Perturbation theory for weakly nonlinear oscillators.

$$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0 \quad \text{——} \ (1)$$

Seek solutions of (1) as a power series in ϵ.

$$x(t, \epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \dots \quad \text{—} \ (2)$$

$x_k(t)$ are to be determined.

→ Idea is that all the important information is contained in the first two terms. Called perturbation theory.

Eg: $\ddot{x} + 2\epsilon\dot{x} + x = 0$. —(3) initial conditions: $x(0) = 0$, $\dot{x}(0) = 1$

The exact sol. is $x(t, \epsilon) = (1-\epsilon^2)^{-1/2} e^{-\epsilon t} \sin\left[(1-\epsilon^2)^{1/2} t\right]$

Using P.T

Substituting (2) in (3)

$$\frac{d^2}{dt^2}(x_0 + \epsilon x_1 + \dots) + 2\epsilon \frac{d}{dt}(x_0 + \epsilon x_1 + \dots) + (x_0 + \epsilon x_1 + \dots)$$

Grouping in terms of ϵ.

$$[\ddot{x}_0 + x_0] + \epsilon[\ddot{x}_1 + 2\dot{x}_0 + x_1] + \underbrace{O(\epsilon^2)}_{\text{Ignore this}} = 0$$

This makes it fail sometimes

Key Idea: Since the equation above is true for all small values of ϵ, each term individually must be 0.

$$O(1): \ \ddot{x}_0 + x_0 = 0$$

$$O(\epsilon): \ \ddot{x}_1 + 2\dot{x}_0 + x_1 = 0$$

Applying initial conditions,

$$x_0(0) + \epsilon\, x_1(0) + \dots = 0 \quad \underset{\substack{\text{argument}\\ \text{as key}\\ \text{idea}}}{\overset{\text{same}}{\Rightarrow}} \quad \begin{aligned} x_0(0) &= 0 \\ x_1(0) &= 0 \dots \end{aligned}$$

Similarly

$$\dot{x}_0(0) + \epsilon\, \dot{x}_1(0) + \dots = 1 \quad \underset{\text{argument}}{\overset{\text{same}}{\Rightarrow}} \quad \begin{aligned} \dot{x}_0(0) &= 1 \\ \dot{x}_1(0) &= 0 \end{aligned}$$

Solve $O(1)$ equation, $\ddot{x}_0 + x_0 = 0$ where $x_0(0) = 0$, $\dot{x}_0(0) = 1$

$$\Rightarrow x_0(t) = \sin t$$

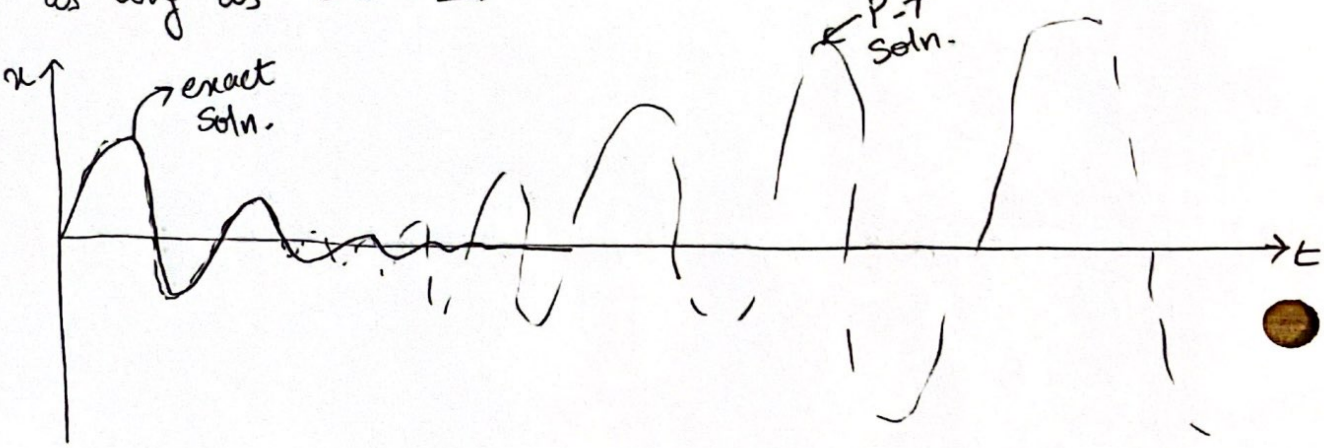
Plug into $O(\epsilon)$ equation $\Rightarrow \ddot{x}_1 + 2\cos t + x_1 = 0$ where $x_1(0) = 0$, $\dot{x}_1(0) = 0$

$$\Rightarrow x_1(t) = -t \sin t \rightarrow \text{blows up as } t \rightarrow \infty$$

$\rightarrow$ a.k.a secular term due to blow up.

$$\therefore \quad x(t, \epsilon) = \sin t - \epsilon t \sin t + O(\epsilon^2).$$

These are infact the first two terms of the power series expansion of the actual solution which converges. This solution alone diverges and therefore fails. It only works as long as $\epsilon t < 1$.



Problems?

① > P.T Solution does not capture slow time scale ⇒ envelope behaviour. To get accurate results we need infinite terms in the series (impractical).

② > Original exact solution give frequency $\simeq 1 - \frac{1}{2}\epsilon^2$ whereas P.T gives frequency of 1. This is an error on a super slow time scale that builds up as $t \uparrow$.

Solution? Two timing.

Let $\tau = t$ denote fast time of $O(1)$.
Let $T = \epsilon t$ denote slow time of $O(\epsilon)$
} Treat them as independant variables.

→ T is considered slow enough that functions in T are constant over τ.

Let $x(t, \epsilon) = x_0(\tau, T) + \epsilon x_1(\tau, T) + O(\epsilon^2)$

Chain rule ⇒ $\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t} = \frac{\partial x}{\partial \tau} + \epsilon \frac{\partial x}{\partial T}$

notation for $\frac{\partial}{\partial t}$

$\dot{x} = \partial_\tau x + \epsilon \partial_T x$

✱ ⇒ $\dot{x} = \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2)$ —— ⑤

Similarly,

✱ → $\ddot{x} = \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + O(\epsilon^2)$ —— ⑥

Also, $x = x_0 + \epsilon x_1 + O(\epsilon^2)$

Example. $\ddot{x} + 2\epsilon \dot{x} + x = 0 \qquad x(0)=0 \qquad \dot{x}(0)=1$

Substituting ⑤ & ⑥ here

$$\partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0) + 2\epsilon\, \partial_\tau x_0 + x_0 + \epsilon x_1 + O(\epsilon^2) = 0$$

$O(1)$: $\partial_{\tau\tau} x_0 + x_0 = 0$

$O(\epsilon)$: $\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0 + 2\partial_\tau x_0 + x_1 = 0$

$O(1) \Rightarrow \quad x_0 = A \sin\tau + B\cos\tau.$

Key Idea.

Here the constants A, B are actually functions of T.

Substituting in $O(\epsilon)$

$$\partial_{\tau\tau} x_1 + x_1 = -2(A' + A)\cos\tau + 2(B' + B)\sin\tau. \quad — ⑦$$

where $A' = \dfrac{\partial A}{\partial T}$

RHS of ⑦ produces secular terms, we therefore use a standard procedure in two timing and set the coefficients to 0

$\Rightarrow$ "Set the coefficients of resonant terms to zero".

$\Rightarrow \quad A' + A = 0; \qquad B' + B = 0. \quad \Rightarrow \quad A(T) = A(0)e^{-T}$

$B(T) = B(0)e^{-T}$

To find $A(0)$ and $B(0)$ we use initial conditions $x(0)=0;\ \dot{x}(0)=1$

$x(0) = x_0(0,0) + \epsilon\, x_1(0,0) + O(\epsilon^2) = 0.$ For all ϵ this is true implies that $x_0(0,0) = 0.$

Similarly, from 5

$$\dot{x}(0) = \partial_\tau x_0(0,0) + \epsilon(\partial_T x_0(0,0) + \partial_\tau x_1(0,0)) + O(\epsilon^2)$$

$$= 1.$$

$$\Rightarrow \quad \partial_\tau x_0(0,0) = 1.$$

$$x_0 = A\sin\tau + B\cos\tau$$

$$x_0(0,0) = \boxed{B(0) = 0}$$

$$\frac{\partial x_0}{\partial \tau} = +A\cos\tau - B\sin\tau$$ since A & B are constant w.r.t τ.

$$\frac{\partial}{\partial \tau} x_0(0,0) = A(0) = 1 \qquad \Rightarrow \begin{array}{l} A(T) = e^{-T} \\ B(T) = 0 \end{array}$$

$$\Rightarrow \boxed{x_0 = e^{-T}\sin\tau}$$

$$\Rightarrow \quad x(t) = e^{-T}\sin\tau + O(\epsilon)$$

$$\boxed{x(t) = e^{-\epsilon t}\sin t + O(\epsilon).}$$

→ Surprisingly this approximates the exact solution very well!

> Could further solve for $x_1(t)$ to get $O(\epsilon)$ terms.

<u>Example.</u> Vander Pol. $\ddot{x} + x + \epsilon(x^2-1)\dot{x} = 0$

$$\dot{x} = \partial_\tau x_0 + \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2).$$

$$\ddot{x} = \partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + \partial_{T\tau} x_0) + O(\epsilon^2).$$

$\Rightarrow$

$$\partial_{\tau\tau} x_0 + \epsilon(\partial_{\tau\tau} x_1 + \partial_{T\tau} x_0) + x_0 + \epsilon x_1 + \epsilon(x^2-1)(\partial_\tau x_0$$
$$+ \epsilon(\partial_T x_0 + \partial_\tau x_1) + O(\epsilon^2) = 0.$$

For any ϵ $\Rightarrow$ $\partial_{\tau\tau} x_0 + x_0 = 0$. — ①

$$\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0 + x_1 + (x_0^2-1)(\partial_\tau x_0) = 0 \text{ — ②}$$

From ①, $x_0 = A \sin \tau + B \cos(\tau)$

or $x_0 = r(T) \cos(\tau + \phi(T))$.

Substituting into ②,

$$\partial_{\tau\tau} x_1 + x_1 = -2\partial_{T\tau}[r(T)\cos(\tau+\phi(T))] - (x_0^2-1)\,\partial_\tau[r(T)\cos(\tau+\phi(T$$

$$= 2\partial_T\, r(T) \sin(\tau + \phi(T)) + (x_0^2-1)\, r(T) \sin(\tau + \phi(T))$$

$$= 2\partial_T\, r(T) \sin(\tau + \phi(T)) + (r^2\cos^2(\tau+\phi) - 1)\, r \sin(\tau+\phi)$$

$$= 2(r' \sin(\tau+\phi) + r\phi' \cos(\tau+\phi))$$
$$+ (r^2 \cos^2(\tau+\phi) - 1)\, r \sin(\tau+\phi)$$

$$\partial_{\tau\tau}x_1 + x_1 = 2(r' \sin(\tau+\phi) + r\phi' \cos(\tau+\phi)) + r^3 \sin(\tau+\phi)\cos^2(\tau+\phi) - r\sin(\tau+\phi).$$

$$\sin\theta\cos^2\theta = \frac{1}{4}(\sin\theta + \sin 3\theta).$$

$$\Rightarrow \partial_{\tau\tau}x_1 + x_1 = \left(2r' - r + \frac{1}{4}r^3\right)\sin(\tau+\phi) + [2r\phi']\cos(\tau+\phi) + \frac{1}{4}r^3 \sin 3(\tau+\phi).$$

To avoid secular terms we have.

$$2r' - r + \frac{1}{4}r^3 = 0$$
$$2r\phi' = 0$$

$$\Rightarrow \quad r' = \frac{1}{8}r(4-r^2) \quad \Rightarrow \text{ limit cycle at } r^* = 2$$

Also $\phi' = 0 \Rightarrow \phi(T) = \phi_0 \Rightarrow \boxed{x(\tau, T) = 2\cos(\tau+\phi_0)}$.

$$\Rightarrow \quad x(t) \longrightarrow 2\cos(\tau+\phi_0) + O(\epsilon). \text{ as } t \to \infty$$

Let $\theta = \tau + \phi(T) \Rightarrow \omega = \frac{d\theta}{dt} = 1 + \frac{d\phi}{dT}\cdot\frac{dT}{dt} = 1 + \epsilon\phi' = \underline{\underline{1}}$

since $\phi \equiv 0$.

$$\therefore \omega = 1 + O(\epsilon^2)$$

General recipe for two timing.

$\ddot{x} + x + \epsilon h(x, \dot{x}) = 0.$

$O(1)$: $\partial_{\tau\tau} x_0 + x_0 = 0$

$O(\epsilon)$: $\partial_{\tau\tau} x_1 + x_1 = -2\partial_{\tau T} x_0 - h.$ where $h = h(x_0, \partial_\tau x_0)$

Solution of $O(1) =$ $x_0 = r(T)\cos(\tau + \phi(T))$.

Substitute into $O(\epsilon)$ & make coefficients of $\cos(\tau+\phi)$ & $\sin(\tau+\phi) = 0$.

RHS of $O(\epsilon)$ is $2[r'\sin(\tau+\phi) + r\phi'\cos(t+\phi)] - h$

where $h = h[r\cos(t+\phi), -r\sin(\tau+\phi)]$

We use Fourier analysis to extract terms in h proportional to $\sin(\tau+\phi)$ & $\cos(\tau+\phi)$.

Let $\theta = \tau + \phi$.

$$h(\theta) = \sum_{k=0}^{\infty} a_k \cos k\theta + \sum_{k=0}^{\infty} b_k \sin k\theta.$$

where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta.$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos k\theta\,d\theta \quad, k \geq 1$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin k\theta\,d\theta \quad, k \geq 1$$

$$\therefore\ 2\left[r'\sin\theta + r\phi'\cos\theta\right] - \sum_{k=0}^{\infty} a_k \cos k\theta - \sum_{k=1}^{\infty} b_k \sin k\theta.$$

$\frac{dr}{dT}$

$T = \epsilon t_0$

$\tau = t$

$$\boxed{\begin{aligned} r' &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\sin\theta\, d\theta = \langle h\sin\theta\rangle \\ r\phi' &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\cos\theta\, d\theta = \langle h\cos\theta\rangle \end{aligned}}$$

avg over one period

$\langle \cos\rangle = \langle \sin\rangle = 0$

$\langle \sin\cos\rangle = 0$

$\langle \sin^3\rangle = \langle \cos^3\rangle = 0$

$\langle \sin^{2n+1}\rangle = \langle \cos^{2n+1}\rangle = 0$

$\langle \sin^2\rangle = \langle \cos^2\rangle = \frac{1}{2}$

$\langle \sin^4\rangle = \langle \cos^4\rangle = \frac{3}{8}$

$\langle \sin^2\cos^2\rangle = \frac{1}{8}$

$\langle \sin^{2n}\rangle = \langle \cos^{2n}\rangle = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)}$

Steps: ① Find r' & $r\phi'$.

② $\omega = 1 + \epsilon\phi'$

$\left.\right\}\ r(0) = \sqrt{x(0)^2 + \dot{x}(0)^2}$

$\phi(0) \simeq \tan^{-1}\left(\frac{\dot{x}(0)}{x(0)}\right) - 1.$

③ solve for $r(T)$ & $\phi(T)$ explicitly using $r(0)$, $\phi(0)$.

④ $x(t,\epsilon) \simeq x_0(\tau, T) + O(\epsilon)$

$\simeq r(T)\cos(t + \phi(T)) + O(\epsilon).$

⑤ f.p of $r(T)$ are limit cycles.

Lec 12

Bifurcations in 2D systems.

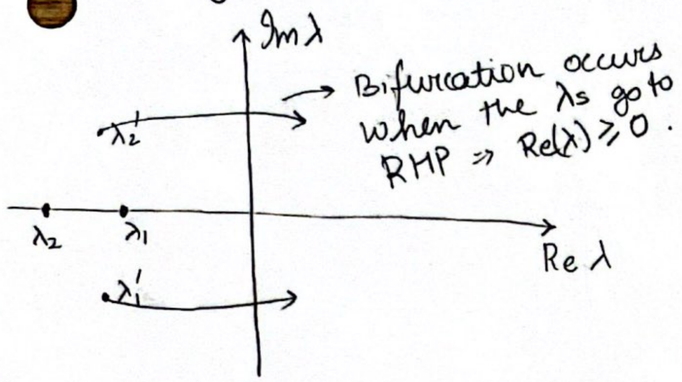
- Suppose ∃ stable equilibrium or closed orbit.
  - How can it vanish or change stability as we vary a parameter?
- In this lecture prototypical examples (normal forms) are discussed.

I. Bifurcations of fixed points.

a) $\lambda = 0$ bifurcations (saddle node, transcritical, pitchfork)

b) $\lambda = \pm i\omega$ (Hopf bifurcations) → lead to creation of closed orbits since complex λs become purely imaginary

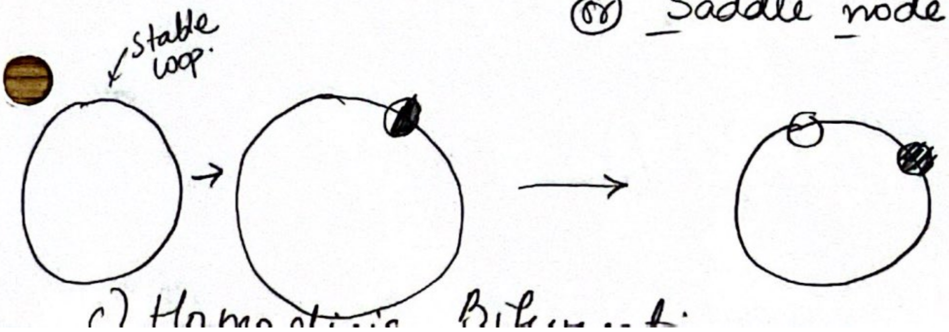
> Importance of eigenvalues: when $\mathrm{Re}\,\lambda_1, \lambda_2 < 0$, fixed point is linearly stable.



II. Bifurcations of closed orbits.

a) Coalessence of cycles → Stable & unstable cycles annihilate. aka saddle node bifurcation of cycles

b) Sniper or Snic → Saddle node infinite period. bif^n (or) Saddle node invariant circle.



> We see a saddle node Bif on a circle.
> At the bifurcation timeperiod goes to ∞.

c) Homoclinic Bifurcati

I.a $\lambda = 0$ Bifurcations.

1) Saddle-node.

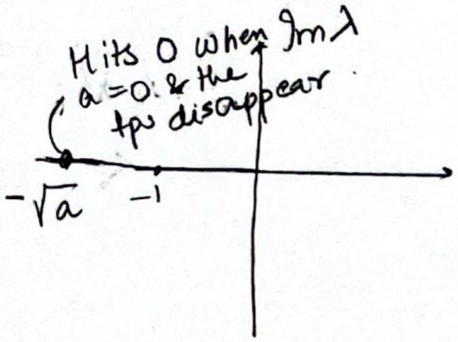
canonical form of saddle node $\rightarrow$ $\dot{x} = a - x^2$
$\dot{y} = -y$

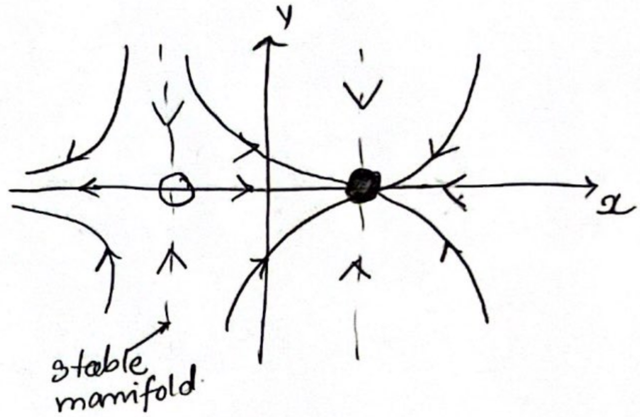
Fixed points: $(\sqrt{a}, 0)$, $(-\sqrt{a}, 0)$ when $a > 0$.

$$A = \begin{bmatrix} -2x^* & 0 \\ 0 & -1 \end{bmatrix} \qquad \lambda_1 = -2x^* \qquad \lambda_2 = -1$$

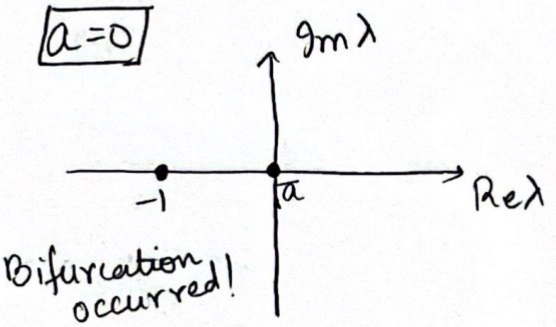
If $\left.\begin{array}{l} x^* = \sqrt{a} \\ y^* = 0 \end{array}\right\}$ stable node. $\qquad \left.\begin{array}{l} x^* = -\sqrt{a} \\ y^* = 0 \end{array}\right\}$ saddle.

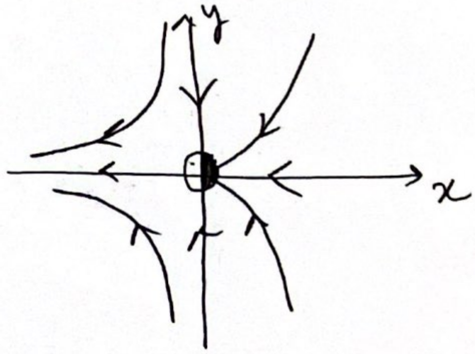
Therefore a saddle and a node are colliding. AKA saddle node node.



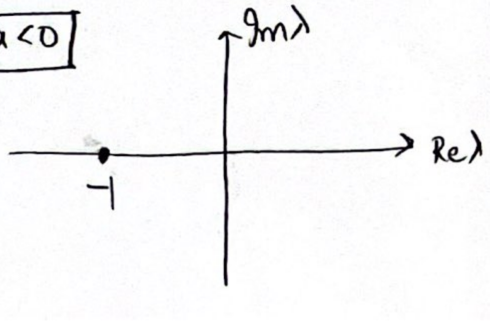


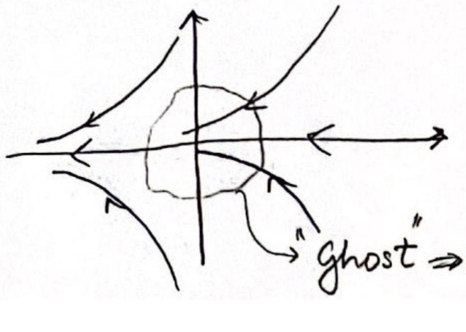
$\boxed{a = 0}$





$\boxed{a < 0}$





② Transcritical (Analyze it yourself)

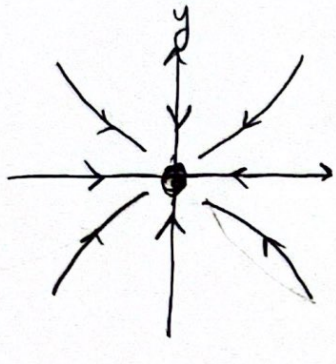
$$\dot{x} = ax - x^2$$
$$\dot{y} = -y$$

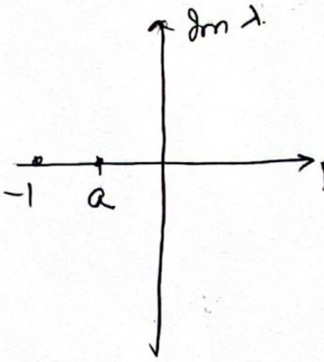
*3) Pitchfork

Supercritical : $\dot{x} = ax - x^3 \quad \dot{y} = -y \quad \bar{x}^* = (\pm\sqrt{a}, 0)$.

$\bar{x}^* = (0, 0)$.

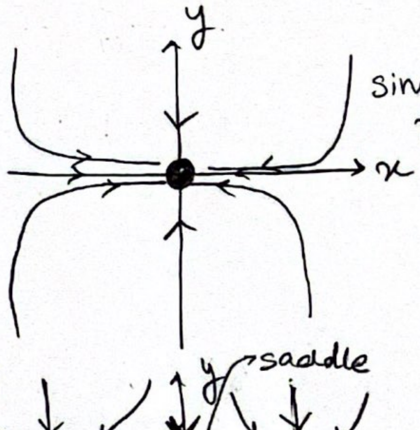
a<0



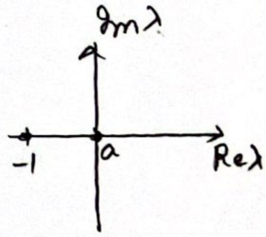


$$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$$

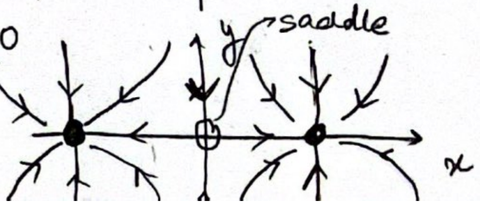
a=0



similar to $\dot{x} = -x^3$ system.
$\Rightarrow$ Called critical slowing down.



a>0



> Supercritical since newly created fp are stable.

Ib Hopf Bifurcation $(\lambda = \pm i\omega)$ → frequency of limit cycle at birth.

Equivalently $\tau = 0$, $\Delta > 0$

Supercritical Hopf bifurcation

→ Stable spiral. ⟶ Unstable spiral. + surrounded by a small amplitude limit cycle that is nearly elliptical!

→ Hopf theorem actually guarantees the existence of a limit cycle & not just closed orbits.

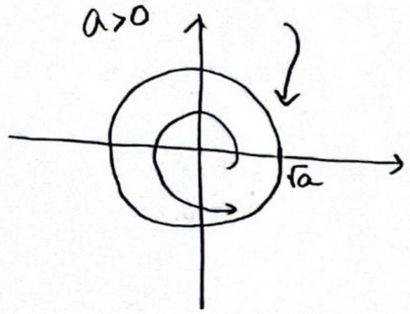
→ At the bifurcation the λs would be $\lambda_1, \lambda_2 = \pm i\omega$.

Ex: $\dot{r} = ar - r^3$, $\dot{\theta} = \omega + br^2$

$r=0$ is a fixed point for the 1D dynamics. $\dot{r} = ar - r^3$.

Stable when $a \leq 0$ } $a=0$ is where bifurcation occurs.
Unstable when $a > 0$ } 1D dynamics ⇒ $r = \sqrt{a}$ is a perfect circle after bifurcation.

→ $a>0$



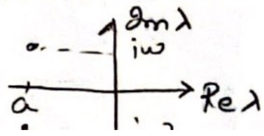
> At birth the limit cycle has 0 amplitude.

→ Parkinson's could actually be a Hopf bifurcation because of the Hopf bifurcation.

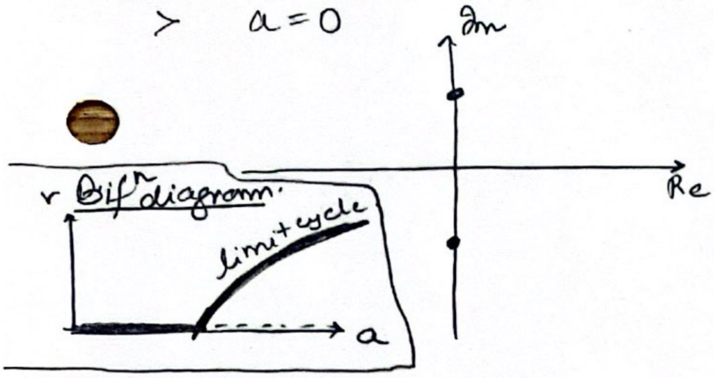
→ As g_m increases inside an electrical oscillator, a Hopf bif^n occurs creating a limit cycle.

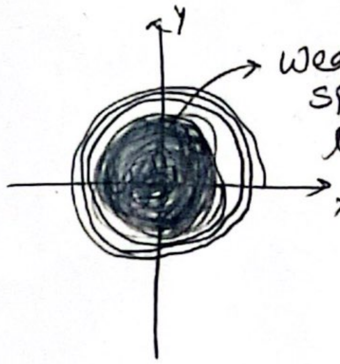
→ $a<0$

→ Convert to cartesian coordinates to calculate Jacobian. $\lambda = a \pm i\omega$



> $a = 0$

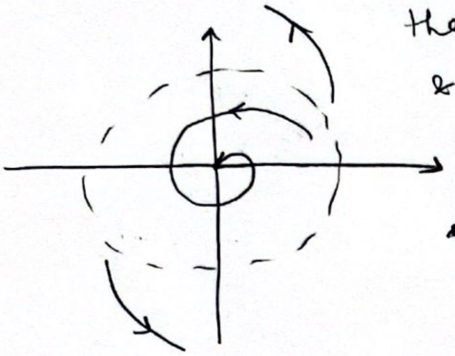


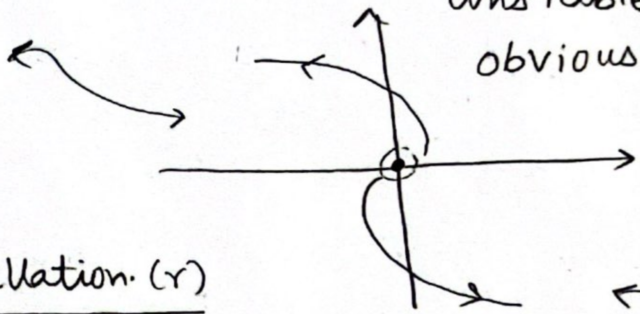


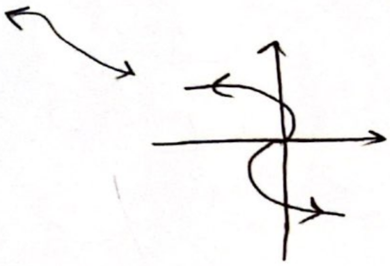
weakly stable spiral although linearity predicts a center.

Lec 13 <u>Subcritical Hopf bifurcation</u> → actually very different from supercritical case.

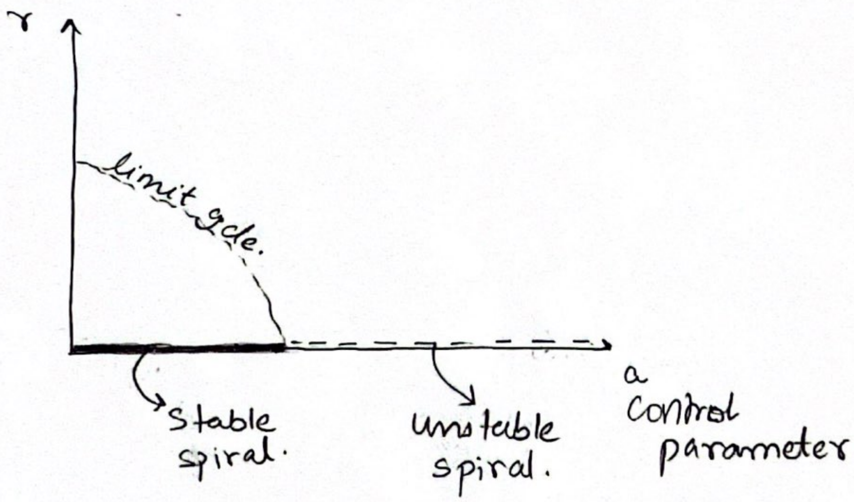
→ In subcritical Hopf, there is an unstable limit cycle that surrounds a stable spiral. As a parameter changes the limit cycle gets smaller & smaller & chokes the limit cycle. At the bif^n point the unstable limit cycle & stable spiral become an unstable spiral. This can obviously go the other way.







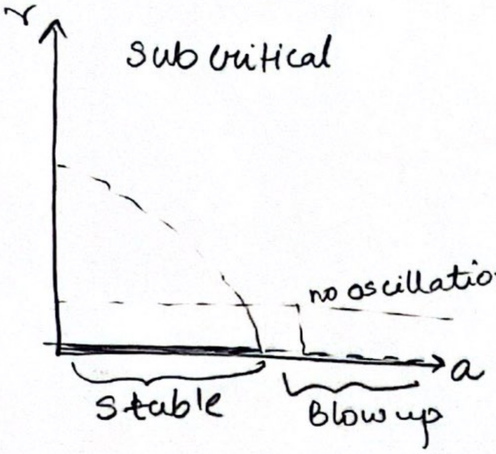
→ <u>Amplitude of oscillation (r)</u>

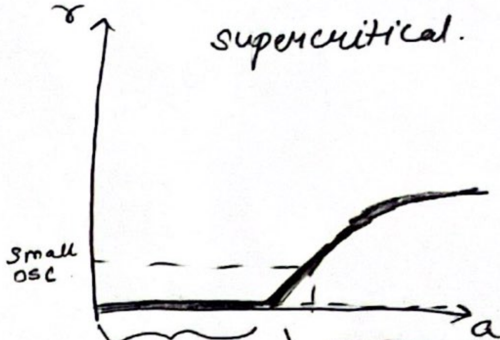


→ Supercritical Hopf aka "soft", "continuous", "safe"

→ Subcritical Hopf aka "hard", "discontinuous", "dangerous".

→ Airplane wings could be stable & suddenly start violently vibrating & break if a subcritical Hopf occurs.

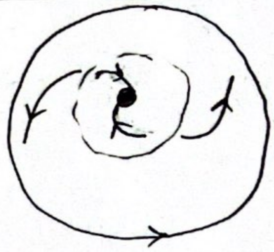




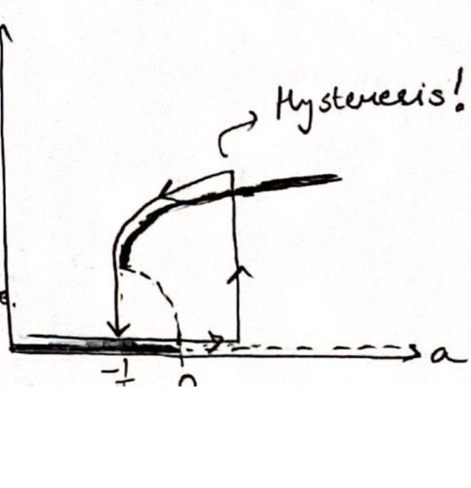
<u>Ex.</u> $\dot{r} = ar + r^3 - r^5$

$\dot{\theta} = 1$

<u>If $-\frac{1}{4} < a < 0$</u> ⇒ Stable Spiral., surrounded by a circular Unstable limit cycle. In this case even outside the limit cycle there is another limit cycle outside that is Stable.



<u>If $a > 0$</u>



> Below $-\frac{1}{4}$ the two limit cycles annihilate.

> Hysteresis of limit cycles is observed.

→ Linearization cannot predict the difference b/w sub & super critical Hopf. Notice at $r \cong 0$ they look the same & in the equations r^3 term sign is where the difference exists which is ignored by the linearization.

→ An analytical criterion to tell the difference exists. But it's complicated. See Guckenheimer & Holmes. § 3.4. Or do exercise § 8.2.12.

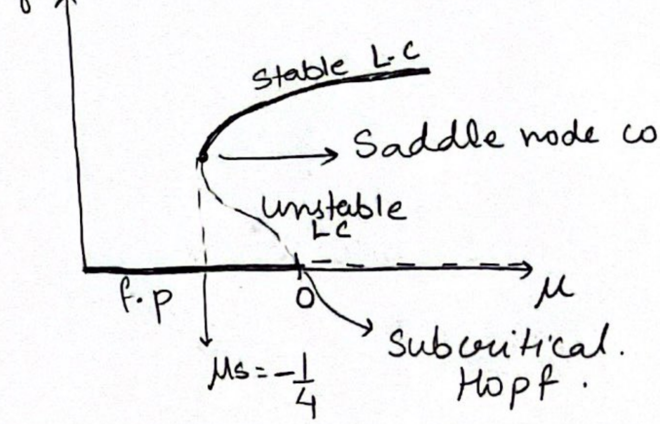
POI

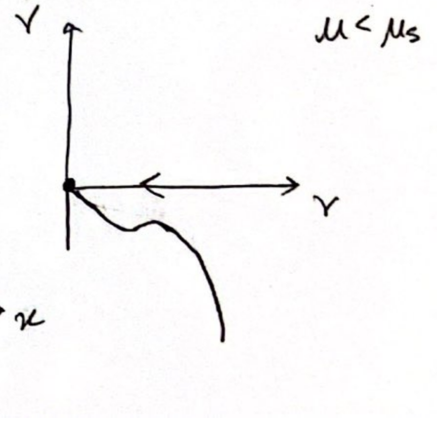
→ This is also subcritical since the bifurcating object is unstable.

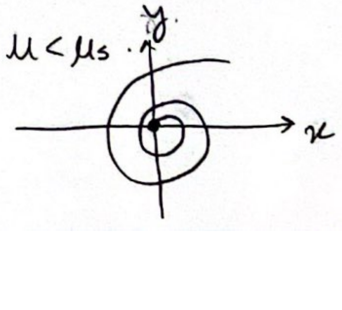
Lec 14 Global Bifurcations in cycles. § 8.4.

→ Saddle node coalescence of limit cycles. Illustrated by

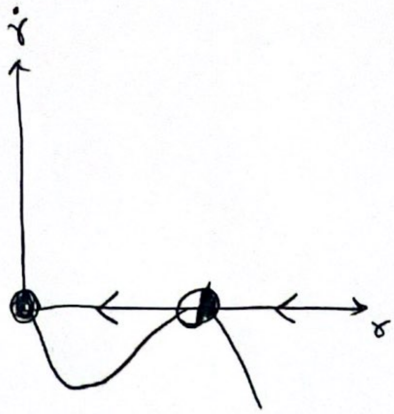
$\dot{r} = \mu r + r^3 - r^5 \qquad \dot{\theta} = 1 + b r^2$

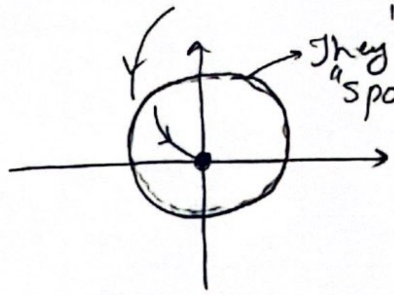






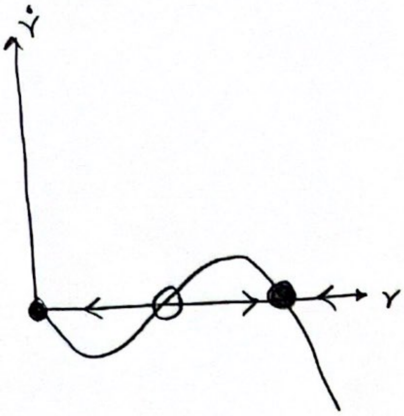
$\underline{\mu = \mu_s}$

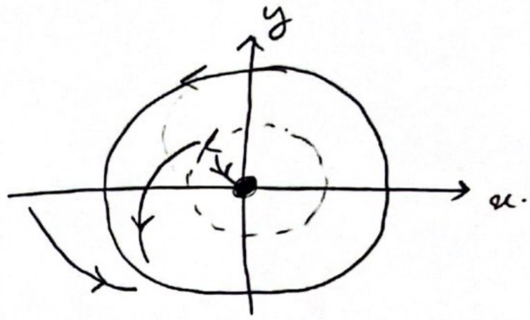




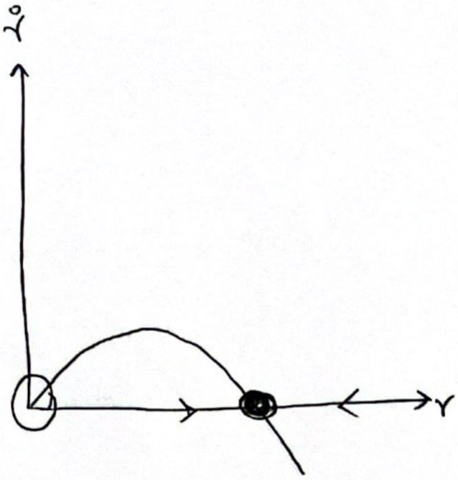
They "collide" or "spontaneously generate" as a half stable limit cycle

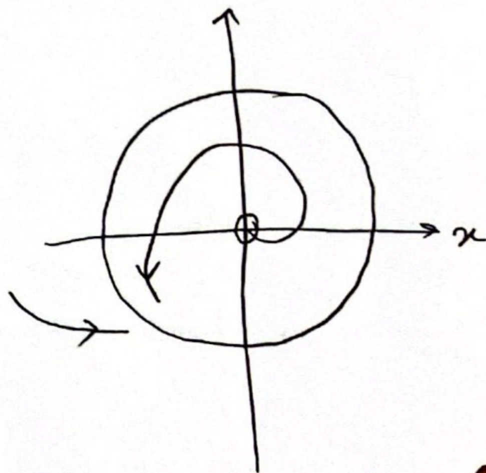
$\underline{0 > \mu > \mu_s}$





$\underline{\mu > 0}$

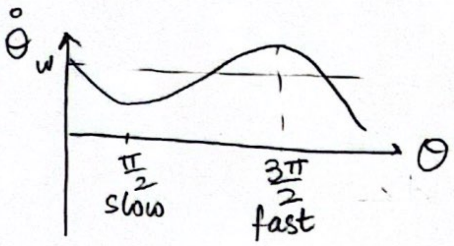


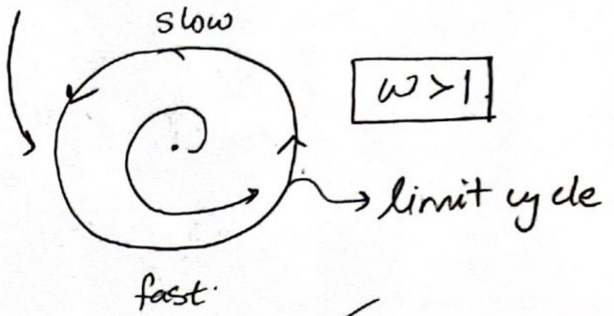


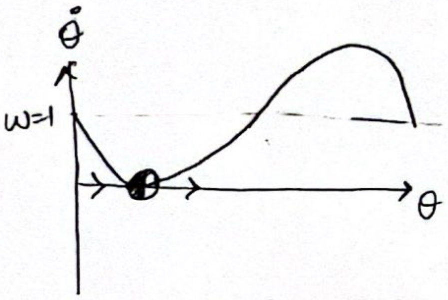
> Cycles born at $\mu = \mu_s$ with an amplitude of $O(1)$.
$O(1)$ is with respect to $|\mu - \mu_s|$. The frequency is also of $O(1)$, since $\dot{\theta} = 1 + b r^{*2}$.

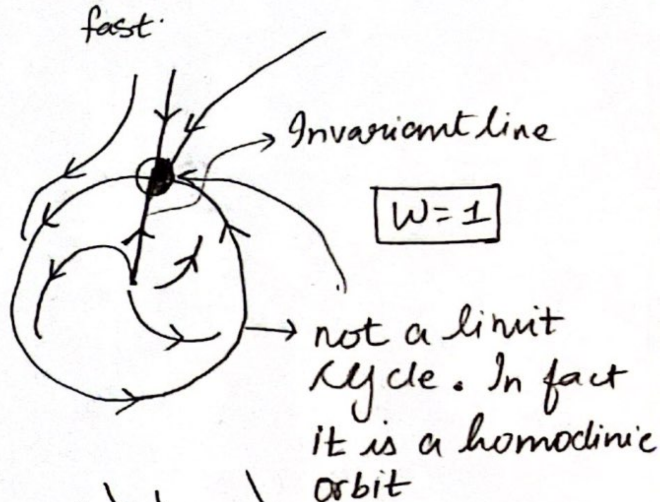
SNIPER. Saddle node infinite period bifurcation occuring on the cycle.

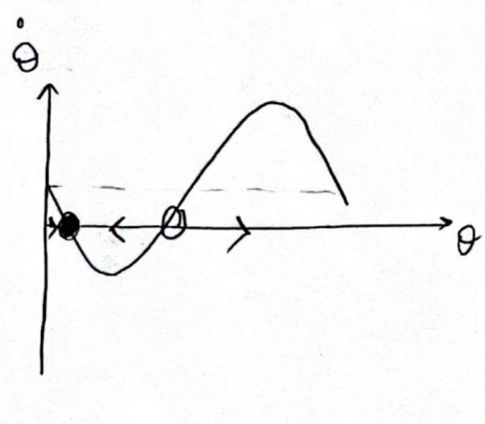
Ex: $\dot{r} = r(1-r^2)$
$\dot{\theta} = \omega - \sin\theta$

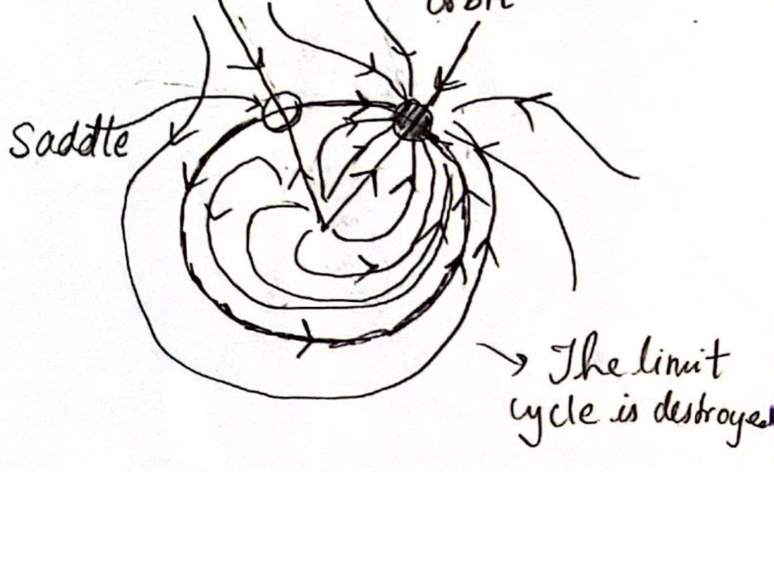


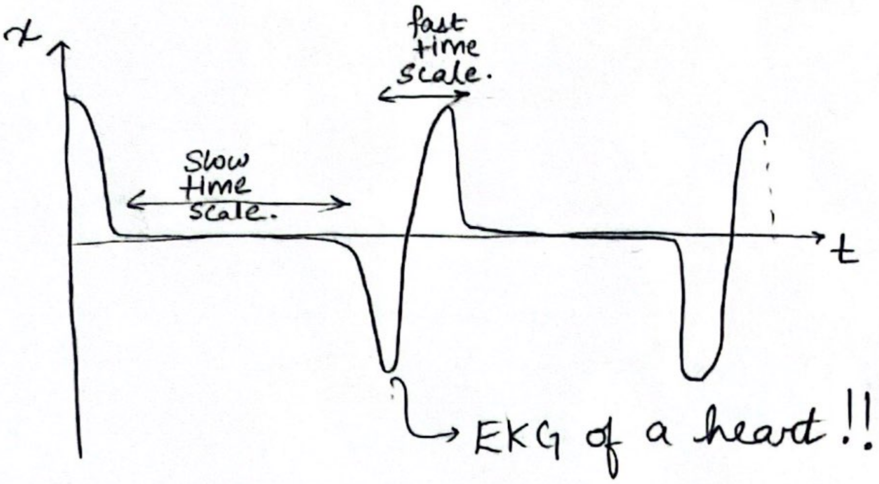


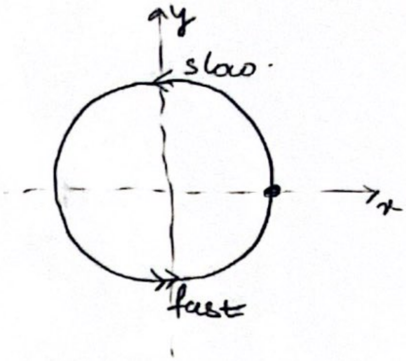












→ This bifurcation occurs in heartbeat firing & nerve cell action potential.

→ <u>Period calculation</u> $T = \int_0^T dt = \int_0^{2\pi} \frac{d\theta}{\omega - \sin\theta}$

Residue Theorem can be used.

Can also be solved using a trig substitution.

<u>Note.</u>

$\int \frac{1}{a-\sin x} dx$ substitute $u = \tan\left(\frac{x}{2}\right)$

$$\Rightarrow du = \frac{1}{2} dx \sec^2\left(\frac{x}{2}\right)$$

$$\cos(x) = \frac{1-u^2}{1+u^2} \quad \& \quad dx = \frac{2du}{u^2+1}$$

$$\sin x = \frac{2u}{u^2+1}$$

$$\Rightarrow \int \frac{2}{(u^2+1)\left(a - \frac{2u}{u^2+1}\right)} du = \frac{-2\tan^{-1}\left(\frac{1-a\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-1}}\right)}{\sqrt{a^2-1}} + c$$

$$\therefore \boxed{T = \frac{2\pi}{\sqrt{\omega^2-1}}} \simeq \frac{2\pi}{\sqrt{2}} \cdot \frac{1}{\sqrt{\omega-1}} \quad \text{as } \omega \to 1$$

In general $T \sim \frac{1}{\sqrt{\mu}}$, In general μ is the distance from saddle node bif^n. Here $\mu = \omega - 1$.

Universal behaviour near bifurcations of cycles.

Let μ = distance from bif^n ($\mu << 1$). ⇒ When thing are just born.

P.

	Amplitude of Stable cycle	Period of cycle
Supercritical Hopf	$O(\mu^{1/2})$ small.	$O(1)$.
Saddle node	$O(1)$	$O(1)$.
SNIPER	$O(1)$	$O(\mu^{-1/2})$ large.
Homoclinic (Not yet discussed. (In sec 8.5))	$O(1)$	$O(\lvert\ln\mu\rvert)$ large.

→ Useful to tell the kind of bif^n by measurement of the these parameters. Which bif^n killed the cycle?

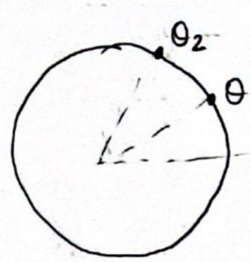
---

§ 8.6 Coupled Oscillators. (and quasiperiodicity).

2D phase spaces: Plane (done a lot ✓), cylinder (done ✓), sphere (like plane), Torus

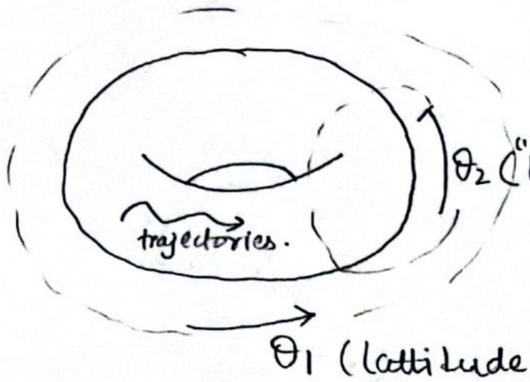
$$\boxed{\begin{aligned}\dot\theta_1 &= f_1(\theta_1, \theta_2)\\ \dot\theta_2 &= f_2(\theta_1, \theta_2)\end{aligned}}$$

fs are 2π periodic in both arguments. These are studied on the Torus (doughnut).

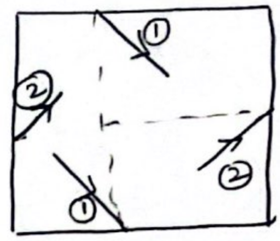


> Here 2 points represent the state of the system.
> However we want one point to represent both states.

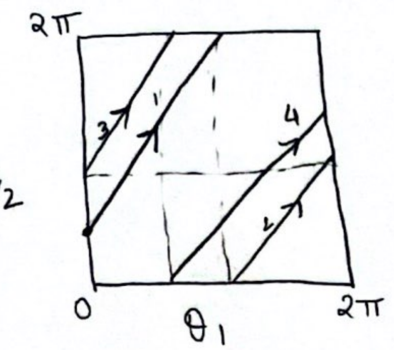
Therefore use a Taurus with coordinates θ_1, θ_2.



or use a square with periodic boundary conditions.



Eg



$$\left.\begin{aligned}\dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2\end{aligned}\right\} \text{No coupling.}$$

$$\boxed{\underline{\underline{\text{In general}}} \quad \begin{aligned}\dot{\theta}_1 &= \omega_1 + \underbrace{K\sin(\theta_2 - \theta_1)}_{\text{coupling.}} \\ \dot{\theta}_2 &= \omega_2 + K\sin(\theta_1 - \theta_2)\end{aligned}}$$

Case 1: $\frac{\omega_1}{\omega_2} = \frac{P}{q}$ where $P, q \in \mathbb{Z}$ & they are relatively prime. $\Rightarrow$ Slope is rational!

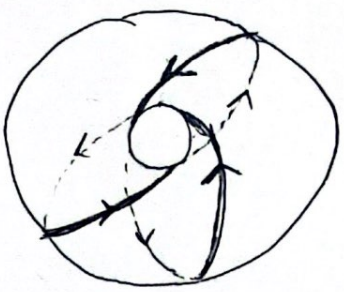
$\rightarrow$ This causes a closed curve on the Torus.

> Basically we go P times around one circle as we go q times around the other circle in the same time.

> Think of them as 2 runners @ 3 laps/time (P) & 2 laps/same time (q).

Eg: $P = 3$, $q = 2$. The trajectory on the Torus would have a knot in it. All trajectories would be closed with knots in them. In Knot theory they are called a "Trefoil Knot".

→



→ Torus Knots occur when P&q are relatively prime.

→ Trefoil knots are 3:2 Torus Knots.

→ However rational numbers are a tiny %(aka countable).

Case 2 : $\frac{\omega_1}{\omega_2}$ = irrational. - get quasiperiodicity.

→ The lines never touch! Every trajectory on the Torus does not ever close and every trajectory is dense on the Torus.

→ Mathematically speaking all points on the Torus are NOT covered by a single trajectory.

→ By picking a point on the Torus. We can show that the trajectory need not hit the point. However, if we pick an ϵ disk around the point where $\epsilon \neq 0$ but ϵ can be arbitrarily small, you can show that the trajectory eventually will penetrate the disk. (Weird!)

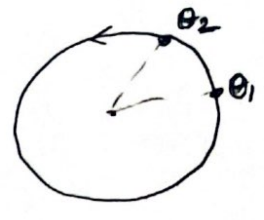
→ Is this Chaos? Not truly! In a chaotic system 2 neighbouring points diverge exponentially fast (not forever though). Here they do not.

Coupled oscillators.

Let $\phi = \theta_1 - \theta_2$

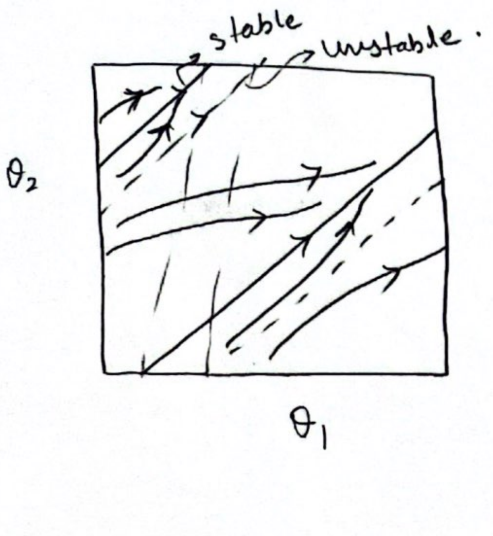
$\dot{\theta}_1 = \omega_1 + K \sin(\theta_2 - \theta_1)$

$\dot{\theta}_2 = \omega_2 + K \sin(\theta_1 - \theta_2)$



If $\theta_2 > \theta_1$, $\dot{\theta}_1 = \omega_1 +$ (positive term) $\Rightarrow$ θ_1 speeds up (& θ_2 slows down) until $\dot{\theta}_1 = \dot{\theta}_2$ and we have $\theta_1 - \theta_2 = \phi$ which would be constant. They are phase locked.

$\dot{\phi} = \omega_1 - \omega_2 - 2K \sin\phi$ → Coupled oscillators also behave like Adler's equation! Therefore there are 2 fixed points. (stable & unstable).



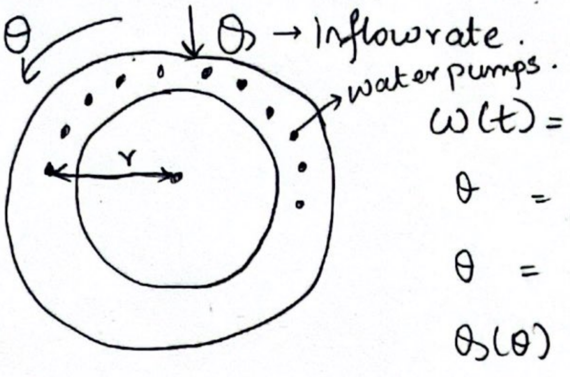
> ϕ = constant $\Rightarrow$ $\theta_1 - \theta_2$ = Constant.
  $\Rightarrow$ diagonal line with a slope of one.

> If $2K > \omega_1 - \omega_2$ $\Rightarrow$ we get a constant phase difference between the 2 oscillators. This is phase locking!
  $\theta_1 - \theta_2 = \phi^*$.

> $\dot{\theta}_1$ & $\dot{\theta}_2$ are equal to ω_1 & ω_2 (the free running frequencies) when coupling is 0. When coupling is introduced, the phase difference between them undergoes a beat phenoma. If coupling is strong enough they "lock" in phase & have the same free running frequency.

Lec 15 Chaotic Waterwheel → Exact analog of the Lorenz equations.

Chapter 9 Lorenz Equations.



Topview of water wheel.

$\omega(t)$ = angular velocity of wheel.

θ = angle in the lab frame.

$\theta = 0 \rightarrow$ 12:00 in the lab frame.

$Q(\theta)$ = rate at which water is pumped in

r = radius.

$m(\theta,t) \rightarrow$ mass distribution of water around the rim.

→ Mass between 2 angles is $\int_{\theta_1}^{\theta_2} M(\theta,t)\,d\theta$.

→ Variables: $\omega(t)$, $m(\theta,t)$.

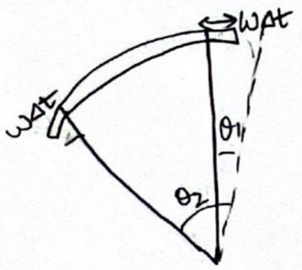
① Conservation of mass:

$$\frac{\partial m}{\partial t} = Q - Km - \omega \frac{\partial m}{\partial \theta}$$

Q ↑ inflow; Km ↑ drainage rate coefficient; $\omega \frac{\partial m}{\partial \theta}$ → Transport term since wheel is moving.

Derivation

Consider any sector $[\theta_1, \theta_2]$



Let $M = \int_{\theta_1}^{\theta_2} m(\theta,t)\,d\theta$ — ①

> In time Δt, $\Delta M \simeq \Delta t \left[\int_{\theta_1}^{\theta_2} Q(\theta)\,d\theta - \int_{\theta_1}^{\theta_2} Km\,d\theta \right]$

ΔM → mass that came in

$+\, m(\theta_1)\,\omega\Delta t$ → mass transported in.

$-\, m(\theta_2)\,\omega\Delta t$ → mass transported out.

$$m(\theta_1) - m(\theta_2) = -\int_{\theta_1}^{\theta_2} \frac{\partial m}{\partial \theta}\, d\theta$$ to make it all in the form $\int \cdots d\theta$

$$\Delta M \cong \Delta t \left[\int_{\theta_1}^{\theta_2} \left(Q - K m - \omega \frac{\partial m}{\partial \theta} \right) d\theta \right]$$

taking the limit

$$\dot{M} = \int_{\theta_1}^{\theta_2} \left[Q - K m - \omega \frac{\partial m}{\partial \theta} \right] d\theta .$$

$$\to \int_{\theta_1}^{\theta_2} \left[\frac{\partial m}{\partial t} \right] d\theta$$ from ①

Since θ_1 & θ_2 are arbitrary

$$\boxed{\frac{\partial m}{\partial t} = Q - K m - \omega \frac{\partial m}{\partial \theta}} \rightarrow \text{(I)}$$

② Newton's Law (expressed as torque balance)

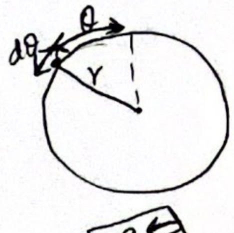
Suppose. $I(t)$ = moment of inertia of wheel. → depends on time since the amount of mass changes.

$(I\omega)^{\cdot} = -\nu\omega$ + torque due to gravity.

(linear damping due to brake + inertial damping)

Note: Water enters with 0 angular momentum but leaves with some non zero angular momentum so it does contribute to the total torque. Luckily it is $\propto \omega$ so include it in ν

Torque due to gravity



In sector $d\theta$, mass $dM = m\, d\theta$

& Torque $d\tau = dM\, g r \sin\theta$. ↳ pendulum Torque.

↳ tilted g.

$g = g_n \sin\alpha$ (g_n = 9.8m/s²)

$$\tau = \int_0^{2\pi} g r m(\theta, t) \sin\theta \, d\theta$$

$$\boxed{\frac{d}{dt}(I\omega) = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta.} \quad \text{II}$$

> We can show that $I(t)$ approaches a constant as $t \to \infty$. regardless of chaotic nature of wheel, since inflow balances outflow

$$\dot{M} = \int_0^{2\pi} \frac{\partial m}{\partial t} d\theta = \int_0^{2\pi}\left[Q - Km - \omega\frac{\partial m}{\partial \theta}\right] d\theta = Q_{total} - KM - 0$$

$m(2\pi) - m(0)$

$$\dot{M} = Q_{total} - KM \implies M \to \frac{Q_{tot}}{K} \text{ as } t \to \infty \left(\text{if } t \gg \frac{1}{K}\right)$$

M : total mass.

$I = Mr^2 + I_{wheel}.$ (parallel axis theorem)

$$(2) \Rightarrow \boxed{I\dot{\omega} = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta} \quad \text{III}$$

$$\frac{\partial m}{\partial t} = Q - Km - \omega\frac{\partial m}{\partial \theta}.$$ nonlinear quadratic term.

$$I\dot{\omega} = -\nu\omega + gr\int_0^{2\pi} m(\theta,t)\sin\theta\, d\theta. \to \text{linear integrodifferential eqn.}$$

> Using Fourier Analysis we could reduce it to 3 nonlinear ODEs.

> Deriving amplitude equations!

Since $m(\theta,t)$ is 2π periodic in θ, write

$$m(\theta,t) = \sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta$$

Derive ODEs for $a_n, b_n, \ldots \infty$, $n = 0, 1, 2 \ldots$ → There are ∞ but we can seperate 3 of them

Let $Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta$ → no sine terms since Q is evenly distributed w.r.t θ. Water is symmetrically added at the top.

$$\frac{\partial}{\partial t}\left(\sum a_n \sin n\theta + b_n \cos n\theta\right) = -\omega \frac{d}{d\theta}\left(\sum a_n \sin n\theta + b_n \cos n\theta\right) +$$

$$+\sum q_n \cos n\theta - K \sum a_n \sin n\theta + b_n \cos n\theta$$

All the sines & cosines are orthogonal ⇒ this gives ∞ equations.
→ Coefficients are only time dependent!

Matching $\sin n\theta$:

$$\dot{a}_n = n\omega b_n - K a_n \quad — (4)$$

Matching $\cos n\theta$

$$\dot{b}_n = -n\omega a_n + q_n - K b_n \quad — (5)$$

Now substitute Fourier for $m(\theta, t)$ into (III):

$$\text{(III)} \rightarrow \quad I\dot{\omega} = -\nu\omega + gr \underbrace{\int_0^{2\pi} \left(\sum a_n \sin n\theta + b_n \cos n\theta\right) \sin\theta \, d\theta}$$

Wow! All these fns are orthogonal to $\sin\theta$ except for $\sin\theta$. Therefore all but $\sin\theta$ integrate to 0.

$$= -\nu\omega + gr \int_0^{2\pi} a_1 \underbrace{\sin^2\theta \, d\theta}_{\to \pi}$$

$$I\dot{\omega} = -\nu\omega + \pi g r a_1 \quad — (6)$$

④ ⑤ & ⑥

$$\therefore \quad \dot{a}_n = n\omega b_n - k a_n$$

$$\dot{b}_n = -n\omega a_n + q_n - k b_n$$

$$I\dot{\omega} = -\nu\omega + \pi g r a_1$$

The next part is truly beautiful.

> $\dot{\omega}$ evolves wrt only a_1
> a_1 evolves wrt only b_1 & a_1
> b_1 evolves wrt only a_1 & b_1

Therefore these equations have decoupled from the rest!!!

first harmonic of inflow.

$$\boxed{\begin{aligned} \dot{a}_1 &= \omega b_1 - K a_1 \\ \dot{b}_1 &= -\omega a_1 + q_1 - K b_1 \\ \dot{\omega} &= -\frac{\nu}{I}\omega + \frac{\pi g r}{I} a_1 \end{aligned}}$$

These 3 coupled parameters are like an engine that determine ω. This ω then defines a_n & b_n for $n \geq 2$. They do not affect this engine on the "bottom

Lec 16 Waterwheel equations and Lorenz equations.

Fixed points: $\dot{a}_1 = 0 \Rightarrow a_1 = \frac{\omega b_1}{K}$ — ①

$\dot{b}_1 = 0 \Rightarrow \omega a_1 = q_1 - K b_1$ — ②

$\dot{\omega} = 0 \Rightarrow a_1 = \frac{\nu\omega}{\pi g r}$ — ③

Eliminate a_1 from ① and ② $\Rightarrow$ $b_1 = \frac{K q_1}{\omega^2 + K^2}$ — ④

① & ③ $\Rightarrow \frac{\omega b_1}{K} = \frac{\nu\omega}{\pi g r}$, $\boxed{\omega = 0}$ or $\boxed{b_1 = \frac{K\nu}{\pi g r}}$ — ⑤

If $\omega = 0 \Rightarrow a_1 = 0$, $b_1 = q_1/K$. (no rotation & leakage balances inflow). → not saying stable or unstable

If $\omega \neq 0$

$$b_1 = \frac{K\nu}{\pi g r} = \frac{K q_1}{\omega^2 + K^2} \Rightarrow \boxed{\omega^2 = \frac{\pi g r q_1}{\nu} - K^2}$$

> Two solutions $\pm\omega$ (corresponding to steady rotation in either direction) exist iff RHS is positive. $\Rightarrow \frac{\pi g r q_1}{K^2 \nu} > 1$.

> This dimensionless group $\boxed{\frac{\pi g r q_1}{K^2 \nu}}$ is the Rayleigh number analog in convection.

> The rotating waterwheel is like a convection cell of air that goes up & down in cycles due to heating & cooling. The steady solution is like heat exchange due to conduction & not convection.

> g, q_1 drive the wheel into rotation (gravity & inflow). K & ν are dampers & oppose the motion.

Edward Lorenz System / Lorenz Equations.

> Derived from simple model of convection.

> First example of a system with a Chaotic attractor. → (self sustaining chaos)

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

$\sigma, r, b > 0$ parameters

$r \rightarrow$ Rayleigh number. → conduction / convection.

$\sigma \rightarrow$ Prandtl number → Viscosity / thermal conduction

$b \rightarrow$ aspect ratio.

> Two quadratic nonlinearities. The waterwheel equations can be mapped to the Lorenz equations.

> Lorenz equations arise in laser dynamics, dynamos etc.

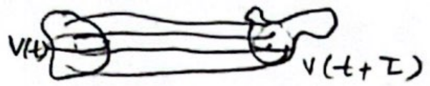
Simple properties of Lorenz equations.

1) Equations are symmetric under $(x, y) \longrightarrow (-x, -y)$

Means if $(x(t), y(t), +z(t))$ is a solution. Then, $(-x(t), -y(t), +z(t))$ is also a solution. Due to symmetry we might expect a pitchfork bifurcation.

2) System is dissipative, in sense that volumes in phase space contract under the flow.

> If we start with a volume of initial conditions where each point is a unique initial condition, this volume over time decreases.



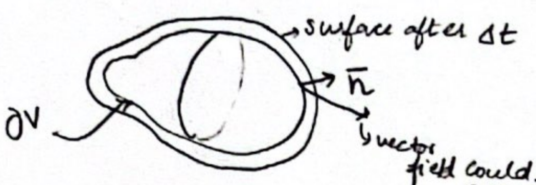
To see this, ask: How do volumes evolve?

> Points on the surface flow according to the vector field.

> What is the volume after Δt.

Let us think of $\vec{x} = (x, y, z)$, $\vec{u} = \dot{\vec{x}} \rightarrow$ velocity in phase space.

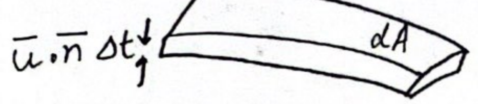
$\vec{u} \cdot \vec{n} = \dot{\vec{x}} \cdot \vec{n}$ = normal outward velocity on boundary $\underset{\sim}{\partial V} = S$ boundary of the vol.



$V(t + \Delta t) = ?$

> The expansion (say) is only due to the normal component of the vector field at each point on the surface.

$V(t+\Delta t) = V(t)$ + volume swept out by tiny patches of the surface integrated over all patches.



$\Delta V = (\bar{u} \cdot \bar{n}\, \Delta t)\, dA$

$$V(t+\Delta t) \simeq V(t) + \Delta t \iint_{\partial V = S} \bar{u} \cdot \bar{n} \, dA$$

$$\Rightarrow \boxed{\dot{V}(t) = \iint_{\partial V} \bar{u} \cdot \bar{n} \, dA}$$

Divergence theorem!

$$\boxed{\dot{V}(t) = \iiint_{V} \nabla \cdot \bar{u} \, dV}$$

True for any vector field in 3D (maybe any dimension)

> In the Lorenz system we want to show RHS < 0.

$$\dot{V} = \iiint_{V} \nabla \cdot \bar{u} \, dV$$

note that $\nabla \cdot \bar{u} = \text{Trace}(\text{Jacobian}(\bar{u}))$.

Lorenz eqns: $\dot{x} = \sigma(y-x)$; $\dot{y} = rx - y - xz$; $\dot{z} = xy - bz$

$$\bar{u} = (\dot{x}, \dot{y}, \dot{z}) \quad \therefore \quad \nabla \cdot \bar{u} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

$$= -\sigma - 1 - b.$$

$\Rightarrow \nabla \cdot \bar{u} < 0$ since σ & b are +ve.

it is also a constant!

$$\Rightarrow \boxed{\dot{V} = -(\sigma + b + 1) V}$$

$\Rightarrow$ volumes shrink exponentially fast in the Lorenz system.

$$V(t) = V(0) \, e^{-(\sigma+b+1)t}$$

→ All trajectories end up on some limiting set of <u>zero volume</u>.

Could be a point, a cycle or a "strange attractor".

Lorenz called it an "infinite complex of surfaces. → FRACTAL.

Fixed points

→ Can show : $(x, y, z) = (0, 0, 0)$ for all values of the parameters.

→ Let us assume r is the only parameter. σ & b are constants.

→ $x = y = \pm\sqrt{b(r-1)}$, $z = r-1$ are fixed points if $r > 1$.

Let $C^+ = \sqrt{b(r-1)}$ & $C^- = -\sqrt{b(r-1)}$

→ As $r \to 1^+$, $(x, y, z) \to 0$ ⇒ these fixed points are born out of the origin.

→ Symmetry & the above point ⇒ pitchfork bifurcation (Supercritical ⇒ f.Ps born are stable

→ Local stability of origin :- Linearization.

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y.$$
$$\dot{z} = -bz$$

⇒ $z(t)$ exponentially decays around the origin

→ In the linearization.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

> What does this imply for xy dynamics.

> $\tau = -\sigma - 1 < 0$

$\Delta = \sigma(1 - r)$ $r > 1$ ⇒ saddle point at the origin.

Saddle point with 2 directions coming in & 1 going out in 3D.

$\tau^2 - 4\Delta = (\sigma - 1)^2 + 4\sigma r > 0$ & $r < 1$ ⇒ stable node. in all directions.

Lec 17 Chaos in Lorenz Equations

Global stability of origin for $r<1$

Every trajectory approaches the origin as $t \to \infty$. Origin is "globally stable" when $r<1$. => There are no other attractors.

Proof: Define $V(x,y,z) = \frac{1}{\sigma}x^2 + y^2 + z^2 \rightsquigarrow$ Lyapunov fn.

> A fn that monotonically decreases over time => x, y, z must reach the origin over time.

> "Energy like" fn in a system with friction.

> $\underbrace{\text{Level sets}}_{\rightsquigarrow \text{set where } V \text{ is constant}}$ of V are concentric ellipsoids centred at the origin.

Idea: Show $\frac{dV}{dt} < 0$ if $r<1$ & $r \neq 0$

Calculate: $\frac{1}{2}\frac{dV}{dt} = \frac{x\dot{x}}{\sigma} + y\dot{y} + z\dot{z}$

use Lorenz equations => $\frac{x}{\sigma}(\sigma(y-x)) + y(rx - y - xz) + z(xy - bz)$

$$= yx - x^2 + ryx - y^2 - \cancel{xyz} + \cancel{xyz} - bz^2$$

$$= xy(1+r) - x^2 - y^2 - bz^2 \longleftarrow \text{complete the square}$$

$$= -\left(x - \frac{r+1}{2}y\right)^2 - \underbrace{\left(1 - \left(\frac{r+1}{2}\right)^2\right)}_{\text{positive if } r<1} y^2 - bz^2$$

$\therefore$ RHS is negative definite! if $r<1$.

> When is $\frac{dV}{dt} = 0$?

Each square must be 0

$\Rightarrow z = 0$ and $y = 0 \Rightarrow x = 0$

$\Rightarrow$ only 0 at the origin.

$\Rightarrow V(x(t), y(t), z(t)) \longrightarrow 0 \Rightarrow (x, y, z)$ must all go to zero.

→ In fluid flow through a pipe if Reynolds' number is below 800 then the only solution is the laminar flow solution, but nobody has found the relevant Lyapunov function to prove it

Suppose $r > 1$ – Origin is a Saddle.

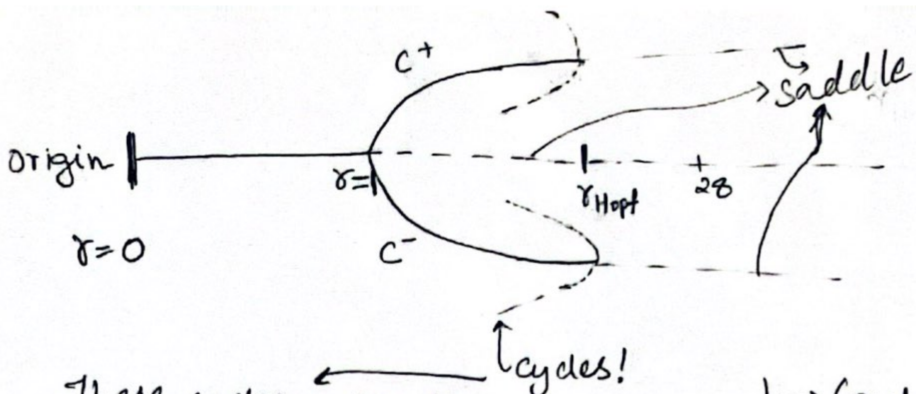
Stability of C^+, C^- is left as a homework exercise.

Can show they are linearly stable for $1 < r < \underbrace{\sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right)}_{r_{Hopf} = r_H}$

→ Also assume $\sigma > b + 1$. Lorenz used $\sigma = 10, b = 8/3$

$r_H = 24.74$.

→ When $r > r_H$, you might suppose there exists a small stable cycle around C^+, C^-. Actually no! The bifurcation turns out to be subcritical. Therefore trajectories must jump to a different attractor. But what can it be?



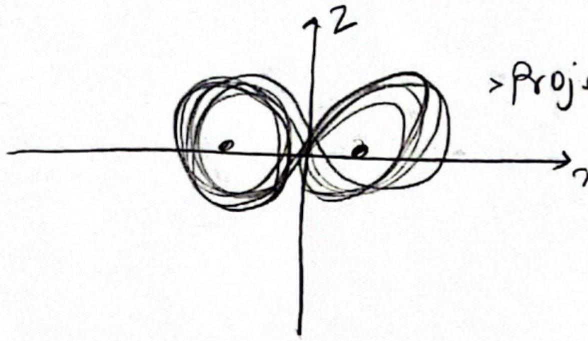
These cycles actually do hit the origin & the dynamics are too complex to analyze here.

→ Could the trajectories go to ∞ when $r > r_{Hopf}$. No. You can prove that there is a large sphere where all trajectories are moving in → trapping sphere

→ Can there be any stable limit cycles for $r > r_{Hopf}$. Lorenz showed there are none for $r = 28$.

→ Can there be quasiperiodicity on an invariant Torus. No because an invariant Torus needs a fixed volume, but we know volume shrinks exponentially fast.

→ Therefore, we have eliminated all known possibilities. In fact what we end up getting is a chaotic strange attractor.



> Projection onto $x\,z$ plane.

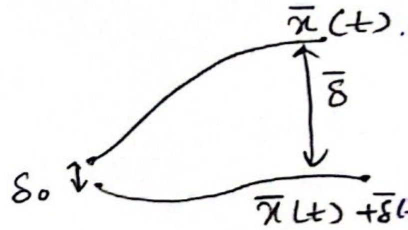
→ As $r \uparrow\uparrow$, we could leave chaos & get a limit cycle & as $r \uparrow\uparrow$ we go back & forth between chaos & limit cycles. This is just like a -ve resistance source on a transmission line.

→ Predictibility horizon for weather is $\simeq$ 10days. For the solar system it is $\simeq$ 5 million years.

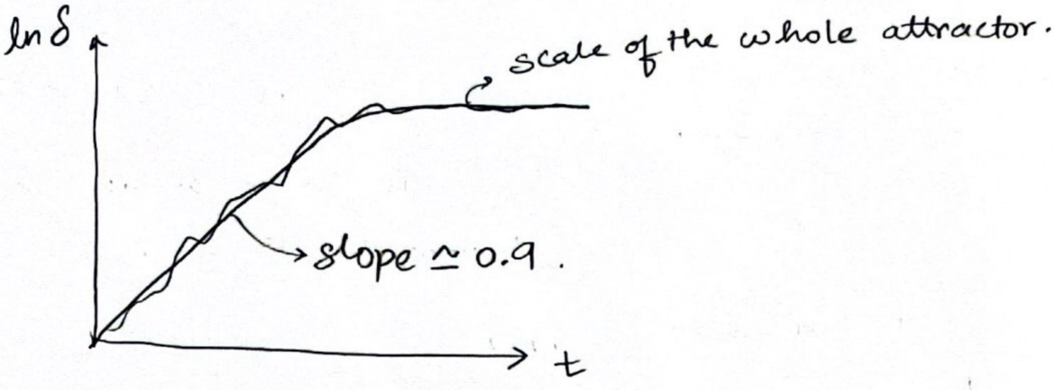
> **On the Lorenz attractor.**

$\delta(t) \rightarrow$ distance between trajectories.

$\delta(t) \cong \delta_0 e^{\lambda t} \rightarrow$ exponential.



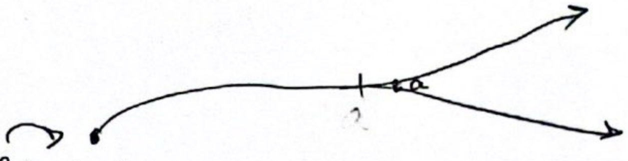
$\lambda \simeq 0.9$



$\rightarrow \lambda =$ Liapunov exponent.

$\rightarrow$ In reality we would have n Liapunov exponents for an n-dimensional system. In 3D like here every initial point $x_0(t)$ would have a sphere of radius δ_0 around it. This sphere grows into an ellipsoid because all the points in the sphere may diverge differently. The largest axis is usually used as the λ, but notice that it has 3 λs corresponding to the 3 axes of the ellipsoid.

$\rightarrow$ $\boxed{\text{positive Liapunov exponent} \Rightarrow \text{Chaos!}}$



2 initial conditions

a = tolerance

$\delta > a$ is when deviation becomes noticeable

$$\boxed{\delta_0 e^{\lambda t} \approx a}$$

$$t \approx \frac{1}{\lambda} \ln\left(\frac{a}{\delta_0}\right)$$

predictibility horizon or Liapunov time!

t is on the order of $\frac{1}{\lambda}$.

→ If we want to predict upto $10t$ we need to make the prediction 10^{10} times better. The logarithm kills the prediction.

Lec 18 **Strange Attractor for the Lorenz Equations**

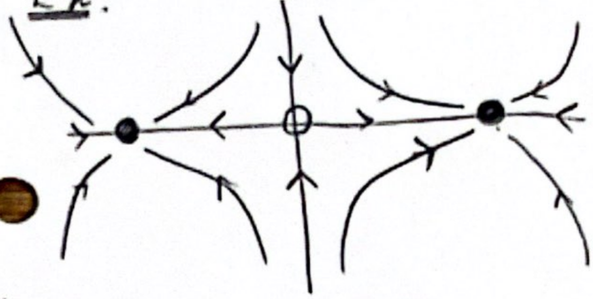
Definitions (conceptual)

Chaos: Aperiodic long term behaviour in a deterministic system (not probabilistic) that exhibits sensitive dependance on initial conditions.
↳ (positive Lyapunov exponent)

Attractor (A):
1) Invariant set (start in A → stay in A forever)
2) Attracts an open set of initial conditions.
3) No proper subset of A satisfies 1) & 2) → "minimal".

Basin of attraction must be an open set

Ex:



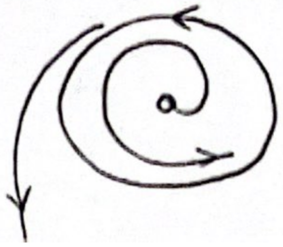
Open set → A disk drawn around a point in the set lies in the set. Just like open brackets ( )

i> Saddle here is not an attractor (Fails 2))
⇒ Origin is not an attractor.
ii> x axis? Does not satisfy 3) since the 2 nodes satisfy 1) & 3). They are in fact attractors.

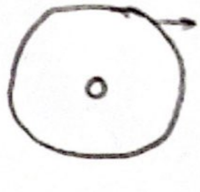
The x axis is called an attracting set but it is not an attractor.

→ Is a stable limit cycle an attractor? Yes.

Is a half stable limit cycle an attractor? NO Because the set that it attracts is the area inside + the cycle itself − the unstable f.p at the center. Since it attracts itself it has a boundary & the set is closed!



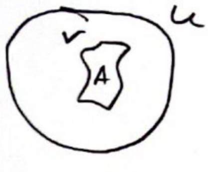
attracting set is →



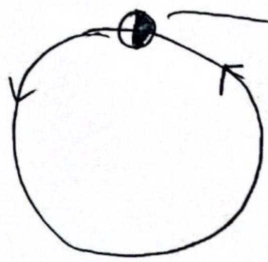
→ not an open set

There is usually a 4th property.

4) Trajectories that start near A, stay near A for all time $t>0$. Given any neighbourhood U of A, there exists another neighbourhood V of A, S.T if you start in V, then you stay in U.



Eg:



→ Is this an attractor?

> It satisfies 1), 2) & 3), but does not satisfy 4)

Strange attractor

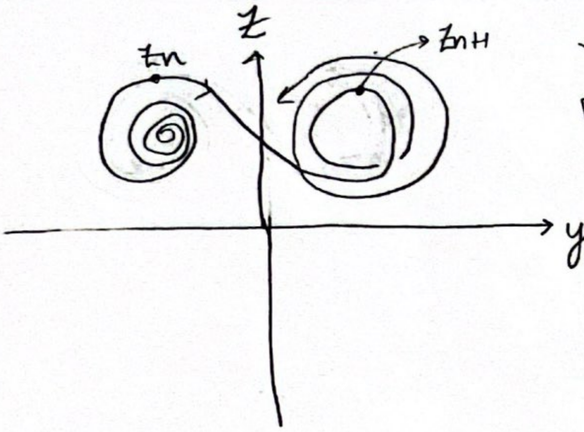
- Attractor that exhibits sensitive dependance on initial condns

(or)

- An attractor whose local structure is a fractal (not smooth)

Dynamics of Lorenz attractor (§9.4)

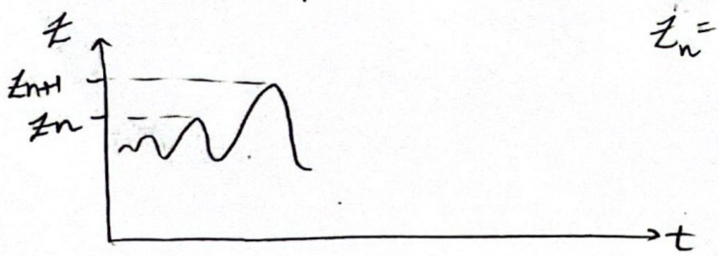
Reduction to a 1D map.



$r = 28$
$\sigma = 10$
$b = 8/3$

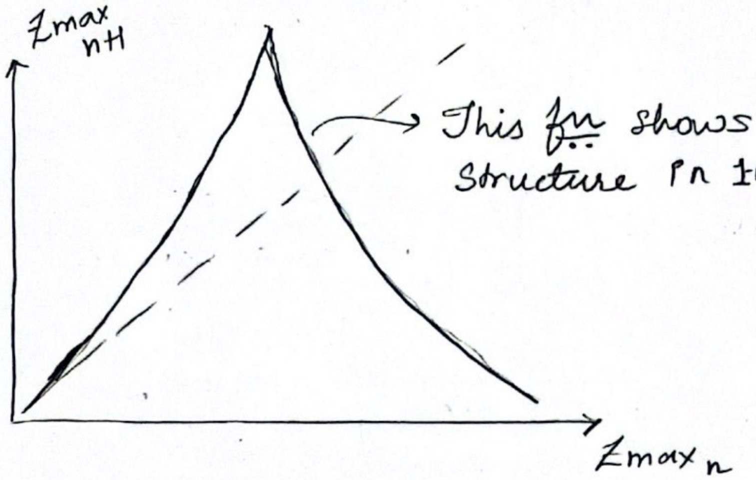
> Once the trajectory reaches a critical value in z, it leaves one spiral & goes to the next. This continues forever.

> Depending on value of z when it leaves the value on the 2nd spiral where it lands is determined.



z_n = nth relative local maximum of the fn $z(t)$.

> How does z_n determine z_{n+1}



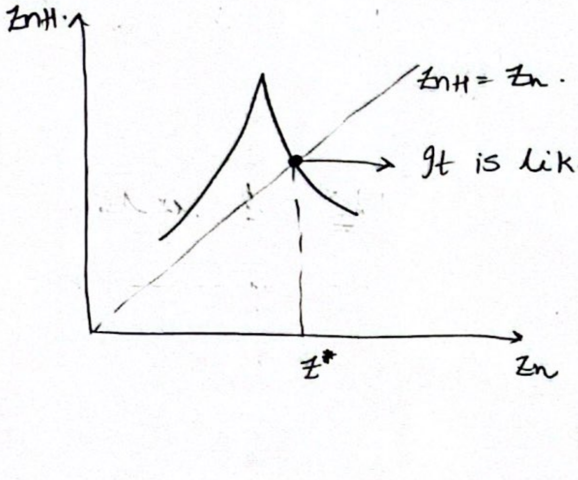
This fn shows some underlying structure in 1D.

→ Looks like $Z_{n+1} = f(Z_n)$ since essentially the curve has no thickness. In reality it is not a fn since for Z_n we could have more than one Z_{n+1}. Remember we are in 3D.

Key: $|\text{slope}| > 1$

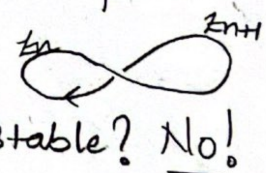
$|f'(z)| > 1 \quad \forall z.$

→ Iterate $f(z)$ to study the dynamics.



It is like a fixed point since $f(z^*) = z^*$

⇒ the trajectory is only in the xy plane.



Is z^* stable? No!

Let $Z_n = z^* + \eta_n$ → tiny perturbation.

Linearizing around z^*

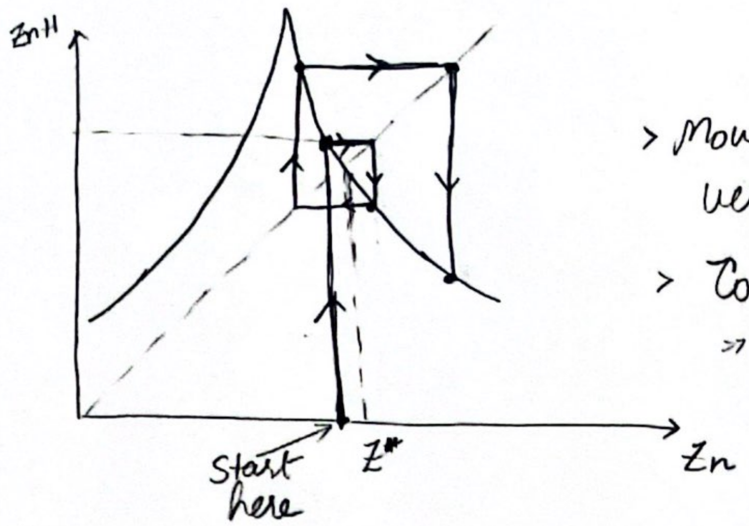
$$Z_{n+1} = \cancel{z^*} + \eta_{n+1} = f(Z_n) = f(z^* + \eta_n) = \cancel{f(z^*)} + \eta_n f'(z^*) + \cdots$$

equal!

$\therefore \eta_{n+1} = \eta_n f'(z^*)$

$\hookrightarrow |f'(z^*)| > 1 \Rightarrow |\eta_{n+1}| > |\eta_n|$

$\Rightarrow z^*$ is unstable.



Cobweb diagram.

> Move Horizontally to the diagonal & vertically to the curve.

> Could the cobweb close?

$\Rightarrow$ Can a periodic orbit be stable.

→ Periodic orbit gives a sequence $z_1, z_2, z_3 \ldots$ with $z_{n+p} = z_n \ \forall n$, where p = period. This is all to explore if the strange attractor is not a long period limit cycle. Need to show that the cobweb is unstable.

$z_2 = f(z_1)$

$z_3 = f(z_2) = f(f(z_1)) = f^2(z_1)$

$\Rightarrow z_4 = f^3(z_1)$

$\Rightarrow \boxed{z_{n+p} = f^p(z_n)}$

Defn: z is a point of period p if $f^p(z) = z$. p is smallest positive integer with this property.

→ We want to show that any period p would be unstable

→ When $p=2$, suppose $f(f(z)) = z \Rightarrow f^2(z) = z$. Therefore, a point of period 2 is a fixed pont for $f^2(z)$.

$\ast\ast$→ In general, a point of period p is a fixed point for f^p.

→ Let's look at stability of f^2 at z.

$\Rightarrow$ we look at $(f^2)' \Rightarrow \frac{d}{dz}(f(f(z))) = f'(f(z))f'(z)$.

We see that $|(f^2)'| = \underbrace{|f'(f(z))|}_{>1} \cdot \underbrace{|f'(z)|}_{>1}$

$$\Rightarrow |(f^2)'| > 1$$

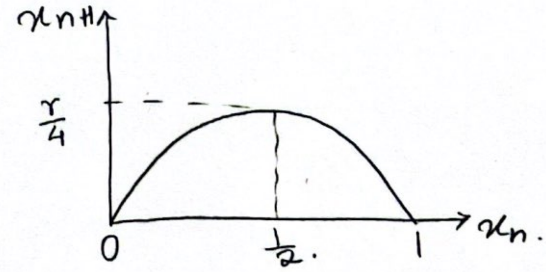
& $(f^2)'$ determines the stability

→ Therefore there are no stable trajectories & therefore no stable limit cycles. This is not a real proof since f is not a function.

Lec 19 - One dimensional maps

$x_{n+1} = f(x_n)$ we leave differential equations for a while and focus on 1D maps as a simpler model of chaos.

Logistic map. $x_{n+1} = r\, x_n (1 - x_n)$.



$x \in [0,1]$
$0 \le r \le 4$

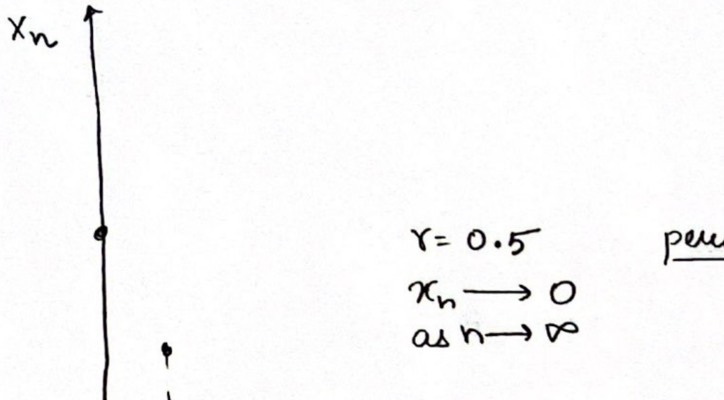
> Robert May (1976) Nature Magazine 261, 459
> Could say it is the simplest possible Nonlinearity.

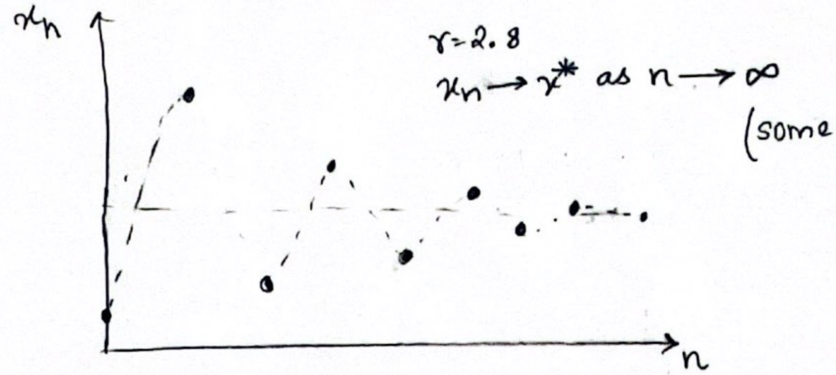
→ If we have a nonlinear system of differential equations we cannot find basis functions since their linear combination will not necessarily be a solution. No basis function or eigenfn
⇒ no Fourier Transform, Laplace transform etc. Complex stuff can happen!

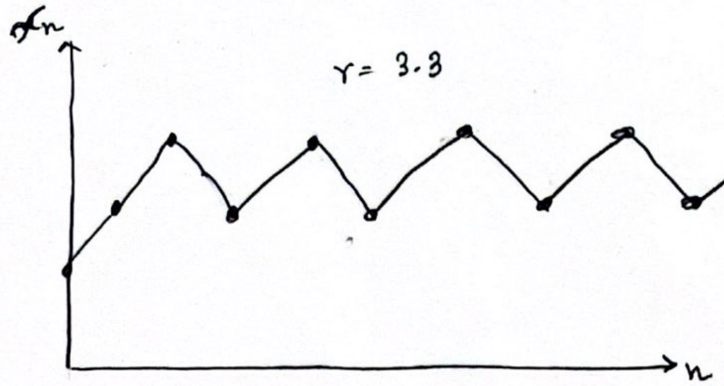
→ $X_1 = r x_0 (1 - x_0)$

$X_2 = r x_1 (1 - x_1)$

⋮







Stable period 2.

$\Rightarrow x_{n+2} = x_n$

Stable period 4

$\Rightarrow x_{n+4} = x_n$

→ Since period doubled as $r \uparrow$, & we can find any power of 2 as a periodic orbit as $r \uparrow$.

→ Where do the bifurcations from period 2^n to 2^{n+1} occur.

→ Let r_n to be the value of r where a stable 2^n cycle first occurs.

<u>Find:</u> Period 2 is born when $r_1 = 3$.

Period 4 " " $r_2 = 3.449$

Period 6 $r_3 = 3.54409$

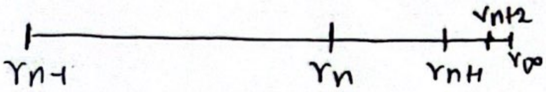
$r_4 = 3.5644$

$r_5 = 3.568759$

→ $r_\infty = 3.569946\ldots \Rightarrow$ period is 2^∞.

→ r_n converges (essentially) geometrically to r_∞.

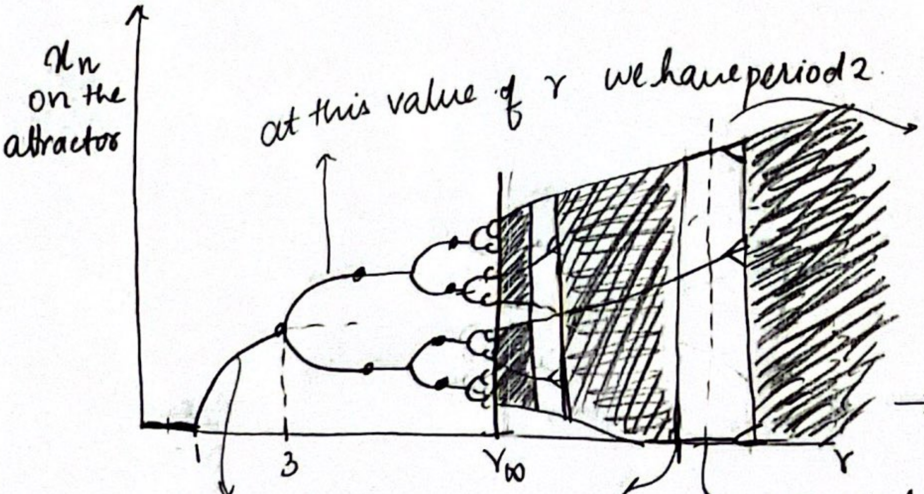
↳ approaches a geometric series as $r_n \uparrow$



$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \longrightarrow \underline{\underline{4.6692\ldots}} \quad \text{as } n \to \infty \quad \to \text{geometric as } n \to \infty$$

↳ $\boxed{\delta}$ → Poster Boy/Girl for Chaos!!

<u>What happens after r_∞?</u>



→ <u>Orbit diagram</u>

→ Inside this period 3 window there is actually a full copy of the orbit diagram. This is true for every window & not just for this period 3 window.

→ <u>Every integer has a window.</u>

↳ Theorem!

This period 1 actually becomes unstable

period 3 → It is born at $1 + 2\sqrt{2}$

→ Periodic windows!

Here we can get period 3!!

→ Try to derive the first few r_ns.

$x_{n+1} = r x_n (1-x_n)$ $\quad$ $x^* = 0$ is always a f.p $\forall r$.

Near $x^* = 0$, map behaves like $x_{n+1} \cong r x_n$

$\Rightarrow$ $x_1 = r x_0$

$x_2 = r x_1 = r^2 x_0$

$\vdots$

$x_n = r^n x_0 \Rightarrow x^* = 0$ is linearily stable if $r < 1$.

→ Stability of x^* depends on looking at $|f'(x^*)| \Rightarrow |f'(x^*)| < 1$ $\Rightarrow$ stable.

Same as Lorenz map.

$f(x) = rx(1-x)$

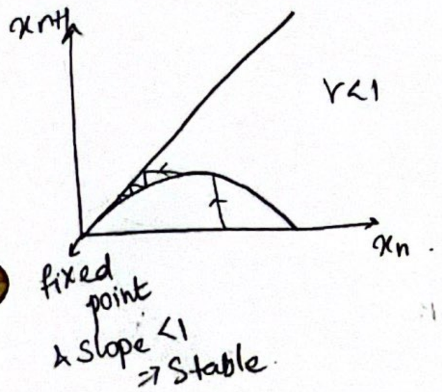
$= rx - rx^2$

$f'(x) = r - 2rx$

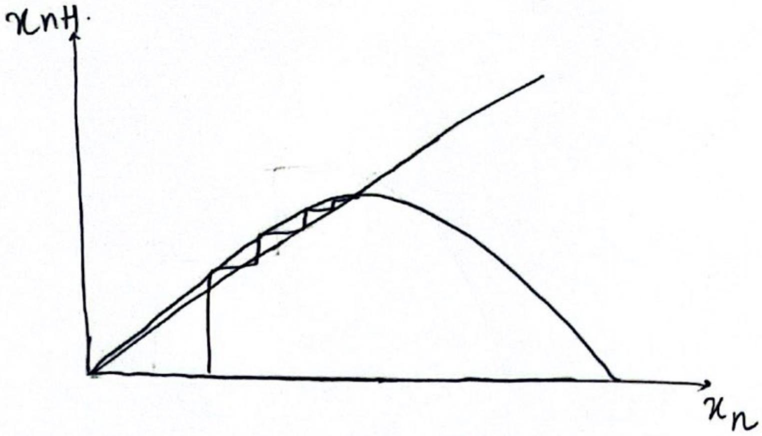
$f'(0) = r \Rightarrow r < 1 \Rightarrow$ linearly stable since r is always positive.

$f(x^*) = x^* \Rightarrow r\cancel{x^*}(1-x^*) = \cancel{x^*} \Rightarrow x^* = 1 - \frac{1}{r}$

$f'(x^*) = r - 2r\left(1 - \frac{1}{r}\right) < 1 \Rightarrow 2 - r < 1 \Rightarrow \boxed{r > 1}$ → notice this on orbit diagram

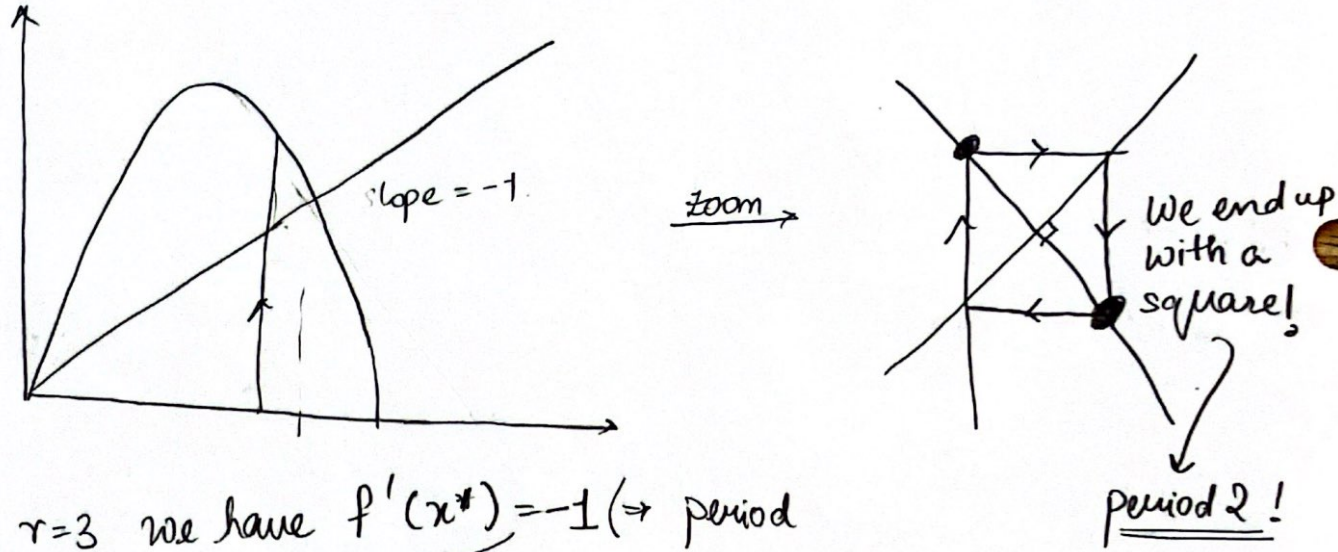
→ 

Also note that from here we see global asymptotic stability not just linear stability.



What happens at $r=3$.

$$f'(x^*) = 2-r = -1$$



→ When $r=3$ we have $f'(x^*) = -1$ (→ period doubling) a condition that leads to periodoubling.

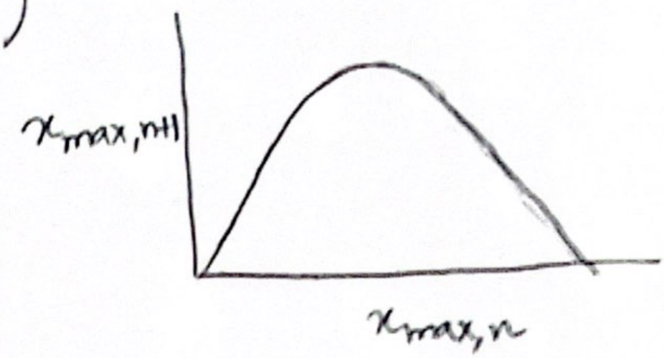
$f'(x^*) = -1$ called flip bifurcation & -1 here is the eigenvalue because it is the multiplier to the deviation for each iteration

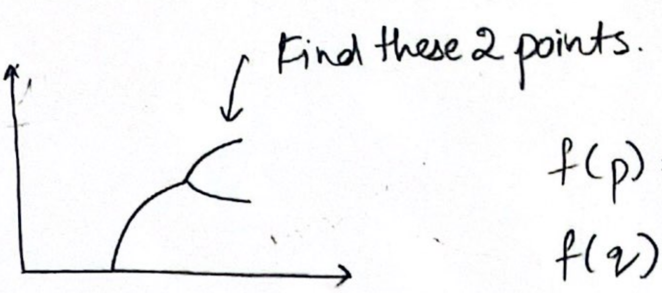
$$\eta_{n+1} = \underbrace{f'(x^*)}_{\text{eigenvalue}} \eta_n.$$

Rossler System

$$\begin{aligned} \dot{x} &= -y-z \\ \dot{y} &= x+ay \\ \dot{z} &= b+z(x-c) \end{aligned}$$

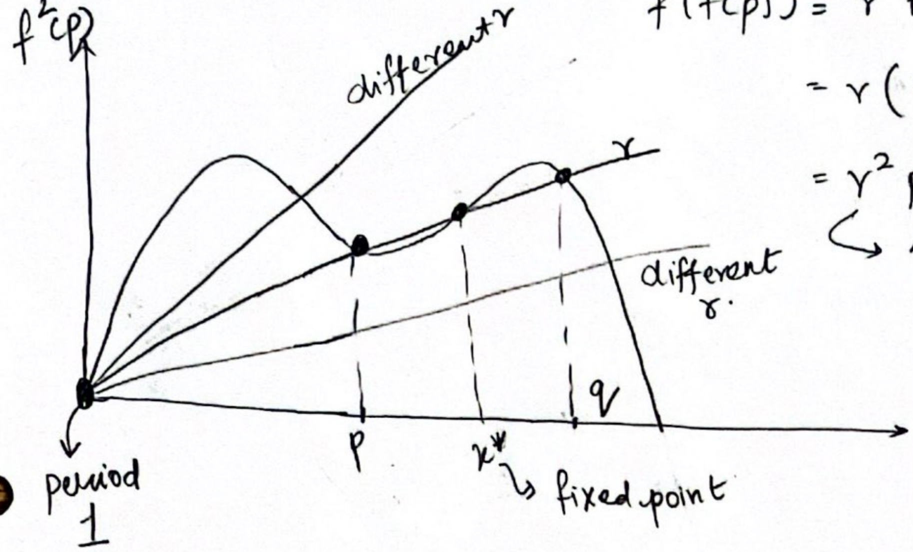
1D map of this is unimodal.





$$\left.\begin{aligned} f(p) &= q \\ f(q) &= p \end{aligned}\right\} \Rightarrow f(f(p)) = p$$

→ Period 2 point is a fixed point of the map $f^2(p)$



$$f(f(p)) = r\, f(p)\, (1 - f(p))$$
$$= r(r\, p(1-p))(1 - rp(1-p))$$
$$= r^2 p(1-p)(1 - rp(1-p))$$

↳ 4th degree polynomial.

$p = f^2(p)$ gives p & q → factoring out the 2 trivial roots
⇒ use quadratic formula.

→ p, q could be explicitly found.

→ When do they go unstable to give period 4 $r_2 = 1 + \sqrt{6} = 3.449\ldots$

→ Cannot really go further than r_3 analytically.

→ p, q are fixed points of f^2. If x_n is at p x_{n+2} is at p, which means at x_{n+1} it must be at q since at x_{n+3} it must be back at q since q is also a f.p of f^2.

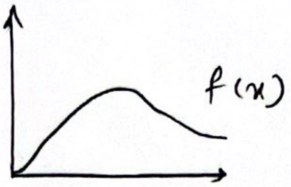
Lec 20 - Universal aspects of Period doubling. → Hardest part of the subject

Hard paper. M. J. Feigenbaum (1978) J. Stat. Physics. vol. 19, Pg 25.

Easier paper. (1983) Physica D vol. 7, Pg 16

1) Studied period doubling in various 1 humped (unimodal) maps

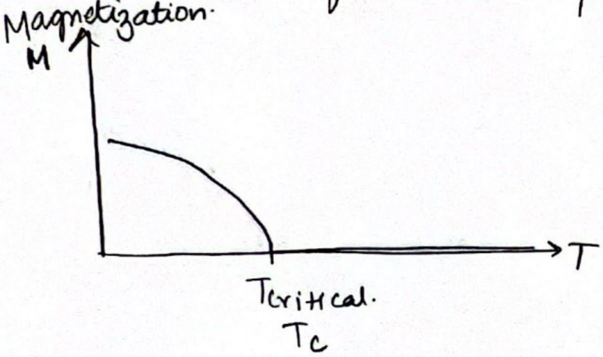
$x_{n+1} = r f(x_n)$



→ He found amazing quantitative laws independant of precise form of f(x)!

2) Connections to statistical physics -

a) Analogy to "universal exponents" observed in 2nd order phase transitions in magnets, fluids, superfluids ...



$M \propto |T - T_c|^\beta$ → exponent rate at which phase transition occurs.

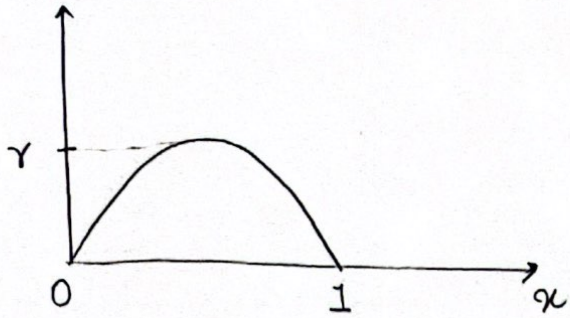
→ Completely different physical systems have the same phase transition exponents. This was explained using "renormalization"

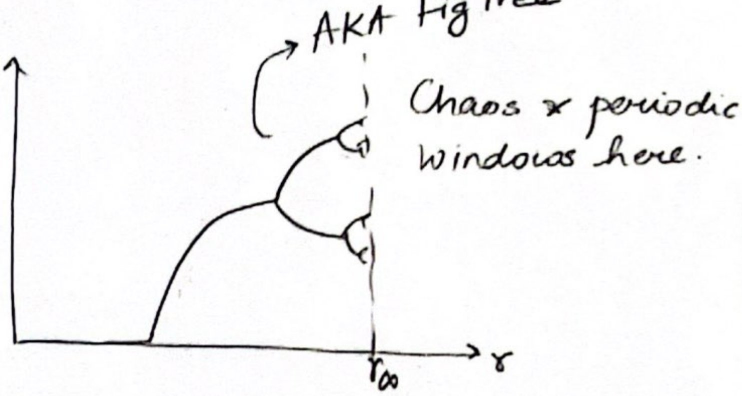
b) → Feigenbaum used renormalization to explain this universality in dynamical chaotic systems.

3) Predictions about route to chaos :- Confirmed in very different experiments in fluids, chem oscillators, semiconductors etc.

<u>Computer experiment</u>

Consider sine map $f(x) = r\sin\pi x$. $x \in [0,1]$.

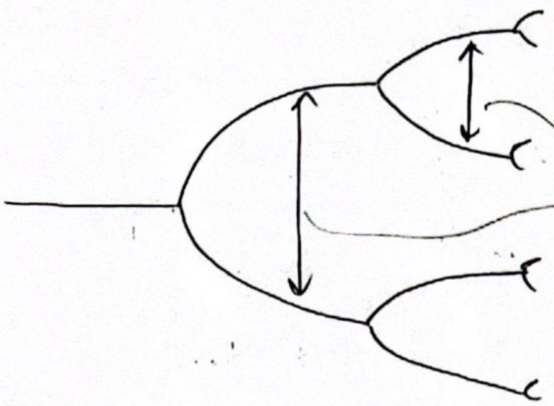




→ Period doubling still occurs. The specific r_ns depend on $f(x)$, but

$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \longrightarrow 4.6692\ldots = \delta \text{ again!}$$

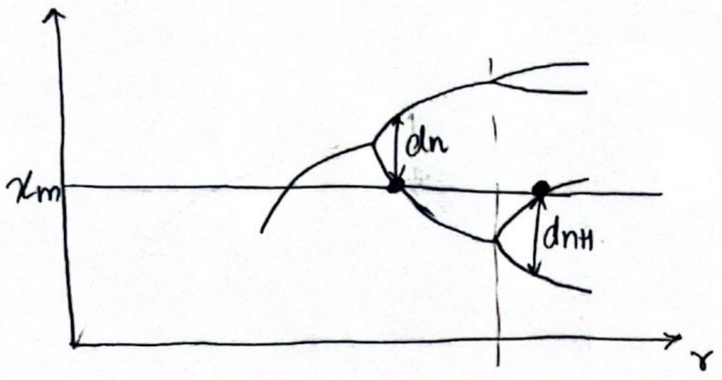
→ There's also universal scaling in the x direction.



He wanted to find what the relationship b/w these 2 heights is.

→ Say x_m is where the maximum is on the map.

→ We want to look at the fig tree near x_m.



$d \rightarrow$ distance from x_m to the branch. We will later show that all periods cut the x_m line.

Ratio $\frac{d_n}{d_{n+1}}$ always converges to $-2.5029\ldots = \alpha$.

<u>Lec 21</u> <u>Feigenbaum's renormalization analysis of period doubling</u>

Define <u>superstable</u> fixed points and cycles.

$x_{n+1} = f(x_n)$. Fixed pt. $f(x^*) = x^*$

$x_n = x^* + \eta_n \qquad \eta_{n+1} = f'(x^*)\,\eta_n + \ldots$ → linear growth

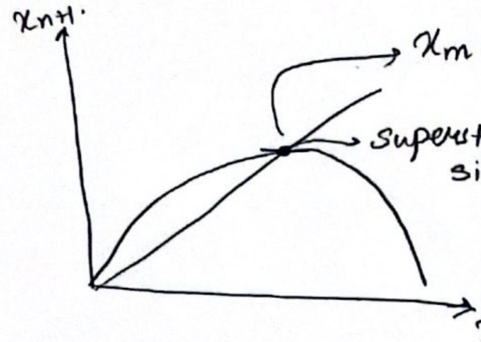
x^* is linearly stable $\Longleftrightarrow |f'(x^*)| < 1$

If $f'(x^*) = 0 \Rightarrow x^*$ is <u>super stable</u>.

$\eta_{n+1} = \frac{1}{2} f''(x^*)\,\eta_n^2 + \ldots \longrightarrow$ quadratic growth.

Say $\eta_{n+1} = \eta_n^2$, if $\eta_1 = 10^{-1} \Rightarrow \eta_2 = 10^{-2} \Rightarrow \eta_3 = 10^{-4}, 10^{-8}, 10^{-16}\ldots$

This is much faster than exponential.



$x_m \Rightarrow$ it happens at the maximum.

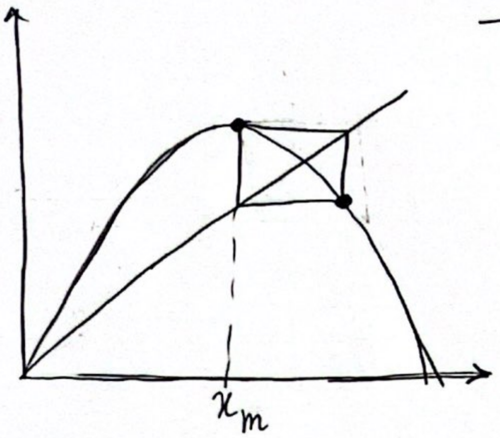
superstable f.p since f' is 0.

$f(x^*) = x^*$

$f'(x^*) = 0$

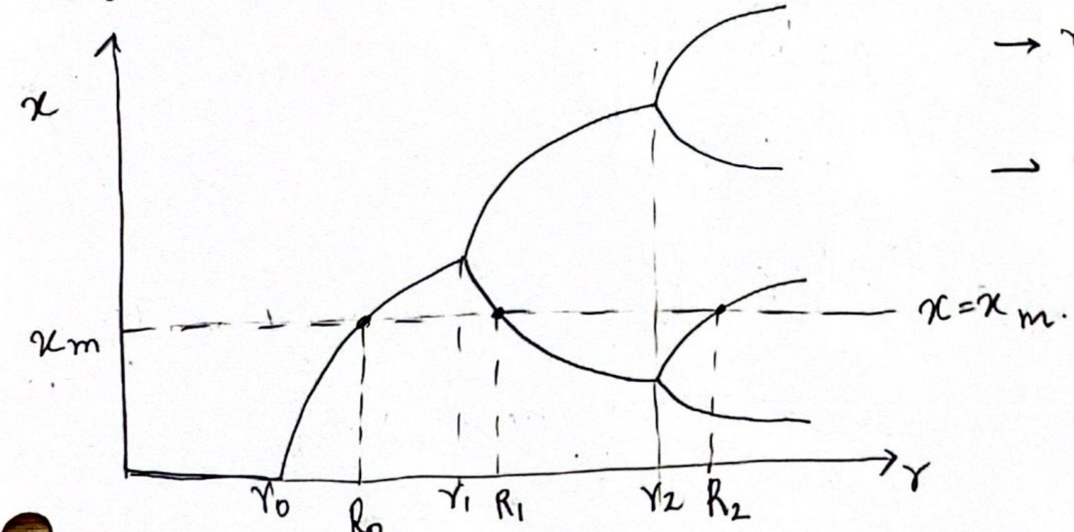
Superstable P-cycle: x_m is a fixed pt. of f^P. Equivalently x_m is one of the points in the P cycle.

Eg: superstable 2 cycle.



→ Finding these points is very fast in convergence.

Figtree.



→ r_n is where a stable 2^n cycle is born.

→ R_n is where the 2^n cycle becomes superstable.

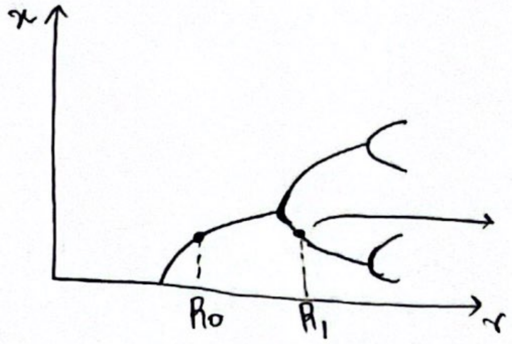
→ R_n occurs between r_n and r_{n+1}.

→ Spacing between the successive R_ns also shrinks geometrically at a rate $\delta = 4.669$ as $n \to \infty$. Since R_n is easier to compute we can study that instead.

Renormalization

→ Picture looks self similar.

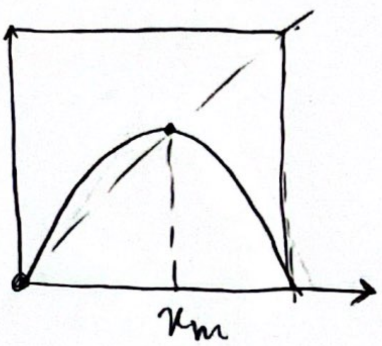
› Compare the situations at R_0 and R_1 and "renormalize" one into the other.

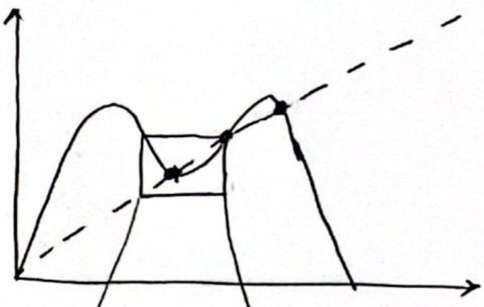


in f this is a point on the superstable cycle. However in f^2 it is a superstable fixed point.

→ Graph of $f(x, R_0)$

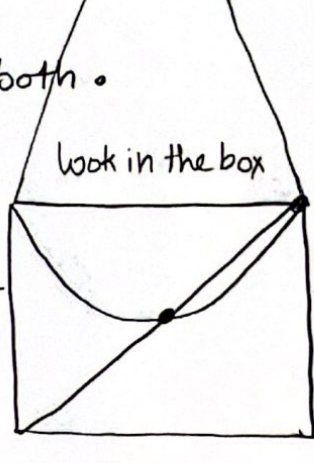
Graph of $f^2(x, R_1)$





→ x_m is a superstable fixed point for both.

look in the box

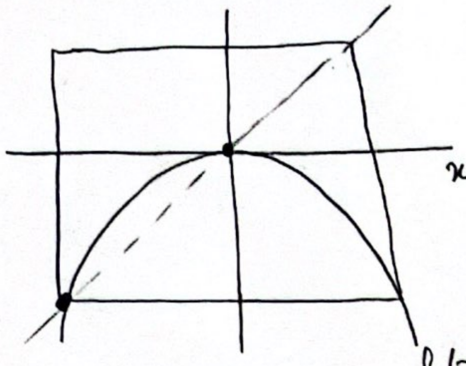


Need to flip ↕ & scale to make the pictures look the same.

→ Same as the box of $f(x, R_0)$.

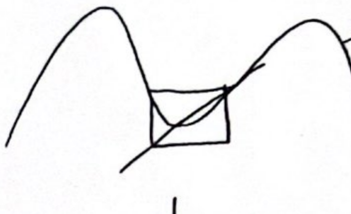
→ $f^2(x, R_1)$ has the same local dynamics as $f(x, R_0)$ except shrunken and flipped.

→ Helpful to translate origin to x_m on diagonal.
⇒ subtract x_m from both x & f ↳ since f is x in the future.



$f(x, R_0)$

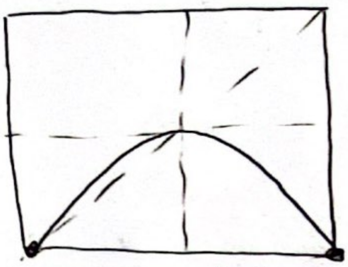
iterate & shift



$f^2(x, R_1)$

↓

Rescale by $\alpha \simeq -2.5$ ↳ Notice!



This shifting & scaling is the renormalization procedure.

In equations.

→ Pictures suggest $f(x, R_0) \simeq \alpha f^2\left(\frac{x}{\alpha}, R_1\right)$

× scales up ÷ scales up

→ Doing it again,

$$f(x, R_0) \simeq \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right)$$

$$\simeq \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right)$$

Renormalization n times!

→ Feigenbaum found numerically that the limit as $n \to \infty$ of

$$\lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x)$$

↳ converged to this $g_0(x)$ which looks

→ What is $g_0(x)$?

→ A universal function with a superstable fixed point. Only works if α is chosen to be $-2.5029...$

→ Universal in the sense that the limit function is 'almost' independent of f. (as long as it is 1 humped)

→ Can now see where universality comes from:

Limiting function $g_0(x)$ is universal because it only depends on the behaviour of the original function f near the maximum x_m.

> The boxes are 'zooming' in around the x_m point with each iteration. Therefore, it doesn't really depend on the fn but only on the nature of the maximum. Global aspects of f are lost & only the order of the maximum survives.

> Different $g_0(x)$s are found for fs with a 4^{th} degree max or 6^{th} degree max etc...

→ To get other universal functions $g_i(x)$ start with $f(x, R_i)$

↳ has a superstable 2^i cycle.

Look for a function $g_i(x) = \lim_{n\to\infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_{n+i}\right)$

→ We want $i = \infty$ ⇒ where we have the onset of chaos.

Notice here R_∞ iterates to $R_{\infty+1}$ which is just R_∞.

→ $R = R_\infty$ is most interesting, since then $f(x, R_\infty) \approx \alpha f^2(\frac{x}{\alpha}, R_\infty)$ ← no shift

⇒ R does not change!

→ Limiting function $g_\infty(x)$ is just called $g(x)$.

$g(x)$ – it satisfies: $\boxed{g(x) = \alpha g^2\left(\frac{x}{\alpha}\right)}$

↳ The functional equation for g and α.

These are like boundary conditions.

→ But we also need $g'(0) = 0$ since max is now at origin.

→ Also need: g has a quadratic max.

→ Can choose $g(0) = 1$ → normalizing x since scaling x still satisfies the fnal. eqn.

→ If $g(x)$ solves the functional eqn. So does $\mu\left(g\left(\frac{x}{\mu}\right)\right)$ for any μ

→ Plug in $g(x) = 1 + c_2 x^2 + c_4 x^4 \ldots$ to solve the functional equation (since $g(x)$ is even → can be shown).

→ <u>Finding α</u>!

→ $g(0) = \alpha g^2(0) \Rightarrow \alpha g(1) = 1 \Rightarrow \boxed{\alpha = \frac{1}{g(1)}}$

Substituting in the functional equation we get,

$c_2 = -1.527, \ldots \quad c_4 = 0.1048 \ldots$

We have found α, now we need to find δ!

Lec 22 Renormalization: Function Space and a hands on Calculation