F}$. ■

__Note__

Take $[v_2] \ldots [v_{i+1}]$; Claim. it is a basis for V_i'.

Argue by induction on i. $[v_2] \neq 0$ in V_1' 1-dim so $[v_2]$ is basis for V_1'. Assume now $[v_2] \ldots [v_i]$ is basis for V_{i-1}'

$v_{i+1} \in V_{i+1} - V_i$ & $[v_{i+1}] \in V_i' - V_{i-1}'$.

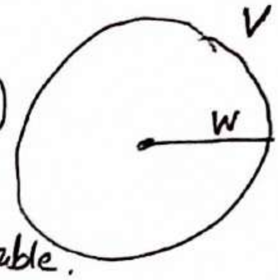
roof of Simultaneous Triangulability.

> Induction on dimension of V.

> $\dim = 1 \Rightarrow$ Trivial.

> $\dim \geq 2 \Rightarrow \exists$ a 1 dim subspace W that is $\mathcal{F}$ invariant

Consider $\bar{\mathcal{F}}$ induced on $\bar{V} = V/W$.



$\bar{F}$ is commuting ($\bar{S}\bar{T}[v] = [STv] = [TSv] = \bar{T}\bar{S}[v]$)

Induction hypothesis $\Rightarrow$ $\bar{F}$ is simultaneously triangulable.

$\Rightarrow \exists$ chain of $\bar{F}$ invariant subspaces s.t. $V_0' \subseteq V_1' \subseteq \dots \subseteq V_{n-1}' = \bar{V}$.

& $\dim(V_i') = i$

Define quotient map: $q : V \longrightarrow \bar{V}$ i.e. $q|_{V_{i+1}} : V_{i+1} \longrightarrow V_i'$

$$\Rightarrow \begin{array}{ccccccc} V_0' & \subseteq & V_1' & \subseteq & V_2' \subseteq & \dots & \subseteq V_{n-1}' \\ \downarrow q^{-1} & & \downarrow q^{-1} & & \downarrow q^{-1} & & \downarrow q^{-1} \\ V_0 \subseteq V_1 & \subseteq & V_2 & \subseteq & V_3 \subseteq & \dots & \subseteq V_n \end{array} \Bigg\} \text{ gives a chain of subspaces.}$$

$V_0 = \{0\}$

Claim: $\dim(V_{i+1}) = i+1$.

$$\left.\begin{array}{l} \operatorname{rank}(q) = \dim(V_i') = i \\ \text{nullity}(q) = \dim(W) = 1 \end{array}\right\} \overset{RNT}{\Rightarrow} \dim(V_{i+1}) = i+1.$$

Claim: Each V_i is T-invariant $\forall\, T \in \mathcal{F}$

$$\begin{array}{ccc} x \in V_{i+1} & \xrightarrow{q} & [x] \in V_i' \\ & & \downarrow \bar{T} \\ \Rightarrow \underbrace{Tx \in V_{i+1}}_{T\text{ invariant}} & \xleftarrow{q^{-1}} & \bar{T}[x] = [Tx] \in V_i' \text{ due to } \bar{T}\text{-inv of } V_i' \end{array}$$

Claim: $v_1, v_2 \dots v_n$ is a basis for V. (Select v_1 from V_1, v_2 from $V_2 - V_1$
v_n from $V_n - V_{n-1}$)

$q(\text{span}\{v_1, v_2, \dots v_{i+1}\}) = V_i'$

$\dim(V_{i+1}) = i+1$ from RNT

$\Rightarrow \text{span}\{v_1, v_2 \dots v_{i+1}\} = V_{i+1}$ //

$\Rightarrow$ The basis $v_1, v_2, \dots, v_n$ triangulize every $T \in \mathcal{F}$. ■

Simultaneous Triangulability & Diagonalizability

Suppose that V fdvs, $T \in L(V,V)$ and T is triangulizable ie. $m_T(x)$ factorizes completely. Will show T can be written in a unique way as $D+N$ where $DN = ND$.

D - diagonalizable & N - Nilpotent.

$\hookrightarrow$ ($\exists\, k \geq 1$ so $N^k = 0$; $m_N(x) = x^k$
$k \leq \dim V$

Proof

Say $m_T(x) = \prod_{i=1}^{r} (x - c_i)^{m_i}$, $m_i \geq 1$

$c_1 \ldots c_r$ distinct

We get $V = V_1 \oplus \ldots \oplus V_r$ as in POT.

, $V_i = \ker((T - c_i)^{m_i})$

Projections E_i are poly expr in T

$1_V = \sum E_i$, $E_i E_j = \delta_{ij} E_i$. $\operatorname{Im}(E_i) = V_i$

Define,

$$N = \sum_{i=1}^{r} E_i \underbrace{(T - c_i)}_{\text{poly in } T}$$

$$N^k = \sum_{i=1}^{r} \left(E_i (T - c_i)\right)^k$$ since $E_i E_j = \delta_{ij} E_i$ & E_i, $T - c_i$ commute.

$$= \sum_{i=1}^{r} E_i (T - c_i)^k = 0 \text{ if } k \geq \max\{m_1 \ldots m_r\}.$$
due to minimal poly.

$\Rightarrow N$ is Nilpotent.

Define $D := \sum_{i=1}^{r} c_i E_i$

$\Rightarrow T = N + D$.

Also D is a $\underbrace{\text{polynomial expression in } T}_{\Rightarrow ND = DN}$. & is diagonalizable since there is a basis of eigenvectors.

Next: Uniqueness

Suppose $T = D' + N'$ & $\underbrace{D'N' = N'D'}$ $\Rightarrow$ D', N' commute with T.

(We want $D = D'$ & $N = N'$)

since D' commutes with $D' + N'$

So they commute with all polys in T. So D', N' commute with each of D, N

$\Rightarrow \quad D + N = D' + N'$

$\Rightarrow \quad D - D' = N' - N$

Claim: LHS is diagonalizable & RHS is nilpotent.

$\Rightarrow \quad D - D' = 0 \quad , \quad N' - N = 0$

(Idea RHS $= 0$ for some power $\Rightarrow \text{LHS}^k = 0 \Rightarrow \text{LHS} = 0$)

LHS: Two commuting diagonalizable endos, can be simultaneously diagonalized.

RHS: In a ring if x, y commute, then $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.
(easy to prove by induction).

So if x, y nilpotent ie $x^p = 0$, $y^q = 0$ $\Rightarrow (x+y)^{p+q-1} = 0$ by bin thm.

So claim LHS ✓

$\Rightarrow$ unique ∎

Proof of $T = D + N$ (where $DN = DN$ & T is triangulable)

$$m_T(x) = \prod_{i=1}^{r} (x - c_i)^{m_i} \ ; \ m_i \geq 1 .$$

PDT $\Rightarrow$ $V = V_1 \oplus \dots \oplus V_r$ where $V_i = \operatorname{Ker}(T - c_i)^{m_i}$

Projections: E_i s.t $V_i = \operatorname{Im}(E_i)$ & $E_i E_j = \delta_{ij} E_i$, $\sum E_i = 1_V$.

Let $\boxed{N = \sum_{i=1}^{r} E_i (T - c_i)}$

$$\Rightarrow N^k = \left[\sum_{i=1}^{r} E_i (T - c_i)\right]^k = \left[\sum_{i=1}^{r} \left(E_i (T - c_i)\right)^k\right] \quad \text{since } E_i E_j = \delta_{ik} E_i$$

$$= \sum_{i=1}^{r} E_i (T - c_i)^k \quad \text{since } E_i^k = E_i .$$

$$N^k v = \left(\sum_{i=1}^{r} E_i (T - c_i)^k\right) v = \sum_{i=1}^{r} (T - c_i)^k v_i \quad \text{where } v_i \in V_i .$$

If $k \geq \max\{m_i\}$ we have $N^k v = \underline{\underline{0}}$ since $v_i \in \operatorname{Null}(T - c_i)^{m_i}$

$\Rightarrow$ N is nilpotent.

$\boxed{D = \sum_{i=1}^{r} c_i E_i}$, D is diagonalizable since it is poly in T & has a basis of eigenvectors.

D, N are polys in T so $DN = ND$.

Uniqueness: D', N' s.t $T = D' + N'$. D', N', D, N all commute.

$D + N = D' + N'$ $\Rightarrow$ $D - D' = N' - N$ $\Rightarrow$ $(D - D')^k = 0$ for $k = p + q - 1$

$(N' - N)^k = 0$. $\Rightarrow$ $D - D' = 0$ $\Rightarrow$ $D = D'$ $\Rightarrow$ $N = N'$. $\begin{pmatrix} N'^p = 0 \ \& \\ N'^q = 0 \end{pmatrix}$.

Canonical Forms.

Rational Canonical Form (RCF).

Thm
V fdvs/F Then $\exists$ a basis so mtx for T is block diagonal.
of companion matrices.

$$\begin{pmatrix} C_{f_1} & & & \\ & C_{f_2} & & \\ & & \ddots & \\ & & & C_{f_n} \end{pmatrix}$$

C_f = Companion mtx for monic poly $f(x)$.
& f_i is divisible by f_{i+1} for $i = 1, \dots r-1$.

Also such a sequence of polys. is uniquely determined by T.

Jordan Canonical Form (JCF)

$T \in L(\overset{\text{fdvs}}{V}, V)$ & $m_T(x)$ factorizes completely. For a scalar $c \in \mathbb{F}$ & integer $k \geq 1$, define $J(k,c) := \begin{pmatrix} c & & & 0 \\ 1 & c & & \\ & 1 & \ddots & \\ 0 & & \ddots \; c & \\ & & 1 & c \end{pmatrix}_{k \times k}$

For T there is a sequence $(k_1, c_1) \dots (k_r, c_r)$. Then for some basis, T is represented by

$$\begin{pmatrix} \boxed{J(k_1,c_1)} & & & \\ & \boxed{J(k_2,c_2)} & & \\ & & \ddots & \\ & & & \boxed{J(k_r,c_r)} \end{pmatrix}$$

(The blocks are unique except for order of the blocks)

RCF: Preparation

V, T as usual.

In V, we have many cyclic subspaces

$Z(T, v) = \text{span}\{v, Tv, T^2v \ldots\}$

If $m_{T,v}(x)$ monic poly which generates the ideal

$\{f(x) \in F[x] \mid f(T)v = 0\}$

then $m_{T,v}(x) \mid m_T(x)$

$$\boxed{\dim Z(v,T) = \deg(m_{T,v}(x))}$$

We want to make a good choice of $v_1 \ldots v_r$ so
should give us the RCF.

$V = Z(T, v_1) \oplus \ldots \oplus Z(T, v_r)$.

Procedure will go by induction on dimension.

Goal : RCF

Two parts: (1) Existence.

$\exists\ V = V_1 \oplus \ldots \oplus V_r$. (all T invariant)

$V_i = Z(T, v_i)$

& $m_{T,v_i}(x) = f_i$

f_{i+1} divides f_i for $i = 1 \ldots r$

(2) Uniqueness (not of the V_i) but of the annihilator poly.

Issue: If $T \in L(V,V)$, W is a T-invariant subspace, is there a subspace W' s.t $V = W \oplus W'$ and W' is T-invariant.

Def: A T-invariant subspace W of V is called T-admissable if whenever $v \in V$ & $f(x) \in F[x]$ so that $f(T)v \in W$, then $\exists\ w \in W$ so that $f(T)w = f(T)v$.

Ex: Suppose $V = W \oplus W'$ where both W, W' are T-invariant then W is admissible. Take $v \in V$; $v = \underset{\underset{W}{\cap}}{v_1} + \underset{\underset{W'}{\cap}}{v_2}$. Take $f(x)$ so

$f(T)v \in W$; $\underbrace{f(T)v}_{\underset{W}{\cap}} = \underbrace{f(T)v_1}_{\underset{W}{\cap}} + \underbrace{\cancel{f(T)v_2}}_{\underset{W'}{\cap}} \to 0$

$\Rightarrow f(T)v = f(T)v_1$ // $\Rightarrow W$ is T admissible.

Thm: Given V fdvs, $T \in L(V,V)$. W an admissible T-invariant subspace s.t V/W is the direct sum of subspaces $Z(\overline{T}; [v_i])$. Then $\exists$ a T-invariant subspace W' of V s.t $V = W \oplus W'$.

$\left(\overline{V} = V/W\right.$ is the set of all $[v]$ & $\left.v \in V\right)$. (v_i is an element in $\overline{V}$)!

Remarks

$Z(T, v)$ has basis $v, Tv, \ldots, T^{d-1}v$.

where $d = \deg(m_{T,v}(x))$.

In $\overline{V}$ we have subspaces $\overline{Z}_1, \ldots, \overline{Z}_r$, each T-inv is a cyclic vector.

$\overline{Z}_i = Z(\overline{T}, [v_i])$ $\quad v_1, \ldots, v_r \in V$

Form $\overbrace{Z(T, v_i)}^{\tilde{Z}_i :=}$, $i = 1 \ldots r$, are T-invariant subspaces of V

$q : V \to \overline{V} = V/W$ quotient map then $q(Z(T, v_i)) = \overline{Z}_i$ in $\overline{V}$.

Then $W + \left(\sum_{i=1}^{r} \tilde{Z}_i\right) = V$

$\left[\ f_i = m_{\overline{T}, [v_i]}(x) \quad ; \quad [f_i(T) v_i] = 0 \quad ; \quad f_i(T) v_i \in W\ \right]$

Now we use admissibility of W in V. So given $f_i(T) v_i \in W$, $\exists$ a $w_i \in W$ s.t $f_i(T) v_i = f_i(T) w_i$. So $f_i(T)(v_i - w_i) = 0$ & $[v_i - w_i] = [v_i]$. So $v_i - w_i$ generates a cyclic subspace.

Proof of complimentary T invariant subspace

- $\bar{V} = V/W$ where W is T-admissible.
- $\bar{V}$ has subspaces $\bar{Z}_1, \ldots, \bar{Z}_r$

  where $\bar{Z}_i = Z(\bar{T}, [v_i]) \longrightarrow \bar{T}$-inv

  Let $\tilde{Z}_i = Z(T, v_i) \longrightarrow T$-inv

- q: quotient map $\tilde{Z}_i \xrightarrow{q} \bar{Z}_i$

  $\Rightarrow W + \sum_{i=1}^{r} \tilde{Z}_i = V$ but we want direct sum.

Let $\boxed{f_i = m_{\bar{T},[v_i]}(x)}$ $\Rightarrow f_i(\bar{T})[v_i] = 0$

$\Rightarrow [f_i(T)v_i] = 0$

$\Rightarrow f_i(T)v_i \in W$.

Using admissibility of W, $f_i(T)v_i = f_i(T)w_i$ for some $w_i \in W$.

$\Rightarrow f_i(T)(v_i - w_i) = 0$ & $[v_i - w_i] = [v_i]$.

$\Rightarrow \bar{Z}_i = Z(\bar{T}, [v_i - w_i])$

Let $Z_i = Z(T, v_i - w_i)$

<u>Claim:</u> $\dim Z_i = \dim \bar{Z}_i$ $\left(m_{T,v_i-w_i} = m_{\bar{T},[v_i-w_i]} \right)$

$f_i = m_{\bar{T},[v_i]}(x) \Rightarrow f_i$ divides $m_{T,v_i}(x)$ (since annihilator of v_i also annihilates $[v_i]$)

Also $f_i(T)(v_i - w_i) = 0 \Rightarrow m_{T,v_i-w_i}(x) \mid f_i(x)$

But $f_i = m_{\bar{T},[v_i-w_i]}(x) \Rightarrow f_i(x) \mid m_{T,v_i-w_i}$

$\Big\} \Rightarrow f_i(x) = m_{T,v_i-w_i}$

(Since $[v_i] = [v_i - w_i]$

$\Rightarrow m_{T,v_i-w_i} = m_{\bar{T},[v_i-w_i]} \Rightarrow \dim Z_i = \dim \bar{Z}_i$

(deg equal $\Leftrightarrow$ dim equal) //

$q(v_i - w_i) = [v_i - w_i] = [v_i]$ & $q(Z_i) = \overline{Z}_i$

$\Rightarrow q$ gives an isomorphism b/w $Z_i \longleftrightarrow \overline{Z}_i$.

Define $W' := Z_1 + Z_2 + \cdots + Z_r$.

<u>Claim:</u> $\dim(W') = \dim(\overline{V})$.

$$\dim(W') \leq \sum \dim(Z_i)$$

$$= \sum \deg(f_i(x)) = \sum \dim(\overline{Z}_i) = \dim(\overline{V}).$$

$q(W') = \overline{V} \Rightarrow \dim(W') \geq \dim \overline{V}$

$\Rightarrow \dim(W') = \dim(\overline{V})$

$\Rightarrow W' = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_r$.

$\Rightarrow W'$ is T-inv.

& $W \oplus W' = V$

$Z_i \to Z_i := Z(T, v_i - w_i)$; $\dim Z_i = \dim \overline{Z}_i$ $\to$ ($m_{T, v_i - w_i}(x) \mid f_i(x)$ but $m_{T, v_i - w_i}(x) = f_i(x)$ since otherwise $m_{T, v_i - w_i}$ annihilates $[v_i]$? (See note below))

& $q(v_i - w_i) = [v_i]$ & $q(Z_i) = \overline{Z}_i$

i.e. $q\big|_{Z_i}$ gives an isomorphism of vector spaces $Z_i \longrightarrow \overline{Z}_i$

We define $W' := Z_1 + Z_2 + \dots + Z_r$.

$$\dim(W') \leq \sum \dim(Z_i) = \sum \deg(f_i(x)) = \sum \dim(\overline{Z}_i) = \dim(\overline{V})$$ Note 2

$q(Z_i) = \overline{Z}_i \Rightarrow q(W') = \overline{V}$. $\Rightarrow \dim W' \geq \dim \overline{V}$

Same dim

equality in 2 directions.

$\Rightarrow$ $q\big|_{W'}$ is an isomorphism. $\Rightarrow W' = Z_1 \oplus \dots \oplus Z_r$.

$\hookrightarrow$ T invariant since any cyclic subspace is T-inv & so is their $\oplus$.

Note:

$f_i = m_{\overline{T}, [v_i]}(x)$ this is the min poly of $[v_i]$ & must divide the min poly of v_i

$\Rightarrow f_i \mid m_{T, v_i}(x)$

Also $f_i(T)(v_i - w_i) = 0 \Rightarrow m_{T, v_i - w_i}(x)$ divides $f_i(x)$

but $f_i(x)$ can also be $m_{\overline{T}, [v_i - w_i]}(x)$ since $[v_i] = [v_i - w_i]$

so $f_i(x) = m_{\overline{T}, [v_i - w_i]}(x)$ //

Note2

$$q(W') = \underset{\overline{Z}}{\underset{\|}{q(Z_1)}} + q(Z_2) + \dots + q(Z_r)$$

$$\overline{Z} \oplus \overline{Z}_2 \oplus \quad \cdots \quad \oplus \overline{Z}_r$$

$$\dim W' \geq \dim(q(W')) = \sum \dim(\overline{Z}_i) = \dim \overline{V}.$$

If $V = V_1 \oplus \dots \oplus V_r$ T-inv.

I will show that

1) $\exists v \in V$ so $m_{T,v}(x) = m_T(x)$

2) Show that $Z(T,v)$ is admissible for v with $m_{T,v}(x) = m_T(x)$.

We can then use induction for RCF.

Proof of 1)

T, $m_T(x)$, V fdvs,

$V = V_1 \oplus \dots \oplus V_r$; $m_T(x) = \prod p_i^{e_i}$; $p_1 \cdots p_r$ distinct monic irreducibles & $e_i \geq 1$.

$V_i = \operatorname{Ker}(p_i(T)^{e_i})$

$T_i \in L(V_i, V_i)$ induced by T.

Then $m_{T_i}(x) = p_i(x)^{e_i}$

Seek $v = v_1 + \dots + v_r$ $\quad$ + $v_i \in V_i$

<u>Claim</u>: In V_i, $\exists\, v_i$ so $m_{T, v_i}(x) = p_i(x)^{e_i}$

(otherwise $p_i(T_i)^{e_i - 1} = 0$ $\divideontimes$).

So get $v_1 \dots v_r$

<u>Lec</u>

If W is T-admissible subspace in V then a decomp of $\bar{V} = V/W$ by <u>cyclic subspaces</u> can be lifted to a T-inv. compliment W' of W in V.

Must show that $Z(T, v)$ is T-admissible if v has $m_{T, v}(x) = m_T(x)$.

Given $T \in L(V, V)$, RCF claims $\exists$ $V = Z(T, v_1) \oplus \cdot \oplus Z(T, v_r)$

$f_i = m_{T, v_i}(x)$.

$f_{i+1} \mid f_i$ for $i = 1, \dots, r-1$.

<u>Mtx technique</u> (Smith normal/canonical form). for matrices / $F[x]$ gets the $f_1, f_2, \dots, f_r$. (Check posted note)

Assume for now, admissibility result for $Z(T, v)$.

<u>Last time</u>: $T \in L(V, V)$; $V = V_1 \oplus \dots \oplus V_r$ by PDT.

$V_i = \ker(p_i(T_i)^{e_i})$

$$m_T(x) = \prod_{i=1}^{r} p_i(x)^{e_i}$$

$p_1 \cdots p_r$ distinct monic irreducibles.

$T_i \in L(V_i, V_i)$

$$m_{T_i, V_i}(x) = p_i(x)^{e_i}$$

Take $v_i \in V_i$ so $m_{T_i, v_i}(x) = p_i(x)^{e_i}$

Form $v = v_1 + \cdots + v_r$

<u>Claim:</u> $m_{T,v}(x) = m_T(x)$

Take $f(x) \in F[x]$ & apply $f(T)v = \underset{V_1}{\underset{\Uparrow}{f(T)v_1}} + \underset{V_2}{\underset{\Uparrow}{f(T)v_2}} + \cdots + f(T)v_r$

So $f(T)v = 0$ iff $f(T)v_i = 0 \ \forall i$

This happens iff $\underbrace{m_{T_i, v_i}(x)}_{m_{T_i}(x)} \mid f(x)$.

So can find such v if we can factorize $m_T(x)$.

If not pick an arbitrary $v \in V$, prob is almost 1 that $m_{T,v}(x) = m_T(x)$.

Ex:

$$D = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \sum \underbrace{a_i e_i}_{"v_i" \, \in \, V_i \; v_i \neq 0}$$

c_i distinct

Note PDT $V_i = \ker(D - c_i) = Fe_i$??

Lemma

$f(x) \in F[x]$, $f(x) = q(x)\, m(x) + r(x)$ (from div. alg). ← Euclidean.

$(m(x) = m_T(x))$.

then $f(T)v = r(T)v \;\; \forall \, v \in V$.

Lemma 2

Suppose $m(x) = h(x)\, g(x)$ factorizes ie; $g(x), h(x) \in F[x]$.
Suppose T has a cyclic vector v on V. Then

(i) $\operatorname{Im}(g(T)) = \ker(h(T))$

(ii) $\ker(h(T))$ has a basis

$g(T)v, \; T g(T)v, \; \ldots, \; T^{s-1} g(T)v$

where $s = \deg(h(x))$. So nullity of $h(T)$ is $\deg(h(x))$

Proof: $0 = m(T) = h(T)g(T) \Rightarrow \operatorname{Im}(g(T)) \subseteq \ker(h(T))$
Let $w \in V$. There is a poly $k(x)$ so $w = k(T)v$. Then
$w \in \ker(h(T)) \Leftrightarrow g(x) \mid k(x) \Leftrightarrow w \in \operatorname{Im}(g(T))$. $k(x) = g(x)\, l(x)$

i) An element $w \in \ker(h(T))$ is a LC of $g(T)v, g(T)Tv, \dots$
$\dots, g(T)T^{s-1}v$

This seq. spans $\ker(h(T))$

LI? Consider $\sum_{i=0}^{s-1} c_i\, g(T) T^i v = 0$.

$g(T)\left(\sum_{i=0}^{s-1} c_i T^i v\right) = 0$ but then $\underbrace{\sum_{i=0}^{s-1} c_i x^i g(x)}_{\text{degree} \leq \deg(m(x))-1}$ is a multiple

of $m(x) = h(x)g(x)$.

So $\sum c_i x^i g(x) = 0 \Rightarrow c_i = 0$.

<u>Notation</u> $v \in V$ so $Z(T, v)$ has dimension $\deg(m_T(x))$ i.e.

$m_{T,v}(x) = m_T(x)$.

<u>Lemma 3</u>: $Z = Z(T, v)$ is T-admissible. → need not be cyclic?

<u>Proof</u>: Suppose $f(x) \neq 0$ in $F[x]$, $w \in V$

so $f(T)w \in Z$. I want $y \in Z$ so $f(T)w = f(T)y$.

Define $d = \gcd(f, m)$. $\exists$ polynomials p, q s.t

$d = pf + qm$. $\Rightarrow \boxed{d(T) = p(T)f(T)} + q(T)\cancel{m(T)}$ 0

Take polynomials $r, e \in F[x]$ so that $f = dr$ & $m = de$.

Now $f(T)w = d(T)r(T)w \in \ker(e(T)) \cap Z$.

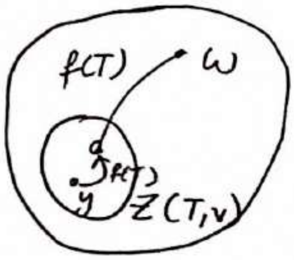
Lemma $v \in V$; $v \neq 0$; $m_{T,v}(x) = m_T(x)$ &

$$Z := Z(v,T) = \text{span}\{v, Tv, \dots\}$$

Then Z is T-admissible.

Proof

Suppose $f(x) \neq 0 \in F[x]$; $w \in V$
such that $f(T)w \in Z$.
We want $f(T)y = f(T)w$ for some $y \in Z$.



Let $d = \gcd(f, m)$

$\Rightarrow \exists\, p, q \in F[x]$ s.t. $d(x) = p(x) f(x) + q(x) m(x)$

$\Rightarrow d(T) = p(T) f(T).$

Let $f(T) = d(T) r(T)$ & $m(T) = d(T) e(T).$

$f(T)w = d(T)\, r(T)w$

from Lemma, $Im(d) = Ker(e)$ since $m = ed$.

$\Rightarrow f(T)w \in Ker(e(T)) \cap Z = d(T)Z$. (?)

$\Rightarrow$ Let $u \in Z$ $\Rightarrow f(T)w = d(T)u = P(T) f(T) u$

Let $y = \underbrace{P(T)u}_{\in Z}$ since Z is T-cyclic.

$\Rightarrow f(T)w = f(T)y$

This equals $d(T)(Z)$ by earlier lemma.

Let $u \in Z$ so $f(T)w = d(T)u$.

Now use $d(T) = p(T)f(T)$.

$\Rightarrow f(T)w = d(T)u = f(T)\underbrace{p(T)u}_{y \in Z}$ // completes proof of existence of RCF.

To get RCF look at $\bar{V} = V/\underset{\text{adm.}}{Z}$ $(Z = Z(T,v))$. $\hookrightarrow m_T(x)$.

$\Rightarrow V/Z$ has decomp with $f_2 \ldots f_r$, can pull back to

$V = Z \oplus Z_1 \oplus \cdots \oplus Z_r$.

$m_T(x) = f_1$

<u>Note</u>: If $(g,h) = 1$ & $f = gh$.

$$\underset{\substack{\downarrow \\ \text{Companion matrix of } f.}}{C_f} \text{ similar to } \left(\begin{array}{c|c} C_g & \\ \hline & C_h \end{array}\right)$$

Knowing $c(x)$, $m(x)$ does not uniquely determine RCF.

Ex: $c = x^7$ $m = x^3$ are shared by $\begin{bmatrix} C_{x^3} & & \\ & C_{x^2} & \\ & & C_{x^2} \end{bmatrix}$ $\begin{bmatrix} C_{x^3} & & & \\ & C_{x^2} & & \\ & & C_x & \\ & & & C_x \end{bmatrix}$

Lecture

JCF: To find JCF decomp, get v so $m_{T,v}(x) = m_T(x)$.
Then study V/Z, $Z = Z(T,v)$. Change basis (expand basis of Z to one of V), get mtx

$$\text{mtx of } Z \quad \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \quad \text{mtx of } \bar{V}.$$

Ex: 1: $\left(\begin{array}{c|c} C_g & 0 \\ \hline 0 & C_h \end{array}\right)$ similar to $\left(C_{gh} \right)$ if g,h are rel. prime & monic $((g,h) = 1.)$

Proof :- $V = V_1 \oplus V_2$; $V_1 = \ker(g(T))$, $V_2 = \ker(h(T))$

take cyclic vector w_1 — Take cyclic vector w_2

Then $w_1 + w_2$ has ann. poly $g(x)h(x)$.
$\Rightarrow Z(T, w_1+w_2)$ has dim $= \deg(g)\deg(h) = \dim(V)$.

So T has mtx C_{gh} since w_1+w_2 is cyclic vector for V.

Jordan Canonical Form for $T \in L(V,V)$ so $m_T(x)$ factors completely into linear factors.

$\exists$ basis so mtx for T is a block diagonal sum of matrices of the form $\underbrace{J(k,c)}_{k \times k \text{ mtx}}$ where c is an eigenvalue.

$$J(k,c) = \begin{pmatrix} c & & & 0 \\ 1 & c & & \\ & 1 & \ddots & \\ 0 & & 1 & c \\ & & & 1 \quad c \end{pmatrix}$$

We first decompose $V = V_1 \oplus \dots \oplus V_r$ as in PDT.

T_i = LT induced by T on V_i. Do RCF decomposition for T_i on $V_i = Z_1 \oplus \dots \oplus Z_s$.

$Z_i = Z(T_i, v_j)$ and annihilator poly is $(x-c_i)^{e_{i,j}}$.

$e_{i,1} \geq e_{i,2} \geq \dots \geq e_{i,s} \geq 1$.

Z_j has dim k_j. We could use basis $v_i, Tv_i, T^2 v_i, \dots$ to get $C_{(x-c_1)^{e_{ij}}}$. This would have nonzero last column.

Instead, $T_i = \underbrace{(T_i - c_i)}_{\text{nilpotent}} + c_i 1_{V_i}$

It's mtx in RCF is $C_{x^{e_{ij}}}$ =

$$\underbrace{\begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & 0 \\ & 1 & \ddots & \vdots \\ & & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix} + \begin{pmatrix} c_i & & & \\ & c_i & & \\ & & \ddots & \\ & & & c_i \end{pmatrix}}_{\text{gives JCF}}$$

<u>Uniqueness for RCF</u> for $T \in L(V,V)$.

<u>Def</u>: Two matrices A, B both $m\times n$ / $F[x]$ are equivalent if $\exists$ $P_{m\times m}$ & $Q_{n\times n}$

So $B = PAQ$. and P = product of $EROM_{m\times m}$ / $F[x]$.

& Q = " " " $n\times n$ " "

<u>Ex</u>: $A = xI - C_f$ $\quad f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i$

$$C_f = \begin{pmatrix} 0 & & 0 & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & & \vdots \\ 0 & & 1 & -a_{n-1} \end{pmatrix}$$

$$A = \begin{pmatrix} x & & & & +a_0 \\ -1 & x & & & +a_1 \\ & -1 & \ddots & & \vdots \\ & & & x & \\ & & & -1 & a_{n-1}+x \end{pmatrix} \times x \text{ \& add} \longrightarrow \begin{pmatrix} x & & & \\ -1 & x & & \\ & -1 & \ddots & \\ & & & a_{n-3} \\ & -1 & 0 & a_{n-2}+xa_{n-1}+x^2 \\ & & -1 & a_{n-1}+x \end{pmatrix} \times x +$$

$$\longrightarrow \begin{pmatrix} x & & & & & a_0 \\ -1 & x & & & & a_1 \\ & -1 & \ddots & & & \\ & & & 0 & 0 & x^3+x^2a_{n-1}+xa_{n-2}+a_{n-3} \\ & & & -1 & 0 & '' \\ & & & & -1 & '' \end{pmatrix} \text{ repeat.}$$

$$\to \begin{pmatrix} 0 & 0 & \cdots & 0 & f(x) \\ -1 & 0 & & & '' \\ & -1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & -1 & \end{pmatrix} \xrightarrow[\text{to remove all other polys in last col}]{\text{column ops}} \begin{pmatrix} 0 & 0 & \cdots & 0 & f(x) \\ -1 & 0 & & & 0 \\ & -1 & 0 & & 0 \\ & & \ddots & \ddots & \vdots \\ & & & -1 & 0 \\ & & & & -1 \quad 0 \end{pmatrix}$$

$$\xrightarrow[\text{col perm.}]{\text{by}} \begin{pmatrix} f & & & \\ & -1 & & \\ & & -1 & \\ & & & \ddots \\ & & & & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} f & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} /\!/$$

This mtx is similar to the C_f of $f(x)$.

Now, given T & a decomposition $V = Z_1 \oplus \dots$ by cyclic subspaces we get mtx for T of form.

$$M = \begin{pmatrix} \boxed{C_{f_1}} & & & \\ & \boxed{C_{f_2}} & & \\ & & \boxed{} & \\ & & & \ddots \end{pmatrix}$$

where $Z_i = Z(T, v_i)$; $f_i = m_{T, v_i}(x)$.

> Now $xI - M$ is equivalent to $\begin{pmatrix} \boxed{\begin{matrix} f_1 & & \\ & 1 & \\ & & \ddots \; 1 \end{matrix}} & & \\ & \boxed{\begin{matrix} f_2 & & \\ & 1 & \\ & & \ddots \; 1 \end{matrix}} & \\ & & \ddots \end{pmatrix}$

using some $P = \begin{pmatrix} \boxed{P_1} & & \\ & \boxed{P_2} & \\ & & \ddots \end{pmatrix}$ & $Q = \begin{pmatrix} \boxed{Q_1} & & \\ & \boxed{Q_2} & \\ & & \ddots \end{pmatrix}$.

Then by equivalence we convert it to $\begin{pmatrix} f_1 & & & & & \\ & f_2 & & & & \\ & & f_r & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$ using col. & row perms.

If in addition T has another RCF decomp. $V = Z_1' \oplus Z_2' \dots \oplus Z_s'$

Where $Z_j' = Z(T, v_j')$ and annihilator polys $g_i(x)$

then get $xI - M$ similar to $\begin{pmatrix} g_1 & & & & & \\ & g_2 & & & & \\ & & g_s & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$.

Given that f_i is divisible by f_{i+1} for $i \le r-1$ & g_j is divisible by g_{j+1} for $j \le s-1$. we shall deduce $f_k = g_k$ for all $k = 1 \dots r = s$.

Def A $m \times n$ matrix $/F[x]$. Say A is in Smith Canonical Form or Smith Normal Form (SNF) iff

i) all off diagonal entries are 0.

ii) nonzero polys $f_1 \dots f_r$ in rows $1, \dots, r$ then only zeroes below row r. and $f_i \mid f_{i+1}$ for $i = 1, \dots, r-1$.

Ex:
$$\begin{pmatrix} x & 0 & 0 \\ 0 & x(x+1) & 0 \\ 0 & 0 & x(x^2-1) \\ 0 & 0 & 0 \end{pmatrix}.$$

> We shall prove that matrices are equivalent to just one mtx in SNF.

We showed RCF gives a seq. of monic polys (by choice of a direct sum decomp) each $f_{i+1} \mid f_i$ for $i = 1 \ldots r$.

For a different direct sum we get $g_1 \ldots g_s$ s.t $g_{j+1} \mid g_j$ for $j = 1 \ldots s$.

We <u>want</u> to show that fs & gs are equal.

Recall SNF $\begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & h_3 & \\ & & & \ddots \end{pmatrix}$ & $h_i \mid h_{i+1}$

If $D = \begin{pmatrix} d_1 & & \\ & d_2 & \\ & & \ddots \end{pmatrix}_{m \times n}$ & $i, j \leq \min\{m, n\}$. We want to switch the entries at (i,i) & (j,j). This can be done by premultiplication with elementary mtxs & post mult by elem. matrices

$$\begin{array}{c} \\ i - \\ \\ j - \\ \\ \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & 1 & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}_{m \times m} = t_{i,j;m}$$

$t_{ij;m} K_{m \times n}$ = switches rows i & j

$K_{m \times n} t_{ij;n}$ = switches cols. i & j

Doing both exchanges elements on a diagonal for a diag. mtx.

Thm $A_{m\times n}$ over $F[x]$. Then A is equivalent to mtx in SNF.
in this sense

Proof: Assume $A \neq 0$. In the eq. class of A, choose a mtx B whose $(1,1)$ entry is monic & has least possible degree.

$$\Rightarrow \quad B = \begin{pmatrix} b_{11} & \cdots \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix}.$$

Claim: B is equivalent to matrix with shape $\begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & ? & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$

$\begin{pmatrix} & b_{ij} & \\ & & \\ & & \end{pmatrix}$ $= b_{11}q + r$ where $r = 0$ since otherwise using col. op we can have $b_{ij} = r$ & r has $\deg < b_{11}$ # Since b_{11} has least degree.

$\Rightarrow b_{ij} = b_{11}q \longrightarrow 0$ from col. op.

$$\Rightarrow \text{So get} \quad B \sim B' = \begin{pmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ 0 & & C' & & \\ \vdots & & (m-1)\times(n-1) & & \\ 0 & & & & \end{pmatrix}$$

By induction this is eq. to a diag mtx given by P' & Q' //

Induction on n.

So $\exists$ P', $(m-1)\times(m-1)$ & Q', $(n-1)\times(n-1)$

So $P'C'Q'$ is in SNF. Set $P = \begin{pmatrix} 1 & \\ & P' \end{pmatrix}$; $Q = \begin{pmatrix} 1 & \\ & Q' \end{pmatrix}$

Then $PB'Q = \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & h_3 & \\ & & & \ddots \end{pmatrix}$ where $h_i \mid h_{i+1}$ $\forall\, i \geq 2$.

(Note: Equivalent is for non sq. mtx & similar is for square mtxs).

Claim : $h_1 | h_2$.

$$PB'Q \sim \begin{pmatrix} h_1 & \overset{= qh_1+r \to 0 \text{ as earlier.}}{h_2} & \cdots \\ 0 & h_2 & \\ & \vdots & \\ & \vdots & \end{pmatrix}$$

(row 1 = row 1 + row 2)

Choice of $b_{11} = h_1$ shows $h_1 | h_2$. //

Uniqueness (After 2 pages).

Def: $A_{m \times n} / F[x]$. A $k \times k$ submatrix is obtained by striking $(m-k)$ rows & $(n-k)$ cols. A $k \times k$ minor is det of $k \times k$ submatrix.

In $F[x]$, we let $\delta_k(A)$ be the monic generator of ideal generated by all $k \times k$ minors. ($\delta_k(A) = 0$ if all $k \times k$ minors are 0)

Thm: If A, B are $m \times n$ & equivalent then $\delta_k(A) = \delta_k(B)$ $\forall k$.

Ex: $\underset{A}{\begin{pmatrix} x^3 & & \\ & x & \\ & & x \end{pmatrix}}$, $\underset{B}{\begin{pmatrix} x^3 & & \\ & x^2 & \\ & & 1 \end{pmatrix}}$ Not eq.

$\delta_1(A) = x$ $\quad$ $\delta_1(B) = 1$

$\begin{pmatrix} x^2 & & & \\ & x^2 & & \\ & & x^2 & \\ & & & x \end{pmatrix}$ $\quad$ $\begin{pmatrix} x^3 & & & \\ & x^3 & & \\ & & x & \\ & & & x \end{pmatrix}$ Not eq.

$\delta_2 = x^3$, $\delta_1 = x$ $\quad$ $\delta_1 = x$, $\delta_2 = x^2$

Proof : Later

Lemma

$S = \begin{pmatrix} h_1 & & \\ & h_2 & \\ & & \ddots \end{pmatrix}$ rank r, in SCF / $F[x]$ then $\delta_k(S) = h_1 h_2 \cdots h_k$

if $k \le r$ & $\delta_k(S) = 0$ if $k > r$. So $h_k = \delta_k(S) / \delta_{k-1}(S)$ for

$k = 1 \ldots r$.

Lemma

(i) If C_f is companion mtx for monic poly $f(x)$, then

$(xI - C_f)$ has SNF $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1_f \end{pmatrix}$ → This was shown earlier (as we did $\begin{pmatrix} f & & \\ & 1 & \\ & & 1 \end{pmatrix}$

(ii) If $A = \begin{pmatrix} (xI - C_{f_1}) & & \\ & (xI - C_{f_2}) & \\ & & (xI - C_{f_r}) \end{pmatrix}$. Then A is equivalent to

$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & f_{\sigma 1} & & \\ & & & & & f_{\sigma 2} & \\ & & & & & & \ddots \; f_{\sigma r} \end{pmatrix}$ for any perm σ of $\{1, 2, \ldots, r\}$

Thm ?
Uniqueness of RCF

V_{fdvs}/F ; $T \in L(V,V)$. $\mathcal{B}, \mathcal{B}'$ bases giving similar matrices

$$A = \begin{pmatrix} C_{f_1} & & \\ & C_{f_2} & \\ & & \ddots \end{pmatrix} ; \quad B = \begin{pmatrix} C_{g_1} & & \\ & C_{g_2} & \\ & & \ddots \end{pmatrix}$$ representing T,

and satisfying $f_{i+1} \mid f_i$ for $i = 1 \ldots r-1$ & $g_{j+1} \mid g_j$ for $j = 1 \ldots s-1$.

Now form $xI - A$ & $xI - B$. Define $A' = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & f_r & \\ & & & & \ddots \\ & & & & & f_1 \end{pmatrix}$

& $B' = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & g_s & \\ & & & & \ddots \\ & & & & & g_1 \end{pmatrix}$

reversing indeces so division goes other way

there are products of elem. matrices P_1, P_2, Q_1, Q_2 so

$$A' = P_1 (xI - A) Q_1 \quad \& \quad B' = P_2 (xI - B) Q_2 .$$

[Note: $xI - A$ is similar to $\begin{pmatrix} (xI - C_{f_1}) & & \\ & (xI - C_{f_2}) & \\ & & \ddots \end{pmatrix} \sim \left(\begin{array}{c|c|c} \begin{smallmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & f_1 \end{smallmatrix} & & \\ \hline & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & f_2 \end{smallmatrix} & \\ \hline & & \ddots \end{array}\right)$]

Now A, B similar, $\exists$ invertible mtx U so $B = U^{-1} A U$.

$\Rightarrow \quad xI - B = U^{-1}(xI - A)U$

$$B' = P_2(xI - B)Q_2 = P_2 U^{-1}(xI - A) U Q_2 = P_2 U (P_1^{-1} A' Q_1^{-1}) Q_2$$

$\Rightarrow$ A', B' are equivalent over $F[x]$ // $\rightarrow$ How does this imply $A \sim B$? (Use Thm from earlier about minors.)

<u>Thm</u> $A \sim B \,/\, F[x] \Rightarrow \delta_k(A) = \delta_k(B)$

<u>Proof</u> I = k-subset of $\{1 \ldots m\}$

J = k-subset of $\{1 \ldots n\}$

$A_{I,J}$ $k \times k$ submatrix ; $\det(A_{IJ})$ is the corresponding minor.

What happens if we do row or col op. to A.

Rec

We looked at $A' = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & f_r & \\ & & & & \ddots \\ & & & & & f_1 \end{pmatrix}$ $\left.\right\}$ p ones, $\left.\right\}$ $p+r=n$.

$B' = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & g_s & \\ & & & & \ddots \\ & & & & & g_1 \end{pmatrix}$ $\left.\right\}$ q $\Rightarrow q+s=n$

We want $p=q$ & $g_i = f_i \ \forall i$. Suppose $p<q$. Note that $\delta_p(A)=1$ $\delta_{p+1}(A') = f_r$. $\delta_p(B')=1$ & $\delta_{p+1}(B')=1$ $\#$ (since $\delta_{p+1}(A') = \delta_{p+1}(B')$)

Similar argument shows $q<p$ is not true.

$\Rightarrow p=q. \Rightarrow r=s.$

Next suppose $\delta_K(A')=1$ for $K \leq p$. If $k=p+t$ then $\delta_K(A') = f_r f_{r-1} \cdots f_{r-t}$

$\delta_k(B') = g_r \, g_{r-1} \cdots g_{r-t+1} \Rightarrow f_r = \delta_{p+1}(A') / \underbrace{\delta_p(A')}_{1}$

$\Rightarrow f_{r-t+1} = \delta_{p+t}(A') / \delta_{p+t-1}(A')$ for $t \geq 1$.

$\Rightarrow f_{r-t+1} = g_{r-t+1} \ \forall \ t=1,\dots,n-r.$

We must prove,

<u>thm</u>: If $A, B / F[x]$ are equivalent then $\delta_K(A) = \delta_K(B) \ \forall k$.

$\delta_0(A) := 1$. $\delta_k(A) = 0$ if $K \geq 1 + \min\{m,n\}$

<u>Proof</u>: $A_{m\times n}$. If I is a k-subset of $\{1 \dots m\}$ & J is a k subset of $\{1, \dots, n\}$. Then A_{IJ} is a $k \times k$ submatrix & $\det(A_{IJ})$ is one of the $k \times k$ minors.

$\delta_k(A)$ = monic generator of ideal generated by all $k \times k$ matrice

> Effect of row ops on $\det(A_{I,J})$.

> Multiply a row by a nonzero scalar, then $\det(A_{I,J})$ is unchanged if that row is not indexed by member of I. If it is, det changes by a non zero scalar.

> Add multiple of row p to row q. No effect if $q \notin I$.

> Suppose $q \in I$, if $p \in I$, no change.

> Suppose $p \notin I$, then we use property of det, ie linearity in argument indexed by q.

$A_{I,J} \rightsquigarrow$ new matrix where row q = ord row q + new row

Then $\det(\text{row } A_{IJ}) = \det(\text{old } A_{IJ}) + \underline{\text{some } k \times k \text{ minor}}$.

$I \rightsquigarrow I'$ where $I' = I - \{q\} \cup \{p\}$

Thm: $(\delta_k(A)) \supseteq (\delta_{k+1}(A))$ ie. $\delta_k(A) \mid \delta_{k+1}(A)$.

Proof: Consider $(k+1) \times (k+1)$ submatrix A_{IJ}. Compute its det by Laplace expansion across row 1.

$$\sum_{j=1}^{n} b_{ij} (-1)^{i+j} \underbrace{\det(A_{IJ}(1 \mid j))}_{\text{all in } (\delta_k(A))}$$

$\Rightarrow \det(A_{I,J}) \in (\delta_k(A))$.

Other uniqueness proof.

$$\overset{m\times n}{A} \rightsquigarrow \underset{\text{inv } m\times m}{P}A\underset{\text{inv } n\times n}{Q} \qquad AB \longrightarrow PABQ$$

$$B \longrightarrow PBQ \qquad \neq PA\underbrace{QP}BQ$$

$QP = I \Rightarrow$ product is respected. but not otherwise -

we look at polys $p(x)$ & nullity $(p(T))$ for a set of polys $p(x)$.

$f_1 \dots f_r$ is determined by this set of nullities for all $p \,/\, m_T(x)$ //

Next: Forms on fdvs.

$V \times V \xrightarrow{g} F$ field . g is bilinear if $g(x,y)$ is linear in each argument if the other one is fixed.

v.s / F

In case $V = \mathbb{F}^{n\times 1}$, g can be described by a matrix

$$A = (g(e_i, e_j)) \; ; \; x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} ; \; y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \text{ then } g(x,y) = \sum_{i,j=1}^{n} x_i y_j \, g(e_i, e_j)$$

$$(x_1 \dots x_n) \left(g(x_i, x_j) \right) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \longleftrightarrow x^t A y.$$

Will look at cases -

→ A is symmetric & $F = \mathbb{R}$ or $\mathbb{C}$!

→ Sesquilinear: $g(cx, y) = c(g(x,y))$ & $g(x, cy) = \bar{c}\, g(x,y)$. (comp - conj)

Inner Product space: A v.s $V / F = \mathbb{R}$ or $\mathbb{C}$ with a sesquilinear positive definite form. //

Def: Sesquilinear form

$f: V \times V \longrightarrow F$ ($\mathbb{R}$ or $\mathbb{C}$).

f linear in first variable & conjugate linear in 2^{nd} variable.

$$f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$$

$$f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$$

$$f(cx, y) = c\, f(x, y)$$

$$f(x, cy) = \bar{c}\, f(x, y).$$

If V basis $v_1 \dots v_n$; $v = \sum c_i v_i$; $w = \sum d_i v_i$

$$f(v, w) = f\left(\sum c_i v_i, \sum d_j v_j\right)$$

$$= \sum_{i,j} c_i \bar{d_j}\, f(v_i, v_j)$$

The scalars $f(v_i, v_j)$ describe f completely.

$X = [v]_{\mathcal{B}}$; $Y = [w]_{\mathcal{B}}$; $G = \overset{n \times n \text{ matrix}}{(g_{ij})}$ where $g_{ij} = f(v_j, v_i)$

We will show $f(v, w) = Y^* G X$ where if M mtx $/F$; $\boxed{M^* = \bar{M}^t.}$

Calculate $Y^* G X = Y^*\left(\sum c_i G_i\right)$ where $G = (G_1 | G_2 | \cdots | G_n)$

$X = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$. $Y^* G X = \sum_i c_i Y^* G_i = \sum_i c_i (\bar{d}_1, \dots, \bar{d}_n) G_i$

$$\Rightarrow \sum_{i=1}^{n} c_i (\bar{d}_1, \ldots, \bar{d}_n) \begin{pmatrix} g_{1i} \\ \vdots \\ g_{ni} \end{pmatrix} = \sum_{i,j}^{n} c_i \bar{d}_j g_{ji} = \underbrace{\sum_{ij}^{n} c_i \bar{d}_j f(v_i, v_j)}_{f(v,w)}$$

<u>Def</u> $G_{n\times n}$ is <u>Hermitian</u> if $G = G^*$.

<u>Def</u> An inner product on a vs. V/F is a sesquilinear function f

$f: V \times V \to F$ which satisfies $f(x,y) = \overline{f(y,x)}$ and

$f(x,x) > 0$ if $x \in V$, $x \neq 0$.

$f(0,0) = 0$.

If V fd & G = mtx for f, inner product on V, with respect to $v_1 \ldots v_n$ then $G = G^*$.

> For now we will study inner product spaces, V with a particular inner product. Notation: $(x|y)$.

> We can derive properties.

<u>Def</u>: The <u>norm</u> (length) of a vector $x \in V$, IPS (inner product space), is $|x| = (x|x)^{1/2}$ nonneg. sq. root.

$(x|x) \in \mathbb{R}$ because $(x|x) = \overline{(x|x)} \Rightarrow \in \mathbb{R}$ (non negative)

$\|x\| = (x|x)^{1/2}$.

↳ notation.

<u>Thm</u>: $|cx| = |c||x|$.

$\hookrightarrow$ abs. value in $\mathbb{C}$.

<u>Cauchy Schwartz inequality</u>.

$$|(x|y)| \leq |x||y|.$$

<u>Triangle inequality</u>.

$$|x+y| \leq |x| + |y|.$$

<u>Proof of CS</u>: x, y & $x \neq 0$; write $z = y - \dfrac{(y|x)}{(x|x)} x$.

$\hookrightarrow$ "projection of y to $F.x$"

$\to$ line of x.

$$0 \leq (z|z) = \left(y - \frac{(y|x)}{(x|x)} x \,\middle|\, y - \frac{(y|x)}{(x|x)} x \right)$$

$$= (y|y) - \frac{(x|y)}{(x|x)}(y|x) - \frac{(y|x)}{(x|x)}(x,y) + \frac{(y|x)(x|y)}{(x|x)^2}(x|x)$$

$$= (y|y) - \frac{(x|y)(y|x)}{(x|x)}$$

$$\Rightarrow (y|y) \geq \frac{(x|y)(y|x)}{(x|y)} \iff \underbrace{(x|y)(y|x)}_{|(x|y)|^2} \leq (x|x)(y|y)$$

Take sq. root for C.S. // $\blacksquare$

Proof of Triangle inequality

$$(x+y \mid x+y) = (x|x) + (x|y) + (y|x) + (y|y)$$

$$= (x|x) + 2\,\mathrm{Re}(x|y) + (y|y)$$

$$\leq (x|x) + 2|x||y| + (y|y)$$

$$= |x|^2 + 2|x||y| + |y|^2$$

$$= (|x| + |y|)^2$$

Take sq. roots.

$$\Rightarrow \quad |x+y| \leq |x| + |y| \;/\!/$$

■

Projections

V IPS & W subspace. $x \in V$. If $\exists\, p \in W$ so that

$$q = x - p \in W^{\perp} = \{ v \in V \mid (v|w) = 0 \ \forall\, w \in W \}.$$

Then p is the projection of x to W.

> In finite dimensional case, the projection exists.

Uniqueness: If $x = p_1 + q_1 = p_2 + q_2$ $(p_1, p_2 \in W,\ q_1, q_2 \in W^{\perp})$

Then $p_1 = p_2$ & $q_1 = q_2$.

$\underbrace{p_1 - p_2}_{\in W} = \underbrace{q_2 - q_1}_{\in W^\perp} \Rightarrow |p_1 - p_2| = 0 = |q_2 - q_1|$ // unique ■

- If $V =$ all smooth functions on $[0,1]$ &

$$(f|g) = \int_0^1 f(t)\overline{g(t)}\,dt, \quad 1, x, x^2, \ldots \in V \ \& \ e^x \ldots \in V.$$

$W_n = \text{span}\{1, x, x^2, \ldots x^n\} \to \dim = n+1.$

Projecting e^x to W_n = expansion of e^x upto deg n.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

<u>Def</u>: $v_1 \ldots v_n$ sequences in V, IPS. It is orthogonal if $(v_i|v_j) = 0$ for $i \neq j$

It is <u>orthonormal</u> if $(v_i|v_j) = \delta_{ij}$ for $i,j = 1 \ldots n$.

<u>Lemma</u>: (1) If $v_1 \ldots v_n$ is orthogonal seq of nonzero vectors, it is LI. (2) If $v_1 \ldots v_n$ is ON & $x = \sum_{i=1}^{n} c_i v_i \in \text{span}\{v_1 \ldots v_n\}$ then $c_i = (x|v_i)$

Proof

$$0 = \sum_{i=1}^{n} c_i v_i \quad \text{L O.}$$

$$0 = (0 | v_k) = \left(\sum c_i v_i \,\middle|\, v_k\right) = \sum_i \big(c_i \underbrace{(v_i | v_k)}_{\delta_{ik}}\big) = c_k (v_k | v_k) \quad \forall k$$

$\neq 0$

$\Rightarrow$ LI.

If $x = \sum c_i v_i$; $(x | v_k) = \left(\sum c_i v_i \,\middle|\, v_k\right) = c_k (v_k | v_k) = c_k$.

Lecture

Recall, projection of $x \in V$ to $\underset{\overset{\shortparallel}{\text{span}(y)}}{Fy}$, $y \neq 0$ is $\dfrac{(x|y)\,y}{(y|y)}$

$\left[\text{not } \dfrac{(y|x)\,y}{(y|y)}\right]$

If W subspace of V, & if $x \in V$ has two expressions.

$x = p + q = p' + q'$

$p, p' \in W$ & $q, q' \in W^{\perp}$ Then $p = p'$ & $q = q'$

$p - p' = q' - q$

$$0 \leq (p-p' \mid p-p') = (\underset{\underset{W}{\cap}}{p-p'} \mid \underset{\underset{0}{\shortparallel}}{\overset{\in W^{\perp}}{q-q'}}) \Rightarrow p = p' \Rightarrow q = q'$$

Check that $x - \dfrac{(x|y)}{(y|y)}\,y$ is in $y^{\perp}$

$$\left(x - \frac{(x|y)}{(y|y)}\,y \,\Big|\, y\right) = (x|y) - \frac{(x|y)(y|y)}{(y|y)} = 0.$$

Def:- A sequence $v_1 \ldots v_r$ in V is orthogonal if $(v_i | v_j) = 0$ for $i \neq j$

A sequence $v_1 \ldots v_r$ is orthonormal (ON) if $(v_i | v_j) = \delta_{ij}$ $\forall i,j$

If $x \in V$, $x = \sum_{i=1}^{r} a_i v_i$; $v_1 \ldots v_r$ ON then $a_i = (x | v_i)$

If $u_1 \ldots u_r$ is ON basis of W then $\sum_{i=1}^{r} (x|u_i) u_i$ is the projection of x to W.

$$\left(x - \sum_i (x|u_i) u_i \,\middle|\, u_j\right) = 0 \quad \forall j.$$

Gram Schmidt Procedure

Given $v_1 \ldots v_r$ LI seq in V,

define $W_k = \text{span}\{v_1 \ldots v_k\}$

Define $u_1 = \dfrac{v_1}{|v_1|}$, For $k \geq 1$ define $v_k' := v_k - P_{W_{k-1}}(v_k)$

(computed with $u_1 \ldots u_{k-1}$)

$u_k := \dfrac{v_k'}{|v_k'|}$ ← nonzero.

Then $\forall k$, $u_1 \ldots u_k$ ON basis of W_k //

In $F^{m \times 1}$ usual inner product.

define the QR factorization of $A_{m\times n}$ mtx with LI cols. to be

$A = Q^{m\times n} R_{n\times n}$, R upper triangular. positive entries on diagonal.

$(v_1 | \ldots | v_n)$ ON cols.

$Q = (u_1 | \cdots | u_n)$ $\quad R = (r_{ij}) = \begin{pmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ & r_{22} & \cdots & & \vdots \\ 0 & & r_{33} & \cdots & \vdots \\ & & & \ddots & r_{nn} \end{pmatrix}$;

$R = (R_1 | \cdots | R_n)$

$QR = (QR_1 | QR_2 | \ldots | QR_n)$

$$QR = (u_1 | u_2 | \ldots | u_n) \overset{R_j}{\begin{pmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{jj} \\ 0 \\ \vdots \\ 0 \end{pmatrix}} = \underbrace{r_{1j} u_1 + r_{2j} u_2 + \ldots +}_{\text{in } W_{j-1}} \underbrace{r_{jj} u_j}_{\text{in } W_{j-1}^{\perp}}$$

We want this to equal v_j. Take $r_{ij} = (v_j | u_i)$ & $u_1 \ldots u_j$ ON basis for W_j created by GS.

$r_{jj} = |v_j'| > 0.$

$r_{jj} u_j = v_j'$

The QR factorization of a given A with LI cols is unique.

<u>Lemma:</u> Suppose R ring, $u \in R$ is nilpotent ($u^n = 0$) Then $1-u$ is a unit.

<u>Proof</u>: $1 = 1 - u^n = (1-u)(1+u+u^2+\ldots+u^{n-1}) = (1+u+u^2+\ldots+u^{n-1})(1-u)$

$\Rightarrow$ $1-u$ is a 2 sided unit //

Remark: If t is a unit, u nilpotent and if $tu = ut$ then $t-u$ is a unit.

Reason: $t-u = t(1-t^{-1}u) \Rightarrow t^{-1}u = ut^{-1}$ is nilpotent

Say we have 2 products, $Q_1 R_1 = Q_2 R_2$, where Q_j ON cols. & R_j is upper triangular with pos diag.

$$R_2 = \begin{pmatrix} r_{11} & r_{12} & \cdots & \\ & r_{22} & \cdots & \\ & & \ddots & \\ & & & \cdot \end{pmatrix} = \underbrace{\begin{pmatrix} r_{11} & & \\ & \ddots & \\ & & r_{nn} \end{pmatrix}}_{\text{inv}} \underbrace{\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}}_{\text{inv}}$$

Furthermore, the inverse is also upper triangular.

→ This is equals $1-u$ where u is strictly upper tri.

$$(1-u)^{-1} = 1 + \underbrace{u + u^2 + \cdots}_{\text{all upper tri.}}$$

So we have $Q_1 (\underbrace{R_1 R_2^{-1}}_{\text{upper tri.}}) = Q_2$

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots \\ & c_{22} & c_{23} & \cdots \\ 0 & & c_{33} & \\ & & & \ddots \end{pmatrix}$$

We argue by induction on j that $c_{jj} = 1$ & $c_{ij} = 0$ if $i<j$.

$\Rightarrow$ Write $Q_1 = (g_1 | g_2 | \dots | g_n)$ ON cols.

Col 1 of $Q_1 R_1 R_2^{-1}$ is $c_{11} g_1 \rightarrow$ unit vector ; $c_{11} = \pm 1$.

$\left(c_{11} = r_{11} r_{11}'^{-1} \right.$ where $R_1 = r_{ij}$, $R_2 = r'_{ij}$; both +ve $\Rightarrow c_{11} = 1 \left. \right)$

Assume true for $1 \dots j-1$, prove for j.

$$R_1 R_2^{-1} = \begin{array}{c} \begin{array}{cccccc} & & & j-1 & j \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & c_{1j} \\ & 1 & 0 & \cdots & 0 & c_{2j} \\ & & 1 & \cdots & \vdots & \vdots \\ & & & & 1 & c_{j-1,j} \\ & & & & & c_{jj} \end{pmatrix} \end{array}$$

$\Rightarrow$ col j of $Q_2 = h_j = c_{1j} g_1 + c_{2j} g_2 + \dots + c_{j-1,j}\, g_{j-1} + c_{jj}\, g_j$.

Let $Q_2 = (h_1 | h_2 | \dots | h_n)$

$$(h_l | h_j) = 0 \quad \text{if } l \neq j$$

$h_1 = g_1$; $h_2 = g_2$; $\dots$; $h_{j-1} = g_{j-1}$

So $(h_j | g_k) = 0$ for $k = 1, \dots, j-1$

$\Rightarrow h_j = c_{jj}\, g_j$; $c_{jj} = +1$ (similar arg as earlier) $c_{jj} = r_{jj}\, r_{jj}'^{-1}$

//

Linear Functionals & Adjoints

To be done: Spectral Thm (complex version)
SVD
Sylvester's Thm about quadratic forms.

Lecture

Thm: 6

V fdips, $f \in V^*$. Then $\exists$ a unique $y \in V$ so

$f(x) = (x|y) \quad \forall\, x \in V$.

Proof

ON basis for V : $v_1 \ldots v_n$

Set $y := \sum_{i=1}^{n} \overline{f(v_i)}\, v_i$

$$\text{Then } (v_i|y) = \sum_j f(v_j)(v_i|v_j)$$

$$= \sum_j f(v_j)\delta_{ij}$$

$$= f(v_i)$$

So the functionals

$x \longmapsto (x|y)$ & f agree on a basis so are equal.

Uniqueness: suppose $y, y' \in V$ satisfy $(x|y) = (x|y') \;\forall\, x \in V$

$\Rightarrow (x|y-y') = 0 \;\forall x$

$\Rightarrow (y-y'|y-y') = 0 \;\Rightarrow\; y-y' = 0$ ∎

Thm 7

V fdips ; $T \in L(V,V)$. There is a unique $T^* \in L(V,V)$ so $(Tx|y) = (x|T^*y)$ $\forall\, x,y \in V$.

Proof Fix $x \in V$. Then $v \longmapsto (Tv|w)$ is functional

so $\exists$ unique $y \in V$ so $(v|y) = (Tv|w)$ $\forall\, v \in V$.

Define a fn. $T^* : V \to V$; $T^*(w) = y$ as before.

Want $T^* \in L(V,V)$

For $v \in V$, $(v\,|\,T^*(cw + c'w'))$

$$= (Tv\,|\,cw + c'w') = \bar{c}(Tv|w) + \bar{c}'(Tv|w')$$

$$= \bar{c}(v|T^*w) + \bar{c}'(v|T^*w')$$

$$= (v\,|\,cT^*w + c'T^*w')$$

$\Rightarrow T^*(cw + c'w') = cT^*w + c'T^*w'$

uniqueness follows from Thm 6.

■

Def. V IPS, $T \in L(V,V)$. Say T has an adjoint if $\exists\, T^* \in L(V,U)$ so $(Tv|w) = (v|T^*w)$ $\forall\, v,w \in V$.

Remark (see text).

If T has an adjoint T^*

then $(T^*)^* = T$

$$(cT)^* = \bar{c}\,T^*$$

$$(TU)^* = U^* T^*$$

Thm 8

V fdips, $\mathcal{B} = \{v_1 \dots v_n\}$ ON basis, $T \in L(V,V)$.

A = mtx for T. Then $a_{ij} = (Tv_j \mid v_i)$ where $A = (a_{ij})$

Proof

If $v \in V$, $v = \sum_{i=1}^{n} (v \mid v_i)\, v_i$

$Tv_j = \sum_i \underbrace{(Tv_j \mid v_i)}_{a_{ij}}\, v_i$ (col j of $A \to$ coeff of Tv_j w.r.t $v_1 \dots v_n$) ∎

Corollary

Mtx for T^* w.r.t same basis is $A^* = \bar{A}^t$.

Proof: i,j entry of mtx T^* is $(T^* v_j \mid v_i) = \overline{(v_i \mid T^* v_j)}$

$= \overline{(Tv_i \mid v_j)} = \overline{a_{ji}}$

Def T Hermitian $\iff T = T^*$

A Mtx $A_{n \times n}$ is Hermitian $\iff A = A^*$.

If $T \in L(V, V)$ has an adjoint T^*;

write
$$\left.\begin{aligned} U_1 &= \frac{1}{2}(T + T^*) \\ U_2 &= \frac{1}{2i}(T - T^*) \end{aligned}\right\} \text{Both are Hermitian.}$$

Then $T = U_1 + i U_2$

$$\rightarrow U_2^* = \left(\frac{1}{2i}(T - T^*)\right)^* = \frac{-1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = U_2$$

If $F = \mathbb{R}$

$A_{n \times n}$; $A = \frac{1}{2}\underbrace{(A + A^t)}_{\text{symmetric}} + \frac{1}{2}\underbrace{(A - A^t)}_{\text{skew symmetric}}$.

Def : V, W Ips/F. An isometry $T : V \to W$, is an invertible LT so that the inner products are preserved.

$$(x|y)_V = (Tx|Ty)_W. \rightarrow \text{(automatically gives a zero kernel for } T \Rightarrow \text{inv.)}$$

- If $V=W$, such a map is called <u>unitary</u>.
  (orthogonal if $F=\mathbb{R}$)

Using ON basis, unitary matrix: $A^*A = I$.
orthogonal matrix: $A^t A = I$.

> $GL(V)$ = units of ring $L(V,V)$ is a group under composition.

$U(V)$ = unitary subgroup of $GL(V)$.
↳ set of all unitary transformations $V \xrightarrow[IPS]{} V$.

- <u>Thm:</u> V, W n-dim IPS/F and $T \in L(V,W)$ then the following are equivalent.

i) T preserves IP_s
ii) T is an isometry.
iii) T takes each ON basis to an ON basis
iv) T takes one ON basis to an ON basis.

<u>Proof</u>: If T preserves $IP_s \Rightarrow Ker(T) = 0 \Rightarrow T$ is isomorphism of v.s $\Rightarrow T$ isometry

- ii) $\Rightarrow$ iii) $\quad v_1 \dots v_n \quad (v_i | v_j) = \delta_{ij} = (Tv_i | Tv_j)$
$Tv_1 \dots Tv_n$ is ON

iii) $\Rightarrow$ iv) Trivial.

Assume iv)

Let $v_1 \ldots v_n$ ON basis so $Tv_1 \ldots Tv_n$ is ON.

Take $x, y \in V$

$x = \sum a_i v_i$, $y = \sum b_j v_j$

$(x|y) = \sum_{i=1}^{n} a_i \overline{b_i}$

$Tx = \sum a_i Tv_i$; $Ty = \sum b_j Tv_j$

$(Tx|Ty) = \left(\sum a_i Tv_i \,\middle|\, \sum b_j Tv_j \right)$

$= \sum_{i,j} a_i \overline{b_j} \underbrace{(Tv_i | Tv_j)}_{(v_i | v_j) = \delta_{ij}}$

$= (x|y)$ ∎

<u>Corollary</u>

V, W fdips, then V, W are isometric iff $\dim(V) = \dim(W)$.

<u>Proof</u>: Existence of isometry $\Rightarrow \dim V = \dim W$.

Conversely assume $\dim(V) = \dim(W) = n$

Take $v_1 \ldots v_n$ ON basis of V

$w_1 \ldots w_n$ ON basis of W.

Then the set map $v_i \longmapsto w_i$; $i = 1 \ldots n$

extends to a unique LT.

It is an isometry because it takes ON basis to ON basis.

Thm 12

$U \in L(V, V)$; V IPS. Then U is unitary iff adj of U exists and is the inverse of U.

Proof

If U is unitary, it is invertible, so U^{-1} exists.

Claim: U^{-1} is an adjoint to U.

Calculate $(Uv | w) = (Uv | UU^{-1}w) = (v | U^{-1}w)$

because U is unitary.

$\Rightarrow U^* = U^{-1}$

Suppose U^* exists & $UU^* = U^*U = 1_V$.

Then $(Uv | Uw) = (v | U^*Uw) = (v | w)$

$\Rightarrow U$ is an isometry.

Ex $V = \mathbb{C}^{n \times 1}$

Define $f(X,Y) = Y^* A Y$ for mtx A is a sesquilinear form.

Define $U \in L(V,V)$ by $UX = AX$.

So $f(UX, UY) = (AX | AY) = (Y^* A^*) AX = Y^* (A^* A) X$.

This equals $f(X,Y)$ $\forall X, Y. \Leftrightarrow A^* A = I_n$.

> Given $J_{m \times n}$ $K_{m \times n}$. If $Y^* J X = Y^* K X$ $\forall X, Y$ Then $J = K$.

Thm 14

For an invertible $B_{n\times n}/F$ there is a unique lower triangular matrix M so MB is unitary.

Proof Follows from QR decomposition + uniqueness of QR.

Write $B^* = \underset{\substack{\text{orthonormal} \\ \text{columns} \Rightarrow \text{unitary}}}{Q} R \rightarrow$ upper tri with pos diag.

$\Rightarrow B = \underset{\substack{\text{lower} \\ \text{triang.} \\ \text{\& inv.}}}{R^*} Q^* \qquad (R^*)^{-1} B = Q^*$

Uniqueness follows from QR uniqueness.

$B^* M_1^* = U_1^* \; ; \; B^* = \underset{\text{unitary}}{U_1^*} \underset{\text{upper tri pos diag.}}{(M_1^*)^{-1}}$ ∎

> $T \in L(V,V)$, V fdips, Ask when T has ON eigenbasis

Def T is normal iff $TT^* = T^*T$

T is self adjoint $\Leftrightarrow T = T^*$

Theorem 15 If $T = T^*$, evals real & e-spaces of distinct evals are orthogonal.

Proof: $v \neq 0$ in V, $|v|=1$, $Tv = cv$

Then $c = (Tv|v) = (v|\underset{\overset{\shortparallel}{T}}{T^*}v) = (v|Tv) = \overline{(Tv|v)} = \bar{c}$.

$\Rightarrow c$ is real.

Take $w \in V$, $|w| = 1$, $Tw = dw$, $d \neq c$.

We have, $c(v|w) = (cv|w) = (Tv|w) = (v|T^*w)$

$$= (v|Tw) = (v|dw) = \bar{d}(v|w) = d(v|w)$$

$$\Rightarrow \quad 0 = \underbrace{(c-d)}_{\neq 0} \Rightarrow \underbrace{(v|w)}_{=0} .$$

<u>Thm 16</u>

$V \neq 0$ fdips, if $T = T^* \in L(V,V)$, Then there is an eigenvector. Also, the characteristic polynomial has real roots.

<u>Proof</u>

Let $\mathcal{B}$ ON basis. T has matrix A, T^* has matrix A^*.

Let $c(x)$ = char poly of A.

If $F = \mathbb{C}$, $c(x)$ has a root in F. so eigenvector exists

Suppose $F = \mathbb{R}$, consider A to be a complex matrix, so there is a scalar $c \in \mathbb{C}$ so $\ker(A - cI) \neq 0$, $\Rightarrow \exists$ a complex eigenvector.

Thm 15 $\Rightarrow$ c is real.

$\Rightarrow$ The linear system $\underbrace{(A - cI)}_{\text{mtx}/\mathbb{R}} x = 0$ has a real solution.

Thm 17

V fdips, $T \in L(V,V)$. If W is a T-inv subspace, $W^\perp$ is T^*-invariant.

Pf: Let $y \in W^\perp$, $w \in W$. We want $(w|T^* y) = 0$.

$(\underset{\underset{W \to T\text{inv}}{\cap}}{Tw}|y) \Rightarrow$ This is 0

Thm 18 Assume V fdips, $T = T^* \in L(V,V)$. Then T has an orthonormal eigenbasis.

Proof: Let v_1 be a unit eigenvector for T (use Thm 16).

Then $v_1^\perp$ is invariant under $T^* = T$.

Also, $(\cdot|\cdot)$ restricted to $v_1^\perp$ is an inner product and S, the LT on $v_1^\perp$ induced by T is self adjoint.

We use induction ($n=1$ trivial). $\text{Dim}(v_1^\perp) = n-1$, where $n = \dim(V)$.

There exists ON eigen basis $v_2 \dots v_n$ for S. So $v_1, v_2, \dots, v_n$ is ON eigenbasis for T.

(Why is S self adjoint? $(Tx|y) = (x|Ty)\ \forall x, y \in V$ & $T = S$ in the restriction to $v_1^\perp$ So S is self adjoint)

Thm 19

V fdips, $T \in L(V,V)$, T normal. If $v \in V$, $v \neq 0$ is an eigenvector for T, $Tv = cv \Leftrightarrow v$ is evec for T^*, $T^* v = \bar{c} v$.

Pf: Let $U \in L(V,V)$ be normal.

Then $|Uv| = |U^* v| \quad \forall\, v \in V$.

Reason: $(Uv | Uv) = (v | U^* U v) = (v | U U^* v)$

$= (U^* v | U^* v)$ (because $(U^*)^* = U$).

Now, take $U := T - c \cdot 1_V$, normal.

$\left(U^* = (T - c 1_V)^* = T^* - (c 1_V)^* = T^* - \bar{c}\, 1_V\right)$

Then $|(T-c)v| = |(T^* - \bar{c})v|$

Get LHS $= 0 \Leftrightarrow$ RHS $= 0$. ∎

Thm 20 V fdips, $T \in L(V,V)$; $\mathcal{B}$ ON basis. Suppose $A =$ mtx of T w.r.t $\mathcal{B}$, is upper triangular. Then T is normal iff A is diagonal.

Pf If A is diagonal so is $A^* \Rightarrow A$ & A^* commute

$\Rightarrow T, T^*$ commute.

Now suppose T is normal. Write $\mathcal{B} = \{v_1 \ldots v_n\}$

Since $A = (a_{ij})$ is upper triangular,

$$Tv_1 = a_{11} v_1 \Rightarrow T^* v_1 = \overline{a_{11}}\, v_1 \qquad A = \begin{pmatrix} \diagdown & \xi \\ 0 & \diagdown \end{pmatrix}$$

Also, since A^* is the mtx for T^*,

$$T^* v_1 = \sum_j (A^*)_{j,1}\, v_j$$

$$A^* \begin{pmatrix} \diagdown & 0 \\ \xi & \diagdown \end{pmatrix}$$ Compare to get $(A^*)_{j,1} = 0$ for $j \geq 2$.

$\overline{a}_{1,j}$

$\Rightarrow a_{1,j} = 0$ for $j = 2, \ldots, n$.

$$\begin{pmatrix} a_{11} & 0 & 0 & \cdots \\ 0 & a_{22} & & \\ \vdots & \bigcirc & & ? \end{pmatrix}$$

Same reasoning shows $A v_2 = a_{2,2} v_2$

$\Rightarrow v_2$ is an evec for A.

$\Rightarrow A^* v_2 = \overline{a}_{22} v_2$

Compare $A^* v_2 = \sum_j (A^*_{j,2})\, v_j$

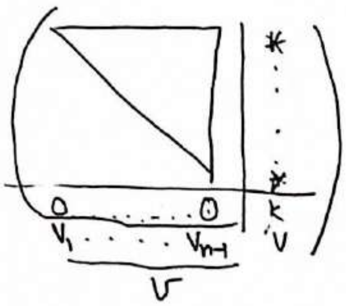
$\Rightarrow A^*_{j,2} = 0$ for $j \geq 3$.

$\overline{a}_{2,j} \Rightarrow a_{2,j} = 0$ for $j \geq 3 \Rightarrow A \begin{pmatrix} a_{11} & 0 & 0 & \cdots \\ 0 & a_{22} & 0 & \cdots \\ 0 & 0 & & \cdots \\ \vdots & \vdots & & \end{pmatrix}$

Keep going ... (To be contd.)

Thm 21 V fdips $/\mathbb{C}$; $T \in L(V,V)$, Then there is an ON basis so that mtx of T is upper Triangular.

Proof There is a unit eigenvector v for T^*.
Then T leaves $v^\perp$ invariant. There is an ON basis for $v^\perp$, Say $v_1 \dots v_{n-1}$ so that mtx S induced on $v^\perp$ by T is upper triangular.



So mtx for T w.r.t $v_1 \dots v_{n-1}, v$ is upper triangular. ∎

Corollary: in Thm 21, if in addition T is normal, the mtx is diagonal. This essentially completes the proof of Spectral Thm.

Spectral Thm (Complex version)

$T \in L(V,V)$ has ON eigenbasis $\iff TT^* = T^*T$ (normal).
(Hermitian is subsumed in this)

Spectral thm (Real version)

$T \in L(V,V)$, V fdips $/\mathbb{R}$ then $T = T^t \iff T$ has ON eigenbasis.

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ x^2+1 diag $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$
$A \neq t$

over $\mathbb{C}$ $A^* = A^t = -A$ is normal. get ON ebasis conj to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Singular Value Decomposition (SVD)

$A_{m\times n}$ /$\mathbb{C}$, $\exists$ unitary matrices $U_{m\times m}$, & $V_{n\times n}$ &

$\Sigma_{m\times n} = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & 0 \\ & & \ddots & \\ 0 & & & \sigma_r \end{pmatrix}$ $r = \mathrm{rank}(A)$ & $\underbrace{\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0}_{\text{Singular values of } A.}$

$$\boxed{A = U \Sigma V^*}$$

A^*A is $n\times n$ & Hermitian. $(A^*A)^* = A^*A \Rightarrow$ use spectral thm.

$\exists$ ON eigenbasis $v_1 \ldots v_n$, evals $\lambda_1 \ldots \lambda_n$. these are real.

Index so $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq 0$.

If v evec. $A^*Av = \lambda v \Rightarrow v^*A^*Av = \lambda \underbrace{v^*v}_{>0}$

$0 \leq Av . Av \Rightarrow \lambda \geq 0$

Let $\lambda_1, \ldots, \lambda_r$ be positive & $\lambda_{r+1} = \ldots = \lambda_n = 0$; $\underbrace{\sigma_i}_{\text{are the singular values of } A} = \lambda_i^{1/2}$ for $i = 1 \ldots r$

We define $u_1 \ldots u_r \in \mathbb{C}^{m\times 1}$, by the equation $\sigma_i u_i = Av_i$. Calculate

$Av_i . Av_j = (Av_j)^* Av_i = v_j^* \underbrace{A^*Av_i}_{\lambda_i v_i} = \lambda_i \underbrace{v_j^* v_i}_{\delta_{ij}}$

$i = j \Rightarrow Av_i\, Av_i = \lambda_i = \sigma_i^2 > 0$

$\Rightarrow u_1, \ldots, u_r$ is ON Seq. } Complete to $u_1 \ldots u_m$ ON basis (using GS.) on $\mathbb{C}^{m\times 1}$.

$$U = (u_1 | u_2 | \dots | u_m)$$

$$V = (v_1 | v_2 | \dots | v_n)$$

$$\Sigma = \begin{pmatrix} \sigma_1 \sigma_2 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{pmatrix}_{m \times n}$$

<u>Example</u> $A = (1, i)$; $A^* A = \begin{pmatrix} 1 \\ -i \end{pmatrix} (1, i) = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

$\exists$ evecs for 0 : $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\left(\begin{pmatrix} p \\ q \end{pmatrix} \Big| \begin{pmatrix} -\bar{q} \\ \bar{p} \end{pmatrix} \right) = -pq + pq = 0$

other evec : $\frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} \rightarrow \lambda = 2$.

$\Rightarrow$ evals : $2, 0$; singular values are $\sqrt{2}$.

$\Sigma = (\sqrt{2}, 0)$; $V = \frac{1}{\sqrt{2}} \begin{pmatrix} \underbrace{\begin{matrix} i \\ 1 \end{matrix}}_{\substack{\text{evec} \\ \text{for } 2}} & \begin{matrix} 1 \\ i \end{matrix} \end{pmatrix}$ $\Rightarrow V^* = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$

$\sigma_1 u_1 = \sqrt{2} u_1 = A v_1 = (1, i) \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix} = \sqrt{2} (i) \Rightarrow u_1 = (i)$

$\Rightarrow U = (i)$

$\Rightarrow (1, i) = (i) (\sqrt{2}, 0) \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} = (i)(-i, 1) = \underline{\underline{(1, i)}}$ ✓

Proof that $A = U\Sigma V^*$ in general.

$$AV = A(v_1|v_2|\ldots|v_n) = (Av_1|Av_2|\ldots|Av_n)$$

$$= (\sigma_1 u_1|\sigma_2 u_2|\ldots|\sigma_r u_r|0|0\ldots|0)$$

$$= (u_1|u_2|\ldots|u_m)\begin{pmatrix}\sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_r\end{pmatrix} = U\Sigma$$

So $AV = U\Sigma$

$$\Rightarrow A\underbrace{VV^*}_{I} = U\Sigma V^*$$

$$\Rightarrow \boxed{A = U\Sigma V^*}$$

Given $A = U\Sigma V^*$ by SVD, apply $A^* = (U\Sigma V^*)^*$

$$= V\Sigma^t U^*$$

So $\begin{pmatrix}1\\-i\end{pmatrix} = (1, i)^*$ has SVD $\frac{1}{\sqrt{2}}\begin{pmatrix}i & 1\\1 & i\end{pmatrix}\begin{pmatrix}\sqrt{2}\\0\end{pmatrix}(-i)$.

$B = \begin{pmatrix}1\\-i\end{pmatrix}$; $B^*B = (2) \Rightarrow v_1 = (1)$ $\lambda_1 = 2 \Rightarrow \sigma_1 = \sqrt{2}$

$Bv_1 = \begin{pmatrix}1\\-i\end{pmatrix}(1) = \begin{pmatrix}1\\-i\end{pmatrix} = \sigma_1 u_1 \Rightarrow u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

Complete u_1 to an ON basis ; say $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ to make it $\perp$ to u_1

$\Rightarrow U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\-i & i\end{pmatrix}$; $\Sigma = \begin{pmatrix}\sqrt{2}\\0\end{pmatrix}$; $V^* = (1) \Rightarrow U\Sigma V^* = \begin{pmatrix}1\\-i\end{pmatrix}$

Applications.

If $r = \text{rank}(A)$, $\sigma_1 \ldots \sigma_r$ are the singular values

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$$

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

Ex: G Strang.

$A_{1000 \times 1000}$ Say $\sigma_1 \ldots \sigma_{10}$ large & σ_j is small $j \geq 11$.

Then $A \approx \sum_{i=1}^{10} \sigma_i \underbrace{u_i}_{10} \underbrace{v_i^*}_{10}$

each has 1000

need 20,000 scalars to describe the $\sigma_i u_i$ & v_i

$$\Rightarrow \frac{20{,}000}{10^6} = 0.02 \Rightarrow 2\%$$

SVD contd.

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^*$$

Proof Enough to show LHS, RHS agree on a basis.

$A v_j = \sigma_j u_j$ from SVD procedure.

$$\sum_{i=1}^{r} \sigma_i u_i \underbrace{v_i^* v_j}_{\delta_{ij}} = \sigma_j u_j$$

$\blacksquare$

Symmetric Bilinear Forms

$f : V \times V \longrightarrow F$

linear in each variable. F field of char $\neq 2$.

A quadratic form is a function $q : V \to F$ of shape

$q(x) = f(x,x)$

$$q(x+y) = f(x+y, x+y) = f(x,x) + f(x,y) + f(y,x) + f(y,y)$$
$$= q(x) + 2f(x,y) + q(y).$$

$$q(x-y) = q(x) - 2f(x,y) + q(y).$$

$$\Rightarrow \quad q(x+y) - q(x-y) = 4f(x,y)$$

Char $F \neq 2$, $\longrightarrow$ (char can only be 0 or prime)

$$f(x,y) = \frac{1}{4}\big(q(x+y) - q(x-y)\big)$$

> A bilinear function (not nec. symmetric) on fdvs is given by a matrix

$\mathcal{B}$ basis $\{v_1 \ldots v_n\}$. $f(v_i, v_j) =: a_{ij}$; $A = (a_{ij})$

$x \in V \quad [x]_{\mathcal{B}} = X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\Longleftrightarrow \quad x = \sum_{i=1}^{n} x_i v_i$$

$$f(x,y) = X^t A Y.$$

(7)

(209)

What if basis changes?

$X = [x]_{\mathcal{B}} \quad X' = [x]_{\mathcal{B}'}$

If $X' = SX$; $Y' = SY$; S invertible.

Then if $A' =$ mtx for f wrt $\mathcal{B}'$.

then $f(x,y) = X'^t A' Y'$

$$= (SX)^t A' (SY)$$

$$= X^t \underbrace{(S^t A' S)}_{A} Y$$

$$\Rightarrow A = S^t A' S$$

$$\Rightarrow \det(A) = \det(S^t) \det(A') \det(S)$$

$$\Rightarrow \boxed{\det(A) = \det(S)^2 \det(A')}$$

Thm: V fdvs /F field of char $\neq 2$. If $f: V \times V \to F$ is a symmetric bilinear form. Then there is a basis which diagonalizes f, i.e. (mtx A is diagonal).

Remark: We saw $\underset{\text{quad forms}}{QFs} \xleftrightarrow[\text{bijection}]{} \underset{\text{sym bil forms.}}{SBFs}$

if $f=0$, then $q=0$.

Assume $q = 0 \Rightarrow 4f(x,y) = 0 \ \forall x, y$

$\Rightarrow f = 0$ since $4 \neq 0$.

Proof of thm:

Say $f = 0$. Trivially $A = 0$.

Next assume $f \neq 0$. Then $\exists v$ s.t $q(v) \neq 0$.

Let $W = v^{\perp} = \{x \in V \mid f(x,v) = 0\}$

This has dim. $n-1$ since $x \longmapsto f(x,v)$ is a nonzero linear functional on V.

Let $v_2 \dots v_n$ be any basis of W. Let $v_1 = v$. Change basis for f to $v_1 \dots v_n$.

mtx A changes to $\underbrace{S^t A S}_{A'}$, also symmetric.

So $A' = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & B & \\ 0 & & & \end{pmatrix}$

$c = f(v,v) \neq 0$.

By induction on dimension

$\exists$ basis $v_2', \dots, v_n'$ of W so if Q represents change of basis

$v_1, \dots, v_n \longrightarrow v_1', \dots, v_n'$ $(v_1' = v_1)$. Then $Q^t B Q = \text{diag}$.

$$\Rightarrow \begin{pmatrix} 1 & \\ & Q \end{pmatrix}^t S^t A S \begin{pmatrix} 1 & \\ & Q \end{pmatrix} = \text{diag}$$

- Take $P := S \begin{pmatrix} 1 & \\ & Q \end{pmatrix}$

$\Rightarrow$ $P^t A P$ is diag ∎

> Take $F = \mathbb{R}$. Then there is a basis so f is represented by matrix

$$\begin{pmatrix} c_1 & & & & \\ & \ddots & & & \\ & & c_\ell & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} \qquad c_1 \neq 0, \dots, c_\ell \neq 0.$$

Also we may arrange for each c_i to be $+1$ or -1.

$\underbrace{f(v,v)}_{q(v)} = c$. Then $q\left(\frac{v}{|c|^{1/2}}\right) = \frac{c}{|c|} = \pm 1$.

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & 0 \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix}$$

This gives subspaces P, Q, R so $f|_{P\times P} : P \times P \to \mathbb{R}$ is positive definite. $f|_{Q\times Q}$ is neg def. $f|_{R\times R} = 0$.

If $\lambda_1 \ldots \lambda_m$ occur in diag mtx for some SBF w.r.t some basis.

$$q\left(\sum x_i v_i\right) = f\left(\sum x_i v_i , \sum x_j v_j\right) = \sum_{i=1}^{n} a_{ii} x_i^2 + \sum_{i \neq j} \underset{\overset{\|}{0}}{a_{ij}} x_i x_j$$

If all λ_i are positive then $q(x) > 0$ for $x \neq 0$. $\rightarrow$ pos def

" $\lambda < 0$ " $\Rightarrow$ $q(x) < 0$ " " $\rightarrow$ neg def

" $= 0$ " $\Rightarrow$ " $= 0$ " " $\rightarrow$ zero

If some λ_i pos, some neg $\Rightarrow$ indefinite ; q takes both $+, -$ values

---

<u>Sylvester Thm.</u>

If V fdvs $/\mathbb{R}$ & s SBF ; f is represented by a mtx

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} \begin{matrix} \}p \\ \}q \\ \}r \end{matrix}$$

& also by matrix

$$\begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ & & & & \ddots & & & \\ & & & & & -1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} \begin{matrix} \}p' \\ \}q' \\ \}r' \end{matrix}$$

. Then $p = p'$, $q = q'$, $r = r'$.

Proof

The two mtx representations refer to orthogonal direct sums

$\underset{\dim \to p}{V} = P \perp \underset{q}{Q} \perp \underset{r}{R}$.

$\underset{\dim p'}{V} = P' \perp \underset{q'}{Q'} \perp \underset{r'}{R'}$.

($W_1 \perp W_2$ means $W_1 \oplus W_2$ & $f(x,y) = 0$ if $x \in W_1$; $y \in W_2$).

Define $N := Q + R$. $\Rightarrow V = P \perp N$.

f is neg semi def on N.

$$q(\underset{\underset{Q}{\cap}}{y_1} + \underset{\underset{R}{\cap}}{y_2}) = \underset{\underset{\text{if } y_1 \neq 0}{\underset{0}{\wedge}}}{q(y_1)} + \underset{\underset{0}{\parallel}}{q(y_2)} \text{ for all } y_2.$$

$\Rightarrow \underset{\underset{\text{pos def}}{q\uparrow}}{P'} \cap \underset{\underset{\text{neg semi}}{q\uparrow}}{N} = 0 \Rightarrow p' + q + r \leq n$

Similarly $p + q' + r' \leq n$

adding $\underbrace{p+q+r}_{n} + \underbrace{p'+q'+r'}_{n} \leq 2n$

$\Rightarrow p' + q + r = n$ & $p + q' + r' = n$

$\Rightarrow p = p'$

Similarly $q = q'$. $\Rightarrow r = r'$

Also $R = R'$ is called the <u>radical of f</u> $= \{x \in V \mid f(x,y) = 0 \ \forall y \in V\}$.

Signature of a q.f here means either (p, q, r) or $p-q$.

END

MATH 420 - Summary notes.

Def: Group ($\cdot$) associative, identity, inverse.

abelian $\Rightarrow$ commutative.

Def: Ring $(+, \cdot)$ $(R, +) \rightarrow$ abelian

$(R, \circ) \rightarrow$ associative, identity, distributive under addition.

R commutative $\Rightarrow$ $(R, \circ)$ commutative

Def: Field $(+, \cdot)$ Commutative ring with multiplicative inverse.

ERO Matrices.

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tc & b+td \\ c & d \end{pmatrix}$$

Here $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ this row $(1, 2)^{+}$ this row $\Rightarrow$ this row.

$\begin{pmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & c_3 \end{pmatrix}$ multiplies each row by c_j.

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ interchanges row 2 & 3.

Def: Inverse pair

A, B $n \times n$ matrices. , $AB = BA = I_n \Rightarrow A, B$ are inverse pairs.

→ Inverse of ERO is another ERO.

→ After forming RREF, leading indeces are dep. var. & others are independant variables.

Thm: $A_{n\times n} \Rightarrow$ following are equivalent.

1) A invertible (has 2 sided inverse)
2) A is row equivalent to I_n.
3) A is a product of EROMs.
4) $A\vec{x}=0$ has only trivial solution.
5) $A\vec{x}=\vec{y}$ solvable for all $\vec{y}$.

Thm: If $AB=I$, then $BA=I$ for A, B $n\times n$ matrices.

thm: Suppose $A_1, \ldots, A_k$ are $n\times n$ matrices, then $A = A_1 A_2 \ldots A_k$ is invertible iff each A_j is invertible.

Def: Vector space: Set V closed under addition & scalar multiplicat s.T $(V,+)$ abelian & $(V,\times)$ associative, identity, distributive.

Def: Subspace: Subset W of V s.T $\vec{0} \in W$ & closed under LC.
($\Rightarrow$ if $\vec{x},\vec{y} \in W$, then $a\vec{x}+b\vec{y} \in W\ \forall a,b \in F$)

Def: Span: Set of all LCs from finite sequences in S.

Lemma: Intersection of subspaces forms a subspace.

Def: Subspace generated by subset $S \subseteq V$: $\bigcap W$ s.T W subspace V and $W \supseteq S$.

Thm: $S \neq 0 \Rightarrow$ span(S) = subspace generated by S.

R3

Def: S, T subsets of V vs/F $\Rightarrow S+T := \{x+y \mid x \in S, y \in T\}$.

Lemma: W_1, W_2 subspaces $\Rightarrow W_1 + W_2$ subspace.

Also $W_1 + W_2 = \text{span}(W_1 \cup W_2)$.

Def: Linearly dependent

A sequence of finite length $\vec{V}_1 \ldots \vec{V}_2$ is LD if $\exists$ scalars $c_1 \ldots c_n$ not all 0 s.t $\sum_{j=1}^{n} c_j \vec{V}_j = 0$.

Def: LD subset S

If $\exists$ a sequence of distinct vectors in S which is LD, then S is LD.

Def Basis

A basis for V is a LI set which spans V.

Def V is finite dimensional if it is spanned by a finite set

Thm: V v.s $\vec{V}_1 \ldots \vec{V}_p$ is LI seq. $\vec{W}_1 \ldots \vec{W}_q$ is a sequence. Suppose $\vec{v}_1 \ldots \vec{v}_p$ is in span $\{\vec{w}_1 \ldots \vec{w}_q\}$ then $p \le q$.

Corollary: If V has basis of length m, any basis of V has length m.

Def: Dimension: Number of elements in a basis for V.

Lemma: **Casting out Lemma.**

V vs and $r \geq 2$. $\vec{v}_1, \dots, \vec{v}_r$ LD sequence. $\exists j$ s.t

$\vec{v}_j$ = L.C of $\vec{v}_1 \dots \vec{v}_{j-1}$ and $\text{span}\{\vec{v}_1 .. \vec{v}_j\} = \text{span}\{\vec{v}_1 .. \vec{v}_{j-1}\}$

and $\text{span}\{\vec{v}_1, \dots \vec{v}_r\} = \text{span}\{\vec{v}_1, \dots, \hat{\vec{v}}_j, \dots \vec{v}_r\}$

$= \text{span}\{\{\vec{v}_1, \dots \vec{v}_r\} - \{\vec{v}_j\}\}$

Corollary: Every f.d.v.s has a basis.

Corollary: $\vec{v}_1 \dots \vec{v}_m$ LI & if $w \notin \text{span}\{\vec{v}_1 .. \vec{v}_m\} \Rightarrow \vec{v}_1, \dots, \vec{v}_m, \vec{w}$ is LI.

Thm: ① V_{fdvs} & W subspace $\Rightarrow$ W_{fdvs} & $\dim W \leq \dim V$

② $W_1 \leq W_2$ are subspaces of V_{fdvs}. then $\dim(W_1) \leq \dim(W_2)$

subspace

& $\underbrace{W_1 = W_2}$ iff $\dim W_1 = \dim W_2$

This is just means W_1 is not a proper subset under the original assumption of $W_1 \leq W_2$

Def: Notation: $[x]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \Rightarrow \vec{x} = \sum_{i=1}^{n} c_i \vec{v}_i$ where v_i form basis $\mathcal{B}$

* **Lemma:** $\vec{v}_1 \dots \vec{v}_p$ basis of subspace W of V. $\vec{v}_1 \dots \vec{v}_p \dots \vec{v}_n$ basis of V.

If $\vec{x} \in V$, $\vec{x} = \sum_{i=1}^{n} c_i \vec{v}_i$. Then $\vec{x} \in W \iff c_i = 0 \ \forall \ i > p$.

Thm (Dimensional Formula)

V(vs), W_1, W_2 fd subspaces. Then $W_1 + W_2$ and $W_1 \cap W_2$ are fd.

and $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

R5

Def Coordinates

- $V_{(v.s)}$ with basis $\vec{v}_1, \ldots, \vec{v}_n$. $\vec{x} \in V \Rightarrow \exists$ a unique sequence $c_1 \ldots c_n$ so $\vec{x} = \sum_{i=1}^{n} c_i \vec{v}_i$. This sequence is called the coordinate sequence for $\vec{x}$ w.r.t basis $\vec{v}_1 \ldots \vec{v}_n$

Lemma: $\vec{v}_1 \ldots \vec{v}_n$ is a basis of V iff every vector in V has a unique coordinate sequence

Notation: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \vec{x} = \sum_{i=1}^{n} x_i \vec{v}_i$.

> Given two bases $\mathcal{B} : \vec{v}_1 \ldots \vec{v}_n$ & $\mathcal{B}' : \vec{v}_1{}' \ldots \vec{v}_n{}'$. Then

- $$\vec{v}_j{}' = \sum_{i=1}^{n} P_{ij} \vec{v}_i{}' \quad \text{where } P_{n \times n} = P_{ij}$$

and $P[\vec{x}]_{\mathcal{B}'} = [x]_{\mathcal{B}}$.

Claim: P is invertible.

Def: Rowspace: Span of Rows of A in $F^{1 \times n}$

Column space: Span of Columns of A in $F^{m \times 1}$

> EROs preserve rowspace and not column space.

- Def: Echelon basis: non zero rows of the RREF of a matrix form an echelon basis.

Thm: W subspace of $F^{1\times n}$ has a unique echelon basis.

Corollary: $A_{m\times n}$ RREF is unique.

Ch 3: Linear Transformations

Def: A function $T: V \longrightarrow W$ is a LT iff

$T(ax+by) = aTx + bTy \quad \forall x, y \in V$, scalars a, b

Thm: $T_{LT}: V \to W \Rightarrow$ kernel $(T) := \{\vec{x} \in V \mid T\vec{x} = 0\}$ is a subspace of V and $\mathrm{Im}(T) := \{T\vec{x} \mid \vec{x} \in V\}$ is a subspace of W.

Thm: V_{fdvs} basis $\vec{v}_1 \ldots \vec{v}_n$; $W_{v.s}$ with elements $\vec{w}_1 \cdots \vec{w}_n$

then $\exists$ a unique LT $T: V \longrightarrow W$ so $T\vec{v}_i = \vec{w}_i$ for $i = 1 \ldots n$.

Def: Restriction $f(X \longrightarrow Y)$, $f_0\,(X_0 \longrightarrow Y_0)$ if $X_0 \subseteq X$ &

$f_0(\vec{x}) = f(\vec{x}) \;\forall\; \vec{x} \in X_0 \quad \Rightarrow f_0$ is a restriction of f.

Def Rank: Dimension of the image of T.

Nullity: Dimension of null space / kernel of T.

Thm (Rank + Nullity Thm): $V_{fd}, W_{v.s}, T \in \overset{\to LT}{L}(V, W)$

$\Rightarrow \dim(V) = \text{rank } T + \text{nullity } T.$

Def: Rowrank: dim(rowspace)

Columnrank: dim(column space)

Rank of A : $x \longrightarrow Ax$ is the column rank.

Thm: Rowrank = column rank.

Thm: $n = \dim V$, $m = \dim W$, then $\mathcal{L}(V, W)$ has dim mn.

Def: $T \in \mathcal{L}(V, W)$ one-one if $x, y \in V$ and $TX = TY \Rightarrow X = Y$.

$\Longleftrightarrow \ker(T) = 0$

(since if $T(u) = T(v)$ and $u \neq v \to T(u-v) = 0$ *)

Def: T is onto $\Rightarrow \operatorname{Im}(T) = W$.

Thm: T is one-one iff whenever $\vec{v}_1, \ldots \vec{v}_r$ is a LI seq then so is $Tv_1, \ldots, Tv_r$.

Thm: V, W fdvs, $\dim V = \dim W$ if $T \in \mathcal{L}(V, W)$ the equivalent are:

1) T is one-one
2) T is onto
3) T is an isomorphism ($\Rightarrow$ LT with 2 sided inverse).

Def: Representation of a LT by a matrix.

V, W fdvs $/F$; Basis $\underbrace{\{v_1 \dots v_n\}}_{\mathcal{B}}$ & $\underbrace{\{w_1 \dots w_m\}}_{\mathcal{B}'}$. $T \in L(V, W)$

$$\boxed{T v_j = \sum_{i=1}^{m} a_{ij} w_i \quad \forall j.}$$ * Here $A = (a_{ij})_{m \times n}$ represents a matrix.

$$A = [T]_{\mathcal{B}\mathcal{B}'}.$$

$$T \longmapsto [T]_{\mathcal{B}\mathcal{B}'} \text{ is an isomorphism.}$$

Thm: $\underset{\{u_j\}}{U} \xrightarrow{S} \underset{\{v_i\}}{V} \xrightarrow{T} \underset{\{w_k\}}{W}$ $\quad \begin{matrix} T \mapsto A \\ S \mapsto B \end{matrix} \quad \Rightarrow TS \mapsto C = AB.$

Note: Relationship between matrices & bases.

$T: L(V, W)$

$$\begin{tikzcd} V \; (\mathcal{B}, \mathcal{B}') \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] & W \; (\mathcal{C}, \mathcal{C}') \end{tikzcd}$$

$\mathcal{B}\mathcal{B}'$ with P; $\mathcal{C}\mathcal{C}'$ with Q.

$[T]_{\mathcal{B},\mathcal{C}} = A \qquad [T]_{\mathcal{B}',\mathcal{C}'} = B.$

P relates $\mathcal{B} \to \mathcal{B}'$, Q relates $\mathcal{C} \to \mathcal{C}'$

$$\Rightarrow \boxed{B = Q^{-1} A P.}$$

$$\begin{tikzcd} V \; \mathcal{B} \arrow[d, "P"'] \arrow[r, bend left, "A"] & \mathcal{C} \; W \arrow[d, "P'"] \\ \mathcal{B}' \arrow[r, bend right, "B"'] & \mathcal{C}' \end{tikzcd}$$

Def: A <u>linear functional</u> on V is a member of $L(V, F)$ where V vs/F.

(Maps vectors to scalars). Ex: $l(g) = \int_a^b g(t)\,dt$

Def: <u>Dual space</u>.

If V is a v.s the collection of all linear functionals on V form a vector space V^* i.e $V^* := L(V, F)$, called the dual space of V.

$\dim(V^*) = \dim(V) \cdot \dim(F) = \dim(V)$

Def: <u>Duality</u>

Sequences $v_1 \ldots v_n$ in V & $f_1 \ldots f_n$ in V^* are in <u>duality</u> if $f_i(v_j) = \delta_{ij}$.

Thm: If $v_1 \ldots v_n$ & $f_1 \ldots f_n$ are in duality, $v_1 \ldots v_n$ is LI in V & $f_1 \ldots f_n$ is LI in V^*.

Def: V fdvs with basis $v_1 \ldots v_n$, the dual basis is $f_1 \ldots f_n$ i.e. basis of V^* S.T $f_i(v_j) = \delta_{ij}$. (By dimension consideration it is a basis of V^* since it is LI

* **Thm:** V fdvs with basis $\vec{v}_1 \dots \vec{v}_n$. Dual space V^* with basis $f_1 \dots f_n$.

For $\vec{v} \in V$, $\vec{v} = \sum_{j=1}^{n} f_j(\vec{v}) \vec{v}_j$ (Vector is Σ of product of basis & dual basis applied on vector)

For $g \in V^*$, $g(\vec{v}) = \sum_{j=1}^{n} g(\vec{v}_j) f_j$ (functional is Σ of product of functional on basis & dual basis).

Note: $f \in V^*$, $f \neq 0$, $Im(f) = F \Rightarrow Ker(f)$ is a proper subspace ($\because f \neq 0$,

$\dim(Ker(f)) = n-1$ since $rank(f) = 1$.

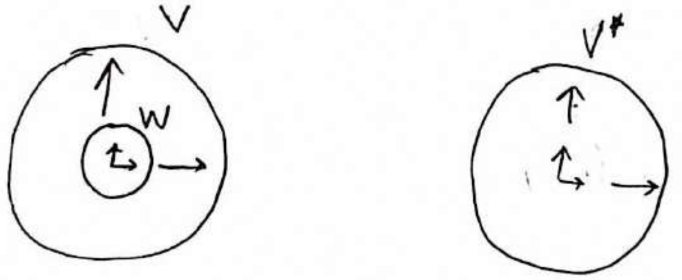
$Codim(Ker(f)) = 1 \rightarrow \dim(V) - \dim(W)$ if W subspace of V.

Hyperspace (subspace with codim $= 1$).

Def: Annihilator

S subset of V, $S^\circ := \{f \in V^* \mid f(\vec{x}) = 0 \ \forall \vec{x} \in S\}$.

thm: $\dim V = n$; W subspace of V; then $\dim(W) + \dim(W^\circ) = n$.
(Proof is elegant).



$\boxed{\text{Key: } g = \Sigma g(w_i) f_i}$

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Note: W (dim m) is the intersection of $n-m$ hyperspaces.

Lemma: W subspace of V, $W \neq V$, $x \in V - W$. Then $\exists f \in V^*$ so $f/W = 0$ and $f(x) \neq 0$.

restriction.

Corollary: W_1, W_2 subspaces of fdvs V. Then $W_1 = W_2 \iff W_1^0 = W_2^0$.

Double dual: $L_v(f) : f(v)$; $L_v \in L(V, V^{**})$

If $L_v = 0 \Rightarrow f(v) = 0 \Rightarrow v = 0 \Rightarrow L$ is an isomorphism.

> E^0 is a subspace of V which annihilates subset E in V^* & is mapped using L^{-1} from V^{**}. $\dim(E) + \dim(E^0) = \dim(V)$.

Thm: $S \subseteq V_{fdvs}$. $S^{00} = (S^0)^0$ is the span of S in V.
$\underset{V^*}{\cap}$

Def: Hyperspace is a maximal proper subspace. i.e. subspace W of V, $W \neq V$ satisfying: if U is a subspace $W \subseteq U \subseteq V$, then $U = W$ or $U = V$.

Lemma: If $f \in V^*$, $f \neq 0$ then $\ker(f)$ is a hyperspace.

Lemma: If W is a hyperspace in V then $\exists f \in V^*$, $f \neq 0$ s.t $W = \ker(f)$. (Pf: write $x = \underset{W}{w} + \underset{F}{cy}$. $f(w+cy) = c \rightarrow$ show this is Linear & $\ker f = W$

Lemma: Let $f, g \in V^*$, then g is a multiple of f $\Leftrightarrow$ $\ker(g) \supseteq \ker(f)$.

Pf: Choose $c = \frac{g(y)}{f(y)}$ where $y \in V \setminus K \rightarrow \ker(f), \ker(g)$

$\Rightarrow h(y) = g(y) - c f(y) = 0$ & $h(K) = 0 \rightarrow h = 0 \Rightarrow g = \underline{\underline{cf}}$

Thm Let $g, f_1, \ldots, f_r \in V^*$. $K = \ker(g)$, $K_j = \ker(f_j)$ then

g is a LC of $f_1 \ldots f_r \Leftrightarrow K \supseteq \bigcap_{j=1}^{r} K_j$.

Def: Transpose.