Nonlinear Dynamics and Chaos.

Physics 413 and Youtube course by Strogatz. MAE 5790

MAE 5790 Lec 1 & Phy413 Lec 1

Logical structure of dynamics.

Differential equations $\dot{x} = f(x)$ $x \in R^n \rightarrow$ Phase/State space.

$x = (x_1, \ldots, x_n)$

→ In components, $\dot{x}_1 = f_1(x_1, \ldots, x_n)$

$\vdots$

$\dot{x}_n = f_n(x_1, \ldots, x_n)$

$f_1, \ldots, f_n$ are given functions.

→ The system is linear if all x_i on RHS appear to first power only (no products, powers, or functions of the x_i).

→ Autonomous systems ⇒ RHS does not have time dependance ⇒ (no external forces)

Eg: Simple harmonic oscillator.

$m\ddot{x} + kx = 0$

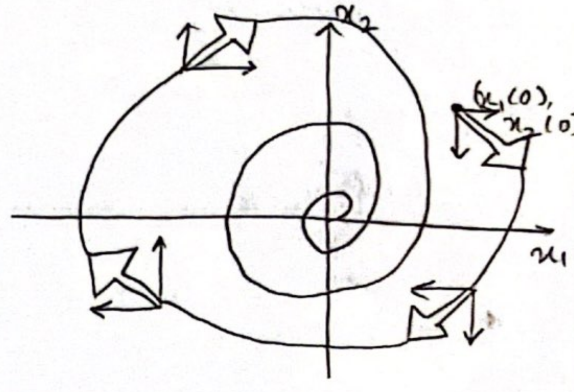
Let $x_1 = x$, $x_2 = \dot{x}$ $\Big\} \Rightarrow$ $\boxed{\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1\end{aligned}}$ $\Big\}$ Linear second order system.

Eg: Damped harmonic oscillator.

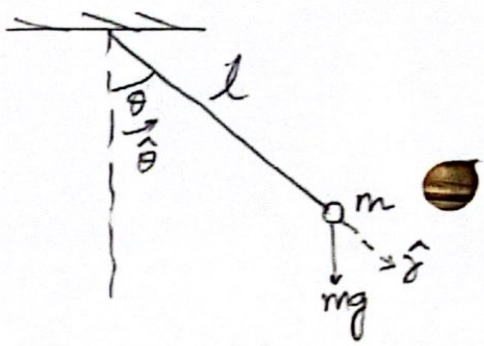
$a = -\frac{k}{m}x - \frac{b}{m}\dot{x}$

$x_1 = x$, $x_2 = \dot{x}$ $\Big\}$ $\boxed{\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{b}{m}x_2\end{aligned}}$

Phase portrait.



Eg:- Pendulum



Angular momentum

$= h = r \times mv = l\hat{r} \times ml\dot{\theta}.\hat{\theta}$

$= ml^2\dot{\theta}\hat{k}$

Torque $= -mgl\sin\theta\ \hat{k}$

$\vec{v} = l\dot{\theta}\hat{\theta}$

$\vec{v}$: linear velocity

$\dot{\theta}$: angular velocity

$\Rightarrow \quad ml^2\ddot{\theta} = -mgl\sin\theta$

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}\right\} \quad \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 \end{aligned}\right\} \text{Small angles} \Rightarrow \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}x_1 \end{aligned}$$

↳ nonlinear, 2nd order.

<u>Including time</u> $\Rightarrow$ add new state $\dot{x} = 1 \Rightarrow x = t$.

→ An external time dependant force is added.

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{m}x_2 - \frac{k}{m}x_1 + F\cos(\omega x_3) \\ \dot{x}_3 &= 1 \end{aligned}\right\}$$

Nonlinear

Autonomous ← (was made into)

<u>Discrete case of dynamical systems.</u>

$$x_1(i+1) = f_1(x_1(i), \ldots, x_n(i))$$
$$\vdots$$
$$x_n(i+1) = f_n(x_1(i), \ldots, x_n(i)).$$

→ Phase portraits are useful in visualizing the behaviour without finding analytical solutions to the given ODEs.

n = no of equations/states.

	$n=1$	$n=2$	$n=3$	$n \gtrsim 3$	$n \gg 1$	"Continuum" $n=\infty$
Linear	RC circuit	Simple harmonic oscillator				Wave equation, Maxwell's equation, Schrodinger's eqn
Nonlinear (This course)	logistic growth, skydiver dynamics	pendulum	iterated maps, Lorenz (Chaos), fractals.		complex	Systems. Turbulence, general relativity, fibrillation in heart

$n=\infty \Rightarrow$ need a continuum of information to know how the next state would look.

$\downarrow$

All PDEs.

Maxwell's $\Rightarrow$ Need to know the fields at every point to know the fields at the next instance.

<u>Chapter 2</u> $\quad \dot{x} = f(x), \quad x \in \mathbb{R}$

<u>1D systems.</u>

<u>Eg:</u> $\dot{x} = \sin x \longrightarrow$ Traditional sol: $\displaystyle\int \frac{dx}{\sin x} = \int dt = t + c.$

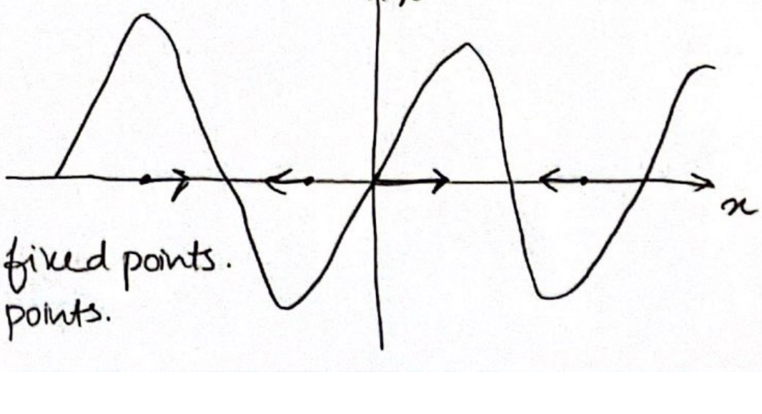
$$\int \operatorname{cosec} x \, dx = -\ln|\csc x + \cot x|$$

Say $x = x_0$ at $t=0 \Rightarrow \quad t = \ln\left|\dfrac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right|$

$\hookrightarrow$ inverting this is painful.

$\rightarrow$ If $x_0 = \frac{\pi}{4}$, what is $\lim\limits_{t\to\infty} x(t)$?

<u>Instead use Phase Portrait.</u>



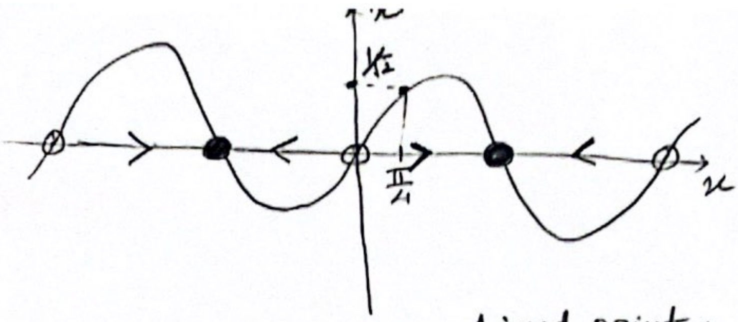
x = position of an imaginary particle

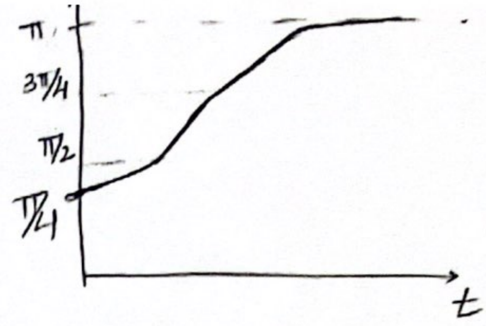
$\dot{x}$ = velocity.

$\Rightarrow \dot{x} = \sin x$ is a velocity vector field on x axis.

When $\dot{x} = 0$ at fixed points.

x^* are fixed points.





$\Rightarrow \lim_{t\to\infty} x(t) = \pi$ fixed point.

→ stable x^* are ● fixed points.

→ unstable x^* are o fixed points.

Eg

Logistic equation in population biology

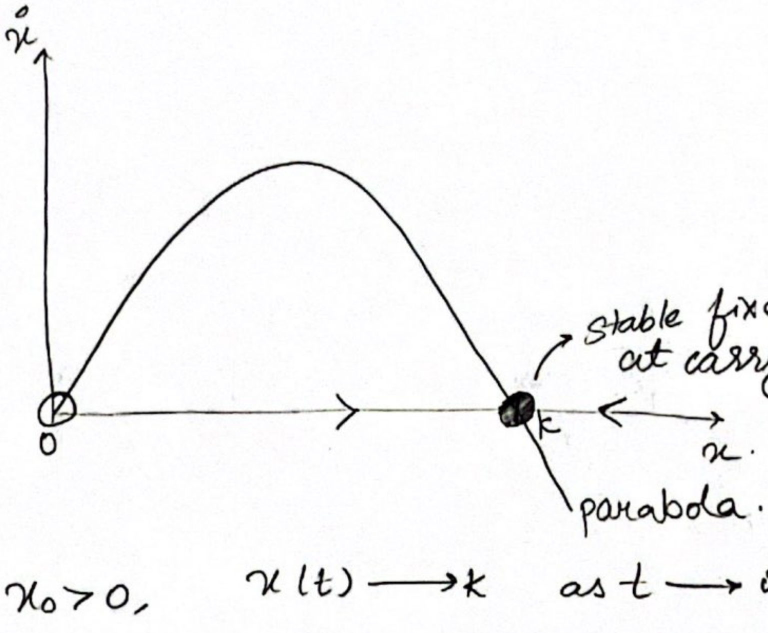
$$\dot{x} = r x\left(1 - \frac{x}{k}\right) \qquad \begin{matrix} r > 0 \\ k > 0 \end{matrix}$$

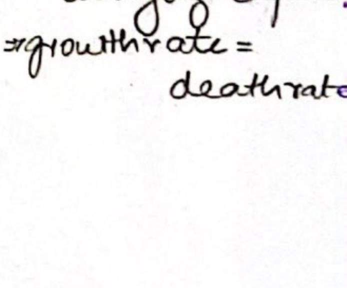
x = population size.

$\dot{x}$ = growth rate

$\frac{\dot{x}}{x}$ = per capita growth rate

Logistically per capita growth rate follows a straight line ⇒ simplest model that accounts for overcrowding





If $x_0 > 0$, $x(t) \longrightarrow k$ as $t \longrightarrow \infty$.

Book sec 2.4 Linearization.

- Examine dynamics close to a fixed point x^* quantitatively.

Let $x(t) = x^* + \eta(t)$
↳ Eta, deviation from x^* & $|\eta| \ll 1$.

$\dot{x} = (\overset{0}{x^*} + \eta)^{\cdot} = \dot{\eta} = f(x) = f(x^* + \eta)$

⇓ use Taylor's formula around x^*

$f(x) = f(x^* + \eta) \doteq \cancel{f(x^*)} + \eta \underbrace{f'(x^*)}_{\text{Slope of f at the fixed pt.}} + \underbrace{\frac{\eta^2}{2} f''(x^*) + \cdots}_{\text{neglect all these terms.}}$

$f(x^*) = 0$

Can only neglect if $f'(x^*)$ is $\neq 0$.

If $f'(x^*) \neq 0$,

$\Rightarrow \boxed{f(x) = \eta f'(x^*).}$ → linearized.

- Neglecting terms of $O(\eta^2)$ yields linearization of system at x^*:

(order)

$\boxed{\dot{\eta} = r\eta}$ where $r = f'(x^*)$ since $f(x) = \dot{x}$

$\Rightarrow$ Exponential growth $\Rightarrow \eta = \eta_0 e^{rt}$ if $r > 0$.

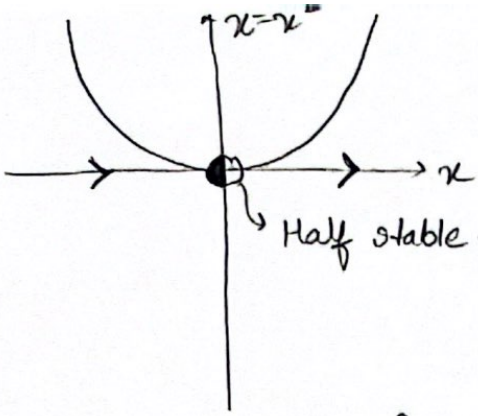
" decay if $r < 0$

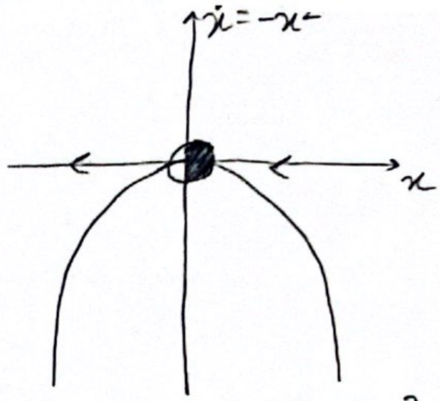
$\Rightarrow$ positive slope at fixed point $\Rightarrow$ exponential growth!

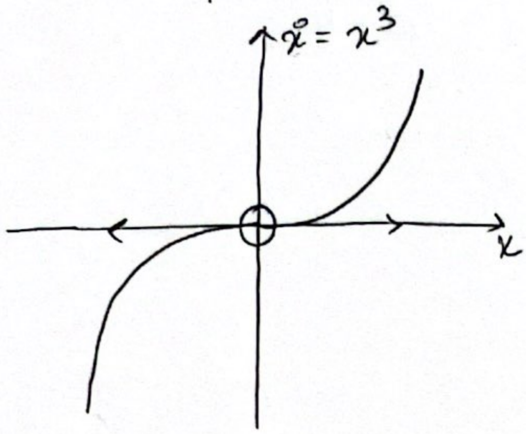
→ If $f'(x^*) = 0$, no information from linearization. ⇒ cannot talk about stability.

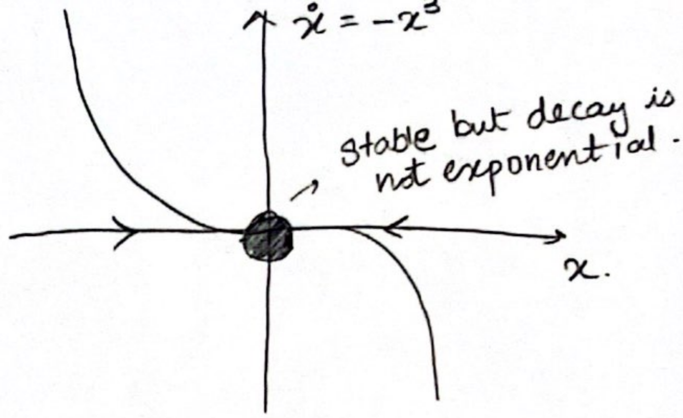
- Eg: $\dot{x} = x^2$, $\dot{x} = -x^2$, $\dot{x} = x^3$, $\dot{x} = -x^3$

$x^* = 0$ is a fixed point for all these. & $f'(x^*) = 0$









Eg: <u>Logistic equation. using linearization.</u>

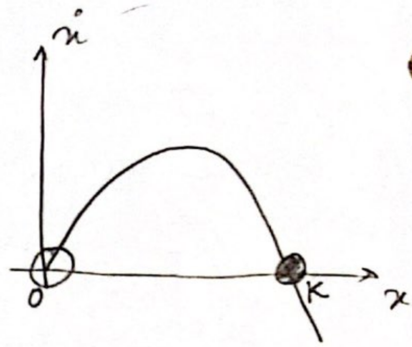
$\dot{x} = rx\left(1 - \frac{x}{K}\right)$

$\dot{x} = 0$ when $x^* = 0$ or $x^* = K$.

$f'(x) = r - \frac{2rx}{K}$

$f'(0) = r > 0 \Rightarrow x^* = 0$ unstable. since r is growth rate.

$f'(K) = r - 2r = -r < 0 \Rightarrow x^* = K$ stable.



→ The value of $|f'(x^*)|$ gives a quantitative value for magnitude of how fast the value of x changes.

⇒ $\frac{1}{|f'(x^*)|}$ is a characteristic time scale.

↳ If this is small, the time required for $x(t)$ is vary signifacantly is small. Like time constant

→ Existance and Uniqueness.

» Solutions to $\dot{x} = f(x)$ do exist and they are unique if $f(x)$ and $f'(x)$ are continuous.

(f is "continuously differentiable").

→ Impossibility of oscillations.

What is the possible behaviour of $x(t)$ as $t \to \infty$, for $\dot{x} = f(x)$?

i) $x(t) \to \pm\infty$ as $t \to \infty$

or ii) $x(t) \to x^*$

⇒ Oscillations are not possible. Not even damped oscillations are possible.

But why?

> All trajectories $x(t)$ increase or decrease monotonically or stay fixed. Since $f(x) = \dot{x}$ is well defined at each point it cannot change value & therefore the trajectory is monotonic.

Ch 3 Bifurcations (in content of 1-D)

> As a parameter changes, qualitative structure of the vector field may change dramatically.

Eg:- A fixed point may be created or destroyed or they may change their stability.

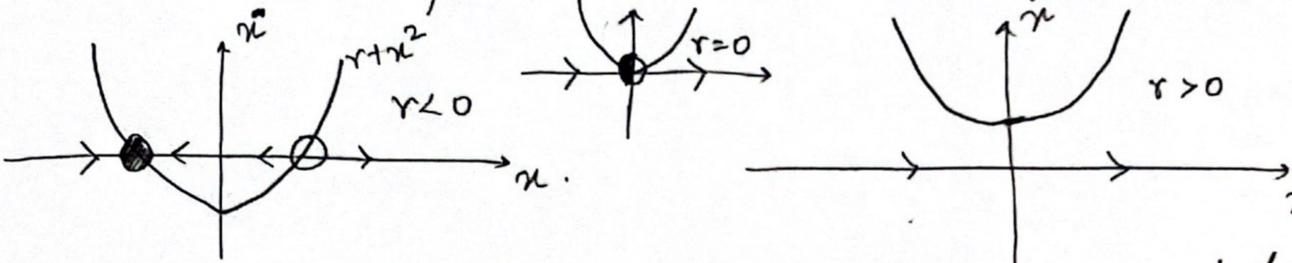
> Bifurcation point or value: Value of the parameter at which the change occurs.

Eg ①

Saddle-node Bifurcation.

> Basic mechanism for creation/destruction of fixed points.

> Standard example: $\dot{x} = r + x^2$ $\quad$ r = control parameter

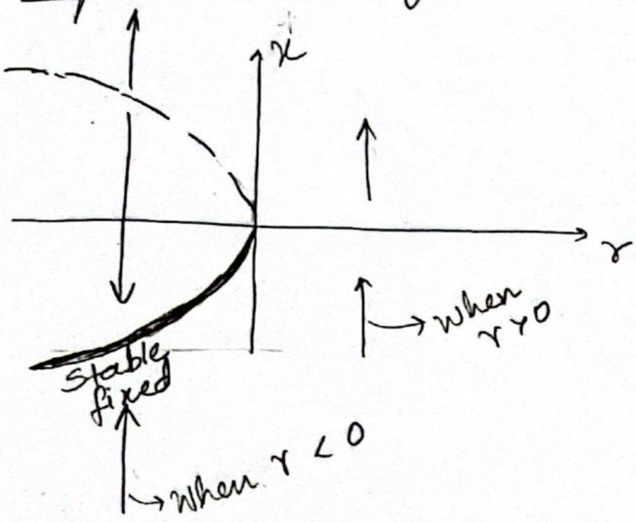


→ Half stable fixed points arise at the point of bifurcations.

Bifurcation diagram. (x^* vs. r)

$\dot{x} = 0 \Rightarrow r + (x^*)^2 = 0$

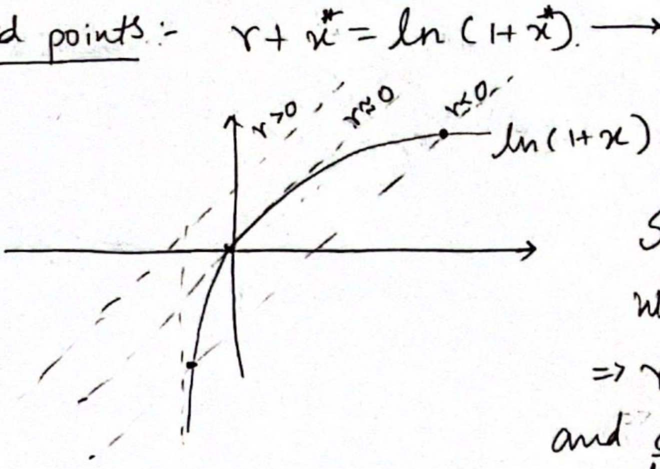
$x^* = \pm\sqrt{-r}$.



↳ AKA fold or turning point bifurcation.

Eg ② : $\dot{x} = r + x - \ln(1+x)$

Fixed points :- $r + x^* = \ln(1+x^*)$ ⟶ can't solve for $x^*(r)$. Use graphical method



⇒ plot $y = r + x$

$y = \ln(1+x)$

Saddle node bifurcation occurs.

when we have a tangential intersection.

$\Rightarrow r + x = \ln(1+x)$

and $\frac{d}{dx}(r+x) = \frac{d}{dx}(\ln(1+x))$

$1 = \frac{1}{1+x} \Rightarrow x^* = 0$ at saddle node bfn.

and then $r + x = \ln(1+x)\Big|_{x=0}$.

$\Rightarrow r = \ln 1 = 0$

$\Rightarrow \boxed{r_c = 0} \longrightarrow$ Critical or bifurcation value.

> Near the bifurcation at $(x, r) = (0, 0)$,

$$\dot{x} = r + x - \ln(1+x)$$
$$\approx r + x - x + \frac{x^2}{2} + \cancel{O(x^3)}^{\,0}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$
when $|x| < 1$

$$\dot{x} \approx \underbrace{r + x^2/2}$$

Since we get $r + x^2$ form again we call this the "normal form".
→ In many cases this form shows up.

Transcritical Bifurcation

Normal form: $\dot{x} = rx - x^2 = x(r - x) \Rightarrow \left.\begin{matrix} x^* = 0 \\ x^* = r \end{matrix}\right\}$ 2 fixed pts.

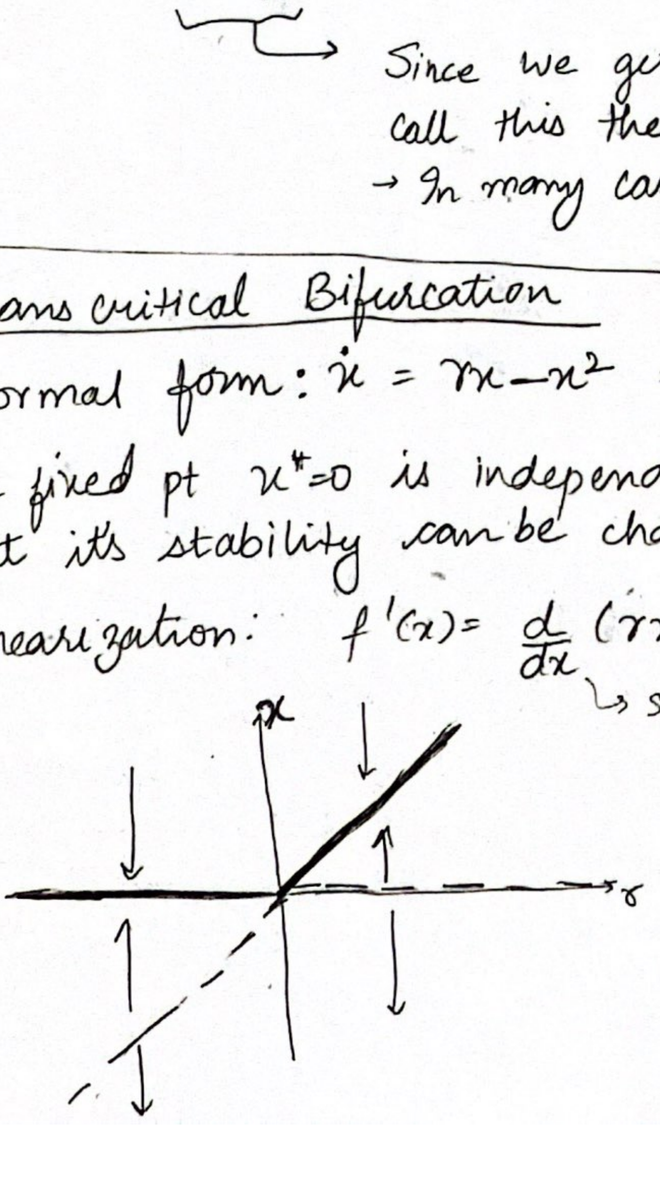
→ The fixed pt $x^* = 0$ is independent of r and cannot be destroyed. But it's stability can be changed.

Linearization: $f'(x) = \frac{d}{dx}(rx - x^2) = r - 2x$

↳ slope -ve ⇒ stable.

$f'(0) = r$.

$x^* = 0 \begin{cases} \text{stable } r < 0 \\ \text{unstable } r > 0 \end{cases}$

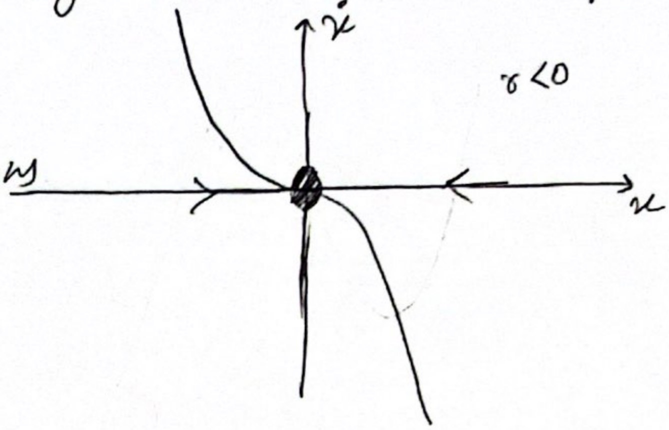


$f'(r) = r - 2r$
$= -r$

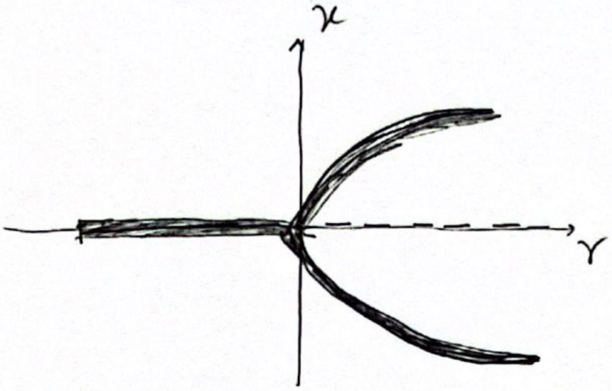
AKA Exchange of stabilities.

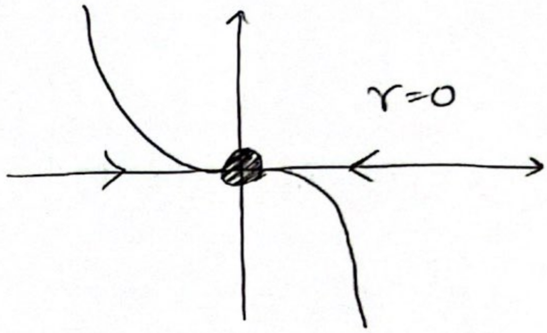
Pitchfork bifurcation – Occurs in systems with symmetry.

$\dot{x} = rx - x^3$ → symmetry b/w x & $-x$ ⇒ mirror symmetry.



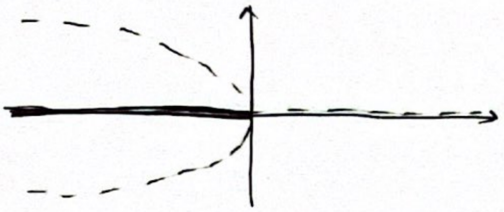
AKA supercritical pitchfork.
↳ Bifurcating solutions are stable.

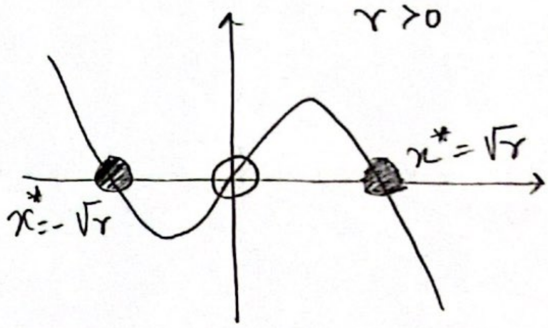




Subcritical

$\dot{x} = rx + x^3$





Normal form of bifurcations.

Saddle node: $\dot{x} = r \pm x^2$

Transcritical: $\dot{x} = rx \pm x^2$ Eg: 3.2.2

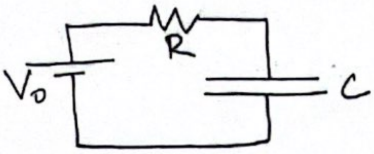
Pitchfork: $\dot{x} = rx - x^3$ ⟶ supercritical.

$\dot{x} = rx + x^3$ ⟶ subcritical.

→ Be careful when $f(x, r)$ is a discontinuous function of r.

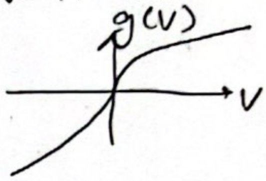
Phy 413 Lec 2

Q



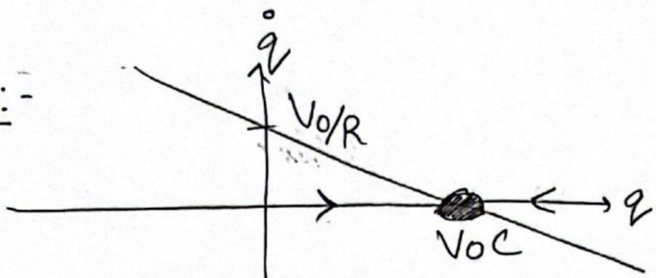
$V_0 = V_R + V_C$

- Plot $\dot{q}$ as a fn of q
- Fixed points?
- $q(t) = ?$
- Solve ODE formula.
- Replace $V = IR$ with $I = g(V)$ where



$$\boxed{\dot{q} = \frac{V_0}{R} - \frac{q}{RC}}$$

Sol:-



Overtime as $t \to \infty$ q reaches V_0C.

$$V_0 = \dot{q}R + q/C$$

$$\frac{dq}{dt} = \frac{V_0}{R} - \frac{q}{RC}$$

$$-R\frac{dq}{dt} = \frac{q}{C} - V_0$$

$$\frac{dq}{q - CV_0} = -\frac{1}{RC}\,dt$$

$$\Rightarrow \ln|q - V_0C| - K = \frac{-t}{RC}$$

$$t = 0 \Rightarrow K = \ln|q_0 - V_0C|$$

$$\Rightarrow \left|\frac{q - V_0C}{q_0 - V_0C}\right| = e^{-t/RC}$$

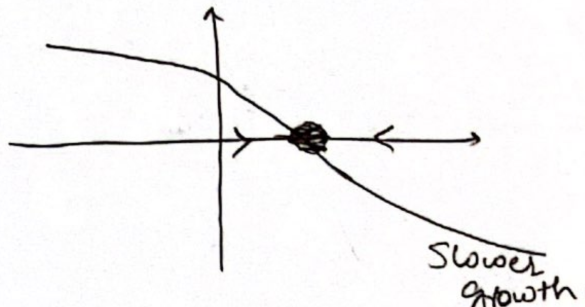
$$\Rightarrow \boxed{q = V_0C + (q_0 - V_0C)e^{-t/RC}}$$

Nonlinear case.

$$I = g(V) \Rightarrow V = g^{-1}(\dot{q})$$

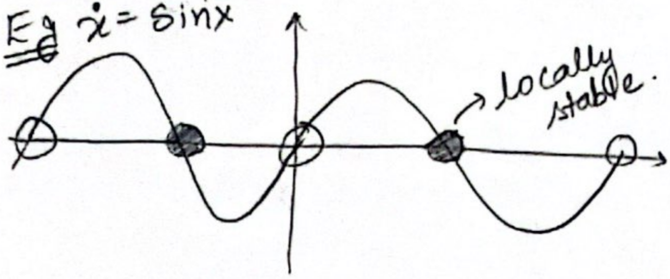
$$\Rightarrow V_0 = g^{-1}(\dot{q}) + \frac{q}{C}$$

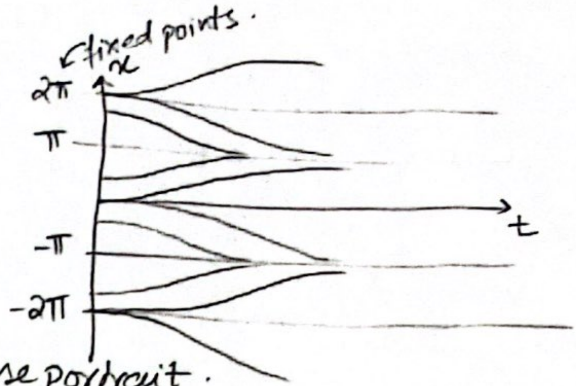
$$\Rightarrow I = \dot{q} = g\left(V_0 - \frac{q}{C}\right).$$

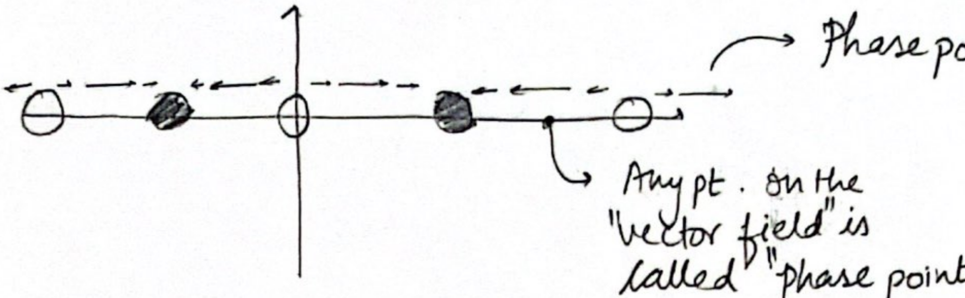


Phase portrait

Eg $\dot{x} = \sin x$

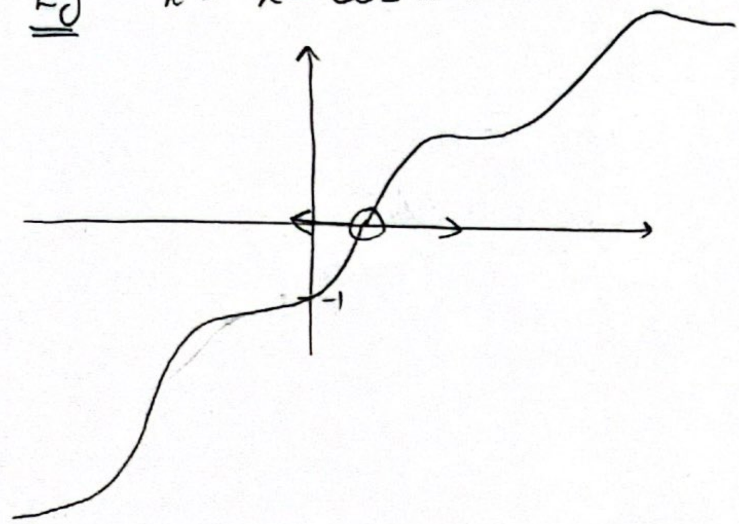


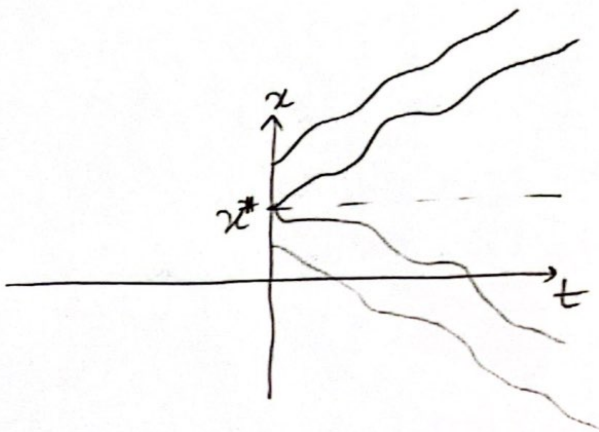




→ Phase portrait.

Eg: $\dot{x} = x - \cos x$.





Lec 3 - Phy 413

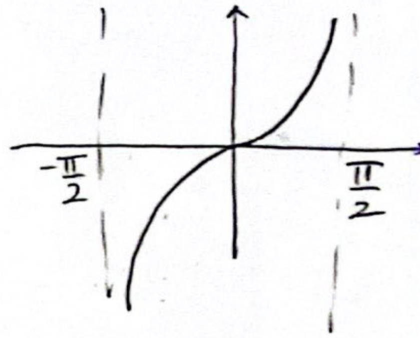
Existence and Uniqueness Theorem.

If $f(x)$ and $f'(x)$ are continuous on an interval R and $x_0 \in R$ and we have $\dot{x} = f(x)$ and $x(0) = x_0$. A solution exists and is unique for t in $(-\tau, \tau)$ for $\tau > 0$.

Eg: $\dot{x} = 1+x^2$, $f(x) = 1+x^2$, $f'(x) = 2x \Rightarrow$ continuous.

$R = (-\infty, \infty)$

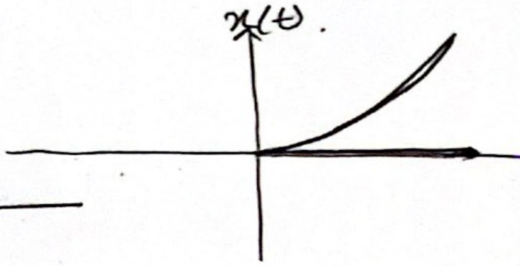
$\int \frac{dx}{1+x^2} = \int dt \Rightarrow x = \tan t$ if $x_0 = 0$.



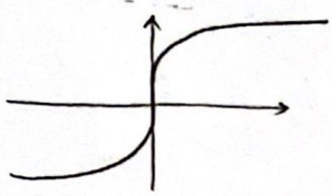
$\Rightarrow I = \frac{\pi}{2}$

Note: R is defined on x axis. I on t axis.

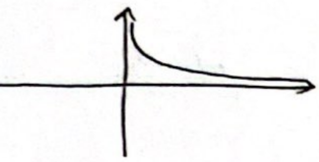
Eg: $\dot{x} = x^{1/3} \Rightarrow x = \left[\frac{2}{3}t\right]^{3/2}$ but notice $x(t) = 0$ is also a solution.



Why? $f(x) = x^{1/3}$ is continuous for $R = (-\infty, \infty)$



$f'(x) = \frac{1}{3} \cdot x^{-2/3} = \frac{1}{3x^{2/3}}$



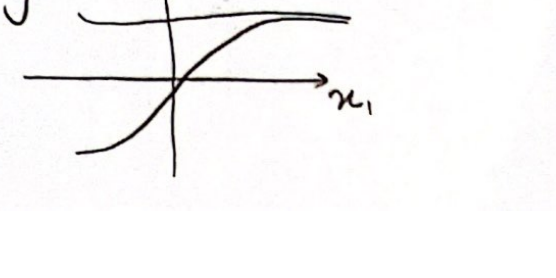
Discontinuous at $x(t) = 0$

Therefore, R cannot include 0. $\Rightarrow x(t) = 0$ is not a solution when $R \in (0, \infty)$.

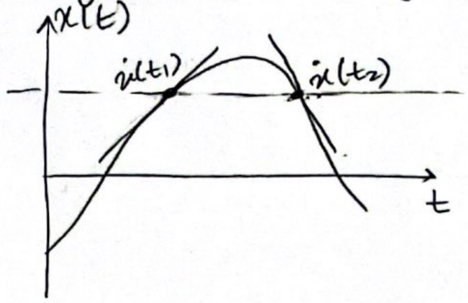
Corollary

① Trajectories cannot cross. Since the phase pt. will not have a unique path as $t \to \infty$

② Trajectories cannot merge in time and space since that would need an ∞ slope $\Rightarrow$ discontinuous. They can only reach the fixed points asymptotically.



Proof of monotonicity.



Assume not monotonic,

$f(x(t_1))$ is +ve.

$f(x(t_2))$ is -ve.

But $x(t_1) = x(t_2)$. $\Rightarrow f(x_1) = f(x_2) = 0$

$\dot{x}(t_1) > 0$

$\dot{x}(t_2) < 0$

$\Rightarrow$ This can only occur at fixed points.

This works because of uniqueness, since $f(x)$ can only have one value for each x.

→ Linearization.

$$f(x) = f(x^*) + \frac{f'(x^*)}{1!}(x-x^*) + O((x-x^*)^2)$$

If $|f'(x^*)| > 0$ then as $x \to x^*$ $O((x-x^*)^2) \to 0$

$\Rightarrow f'(x^*) > 0 \Rightarrow$ exp. growth.

$f'(x^*) < 0 \Rightarrow$ exp. decay.

$\frac{1}{|f'(x^*)|}$ → Characteristic time scale ⇒ ↑ or ↓ by e after on timescale.

Half life $= \lambda = \frac{\ln 2}{|f'(x^*)|}$

Potentials.

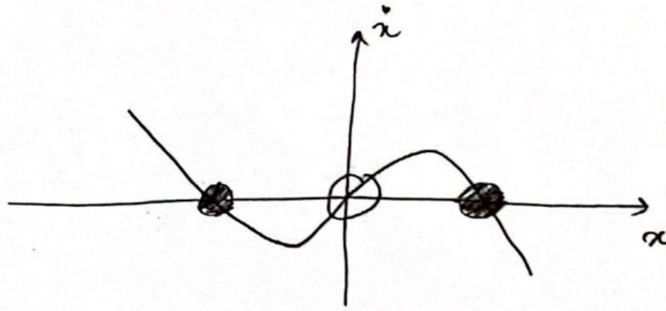
In math $f(x) = -\frac{dV}{dx}$. ← force.

$\Rightarrow$ $f(x) = \dot{x} = -\frac{dV}{dx}$ → solve for a V, that gives a visual representation of fixed points.

Maxima in $V \Rightarrow$ Unstable x^*

Minima in $V \Rightarrow$ Stable x^*

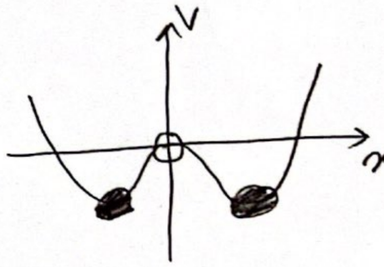
Eg:- $\dot{x} = x - x^3$



$$\frac{dV}{dx} = x^3 - x.$$

$$V = \int x^3 dx - \int x \, dx.$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + c \qquad \text{let } c = 0$$

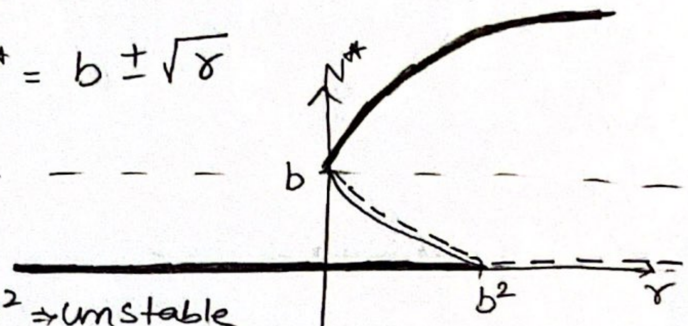


Phy 413 Lec 4 → Bifurcations.

Population growth example for bifurcations.

$$\frac{\dot{N}}{N} = r - (N-b)^2 \Rightarrow \dot{N} = N(r - (N-b)^2) \Rightarrow N^*(r - (N^*-b)^2) = 0$$

$\Rightarrow N^* = 0$ and $(N^*-b)^2 = r$ $\Rightarrow$ $N^* = b \pm \sqrt{r}$



$$\frac{d}{dN}\dot{N} = r - b^2 + 4Nb - 3N^2$$

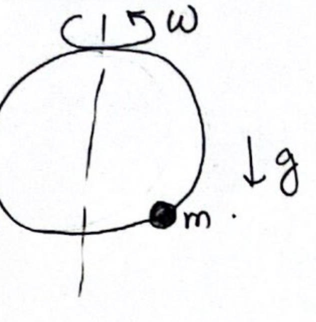
for $N = N^* = 0$ $\quad \frac{d}{dN}\dot{N} = r - b^2 \Rightarrow r > b^2 \Rightarrow$ unstable
$r < b^2 \Rightarrow$ stable

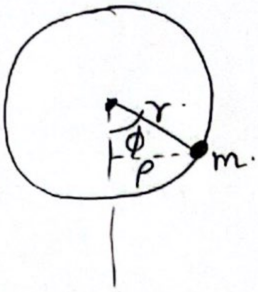
for $N = N^* = b + \sqrt{r} \Rightarrow \frac{d}{dN}\dot{N} = -2r - 2b\sqrt{r} \Rightarrow r > 0 \Rightarrow$ stable.

$N = N^* = b - \sqrt{r}$
$\Rightarrow \frac{d}{dN}\dot{N} = -2r + 2b\sqrt{r}$

MAE5790 - Lec 3 Overdamped bead on a rotating hoop.

Sec 3.5

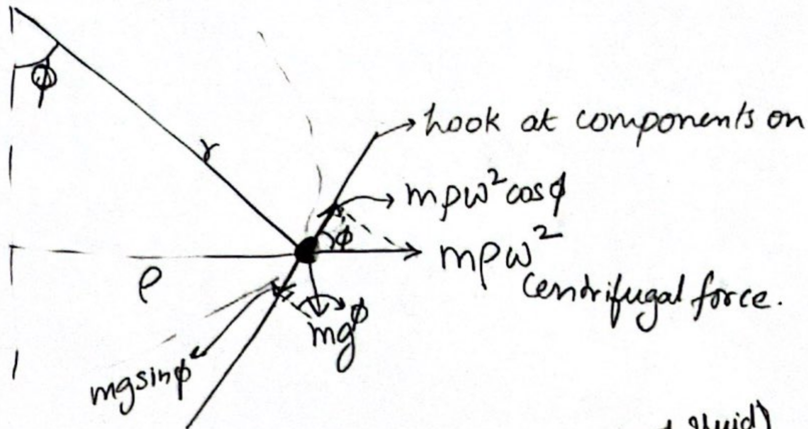




Newton's law.

$F = ma$

do the problem in a frame co-rotating with the hoop.



> To model friction imagine the bead is rotating in honey at same ω. So it cannot slide up & down easily.

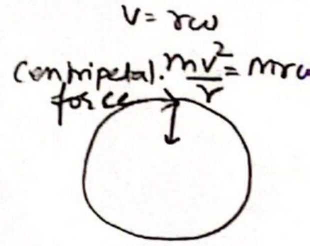
$\rho = r\sin\phi$

constant {property of fluid & bead only.}

$$\Rightarrow \underbrace{mr\ddot{\phi}}_{ma} = \underbrace{-br\dot{\phi}}_{\text{viscous force}} - mg\sin\phi + mr\omega^2\cos\phi\sin\phi$$

rotational velocity

Translational velocity



> Coriolis force is along r so it does not have any effect

$$mr\ddot{\phi} = -br\dot{\phi} - mg\sin\phi + mr\omega^2\cos\phi\sin\phi.$$

> Start by ignoring $mr\ddot{\phi}$ term. (Discussed later)

> The dynamics would be governed by $br\dot{\phi} = mg\sin\phi\left[\frac{r\omega^2}{g}\cos\phi - \right.$

$$\Rightarrow \quad br\dot{\phi} = mg\sin\phi\left[\frac{r\omega^2}{g}\cos\phi - 1\right]$$

Fixed points $\Rightarrow$ $\sin\phi = 0$ or when $\cos\phi = \dfrac{g}{r\omega^2}$

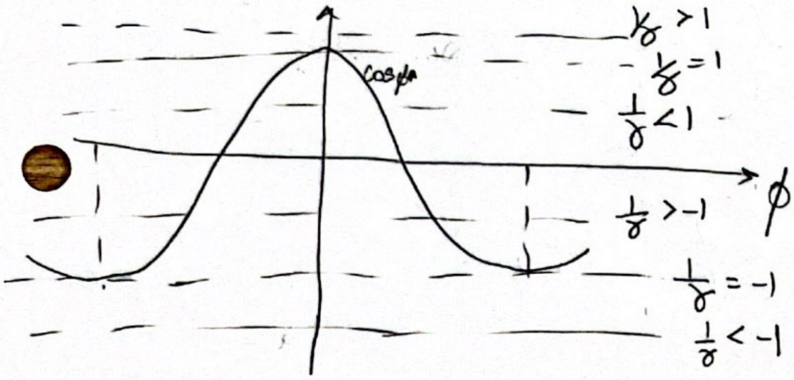
$\sin\phi^* = 0 \Rightarrow \phi = 0$ or $\pi \to$ top
$\hookrightarrow$ bottom.

Check: $\phi^* = \pi \to$ unstable for all $m, g, r, \omega, b > 0$
$\phi^* = 0$ may or maynot be stable $\to$ Try it.

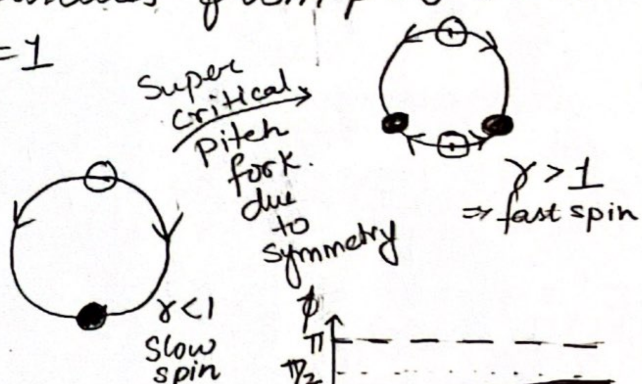
$\cos\phi^* = \dfrac{g}{r\omega^2}$ has solutions iff $g \le r\omega^2 \Rightarrow$ hoop needs to be fast enough for hoop to leave the bottom.

Let $\gamma = \dfrac{r\omega^2}{g}$ (dimensionless - no units).

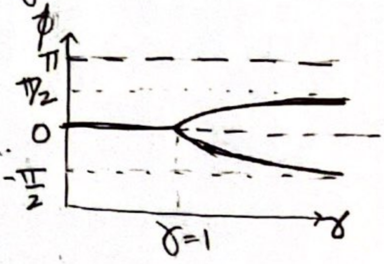
Then $\cos\phi^* = \dfrac{1}{\gamma}$ must be solved.



A pair of fixed points $\pm\phi^*$ bifurcates from $\phi = 0$ when $\gamma = 1$



> Supercritical pitchfork bifurcation at $\gamma = 1$.



$\to$ At high γ or ω, the 2 fixed points go to $\frac{\pi}{2}, -\frac{\pi}{2}$

$\to$ Going back to the second derivative term $mr\ddot{\phi}$?
When is $mr\ddot{\phi}$ negligible?
> $m \to 0$, is too crude. Other terms also go to 0.
> Dimensional analysis $\to$ To reduce the no. of parameters.
> Suppose T = timescale s.t. $\dot{\phi}$ is of the order $\frac{1}{T}$ and $\ddot{\phi} \approx \frac{1}{T^2}$. T is chosen later.

Better way:

Let $\tau =$ dimensionless time.

$$\tau = \frac{t}{T}$$

Then $\frac{d\phi}{dt} = \frac{d\phi}{d\tau} \cdot \frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$ → notice $\dot{\phi} \approx \frac{1}{T}$

→ Find T s.t. $\frac{d\phi}{d\tau} \xrightarrow[\text{order of } 1]{\text{on the}} 1$ as a parameter goes to 0 to $\pm\infty$.

→ $\frac{d^2\phi}{dt^2} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}$. Let $\phi' = \frac{d\phi}{d\tau} \Rightarrow \phi'' = \frac{d^2\phi}{d\tau^2}$

$\Rightarrow \quad \ddot{\phi} = \frac{1}{T^2}\phi''$

$\Rightarrow$ Governing equation becomes:

$$\frac{mr}{T^2}\phi'' = -\frac{br}{T}\phi' + mg\sin\phi\,(\gamma\cos\phi - 1)$$

Make the equation dimensionless by dividing by mg.

Why dimensionless? We know when something is small if we compare it to 1.

$$\Rightarrow \underbrace{\left[\frac{r}{gT^2}\right]}_{\text{We want to neglect this term.}}\phi'' = \left[-\frac{br}{mgT}\right]\phi' + f(\phi)$$

$f(\phi) \longrightarrow \sin\phi\,(\gamma\cos\phi - 1)$

→ ϕ'' & ϕ' are order $O(1)$. Even $f(\phi)$ is order $O(1)$ since γ does not change by changing the parameters (mass, viscocity).

> We want to choose T s.t $\frac{br}{mgT} \sim O(1)$ but $\frac{r}{gT^2} \simeq 1$

> Choose $\boxed{T = \frac{br}{mg}}$

> $\frac{r}{gT^2} = \frac{r}{g\left(\frac{br}{mg}\right)^2} = \frac{\cancel{r}\, m^2 g^{\cancel{2}}}{\cancel{g}\, b^2 r^{\cancel{2}}} = \boxed{\frac{m^2 g}{b^2 r} \ll 1}$

$\Rightarrow$ $\boxed{\boxed{m^2 \ll \frac{b^2 r}{g}.}}$ $\longrightarrow$ Only now we can ignore m.

Looking back: $\epsilon \phi'' = -\phi' + f(\phi)$.

$0 \Rightarrow \phi' = f(\phi) \Rightarrow \phi'$ is order 1 as was assumed.

→ What if ϕ'' is very large? Then $\epsilon \to 0$ does not mean $\epsilon \phi'' \to 0$

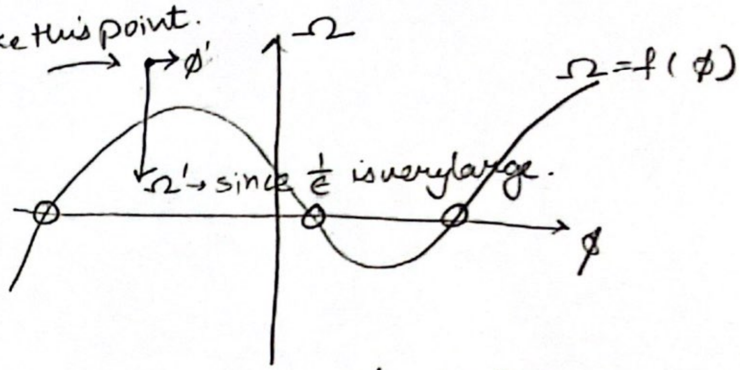
→ <u>A singular limit:</u> As $\epsilon = \frac{m^2 g}{b^2 r} \longrightarrow 0$ we lose the highest order derivative $(\epsilon \phi'')$. Then we can't satify both the initial velocity and position. Even when $\epsilon \ll 1$, our approach is only valid <u>after a rapid initial transient.</u>

→ That transient time would require a different timescale. So our scaling (T) fails during the transient but correct after that.

→ $\epsilon\phi'' + \phi' = f(\phi)$

→ Let $\Omega = \phi'$; $\Omega' = \phi'' = \frac{1}{\epsilon}\left[f(\phi) - \Omega\right]$

$$\phi' = \Omega$$
$$\Omega' = \frac{1}{\epsilon}\left[f(\phi) - \Omega\right]$$



→ ∴ If the behaviour quickly relaxes to the curve $\Omega = f(\phi)$, we would be OK.

$\Omega = f(\phi) \Rightarrow \phi' = f(\phi)$

↳ this is the one dimensional curve we have been analysing.

→ Ω' is very large when $f(\phi)$ is very different from Ω. This is not true when $f(\phi)$ is close to Ω. When this happens the point stays very close to $f(\phi)$ line & moves along to the fixed points.

→ Therefore there is an initial "jump" at a fast time scale & then a "normal" behaviour which we have analysed.

MAE

Lec 4: Model of an insect outbreak Sec: 3.7

- → A bifurcation problem, involving 2 parameters.
- → Discontinuous jumps are observed.
- → Ludwig et al. (1978) Journal of Animal Ecology. 47, 315

- → Spruce budworms pest. - Canadian timber problems and also in Maine (US).

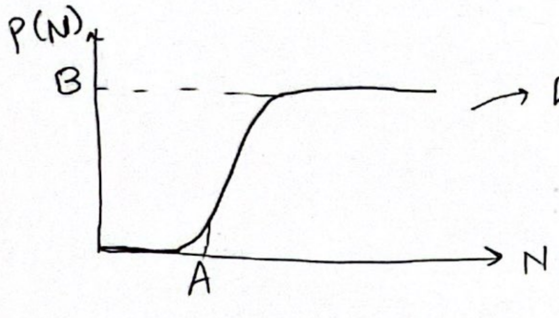
Can destroy an entire forest in 4 years.

- → $N(t)$ = population of budworms.

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) \longrightarrow \text{logistic model.}$$

- → Including predation by birds.

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - p(N).$$



→ Empirical model of $P(N)$

$$P(N) = \frac{BN^2}{A^2 + N^2}$$

- → Four parameters: A, B, R, K all have units.
- → What are the dynamics of this system for various parameters?
- → Helps to non-dimensionalize ⇒ reduces from 4 to 2 dimensions.
- → Both A & K have the same dimensions as N. What should we use for nondimensionalizing?

Choose scale of N to make the nonlinear $p(N)$ term have no parameters (since it is more complex).

→ Let $x = \frac{N}{A}$. $N = Ax$.
$\dot{N} = A\dot{x}$

$$\therefore A\frac{dx}{dt} = RAx\left(1 - \frac{Ax}{K}\right) - \frac{B\cancel{A^2}x^2}{\cancel{A^2} + \cancel{A^2}x^2}$$

$$\Rightarrow \frac{A}{B}\frac{dx}{dt} = \frac{RA}{B}x\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}$$

→ We want to introduce a dimensionless time since x is dimensionl.

Let $\tau = \frac{Bt}{A}$. This choice makes LHS dimensionless.

$\Rightarrow dt = \frac{A}{B}d\tau$ Let $x' = \frac{d}{d\tau}$

→ Notice $\frac{A}{K}$ is dimensionless. Let $\frac{A}{K} = \frac{1}{k}$

$\frac{RA}{B} = r$. These choices make it look like the logistic model.

$$\Rightarrow \boxed{x' = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}}$$ ⇒ We see now we have only 2 parameters.

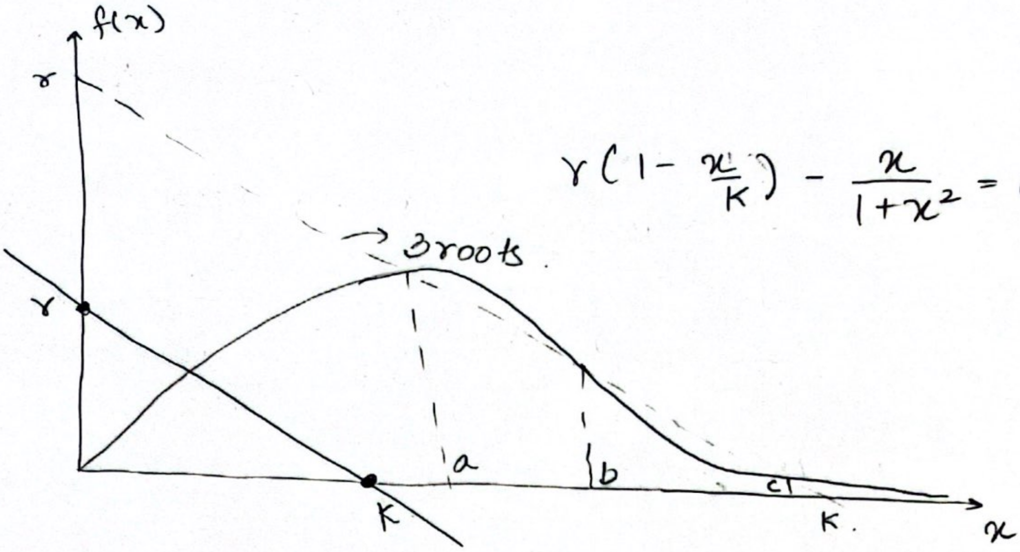
→ Scale models preserve all the dynamics of the system.
The behaviour of the birds has been included into the r & k.

<u>Fixed points.</u>

$x' = 0 \Rightarrow x^* = 0$; No budworms ⇒ no growth rate.

Other x^*'s satisfy $r\left(1 - \frac{x}{k}\right) = \frac{x}{1+x^2}$

Now you see why the parameters were lumped into the Simpler equation on the left.

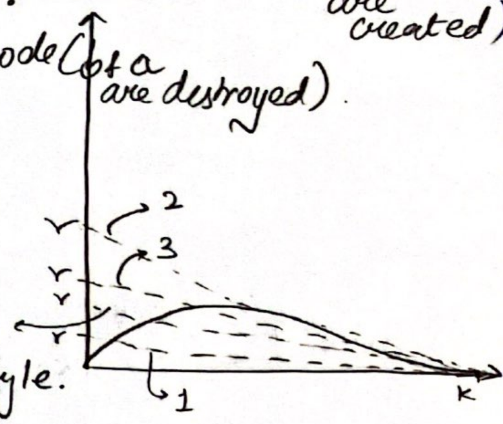


$$r\left(1-\frac{x}{K}\right)-\frac{x}{1+x^2}=0$$

Steep line ⇒ one root
Shallow line ⇒ 3 roots.

> We get one intersection for small K, for any r.

> When K is very large and r is small only one intersection.
> Now as r increases we see a bifurcation (saddle node) (b & c are created)
> Again as r increases we see another saddle node (b & a are destroyed).
> For very large r we see only c,

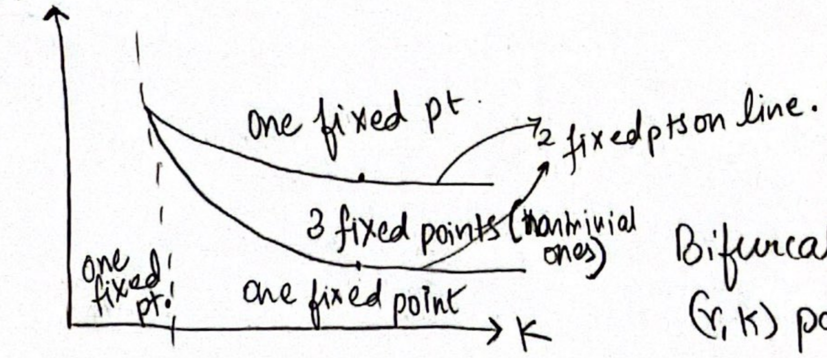


>

1 ⟶ 2 ⟶ 3 ⟶ 2 ⟶ 1

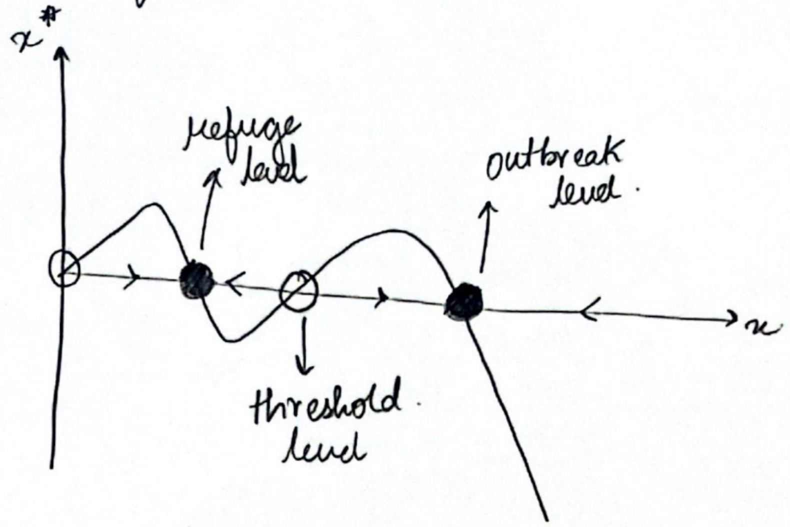
a   a bc   a b c   ab c   c

> b is colliding with a or c in saddle node style.



Bifurcations occur at certain (r, K) pairs.

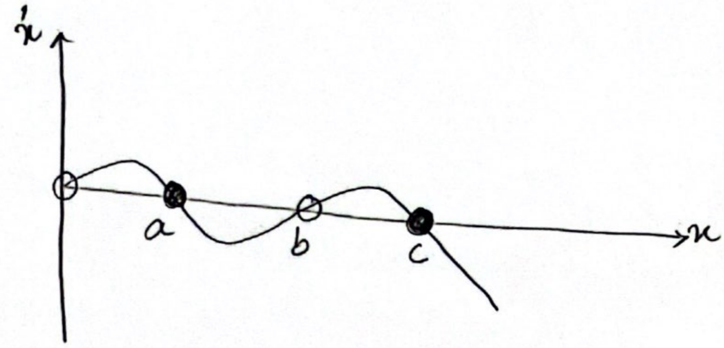
> Suppose 3 fixed points besides $x^* = 0$.



> When x is small
$x' \simeq rx$

> Assuming stability of other 3 fixed pts. alternate.
↳ Topologically this is always true.

> Can get jump phenomenon after a saddle node. Suppose x is at 'a' and the parameters start to drift.



> As the forest ages, r increases slowly.

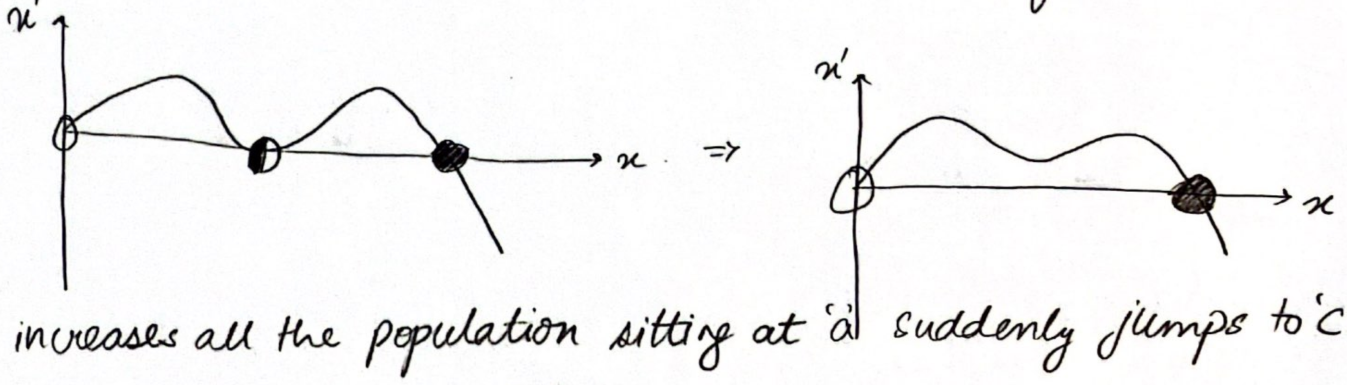
$r = \frac{RA}{B}$

$A \propto$ surface area of foliage

$= A'S$ → critical density of budworms per leaf.

S → # of leaves.

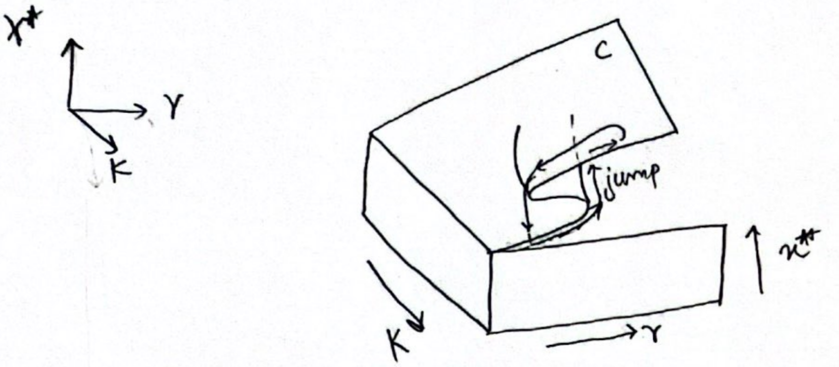
> As r grows slows, a & b approach each other & annihilate.



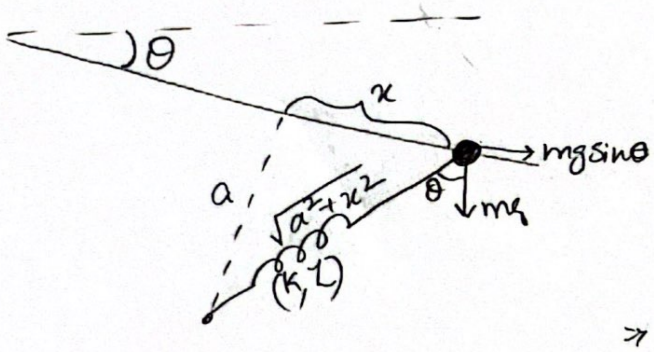
> As r increases all the population sitting at 'a' suddenly jumps to 'c', when the fps a & b annihilate.

→ Now even if the system goes back to 3 fixed points the population that was at c does not go back to 'a'. Hysteresis. Unless b & c annihilate which would be quite hard.

→ $\boxed{\text{Hysteresis}}$ → Can visualize this as a cusp catastrophe



Lect - Phy 413



Force balance

$$mg\sin\theta = -K\left(L-\sqrt{a^2+x^2}\right)\cdot\frac{x}{\sqrt{x^2+a^2}}$$

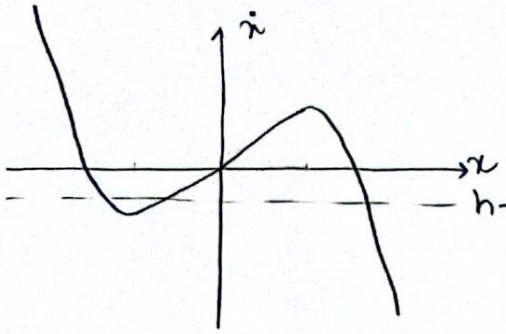
Let $u = \frac{x}{a}$ to non dimensionalize.

$$\Rightarrow mg\sin\theta = -Kau\left(\frac{L}{a}\frac{1}{\sqrt{1+u^2}}-1\right)$$

$$\Rightarrow \underbrace{\frac{mg\sin\theta}{Ka}}\cdot\frac{1}{u} = 1-\underbrace{\frac{L}{a}}\left(\frac{1}{\sqrt{1+u^2}}\right)$$

2 Nondimensional groups ⇒ only 2 parameters control the dynamics.

Eg: $\dot{x} = h + rx - x^3 \rightarrow$ 2 parameters



What value of h gives half stable fp?

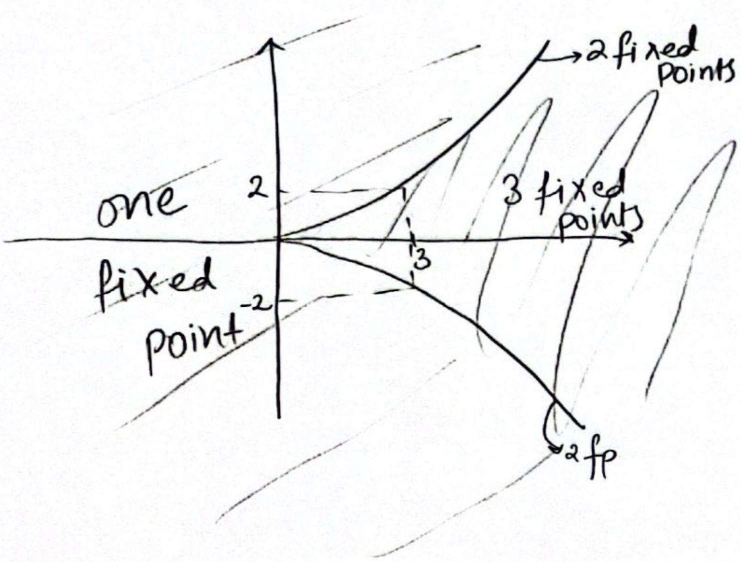
$\dot{x} = rx - x^3$

$f'(x) = r - 3x^2 = 0$

$\Rightarrow x = \sqrt{\frac{r}{3}}$

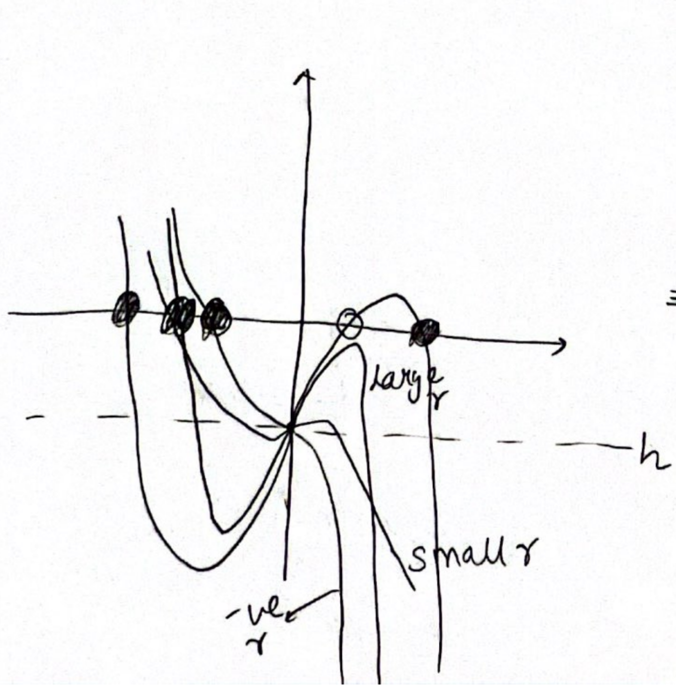
$\Rightarrow \dot{x} = -\frac{2}{3}\sqrt{\frac{r^3}{3}}$ are the y axis values at which slope is 0. These are the values of h that give half stable points.

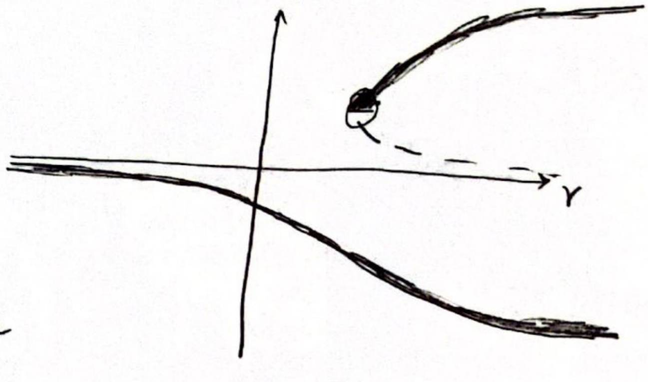
$\Rightarrow h_c = \frac{2}{3}\sqrt{\frac{r^3}{3}}$



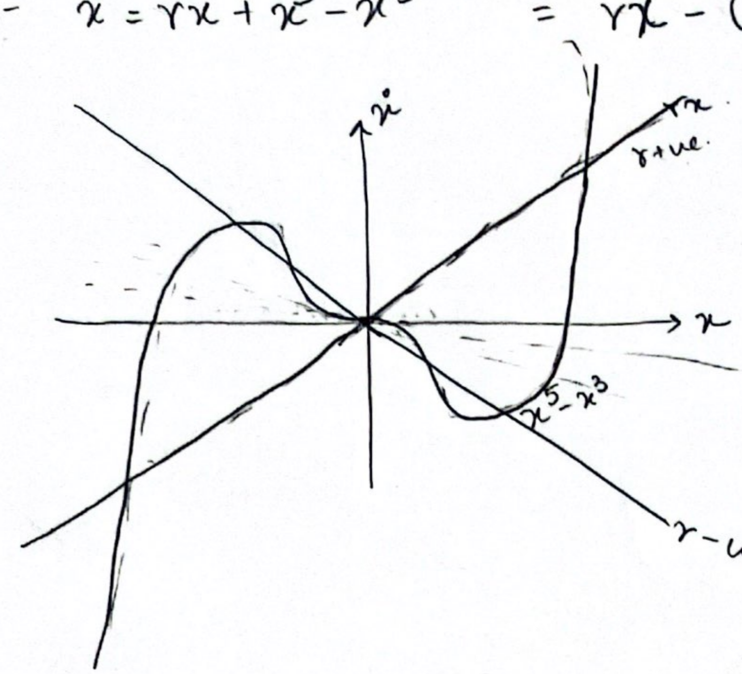
> When h=0 and r=0 we get pitchfork (supercritical)

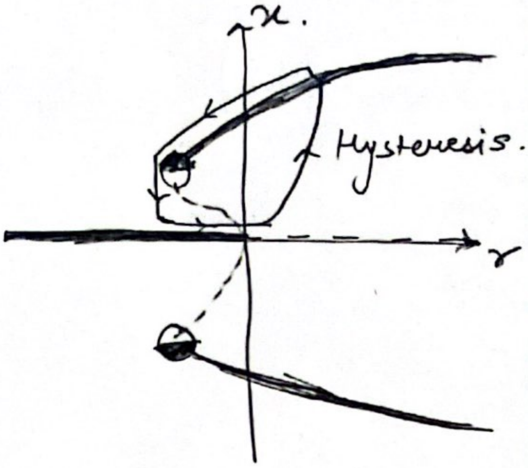
> $h \neq 0, r \neq 0 \Rightarrow$ let h is positive $\Rightarrow \dot{x} = h + rx - x^3$





Eg:- $\dot{x} = rx + x^3 - x^5 \quad = rx - (x^5 - x^3)$





Hysteresis or Bistable switching.

Eg: Going back to the spring bead example

$$\frac{mg\sin\theta}{Ka} \cdot \frac{1}{u} = 1 - \frac{L}{a}\left(\frac{1}{\sqrt{1+u^2}}\right) \qquad \text{Let } \frac{mg\sin\theta}{Ka} = \text{"}\theta\text{"}$$

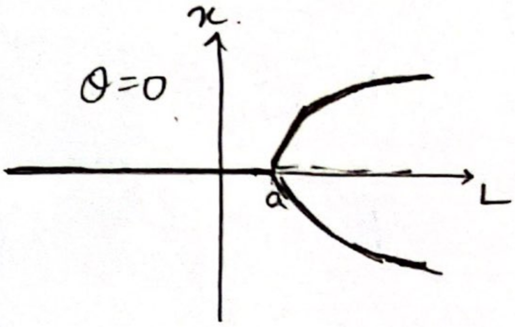
$$\frac{L}{a} = \text{"}L\text{"}$$

$\underline{\theta = 0}$ $\qquad \theta \cdot \frac{1}{u} = 1 - L\left(\frac{1}{\sqrt{1+u^2}}\right)$

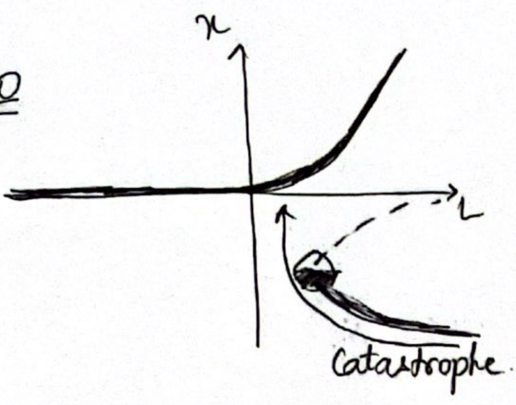
$\frac{L}{a} < 1 \Rightarrow$ only one stable equilibrium

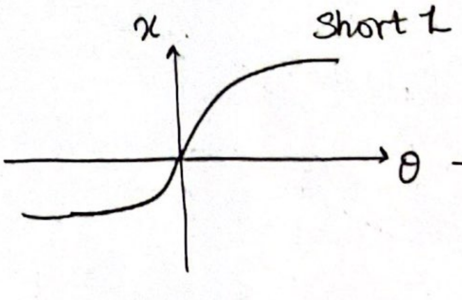
$\frac{L}{a} > 1 \Rightarrow$ spring pushing $\Rightarrow$ 3 equilibrium.

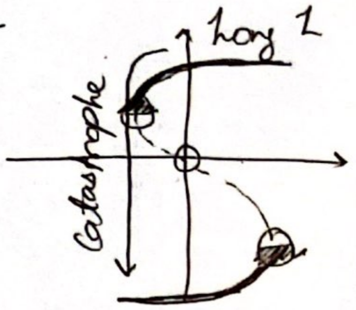
$\theta = 0$



$\underline{\theta \neq 0}$



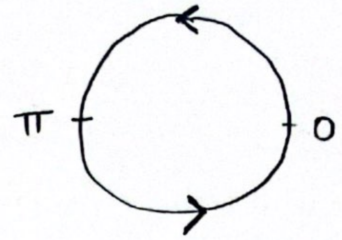




Lec 8 Phy415

Chapter 4 Flows on a Circle

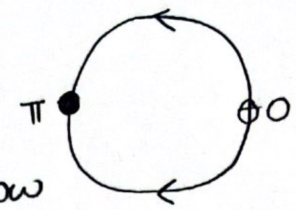
$\dot{\theta} = f(\theta)$

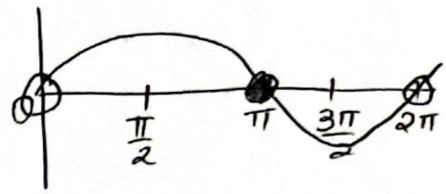


Systems like this can oscillate.

Eg

$\dot{\theta} = \sin\theta$





$\dot{\theta} > 0 \Rightarrow$ counterclockwise flow

$\dot{\theta} < 0 \Rightarrow$ clockwise flow.

→ $f(\theta)$ must be a real valued, 2π periodic function.

Eg:- Uniform oscillator.

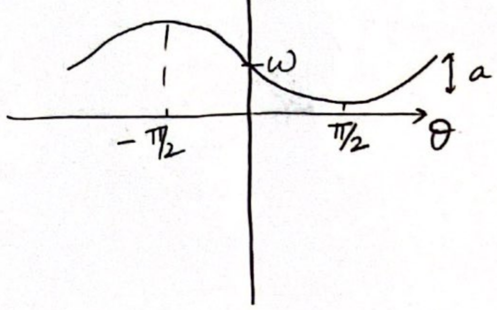
$\dot{\theta} = \omega \quad \Rightarrow \quad \theta(t) = \omega t + \theta_0$. $\Rightarrow$ uniform motion @angular frequ

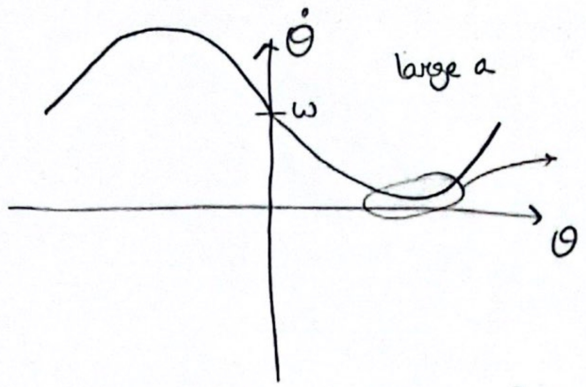
$T = \frac{2\pi}{\omega}$

$$T_{beat} = \frac{2\pi}{\omega_1 - \omega_2} = \left[\frac{1}{T_1} - \frac{1}{T_2}\right]^{-1} = \frac{T_1 T_2}{T_2 - T_1}$$

Eg:- Non uniform oscillator.

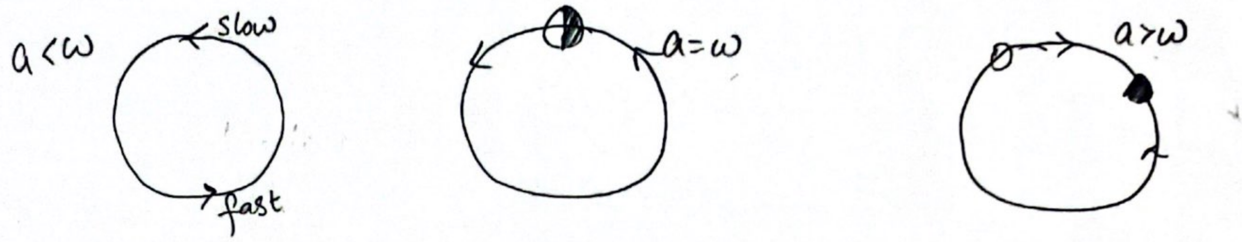
$\dot{\theta} = \omega - a\sin\theta$.



→


Bottleneck where the phase points spends a long time.

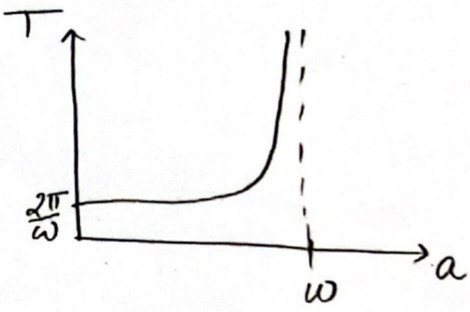
→ $a = \omega \Rightarrow$ No oscillation $\Rightarrow$ Saddle node bifurcation at $\pi/2$.



→ $a<\omega \rightarrow$ Find oscillation period.

$$T = \int dt = \int_0^{2\pi} \frac{dt}{d\theta} \cdot d\theta = \int_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta}$$ → Evaluated using $u = \tan\frac{\theta}{2}$.

$$\Rightarrow \quad T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$



Since $\sqrt{\omega^2 - a^2} = \sqrt{(\omega+a)(\omega-a)}$

$$\simeq \sqrt{(2\omega)\cdot(\omega-a)}$$

As $a \to \omega$ $\quad T \approx \frac{\pi\sqrt{2}}{\sqrt{\omega}} \cdot \frac{1}{\sqrt{\omega - a}}$

$\Rightarrow$ T blows up at $(a_c - a)^{-1/2}$: Called the square root scaling law.

→ This square root scaling law is very general to saddle node bifurcations.

> Ghosts

Just after fps collide there is a 'ghost' of the saddle node when the passage of dynamics are very slow.

> $\dot{\theta} = \omega - a\sin\theta$

Let $\phi = \theta - \frac{\pi}{2}$

$$\Rightarrow \dot{\phi} = \omega - a\cos\phi.$$

$$= \omega - a\left(1 - \frac{\phi^2}{2}\right)$$

$$= \omega - a + \frac{a}{2}\phi^2$$

Let $x = \sqrt{\frac{a}{2}}\ \phi^2 \Rightarrow \dot{x}\sqrt{\frac{2}{a}} = r + x^2$ and $\omega - a = r$.

Recall $T = \int_{-\infty}^{\infty} \frac{dx}{r+x^2} = \frac{\pi}{r} = \frac{\sqrt{2}\,\pi}{\sqrt{\omega}\sqrt{\omega - a}}$ which takes the same form as what we saw earlier. Therefore, the trajectory spends most of its' time in the ghost.

Two Dimensional Flows - Linear - MAE5790

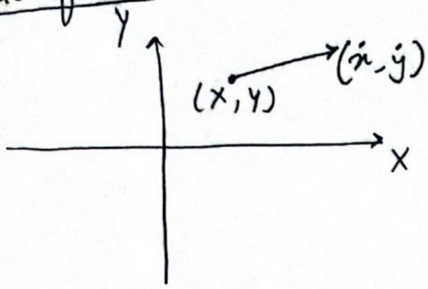
Phase plane analysis

for: $\dot{x} = f(x,y)$ $\qquad (x,y) \in \mathbb{R}$

$\dot{y} = g(x,y)$ $\qquad (\dot{x},\dot{y})$ are velocity vectors (not actually velocity).

Vector field.

$\Rightarrow$ gives a vector field on the phase plane



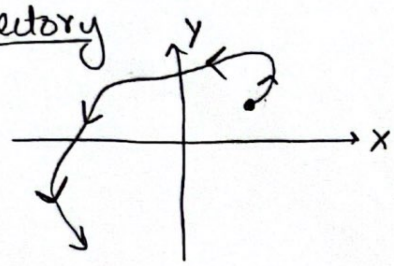
Vector form

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\dot{\vec{x}} = \vec{f}(\vec{x}) \quad \text{where } x \in \mathbb{R}^2$$

$\rightarrow$ If f is continuously differentiable, then solutions $\vec{x}(t)$ exist and are unique, for any initial conditions.

Trajectory



Implication of uniqueness:

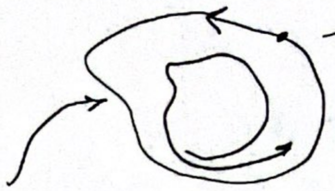
> Trajectories cannot cross!

> But trajectories can approach the same fixed point.

> Fixed point: A point where both $\dot{x} = 0$ and $\dot{y} = 0$.

$\Rightarrow \vec{f}(x^*) = 0$.

> Strong topological consequences of noncrossing trajectories in $\mathbb{R}^2$



$\rightarrow$ A closed orbit must be a periodic behaviour

$\Rightarrow$ Trajectories outside can't get inside.

$\Rightarrow$ Trajectories inside are like playing "Snake"

Goal: Given $\dot{\vec{x}} = \vec{f}(\vec{x})$, deduce phase portrait (picture of all qualitatively different trajectories), extract qualitative info. from them (existence, stability of fixed points and closed orbits).

Chapter 5

$\dot{\vec{x}} = A\vec{x}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ → real.

→ Only choosing homogeneity.

$\vec{x}^* = \vec{0}$ is always a fixed point.

Phase portrait: determined by eigenvalues and eigenvectors of A.

> Seek straight line solutions.

$\vec{x}(t) = \vec{v} e^{\lambda t}$ → assume this is a straight line solution.

$$\left.\begin{aligned} \dot{\vec{x}}(t) &= \vec{v}\lambda e^{\lambda t} \\ A\vec{x} &= A(\vec{v} e^{\lambda t}) = e^{\lambda t} A\vec{v} \end{aligned}\right\} \text{Equating the two}$$

$\dot{\vec{x}}(t) = A\vec{x} \Rightarrow (\vec{v}\lambda)\cancel{e^{\lambda t}} = (A\vec{v})\cancel{e^{\lambda t}}$

⇒ Solutions of the type $\vec{v}e^{\lambda t}$ exist if $A\vec{v} = \lambda\vec{v}$.

Therefore a linear differential equation system $\dot{\vec{x}} = A\vec{x}$ has the general solution $\vec{x}(t) = \vec{v}e^{\lambda t}$ if $A\vec{v} = \lambda\vec{v}$.

λ is given by $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$.

$\Rightarrow \lambda^2 - \tau\lambda + \Delta$ where τ is the trace $A = a+d$.
Δ is the det $A = ad - bc$.

$\therefore$ $\boxed{\lambda^2 - \tau\lambda + \Delta = 0}$ is the characteristic equation.

$$\Rightarrow \boxed{\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$$

Properties

$\tau = \lambda_1 + \lambda_2$ } τ & Δ give a lot of information
$\Delta = \lambda_1 \lambda_2$

Classification of fixed points for $\dot{\vec{x}} = A\vec{x}$, $\vec{x} \in \mathbb{R}^2$.

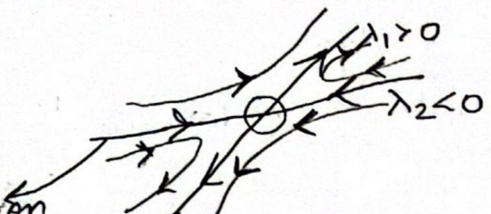
Cases: Case 1) Saddle points $\rightarrow$ $\boxed{\Delta < 0}$

Then $\lambda_1 > 0$ and $\lambda_2 < 0$ $\Rightarrow$ $\lambda_1 \neq \lambda_2$ $\Rightarrow$ distinct.

> Distinct eigenvalues $\Rightarrow$ we get 2 eigenvectors that are linearly independent, and can be orthogonalized using Gram Schmidt.

General solution: $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

= (linear combination of the eigen solutions)



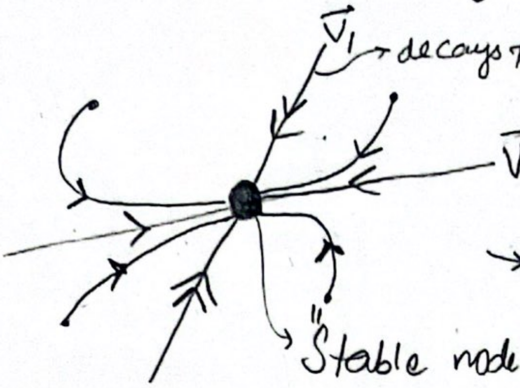
Case 2) Attracting (and Repelling) Fixed points.

Focus on Attracting $\Rightarrow$ $\boxed{\Delta > 0, \tau < 0}$ Repelling $\boxed{\Delta > 0, \tau > 0}$

2a) Nodes : $\tau^2 - 4\Delta > 0$ $\Rightarrow$ Real λ, same sign.

Suppose both λ_1 & $\lambda_2 < 0$ for attracting case and $\lambda_1 < \lambda_2 < 0$

Again $\bar{V}_1$ & $\bar{V}_2$ are linearly independant.



$\vec{V}_1 \rightarrow$ decays faster

$\vec{V}_2 \rightarrow$ slow decay λ_2 is closer to 0.

→ "pplane" is a good program for graphing this stuff

"Stable node"

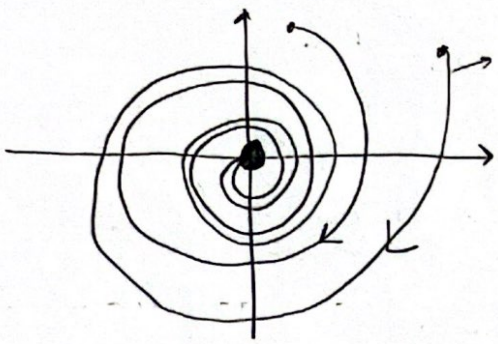
> As $t \to \infty$ (typical) trajectories approach $\vec{x}^*$ tangent to the slow direction. In backwards time $t \to -\infty \Rightarrow \vec{x}^*$ is ∥del to fast direction

> 2b) Spirals: $\Delta > 0$ $\tau < 0$, $\tau^2 - 4\Delta < 0 \Rightarrow$ complex λs.

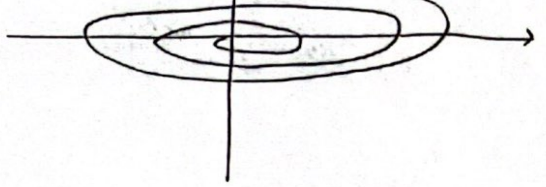
No real eigenvectors. $\lambda_1 \neq \lambda_2$ let $\lambda = \mu \pm i\omega$.

Say $\mu < 0$ ($\Rightarrow$ attracting) controls the decay rate, ω controls the rotation rate.

From Linear Algebra → $x(t) \longrightarrow$ each component of $\vec{x}(t)$ is a linear combination of $e^{\mu t}\cos\omega t$ and $e^{\mu t}\sin\omega t$.



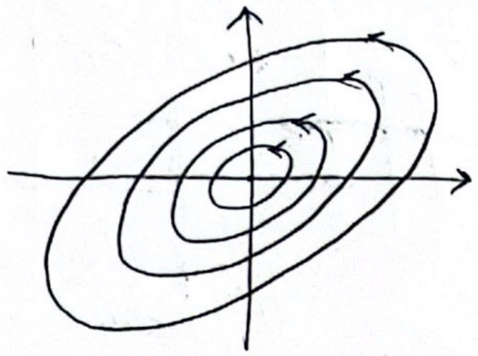
→ Direction of the spiral is not given by τ & Δ, we should just calculate it at one point and see.



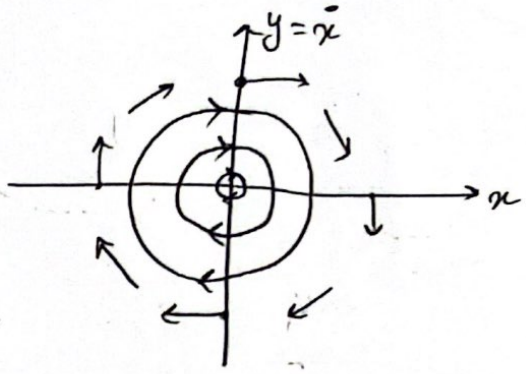
→ usually they are elliptical.

Case 3) **Center.** $\Delta > 0,\ \tau = 0,\ \lambda = \pm i\omega$

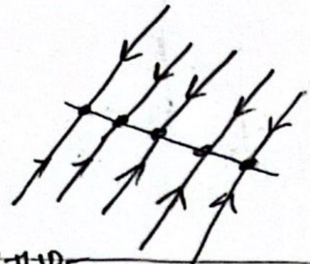
⇒ Every trajectory is closed.



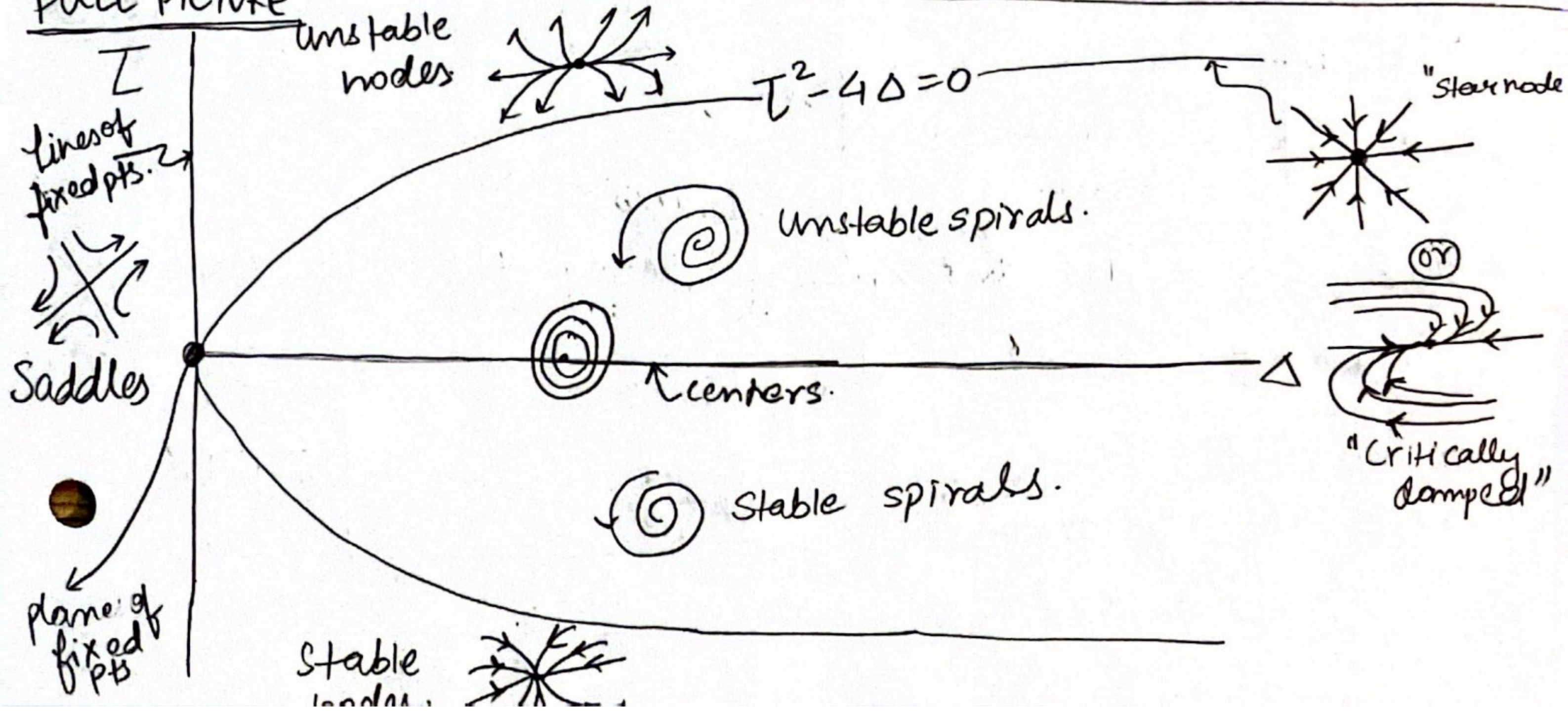
Eg: Undamped oscillator.
$\ddot{x} + x = 0$ could be written as. $\dot{x} = y$.
$\dot{y} = -x$.



Case 4) $\Delta = 0$ ⇒ $Ax = 0$ doesn't have a unique solution.
⇒ line or plane of fixed.

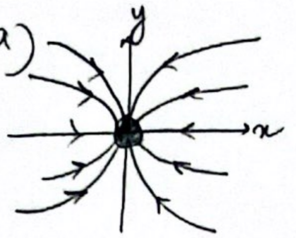


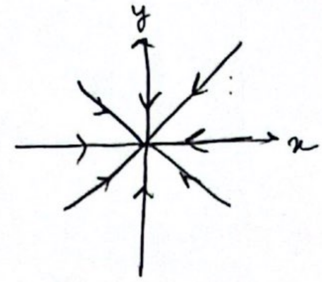
FULL PICTURE

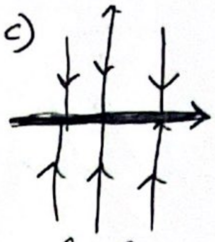


Phy413 Two dimensional linear systems.

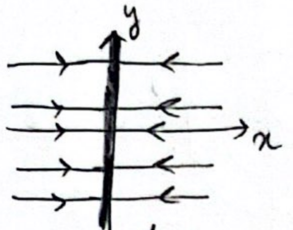
Eg: $\dot{x} = ax$
$\dot{y} = dy$ $\rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

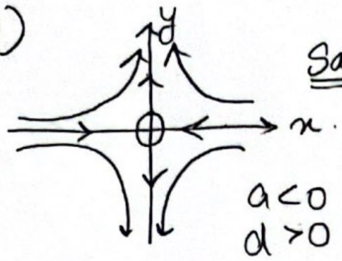
a)


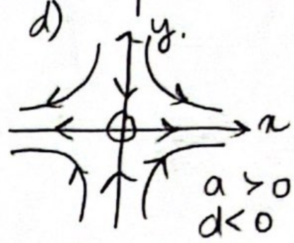
b)


c)

$a=0$

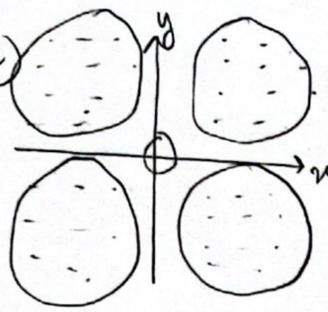
d \ a	<0	$=0$	>0
<0	$a\neq$ / $=$ (b)	(c)	(d)
$=0$	(c)	(e)	(f)
>0	(d)	(f)	(g) $\neq$ / $=$ (h)

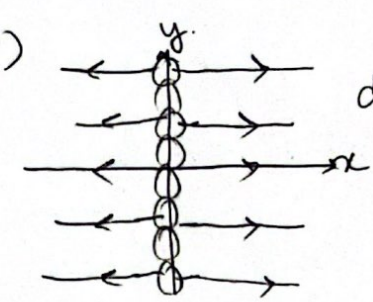
c)

$d=0$

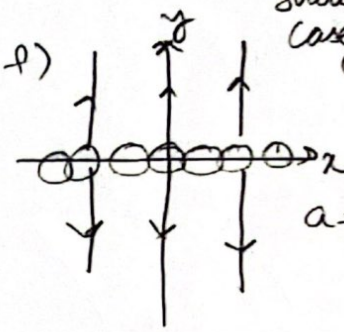
d) Saddle.

$a<0$
$d>0$

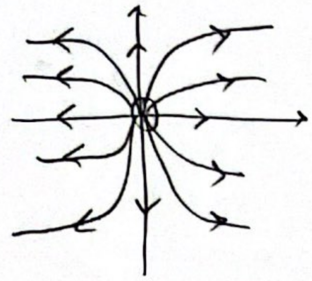
d)

$a>0$
$d<0$

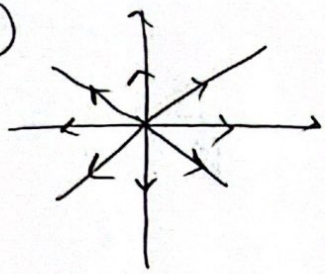
Asymptotes approach the slower side. here we are showing $|a|=|d|$ case so it's not clear.

e)


f)

$d=0$

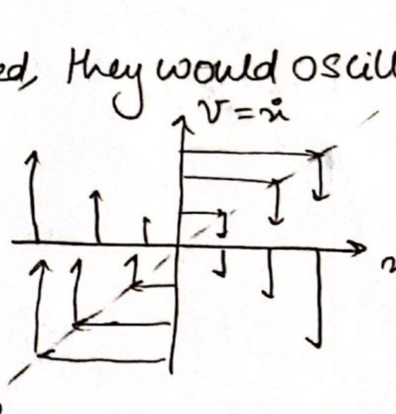
f)

$a=0$

g)


h)


→ If $\dot{x}, \dot{y}$ are coupled, and cannot be decoupled, they would oscillate.

Eg:- $m\ddot{x} = -kx \Rightarrow \dot{x} = v, \; \dot{v} = \frac{-k}{m}x \Rightarrow \dot{v} = -\omega^2 x$



If $\omega=1$ trajectories are circles because they are $\perp$lar to line through origin.

$\frac{d}{dt}(\omega^2x^2+v^2) = 2\omega^2 x\dot{x} + 2v\dot{v} = 2\omega^2 xv - 2v\omega^2 x = 0$

$\Rightarrow \omega^2x^2+v^2 = $ constant, $\Rightarrow$ Energy $= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \Rightarrow \frac{2E}{m} = \omega^2x^2+v^2$ defines constant

MAE - Lec 6 Two dimensional Nonlinear Systems.

$$\dot{x} = f(x)$$

Suppose (x^*, y^*) fixed pts. of $\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$

To classify it, consider small deviations

$$u(t) = x(t) - x^*$$

$$v(t) = y(t) - y^*$$

$$\dot{u} = \dot{x} = f(x,y) = f(x^*+u,\ y^*+v)$$

$$= \cancelto{0}{f(x^*,y^*)} + u\frac{\partial f}{\partial x}\Big|_{\vec{x}^*} + v\frac{\partial f}{\partial y}\Big|_{\vec{x}^*} + \text{H.O.T}$$

$$\dot{v} = u\frac{\partial g}{\partial x}\Big|_{\vec{x}^*} + v\frac{\partial g}{\partial y}\Big|_{\vec{x}^*} + \text{H.O.T}$$

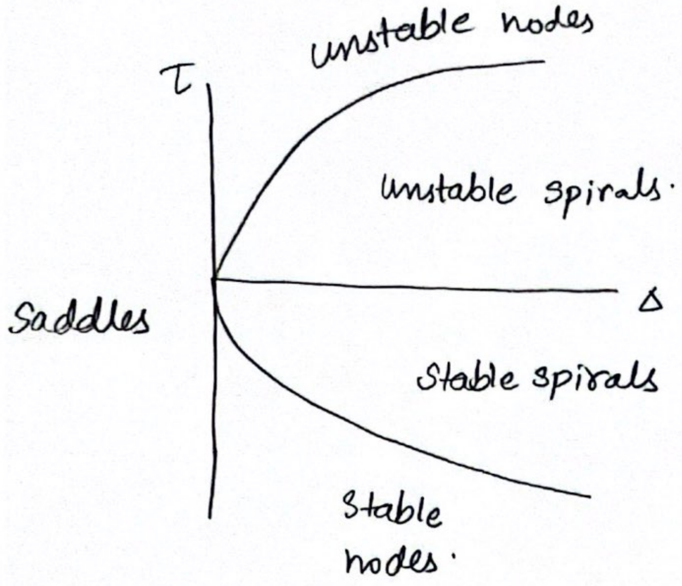
$$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix}_{(x^*,y^*)} \begin{pmatrix}u\\ v\end{pmatrix} + \text{H.O.T}$$

Linearizing $\Rightarrow$ Ignore H.O.T

$$\rightarrow \boxed{\dot{\vec{u}} = A\vec{u}}$$

$A \rightarrow$ Jacobian. $\rightsquigarrow$ gives the linearization around the fixed point.

- If $\vec{x}^*$ is a saddle, node or spiral then the linearization works.
- But boderline cases (degenerate node, star, center, nonisolated fixed pt) can be altered.



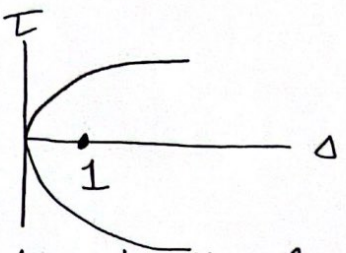
- All these fixed points that line in open spaces can be linearized.
- The "border" line cases lie on the borders! They cannot tolerate small perturbations from H.O.T

Eg:

$\dot{x} = -y + ax(x^2+y^2) \qquad \dot{y} = x + ay(x^2+y^2) \qquad , (x^*, y^*)_1 = (0,0)$

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightarrow$ This is obvious since if we ignore the H.O.T we just get $\dot{u} = -v$, $\dot{v} = u$ when the fixed point is $(0,0)$.

$\tau = 0, \quad \Delta = 1$



$\therefore$ $(0,0)$ is a center, according to the linearization for any 'a'. But this is wrong! It only works when $a=0$. But linearization ignores a.

Let $x = r\cos\theta$, $y = r\sin\theta$
$r = r(t)$, $\theta = \theta(t)$.

$\Big\}$ $x^2+y^2 = r^2 \Rightarrow 2x\dot{x} + 2y\dot{y} = 2r\dot{r}$

$\Rightarrow \boxed{r\dot{r} = x\dot{x} + y\dot{y}}$

$\dot{x} = -y + axr^2 \, ; \, \dot{y} = x + ayr^2 \, ; \, r\dot{r} = x(-y+axr^2) + y(x+ayr^2)$

$\Rightarrow r\dot{r} = ar^2(x^2+y^2) = ar^4 \qquad \Rightarrow \boxed{\dot{r} = ar^3}$

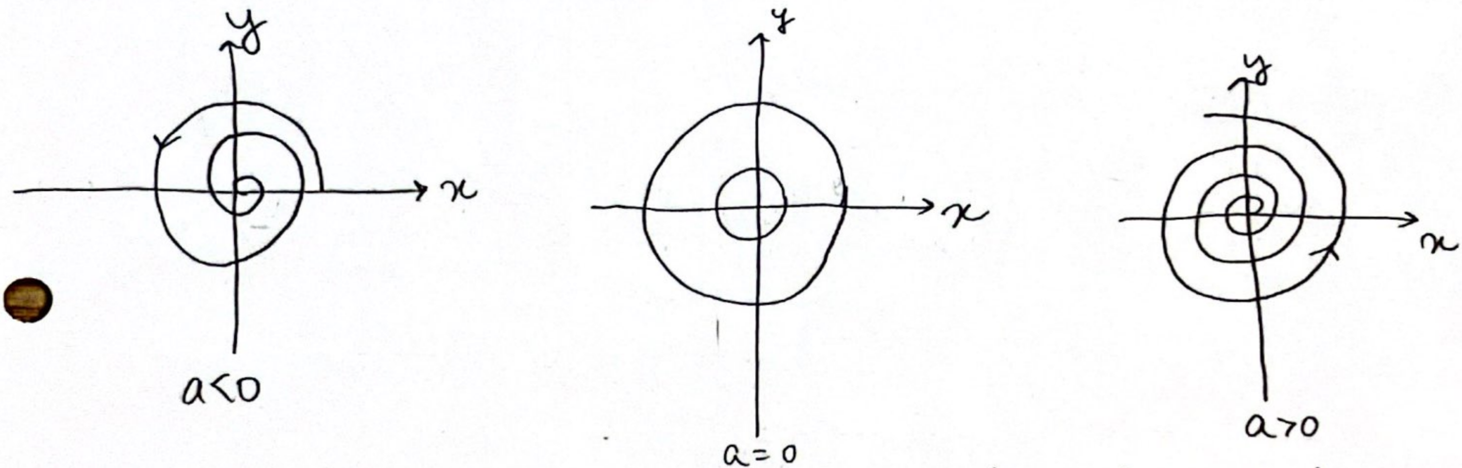
$\theta = \tan^{-1} \frac{y}{x}$

Solving,

$\Rightarrow \boxed{\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}} = \frac{x(x + ayr^2) - y(-y + axr^2)}{r^2}$

$$\boxed{\dot{\theta} = 1}$$

$\Rightarrow \left.\begin{aligned} \dot{r} &= ar^3 \\ \dot{\theta} &= 1 \end{aligned}\right\}$ 2 uncoupled one dimensional systems.



→ The H.O.T are pushing the centre away from it's border!

§ 6.4 Rabbits Vs. Sheep. Lotka-Volterra model of Competitions.

x = popn. of rabbits.
y = popn. of sheep
$x, y \geq 0$

$\dot{x} = x(3 - x - 2y)$ If $y = 0$ $\dot{x} = x(3-x)$

Sheep are mean ⇒ $-2xy$
Rabbits reproduce fast ⇒ 3 is the carrying capacity.

Logistic model!   Carrying capacity.

Fixed points : $(0,0), (3,0), (0,2), (1,1)$.

$$A = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

Classifying fixed points.

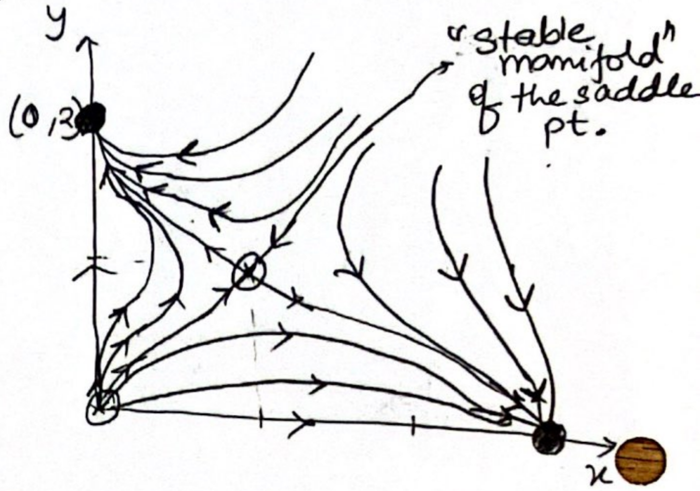
$(0,0) \Rightarrow A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ $\lambda_1 = 3$, $\lambda_2 = 2$ — unstable node. $\vec{v}_1 = (1,0)$, $\vec{v}_2 = (0,1)$

Trajectories leave along slow direction $\Rightarrow \vec{v}_2$

$(0,2) \Rightarrow A = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow$ $\lambda_1 = -1$, $\lambda_2 = -2$ — Stable node. $\vec{v}_1 = (1,-2)$, $\vec{v}_2 = (0,1)$

$(3,0) \Rightarrow A = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ $\lambda = -3, -1$ stable node.

$(1,1) \Rightarrow A = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ $\Delta = -1$ $\Rightarrow$ saddlept.



$$\dot{x} = (3 - x - 2y)x$$
$$\dot{y} = y(2 - x - y)$$

If $x = 0 \Rightarrow \dot{x} = 0 \Rightarrow$ x axis & similarly y axis is invariant.

→ Below stable manifold ⇒ rabbits win, Above stable manifold ⇒ sheep win.

What happens if the competition is reduced?

$\dot{x} = x(3 - x - y)$ $\quad \dot{y} = y(2 - y - \frac{1}{2}x)$

⇒ Fixed points at $(0,2)$, $(3,0)$, $(0,0)$, $(2,1)$

But now $(2,1)$ is a stable node and the 2 species can coexist

PHY413

Stability definitions.

Attracting. $\vec{x}(t) \longrightarrow \vec{x}^*$ as $t \to \infty$

" $\exists\ \delta > 0$ s.t $\|\vec{x}_0 - \vec{x}^*\| < \delta \Rightarrow \vec{x}(t) = \vec{x}^*$ as $t \to \infty$

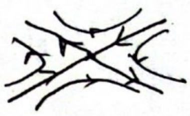
Lyapunov:
" $\forall\ \epsilon > 0\ \exists\ \delta > 0$ s.t $\|\vec{x}(t) - \vec{x}^*(t)\| < \epsilon$ if $\|\vec{x}_0 - \vec{x}^*\| < \delta$.
at all times".

Fixed points.

Purely real $[\lambda_1 > 0 \quad \lambda_2 > 0] \Rightarrow$ Unstable node (repeller) 

Purely real $[\lambda_1 < 0 \quad \lambda_2 < 0] \Rightarrow$ Stable node (attractor) 

$\lambda_1 = \lambda_2 \Rightarrow$ Stars 

Purely real. $[\lambda_1 > 0 \quad \lambda_2 < 0] \Rightarrow$ Saddle 

λ_1, λ_2 are purely imaginary $\Rightarrow$ Centers 

- λ_1, λ_2 are complex $\Rightarrow$ spirals. (stable $\Rightarrow \tau < 0$, Unstable $\Rightarrow \tau > 0$).
  attractors / repellers.

Hartman Grobman Theorem (When does linearization work?)

→ "If fixed point is hyperbolic, the local phase portrait is topologically equivalent to the linearization".

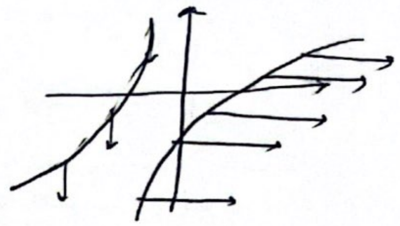
1) F.P is "hyperbolic" if $\forall\ \lambda_i$, $Re(\lambda_i) \neq 0$

Topologically equivalent ⇒ There exists a homeomorphism from the linearization to the nonlinear system close to the fixed pt.

→ Stars $\not\Longleftrightarrow$ Nodes $\Longleftrightarrow$ Spirals.

Nullclines.

Lines along which $\dot{x}=0$ or $\dot{y}=0$



In summary: 2D LSA works if $Re(\lambda_{1,2}) \neq 0$.

MAE Lec 7 - Conservative Systems.

- Consider mechanical system, 1 degree of freedom.

$$m\ddot{x} = \underbrace{F(x)}_{\text{Force}} = -\frac{dV}{dx} \text{ (potential energy)} \Rightarrow m\ddot{x} + \frac{dV}{dx} = 0 \quad \text{— (1)}$$

- F independant of $\dot{x}$ and t $\Rightarrow$ no damping & external drive.

$\Rightarrow$ The energy $E = \frac{1}{2}m\dot{x}^2 + V(x)$ is conserved.

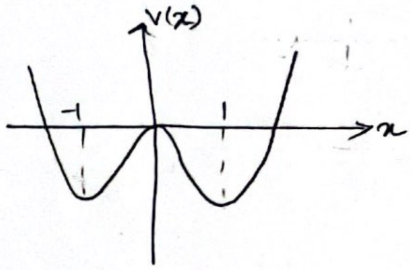
Proof: From (1) $\rightarrow$ $m\ddot{x}\dot{x} + \frac{dV}{dx}\dot{x} = 0$ $\Rightarrow$ $\frac{d}{dt}\Big(\underbrace{m\frac{\dot{x}^2}{2} + V(x(t))}_{\text{constant}}\Big) = 0$

$\Rightarrow$ E is conserved on trajectories.

- More generally, $\dot{\vec{x}} = \vec{f}(x)$ is conservative if it has a "conserved quantity" $E(\vec{x})$.
  - $E(\vec{x})$ is a continuous real valued fn that is constant on trajectories.
  - Also, this requires $E(\vec{x}) \not\equiv$ constant on any open set. Otherwise something like $E(\vec{x}) = 17$ would be a conserved quantity for every system.

Eg: Particle in a double well potential.

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$



Suppose $m=1$ for simplicity.

$$\ddot{x} = -\frac{dV}{dx} = x - x^3 \quad , \quad \text{Let } \dot{x} = y, \ \dot{y} = x - x^3$$

$$A = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

F.P: $x^*, y^* = 0, 0$
$x^*, y^* = 1, 0$
$x^* \, y^* = -1, 0$

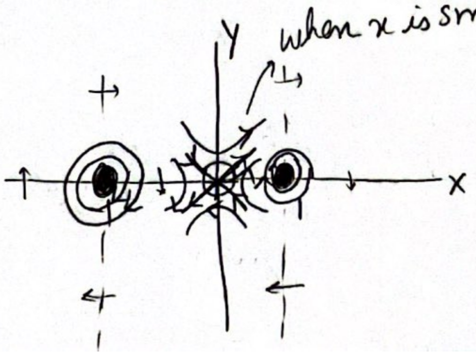
- $x^*, y^* = (0,0) \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\tau = 0$, $\Delta = -1$ $\Rightarrow$ Saddle.

$x^*, y^* = (\pm 1, 0) \Rightarrow A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \Rightarrow \Delta = 2$, $\tau = 0$ $\Rightarrow$ Linearization gives a center.

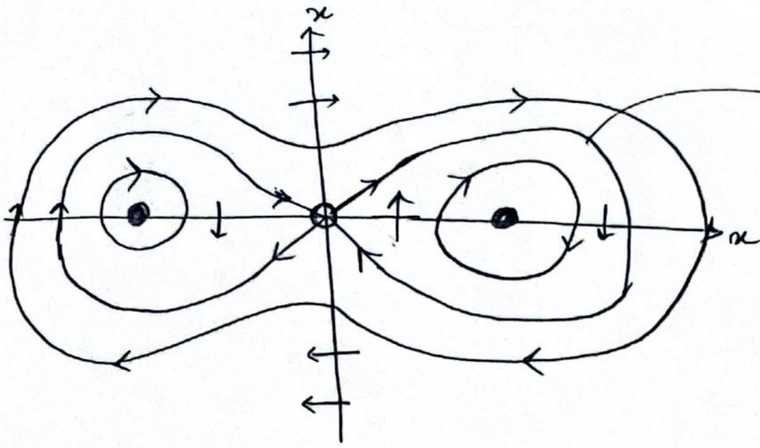
> In fact $(\pm 1, 0)$ are truly nonlinear centres in the case of conserved energy systems!

> $E = \underbrace{\frac{1}{2}y^2}_{\frac{1}{2}m\dot{x}^2} \overbrace{-\frac{1}{2}x^2 + \frac{1}{4}x^4}^{V(x)}$ = constant. These represent closed curves.

when x is small $\frac{1}{2}y^2 - \frac{1}{2}x^2 = \text{const.}$ → hyperbolas.

- 

$\dot{x} = y$
$\dot{y} = x - x^3$
$\Rightarrow$ nullclines $\dot{y} = 0$ are y axis & $x = 1, -1$.
nullclines $\dot{x} = 0 \Rightarrow$ x axis

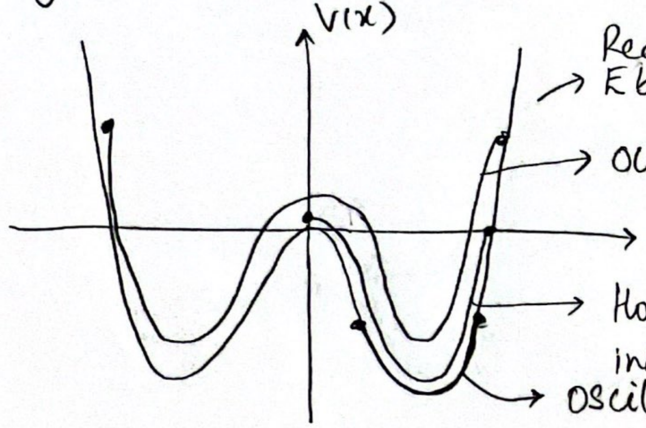


"Homoclinic orbit"
↳ same ↳ slope (incline)

> inclined to go back to same place.

⇒ A trajectory that kind of Starts & ends at the same fixed point

> Homoclinic orbits are not periodic or $T = \infty$. All other trajectories here are periodic.



Because here we are ignoring the y axis!

Really these trajectories move on constant E boundaries - This picture is just for intuition.

→ outer oscillatory orbits (sample both wells)

→ Homoclinic trajectory.

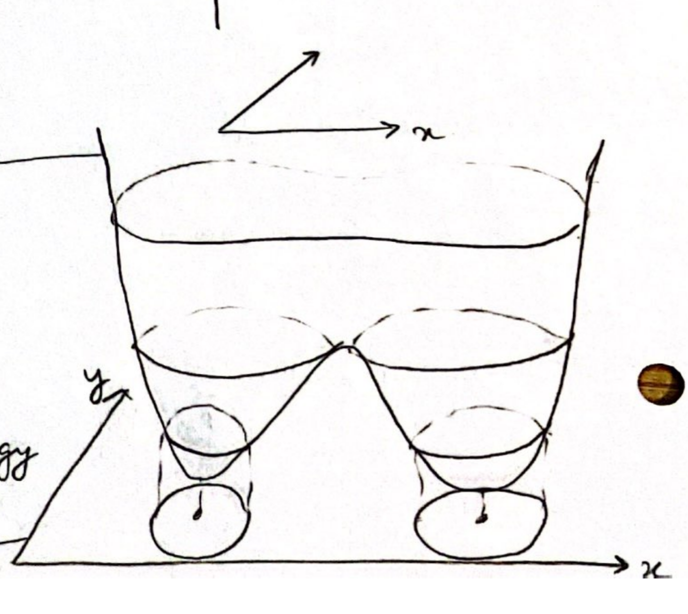
→ inner oscillatory trajectories.

Energy Surface.

$$E(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

→ On the energy surface E is conserved so particle move around with a constant height.

Energy Surface.



Projection of Energy Surface onto x,y plane gives the phase portrait.

- <u>Openset</u> : A set which doesn't include its boundary. Similar to open interval but in 2 dimensions.

*→ In conservative systems, we cannot have any attracting fixed points!

<u>Theorem</u>: (Nonlinear centres in 2D conservative systems)

Suppose $\dot{\vec{x}} = \vec{f}(\vec{x})$ = conservative. f is continuously differentiable

$\vec{x} \in \mathbb{R}^2$, $E(x)$ is a conserved quantity.

> $\vec{x}^*$ is an isolated fixed point => No other fixed points near it.

or

> If that fixed point is a local min or local max of $E(x)$, then $\vec{x}^*$ is a centre (all trajectories close to $\vec{x}^*$ are closed).

<u>Idea of proof</u>: > E is constant on trajectories → Trajectories lie <u>in</u> contours of E.

It is "in" the contour & not the whole contour since for homoclinic case it occupies only half the contour.

> Contours are closed curves near a minimum or a maximum. (needs proof) but intuitively cutting near a max or a min gives contours. 

> They need to be closed trajectories since there are no fixed points near by.

> Saddle points for example are fixed pts. without being a min. or max.

Eg: <u>Pendulum</u> (dimensionless)

$\ddot{\theta} + \sin\theta = 0$, Let $v = \dot{\theta}$ = angular velocity.

$$\left.\begin{aligned} \dot{v} &= -\sin\theta. \\ \dot{\theta} &= v \end{aligned}\right\}$$ No small angle approximation.

$$A = \begin{bmatrix} \frac{d\dot{\theta}}{d\theta} & \frac{d\dot{\theta}}{dv} \\ \frac{d\dot{v}}{d\theta} & \frac{d\dot{v}}{dv} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\cos\theta & 0 \end{bmatrix} \qquad (0,0) \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\Rightarrow \tau = 0, \Delta = 1 \Rightarrow$ Linear centre.

→ This is a conservative system $E = \frac{1}{2}v^2 - \cos\theta$ = constant, has a local min at $(0,0)$

$$E = \frac{1}{2}v^2 - \left[1 - \frac{\theta^2}{2} + \dots\right]$$

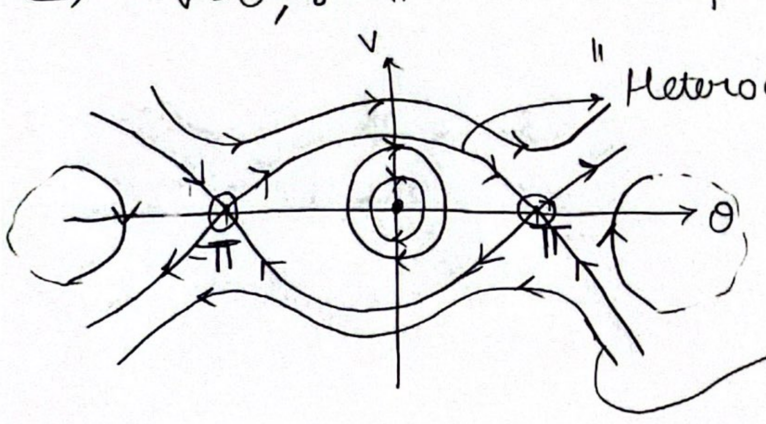
$$= \frac{1}{2}(v^2 + \theta^2) + \text{constant}$$

↳ paraboloid with min at $(0,0)$ & contours are circles.

⇒ Nonlinear centre at $(0,0)$

Makes sense because close to 0,0 the pendulum oscillates!

→ $v^* = 0, \theta^* = \pi \Rightarrow$ inverted pendulum. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow$ saddle.



"Heteroclinic orbit" ↳ Connects 2 saddles.

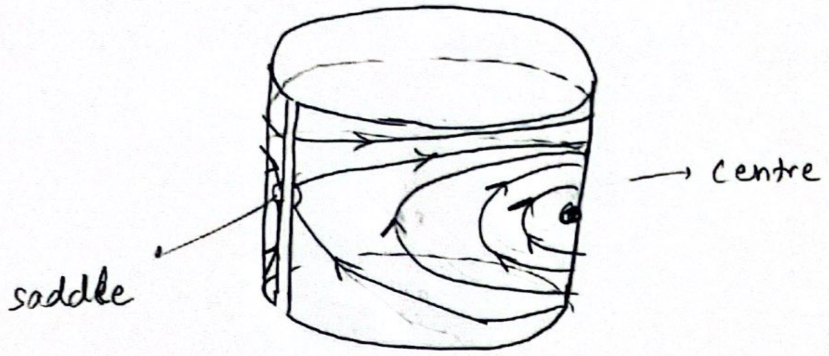
⇒ Pendulum leaves the top, makes a full circle & stops at the top.

Pendulum goes around in circles.

> If we regard all θ mod 2π $\Rightarrow$ The plane becomes a cylinder.

Cylindrical Phase Space

>



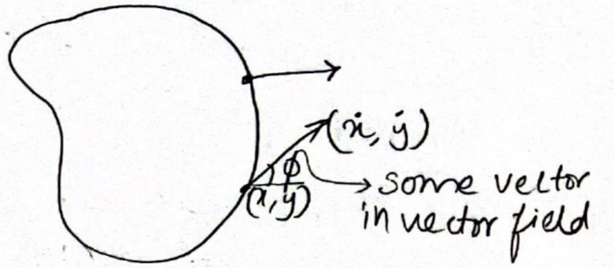
> Only one centre and one saddle.

> On the homoclinic orbit the trajectory takes infinite time to go from the start to the stop. However all other orbits go around their trajectories in finite time. The closer the particle gets to the homoclinic orbit's fixed point the slower it gets.

MAE - Lec 8 Index Theory

> Provides "global" info. about phase portraits.

> Index of a closed curve C: C = simple closed curve, not necessarily a closed trajectory, should not pass through a fixed point.



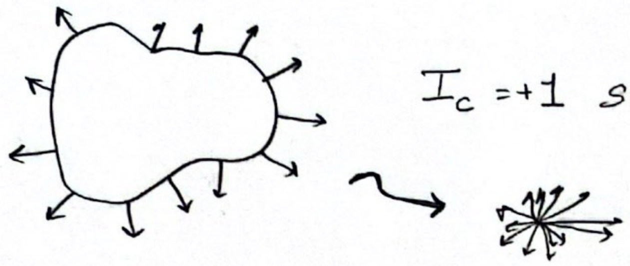
$\phi = \tan^{-1} \frac{\dot{y}}{\dot{x}}$

As $\vec{x} = (x, y)$ goes around C once counterclockwise, then ϕ changes continuously (based on the vector field at those points) if $\dot{x}, \dot{y}$ are continuous functions of $\vec{x}$.

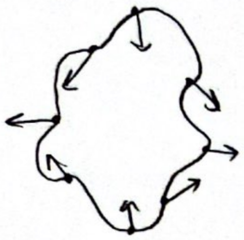
> Let $[\phi]_c$ = net change in ϕ when we go around C.
Divide by 2π $\Rightarrow$ $I_c = \frac{1}{2\pi}[\phi]_c$ is the index of C w.r.t the vector field $(\dot{x}, \dot{y})$

> It is basically the no. of times a chalk along the vector rotates as it is slid around the contour.

Eg:



$I_c = +1$ since chalk goes around counterclockwise



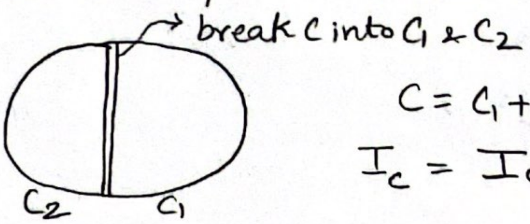
$\Rightarrow I_c = -1$

Index is always an integer? Yes!!

Properties.

i) Index of a closed trajectory. $\Rightarrow$ The vectors would be tangent to the trajectory/curve $\Rightarrow$ $\boxed{I_c = +1}$ "Cannot be -1".

ii) Index is additive if we subdivide the curve 'C'.



$C = C_1 + C_2$

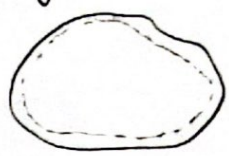
$I_c = I_{c_1} + I_{c_2}$

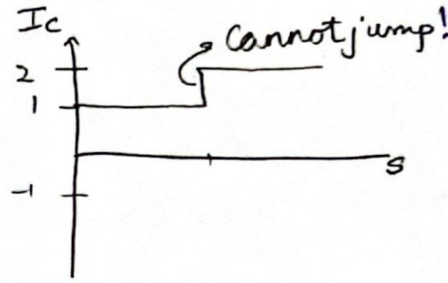
The angles on the bridge cancel since one goes ↓ & other goes ↑.

iii) If C is deformed continuously into C' without passing through a fixed pt., then $I_c = I_{c'}$. This is like Gauss' Law, Flux through a surface doesn't change if charge doesn't change.

Proof: I_c depends continuously on C. Eg: shrink it a little.
$\Rightarrow$ vectors change a little & ϕ also changes very little.

But I_c is an integer $\Rightarrow$ if we have a continuous function that is integer valued. $\Rightarrow$ it must be a constant.





> Fixed pts. cause problems because the vector there has no ϕ so it is undefined.

iv) If C does not enclose a fixed point $\Rightarrow I_c = 0$.

On a small closed curve vector field is constant $\Rightarrow I_c = 0$. Any curve can be shrunk down to a small curve if there are no fixed points inside.

v) If $t \to -t$ all the arrows rotate by $180°$ $\Rightarrow \vec{x}$ now is $-\vec{x} \Rightarrow$ ϕ at each point goes to $\phi + \pi \Rightarrow [\phi]_c$ unchanged. Index is invariant to reversing the time. It doesn't say anything about stability.

<u>Index of a point P</u> = I_c, for any C that encloses P and no other fixed points. (By iii))

Index of points.

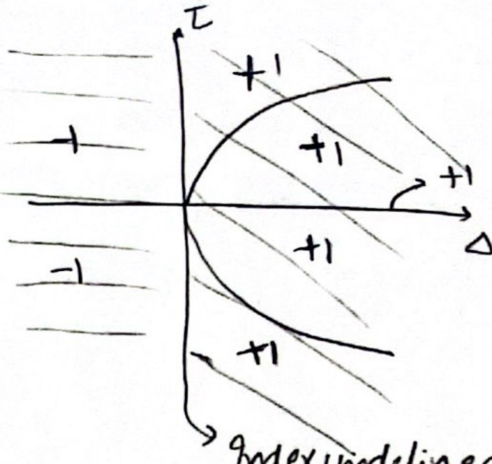
Index of node $I_{node} = +1$ for both stable & unstable.

$I_{saddle} = -1$

$I_{spiral} = +1$

$I_{center} = +1$

$I_{ordinary\ pt.} = 0$



Index undefined since FPs are non isolated

Eg:- $\dot{z} = z^2$ & take $z = x+iy$.

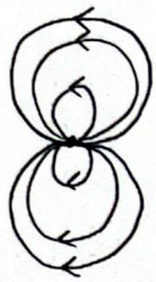
$\Rightarrow \dot{x}+i\dot{y} = x^2-y^2+i(2xy)$

$\Rightarrow \left.\begin{array}{l}\dot{x} = x^2-y^2\\ \dot{y} = 2xy\end{array}\right\}$ linearization gives $\begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow$ useless!

There is a single point with Index = 2 or -2 (not sure)

In general $\dot{z} = z^n$ gives more exotic fixed points. Other case is $\dot{z} = \bar{z}^n$

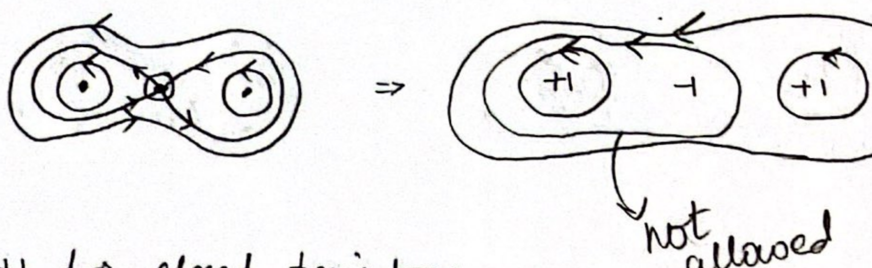
Eg:-



$I = 2$

Theorem: Any closed trajectory on $\mathbb{R}^2$ must enclose fixed points whose indices obey $\sum_{k=1}^{N} I_k = +1$. Fixed points must be isolated

Eg:-



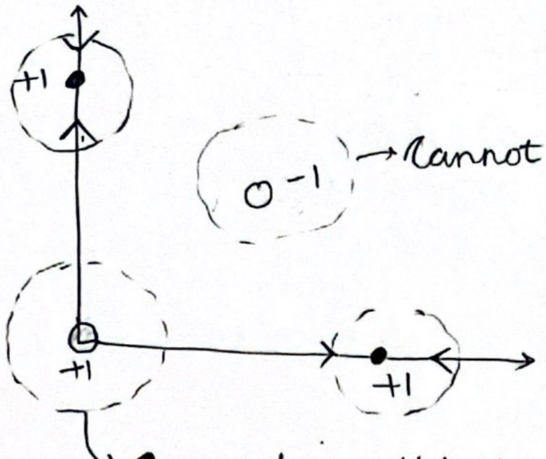
not allowed

Proof: $I_c = +1$ for closed trajectory & rule iii)

Eg: Can use index theory sometimes to rule out closed trajectories.

Rabbits vs. sheep.

$\dot{x} = x(3-x-2y)$ $\dot{y} = y(2-x-y)$



→ Cannot have this since $I = -1$

Cannot have this since it crosses a trajectory & same for any other closed curves.

→ L. Glass, Science, 1977, Vol. 198, 321.

> If you cut off salamander hands they grow back. If you cut off the right hand & attach it to the left side (after cutting the hand off on the left side), the salamander grows 2 left hands to preserve the "index" of the left arm.

→ Hairy Ball Theorem

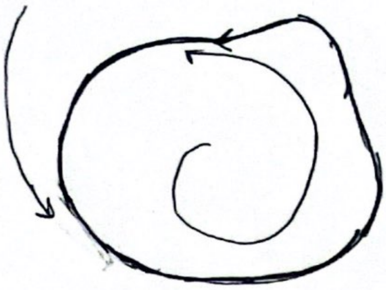
→ The index of a spherical surface is always 2.

→ ⇒ If the vectors were hair, they could not be combed down without zeros ⇒ no vectors there. In a taurus (dough nut) shape you can.

→ In general the index of a surface $I = 2 - 2g$. Where g is the genus. g = # handles. Sphere ⇒ no handles. Taurus ⇒ $g = 1$

→ Poincare Hopf index theorem: $\sum I_k = 2 - 2g$.

Chapter 7 Limit Cycles - Isolated closed trajectories!



→ Could have unstable & half stable cycles.

Eg: Heartbeat, biological rythims (bodytemperature, hormones), feedback control systems, aeroelastic flutter. , mechanical vibrations, chemical oscillations.

* Linear systems DO NOT have limit cycles.
↳ $\dot{\vec{x}} = A\vec{x}$. ⇒ periodic solutions are not isolated.

Since if $\vec{x}(t)$ is periodic, so is $c\,\vec{x}(t)$ for any c.
Eg:- Simple Harmonic Oscillator. (remembers variations forever).

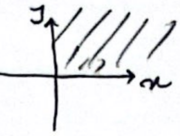
Lec 9 - MAE: Testing for closed orbits.

Ruling out closed orbits.

a) Index theory:- A closed orbit in $\mathbb{R}^2$ must encircle fixed points whose indices add up to +1.
Less powerful.

b) Dulac's criterion:- Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ smooth, $\vec{x} \in \mathbb{R}^2$, R = "simply connected" region in $\mathbb{R}^2$. Simply connected ⇒ "no holes".
If ∃, a smooth function $g(\vec{x})$ such that $\nabla\cdot(g\,\dot{\vec{x}})$ has one sign in R, then ∄ a closed orbit within R.

⇒ $\nabla\cdot(g\dot{\vec{x}})$ is strictly +ve or strictly -ve in R. Cannot be 0.

Ex: Show that, $\dot{x} = x(2-x-y)$
$\dot{y} = y(4x - x^2 - 3)$ has no closed orbits in the region $x>0, y>0$.  . The problem is coming up with a function g!

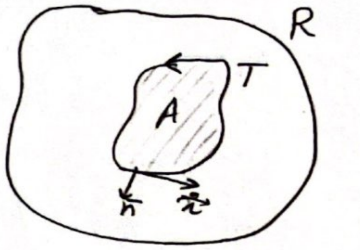
Solution: Pick $g = \frac{1}{xy}$ (usually through trial & error).

$$\nabla \cdot (g\vec{\dot{x}}) = \frac{\partial}{\partial x}(g \cdot \dot{x}) + \frac{\partial}{\partial y}(g \cdot \dot{y}) = \frac{\partial}{\partial x}\left(\frac{2-x-y}{y}\right) + \underbrace{\frac{\partial}{\partial y}\left(\frac{4x-x^2-3}{x}\right)}_{0}$$

$\Rightarrow \nabla \cdot (g\vec{\dot{x}}) = -\frac{1}{y} < 0$ in R $\Rightarrow$ no closed orbits.

Achilles heel: Hard to guess g. Try $g = 1$, $g = \frac{1}{xy}$, $g = \frac{1}{x^a y^b}$, e^{kx}, e^{ky}

Proof of Dulac: Proof by contradiction.
Let T = closed orbit in the region R, A = region inside T.
$\vec{\dot{x}}$ is tangential to T



Green's Theorem / Div Thm: $\iint_A \nabla \cdot \vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, dl$.

$$= \underbrace{\iint_A \underbrace{\nabla \cdot (g\vec{\dot{x}})}_{\text{has one sign}}\, dA}_{\text{has same sign}} = \oint_C g \underbrace{\vec{\dot{x}} \cdot \vec{n}\, dl}_{\text{always 0 since } \vec{\dot{x}} \,\&\, \vec{n} \text{ are } \perp^{lar}}$$

$\Rightarrow$ There cannot be such a closed trajectory since sign = 0 is a contradiction.

Ex: $\dot{x} = y$; $\dot{y} = -x - y + x^2 + y^2$, let $g = e^{-2x}$ (magic!)

$$\frac{\partial}{\partial x} y e^{-2x} + \frac{\partial}{\partial y}\left[(\cancel{-x} - y + \cancel{x^2} + y^2)e^{-2x}\right] = -2ye^{-2x} + e^{-2x}(+2y-1)$$

$= -e^{-2x}$ always $-ve \Rightarrow$ this system has no closed orbits anywhere!

Section 7.3

How to prove the existence of a closed orbit

Poincaré – Bendixson Theorem (valid in 2-D).

i) Suppose we have a closed bounded region in $\mathbb{R}^2$.

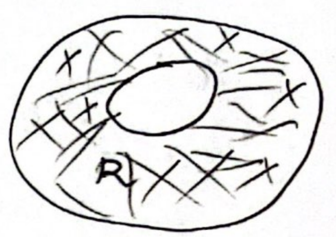
ii) $\dot{\vec{x}} = \vec{f}(\vec{x})$ is smooth.

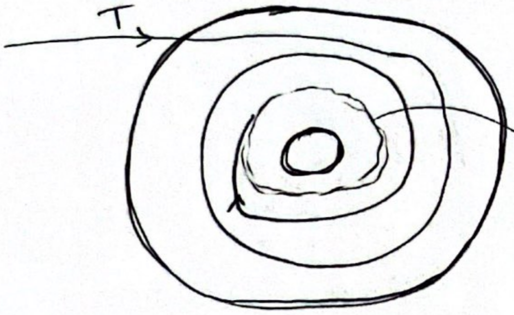
iii) There are no fixed points in the region R.

iv) $\exists$ a trapped trajectory T, i.e. $T = (x(t), y(t))$ lies in R for $t = t_0$ and stays in R for all $t > t_0$.

Then: T is either a closed trajectory, or T spirals toward a closed trajectory as time goes to ∞. Either way there is a closed trajectory in the region.

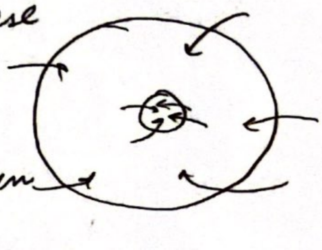
- The region R must be "washer" (annulus) shaped to contain trapped trajectories without containing fixed points.





→ closed orbit.

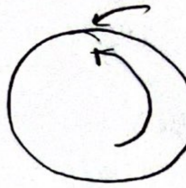
Standard trick: Find an annulus such that the vector field points into the annulus on its boundary. In this case all trajectories are stuck inside.



Ex: Consider $\dot{r} = r(1-r^2) + \mu r\cos\theta$ and $\dot{\theta} = 1$, when

$\mu = 0$; $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1 \Rightarrow$ decoupled.

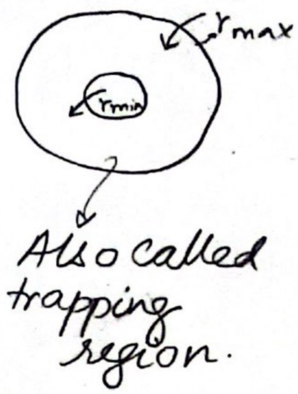
$\Rightarrow$



$\Rightarrow$ stable limit cycle at $r = 1$

> Show that the closed orbit still exists if μ is small but not zero.

Soln :- Find 2 circles with inward flow towards R.



Choose r_{max} s.t $\dot{r}<0$ when $r=r_{max}$

$$\dot{r} = r\left((1-r^2)+\mu\cos\theta\right)$$

$r_{max} > \sqrt{1+\mu} \Rightarrow \dot{r}<0$ since $\max(\cos\theta)=1$

$r_{min} < \sqrt{1-\mu} \Rightarrow \dot{r}>0$ & μ is small $\Rightarrow \mu<1$.

$\Rightarrow$ There is a closed orbit between r_{max} & r_{min}.

Ex: Glycolysis

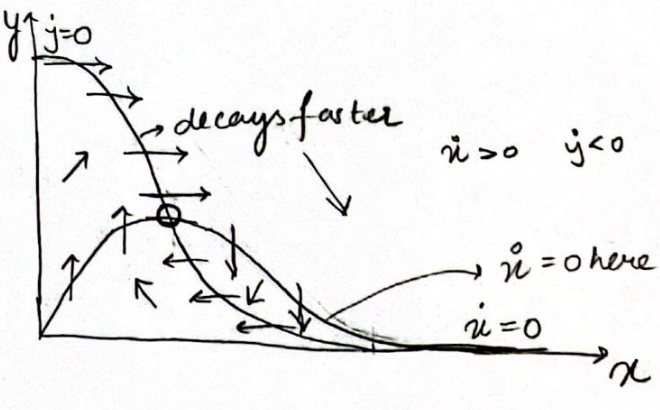
$$\dot{x} = x + ay + x^2y$$
$$\dot{y} = b - ay - x^2y$$

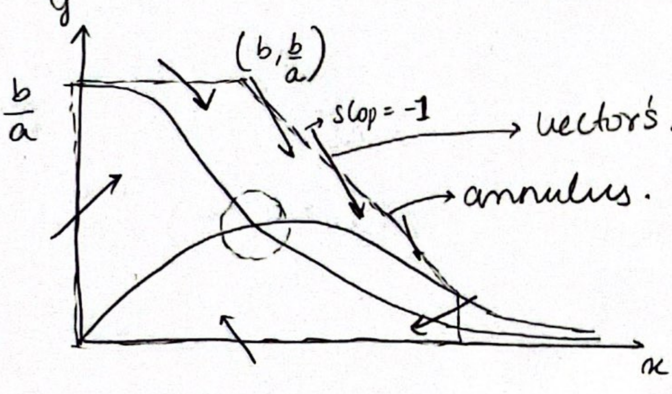
$x = [ADP] \rightarrow$ adinosene diphosphate.
$y = [F6P] \rightarrow$ fructose 6 phosphate
$a, b > 0$ kinetic parameters.
$x, y > 0$ ($\because$ conc.)

① Construct a trapping region, using nullclines.

$\dot{x}=0 \Rightarrow y = \dfrac{x}{a+x^2}$

$\dot{y}=0 \Rightarrow y = \dfrac{b}{a+x^2}$





> Intuition on how to arrive at the annulus/trapping region

$x, y \gg 1 \Rightarrow \dot{x} \simeq x^2 y \quad \dot{y} = -x^2 y$

$\Rightarrow \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \simeq -1 \Rightarrow \frac{dy}{dx} \simeq -1$ when $x, y \gg 1$

Compare $\dot{x}$ and $-\dot{y}$: $\dot{x} - (-\dot{y}) = \dot{x} + \dot{y} = b - x$

$\Rightarrow -\dot{y}$ is $>$ $\dot{x}$ if $x > b$. $\Rightarrow$ when $x > b \Rightarrow$ vectors have a slope steeper than -1.

> What is happening around the hole? We want the fixed pt. to be a repeller.

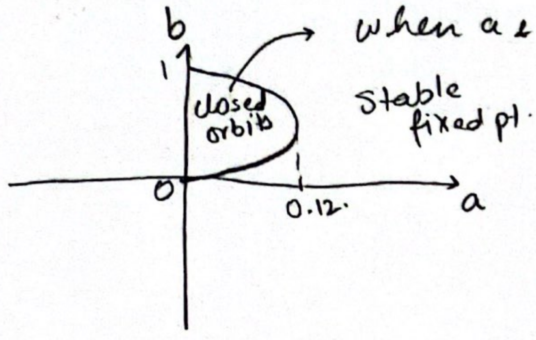
$$A = \begin{bmatrix} -1+2xy & a+x^2 \\ -2xy & -a+x^2 \end{bmatrix} \qquad x^* = b, \quad y^* = \frac{b}{a+b^2}$$

Jacobian

$\Rightarrow \Delta = (a+b)^2, \quad \tau = -\frac{(b^4 + (2a-1)b^2 + (a+a^2))}{a+b^2}$, we want $\tau > 0$

$\Delta > 0$

since we want a repeller. $\Rightarrow b^2 = \frac{1}{2}(1 - 2a \pm \sqrt{1-8a})$. is when $\tau = 0$



when a & b lie here $\tau > 0 \Rightarrow$ hole is repelling $\Rightarrow$ we get a closed orbit.

Lec-10 Van der Pol oscillator.

> $\ddot{x} + \mu \dot{x}(x^2-1) + x = 0$

> Nonlinear damping, x is large $\Rightarrow$ damping is +ve & oscillations decay. x is small $\Rightarrow$ negative damping $\Rightarrow$ pumping!

> Using Poincaré-Bendixson, we can prove $\exists!$ stable limit cycle for all $\mu > 0$.
uniquely exists

> Here let us look at limiting cases for μ to make it simpler.
$\Rightarrow$ Very nonlinear and weakly nonlinear cases.

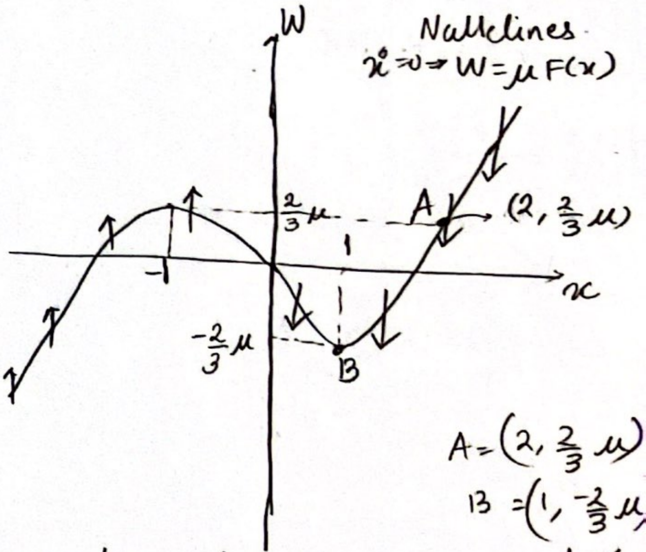
Very nonlinear, μ large $\Rightarrow$ relaxation oscillators § 7.5

> Use the Liénard Transformation. Causes the limit cycle to approach a constant shape as $\mu \to \infty$. (Very useful!)

> Note: $\ddot{x} + \mu \dot{x}(x^2-1) = \frac{d}{dt}(\dot{x} + \mu \frac{1}{3}x^3 - \mu x)$. Let $w = \dot{x} + \mu F(x)$

$\Rightarrow \dot{w} = -x$ from VdP

$$\Rightarrow \boxed{\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x \end{aligned}}$$



$A = (2, \frac{2}{3}\mu)$
$B = (1, -\frac{2}{3}\mu)$

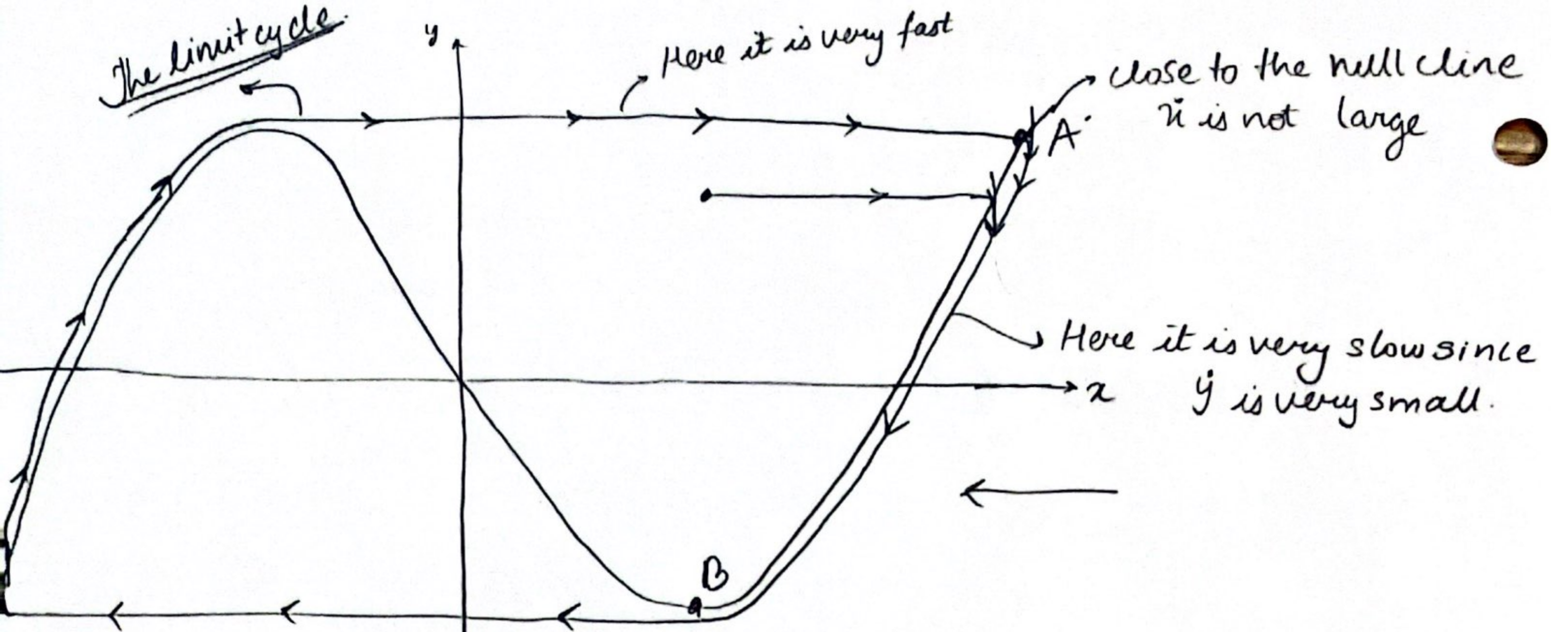
→ Let $\boxed{y = \frac{w}{\mu}}$ to rescale, since A, B have μ in their y coordinate & $\mu \to \infty$.
$y \sim O(1)$ for large μ

$$\dot{x} = \mu(y - F(x))$$
$$\dot{y} = -\frac{1}{\mu}x$$

$\Rightarrow \dot{x} \sim O(\mu) \to$ very large when y is not $\cong F(x) \Rightarrow$ far from nullcline
$\dot{y} \sim O(\frac{1}{\mu}) \to$ very small. also $\dot{y}$ is -ve when x is +ve

$\Rightarrow$ all vectors basically point $\longrightarrow$

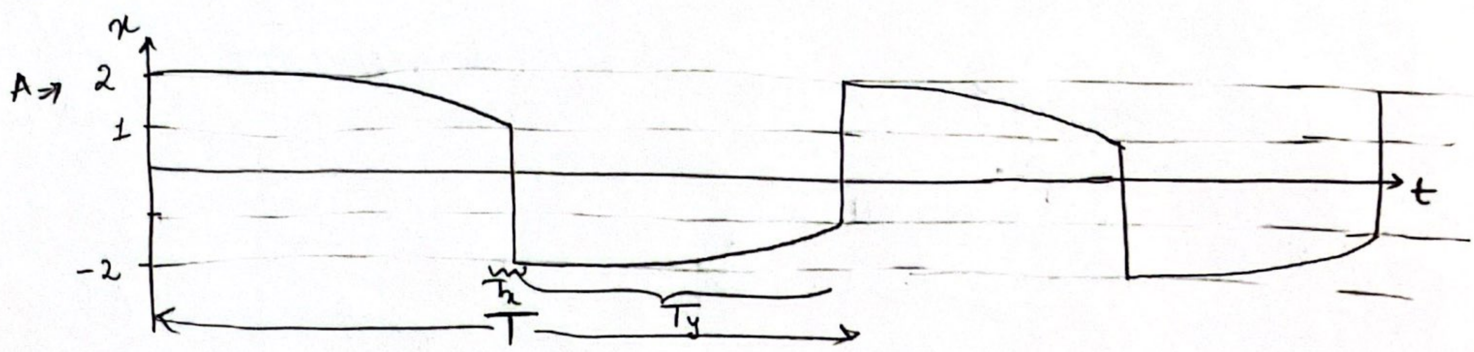
The limit cycle



> Since there is a slow build up & fast release we see relaxation oscillation.

$\dot{x} = \mu(y - F(x)) \Rightarrow \dot{x} < 0$ iff $y < F(x)$. $\Rightarrow$ x moves to the left only when $y < F(x)$ therefore the trajectory hugs the curve on the outer side.

> Waveform!



> y has velocity $\sim O(\frac{1}{\mu})$ & distance $\sim O(1) \Rightarrow \underset{T_y}{\text{Time}} \sim O(\mu)$,
& $T_x \sim O(\frac{1}{\mu}) \Rightarrow$ Period of oscillation is $\sim O(\mu)$.

→ Estimate period $T(\mu)$: $T \sim 2(\text{time to go from } A \text{ to } B)$.

Use fact that $w \simeq \mu f(x) \Rightarrow$ we are essentially on the nullcline.

$$T = 2\int_{t_A}^{t_B} dt \approx -2 \int_1^2 \frac{dt}{dw}\frac{dw}{dx}\, dx$$

$\dot{w} = -x$; $\frac{1}{\dot{w}} = \frac{-1}{x}$; $\frac{dw}{dx} \simeq \frac{d}{dx}(\mu F(x)) = \mu F'(x) = \mu(x^2-1)$

$$\Rightarrow T \simeq -2\int_1^2 \frac{-1}{x}\,\mu(x^2-1)\,dx$$

$$= 2\mu\left(\tfrac{1}{2}x^2 - \ln x\right)\Big|_1^2$$

$$T = \boxed{2\mu\left(\tfrac{3}{2} - \ln 2\right)} = O(\mu)$$

Weakly Nonlinear Van der Pol (§7.6) $\mu \ll 1$ More mathy

$\ddot{x} + x + \epsilon\dot{x}(x^2-1) = 0$ $\quad 0 \le \epsilon \ll 1$

- When $\epsilon = 0$, we have harmonic oscillator. | Here freq is 1.
  & all orbits are circular, period $= 2\pi$.

  $x(t) = A\cos t$, $A =$ constant & depends on initial condition.

- For small ϵ, except all orbits to be nearly circular & since centers are so sensitive, they are spirals that repeat every 2π.
- How to find the limit cycle + what's the amplitude.

Method 1 (crude): Look at the change in energy over 1 cycle $\to \Delta E$.
On cycle, we have $\Delta E = 0$. Other trajectories have $\Delta E > 0$, $\Delta E < 0$.

- $E = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2$; $\frac{dE}{dt} = x\dot{x} + \dot{x}\ddot{x} = \dot{x}(x + \ddot{x}) = \dot{x}(-\epsilon\dot{x}(x^2-1))$

- Now suppose $x = A\cos t + O(\epsilon)$, here A to be determined.
  $\Rightarrow \dot{x} = -A\sin t + O(\epsilon)$

The change in energy over 1 cycle; $\Delta E = \int^{T} \frac{dE}{dt}\,dt = -\epsilon \int^{2\pi + O(\epsilon)} A^2\sin^2 t\,(A^2\cos^2 t - 1)$

Due to $O(\epsilon^2)$ we can ignore $O(\epsilon)$ in the limit of integration.

$$\Rightarrow \quad \Delta E = -\epsilon\left[A^4 \underbrace{\langle \sin^2 t \cos^2 t\rangle}_{\frac{1}{8}} 2\pi - A^2 \underbrace{\langle \sin^2 t\rangle}_{\frac{1}{2}} 2\pi\right]$$

$$= -2\pi\epsilon A^2\left[\frac{A^2}{8} - \frac{1}{2}\right]$$

$\Rightarrow \quad A^2 = 4 \Rightarrow \boxed{A = 2}$ gives $\Delta E = 0 \Rightarrow$ Energy is conserved $\Rightarrow$ limit cycle.

$\Rightarrow$ VdP for weakly nonlinear case has an amplitude 2.

$$\Rightarrow \boxed{x(t) = 2\cos t + O(\epsilon).}$$

Lec 11 <u>Averaging Theory for weakly Nonlinear Oscillators.</u>

> Consider: $\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \quad , \quad \epsilon \ll 1$

$\hookrightarrow$ Small perturbation to linear SHO.

$\hookrightarrow$ In the book he presents seperation of timescales which is different.

$\dot{x} = y$

$\dot{y} = -x - \epsilon h(x, y)$

> When $\epsilon = 0$, soln's of the form $\left.\begin{array}{l} x(t) = r\cos(t+\phi) \\ y(t) = -r\sin(t+\phi)\end{array}\right\}$ r, ϕ are constant on trajectories.

> When $\epsilon \neq 0$, we expect slow drift of r, ϕ, but trajectories will stay nearly circular with a "period" $\simeq 2\pi$.

* > Let $\left.\begin{array}{l} x(t) = r(t)\cos(t+\phi(t)) \\ y(t) = -r(t)\sin(t+\phi(t))\end{array}\right\}$ View this as a definition of $r(t)$ and $\phi(t)$. i.e. $r(t) = \sqrt{x^2(t) + y^2(t)}$

$\tan(t+\phi(t)) = \dfrac{-y(t)}{x(t)}$

> Find eqns for $\dot{r}, \dot{\phi}$: $\quad r^2 = x^2 + y^2 \Rightarrow r\dot{r} = x\dot{x} + y\dot{y}$

$$= xy + y(-x - \epsilon h)$$

$$= -\epsilon y h = -\epsilon h(-r\sin(t+\phi))$$

$$\Rightarrow \boxed{\dot{r} = \epsilon h \sin(t+\phi)}$$

Similarly,

since $t + \phi(t) = \arctan^{-1} \frac{y(t)}{x(t)}$.

$\frac{d}{dt}$ on both sides gives,

$$** \boxed{\dot{\phi} = + \frac{\epsilon h}{r} \cos(t+\phi)} **$$

Both $\dot{r}$ and $\dot{\phi}$ are order $\epsilon \Rightarrow$ evolution is slow..

$\Rightarrow h = h(x,y) = h(r\cos(t+\phi), -r\sin(t+\phi)) \longrightarrow$ it is time dependant

$\Rightarrow$ We have made our system <u>non autonomous</u>! $\Rightarrow$ time dependant

$\Rightarrow$ The arrows in phase plane are no longer frozen.

> To fix this we use seperation of time scales. Since time scale for oscillations are much faster than time scale for $\dot{r}$ & $\dot{\phi}$.

→ Iron out fast oscillations by averaging over one cycle of length 2π. Given $g(t)$, define average over one cycle about the point t as

$$\bar{g}(t) = \langle g \rangle (t) = \frac{1}{2\pi} \int_{t-\pi}^{t+\pi} g(s)\, ds.$$

→ Time scale of the faster time

→ This is a running average. As t moves along we have a window around it that averages things. "Moving avg"

→ Taking my height as an example. the instantaneous value of my height is equal to the moving average if the time scale is on the order of seconds or higher. Therefore as long as the timescale is approriately chosen we can replace a variable by its average. The error here is within an epsilon order in one cycle.

Observe: $\overline{\dot{g}} = \dot{\bar{g}}$ because $\overline{\dot{g}} = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} \frac{dg}{ds}\,ds$

$$= \frac{1}{2\pi}\left[g(t+\pi) - g(t-\pi)\right]$$

> Recall the formula of taking time derivative when limits have 't'. from
> Fundamental theorem of Calculus.

$$\& \ \dot{\bar{g}} = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{t-\pi}^{t+\pi} g(s)\,ds\right]$$

$$= \frac{1}{2\pi}\left[g(t+\pi) - g(t-\pi)\right]$$

> <u>Derive equations for $\bar{r}$, $\bar{\phi}$.</u>

$$\overline{\dot{r}} = \dot{\bar{r}} = \langle \epsilon h \sin(t+\phi)\rangle_{t \to \text{avg. w.r.t } t.}$$

$$\overline{\dot{\phi}} = \dot{\bar{\phi}} = \left\langle \frac{\epsilon h}{r} \cos(t+\phi)\right\rangle_t$$

So far everything is exact.

> But we want $\dot{\bar{r}}$ & $\dot{\bar{\phi}}$ to depend on $\bar{r}$ and $\bar{\phi}$. This would not be exact.

> Let us now replace r & ϕ by $\bar{r}$ and $\bar{\phi}$ which incurs some error

> Over one cycle, $r = \bar{r} + O(\epsilon)$, $\phi = \bar{\phi} + O(\epsilon)$.

> Replacing r, ϕ by $\bar{r}$ & $\bar{\phi}$ everywhere & the $O(\epsilon)$ error becomes $O(\epsilon^2)$ since it multiplies with ϵ everywhere.

> <u>Beauty of it:</u> We now have autonomous equations we can analyze using phase plane methods.

> We can treat $\bar{r}$ & $\bar{\phi}$ as not only equal to r & ϕ but they are also constant within the integral over one cycle of short or fast time scale. Like the height over one second is constant but over 5 years is not.

Approximation

- Treat $\bar{r}, \bar{\phi}$ as <u>constants</u> when performing these averages $\langle \cdot \rangle_t$.

Eg: Van der Pol.

$$\ddot{x} + x + \epsilon \dot{x}(x^2 - 1) = 0 \quad \Rightarrow h = \dot{x}(x^2-1) = y(x^2-1)$$

$$h = y(x^2-1)$$

$$= -r\sin(t+\phi)\left[r^2\cos^2(t+\phi) - 1\right]$$

$$h = -\bar{r}\sin(t+\bar{\phi})\left[\bar{r}^2\cos^2(t+\bar{\phi}) - 1\right] + O(\epsilon).$$

↳ Notice it is explicitly time dependent

- $$\dot{\bar{r}} = \langle \epsilon h \sin(t+\bar{\phi})\rangle + O(\epsilon^2)$$

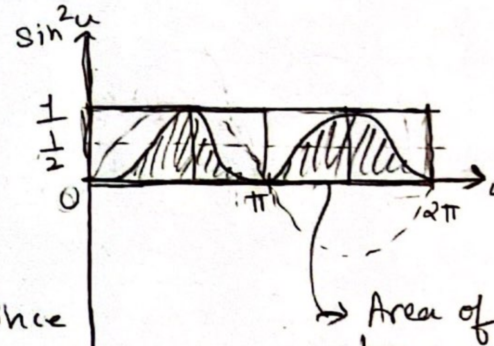
This also includes $O(\epsilon)$ from h since it multiplies with ϵ

$$= \langle -\epsilon \bar{r}\sin^2(t+\bar{\phi})\left[\bar{r}^2\cos^2[t+\bar{\phi}] - 1\right]\rangle + O(\epsilon^2).$$

$$= -\epsilon\bar{r}\left(\bar{r}^2\langle \sin^2\cos^2\rangle - \langle \sin^2\rangle\right) + O(\epsilon^2).$$

<u>Average of $\sin^2$</u>

$$\langle \sin^2\rangle = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} \sin^2 s\, ds = \frac{1}{2}$$



→ Area of box $= 2\pi$
⇒ Area of integral $= \pi$ from Symmetry in each box of $\frac{\pi}{2}$ width
⇒ $\langle \sin^2\rangle = \frac{1}{2}$

Another way to see it is $\langle \sin^2\rangle = \langle \cos^2\rangle$ since they are only out of phase. Therefore $2\langle \sin^2\rangle = 1$

$$\Rightarrow \langle \sin^2\rangle = \frac{1}{2}$$

- $$\langle \sin u \cos u\rangle = \frac{1}{2}\langle \sin 2u\rangle$$

$$\Rightarrow \langle \sin^2 u \cos^2 u\rangle = \frac{1}{4}\langle \sin^2 2u\rangle = \frac{1}{4}\left[\frac{1}{2}\right] = \underline{\underline{\frac{1}{8}}}$$

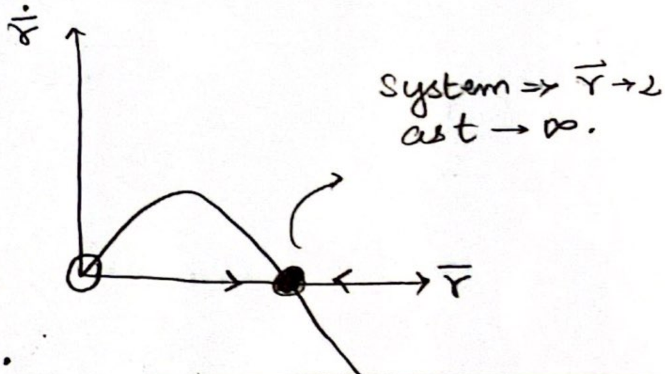
$$\therefore \dot{\bar{r}} = -\epsilon \bar{r}\left(\bar{r}^2 \cdot \frac{1}{8} - \frac{1}{2}\right) + O(\epsilon^2).$$

$$\therefore \dot{\bar{r}} = -\frac{\epsilon}{8}\bar{r}^3 + \frac{\epsilon}{2}\bar{r}$$

<u>Idea Summary</u>

> Since we only want to look at dynamics of $\dot{r}$ & $\dot{\theta}$ which change over a slow time scale (like height over a year) we can replace them with $\dot{\bar{r}}$ & $\dot{\bar{\phi}}$ $(= \dot{\bar{r}}$ & $\dot{\bar{\phi}})$ where the average is taken over the fast time scale (like height replaced by height over one second). This averaging does not really affect r & θ but kills the fast timescale fluctuations (which was the oscillation). Basically by replacing $r \to \langle r \rangle$ or $\bar{r}$ and $\phi \to \langle \phi \rangle$ or $\bar{\phi}$ they are "protected" since averaging does not affect them. It does however affect stuff that changes fast & irons them out.

$$\boxed{\dot{\bar{r}} = \frac{\epsilon \bar{r}}{8}(4 - \bar{r}^2)}$$



$\Rightarrow$ Amplitude of vdP $= 2$.

> We could also explicitly solve for $\bar{r}$ as a fn of t which is very powerful.

> Eg: $x(0) = 1$; $\dot{x}(0) = 0$, then $\bar{r}(0) = 1 \Rightarrow \boxed{\bar{r}(t) = \frac{2}{\sqrt{1 + 3e^{-\epsilon t}}}}$

notice t is order $\frac{1}{\epsilon}$ to show results.

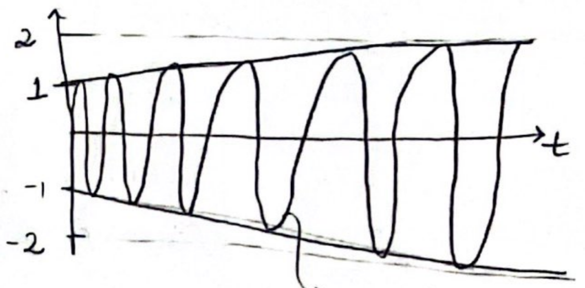
$$\dot{\bar{\phi}} = \left\langle \epsilon \frac{h}{r} \cos(t+\phi) \right\rangle = -\epsilon \langle r^2 \langle \cos^3 \sin \rangle - \langle \sin \cos \rangle$$

$$= \int_0^{2\pi} \cos^3 u \sin u \, du = -\cos^4 u \Big|_0^{2\pi} = 0$$

$\Rightarrow \boxed{\dot{\bar{\phi}} = 0}$ $\Rightarrow \dot{\phi} = O(\epsilon^2) \Rightarrow$ superslow timescale. It is basically constant. Therefore the <u>period of the</u> vdP is almost 2π

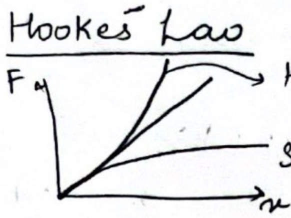
$x(t) = r(t)\cos(t+\phi(t))$.

$\Rightarrow x(t) = \dfrac{2}{\sqrt{1+3e^{-\epsilon t}}}\cos t + O(\epsilon^2)$



Eg: Duffing's Equation.

$\ddot{x} + x + \epsilon x^3 = 0$



$h = x^3$

$= r^3\cos^3(t+\phi)$.

$\dot{r} = \langle \epsilon h \sin(t+\phi)\rangle$

$= \langle \epsilon \underbrace{\cos^3(t+\phi)\sin(t+\phi)\rangle}_{=0} r^3$

Physically if $\epsilon > 0$
$\Rightarrow$ harder spring decreases timeperiod $\Rightarrow$ ϕ changes faster as r is increased

$\Rightarrow \dot{r} = O(\epsilon^2)$.

$\dot{\phi} = \langle \epsilon \frac{h}{r}\cos(t+\phi)\rangle = \epsilon r^2 \langle \cos^4 \rangle + O(\epsilon^4)$

$= \frac{3}{8}\epsilon r^2$

$\theta = \phi + t$ — θ fast, ϕ slow, t fast
$\dot{\theta} = 1$ since $\dot{\phi} \doteq 0$ over fast time scale.

$\boxed{\dot{\phi} = \frac{3}{8}\epsilon r^2}$

$\omega = \frac{d}{dt}[\underbrace{t+\phi}_{\theta}] = 1 + \dot{\phi} = 1 + \frac{3}{8}\epsilon r^2$

$\Rightarrow$ Frequency increases by $\frac{3}{8}\epsilon r^2$ due to cubic term. $\Rightarrow T = 2\pi\left(1 - \frac{3}{8}\epsilon r^2\right) + O(\epsilon^2)$

→ Averaging irons out the fast oscillations contributions to $\dot{r}$ and $\dot{\phi}$ and keeps the slow variations of r & ϕ.

Recipe: $x(t) = r(t)\cos(t+\phi(t))$ $\quad y(t) = -r(t)\sin(t+\phi(t))$

$\boxed{\dot{r} = \langle \epsilon h \sin(t+\phi)\rangle \quad \dot{\phi} = \langle \frac{\epsilon h}{r}\cos(t+\phi)\rangle}$

where h is coefficient of ϵ in $\ddot{x} + x + \epsilon h(x,\dot{x}) = 0$.

We saw $\dot{r} = \epsilon h \sin(\phi + t)$

$\dot{\phi} = \frac{\epsilon h}{r} \cos(\phi + t)$.

$\bar{\dot{r}} = \langle \epsilon h \sin(\phi + t) \rangle$

$\bar{\dot{\phi}} = \langle \frac{\epsilon h}{r} \cos(\phi + t) \rangle$

$\therefore \bar{\dot{r}} = \dot{\bar{r}}$

$\Rightarrow \dot{\bar{r}} = \langle \epsilon h \sin(\phi + t) \rangle$

$\dot{\bar{\phi}} = \langle \frac{\epsilon h}{r} \cos(\phi + t) \rangle$

↳ average over fast time

$T = \epsilon t$

slow fast

$\therefore \quad r' = \frac{dr}{dT} = \frac{dr}{d\epsilon t} = \frac{1}{\epsilon} \frac{dr}{dt} = \frac{1}{\epsilon} \dot{r} = \frac{1}{\epsilon} \dot{\bar{r}}$

↳ replace instantaneous with average fast time T.

$\phi' = \frac{1}{\epsilon} \dot{\bar{\phi}}$

$$\Rightarrow \boxed{\begin{aligned} r' &= \langle h(\theta) \sin(\theta) \rangle \\ \phi' &= \langle \frac{h(\theta)}{r} \cos(\theta) \rangle \end{aligned}}$$

$\langle \ \rangle = \frac{1}{2\pi} \int_0^{2\pi} \dots \, d\theta$

$\theta = \phi + t$

fast slow fast

derivative w.r.t slow time.

→ How do r & φ change over long periods of time.