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Complex Analysis

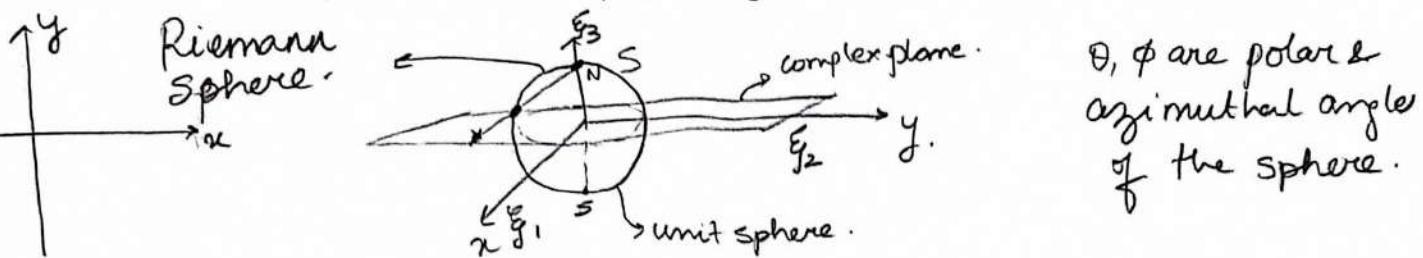
Rec 1

$$\rightarrow z = x + iy \quad (x, y \in \mathbb{R})$$

$\rightarrow z^* = x - iy \rightarrow$ linearly independent of $x + iy$

Analytic functions are fns of only $z = x + iy$ & not of z^* .

Stereographic projection. \rightarrow projecting sphere onto plane.



Let ξ_1, ξ_2, ξ_3 be coordinates of the sphere. $\Rightarrow \xi_1 = \sin \theta \cos \phi$

$$\xi_2 = \sin \theta \sin \phi$$

$$\xi_3 = \cos \theta$$

$$\& \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$$

\rightarrow Projection from North pole onto complex plane is always associated to one point on the sphere

\rightarrow map $x, y \rightarrow \xi_1, \xi_2, \xi_3$ is not independent.

$$\text{It is easy to show that } x = \frac{\xi_1}{1 - \xi_3} ; y = \frac{\xi_2}{1 - \xi_3} \Rightarrow \overset{\text{not } z \text{ axis.}}{z} = \frac{\xi_1 + i \xi_2}{1 - \xi_3}$$

$$\Rightarrow x = (\cot \frac{\theta}{2}) \cos \phi ; y = (\cot \frac{\theta}{2}) \sin \phi ; z = e^{i\phi} \cot \frac{\theta}{2}$$

$$\xi_1 = \frac{2x}{x^2 + y^2 + 1} \quad \xi_2 = \frac{2y}{x^2 + y^2 + 1} \quad \xi_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \Rightarrow \xi_1 = \frac{z + z^*}{|z|^2 + 1} ; \xi_2 = \frac{z - z^*}{i(|z|^2 + 1)} ; \xi_3 = \frac{z^2 - 1}{|z|^2 + 1}$$

The origin is mapped to ∞ , all points at ∞ are mapped to N.P

$\Rightarrow F : \{ |z| < \infty \} \rightarrow \text{complex plane.}$

$\hat{F} : \{ |z| \leq \infty \} \rightarrow \text{extended complex plane}$

If z_1, z_2 are two points on the complex plane the chordal distance b/w the corresponding points on the sphere is $\frac{2|z_1 - z_2|}{\sqrt{(|z_1|^2 + 1)(|z_2|^2 + 1)}}$

Now distance between any 2 pts. becomes finite.

Distance (chordal) from z to ∞ = $\frac{2}{\sqrt{|z|^2 + 1}}$.

Analytic functions

$f(z) = u(x, y) + i v(x, y)$ is analytic if the Cauchy Riemann conditions are satisfied.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$$x = \frac{z+z^*}{2}, \quad y = \frac{z-z^*}{2i}$$

We basically say $\frac{\partial f}{\partial z^*} = 0$ since f is not a fn of z^*

Wirtinger derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \quad ; \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

$$\Rightarrow \frac{\partial f}{\partial z^*} = 0 \Rightarrow \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \quad \& \quad \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} = 0$$

Eg: $f = x \rightarrow$ not analytic
 $f = y \rightarrow$ No

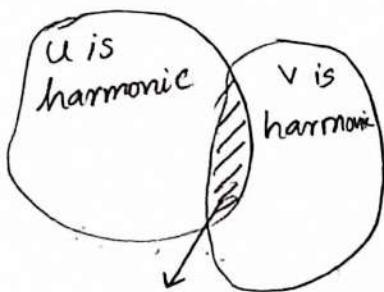
$$f = r = \sqrt{x^2 + y^2} \rightarrow \text{No since } r = (zz^*)^{1/2}$$

$$f = \theta = \tan^{-1} \frac{y}{x} = \frac{1}{2i} \ln \left(\frac{z}{z^*} \right) \rightarrow \text{No}$$

From CR equations we can see that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\Rightarrow u, v$ are harmonic functions in the region where f is analytic.

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Given u we can find v from CR eqns where $u+iv$ is analytic.

→ If f is analytic in \mathbb{C} then f is called an entire function.

$$f(z) = z^2 \rightarrow \text{Yes! Entire!}$$

$$f(z) = z^n \rightarrow \text{Yes, } f(z) = P_n(z) \xrightarrow{\text{polynomial of degree } n} \text{Yes.}$$

$$f(z) = e^z \rightarrow \text{Yes}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \rightarrow \text{all finite } z, \text{ this is convergent} \Rightarrow |z| < \infty$$

$$e^{-z} \rightarrow \text{Yes.}$$

$$\frac{e^z + e^{-z}}{2} = \cosh(z) \rightarrow \text{Yes}; \quad \sinh(z) \rightarrow \text{Yes}; \quad \cos(z) \rightarrow \text{Yes}, \\ \sin(z) \rightarrow \text{Yes.}$$

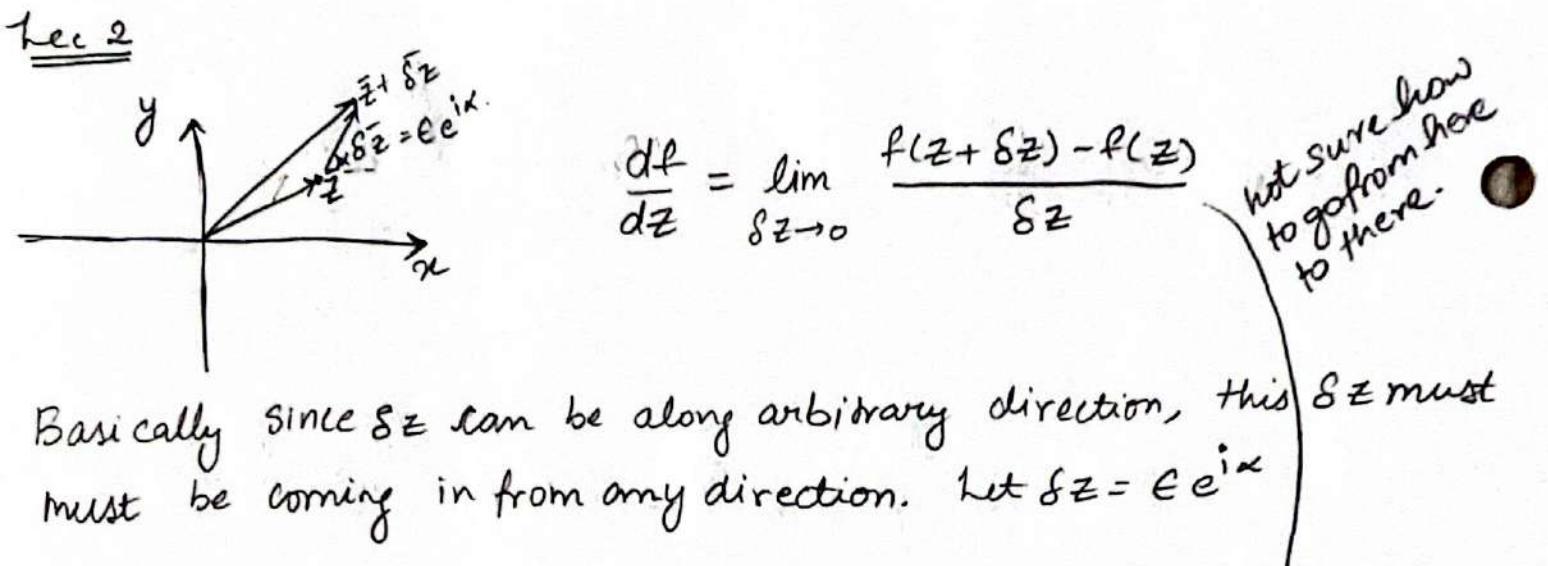
$\tan z \rightarrow$ not singularities \Rightarrow not differentiable
entire.

$z^{-1} \rightarrow$ not entire (singularity at $z=0$).

→ Any fn. that is entire in the extended complex plane, must be a constant.

Liouville's Theorem: A function that is bounded & entire must be constant.

⇒ If it is entire & not constant, it must be singular at ∞ .



Basically since δz can be along arbitrary direction, this δz must be coming in from any direction. Let $\delta z = e^{i\alpha}$

$$\Rightarrow \frac{df}{dz} = e^{-i\alpha} \left\{ \left(\frac{\partial u}{\partial x} \cos \alpha + i \frac{\partial v}{\partial y} \sin \alpha \right) - i \left(\frac{\partial u}{\partial y} \cos \alpha + i \frac{\partial v}{\partial x} \sin \alpha \right) \right\}$$

$$= e^{-i\alpha} \left\{ \frac{\partial u}{\partial x} \cos \alpha + i \frac{\partial v}{\partial y} \sin \alpha - i \left(\frac{\partial u}{\partial y} \cos \alpha - i \frac{\partial v}{\partial x} \sin \alpha \right) \right\}$$

Apply CR

$$= e^{-i\alpha} \left\{ \frac{\partial u}{\partial x} [\cos \alpha + i \sin \alpha] - i \frac{\partial u}{\partial y} [\cos \alpha + i \sin \alpha] \right\}$$

$$= e^{-i\alpha} \left\{ e^{i\alpha} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right) \right\}$$

$$\boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}}$$

$$\text{or } \frac{df}{dz} = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

$$\frac{df}{dz} = i \left(\frac{\partial v}{\partial x} - i \frac{\partial v}{\partial y} \right)$$

\Rightarrow Only when CR conditions are applied the derivative is independent of α .

$$\Rightarrow 2 \frac{df}{dz} = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) - i \left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right).$$

$$\Rightarrow \frac{df}{dz} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial v}{\partial y} \right) f \rightarrow \text{Wirtinger derivative from earlier.}$$

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$f(x) = x|x|$ has first derivative at 0 but no 2nd derivative.

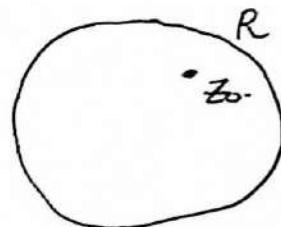
$$f(x) = \begin{cases} -x^2 & x > 0 \\ x^2 & x < 0 \end{cases}$$

however if $f(z)$ is analytic \Rightarrow first derivative \Rightarrow all derivatives exist.

$\Rightarrow f(z)$ can be represented as a power series in the region where it is analytic

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \rightarrow \text{Power series (Taylor series).}$$

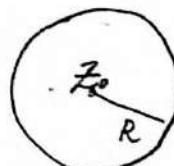
$$a_n = \left. \frac{1}{n!} \frac{d^n}{dz^n} f(z) \right|_{z=z_0}$$



Worst case convergence check is absolute convergence $\Rightarrow \sum_{n=0}^{\infty} |a_n| r^n < \infty$

Every power series about z_0 has a circle of convergence with radius R . inside which the series converges absolutely.

$$\Rightarrow \sum_{n=0}^{\infty} |a_n (z - z_0)^n| < \infty$$



R = radius of convergence.

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

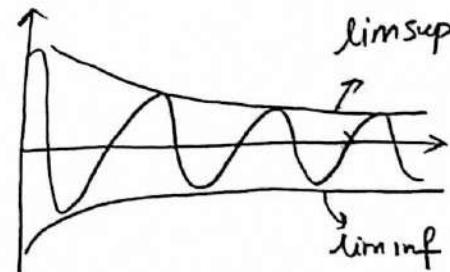
Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1} (z - z_0)^{n+1}}{a_n (z - z_0)^n} \right| = L$
 $L < 1 \Rightarrow \text{absolute conv.}$

Better defn.,

$$R = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$$

least upper bound.

$R = \infty \Rightarrow \text{entire function.}$



Eg: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\Rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{z^{n+1} n!}{(n+1)! z^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{z}{n+1} \right| \rightarrow 0$ for all $|z| < \infty$

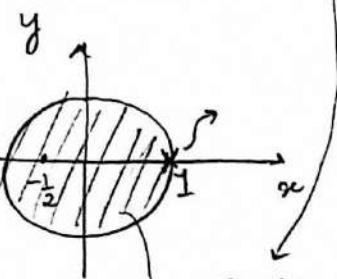
$\Rightarrow e^z$ is an entire function

Eg: $f(z) = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n \rightarrow$ converges when $|z| < 1$

$$R = 1$$

$$f(z) - 1 = z f(z) \Rightarrow f(z)(1-z) = 1$$

$$\Rightarrow f(z) = \frac{1}{1-z} \text{ when } |z| < 1$$



Series converges to $\frac{1}{1-z}$ value inside but diverges outside.

But, $\frac{1}{1-z}$ can have values at any point outside the region of convergence of the series. The Taylor series & the fn $f(z)$ match inside the ROC.

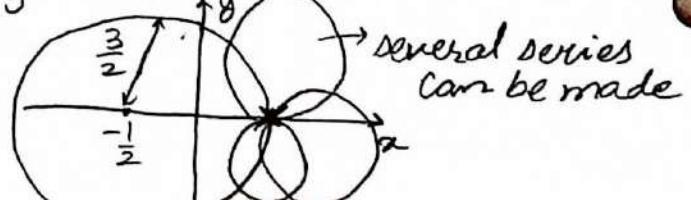
$\Rightarrow \frac{1}{1-z}$ blows up only at $z = 1$. In fact $\frac{1}{1-z}$ is analytic everywhere except at $z = 1$.

$\frac{1}{1-z}$ is the analytic continuation of the series $\sum_{n=0}^{\infty} z^n$ since it has values outside the ROC & matches the series inside the ROC.

\Rightarrow Say we want a power series for $\frac{1}{1-z}$ about $-\frac{1}{2}$

$$\frac{1}{1-z} = \frac{1}{\frac{3}{2} - (z + \frac{1}{2})} = \frac{2}{3} \cdot \frac{1}{1 - \frac{2}{3}(z + \frac{1}{2})} = \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \left(z + \frac{1}{2}\right)^n$$

This converges when $|z + \frac{1}{2}| < \frac{3}{2}$

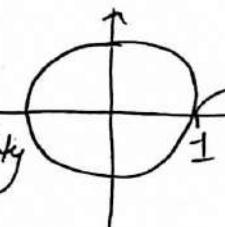


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- > Therefore a given analytic function can have an infinite number of series representations each valid in a different region. Always specify ROC. Each of these series are analytic continuations of each other.
- > For every power series there is a circle of convergence (may be ∞) inside which it is absolutely convergent, outside which it is divergent & on which nothing can be said & investigation on a case by case basis is required.
- > Let $z = i \Rightarrow \sum z^n = 1 + i - 1 - i + 1 + \dots$ Therefore we get 4 partial sums, 1, 1+i, i, 0 depending on where you end the series. The arithmetic avg of these four sums is $\frac{1+i}{2}$ which is also given by substituting $z=i$ in $\frac{1}{1-z}$.
- > This arithmetic mean of partial sums is guaranteed to match the value given by the analytic contr. fn. This arithmetic mean of partial sums is called Cesaro sum.
- > For every power series, the function that represents the power series must have atleast one singularity on its circle of convergence.

Eg : $\sum_{n=1}^{\infty} \frac{z^n}{n^2} \rightarrow$ converges even on the circle of convergence at all points.

If we find $f(z)$ for this series, we can show that there is indeed a 'subtle' singularity at $z=1$.



here value is
max on coc
& value is
 $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Eg:

$\lim_{z \rightarrow 1} (z-1) \ln(z-1) \rightarrow 0$ but there is still a singularity.
Therefore singularity does not mean ∞ .
Here there is a logarithmic branch cut.

Eg:

$f(z) = z + z^2 + z^4 + z^8 + \dots = \sum_{n=0}^{\infty} (z)^{2^n}$ → diverges everywhere
on the COC. (shown below)

$f(z) = z + f(z^2) \Rightarrow f(z^2)$ is singular at both $+1, -1$. for sure

$f(z) = z + z^2 + f(z^4) \Rightarrow$ singular at $+1, -1, +i, -i$ for sure.

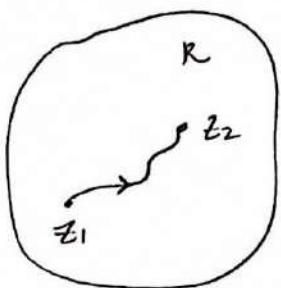
→ we can show there is a dense set of singularities on the
unit circle & therefore, there is no way this series can
be analytically continued outside the COC. The COC forms a "natural
boundary". $f(z)$ has a "natural boundary".

Such a series with no analytic continuation outside the COC
is called a "lacunary series." Large gaps in powers
give lacunary series.

Eg: $\sum_{n=0}^{\infty} z^{n!}$

Lec 3 Calculus of Residues

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$\int_{z_1}^{z_2} f(z) dz$ is independant of the actual path from z_1 to z_2 .

> looks like a conservative vector field

$$\vec{u} = \nabla \phi(\vec{r})$$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{u} \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} \nabla \phi \cdot d\vec{r} = \phi(\vec{r}_2) - \phi(\vec{r}_1)$$

> This is why potentials (potenergy) are used to describe a conservative force.

$$\Rightarrow \oint \nabla \phi \cdot d\vec{r} = 0.$$

Cauchy's Theorem.

$$\Rightarrow \boxed{\oint_C f(z) dz = 0}$$

The integral of an analytic function over a closed contour is identically zero.

> This is also why potentials are used to describe E field.

$$V = \int E \cdot dl \quad & \oint E \cdot dl = 0 \text{ when } \frac{dB}{dt} = 0 \text{ - Recall that} \\ E = -\frac{\partial A}{\partial t} - \nabla \phi \quad \Rightarrow \text{if } \frac{\partial A}{\partial t} = 0 \text{ E becomes a conservative field}$$

> can be written as $V_2 - V_1 \rightarrow \text{Voltage!}$

Singularities

> Removable singularities.

By defining the fn appropriately we can remove it.

Eg: $\frac{\sin z}{z} \rightarrow$ indeterminate at $z=0$ but limit as $z \rightarrow 0$ is 1

$$\Rightarrow \text{define as } f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z=0 \end{cases} \Rightarrow f(z) \text{ becomes continuous.}$$

Other singularities

singular part regular part.

$$f(z) = \frac{c_1}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n \rightarrow \text{Simple pole at } z=a$$

Residue at $z=a$ = c_1 ! Super essential.

$$f(z) = \frac{\sin z}{z^2} = \frac{z - \frac{z^3}{3!} + \dots}{z^2} = \underbrace{\frac{1}{z}}_{\text{simple pole with residue of 1.}} - \frac{z}{3!} + \dots$$

$$f(z) = \frac{g(z)}{h(z)} ; \quad g(a) \neq 0 \quad h(a) = 0$$

$$f(z) = \frac{g(a) + (z-a)g'(a) + \dots}{h(a) + (z-a)h'(a) + \dots} \Rightarrow \frac{g(a)}{(z-a)h'(a)} + \frac{g'(a)}{h'(a)} + \dots$$

\Rightarrow Simple pole with residue of $\boxed{g(a)/h'(a)}$

$\rightarrow f(z)$: simple pole at $z=a$

$$\Rightarrow \text{Residue} : \boxed{\lim_{z \rightarrow a} (z-a)f(z)}$$

since this would extract c_1 & kill all other terms.

Eg: $\frac{1}{\sin \pi z} \rightarrow$ Singularities at all integers! All singularities are simple poles.

At $z \rightarrow 0$ $\frac{1}{\sin \pi z} \rightarrow \frac{1}{\pi z} \Rightarrow$ residue is $\frac{1}{\pi}$.

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→ Residue of $\frac{1}{\sin \pi z}$ at $z=n$?

$$\lim_{z \rightarrow n} \left(\frac{z-n}{\sin \pi z} \right) = \frac{1}{\pi \cos \pi z} = \frac{(-1)^n}{\pi}$$

→ How about funs with multiple poles?

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \dots + \underbrace{c_{-1}}_{\text{Residue}} + \sum_{n=0}^{\infty} c_n (z-a)^n.$$

└ pole of order m at $z=a$. (more singular)

The residue is c_{-1} not c_{-2}, c_{-3}, \dots so we need to extract it.

$$c_{-1} = \text{Res}(f(z)) \Big|_{z=a}$$

$$c_{-1} = \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

→ m is order of pde.

Essential singularity.

If $f(z)$ contains all negative powers of $(z-a)$ it has an essential singularity at $z=a$. if $f(z)$ can be expressed as,

$$f(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} + \sum_{n=0}^{\infty} c_n (z-a)^n. \text{ Again } c_{-1} \text{ is the residue of } f(z).$$

Such a series representation is called a Laurent series.

→ Say the regular part converges inside a ROC at a with radius r .



The singular part also converges outside some other ROC around a .

If there is no overlap b/w these two ROCs, the Laurent series representation cannot be used.

→ If $r_1 > r_2$ we can use the Laurent series... "

$\gamma_2 < |z| - \alpha < \gamma_1 \Rightarrow$ Laurent series converges.

$$\text{Eg: } e^{\frac{1}{z}} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \dots$$

$$= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n! z^n}}_{\text{singular}} + \underbrace{\frac{1}{z}}_{\text{regular.}}$$

\rightarrow converges in $|z| > 0$

Residue is ± 1

$$\Rightarrow \gamma_2 = 0, \gamma_1 = \infty$$

in fact it also converges at ∞ .

$$\text{Eg: } e^{\frac{1}{z}} + e^z = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n! z^n}}_{|z| > 0} + 2 + \underbrace{\sum_{n=1}^{\infty} \frac{z^n}{n!}}_{|z| < \infty} \rightarrow \text{converges } 0 < |z| < \infty$$

Eg: $e^{\frac{1}{z^2}} \rightarrow$ essential singularity, residue = 0.

Eg: $e^{-\frac{1}{(z-2)^2}} \rightarrow$ " " " " "

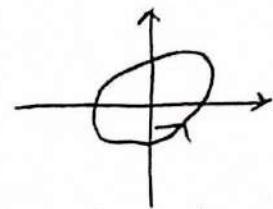
Eg: $(z-2) e^{-\frac{1}{(z-2)^2}} \rightarrow$ essential singularity, residue = -1

Eg: $(z-2)^{-1} e^{-\frac{1}{(z-2)^2}} \rightarrow$ essential, residue = 0.

Residues → Why is it so important?

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Consider $\oint_C z^n dz \rightarrow$ entire function $\Rightarrow = 0.$



$z = e^{i\theta}$ on the unit circle (we can distort the shape to make it a unit circle)

$$dz = e^{i\theta} d\theta i$$

$$\Rightarrow \oint_0^{2\pi} e^{(n+1)i\theta} i d\theta = 0$$

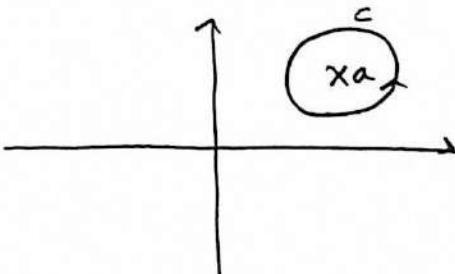
Consider $\oint \frac{dz}{z^{n+1}}$ where $n=0, 1, 2, 3, \dots$

$$\Rightarrow \oint \frac{dz}{z^{n+1}} = i \int_0^{2\pi} \frac{e^{i\theta} d\theta}{e^{(n+1)i\theta}} = i \int_0^{2\pi} d\theta e^{-ni\theta} = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

$$\rightarrow \boxed{\frac{1}{2\pi i} \oint_C \frac{dz}{z^{n+1}} = S_{n,0}} \quad \text{Kronecker delta} \Rightarrow \begin{cases} 1 & \text{when } n=0 \\ 0 & \text{when } n \neq 0 \end{cases}$$

Generalising

$$\boxed{\frac{1}{2\pi i} \oint_C \frac{dz}{(z-a)^{n+1}} = S_{n,0}}$$



Therefore when integrating a pole in counter clockwise direction, the only term that contributes is the $\frac{1}{z-a}$ term + hence that is the residue.

$$\boxed{\frac{1}{2\pi i} \oint_C f(z) dz = \operatorname{Res}_{z=a} [f(z)]}$$

→ Cauchy's integral formula!

Eg:

$$\oint_C e^{\frac{1}{z}} dz = 2\pi i$$

Despite having a horrible essential singularity at $z=0$, the value depends only on the residue \Rightarrow only on $\frac{1}{z}$ term

$$\oint_C e^{z^2} dz = 0$$

Recursion relations Lec-4

$$C_{n+1} = \frac{C_n + C_{n+2}}{2}; n=0, 1, 2, 3, \dots \rightarrow C_{n+1} \text{ is the mean} \Rightarrow \text{solution must be a linear function} \rightarrow \text{equivalent of a harmonic function.}$$

$$C_{n+2} - 2C_{n+1} + C_n = 0; C_0, C_1 \text{ are given.}$$

Define, $f(z) \rightarrow$ a generating function.

$$f(z) \equiv \sum_{n=0}^{\infty} C_n z^n \quad \text{use coefficients to define this series.}$$

$$\Rightarrow \underbrace{\sum_{n=0}^{\infty} z^n C_{n+2}}_{\frac{1}{z^2}(f(z)-C_0-C_1z)} - 2 \underbrace{\sum_{n=0}^{\infty} z^n C_{n+1}}_{\frac{2}{z}(f(z)-C_0)} + \underbrace{\sum_{n=0}^{\infty} z^n C_n}_{f(z)} = 0$$

$$\Rightarrow \frac{1}{z^2} (f(z) - C_0 - C_1 z) - \frac{2}{z} (f(z) - C_0) + f(z) = 0$$

$$\Rightarrow f(z) \left(\frac{1}{z^2} - \frac{2}{z} + 1 \right) = \frac{C_0 + C_1 z}{z^2} - \frac{2C_0}{z}$$

$$\Rightarrow f(z) = \frac{C_0 + (C_1 - 2C_0)z}{(z-1)^2}$$

\Rightarrow Substitute ICs C_0, C_1 & expand $f(z)$ as a power series,
the coefficients now give C_n .

$$\rightarrow C_n = \left[\frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z=0} \right] \rightarrow \text{directly gives } C_n. \text{ But doesn't work if negative powers of } z \text{ exist.} \Rightarrow \text{pdes.}$$

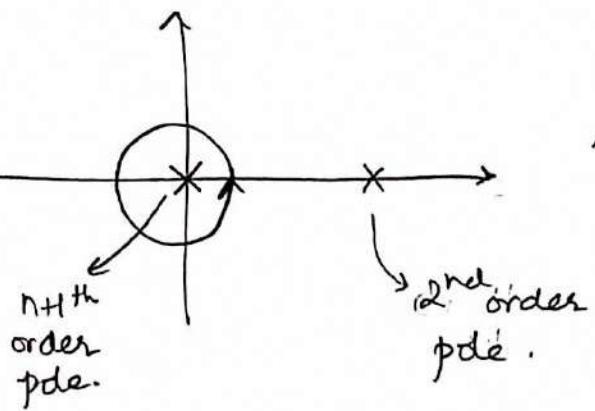
\Rightarrow To find C_1 say we can simply find the residue $\frac{1}{2\pi i} \oint_C f(z) dz$.

To find C_3 I can find the residue of $\frac{1}{2\pi i} \oint_C \frac{f(z)}{z^4} dz$ since

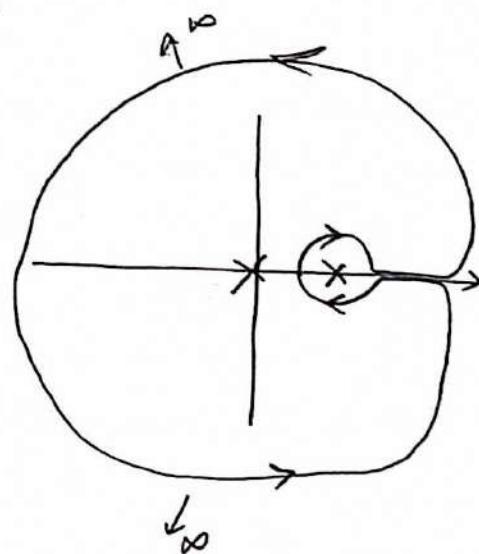
C_3 goes from being the coefficient of z^3 to z^{-1} which is the residue.

$$\Rightarrow C_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}} dz \rightarrow \text{Always works even with singularities as long as series is Laurent.}$$

$$\Rightarrow C_n = \frac{1}{2\pi i} \oint_{\text{origin.}} \frac{dz}{z^{n+1}} \cdot \frac{C_0 + (C_1 - 2C_0)z}{(z-1)^2}$$



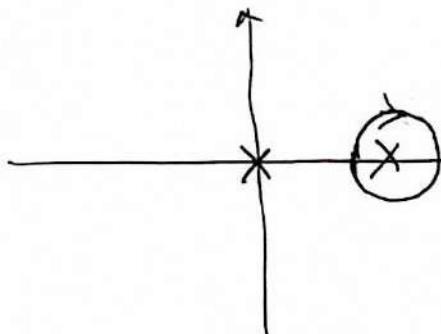
deform contour.



$$\text{At infinity, } dz \rightarrow R \quad z \rightarrow R \\ z^{n+1} \rightarrow R^{n+1} \quad (z-i)^2 \rightarrow R^2 \Rightarrow \frac{R \times R}{R^{n+1} \times R^2} = \frac{1}{R^{n+1}}$$

goes to 0 (even) for $n=0, 1, \dots \Rightarrow$ contour at $\infty \rightarrow 0$.

\Rightarrow



Therefore we only need to find
-ve of residue of $(z-1)^{-n}$ pole.

Finding residue at origin \Rightarrow we would need
to differentiate n times since pole there is of
order $n+1$.

$$\Rightarrow C_n = \frac{1}{2\pi i} (-2\pi i) \left[\text{Residue of } \frac{C_0 + (C_1 - 2C_0)z}{z^{n+1} (z-1)^2} \right] \text{ at } z=1$$

$$= -\frac{2\pi i}{2\pi i} \left[\lim_{z \rightarrow 1} \frac{1}{(2-1)!} \frac{d}{dz}^{2-1} \left[\frac{(z-1)^2 C_0 + (C_1 - 2C_0)z}{z^{n+1} (z-1)^2} \right] \right]$$

$$= -\lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{C_0 + (C_1 - 2C_0)z}{z^{n+1}} \right]$$

$$= -\lim_{z \rightarrow 1} C_0(n-1) z^{-n-2} + -n(C_1 - 2C_0) z^{-n-1}$$

$$= (n+1)C_0 + n(C_1 - 2C_0)$$

$$-nC_n = -nC_0 + C_0 + nC_1 = n(C_1 - C_0) + C_0.$$

$$\Rightarrow C_n = n(C_1 - C_0) + C_0$$

(17)

Eg: Try $C_{n+2} = C_n + C_{n+1}$. $C_0 = 0$; $C_1 = 1$. \Rightarrow Fibonacci!

Steps: 1 > Define generating function.

2 > Apply summation to given Diff. Eq.

3 > Write in terms of $f(z)$.

4 > Use $C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{z^{n+1}} dz$

5 > Distort contour to find residue over finite poles.

$$C_1 = \lim_{\substack{z \rightarrow a \\ \text{poles}}} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \Delta \text{residue} = 2\pi i (C_1)$$

Soln: 1 > $f(z) = \sum_{n=0}^{\infty} C_n z^n$

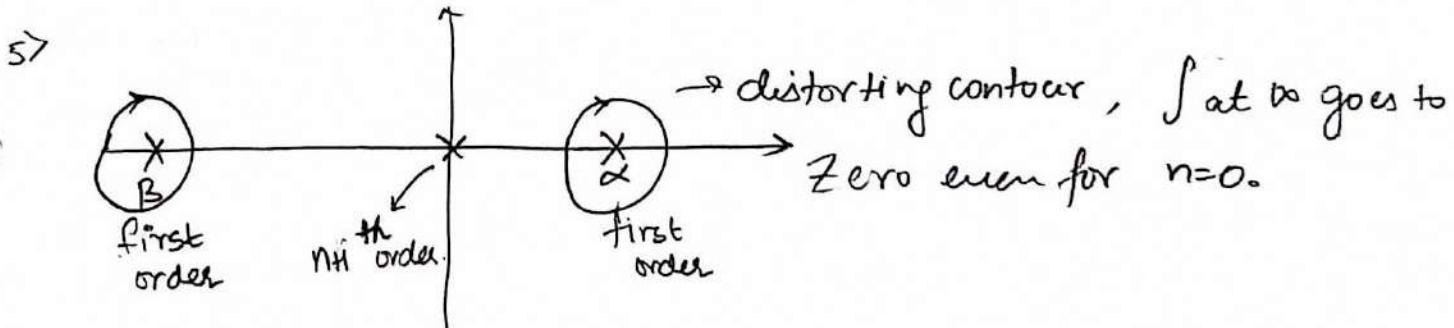
2 > $\underbrace{\sum_{n=0}^{\infty} z^n C_{n+2}} - \underbrace{\sum_{n=0}^{\infty} z^n C_{n+1}} - \underbrace{\sum_{n=0}^{\infty} z^n C_n}_{P(z)}$
 $\frac{1}{z^2} [f(z) - C_0 - C_1 z] \quad \frac{1}{z} [f(z) - C_0]$

3 > $f(z) \begin{bmatrix} \frac{1}{z^2} & -\frac{1}{z} & -1 \end{bmatrix} = \frac{C_0 + C_1 z}{z^2} - \frac{C_0}{z}$

$\Rightarrow f(z) \begin{bmatrix} 1-z-z^2 \\ z^2 \end{bmatrix} = \frac{C_0 + C_1 z - C_0 z}{z^2}$

$\Rightarrow f(z) = \frac{C_1 z + C_0 (1-z)}{1-z-z^2} \rightarrow$ Could use a binomial expansion
& write it as $\sum_{n=0}^{\infty} C_n z^n$ to find C_n
but let us use Complex Analysis.

4 > $C_n = -\frac{1}{2\pi i} \oint \frac{1}{z^{n+1}} \cdot \frac{C_1 z + C_0 (1-z)}{z^2 + z - 1} dz$
 \rightarrow roots $= \alpha, \beta = \frac{-1 \pm \sqrt{5}}{2}$



$$\Rightarrow C_n = -\frac{1}{2\pi i} \oint_{\text{clockwise}} \lim_{z \rightarrow \alpha} (z-\alpha) \cdot \frac{C_1 z + C_0(1-z)}{z^{n+1}(z-\alpha)(z-\beta)} + \lim_{z \rightarrow \beta} (z-\beta) \cdot \frac{C_1 z + C_0(1-z)}{(z-\alpha)^{n+1}(z-\beta)}$$

$$C_n = \left[\frac{C_1 \alpha + C_0(1-\alpha)}{\alpha^{n+1}(\alpha-\beta)} + \frac{C_1 \beta + C_0(1-\beta)}{\beta^{n+1}(\beta-\alpha)} \right] \quad \begin{aligned} \alpha &= \frac{\sqrt{5}-1}{2} \\ \beta &= -\frac{\sqrt{5}-1}{2} \end{aligned}$$

Fibonacci $\Rightarrow C_0 = 0 ; C_1 = 1$

$$\Rightarrow C_n = - \left[\frac{\alpha^n}{\alpha^{n+1}(\alpha-\beta)} - \frac{\beta^n}{\beta^{n+1}(\alpha-\beta)} \right] = \frac{1}{\alpha-\beta} \left[\frac{1}{\alpha^n} - \frac{1}{\beta^n} \right]$$

$$= \frac{1}{\sqrt{5}} \left[\frac{2^n}{(\sqrt{5}-1)^n} - \frac{2^n}{(-\sqrt{5}-1)^n} \right] = \frac{1}{\sqrt{5}} \left[\frac{2^n}{(\sqrt{5}-1)^n} - \frac{1}{(-1)^n} \cdot \frac{2^n}{(\sqrt{5}+1)^n} \right]$$

$$= \frac{1}{\sqrt{5}} \left[\frac{2^n (\sqrt{5}+1)^n}{4^n} - \frac{(-1)^n 2^n (\sqrt{5}-1)^n}{4^n} \right]$$

$$C_n = \frac{1}{2^n \sqrt{5}} \left[(\sqrt{5}+1)^n - (-1)^n (\sqrt{5}-1)^n \right]$$

$\frac{1+\sqrt{5}}{2} \rightarrow \phi$
Golden ratio.
1.618

Lec 5 Summation of Series

$$S(a) = \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} \quad (a > 0, \text{ say})$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} \quad \text{since } n \text{ is even.}$$

Note, the function $\pi \cot \pi z$ has simple poles at all integers & the residue is 1.

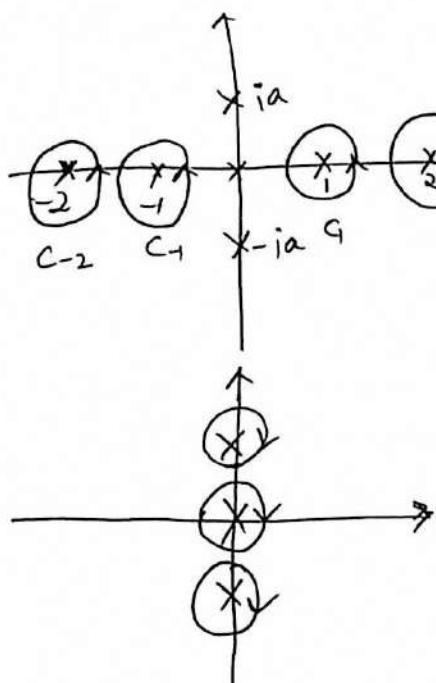
$$\pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z} \xrightarrow[z \rightarrow n]{} \frac{\frac{\pi(-1)^n}{(z-n)(-1)^n}}{\underbrace{(z-n)}_{\text{simple poles for all integers}}} + \text{regular part}$$

∴ Consider $f(z) = \frac{\pi \cot \pi z}{z^2 + a^2} \rightarrow$ this fn at $z=n$ would have a residue of $\frac{1}{n^2 + a^2}$.

$$\Rightarrow S(a) = \frac{1}{2} \cdot \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C \frac{dz}{z^2 + a^2} \frac{\pi \cot \pi z}{z^2 + a^2} \xrightarrow[R]{\text{remains bounded}} \text{NUM}$$

could be imag!

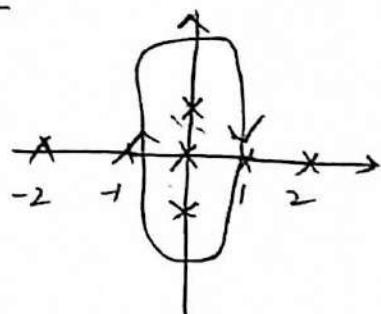
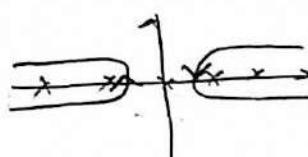
$$\cot \pi z = \frac{e^{iz} + e^{-i\pi z}}{e^{iz} - e^{-i\pi z}} \xrightarrow[\text{DEN}]{\text{compensate}}$$



$$\Rightarrow S(a) = \frac{1}{4\pi i} (-2\pi i) \left[\frac{1}{a^2} + \frac{\pi \cot i\pi a}{2ia} + \frac{\pi \cot -i\pi a}{-2ia} \right]$$

$$= -\frac{1}{2a^2} - \frac{\pi}{2\pi a} \left[e^{i(i\pi a)} + e^{-i(i\pi a)} \right]$$

$$= -\frac{1}{2a^2} - \frac{\pi}{2\pi a} \left[\frac{e^{i(i\pi a)} - e^{-i(i\pi a)}}{e^{i(i\pi a)} + e^{-i(i\pi a)}} \right]$$



$$\Rightarrow S(a) = \frac{\pi}{2a} \left[\coth \pi a - \frac{1}{\pi a} \right]$$

$\coth x - \frac{1}{x} \rightarrow$ hankel func.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \left[\coth \pi a - \frac{1}{\pi a} \right]$$

$$\coth z \underset{z \rightarrow 0}{\longrightarrow} \frac{\text{Residue}}{z} + \underbrace{\text{reg part}}_{\sum_{n=0}^{\infty} a_n z^n} \rightarrow \begin{array}{l} \text{first term must be } z' \text{ not } z^0 \text{ since} \\ \coth \text{ is odd} \end{array}$$

$$\frac{\cosh z}{\sinh z} = \frac{1 + \frac{z^2}{2} + \dots}{z(1 + \frac{z^2}{6} + \dots)} \approx \frac{(1 + \frac{z^2}{2} + \dots)(1 - \frac{z^2}{6})}{z} = \frac{1}{z} + \frac{z}{3} + \dots$$

$$\Rightarrow S(a) \rightarrow \frac{\pi}{2a} \left[\frac{1}{\pi a} + \frac{\pi a}{3} + \dots - \frac{1}{\pi a} \right]$$

$$\Rightarrow S(a) = \frac{\pi^2}{6} \rightarrow \zeta(2) \rightarrow \text{Zeta fn of 2.}$$

We already know this value.

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

$$\text{Eg: } \sum_{n=1}^{\infty} \frac{(-1)^n}{(n^2 + a^2)} \Rightarrow \text{we need a fn whose residue at each integer is } (-1)^n$$

$$f(z) = \frac{\pi \csc \pi z}{z^2 + a^2} \text{ be careful at } \infty$$

$$\begin{aligned}
 \text{Eg: } \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} &= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots \\
 &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \\
 &- 2 \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} \right] \\
 &= \zeta(2) - \frac{2}{4} \zeta(2) \\
 &= \frac{\pi^2}{12}
 \end{aligned}$$

Eg: $\sum_{n=1}^{\infty} \frac{1}{n^4} \rightarrow \frac{\pi \cot \pi z}{z^4} \rightarrow$ bit messy $\rightarrow 5^{\text{th}}$ order pole at origin.

However $\sum_{n=1}^{\infty} \frac{1}{n^4+a^4} \rightarrow \frac{\pi \cot \pi z}{z^4+a^4} \rightarrow$ 5 simple poles \Rightarrow easier than set $a=0$.

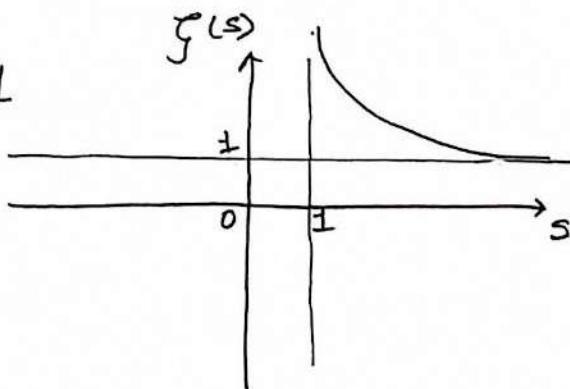
> Also note that differentiating $S(a) = \sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)}$ gives $-\sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2}$

then setting $a=0$ (carefully) we can find $\zeta(4)$.

$\Rightarrow \frac{d}{da} \left[\frac{\pi}{2a} \left(\coth \pi a - \frac{1}{\pi a} \right) \right]$. Similarly further differentiation gives all even positive integer argument zetas.

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\lim_{K \rightarrow \infty} \zeta(K) \rightarrow 1$$



Lec 6 Dirichlet integral

$$\int_0^\infty dx \frac{\sin x}{x} = \frac{\pi}{2}$$

Note that $|\frac{\sin x}{x}|$ does not converge since the $\frac{dx}{x}$ grows logarithmically. But the + & - signs in $\sin x$ make it finite.

Also,

$$\int_0^\infty dx \frac{\sin bx}{x} \begin{cases} = \frac{\pi}{2} & \text{for all } b > 0 \\ = -\frac{\pi}{2} & \text{for all } b < 0 \end{cases}$$

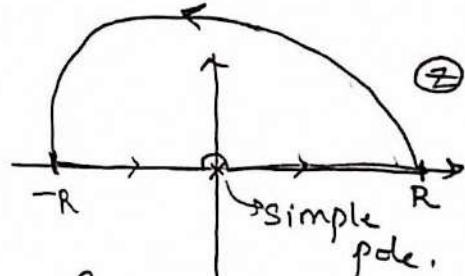
$$\Rightarrow \int_0^\infty dx \frac{\sin bx}{x} = \frac{\pi}{2} \operatorname{sgn}(b)$$

Derivation

$$= \frac{1}{2} \int_{-\infty}^{\infty} dx \frac{\sin bx}{x} \rightarrow \frac{e^{ibx} - e^{-ibx}}{2i}, \text{ but one of these will always blow up if we close the contour in upper or lower Half plane. So we need to eliminate } \sin x.$$

$$= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} dx \frac{e^{ibx}}{x} \quad \text{But this integral blows up at the origin, unlike the case where we had } \frac{\sin x}{x}$$

Consider, $\int_C dz \frac{e^{ibz}}{z}$ where C is



$$\Rightarrow 0 = \int_C dz \frac{e^{ibz}}{z} = \int_{-R}^R dx \frac{e^{ibx}}{x} + \int_R^R dx \frac{e^{ibx}}{x} + \text{Small contour.} + \text{Large contour.}$$

where on the small contour $z = e^{i\theta} \Rightarrow dz = e^{i\theta} i d\theta$.

$$\Rightarrow \text{Small contour} = \int_0^\pi e^{ib\theta} i d\theta \frac{e^{ib\theta}}{e^{i\theta}} = -i\pi \text{ as } \epsilon \rightarrow 0$$

$$\text{Large contour} = \int_0^\pi \frac{Re^{i\theta}}{R e^{i\theta}} i d\theta e^{ibRe^{i\theta}}, \dots z = Re^{i\theta} \text{ has the imaginary part } b \text{ should be positive.}$$

(23)

- On the line of integration if a singularity exists & a symmetric neighbourhood (infinitesimal) is excluded, the integral is called the principal value integral.

$$\Rightarrow P \int_{-R}^R dx \frac{e^{ibx}}{x} \quad \text{or} \quad \int_{-R}^R dx \frac{e^{ibx}}{x}$$

$$\Rightarrow \lim_{R \rightarrow \infty} P \int_{-R}^R dx \frac{e^{ibx}}{x} = i\pi$$

Equate imaginary parts $\Rightarrow P \int_{-\infty}^{\infty} dx \frac{\sin bx}{x} = \pi$

But now we don't have a singularity since $\frac{\sin bx}{x}$ has a defined value at 0 so we can remove the P.

$$\Rightarrow \int_{-\infty}^{\infty} dx \frac{\sin bx}{x} = \pi \xrightarrow{\text{even fm}} \int_0^{\infty} dx \frac{\sin bx}{x} = \frac{\pi}{2}$$

What is the physical significance of residues?

Eg: For a resonator with multiple natural frequencies, resonance frequencies correspond to the poles. At each pole the residue gives the "strength" of the resonance at that frequency.

Meromorphic function.

- "A function whose only singularities in the finite part of the complex plane are poles".
- Recall that branch points are also singularities.
 - Could be an entire function.
 - You could express meromorphic functions as the sum of a singular part and an entire function.

Mittag-Leffler representation.

Recall $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a} \left[\coth \pi a - \frac{1}{\pi a} \right]$, a is real.

➤ a can be complex but the equality is only valid where the fn has no poles.

➤ Let $a = iz$

$$\Rightarrow \frac{\pi}{2iz} \coth(i\pi z) + \frac{1}{2z^2} = \sum_{n=1}^{\infty} \frac{1}{n^2-z^2}$$

\downarrow

$$\frac{\cos \pi z}{i \sin \pi z} = -i \cot \pi z$$

$$\Rightarrow -\frac{\pi \cot(\pi z)}{2z} + \frac{1}{2z^2} = \sum_{n=1}^{\infty} \frac{1}{n^2-z^2}$$

$$\Rightarrow \boxed{\pi \cot(\pi z) = \frac{1}{z} + 2 \sum_{n=1}^{\infty} \frac{z}{z^2-n^2}}$$

\downarrow

pole at zero

$\rightarrow (z+n)(z-n)$
 \Rightarrow at all non-zero integers there are simple poles which is what $\cot \pi z$ has.

In the Mittag-Leffler representation,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left[\frac{1}{(z-n)} + \frac{1}{(z+n)} \right]$$

This is not the
ML representation.

→ Here we cannot simply open the square brackets since $\sum \frac{1}{z^n}$ diverges as $\sum \frac{1}{n}$.

Eg: $\sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n} \right]$ cannot be separated (my eg).

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left[\left(\frac{1}{z-n} + \frac{1}{n} \right) + \underbrace{\left(\frac{1}{z+n} - \frac{1}{n} \right)}_{\text{Residues}} \right]$$

Now as $n \rightarrow \infty$ $\frac{1}{z-n} \rightarrow \frac{1}{-n}$ & it cancels with $\frac{1}{n}$ so the remainder grows as $\frac{1}{n^2}$ which converges. So we can now separate them.

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} \right) + \sum_{n=-\infty}^{-1} \frac{1}{z-n} + \frac{1}{n}$$

→ replace $n \rightarrow -n$

$$\Rightarrow \text{Res}(\pi z) = \frac{1}{z} + \sum_{n=-\infty}^{\infty} \left[\frac{1}{z-n} + \frac{1}{n} \right] \quad \Rightarrow n=0 \text{ is excluded.}$$

→ This is the ML

Representation in the
correct form.

- The correct representation must include a term which regularizes the series (ie $\frac{1}{n}$) & must contain an entire fun (ie 0 in this case).

Differentiating both sides:

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

just a sum of all the double poles!!

Lec 7 Linear response, dispersion relations.

Response $R(t)$, Stimulus/Force $F(t)$

$$R(t) = [] F(t).$$

$\xrightarrow{t \rightarrow \text{ causality}}$ force till t produces response at t .

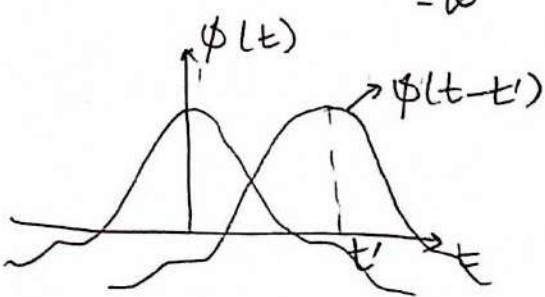
$$R(t) = \int_{-\infty}^t dt' \phi(t, t') F(t')$$

\curvearrowleft linearity due to superposition integral.

t' is when the force is applied & t is when the response is measured. As long as system is invariant with time ϕ is only a function of $t - t'$ since it does not matter when t' is applied.

$$\Rightarrow \text{LTI} \Rightarrow R(t) = \int_{-\infty}^t dt' \phi(t-t') F(t')$$

\curvearrowleft retarded response.
 \curvearrowleft response fn.



Take input at t' as an impulse & multiply with the impulse response shifted to t' , this shifting of the impulse response & weighting by the input & summing is the convolution.

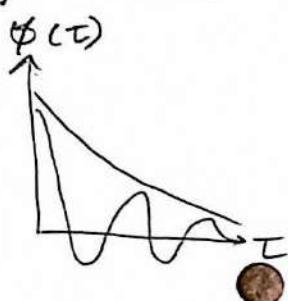
$\rightarrow F, R$ could be vectors & ϕ would be a 2nd order tensor.

$\rightarrow \phi$ here is called the "response functions" in this framework.

Let $T = t - t'$ be time after force is applied.

$$\text{Let } R(t) = \int_{-\infty}^{\infty} dw e^{-i\omega t} \tilde{R}(\omega)$$

$$\Rightarrow \tilde{R}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} R(t)$$



$$F(t) = \int_{-\infty}^{\infty} dw e^{-i\omega t} \tilde{F}(w) \quad & \tilde{F}(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i\omega t} F(t)$$

$$\Rightarrow \int_{-\infty}^{\infty} dw e^{-i\omega t} \tilde{R}(w) = \int_{t'=-\infty}^t dt' \phi(t-t') \int_{w=-\infty}^{\infty} dw e^{-i\omega t'} \tilde{F}(w)$$

$$= \int_{-\infty}^{\infty} dw \tilde{F}(w) \int_{-\infty}^t dt' \phi(t-t') e^{-i\omega t'}$$

$$= \int_{-\infty}^{\infty} dw \tilde{F}(w) \int_0^{\infty} d\tau \phi(\tau) e^{-i\omega(t-\tau)}$$

$$= \int_{-\infty}^{\infty} dw e^{-i\omega t} \tilde{F}(w) \int_0^{\infty} d\tau \phi(\tau) e^{+i\omega\tau}$$

$$\Rightarrow \int_{-\infty}^{\infty} dw e^{-i\omega t} \left\{ \tilde{R}(w) - \left(\int_0^{\infty} d\tau e^{i\omega\tau} \phi(\tau) \right) \tilde{F}(w) \right\} = 0$$

$e^{-i\omega t}$ is a complete orthonormal set of functions of t ,

$$\Rightarrow \tilde{R}(w) = \left(\int_0^{\infty} d\tau e^{i\omega\tau} \phi(\tau) \right) \tilde{F}(w) \text{ for each real } w.$$

$\tilde{R}(w) = X(w) \tilde{F}(w)$

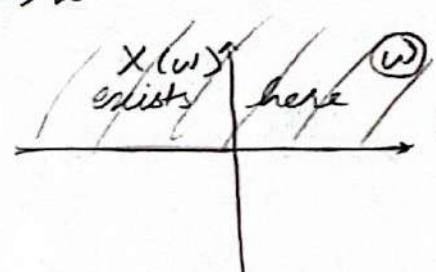
Called Generalized Susceptibility!

$X(w) = \int_0^{\infty} dt e^{i\omega t} \phi(t)$

only defined for $t > 0$ due to causality.
Neither a FT or LT. \rightarrow if it were e^{-st} it would be LT

- lets call it a Fourier Laplace transform.
 - > Assume the integral exists $\Rightarrow X(\omega)$ is a complex number.
- $$X(\omega) = \int_0^\infty dt e^{i\omega t} \phi(t).$$
- $\Rightarrow \begin{cases} \operatorname{Re}\{X(-\omega)\} = \operatorname{Re}\{X(\omega)\} & \text{since } \cos(\omega t) \\ \operatorname{Im}\{X(-\omega)\} = -\operatorname{Im}\{X(\omega)\} & \text{since } \sin(\omega t) \end{cases} \quad \left. \begin{array}{l} \text{For real } \omega. \\ \text{Later we see that } X(\omega) \text{ is analytic} \Rightarrow \text{cannot be purely real or imaginary.} \\ \Rightarrow \text{dissipation/energy storage always exists.} \end{array} \right.$
- $X(\omega)$ also exists for complex ω as long as $\operatorname{Im}(\omega) \geq 0$ since that would be give decaying exponentials as $t \rightarrow \infty$.

" $X(\omega)$ is an analytic function of ω in the UHP including real axis".

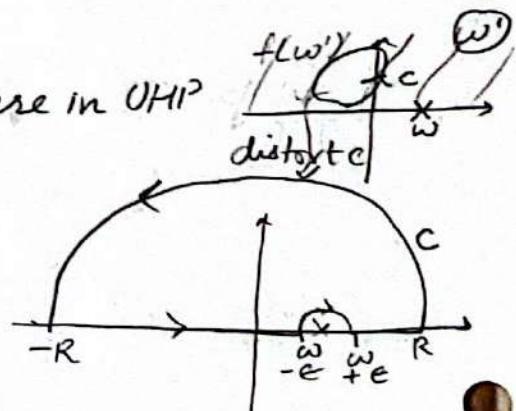


$\int_C dw X(\omega) = 0$ if C is entirely in the UHP including real axis.

Define $f(w') = \frac{X(w')}{w' - w}$ where w is some fixed real value.

\Rightarrow In w' plane $f(w')$ is analytic everywhere in UHP except at point w .

$$\oint_C f(w') dw' = 0 = \int_C dw' \frac{X(w')}{w' - w}$$



$$\int_{-R}^R \frac{dw' X(w')}{w' - w} + \underset{\text{small semi circle}}{\text{Small}} + \underset{\text{large semi circle}}{\text{large}} = 0$$

> On small semicircle $\omega' = \omega + \epsilon e^{i\theta} \Rightarrow d\omega' = \epsilon e^{i\theta} i d\theta$.

$$\Rightarrow \int_{-R}^R \frac{d\omega' X(\omega')}{\omega' - \omega} + \int_{\pi}^0 \frac{\cancel{\epsilon e^{i\theta}} i d\theta X(\omega + \epsilon e^{i\theta})}{\cancel{\epsilon e^{i\theta}}} + \int_0^R d\omega' \dots = 0$$

large

Large integral = $\int_0^\pi \frac{Re^{i\theta} i d\theta \cdot X(Re^{i\theta})}{Re^{i\theta} - \omega}$ as $R \rightarrow \infty$ if $X(Re^{i\theta})$

goes to 0, the integral goes to 0 \Rightarrow Susceptibility must vanish at ∞ frequency.

Small integral = $\lim_{\epsilon \rightarrow 0} \int_{-\pi}^0 i d\theta X(\omega + \epsilon e^{i\theta}) = i\pi X(\omega)$

$$\Rightarrow X(\omega) = \frac{-i}{\pi} \int_{-\infty}^{\infty} \frac{d\omega' X(\omega')}{\omega' - \omega} \quad \begin{array}{l} \text{real freq again.} \\ \text{Note that these are} \\ \text{the Kramers-Kronig} \\ \text{relations.} \end{array}$$

$$\Rightarrow \text{Re}(X(\omega)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega' \text{Im } X(\omega')}{\omega' - \omega}$$

$$\text{Im}(X(\omega)) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega' \text{Re } X(\omega')}{\omega' - \omega}$$

We see here that the Real and Imaginary parts of X are Hilbert transforms of each other.

Hilbert transform: $H(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\tau)}{t - \tau} d\tau$, it is the convolution

of $u(t)$ with $\frac{1}{\pi t}$.

⇒ $\text{Re}(X)$ & $\text{Im}(X)$ comprise a Hilbert transform pair!

* For stable systems, causality \Rightarrow analyticity. Since causality only holds in UHP, the X is analytic in UHP. For functions that are analytic in UHP, the Kramers-Kronig relations are true.

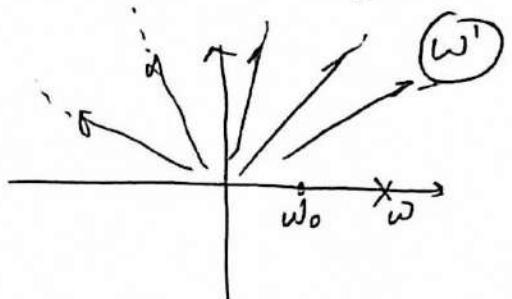
Lec 8 The Hilbert transform relations aka Dispersion relations.

If $X(w') \rightarrow X_\infty (\neq 0)$ as $|w'| \rightarrow \infty$ in the UHP we can account for this since $\int_{\text{large semi circle}} \frac{d\omega' X(w')}{w'-w} \xrightarrow[\substack{\text{large semi circle}}]{\substack{d\omega' \\ w' \rightarrow \infty}} \lim_{R \rightarrow \infty} i\pi X_\infty$

Subtracted dispersion relations.

If $X(w') \neq 0$ & goes to a different constant along different rays reaching ∞ or if it grows to ∞ but slower than linear growth we can define,

$$f(w') = \frac{X(w') - X(w_0)}{(w' - w)(w' - w_0)}$$



where $X(w_0)$ for some w_0 must be known.

$d\omega'$ cancels with $w' - w$ & $X(w')$ must grow slower than $w' - w_0$.

For faster growth we can add more terms if we know X at other points.

Eg: LCR series circuit

$$L \frac{dI}{dt} + RI + \frac{1}{C} \int_{-\infty}^t dt' I(t') = V(t)$$

$$\text{FT} \Rightarrow \left(-i\omega L + R + \frac{1}{i\omega C} \right) \tilde{I}(w) = \tilde{V}(w)$$

$$\Rightarrow \tilde{\frac{I}{V}}(w) = Y(w) \tilde{V}(w).$$

Admittance.

$$\Rightarrow Y(w) = \frac{i\omega}{L(\omega^2 + i\gamma\omega - \omega_0^2)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

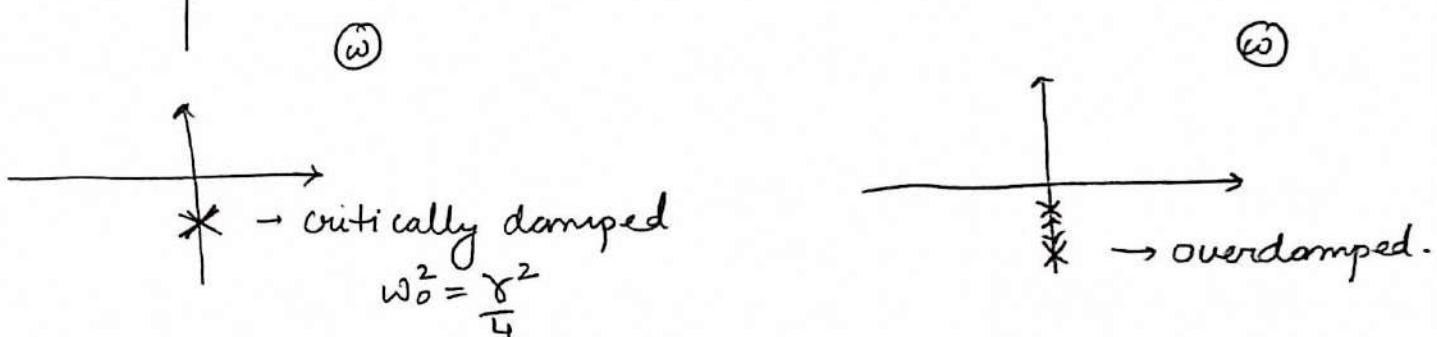
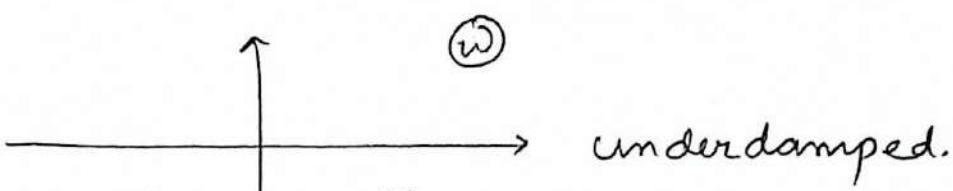
$$\gamma = \frac{R}{L}$$

inverse of time const.

This looks like the form $\tilde{R}(w) = X(w) \tilde{F}(w)$. $\Rightarrow Y(w)$ cannot have poles in UHP. It is analytic there.

$$Y(w) = \frac{i\omega}{L(\omega^2 + R\frac{1}{L}\omega - \frac{1}{C})}$$

> Poles of $\gamma(\omega)$ at $\omega = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$ → underdamped case ($\Rightarrow \omega_0 > \frac{\gamma}{2}$)

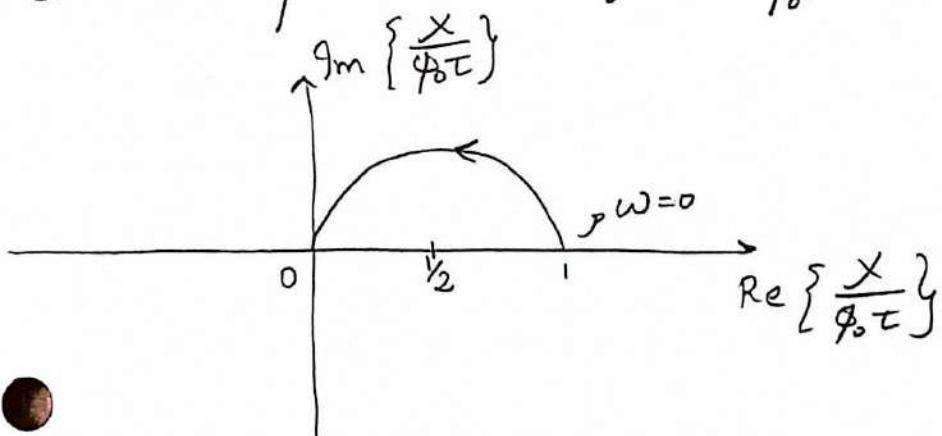


$\gamma(\omega)$ should satisfy K K relations

$$\rightarrow X(\omega) = \int_0^\infty dt e^{i\omega t} \phi(t) \quad \text{say, } \phi(t) = \phi_0 e^{-t/\tau} \quad \begin{array}{l} \text{relaxation time} \\ \hookrightarrow \text{Debye relaxation} \Rightarrow \text{single order relaxation} \end{array}$$

$$X(\omega) = \frac{\phi_0 \tau}{1 - i\omega \tau} = \phi_0 \tau \left(\frac{1}{1 + \omega^2 \tau^2} + \frac{i\omega}{1 + \omega^2 \tau^2} \right)$$

Cole-Cole plot → Plot of $\text{Im}\left\{\frac{X}{\phi_0 \tau}\right\}$ vs. $\text{Re}\left\{\frac{X}{\phi_0 \tau}\right\}$



$$\text{Re}\frac{X}{\phi_0 \tau} = u \quad \& \quad \text{Im}\frac{X}{\phi_0 \tau} = v$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$$

If measured u, v fall on this semicircle we can say that the relaxation process has a single relaxation time.

Lec 9 Analytic Continuation & Γ function.

Euler integral of second kind.

Euler noticed, $\int_0^\infty dt t^n e^{-t} = n!$ for $n=1, 2, 3 \dots$

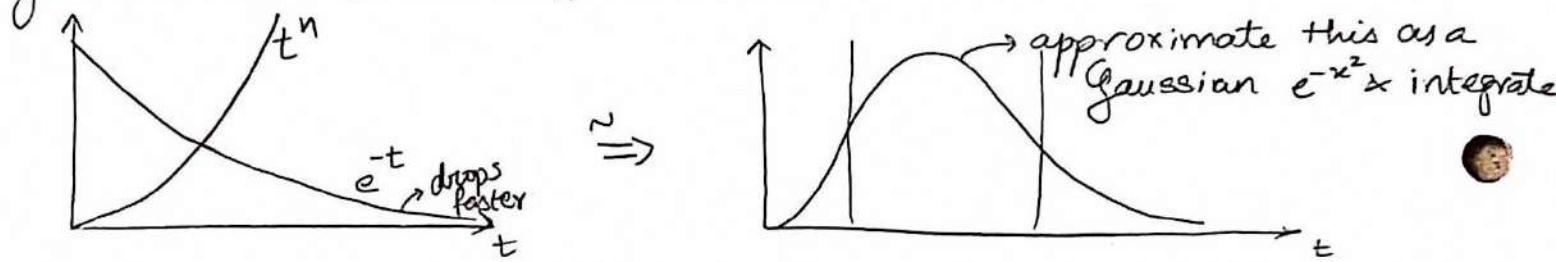
It is easy to show this using integration by parts. $\int u v dt = u \int v dt - \int u' (\int v dt) dt$

For $n=0 \Rightarrow \int_0^\infty dt t^0 e^{-t} = 1 \rightsquigarrow$ define this as $0!$.

For large n use Stirling's formula.

$$n! = n^n e^{-n} \sqrt{2\pi n} \left\{ 1 + \frac{1}{12n} + O(n^{-2}) \right\}$$

Gaussian integration approximation



Note, $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$ & $\int_{-\infty}^\infty e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}$

$$\Rightarrow \int_0^\infty dt e^{-t} t^n = \int_0^\infty dt e^{-(t-n \log t)}$$

where does $f(t) = t - n \log t$.

$$f'(t) = 1 - \frac{n}{t} = 0 \Rightarrow t = n$$

$$= \int_0^\infty dt e^{-[(n-n \log n) + \frac{(t-n)^2}{2n} + \dots]}$$

expanding around maxima

$$f''(t) = \frac{n}{t^2} = \frac{1}{n} \text{ at } t=n.$$

$$\Rightarrow n! = n^n e^{-n} \int_0^\infty dt e^{-\frac{(t-n)^2}{2n}} \left\{ 1 + \dots \right\}$$

could show that $\int_{-\infty}^\infty$ yield some result (not shown here though)

$$\Rightarrow \int_{-\infty}^{\infty} e^{-at^2} = \sqrt{\frac{\pi}{a}}$$

$$\Rightarrow n! \simeq n^n e^{-n} \sqrt{2\pi n} \rightarrow \text{Sterling's approximation}$$

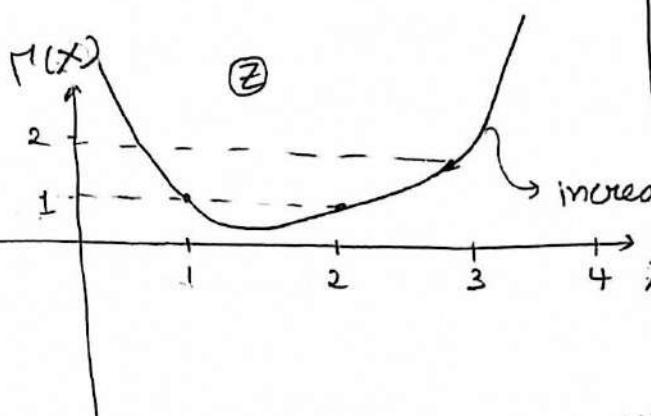
Going back;

$$\int_0^{\infty} dt t^n e^{-t} = n!$$

Historical reason to make it $n-1$

$$\boxed{\int_0^{\infty} dt t^{n-1} e^{-t} = \Gamma(n)} \rightarrow \text{Gamma fn}$$

$$\Gamma(n) = (n-1)! \text{ for } n=1, 2, 3, \dots$$



$$\Gamma(z) = \int_0^{\infty} dt t^{z-1} e^{-t}$$

increases as x^x faster than exponentials

real.

If $x=0$ $\int_0^{\infty} \frac{dt}{t e^t}$ does not converge for $t=0$. Same for $x < 0$

- $\Gamma(x)$ converges when $x-1 > -1 \Rightarrow x > 0$
- $\Gamma(z)$ converges when $x > 0$ for $z=x+iy \Rightarrow$ RHP
- > $\Gamma(z)$ is analytic in RHP + there are singularities on Im axis .

$$\Gamma(z) \equiv \int_0^{\infty} dt t^{z-1} e^{-t} \quad \text{Re } z > 0$$

Integrate \Rightarrow by parts

$$\int_0^{\infty} \frac{t^z}{z} e^{-t} dt + \frac{1}{z} \int_0^{\infty} dt t^{z-1} e^{-t} \quad \text{Re } z > 0$$

Vanishes at $t=\infty$ iff $\text{Re } z > 0 \Rightarrow$ first term = 0

$$\Rightarrow \Gamma(z) = \frac{1}{z} \int_0^\infty dt t^z e^{-t} \rightarrow \text{This representation is exact & equally valid.}$$

> However now we have exposed the simple pole at $z=0$ that has a residue of 1. Also note that this integral is valid for all $\operatorname{Re}(z) > -1$

> Keep integrating by parts to move further into LHP.

$$\Gamma(z) = \frac{1}{z(z+1)} \int_0^\infty dt t^{z+1} e^{-t} \Rightarrow \begin{array}{l} \text{Another simple pole with} \\ \text{residue} = -1 \end{array}$$

> Therefore this function has simple poles at all non positive integers and residue is $(-1)^n$.

> Therefore, $\Gamma(z)$ is a meromorphic function with simple poles at $z = -n$ ($n = 0, 1, 2, 3, \dots$) and residue $\frac{(-1)^n}{n!}$.

$$\Gamma(z) = \frac{1}{z(z+1)\dots(z+n)} \int_0^\infty dt t^{z+n} e^{-t} \quad \operatorname{Re} z > -n-1$$

$\Gamma(z+1) = z \Gamma(z)$

→ Functional equation for $\Gamma(z)$. This enables the analytic continuation. since $\Gamma(z+1)$ is valid for $\operatorname{Re}(z) > -1$. This can be continued.

Define $\Psi(z) = \frac{d}{dz} \ln \Gamma(z)$

$$= \frac{1}{\Gamma(z)} \frac{d}{dz} \Gamma(z)$$

$$\Psi(z+1) - \Psi(z) = \frac{1}{z} \rightarrow \text{Functional equation.}$$

Near $z = -n$ $\Gamma(z) = \frac{(-1)^n}{n!(z+n)} + \text{regular part}$

$\Rightarrow \Psi(z) = \frac{-1}{z+n} + \text{regular part}$

(diff & divide by itself
to give $\frac{-1}{z+n}$. \Rightarrow deriv of
regular part in regular)

$\Rightarrow \Psi(z)$ is meromorphic with simple poles at $-n$ but residue is always -1 .

Can we write a Mittag Leffler representation since it is meromorphic and we have the poles?

Note that in $\Gamma(z) = \frac{1}{z(z+1)(z+2)\dots(z+n)} \int_0^\infty dt t^{z+n} e^{-t}$ we only

have trouble in the \int_0^∞ region since this is where t^{z+n} was blowing up.

Therefore, splitting the integral we can find the exact ML representation

$$\Gamma(z) = \frac{1}{z(z+1)\dots(z+n)} \left[\int_0^\infty dt t^{z+n} e^{-t} + \underbrace{\int_1^\infty dt t^{z+n} e^{-t}}_{\text{regular part.}} \right]$$

Beta function - Euler integral of first kind.

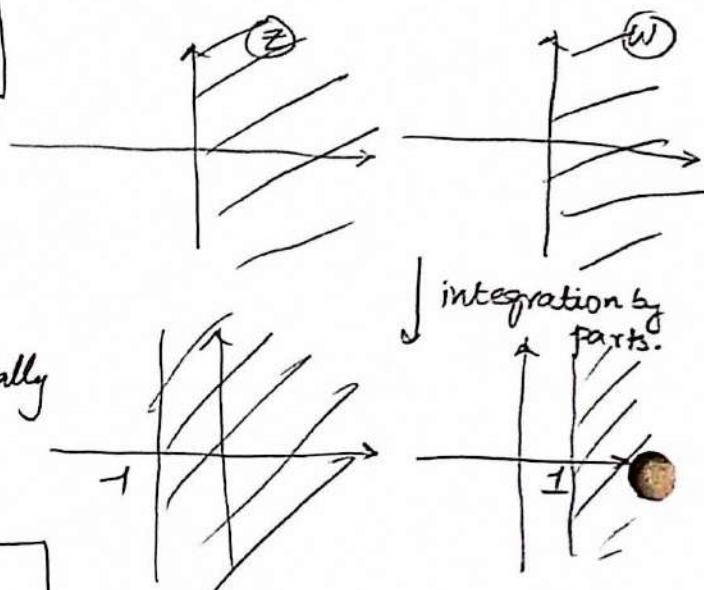
$$B(m, n) \equiv \int_0^1 dt t^{m-1} (1-t)^{n-1} \quad m, n = 1, 2, 3, \dots$$

↑ ↑
diverges at 0 diverges at 1
if $m=0$ if $n=0$

$$B(z, w) = \int_0^1 dt t^{z-1} (1-t)^{w-1}$$

$$\Rightarrow \operatorname{Re}\{z\} > 0 \text{ and } \operatorname{Re}\{w\} > 0.$$

Integration by parts with $u = t^{z-1}$ &
 $v = (1-t)^{w-1}$ would improve for z but
 worsen for w . Therefore integration
 by parts cannot be used to analytically
 continued.



But

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

can be used.

Gaussian Integrals

$$\int_0^\infty du e^{-au^2} u^r \quad \text{where } \operatorname{Re} a > 0 ; r > -1$$

$$\text{Let } au^2 = t ; u = \frac{\sqrt{t}}{\sqrt{a}} \Rightarrow du = \frac{dt}{2\sqrt{at}}$$

$$\Rightarrow \int_0^\infty \frac{dt}{2t^{1/2} a^{1/2}} e^{-t} \frac{t^{\frac{r}{2}}}{a^{\frac{r+1}{2}}} = \frac{1}{2a^{\frac{r+1}{2}}} \int_0^\infty dt t^{\frac{r-1}{2}} e^{-t}$$

$$\Rightarrow \boxed{\int_0^\infty du e^{-au^2} u^r = \frac{\Gamma(\frac{r+1}{2})}{2a^{\frac{r+1}{2}}}} \quad \Rightarrow \int_0^\infty du e^{-au^2} = \frac{\Gamma(\frac{1}{2})}{2a^{1/2}} = \frac{\sqrt{\pi}}{2\sqrt{a}}$$

↳ since integral from 0 to ∞ .

$$\text{Also, } \Gamma(\frac{1}{2}) = \sqrt{\pi}; \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}; \Gamma(\frac{5}{2}) = \frac{3}{4}\sqrt{\pi}; \dots$$

Mittag-Leffler representation of Γ

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} + \int_1^\infty dt e^{-t} t^{z-1} \rightarrow \text{Only this break up gives the right pole terms.}$$

expand

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\infty dt t^{n+z-1} + \int_1^\infty dt e^{-t} t^{z-1}$$

$$\Gamma(z) = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n}}_{\text{All poles with corresponding residues}} + \underbrace{\int_1^\infty dt e^{-t} t^{z-1}}_{\text{regular part}}$$

All poles
with corresponding
residues

regular part

Recall, $\Psi(z) = \frac{d}{dz} \ln \Gamma(z)$ & $\Psi(z+1) = \frac{1}{z} + \Psi(z)$

Near the poles $\Gamma(z) \approx \frac{(-1)^n}{n!(z+n)} + \text{reg part}$

$$\Rightarrow \Psi(z) \approx \frac{(-1)}{z+n} + \text{reg part} \Rightarrow \text{constant residue.}$$

$$\Gamma(z) \underset{z \approx 0}{\approx} \frac{1}{z} + \sum_{n=0}^{\infty} c_n z^n \underset{\text{reg part}}{\approx} \frac{1}{z} - \gamma + c_1 z + \dots$$

Euler - Mascheroni constant

$$\gamma = \lim_{N \rightarrow \infty} \left(\sum_{n=1}^N \frac{1}{n} - \ln N \right)$$

↓ diverges as $\ln N$ ↓ corrects this & makes this series converge

and $\gamma = 0.5772\dots \rightarrow$ conjectured to be irrational & also transcendental.

$$\Psi(z) \underset{\text{singular}}{\approx} -\frac{1}{z} - \gamma + \dots \quad \& \quad \Psi(z+1) = \frac{1}{z} + \Psi(z).$$

$$\Rightarrow \boxed{\Psi(1) = -\gamma} \quad \text{← } \Psi(1) = \frac{1}{1} - \frac{1}{2} - \gamma + \dots \underset{z=0}{\text{goes to 0 at}}$$

$$\Psi(2) = 1 - \gamma$$

$$\Psi(n+1) = -\gamma + \sum_{j=1}^n \frac{1}{j}$$

- > $\Gamma(z)$ is meromorphic. & $\frac{1}{\Gamma(z)}$ is entire $\Rightarrow \Gamma(z)$ has no zeroes
- > Is there a representation such that all zeroes can be explicitly written down for $\frac{1}{\Gamma(z)}$? Yes.

Weierstrass Representation.

$$\frac{1}{\Gamma(z)} = \underbrace{z e^{xz}}_{\text{entire fn.}} \underbrace{\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}}_{\text{zeroes.}}$$

Beta function

$$B(z, w) = \int_0^1 dt t^{z-1} (1-t)^{w-1} \quad \begin{array}{l} \text{Re } z > 0 \\ \text{Re } w > 0 \end{array} \quad \text{converges}$$

Consider,

$$I(z, w) = \int_0^\infty du \int_0^\infty dv e^{-(u^2+v^2)} u^{z-1} v^{w-1}$$

$$= \int_0^\infty du e^{-u^2} u^{z-1} \quad \text{let } u^2 = t ; u = \sqrt{t}$$

$$= \frac{1}{2} \int_0^\infty dt e^{-t} t^{z-1} = \frac{1}{2} \Gamma(z)$$

$$\Rightarrow I(z, w) = \frac{1}{4} \Gamma(z) \Gamma(w)$$

Evaluating the integral in polar coordinates

$$u = r \cos \theta; v = r \sin \theta.$$

$$\begin{aligned} \Rightarrow I(z, w) &= \int_0^\infty dr r e^{-r^2} \int_0^{\pi/2} d\theta r^{2z+2w-2} (\cos \theta)^{2z-1} (\sin \theta)^{2w-1} \\ &= \frac{1}{2} \int_0^\infty 2r dr e^{-r^2} r^{2(z+w-1)} \int_0^{\pi/2} d\theta (\cos \theta)^{2z-1} (\sin \theta)^{2w-1} \\ \text{Let } r^2 &= s \Rightarrow \frac{1}{2} \int_0^\infty ds e^{-s} s^{z+w-1} \underbrace{\int_0^1 \frac{1}{2} \cdot d\xi \xi^{z-1} (1-\xi)^{w-1}}_{B(z, w)} \\ &\quad & & \underbrace{\qquad\qquad\qquad}_{2 \cos \theta \sin \theta} \\ \Rightarrow 2 \cos \theta \sin \theta &= d\xi \\ &= d\xi \end{aligned}$$

$$\Rightarrow I(z, w) = \frac{1}{4} \Gamma(z+w) B(z, w)$$

$$\Rightarrow B(z, w) = \frac{\Gamma(z) \Gamma(w)}{\Gamma(z+w)}$$

Now we can analytically continue the Beta function.

$$\text{Put } w=1-z \Rightarrow B(z, 1-z) = \frac{\Gamma(z) \Gamma(1-z)}{\Gamma(1)}$$

$$\Rightarrow B(z, 1-z) = \frac{\Gamma(z) \Gamma(1-z)}{\Gamma(1)}$$

Could show

$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}$	+	$\psi(1-z) - \psi(z) = \pi \cot(\pi z)$
--------------------------------------------------	---	-----------------------------------------

negative
z poles
at integers

positive
z poles.
at integers

all integer
poles.

also,
 Reflection
formula for
 $\Gamma(z)$.

$$B(z, z) = \frac{\Gamma(z) \Gamma(z)}{\Gamma(2z)}$$

evaluate
this

Doubling formula.

$$\boxed{\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2})}$$

$$\ln(\Gamma(2z)) = (2z-1) \ln 2 + \ln \Gamma(z) + \ln \Gamma(z + \frac{1}{2}) - \frac{1}{2} \ln \pi$$

$$\Rightarrow 2\psi(2z) = 2\ln 2 + \psi(z) + \psi(z + \frac{1}{2}) \rightarrow \text{Gives } \psi(\frac{1}{2}), \psi(\frac{3}{2})$$

& so on.
Check the signs maybe wrong.

Möbius Transformation (lec 11)

- > If $w = f(z) = u + iv$ if $f(z)$ is analytic in some region, curves in this region in z plane preserve angles as they are mapped to w plane.
 - > What is the most general mapping from $z \rightarrow w$ such that the mapping is one to one \Rightarrow it cannot involve higher powers \Rightarrow must be a linear map $\Rightarrow w = az + b$. This mapping preserves the 'point' at infinity. It is basically a mapping which preserves the north pole of the Riemann sphere. However, if we are not restricted to this & we can map "all" points on one Riemann sphere to another Riemann sphere we get the Möbius transformation.
- $$w = \frac{az+b}{cz+d}$$

Where $a, b, c, d \in \mathbb{C}$ & the dot must be nonzero to ensure map $\frac{\text{doesn't become}}{\text{degenerate}} \Rightarrow ad - bc \neq 0 \Rightarrow$ constant.

- The Möbius transformation is analytic and one to one.
- If $ad-bc = k$, divide a, b, c, d by \sqrt{k} & we get $ad-bc=1$
 this does not change the transform. Therefore we can always
 enforce $ad-bc=1$.

$$\rightarrow z = -\frac{d}{c} \rightarrow w = \infty$$

$$\rightarrow z = \infty \rightarrow w = \frac{a}{c}$$

$$z = \frac{dw-b}{-cw+a}$$

Fixed Points

$$\Rightarrow \text{Solutions to } z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0$$

$$\Rightarrow \xi_{1,2} = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c}$$

$$\boxed{\xi_{1,2} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}}$$

since $ad-bc=1$

Case

(1) ξ_1, ξ_2 are finite and distinct.

$$\Rightarrow c \neq 0 \& ad \neq \pm 2$$

Case
(2) $\xi_1 = \xi_2 = \xi'$

$$\Rightarrow c \neq 0 \& ad = \pm 2 \Rightarrow \xi' = \frac{a-d}{2c}$$

Case (3) ξ_1 finite, $\xi_2 = \infty$ \Rightarrow linear translation

$$\Rightarrow C=0 \quad \& \quad ad=1 = d=\frac{1}{a}$$

$$\Rightarrow z = \frac{az+b}{d} = a^2z + ab$$

$$\Rightarrow \xi_1 = \frac{ab}{1-a^2}; \xi_2 = \infty$$

Case (4) $\xi_1 = \xi_2 = \infty$

$$\Rightarrow C=0 \quad \& \quad a=d \quad \& \quad ad=1 = a = \pm 1$$

$$\Rightarrow iz \rightarrow z + ab$$

$$\Rightarrow z \rightarrow z \pm b \Rightarrow \text{trivial translation}$$

Cross ratio of 4 points.

Given z_1, z_2, z_3, z_4 ,

$$[z_1, z_2, z_3, z_4] \equiv \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

$$\text{Let } z_i \rightarrow \frac{az_i+b}{cz_i+d} \quad \& \quad z_j \rightarrow \frac{az_j+b}{cz_j+d}.$$

$$\Rightarrow z_i - z_j \rightarrow \frac{acz_i z_j + adz_i + bz_j + bd - acz_i z_j - adz_j - bcz_j - bd}{(cz_i + d)(cz_j + d)}.$$

$$\Rightarrow z_i - z_j \longrightarrow \frac{z_i - z_j}{(cz_i + d)(cz_j + d)}$$

Now take $[z_i, z_j; z_k, z_l]$ & substitute difference values
 \Rightarrow cross ratio doesn't change.

\Rightarrow If $z_1 \rightarrow w_1, z_2 \rightarrow w_2, z_3 \rightarrow w_3, z_4 \rightarrow w_4$

$$[w_1, w_2; w_3, w_4] = [z_1, z_2; z_3, z_4]$$

- > Choose 4 arbitrary points & find the cross ratio, & find the cross ratio of the mapped points. They would be equal.
- > From a set of 4 points we could write 24 cross ratios?
- > Represent $[z_i, z_j; z_k, z_l] \rightarrow [i, j; k, l]$

$$\begin{aligned} \text{Note, } [i, j; k, l] &= [k, l; i, j] \\ &= [j, i; l, k] \\ &= \frac{1}{[j, i; l, k]} \\ &= \frac{1}{[i, j; l, k]} \\ &= , 1 - [i, k; j, l] \end{aligned}$$

> All 24 cross ratios are related like above so only one cross ratio is independent.

$$[\infty, z_2; z_3, z_4] = \frac{z_2 - z_4}{z_2 - z_3}$$

- > Given z_1, z_2, z_3 & their images w_1, w_2, w_3 we can show that there exists a unique transformation that satisfies this mapping. Therefore, it maps circles to circles.
- > $[z_1, z_2, z_3] = [w_1, w_2, w_3]$ & solve for w in terms of z & you will see that it is unique.
- > ξ_1, ξ_2 distinct, finite.

$$\xi_{1,2} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

$$> \text{Consider } [z, \infty; \xi_1, \xi_2] \Rightarrow [w, \frac{a}{c}; \xi_1, \xi_2]$$

$$\Rightarrow \frac{(z - \xi_1)}{(z - \xi_2)} = \frac{(w - \xi_1)(\frac{a}{c} - \xi_2)}{(w - \xi_2)(\frac{a}{c} - \xi_1)}$$

$$\Rightarrow \boxed{\frac{w - \xi_1}{w - \xi_2} = \left(\frac{a - c\xi_1}{a - c\xi_2} \right) \left(\frac{z - \xi_1}{z - \xi_2} \right)}$$

$\downarrow k \rightarrow \text{The multiplier.}$

$$\Rightarrow \boxed{\frac{w - \xi_1}{w - \xi_2} = k \cdot \frac{z - \xi_1}{z - \xi_2}}$$

\rightarrow The normal form of
the Möbius Transform.

'A more natural representation'

→ Making n successive Möbius Transformation

$$\frac{z^{(n)} - \xi_1}{z^{(n)} - \xi_2} = k^n \underbrace{\frac{(z^{(0)} - \xi_1)}{(z^{(0)} - \xi_2)}}_{\text{net } = A}$$

$$\Rightarrow z^{(n)} - \xi_1 = A (z^{(0)} - \xi_2)$$

$$\Rightarrow z^{(n)} = \xi_1 + A \xi_2$$

$$\boxed{z^{(n)} = \frac{\xi_1 - \xi_2 k^n \left(\frac{z^{(0)} - \xi_1}{z^{(0)} - \xi_2} \right)}{1 - k^n \left(\frac{z^{(0)} - \xi_1}{z^{(0)} - \xi_2} \right)}}$$

→ After n iterations point $z^{(0)}$ goes to $z^{(n)}$.

→ This also works for negative n values.

$$k = \frac{a - c \xi_1}{a - c \xi_2} = \frac{a - c \left[\frac{a - d + \sqrt{(a+d)^2 - 4}}{2c} \right]}{a - c \left[\frac{a - d - \sqrt{(a+d)^2 - 4}}{2c} \right]}$$

$$\boxed{k = \frac{T - \sqrt{T^2 - 4}}{T + \sqrt{T^2 - 4}}}$$

Where $T = a+d$ is the trace of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\text{If } |k| > 1 \text{ as } n \rightarrow \infty, A \rightarrow \infty \Rightarrow \boxed{z^{(n)} \underset{n \rightarrow \infty}{\lim} \xi_2}$$

⇒ Then every point except ξ_1 falls on ξ_2 as $n \rightarrow \infty$

> If $|k| < 1$, $\tilde{\gamma}_1$ becomes attractor & $\tilde{\gamma}_2$ becomes repeller.

> If $|k| = 1$, we will see.

> If $\tilde{\gamma}_1 = \tilde{\gamma}_2 = \tilde{\gamma} = \frac{a-d}{2c}$ ($a+d = \pm 2$)

We can show that

$$\frac{1}{w-\tilde{\gamma}} = \frac{1}{z-\tilde{\gamma}} \pm c \quad \begin{array}{l} \text{depends on} \\ \rightarrow \text{Normal form.} \end{array}$$

> If $c=0 \Rightarrow d=\frac{1}{a}$; $\tilde{\gamma}_1 = \frac{ab}{1-a^2}$; $\tilde{\gamma}_2 = \infty \Rightarrow z \rightarrow a^2 z + ab$.

$$w - \tilde{\gamma}_1 = k(z - \tilde{\gamma}_1) \quad \begin{array}{l} \downarrow a/d \\ \rightarrow \text{Normal form.} \end{array}$$

> If $c=0 \Rightarrow a=d \Rightarrow a = \pm 1 \Rightarrow z \rightarrow z \pm b$.

$$\Rightarrow z^{(n)} = z \pm nb$$

Note; $w = \frac{az+b}{cz+d} \Rightarrow \left| \frac{dw}{dz} \right| = \left| \frac{a(cz+d) - c(az+b)}{(cz+d)^2} \right| = \frac{1}{|cz+d|^2} \rightarrow \text{Stretch factor.}$

If $|cz+d| = 1 \Rightarrow$ This curve is not stretched.

Note that this is a circle with radius = 1 &

centre at $-\frac{d}{c}$. This is called the isometric circle.

Lec 13 Recall, $k = \frac{T - \sqrt{T^2 - 4}}{T + \sqrt{T^2 - 4}}$ where, $T = a + d$

i) Note, if $-2 < T < 2 \Rightarrow |T - \sqrt{T^2 - 4}| = |T + \sqrt{T^2 - 4}| \Rightarrow |\underline{k}| = 1, k \neq 1$

Then we get an elliptic transformation.

ii) Note if $T = \pm 2 \Rightarrow k = 1 \Rightarrow$ parabolic transformation.

iii) T is real & $|T| > 2 \Rightarrow k$ is real & not 1 \Rightarrow hyperbolic transformation.

iv) T is complex, $T \notin [-2, 2] \Rightarrow$ loxodromic transformation.

Note iii) is a special case of iv).

- The isometric circle $|cz+d|=1$ is mapped onto $|c\bar{w}+\bar{d}|=1$
- The centre of $|cz+d|=1$ is mapped onto ∞ since if $z = -\frac{d}{c}$, $w = \infty$.
Same goes for centre of $|c\bar{w}+\bar{d}|=1$ in the inverse transform.
- Points inside $|cz+d|=1$ are mapped outside circle $|c\bar{w}+\bar{d}|=1$ & vice versa.

→ To map $|z|=1$ to $|w|=1$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ must be of the form

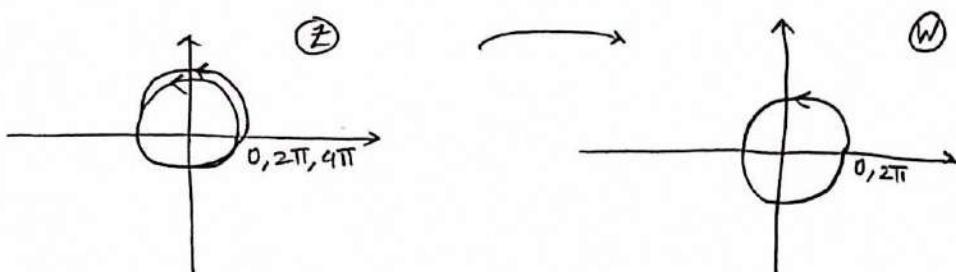
$$\begin{pmatrix} a & b \\ a^* & b^* \end{pmatrix} \text{ with } |a|^2 - |b|^2 = 1 \quad \text{or} \quad \begin{pmatrix} a & b \\ -b^* & -a^* \end{pmatrix} \text{ with } -|a|^2 + |b|^2 = 1.$$

Unitary matrix $\Rightarrow U^T U = U U^T = I \Rightarrow U^+ = U^T$ where U^+ is the conjugate transpose (or) Hermitian adjoint.

Also, $|\det U| = 1 \Rightarrow \det U = e^{i\theta}$ where $\theta \in \mathbb{R}$.

Lec 14 Multivalued Functions; Integral representations.

Eg: $f(z) = z^{1/2} = w \Rightarrow w = \sqrt{r} e^{i\frac{\theta}{2}}$



- > θ must go from 0 to 4π to map onto the full W plane.
- > The 2 copies of the Z plane are called Riemann sheets.
- > For $z^{1/3}$ we need 3 Riemann sheets.

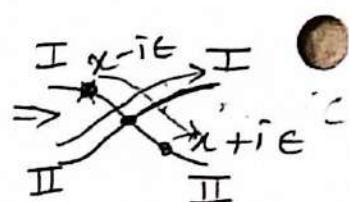
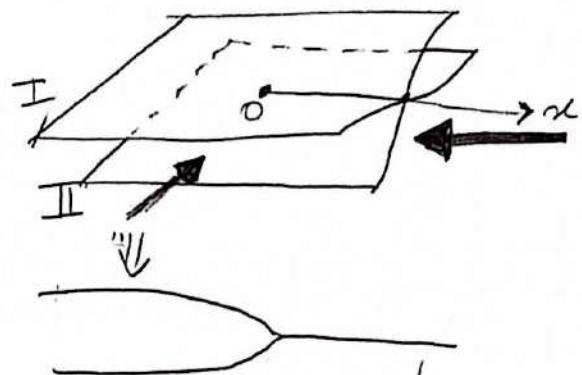
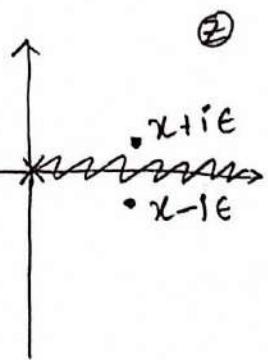
Topology of Riemann Sheets

$w = z^{1/2}$ → on top sheet,

$$w = e^{i\pi/2} z^{1/2} \rightarrow \text{on the bottom sheet} \quad \text{since } \theta \rightarrow \theta + 2\pi \leftarrow \frac{\theta}{2} \rightarrow \frac{\theta}{2} + \pi$$

$$\Rightarrow w = -z^{1/2}$$

- > At $z = 0$ the 2 branches coincide & this point is called a "branch point" for this function. Note that $z = \infty$ is also a branch point.
- > The two sheets must be glued together at the branch points since outputs coincide. Draw a line joining the branch points, this line can go to ∞ in any direction, but let's choose along positive x. This line joining the 2 branch points is called a "branch cut". The trajectory descends/ascends onto the next sheet at the branch cut.



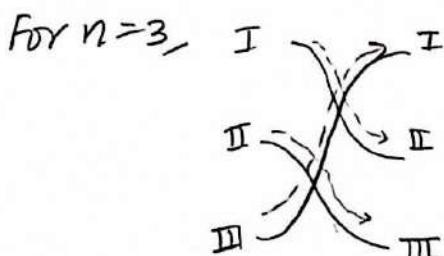
By continuity $\lim_{\epsilon \rightarrow 0} f_I(x - i\epsilon) = \lim_{\epsilon \rightarrow 0} f_{II}(x + i\epsilon)$

Now we don't need to specify $\pm\sqrt{z}$. We have a unique value for \sqrt{z} from $0 \rightarrow 2\pi$ in W plane.

$$\rightarrow \text{dis } f(z) \Big|_{z=x>0} = f_I(x+i\epsilon) - f_I(x-i\epsilon) = f_I(x+i\epsilon) - f_{II}(x+i\epsilon)$$

Therefore there is a discontinuity as we jump b/w the sheets.
The disc vanishes on the branch cut & increases as $\epsilon \uparrow$.

Eg: $f(z) = z^n$ where $n = 2, 3, \dots$ all have branch points at ∞ .
These branch points are called algebraic branch point



Eg: $z^{\frac{p}{q}} = f(z) \Rightarrow q$ sheets

Eg: $f(z) = z^\alpha$; α is real. If it is irrational we have an infinite sheeted structure & $z=0$ is a "winding branch point"

Eg: $f(z) = \ln z = \ln r + i\theta \Rightarrow$ we get an additive constant of $2\pi ni$ for each sheet. $n=0$ sheet is called the principle sheet.

$\Rightarrow z=0, \infty$ are "logarithmic branch points"

Eg: $f(z) = \ln(z-1) \Rightarrow$ branch point (log) at $z=1, \infty$.

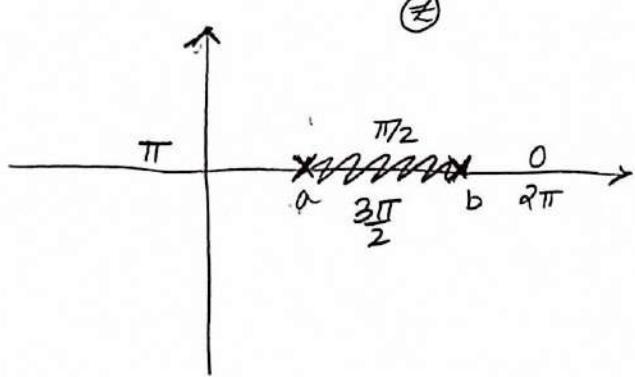
Eg: $f(z) = \frac{1}{z} \ln(1-z)$ At $z=0$ $\ln(1-z) \approx -z - \frac{z^2}{2} - \dots$

\Rightarrow it looks like the singularity at $z=0$ is removable.

But, remember that this is only true on the principle sheet. For all other sheets there is a $2\pi ni$ factor.
 $\ln 1 = 0$ only on the principle sheet.

$\Rightarrow f(z) = \frac{1}{z} \ln(1-z)$ has a simple pole on every sheet except the principle sheet.

Eg: $f(z) = (z-a)^{\frac{1}{2}} (z-b)^{\frac{1}{2}}$ $0 < a < b < \infty \in \mathbb{R}$



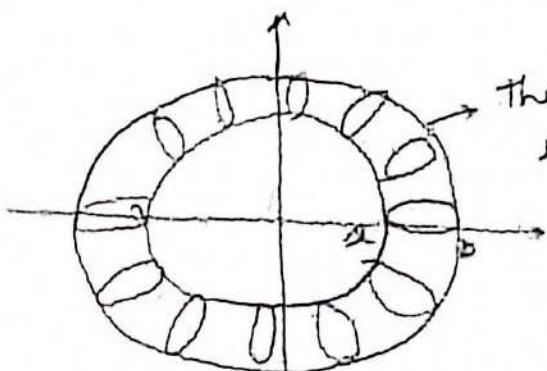
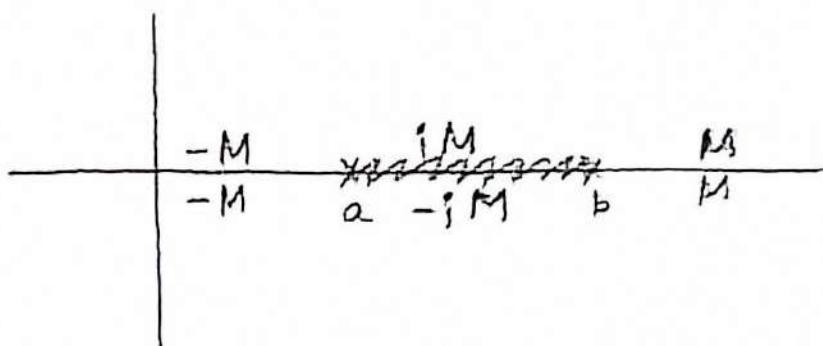
$$(z-a)^{\frac{1}{2}}$$

$$(z-b)^{\frac{1}{2}}$$

↓
 $\pi/2 + \pi/2$
 $a \frac{\pi+\pi/2}{2} \quad b \frac{\pi+\pi}{2} \quad 0+0 \rightarrow$ same so no branch cut
phase adds since we have a product.

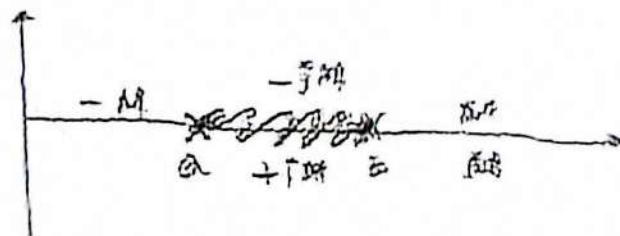
$$\text{Let } M = |(z-a)(z-b)|^{1/2}.$$

= no branch point at ∞ .



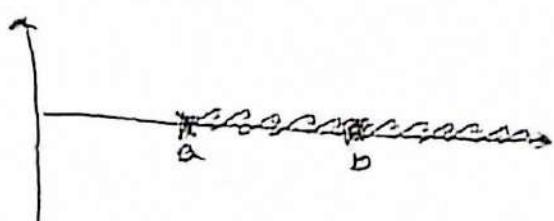
The 2 surfaces form a 'torus' & are attached elsewhere.

$$\text{Eg: } f(z) = \frac{(z-a)^{1/2}}{(z-b)^{1/2}}$$



$$\text{Eg: } f(z) = (z-a)^{1/2} (z-b)^{1/2} \rightarrow \text{Here branch cut goes from } a \rightarrow \infty \text{ & } b \rightarrow \infty$$

Discontinuities are diff
b/w $a \neq b$ & b/w $b = \infty$.



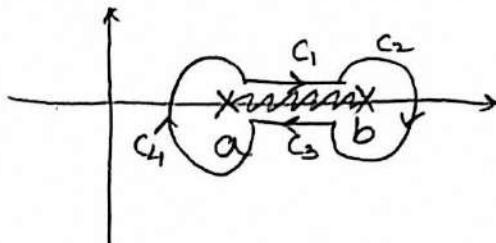
$$\text{Eg: } f(z) = (z-a)^{1/3} (z-b)^{-1/2} \Rightarrow \text{cut only b/w } a \neq b.$$

\rightarrow Drawing closed contours must take into account the sheeted structure.

lec 15

Eg: $\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}}$ using contour integration.

Consider $\frac{1}{\sqrt{(z-a)(z-b)}}$

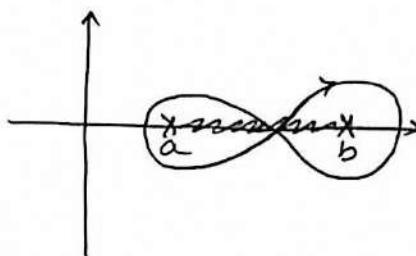


Answer is $2\pi i \rightarrow$ Solve it.

Note that C_2 & C_4 should evaluate to zero. Then blow up the contour to ∞ & evaluate the integral.

> Could choose another contour.

Try this

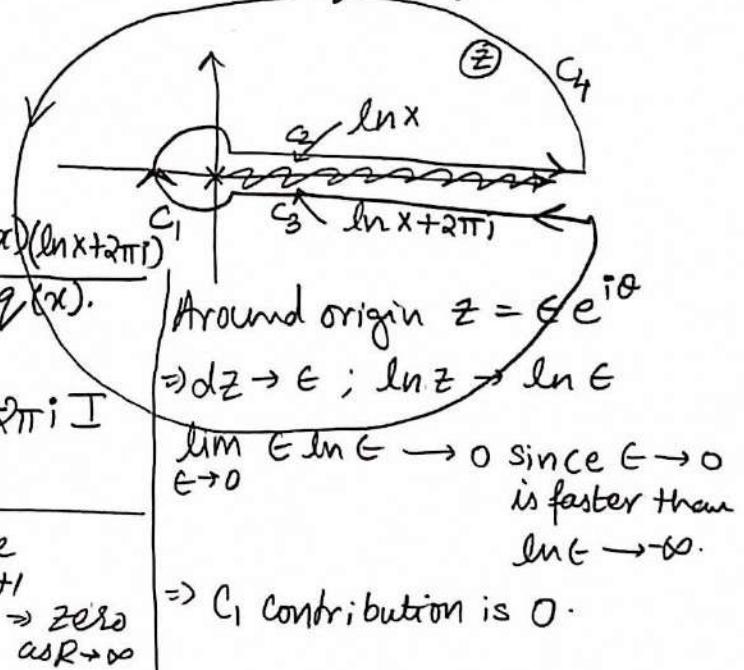


Eg: $I = \int_0^\infty dx \frac{P(x)}{q(x)}$ where $q(x) & p(x)$ are polynomials & $\deg(q) - \deg(p)$ is ≥ 2 & no real positive zeroes of q .
on the line of integration.

Consider, $\int_C dz \frac{P(z) \ln z}{q(z)}$

$$\Rightarrow \int_0^\infty dx \frac{P(x) \ln x}{q(x)} + \int_\infty^0 dx \frac{P(x)(\ln x + 2\pi i)}{q(x)}$$

$$= \int_0^\infty dx \left[\frac{P(x) \ln x}{q(x)} - \frac{P(x)(\ln x + 2\pi i)}{q(x)} \right] = -2\pi i I$$



C_4 also has zero contribution since den has $R^{\deg(q)}$ & num has $R^{\deg(p)+1}$ \rightarrow zero as $R \rightarrow \infty$

$$\Rightarrow I = -\frac{1}{2\pi i} \oint_C \frac{dz P(z) \ln z}{q(z)} = -\frac{1}{2\pi i} \times 2\pi i \sum_{\text{poles}} \text{Residues of } \frac{P(z)}{q(z)} \ln(z).$$

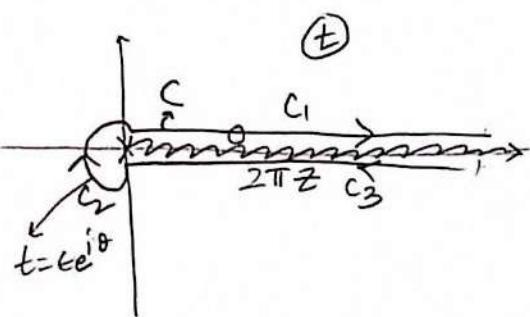
→ Note that $g(z)$ has zeroes outside of the real positive axis.

$$\boxed{I = -\sum \text{residues of } \frac{P(z) \ln z}{g(z)}}$$

Eg: $\int_0^\infty \frac{dx}{x^n+1}$ where ($n \geq 3$) \Rightarrow poles at zeroes of $z = \sqrt[n]{-1}$.

Going back to $\Gamma(z)$

$$\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t} \quad (\operatorname{Re} z > 0)$$



$$t^{z-1} = |t|^{z-1} e^{i\theta} \quad \text{above.}$$

$$|t|^{z-1} e^{2\pi i(z-1)} \quad \text{below}$$

$$\hookrightarrow e^{2\pi i z}$$

$$\int_C dt t^{z-1} e^{-t} = \int_0^\infty dt t^{z-1} e^{-t} + \int_{C_1}^\infty dt t^{z-1} e^{2\pi i z} e^{-t}$$

$$+ \int_{2\pi}^{\infty} \underbrace{te^{i\theta} id\theta}_{dt} (e^{i\theta})^{z-1} e^{-te^{i\theta}}$$

$$\hookrightarrow e^z \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

since ($\operatorname{Re} z > 0$) for $\Gamma(z)$.

$$\Rightarrow \int_C dt t^{z-1} e^{-t} = (1 - e^{2\pi i z}) \Gamma(z), \quad \operatorname{Re}(z) > 0$$

$$\boxed{\Gamma(z) = \frac{1}{1 - e^{2\pi i z}} \int_C dt t^{z-1} e^{-t}}$$

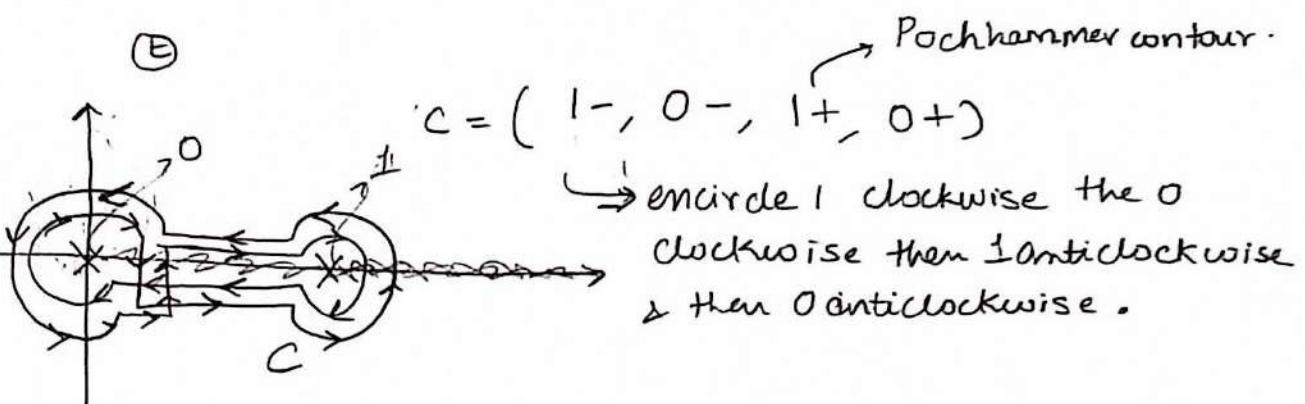
For ALL z since this integral is valid for all z it is the analytic continuation.

"Master representation of $\Gamma(z)$ " → This contour integral converges for all z since the problem with $\operatorname{Re} z < 0$ only arose at $t=0$ (ie C_2). However, this contour does not touch that point. Therefore this representation is valid for all values of z . When z is an integer the branch cut disappears $\rightarrow 1 - e^{2\pi i z} \Rightarrow$ we get poles at -ve integers.

Tec 16 Beta function.

$$B(z, w) = \int_0^1 dt t^{z-1} (1-t)^{w-1} \quad (\operatorname{Re} z > 0, \operatorname{Re} w > 0)$$

- > Recall integration by parts doesn't improve overall ROC.
- > There are branch points at $t=0, 1$ since z, w can take on arbitrary values. In fact this branch pt. is a winding point since z, w are arbitrary.



1 - picks up $-2\pi i w$ from $(1-t)^{w-1}$ term
 \downarrow since clockwise

0 - picks up $-2\pi i z$ from t^{z-1} term.

1+ picks up $+2\pi i w$ & 0+ picks up $+2\pi i z$

$\Rightarrow M \rightarrow M e^{-2\pi i w} \rightarrow M e^{-2\pi i z}$
 $\Rightarrow M e^{-2\pi i z} \rightarrow M$.
 So we are sure that despite crossing branch cuts we are back to the original plane.

> $dt \rightarrow \epsilon$; $t^z \rightarrow \epsilon^{z-1}$; $(1-t)^{w-1} \rightarrow 1 \Rightarrow$ integrand $\rightarrow \epsilon^z$ which goes to 0 as $\epsilon \rightarrow 0$ for $\operatorname{Re} z > 0$. Similarly contribution from circles around 1 also give 0.

$$\Rightarrow B(z, w) = \frac{1}{(1 - e^{-2\pi i z})} \frac{1}{(1 - e^{2\pi i w})} \int_C dt t^{z-1} (1-t)^{w-1}$$

\hookrightarrow This is valid for all z, w since this representation converges everywhere. Note that C never hits the poles it is always arbitrarily close to them so we are fine.

Riemann Zeta Function $\zeta(z)$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \operatorname{Re}(z) > 1 \quad \rightarrow \text{Analytic Continuation?}$$

Start with,

$$\Gamma(z) \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \int_0^{\infty} du u^{z-1} e^{-u}, \quad \operatorname{Re}(z) > 1$$

Put $u = nt$

$$\Rightarrow \Gamma(z) \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \int_0^{\infty} y dt n^{z-1} t^{z-1} e^{-nt}$$

$$= \int_0^{\infty} dt t^{z-1} \sum_{n=1}^{\infty} e^{-nt}; \quad \sum_{n=1}^{\infty} (e^{-t})^n \text{ is a geometric series.}$$

$$= \int_0^{\infty} \frac{dt t^{z-1}}{e^t - 1} \quad \begin{aligned} \operatorname{Re}(z) &> 0 \text{ still} \\ &\text{since } t \rightarrow \infty \text{ at } t=0 \end{aligned}$$

$$= \frac{1}{1-e^t} - 1$$

$$= \frac{e^{-t}}{1-e^t} \quad \begin{aligned} \text{since } e^t &> 1 \\ \Rightarrow t &> 0 \end{aligned}$$

in integrand.

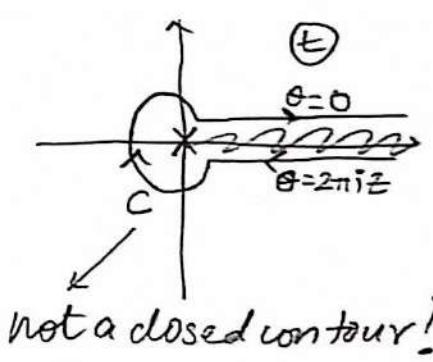
But $\operatorname{Re}(z) > 1$ still since $e^{t-1} \approx t(1 + \frac{t}{2} + \dots)$

$\frac{t^{z-1}}{e^{t-1}} \approx \frac{t^z}{t^2(1\dots)} \Rightarrow z=1 \Rightarrow \frac{1}{t} \text{ blows up since } \ln t \rightarrow \infty \text{ at } 0.$

$\therefore z > 1 \Rightarrow \frac{t^z}{t^2(1\dots)}$ does not blow up at $t=0$

$$\Gamma(z) \zeta(z) = \int_0^{\infty} \frac{dt t^{z-1}}{e^t - 1}, \quad \operatorname{Re}(z) > 1$$

Now use contour integral in t plane.



$$= \frac{1}{1 - e^{2\pi i z}} \int_C dt \frac{t^{z-1}}{e^t - 1}$$

same as earlier.

$$\Rightarrow \boxed{\zeta(z) = \frac{1}{\Gamma(z)(1 - e^{2\pi i z})} \int_C dt \frac{t^{z-1}}{e^t - 1}}$$

C does not touch z so this rep is valid for all z .

$$\text{Recall } \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

Rewriting,

$$\zeta(z) = \frac{i e^{-i\pi z}}{2\Gamma(z)\sin \pi z} \int_C dt \frac{t^{z-1}}{e^t - 1}$$

$$\boxed{\zeta(z) = \frac{i e^{-i\pi z}\Gamma(1-z)}{2\pi} \int_C \frac{dt t^{z-1}}{e^t - 1}}$$

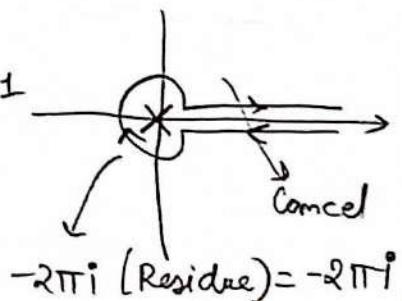
has singularity at $z=1, 2, 3, \dots$
but Zeta has pole only at $z=1$.
So why the discrepancy.

$$\zeta(z) \xrightarrow[z \rightarrow 1]{} \frac{i e^{-i\pi}}{2\pi} \cdot \frac{1}{1-z} \cdot \int_C \frac{dt t^0}{e^t - 1} \xrightarrow[t \rightarrow 0]{} 1 \Rightarrow \text{branch point disappears.}$$

\sim
Gamma
fun.

but $e^t - 1 \rightarrow t(\dots)$ which
has a singularity at $t=0$.

$$= \frac{i e^{-i\pi}}{2\pi} \cdot \frac{-1}{z-1} \cdot \int_C \frac{dt}{t} \left(1 + \frac{1}{2} + \dots\right)^{-1}$$



$$\Rightarrow \zeta(z) = \frac{i}{2\pi} \cdot \frac{1}{z-1} \cdot (-2\pi i) + \text{Reg part.}$$

$$\zeta(z) = \frac{1}{z-1} + \text{Reg part}$$

\Rightarrow Simple pole at $z=1$ with residue of 1

For $z = n$; $n = 2, 3, 4, \dots$

$\Gamma(1-z)$ has poles,

$$\begin{aligned} \Rightarrow \zeta(z) &= \frac{i e^{-i\pi z}}{2\pi} \Gamma(1-z), \quad \int_C \frac{dt}{e^t - 1} t^{z-1} \quad \text{let } t^{z-1} = e^{(z-1)\ln t} \\ &= \frac{i}{2\pi} \frac{(z-1)^{n+1}}{(n-1)!} \cdot \frac{(z-1)^{n+1}}{z-n} \left[\int_C \frac{dt}{e^t - 1} t^{n+1} + (z-n) \int_C \frac{dt t^n \ln t}{e^t - 1} \right] \\ &= \frac{i}{2\pi(n-1)!} \left[\int_C \frac{dt t^n \ln t}{e^t - 1} \right] \quad \text{go to zero as } z \rightarrow n \\ &\quad \hookrightarrow \text{recall that this integral} \\ &\quad \text{the } \ln t - \ln t - 2\pi i \rightarrow -2\pi i \end{aligned}$$

$$\Rightarrow \frac{i}{2\pi(n-1)!} \int_C \frac{dt t^n (2\pi i)}{e^t - 1} = \frac{1}{(n-1)!} \int_C \frac{dt t^n}{e^t - 1}$$

$$\Rightarrow \boxed{\zeta(n) = \frac{1}{(n-1)!} \int_C \frac{dt t^n}{e^t - 1}} \quad (n=2, 3, \dots)$$

\Rightarrow the poles from Γ function disappear

What happens at $n=0, -1, -2 \dots$? Let's look at Bernoulli numbers.

We know, $\zeta(2) = \frac{\pi^2}{6}$; $\zeta(4) = \frac{\pi^4}{90}$; ...

Bernoulli numbers

$$\text{Consider, } \frac{t}{e^t - 1} = \frac{t}{(1 + t + \frac{t^2}{2!} + \dots) - 1} = \frac{t}{t(1 + \frac{t}{2!} + \dots)} = \frac{1}{1 + \frac{t}{2!} + \frac{t^2}{3!} + \dots}$$

$$\Rightarrow \left[\frac{t}{e^t - 1} \equiv \sum_{n=0}^{10} B_n \frac{t^n}{n!} \right]$$

B_n are the Bernoulli numbers.

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}$$

$$B_6 = \frac{1}{42} \dots$$

> All odd $B_n = 0$; $n \geq 3$.

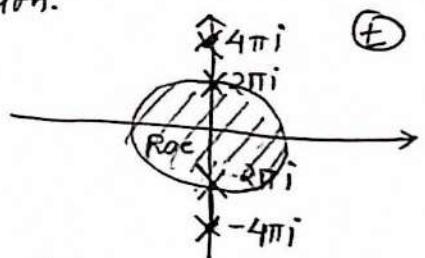
> looks like RHS converges for all t since B_n seems to decrease.

> $n!$ blows up.

> But on LHS pole at $t=0$ is removable. But $e^t = 1$ for $t = 2\pi n i$

> Therefore ROC is only 2π . for the series expansion.

> in fact the Bernoulli numbers blow up after few values! (faster than $n!$).



Going back to $\zeta(z)$ $\zeta(z) = \frac{i e^{-iz}}{2\pi} \Gamma(1-z) \int_C \frac{dt}{e^t - 1} t^{-z}$

$$\begin{aligned} \zeta(0) &= \frac{i}{2\pi} \oint_C \Gamma(1) \int_C \frac{dt}{e^t - 1} \frac{t}{t^z} = \frac{i}{2\pi} \oint_C \frac{dt}{t^2} \cdot [B_0 + B_1 t + B_2 t^2 + \dots] \\ &\quad \xrightarrow{\text{contour around origin}} \xrightarrow{\text{residue} = B_1} \\ &= \frac{i}{2\pi} (-2\pi i) B_1 = B_1 \Rightarrow \boxed{\zeta(0) = -\frac{1}{2}} \end{aligned}$$

trivial zeroes of the zeta function.

residue $\Rightarrow \zeta(-2n)$ for $n = 1, 2, 3, \dots = 0$. These are the

Since there are no odd powers in the B_n terms there is no

$$\left[-\frac{B_0}{2} + B_1 t + \frac{B_2}{2!} t^2 + \frac{B_3}{3!} t^3 + \dots \right] = (-2)^n \zeta(n+1) \int \frac{dt}{t^{n+2}} [B_0 + B_1 t + \frac{B_2}{2!} t^2 + \frac{B_3}{3!} t^3 + \dots]$$

$$\boxed{\zeta(-1) = -\frac{1}{12}}$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-it} \frac{dt}{t^3}$$

$$\zeta(-1) = \frac{i}{2\pi} \oint \frac{dt}{t^3} [B_0 + B_1 t + \frac{B_2}{2!} t^2 + \dots]$$

Similarly,

Legendre Functions

$$\left[(1-z^2) \frac{d^2}{dz^2} - 2z \frac{d}{dz} + \nu(\nu+1) \right] \phi(z) = 0 \rightarrow \text{Legendre's Equation.}$$

where, $\nu \in \mathbb{C}$

> The 2 solutions are $P_\nu(z)$, $D_\nu(z)$. They are linearly independent.

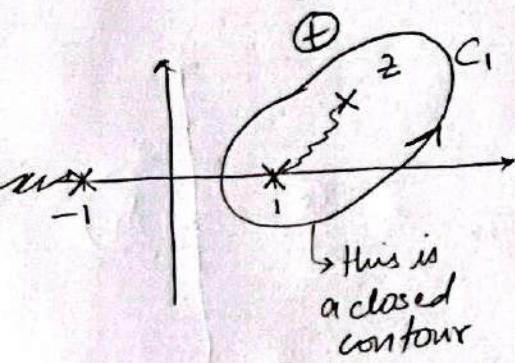
> For $-1 \leq z \leq 1$ for $\nu = n$ where $n \in \mathbb{Z}$, $P_n(z)$ are the Legendre Polynomials.

> Legendre Eq. can be recast as an Eigenvalue problem.

$$\frac{d}{dz} \left[(1-z^2) \frac{d\phi}{dz} \right] = -\nu(\nu+1)\phi$$

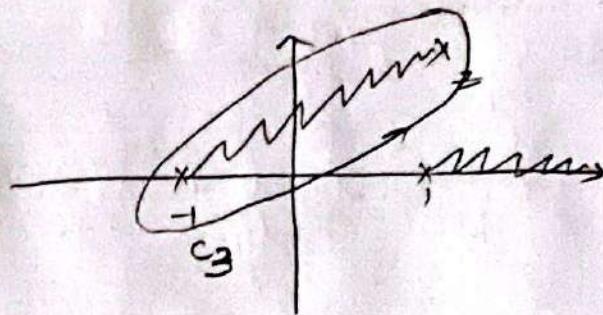
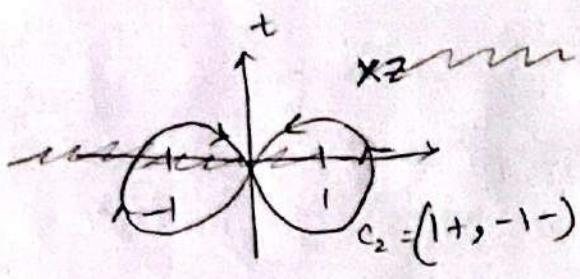
Consider, $\left[\frac{t^2-1}{2(t-z)} \right]^\nu$ as a fn. of ν for a given z . This function

has winding points at $t=\pm 1$ & $t=z$ since ν is arbitrary.



$$C_1 = (-1, z+)$$

Encircling $t=1$ or $t=-1$ counter-clockwise gives $e^{2\pi i \nu}$
whereas encircling $t=z$ counter-clockwise gives $e^{-2\pi i \nu}$. So choosing the contour wisely can form a closed contour. This allows us to choose cuts appropriately.



> C_1, C_2, C_3 are the only 3 independent cuts. $C_1 \cup C_3$ are almost same

$$\rightarrow P_V(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{dt}{(t-z)} \cdot \left[\frac{t^2-1}{2(t-z)} \right]^\vee$$

Contour integral representation

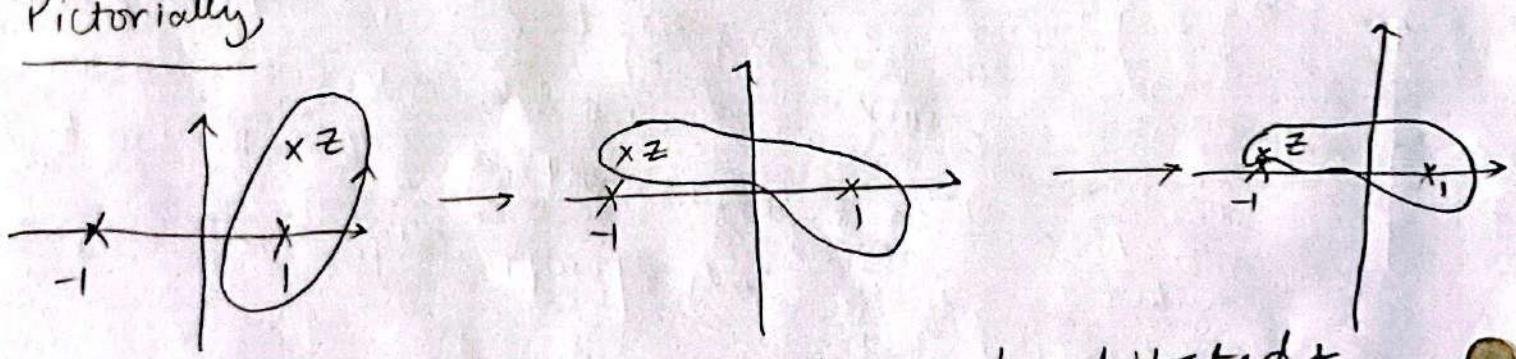
$$Q_V(z) = \frac{1}{\pi(e^{2\pi i v} - 1)} \oint_{C_2} \frac{dt}{(t-z)} \left[\frac{t^2-1}{2(t-z)} \right]^\vee$$

Let $v \neq \mathbb{Z}$,

When $z = -1$ the contour is "pinched b/w the 2 poles (in t plane)" & we get a pole in the z plane.

$\Rightarrow P_V(z)$ is singular at $z = -1$

Pictorially,

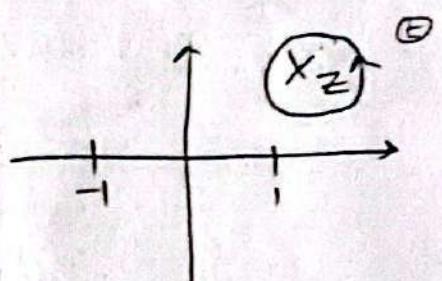


When $z \rightarrow -1$, the contour can no longer be distorted & gets pinched b/w the 2 poles in t plane. We can show that this produces a singularity in the z plane at $z = -1$.

> Similarly a singularity occurs at $z = \pm i$, for $Q_{\nu}(z)$
Since the contour C_2 is pinched when $z \rightarrow \pm i$.

i) > Let $\nu = l = 0, 1, 2, \dots$ the branch cut $b/w 1 \leftarrow z$ disappears, since it was only there for fractional ν .

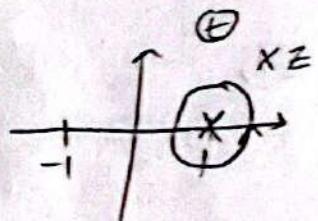
$$P_l(z) = \frac{1}{2\pi i} \oint_C \frac{dt}{2^l} \frac{(t^2 - 1)^l}{(t - z)^{l+1}} \Rightarrow \text{pole of order } l+1 \text{ at } t = z$$



$P_l(z)$ is a polynomial

$$P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dt^l} (t^2 - 1)^l \Big|_{t=z}$$

$$\Rightarrow P_l(z) = \frac{1}{2^l l!} \frac{d^l}{dz^l} (z^2 - 1)^l \rightarrow \text{Rodrigues formula for } P_l(z) \text{ true for all } z.$$



ii) > $P_{-l-1}(z)$ for $l = 0, 1, 2, \dots \Rightarrow$ pole at $t = -1$

$$\Rightarrow P_{-l-1}(z) = P_l(z)$$

> Recall $l(l+1) = \left(\frac{l+1}{2}\right)^2 - \frac{1}{4} \Rightarrow$ Symmetry around $-\frac{1}{2}$ since $\left(\frac{l+1}{2}\right)^2 - \frac{1}{4}$ is even around $-\frac{1}{2}$.
from Diff Eq.

\Rightarrow Symmetry about $-\frac{1}{2}$ is true for any $\nu \Rightarrow P_{-l-1}(z) = P_\nu(z)$

iii) Let's look at $\theta_v(z)$.

> $v = l = 0, 1, 2 \dots$

Since the poles at $t = \pm i$ vanish, the contour integral $\rightarrow 0$ but $(e^{v\pi N} - 1)$ also vanishes. So we need to look at the limiting behaviour to find the value.

$$\theta_l(z) = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right) P_l(z) + R_{l-1}(z)$$

↪ polynomial of degree $l-1$
in z .

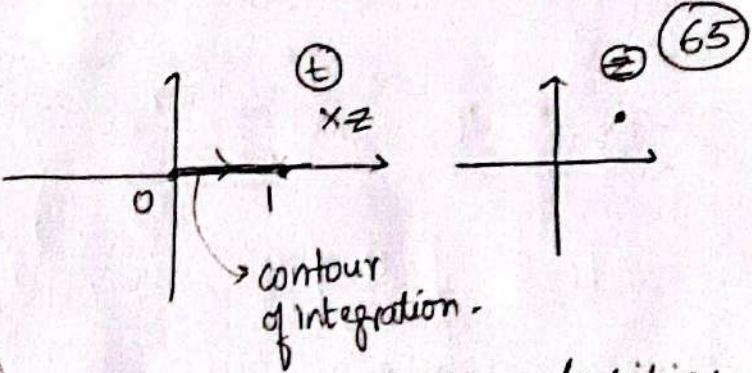
→ θ_l has a logarithmic singularity at $z = \pm i$. Remember pinching.

> $\theta_l(z)$ is singular for negative l .

> Also, $\theta_v(z) = \frac{P_l(z)}{v+l+1} + \text{reg part}$
↪ there are singularities in v plane.

$$> \theta_l(z) = \frac{1}{2} \int_{-1}^1 dt \frac{P_l(t)}{z-t}. \quad (l=0, 1, 2 \dots)$$

$$\text{Eg: } f(z) = \int_0^z \frac{dt}{t-z}$$

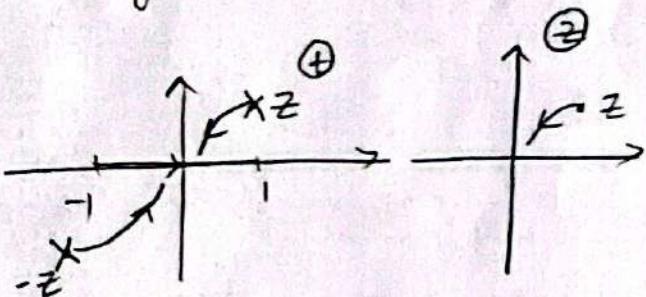


- > As pt. in z plane moves, the pole int plane moves. As long as the pole does not hit $0, 1$, the contour can be distorted to maintain the integral. If $z \rightarrow 0$ the contour goes to ∞ & this may cause the integral to diverge.
- > Such singularities are called "end point singularities". When a singularity of the integrand attacks an end point of integration

Solving, $\ln|t-z| \Big|_{t=0}^{t=1} = \ln \frac{z-1}{z} \rightarrow$ This explicitly shows the 2 singularities.

Therefore, we can recognize end point singularities without explicitly solving the integral.

$$\text{Eg: } f(z) = \int_{-1}^1 \frac{dt}{t^2 - z^2}$$



As $z \rightarrow \pm 1$ we have end point singularities. However even when $z \rightarrow 0$ the 2 poles "pinch off" the contour. This issue only occurs at $z \rightarrow 0$. This singularity is called a "pinched singularity".

Solving the integral,

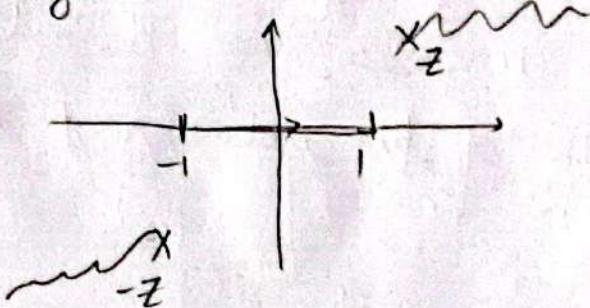
$$f(t) = \int dt \left(\frac{1}{t-z} - \frac{1}{t+z} \right)$$

$$f(t) = \frac{1}{z} \ln \frac{1-z}{1+z} \Rightarrow \text{singularities at } z = \pm 1 \neq 0.$$

But the simple pole occurs for all sheets except the principle sheet.

Eg: $f(t) = \int_{-1}^1 \frac{dt}{\sqrt{z^2-t^2}}$ singularities at $\pm z$ are branch points.

Try solving it.



$$\rightarrow \theta_l(z) = \frac{1}{2} \int_{-1}^1 \frac{dt P_l(t)}{z-t} \xrightarrow{\text{polynomial}} \text{no singularities.}$$

$l = 0, 1, 2, \dots$

\Rightarrow we have end point singularities that end up as log singularities.

Partial Differential Equations - V. Balakrishnan

Lec 23 Poisson's Equation

$$\nabla^2 f(\vec{r}) = g(\vec{r}) \rightarrow \text{eg: electrostatics } \nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r}')}{\epsilon_0}$$

∇^2 is a scalar operator \Rightarrow invariant under rotations.
 ↳ aka elliptic partial differential operator.

Green's function.

Let D_x be a differential operator acting on $f(x)$.

$D_x f(x) = g(x)$ where $g(x)$ is given.

In abstract notation, $D_x \rightarrow D$
 $f(x) \rightarrow |f\rangle$ Ket vector notation
 $g(x) \rightarrow |g\rangle$ from Bra-ket notation.
 Kets $\rightarrow | \rangle$ are vectors.
 Bras $\rightarrow \langle |$ are functionals.

$$\Rightarrow |f\rangle = D^{-1} |g\rangle + \sum_i c_i |h_i\rangle$$

where $D|h_i\rangle = 0$ ↳ linear comb. of all vectors $|h_i\rangle$
 such that when D acts on it, it is annihilated.

Therefore $D^{-1}|g\rangle$ is the particular integral.

$\& \sum_i c_i |h_i\rangle$ is the complementary fn (since $D|h_i\rangle = 0$ is homogeneous).

Going back to position space $f(x)$ is $\langle x | f \rangle$ \Rightarrow representative of f vector in x basis.

$$\Rightarrow f(x) = \int dx' G(x, x') g(x') + \sum_i c_i h_i(x)$$

Here $G(x, x')$ represents the D_x^{-1} and is called the Green's function of D_x . Explicitly,
 $G(x, x') = \langle x | D^{-1} | x' \rangle$

We have, $|f\rangle = D^\dagger |g\rangle + \sum_i c_i |h_i\rangle$

$$\Rightarrow \langle x | f \rangle = f(x) = \langle x | D^\dagger |g\rangle + \sum_i c_i \underbrace{\langle x | h_i \rangle}_{h_i(x)}$$

↓
insert a complete set of states
 $\int dx' |x'\rangle \langle x'| \rightarrow \text{Identity operator.}$

$$\Rightarrow \int dx' \underbrace{\langle x | D^\dagger |x'\rangle}_{G(x, x')} \underbrace{\langle x' | g \rangle}_{g(x')}$$

$$= \int dx' G(x, x') g(x')$$

Finding $G(x, x')$

$DD^\dagger = \mathbb{I} \rightarrow \text{identity operator.}$

$$\Rightarrow D_x G(x, x') = \langle x | x' \rangle = \delta(x - x').$$

$\Rightarrow D_x f(x) = g(x)$ can be fully solved by finding

$D_x G(x, x') = \delta(x - x')$ → Basically an impulse response.

The Boundary conditions would determine how much of the CF must be used in the full soln.

Eg: $\frac{d^2 f(x)}{dx^2} = g(x) \quad x \in [0, 1] ; f(0) = a, f(1) = b$

$$\frac{d^2 G(x, x')}{dx^2} = \delta(x - x') \Rightarrow \begin{aligned} &\text{in } 0 \leq x \leq x' \\ &\Rightarrow G(x, x') = A_1 x + A_2 \text{ since } \frac{d^2 G}{dx^2} = 0. \end{aligned}$$

2 BC & 4 constants.
But we know more about G .

$$\begin{aligned} &\text{in } x' < x \leq 1 \\ &\Rightarrow G(x, x') = A_3 x + A_4 \end{aligned}$$

→ We can also show that $G(x, x')$ is continuous. The second derivative has an ∞ jump & first derivative has a finite jump.

$$\frac{d^2 G(x, x')}{dx^2} = g(x - x')$$

$$\Rightarrow \int_{x'-\epsilon}^{x'+\epsilon} dx' \frac{d^2 G(x, x')}{dx^2} = \left. \frac{dG}{dx} \right|_{x'+\epsilon} - \left. \frac{dG}{dx} \right|_{x'-\epsilon} = 1 \quad \text{--- (1)}$$

→ Apply (1) to A_1, A_2, A_3, A_4 eqns. to get another B.C.

→ G must have a cusp (it is still continuous)
this gives the fourth B.C i.e continuity of G at x' .

$G \rightarrow \Lambda$
 $G' \rightarrow \text{?}$
 $G'' \rightarrow \text{?}$

$$A_1 x' + A_2 = A_3 x' + A_4 \quad \text{BC 4}$$

$$A_1 - A_3 = 1 \quad \text{BC 3}$$

Going back to Poisson's Equation.

$\nabla^2 \phi(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$; we know the soln. is Coulomb's law,

$$\Rightarrow \phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\vec{r}'} d^3 \vec{r}' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

Therefore in general

$$f(\vec{r}) = \frac{-1}{4\pi} \int_{\vec{r}'} d\vec{r}' \frac{g(\vec{r}')}{|\vec{r} - \vec{r}'|} \rightarrow \text{Need to derive this more formally.}$$

↓
B.C used is $\vec{r} \rightarrow \infty \Rightarrow G \rightarrow 0$

Formal solution to Poisson's Equation.

$$\nabla^2 f(\bar{r}) = g(\bar{r})$$

$$\Rightarrow f(\bar{r}) = \int d\bar{r}' G(\bar{r}, \bar{r}') g(\bar{r}'). + \text{if } \cancel{f} \text{ ignoring this}$$

$$\nabla_{\bar{r}}^2 G(\bar{r}, \bar{r}') = \delta^{(3)}(\bar{r} - \bar{r}')$$

> Since $\nabla_{\bar{r}}^2$ is a scalar function if $\bar{R} = \bar{r} - \bar{r}'$ we can write,

$\nabla_{\bar{R}}^2 = \nabla_{\bar{r}}^2$. Invariant to translation + rotation.

> RHS is a fun of only $\bar{r} - \bar{r}'$ so is G also only a fun of $\bar{r} - \bar{r}'$?

We need to ensure that the B.C is also translation invariant.

Therefore we need $\bar{r} \rightarrow \infty \Rightarrow \bar{G} \rightarrow 0$ ie natural B.C.

$$\Rightarrow \nabla_{\bar{r}}^2 G(\bar{r} - \bar{r}') = \delta^{(3)}(\bar{r} - \bar{r}')$$

$$\Rightarrow \boxed{\nabla_{\bar{R}}^2 G(\bar{R}) = \delta^{(3)}(\bar{R})} \quad \text{--- (1)}$$

$$G(\bar{R}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\bar{K} \cdot \bar{R}} \tilde{G}(\bar{K}) \xrightarrow{\text{defining a fourier transform in 3D.}}$$

$$\Rightarrow \tilde{G}(\bar{K}) = \int d^3R e^{-i\bar{K} \cdot \bar{R}} G(\bar{R})$$

$$\nabla_{\bar{R}} e^{i\bar{K} \cdot \bar{R}} = i\bar{K} e^{i\bar{K} \cdot \bar{R}} \Rightarrow \nabla^2 = -|k|^2 e^{i\bar{K} \cdot \bar{R}}$$

Remember,

$$\nabla_{\bar{r}} \bar{a} e^{i\bar{K} \cdot \bar{R}} = i\bar{K} \cdot \bar{a} e^{i\bar{K} \cdot \bar{R}}$$

$$\nabla_{\bar{x}} \bar{a} e^{i\bar{K} \cdot \bar{R}} = i\bar{K} \times \bar{a} e^{i\bar{K} \cdot \bar{R}}$$

Replace LHS in ① by ② & replace RHS by its analog of Eq ②. (5)

$$\Rightarrow \frac{1}{(2\pi)^3} \int_{\mathbf{k}} d^3k e^{i\bar{\mathbf{k}} \cdot \bar{\mathbf{R}}} [-\mathbf{k}^2 \tilde{G}(\mathbf{k})] = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} d^3k e^{i\bar{\mathbf{k}} \cdot \bar{\mathbf{R}}} - ③$$

Since $\bar{\mathbf{k}}$ forms a complete set in $\bar{\mathbf{R}}$ space they must be equal by components.

$$\Rightarrow -\mathbf{k}^2 \tilde{G}(\mathbf{k}) = 1$$

$$\Rightarrow \boxed{\tilde{G}(\mathbf{k}) = -\frac{1}{\mathbf{k}^2}}$$

③ can be rewritten as

$$\frac{1}{(2\pi)^3} \int_{\mathbf{k}} d^3k e^{i\bar{\mathbf{k}} \cdot \bar{\mathbf{R}}} [\tilde{G}(\mathbf{k})] = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} \frac{d^3k}{\mathbf{k}^2} e^{i\bar{\mathbf{k}} \cdot \bar{\mathbf{R}}}$$

$$\Rightarrow \boxed{G(\bar{\mathbf{R}}) = \frac{1}{(2\pi)^3} \int_{\mathbf{k}} \frac{d^3k}{\mathbf{k}^2} e^{i\bar{\mathbf{k}} \cdot \bar{\mathbf{R}}}}$$

Since $G(\bar{\mathbf{R}})$ is a scalar, we can use rotational invariance property & choose polar axis of $\bar{\mathbf{k}}$ space (ie $\hat{\mathbf{k}}_z$) along $\bar{\mathbf{R}} \Rightarrow \bar{\mathbf{k}} \cdot \bar{\mathbf{R}} = k \cos \theta$

$$\Rightarrow G(\bar{\mathbf{R}}) = \frac{1}{(2\pi)^3} \int_0^\infty \frac{k^2 dk}{k^2} \int_0^{2\pi} d\phi \int_0^\pi e^{ikR \cos \theta} \sin \theta d\theta.$$

$$\downarrow \quad \quad \quad \downarrow$$

$$-\int_1^{-1} e^{ikR \cos \theta} d(\cos \theta)$$

\downarrow case is even

$$\frac{e^{ikR} - e^{-ikR}}{ikR} \leftarrow \int_{-1}^1 e^{ikRx} dx. \quad \leftarrow \int_{-1}^1 e^{ikR \cos \theta} d(\cos \theta)$$

$$\Rightarrow G(\bar{R}) = -\frac{1}{2\pi^2 R} \underbrace{\int_0^\infty dk \frac{\sin kR}{kR}}_{\text{Dirichlet's integral}}$$

$$= -\frac{1}{4\pi R}$$

$$G(\bar{R}) = \boxed{-\frac{1}{4\pi |\bar{r}-\bar{r}'|}} \rightarrow \begin{aligned} &\text{Coulomb potential!} \\ &\text{Free space green's function} \\ &\text{for Poisson Equation.} \end{aligned}$$

$$\Rightarrow f(\bar{r}) = \boxed{-\frac{1}{4\pi} \int d^3 r' \frac{g(\bar{r}')}{|\bar{r}-\bar{r}'|}}$$

$\text{CF} \rightarrow 0$ due to BC that
 $G \rightarrow 0$ as $\bar{r} \rightarrow \infty$

$\frac{1}{|\bar{r}-\bar{r}'|}$ is called the Coulomb kernel.

$$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{r_s} \sum_{l=0}^{\infty} \left(\frac{r'}{r} \right)^l P_l(\cos \gamma) \quad \begin{aligned} &\text{where } \gamma \text{ is angle b/w } \bar{r}, \bar{r}' \\ &\& r_s \text{ is } r \text{ if } r > r' \text{ or} \\ &\& r_s \text{ is } r' \text{ if } r' > r. \end{aligned}$$

Legendre polynomial.

$$= \sum_{l=0}^{\infty} \frac{1}{r_s} \left(\frac{r_s}{r} \right)^l \sum_{m=-l}^l Y_{lm}(\theta, \phi) \underbrace{Y_{lm}^*(\theta', \phi')}_{\text{spherical harmonics.}}$$

Greens function of Poisson's Eqn. in d spatial dimensions

- Harmonic fns in 1D are lines $ax+b$ since $\frac{d}{dx^2}(ax+b)=0$.
- These fns have the property that their value at any pt is the mean of all points around them that lie on a sphere centered at that point. In higher dimensions the shapes are not trivial like in 1D. Since we can have $\frac{d^2f}{dx^2} = -\frac{d^2f}{dy^2}$

$$f(x,y) = \frac{f(x+\Delta x, y) - f(x-\Delta x, y) + f(x, y+\Delta y) - f(x, y-\Delta y)}{4}$$

from this we can derive $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ in the limit. (Proof)

- $\nabla^2 G(\underline{R}) = \delta^{(d)}(\underline{R})$ \underline{R} is a vector in d dimensions.

$$G(\underline{R}) = \frac{1}{(2\pi)^d} \int \frac{dk e^{i\underline{k} \cdot \underline{R}}}{k^2}$$

$$d=2 \Rightarrow dk = k dk d\phi \Rightarrow \frac{1}{(2\pi)^d} \int_0^\infty \frac{k dk e^{i\underline{k} \cdot \underline{R}}}{k^2}$$

when $k=0$, \int is unbounded. \Rightarrow Blows up. aka infrared divergence since it happens at low frequencies $\Rightarrow k \downarrow$
similarly we could have a UV divergence.

Polar coordinates in d dimensions. (Hyperspherical polar coordinates)

$$K, \quad 0 \leq K < \infty$$

$$\theta_1, \theta_2, \theta_3, \dots, \theta_{d-2} \in [0, \pi]$$

$$\phi \quad 0 \leq \phi \leq 2\pi$$

$$d^d K = K^{d-1} (\sin \theta_1)^{d-2} (\sin \theta_2)^{d-3} \dots (\sin \theta_{d-2})^1 dK d\theta_1 d\theta_2 \dots d\theta_{d-2} d\phi$$

Choose \underline{R} along polar axis of \underline{k}

$$\Rightarrow \underline{K} \cdot \underline{R} = |K| |R| \cos \theta_1$$

$\Rightarrow \int_0^\pi d\theta (\sin \theta)^r$ is left for all other angles & it is solvable.

Only θ_1 is problematic since $e^{iK \cos \theta_1}$ is also present & it can be written in terms of Bessel fns. Then put that in $\int dk$.

In the end we get,

$$G(\underline{R}) = (\text{const}) \int_0^\infty dk \ k^{\frac{d-4}{2}} J_{\frac{d-1}{2}}(KR) \quad \xrightarrow{\text{Bessel fn of first kind}}$$

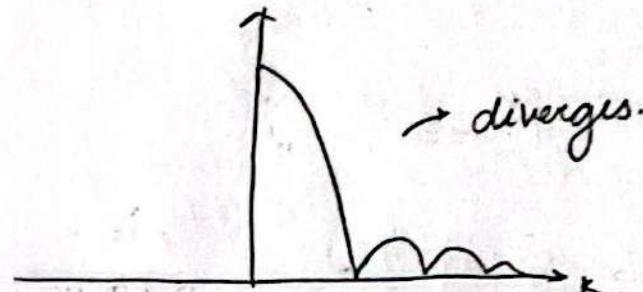
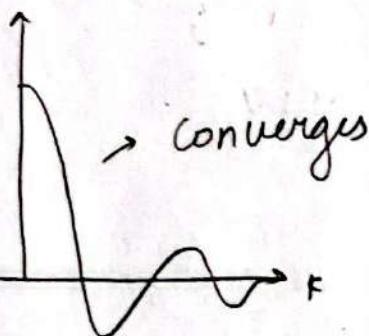
$$\int_0^\infty dk \ k^r < \infty \text{ if } r < -1 ; \text{ if } r = -1 \Rightarrow \log \text{divergence.}$$

$$\text{When } k \rightarrow \infty \quad J \propto \frac{1}{\sqrt{k}} \rightarrow \frac{d-4-1}{2} \frac{1}{2} < -1 \text{ for convergence -}$$

$$\Rightarrow d < 3 \quad \text{but we already saw that for } d=3 \text{ we have a convergence.}$$

Therefore, the problem is that the power counting approach ignores the changes in sign of the Bessel fn.

In 3D, $\int_0^\infty \frac{\sin kx}{k} dk$ is a sinc fn & the integral is finite. However if we ignore the sign of sinc, say $\int_0^\infty \frac{|\sin kx|}{k} dk$, num is bounded by 1 & the integral diverges logarithmically due to $\frac{1}{k}$.



→ So if $d=3$ we barely made it, when $d>3$ it diverges & it's called an ultraviolet divergence since it diverges for large k .

How about small k ?

$\int_0^\infty dk k^r$ converges when $r>-1$ since $r=-1 \Rightarrow \log k \rightarrow -\infty$

→ looks like $\int_0^\infty dk k^r$ never exists.

But here the Bessel fn saves us.

$$J_r(z) = \sum_{n=0}^{\infty} \frac{(\frac{z}{2})^{r+2n}}{\Gamma(r+n+1)} (-1)^n \text{ which for small } k \text{ is } \propto k^{\frac{d-4}{2}-1}$$

$$\Rightarrow r = \frac{d-4}{2} + \frac{d}{2} - 1 > -1 \Rightarrow d > 2$$

\therefore Roughly speaking or naive power counting implies the integral for $G(R)$ is convergent when $2 < \text{Re}(d) < 3$. Here d is replaced with $\text{Re}(d)$ since its imag. part corresponds to some oscillatory term. Let d be a complex variable although we only care about $d=2, 3, 4, 2, \dots$

Therefore we solve for $G(R)$ b/w 2 & 3 & use analytic continuation for higher orders if possible. This is called dimensional regularization.

Dimensional Regularization

Find $G(R)$ as a fn. of d in $d \in (2, 3)$ & use analytic continuation for all other values.

Going back to $\nabla^2 G^{(d)}(R) = \delta^{(d)}(R)$ \rightarrow translation invariant since $\delta(d)$ is spherically symmetric.

Integrating over a hypersphere

$$\int dV \nabla \cdot \nabla G^{(d)} = 1$$

Use Div Thm. $\int_S \nabla G^{(d)} \cdot dS$ & ∇G we know is radially outward due to spherical symmetry. Therefore $\nabla G \cdot dS$ has no cosine term.

$$\Rightarrow \int_S \nabla G^{(d)} \cdot dS = \frac{\partial G^{(d)}}{\partial R} \int_S dS \hookrightarrow \text{integral of hypersphere of radius } R.$$

Since radial component of gradient is the only one that survives due to spherical symmetry of G

$$\frac{dG}{dR} \stackrel{(d)}{\int} ds = \frac{dG}{dR} \stackrel{(d)}{\cdot} \frac{2\pi^{\frac{d}{2}} R^{d-1}}{\Gamma(\frac{d}{2})}$$

\longrightarrow surface area of sphere of d dimensions.

$$\Gamma(z+1) = z \Gamma(z)$$

$$\Gamma(n) = (n-1)!$$

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \text{ if } \operatorname{Re}(z) > 0$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 1 \quad \text{Therefore } 0! = 1$$

$$\Gamma(\frac{1}{2}) = \sqrt{\pi} \Rightarrow \Gamma(\frac{3}{2}) = \frac{1}{2} \sqrt{\pi}$$

$$d=3 \Rightarrow \frac{2\pi^{\frac{3}{2}} \cdot R^2}{\frac{1}{2} \Gamma(\frac{1}{2})} = 4\pi R^2$$

$$d=2 \Rightarrow 2\pi R$$

$$\Rightarrow \frac{dG}{dR} \stackrel{(d)}{=} \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}} R^{d-1}}$$

What is the volume of hypersphere?

$$S_d(R) = \frac{2\pi^{\frac{d}{2}} R^{d-1}}{\Gamma(\frac{d}{2})}$$

$$V_d(R) = \frac{\pi^{\frac{d}{2}} R^d}{\Gamma(\frac{d}{2}+1)}$$

$$\frac{dV_d(R)}{dR} = \frac{d + \frac{d}{2} R^{d-1}}{\Gamma(\frac{d}{2}) \cdot \frac{d}{2}}$$

$$= \frac{2\pi^{\frac{d}{2}} R^{d-1}}{\Gamma(\frac{d}{2})} = S_d(R)$$

$$\Rightarrow \int_R^\infty dR' \frac{dG(R')}{dR'} = -G(d)(R) \quad \text{since } G \text{ is 0 at infinity.}$$

$$\Rightarrow G(d)(R) = - \int_R^\infty dR' R'^{1-d} \cdot \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}$$

$$\Rightarrow G(d)(R) = - \frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}} \frac{(R')^{2-d}}{2-d} \Big|_{R'=R}^{R'=\infty}$$

$d \geq 2 \Rightarrow S$ vanishes at ∞ .

$$\Rightarrow G^{(d)}(R) = \frac{-\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}(d-2)R^{d-2}}$$

Free space
Green's function for
Laplace's Equation
in d dimensions.

It is an analytic function in d with $d=2$ pde \rightarrow IR divergence.
This is only valid when $2 < d$. UV divergence disappeared? Yes.

$$d=3 \Rightarrow G^{(3)}(R) = -\frac{1}{4\pi R}$$

$$\text{We know } \frac{dG^{(2)}}{dR} = \frac{1}{2\pi R}$$

$$\Rightarrow G^{(2)} = \frac{\log e^R}{2\pi}$$

potential of an ∞ line of charge.

$$G(R) = \int \frac{1}{S(R)} dR.$$

Applying the prescription of regularization for $d=2$.
Expand $G^{(d)}(R)$ in terms of the singular part & a regular part.
Which is essentially a Taylor expansion of $\frac{1}{R^{d-2}}$ around $d=2$.

Let $R^{-(d-2)} = e^{-(d-2)\log R}$ \rightarrow expand as Taylor series.

$$\Rightarrow G^{(d)}(R) = \frac{-1}{2\pi(d-2)} \left[1 - (d-2)\log R + O((d-2)^2) \right]$$

$$= -\frac{1}{2\pi(d-2)} + \frac{\log R}{2\pi}$$

Regularization says ignore singular part.

$$\Rightarrow G^{(d)}(R) = \frac{\log R}{2\pi}$$

The Diffusion Equation -lec 25

$p(\vec{r}, t)$ is the conc. of solute in solvent. Follows certain laws.

$$① \frac{\partial p}{\partial t} + \nabla \cdot \vec{J} = 0 \rightarrow \text{Ficks I Law} \quad (\text{Eqn. of continuity}).$$

$$② \vec{J}(\vec{r}, t) = -D \nabla p(\vec{r}, t) \rightarrow$$

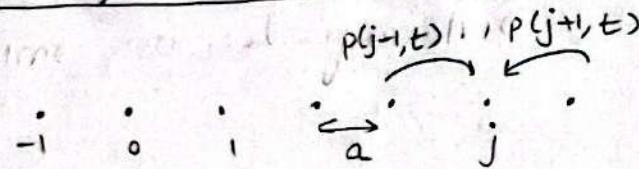
Flow of solute current is from high conc. to low concentration. $D \rightarrow$ diffusion constant. (L^2/t dimension)

$$\Rightarrow \boxed{\frac{\partial p}{\partial t} = D \nabla^2 p} \rightarrow \text{Diffusion Equation. (macroscopic perspective)}$$

Since it is first order in time, it is not time reversible. Note that M.E are also f.o in time but also f.o in space \Rightarrow time reversible.
 1D in time & 2D in space \Rightarrow parabolic DE \Rightarrow not time reversible.
 2D in time & 2D in space \Rightarrow elliptic DE \Rightarrow time reversible

p may be temperature, D the diffusivity $\Rightarrow \frac{\partial u}{\partial t} = \alpha \nabla^2 u$ $\xrightarrow{\text{temp}} \text{diffusivity}$.

Derivation / Relationship with Random Walks



$$\frac{dP(j, t)}{dt} = \frac{\lambda}{2} [P(j+1, t) + P(j-1, t) - 2P(j, t)]$$

50% prob from left or right.

$\lambda \rightarrow$ rate at which jumps occur

$P(j, t)$ goes from probability to probability density.
 When $a \rightarrow 0$ & $\lambda \rightarrow \infty$

$$\begin{aligned} \Rightarrow \frac{dP(ja, t)}{dt} &= \frac{\lambda}{2} [P(ja+a, t) + P(ja-a, t) - 2P(ja, t)] \\ &= \frac{a^2 \lambda}{2} \left[\frac{P(ja+a, t) - P(ja, t)}{a} - \frac{(P(ja, t) - P(ja-a, t))}{a} \right] \end{aligned}$$

$$= \frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 p(x, t)}{\partial x^2} \text{ where, } D = \lim_{\substack{\lambda \rightarrow \infty \\ a \rightarrow 0}} \left(\frac{1}{2} \lambda a^2 \right)$$

In 3D $\frac{\partial P(\vec{r}, t)}{\partial t} = D \nabla^2 P(\vec{r}, t) \rightarrow \text{microscopic perspective.}$

B.C. $\Rightarrow p(\vec{r}, t) \rightarrow 0$ as $\vec{r} \rightarrow \infty$.

I.C. $\Rightarrow p(\vec{r}, 0) = \delta^{(d)}(\vec{r})$ in d dimensions \rightarrow gives the Green's function.

> Since t goes from 0 to ∞ we use Laplace transform in t .

$$\mathcal{L}[p(\vec{r}, t)] = \tilde{p}(\vec{r}, s)$$

$$\Rightarrow s\tilde{p}(\vec{r}, s) - p(\vec{r}, 0) = D \nabla^2 \tilde{p}(\vec{r}, s)$$

$$\Rightarrow (s - D \nabla^2) \tilde{p}(\vec{r}, s) = \delta^{(d)}(\vec{r}).$$

in d dimensions.

> Since \vec{r} goes from $-\infty$ to ∞ we use Fourier transform in \vec{r}

$$\Rightarrow \tilde{p}(\vec{r}, s) = \frac{1}{(2\pi)^d} \int d^d k \tilde{f}(\vec{k}, s) e^{i\vec{k} \cdot \vec{r}}$$

$$\tilde{f}(\vec{k}, s) = \int d^d r \tilde{p}(\vec{r}, s) e^{-i\vec{k} \cdot \vec{r}}$$

$$\Rightarrow (s + D k^2) \tilde{f}(\vec{k}, s) = 1$$

$$\Rightarrow \tilde{f}(\vec{k}, s) = \frac{1}{s + D k^2}$$

> Inverting F.T would be painful.

> It is much easier first

$$\Rightarrow f(\vec{k}, t) = \mathcal{L}^{-1} \frac{1}{s + D k^2} = e^{-D k^2 t}$$

$$p(r, t) = \frac{1}{(2\pi)^d} \int d^d k e^{ik \cdot r} e^{-Dk^2 t}$$

$$k^2 = k_1^2 + k_2^2 + \dots + k_d^2$$

It is much easier in Cartesian.

$$\Rightarrow p(r, t) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} dk_1 e^{ik_1 x_1 - Dk_1^2 t} \cdot \int_{-\infty}^{\infty} dk_2 e^{ik_2 x_2 - Dk_2^2 t} \cdots \int_{-\infty}^{\infty} dk_d e^{ik_d x_d - Dk_d^2 t}$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_1 e^{-Dt} \left(k_1^2 - i \frac{k_1}{Dt} x_1 + \frac{i^2 x_1^2}{4(Dt)^2} - \frac{i^2 x_1^2}{4(Dt)^2} \right)$$

$$\Rightarrow \frac{e^{-x_1^2}}{2\pi} \int_{-\infty}^{\infty} dk_1 e^{-Dt \left(k_1 - \frac{i x_1}{2Dt} \right)^2} \quad \text{exchange variables}$$

$$= \frac{e^{-\frac{x_1^2}{4Dt}}}{2\pi} \cdot \frac{\sqrt{\pi}}{\sqrt{Dt}}$$

$$= \frac{e^{-\frac{x_1^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

$$\Rightarrow p(r, t) = \boxed{\frac{e^{-\frac{r^2}{4Dt}}}{(4\pi Dt)^{d/2}}}$$

This is in fact the Green's function of the diffusion equation.

Gaussian in r & the variance increases linearly in time. Wow.

$$\text{Recall Gaussian } f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2}$$

$$\boxed{\langle r(t)^2 \rangle = 2dDt}$$

$$\text{Zero mean Gaussian} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

mean square displacement $\propto t$.

$$\Rightarrow \sigma^2 = 2Dt \text{ in 2 dimensions.}$$

$$\text{In general } \sigma^2 = 2dDt$$

Green's function for diffusion equation.

$$\left(\frac{\partial}{\partial t} - D \nabla^2 \right) G(\vec{r}, t) = \delta^{(d)}(\vec{r}) \delta(t) \rightarrow \text{But this is what we just solved.}$$

Suppose $p_{\text{init}}(\vec{r}, 0)$ given. Integrate $G(\vec{r}, t)$ to get $p(\vec{r}, t)$

$$\Rightarrow p(\vec{r}, t) = \frac{1}{(4\pi D t)^{d/2}} \int_{\vec{r}'} e^{-\frac{(\vec{r}-\vec{r}')^2}{4Dt}} p_{\text{init}}(\vec{r}', 0) d^d r'$$

In general given $p_{\text{init}}(\vec{r}, t_0)$ if $t > t_0$.

$$\Rightarrow p(\vec{r}, t) = \frac{1}{(4\pi D(t-t_0))^{d/2}} \int_{\vec{r}'} e^{-\frac{(\vec{r}-\vec{r}')^2}{4D(t-t_0)}} p_{\text{init}}(\vec{r}', t_0) d^d r'$$

- > Say we are in 1D & want a finite B.C. \Rightarrow particles reflect off the boundary \Rightarrow current at boundary = 0 like open circuit. $\Rightarrow \frac{\partial P}{\partial x} = 0$ or in 3D $\nabla P = 0$ at boundary.
- > If particle disappears at the ends $\Rightarrow P=0$ at the boundary like S.C.
- > If P at x_{boundary} is 0 & $\frac{\partial P}{\partial x}$ is 0 at boundary is P identically zero at the boundary? I think yes although just inside the boundary it is not 0. It could be a discontinuous jump maybe?

Rec 26 - Diffusion Equation contd.

$$\frac{\partial p(\bar{r}, t)}{\partial t} = D \nabla^2 p(\bar{r}, t)$$

$$p(\bar{r}, t) = \frac{1}{(4\pi D t)^{3/2}} e^{-\frac{r^2}{4Dt}}$$

Given $p_{\text{init}}(\bar{r}, t')$

$$p(\bar{r}, t) = \frac{1}{[4\pi D(t-t')]}^{1/2} \int d^3 r' e^{-\frac{-(\bar{r}-\bar{r}')^2}{4D(t-t')}} p_{\text{init}}(\bar{r}', t')$$

For a free particle the Schrodinger Eqn is the same.

$$\frac{\partial \psi(\bar{r}, t)}{\partial t} = \frac{i\hbar}{2m} \nabla^2 \psi(\bar{r}, t) \quad D = \frac{i\hbar}{2m}$$

$$\Rightarrow \psi(\bar{r}, t) = \int d^3 r' \underbrace{\frac{e^{\frac{i m (\bar{r}-\bar{r}')^2}{2\hbar(t-t')}}}{[2\pi \frac{i\hbar}{m}(t-t')]}^{3/2}}_{\text{Propagator } k(\bar{r}, t; \bar{r}', t')} \psi(\bar{r}', t')$$

Applying finite Boundaries.

1-D

$$\frac{\partial p(x, t)}{\partial t} = D \frac{\partial^2 p(x, t)}{\partial x^2}$$

$$\begin{array}{c} \leftarrow \\ \boxed{1} \\ a \\ \text{absorbing B.C} \\ p|_{x=b} = 0 \end{array}$$

$$\begin{array}{c} \vdots \\ b \\ \text{No flux} \quad \left\{ \begin{array}{l} \text{Reflecting B.C} \\ \Rightarrow \frac{\partial p}{\partial x}|_{x=b} = 0 \end{array} \right. \end{array}$$

Semi infinite region ($-\infty \leq x \leq b$)

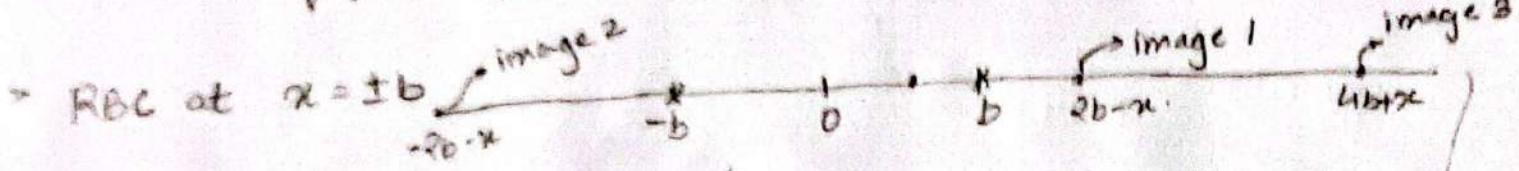
Use method of images & uniqueness.

$$\text{RBC at } x=b \Rightarrow p(\bar{r}, t) = \frac{1}{\sqrt{4\pi D t}} \left[e^{-\frac{x^2}{4Dt}} + e^{-\frac{(2b-x)^2}{4Dt}} \right]$$

↳ satisfies the B.C

Absorbing BC at b

$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \left[e^{\frac{x^2}{4Dt}} - e^{-\frac{(2b-x)^2}{4Dt}} \right] \quad x=b \Rightarrow \text{annihilation.}$$



We have ∞ no. of images.

$$\int dx p(x,t) = 1 \text{ for all } t > 0$$

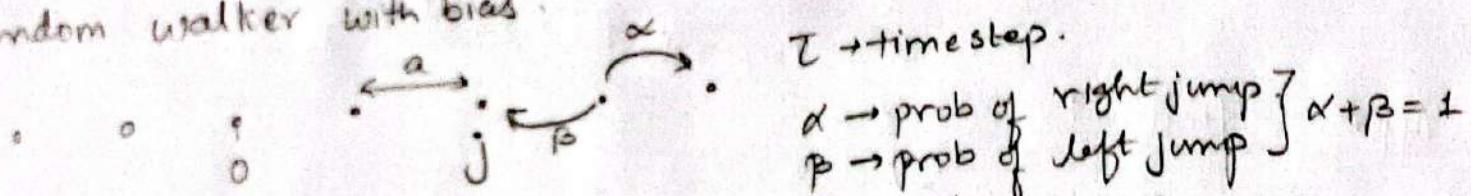
$$p(x,t) = \frac{1}{\sqrt{4\pi Dt}} \sum_{n=-\infty}^{\infty} e^{-(x+2nb)^2/4Dt} \quad \text{for RBC}$$

$$p(x,t) = \frac{1}{4\pi Dt} \sum_{n=-\infty}^{\infty} (-1)^n e^{-(x+2nb)^2/4Dt} \quad \text{for ABC} \rightarrow \int_{-b}^b dx p(x,t) = S(t)$$

(survival probability)
 $t=0 \Rightarrow S(t)=1$
 $t=\infty \Rightarrow S(t)=0$

Diffusion with drift \Rightarrow (diffusion under a constant force)

Random walker with bias



$$p(ja, n\tau) = \alpha P(ja-a, (n-1)\tau) + \beta P(ja+a, n\tau-\tau)$$

$$p(ja, n\tau) - p(ja, n\tau-\tau) = \alpha P(ja-a, (n-1)\tau) + \beta P(ja+a, n\tau-\tau) - P(ja, n\tau-\tau)$$

divide by τ & apply $\alpha \rightarrow 0$; $\tau \rightarrow 0 \Leftrightarrow \alpha - \beta \rightarrow 0$ (bias $\rightarrow 0$).

$$\Rightarrow \frac{\alpha^2 \alpha}{\tau} \rightarrow D \Rightarrow \frac{\alpha(\alpha - \beta)}{\tau} \rightarrow c \quad ja \rightarrow x, n\tau \rightarrow t$$

$$\boxed{\frac{\partial p(x,t)}{\partial t} = -c \frac{\partial p(x,t)}{\partial x} + D \frac{\partial^2 p(x,t)}{\partial x^2}}$$

avg. drift — avg. diffusion constant
 velocity

Smoluchowski equation

Sedimentation (Diffusion with drift)

$$\frac{\partial p(z,t)}{\partial t} = -c \frac{\partial P}{\partial z} + D \frac{\partial^2 p}{\partial z^2}$$

Corresponding continuity equation.

$$\frac{\partial P}{\partial t} + \frac{\partial j}{\partial z} = 0 ; \text{ where } j(z,t) = c p(z,t) - D \frac{\partial p(z,t)}{\partial z}$$

Drift velocity $c = -v$

$$\text{B.C. at } z=0 \text{ is } j(0,t) = 0 \Rightarrow v P(z,t) - D \frac{\partial p(z,t)}{\partial z} \Big|_{z=0} = 0$$

In thermal equilibrium $p(z,t) \rightarrow p(z) \Rightarrow \frac{\partial p(z,t)}{\partial t} = 0$.

$\Rightarrow \frac{\partial j}{\partial z} = 0 \Rightarrow j \text{ is independent of } z \Rightarrow \text{current is same everywhere.}$
 but at $z=0$ it is 0 so it must be 0 everywhere.

$$D \frac{dp}{dz} + V p = 0 \Rightarrow p \propto e^{-V/D z}$$

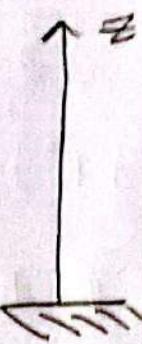
$$\Rightarrow p(z) \propto e^{-\frac{mgz}{k_B T}} \rightarrow \text{Barometric distribution of air.}$$

$$\Rightarrow \frac{V}{D} = \frac{mg}{k_B T} \Rightarrow D = \frac{V k_B T}{mg}$$

Using Stokes' relation $6\pi R \eta v = mg$.

$\Rightarrow D = \frac{k_B T}{6\pi R \eta} \rightarrow$ derived by Einstein & earlier by Sutherland & Nernst.

where $\eta \rightarrow \text{viscosity}$
 $R \rightarrow \text{radius}$
 $v \rightarrow \text{terminal vel.}$
 for spherical particles.
 with stick bound. cond.



Helmholtz Equation - Lec 29

$$(\nabla^2 + k^2) f = g$$

- >If $\bar{r} \in R$ then the homogeneous HH equation $(\nabla^2 + k^2) u = 0$ gives the normal modes of vibration for given B.C (say $u=0$ on the boundary of $R \rightarrow$ Dirichlet boundary condition).
- Say u is the z displacement of points on a drum, $\exists u=0 \Rightarrow$ clamping the boundary of the drum. Normal modes are the possible values of $k^2 \rightarrow$ eigenvalue problem. $\nabla^2 u = -k^2 u$.

Green's Function (free space) for HH Eqn

The physical problem is of non-relativistic (low speed) scattering of a quantum mechanical particle from a fixed static potential.

Assume there is a scattering face center with some central potential around it. The incoming particle must be described by a wave function with some incident wave vector. Let incident Ψ_{inc} be a plane wave state.

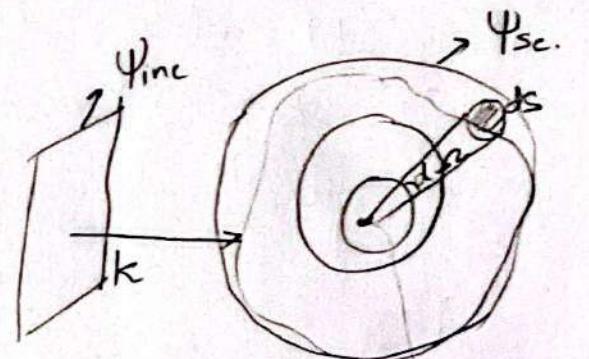
$$\Rightarrow \Psi_{\text{inc}}(\bar{r}) = e^{i\bar{k} \cdot \bar{r}}$$

Outgoing $\Psi_{\text{sc}}(\bar{r})$ are spherical waves.

$$\Psi(\bar{r}) = \Psi_{\text{inc}}(\bar{r}) + \Psi_{\text{sc}}(\bar{r})$$

We want to know what is the flux at some solid angle $d\Omega$.

$$I = \frac{\pi^2 k^2}{2m}$$



$$\Psi_{\text{sc}}(\bar{r}) \xrightarrow{\bar{r} \rightarrow \infty} \frac{e^{ik\bar{r}}}{\bar{r}} f(k, \theta, \phi)$$

since $\bar{k} \parallel \bar{r}$ as $\bar{r} \rightarrow \infty$ for spherical waves.

If \vec{k} is $= +\hat{z} \Rightarrow \psi_{sc}$ has no dependence on ϕ . (Symmetry). (2)

$$r \rightarrow \infty \Rightarrow kr \gg 1$$

$$\Rightarrow \psi(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} + f(k, \theta) \frac{e^{ikr}}{r} \rightarrow \text{soln. of Sch. E we are looking for.}$$

↑
Scattering amplitude.

$$\frac{\text{Flux through } ds}{\text{incident flux}} = d\sigma \rightarrow \text{differential cross section.}$$

$$\bar{j}_{inc} = \frac{\hbar}{2mi} (\psi_{inc}^* \nabla \psi_{inc} - \psi_{inc} \nabla \psi_{inc}^*) \rightarrow \text{std. formula in BM.}$$

S.E (time dep version)

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{i\hbar}{2m}\right)^2 \nabla^2 \psi + V(r) \psi \times \psi^* \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Subtract}$$

$$\Rightarrow i\hbar \frac{\partial \psi^*}{\partial t} = \left(\frac{i\hbar}{2m}\right)^2 \nabla^2 \psi^* + V(r) \psi \times \psi$$

$$\Rightarrow |\psi|^2 = \left(\frac{i\hbar}{2m}\right)^2 (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = -\nabla \cdot j \Rightarrow j \text{ is as above.}$$

prob
dens.

$$\Rightarrow \bar{j}_{inc} = \frac{\hbar \vec{k}}{m} \text{ since } \psi_{inc} = e^{i\vec{k} \cdot \vec{r}} \quad \begin{array}{l} \text{momentum } \vec{p} \\ \text{and} \end{array} \quad \nabla \psi_{inc} = i\vec{k} e^{i\vec{k} \cdot \vec{r}}$$

$$\nabla \psi_{inc}^* = -i\vec{k} e^{i\vec{k} \cdot \vec{r}}$$

Incident
flux

Scattered flux (only looking at radial component).

$$\bar{j}_{sc} = \frac{\hbar}{2mi} \left(\psi_{sc}^* \frac{\partial \psi_{sc}}{\partial r} - \psi_{sc} \frac{\partial \psi_{sc}^*}{\partial r} \right) \quad \psi_{sc} = f(k, \theta) \frac{e^{ikr}}{r}$$

$$\Rightarrow \bar{j}_{sc} = \frac{\hbar}{2mi} |f(k, \theta)|^2 \cdot \left[\frac{e^{-ikr}}{r} \left\{ ike^{\frac{ikr}{r}} - \frac{e^{ikr}}{r^2} \right\} - \frac{e^{ikr}}{r} \left\{ -ike^{-\frac{ikr}{r}} - \frac{e^{-ikr}}{r^2} \right\} \right]$$

$$\bar{j}_{sc} = \frac{\hbar^2}{2m\epsilon} |f|^2 \left[\frac{2ik}{r^2} \right]$$

$$\bar{j}_{sc} = \frac{\hbar k}{m} \frac{|f|^2}{r^2}$$

$$\bar{j}_{sc} \cdot dS = \frac{\hbar k}{m} |f|^2 d\Omega$$

$$\Rightarrow d\sigma = \frac{\frac{\hbar k}{m} |f|^2 d\Omega}{\frac{\hbar k}{m}} = |f|^2 d\Omega$$

Therefore it does not depend on amplitude of incident wave & we could just assume that it is 1 which is what we did here.

\therefore Differential cross section $\boxed{\frac{d\sigma}{d\Omega} = |f|^2}$

find this.

Total cross section $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_{-1}^{+1} |f|^2 d(\cos\theta)$

(notice it has dimension of area) \hookrightarrow same as $\sin\theta d\theta$ from $0 \rightarrow \pi$

SE $\hookrightarrow \frac{\hbar^2 k^2}{2m}$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}) + \lambda V(\vec{r}) \psi(\vec{r}) = E \psi(\vec{r})$$

\hookrightarrow coupling constant

$$\nabla^2 \psi(\vec{r}) + \frac{2mE}{\hbar^2} \psi(\vec{r}) = \lambda \left(\frac{2m}{\hbar^2} V(r) \right) \psi(\vec{r})$$

$$\hookrightarrow (\nabla^2 + k^2) \psi(\vec{r}) = \lambda U(r) \psi(\vec{r}) \rightarrow \text{looks like Helmholtz. but RHS is unknown.}$$

Say we know RHS.

$$\Rightarrow \psi(\vec{r}) = \underbrace{e^{i\vec{k} \cdot \vec{r}}}_{C.F} + \underbrace{\lambda \int d^3 r' G(\vec{r}, \vec{r}') U(\vec{r}') \psi(\vec{r}')}_{P.I}$$

→ We are able to write down the C.F because we know that we are looking for solns. of the form $e^{i\vec{R} \cdot \vec{r}} + f(k, \theta) \frac{e^{ikr}}{r}$.

→ So now we find $G(\vec{r}, \vec{r}')$. This does not solve for $\Psi(\vec{r})$ but gives us an integral equation for $\Psi(\vec{r})$.

Finding the Green's Function

$$(\nabla^2 + k^2) G(\vec{r}, \vec{r}') = \delta^{(3)}(\vec{r} - \vec{r}') \quad \text{with } G(\vec{r}, \vec{r}') \rightarrow 0 \text{ as } r \rightarrow \infty$$

Translation inv

$$\Rightarrow G(\vec{r}, \vec{r}') \rightarrow G(\vec{r} - \vec{r}')$$

$$\Rightarrow (\nabla^2 + k^2) G(\vec{R}) = \delta^{(3)}(\vec{R}).$$

Use Fourier Transform

$$\Rightarrow G(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3 q e^{i\vec{q} \cdot \vec{R}} \tilde{G}(\vec{q}).$$

$$\delta^{(3)}(\vec{R}) = \frac{1}{(2\pi)^3} \int d^3 q e^{i\vec{q} \cdot \vec{R}}$$

→ it converts ∇^2 to $-q^2$ since $e^{i\vec{q} \cdot \vec{R}}$ forms a complete set we can equate the coefficients.

$$\Rightarrow (-q^2 + k^2) \tilde{G}(\vec{q}) = 1$$

$$\Rightarrow \tilde{G}(\vec{q}) = \frac{-1}{q^2 - k^2}$$

$$\Rightarrow G(\vec{R}) = \frac{-1}{(2\pi)^3} \int d^3 q \frac{e^{i\vec{q} \cdot \vec{R}}}{q^2 - k^2}$$

Choose polar axis in \vec{q} space along \vec{R}

$$= -\frac{1}{4\pi^2} \int_0^\infty \frac{q^2 dq}{q^2 - k^2} 2i \frac{\sin qR}{iqR} = \frac{-1}{2\pi^2} \int_0^\infty dq \frac{q \sin qR}{q^2 - k^2}$$

We need to shift the poles off the real axis by maintaining the boundary conditions.

→ poles at $+k$ & $-k$.

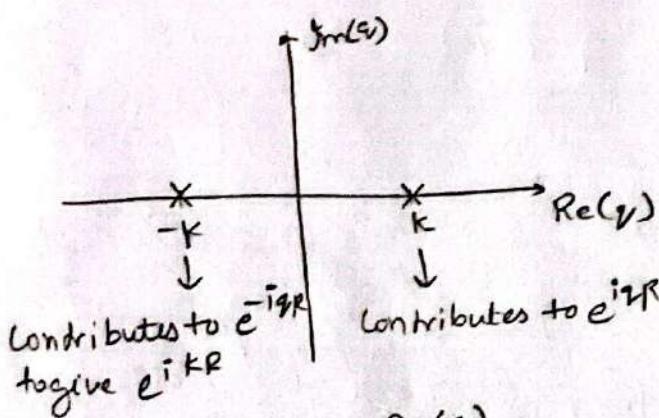
$$\Rightarrow G(R) = \frac{-1}{2\pi^2 R} \int_0^\infty dq \frac{q \sin qR}{q^2 - k^2} \xrightarrow{\text{even } f(q)} \text{even } f(q) \Rightarrow \text{change limits to } -\infty \text{ to } \infty$$

no longer
af. of R

The solution from here on is complex analysis magic...

We are looking for solns. of the form e^{ikR} .

$$\text{Rewriting sing} \rightarrow \frac{e^{iqR} - e^{-iqR}}{2iq}$$

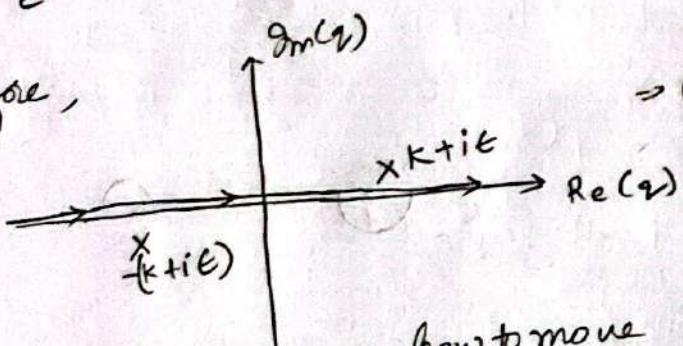


$$G(R) = \frac{-1}{8\pi^2 R i} \int_{-\infty}^{\infty} dq \frac{q (e^{iqR} - e^{-iqR})}{q^2 - k^2}$$

need to close
contour in UHP
due to f_i

need to
close in
LHP

Therefore,



$$\Rightarrow G(R) = \frac{-1}{8\pi^2 R i} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} dq \frac{q (e^{iqR} - e^{-iqR})}{q^2 - (k+ie)^2}$$

There are 4 choices on how to move these poles but only the one chosen here gives $e^{ikR} \rightarrow$ outgoing waves. Using Residue thm after closing contours over UHP & LHP for the Q_2 terms.

$$\Rightarrow G(R) = -\frac{1}{8\pi^2 R i} \left[\frac{2\pi i k e^{ikR}}{2k} - \frac{(-2\pi i)(-k) e^{ikR}}{-2k} \right]$$

$$G(R) = -\frac{e^{ikR}}{4\pi R}$$

$$\Rightarrow G(R) = \frac{-1}{4\pi} \frac{e^{ikR(\bar{r}-\bar{r}')}}{|\bar{r}-\bar{r}'|} \rightarrow \text{in 530 we got } \frac{1}{4\pi R} e^{ikR} \text{ since RHS of MH was chosen as } -S(\bar{r}-\bar{r}')$$

$$\therefore \psi(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} - \frac{\lambda}{4\pi} \int d^3 r' \frac{e^{i\vec{k} \cdot \vec{r}'}}{|\vec{r} - \vec{r}'|} U(r') \psi(r')$$

$\frac{2m}{\hbar^2} V(r)$ (Central potential)

incident wave
Integral equation for scattering. (inhomogeneous)

Lec 31

The above equation can be written as,

$$f(x) = g(x) + \int_a^b K(x,y) f(y) dy \rightarrow \text{Fredholm equation of } 2^{\text{nd}} \text{ kind.}$$

(inhomogeneous)

given fn. a Kernel. fn.

- > If $f(x) = 0 \Rightarrow$ we get Fredholm eqn. of first kind.
- > If $g(x) = 0 \Rightarrow$ we get an eigenvalue equation (homogeneous Fredholm eqn.).
- > In abstract notation $|f\rangle = \lambda K |f\rangle$ where K is the integral operator.

$|\vec{r} - \vec{r}'| \xrightarrow[r \rightarrow \infty]{} r$ in amplitude factor.

$$|\vec{r} - \vec{r}'| = (\vec{r}^2 + \vec{r}'^2 - 2\vec{r} \cdot \vec{r}')^{1/2} = r \left(1 + \frac{\vec{r}^2 - 2\vec{r} \cdot \vec{r}'}{r^2} \right)^{1/2}$$

assuming $r \gg r'$

binomial expansion

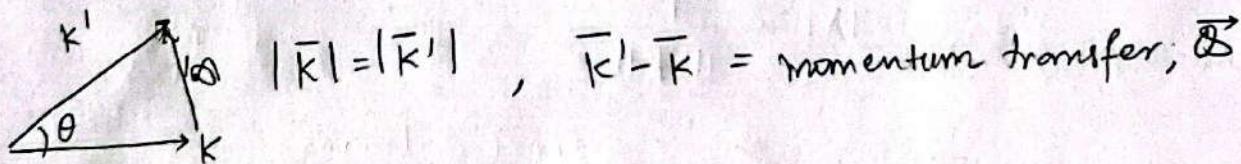
$$\approx r \left(1 - \frac{\vec{r} \cdot \vec{r}'}{r^2} \right) \rightarrow \text{same as antenna far field approximation.}$$

$$\Rightarrow k|\vec{r} - \vec{r}'| = kr - k\vec{r} \cdot \vec{r}'$$

wave vector of scattered wave along observation direction.

$$\Rightarrow \psi(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} + \frac{e^{ikr}}{r} \left\{ -\frac{\lambda}{4\pi} \int d^3 r' e^{-i\vec{k}' \cdot \vec{r}'} U(r') \psi(r') \right\}$$

$f(k, \theta) \rightarrow \text{scattering amplitude.}$



$$f(k, \theta) = -\frac{\lambda}{4\pi} \int d^3 r' e^{-i\vec{k}' \cdot \vec{r}'} U(\vec{r}') \psi(\vec{r}') \longrightarrow \text{exact not approximate.}$$

$$\Omega^2 = k^2 + k'^2 - 2kk' \cos\theta$$

$$\boxed{\Omega^2 = 2k^2(1-\cos\theta)} \quad \frac{2mE}{\hbar^2} = k^2 \Rightarrow \boxed{\Omega = 2k \sin \frac{\theta}{2}}$$

Going back to Fredholm

$$|f\rangle = |g\rangle + \lambda |\Phi|^2 |f\rangle$$

$$\Rightarrow |f\rangle = (1 - \lambda |\Phi\rangle)^{-1} |g\rangle$$

$$\text{Using binomial.} \Rightarrow (1 + \lambda k + \lambda^2 k^2 + \lambda^3 k^3 + \dots) |g\rangle$$

this is convergent when norm of $\lambda k < 1$

\hookrightarrow Neumann series.

$$\Rightarrow |\lambda| \|\Phi\| < 1$$

$$\text{Norm of } \|\Phi\|^2 \equiv \limsup_{|\Psi\rangle \in V} \frac{\langle \Phi \Psi | \Phi \Psi \rangle}{\langle \Psi | \Psi \rangle}$$

$$\langle x | \Phi^2 | f \rangle \rightarrow \int dz K(x, z) \int dy K(z, y) f(y)$$

,, Formal Solution.

$$\begin{aligned} \therefore \Psi(\vec{r}) &= e^{i\vec{k} \cdot \vec{r}} - \frac{\lambda}{4\pi} \int d^3 r' \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{|\vec{r} - \vec{r}'|} U(\vec{r}') e^{i\vec{k} \cdot \vec{r}'} \\ &\quad + \left(-\frac{\lambda}{4\pi}\right)^2 \int d^3 r' \int d^3 r'' \frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} + e^{i\vec{k} \cdot (\vec{r}' - \vec{r}'')}}{|\vec{r} - \vec{r}'| |\vec{r}' - \vec{r}''|} U(\vec{r}') U(\vec{r}'') e^{i\vec{k} \cdot \vec{r}''} + \dots \end{aligned}$$

$$\text{Converges when } |\lambda| < \frac{1}{\|\Phi\|}$$

\downarrow
Born series

1st Born approximation \Rightarrow retaining first 2 terms.

$$f(K, \theta) = -\frac{\lambda}{4\pi} \int d^3 r' e^{-i \vec{K} \cdot \vec{r}'} U(r') \psi(\vec{r}')$$

Using Born approximation. (1st)

$$\begin{aligned} f_B(K, \theta) &= -\frac{\lambda}{4\pi} \int d^3 r' e^{i \vec{K} \cdot \vec{r}'} U(r') e^{i \vec{K} \cdot \vec{r}'} \\ &= -\frac{\lambda}{4\pi} \int d^3 r' e^{-i \vec{Q} \cdot \vec{r}'} U(r') \end{aligned}$$

$\vec{n} \vec{Q} = \vec{n}(\vec{K}' - \vec{K})$

↳ looks like Fourier Transform of $U(r')$

What is $\langle \vec{p}' | V | \vec{p} \rangle$? Inserting a couple of complete set of states.

$$\begin{aligned} \langle \vec{p}' | V | \vec{p} \rangle &= \langle \vec{p}' | \underbrace{\int d^3 r_1 | \vec{r}_1 \rangle}_{\text{II}} \underbrace{\langle \vec{r}_1 | V \int d^3 r_2 | \vec{r}_2 \rangle}_{\text{II}} \langle \vec{r}_2 | \vec{p} \rangle \\ &= \int d^3 r_1 \int d^3 r_2 \underbrace{\langle \vec{p}' | \vec{r}_1 \rangle}_{e^{-i \vec{K} \cdot \vec{r}_1}} \underbrace{\langle \vec{r}_1 | V | \vec{r}_2 \rangle}_{V(r_2) \delta^{(3)}(\vec{r}_1 - \vec{r}_2)} \underbrace{\langle \vec{r}_2 | \vec{p} \rangle}_{e^{i \vec{K} \cdot \vec{r}_2}} \\ &\Rightarrow \int d^3 r_1 e^{-i \vec{Q} \cdot \vec{r}_1} V(r_1) \end{aligned}$$

$$\Rightarrow f_B(K, \theta) = -\frac{\lambda}{4\pi} \frac{2m}{\hbar^2} \langle \vec{p}' | V(r) | \vec{p} \rangle$$

$$\text{lets choose } \lambda V(r) = \frac{\lambda e^{-r/\xi}}{r} \quad \text{where } \xi > 0$$

Called Yukawa potential

lets first try to simplify f_B .

$$f_B(k, \theta) = -\frac{m\lambda}{2\pi\hbar^2} \int d^3r e^{-i\vec{k}\cdot\vec{r}} V(r)$$

Choose polar along \vec{k}

$$\begin{aligned} &= -\frac{m\lambda}{\hbar^2} \int_0^\infty dr r^2 \int_{-1}^1 e^{-i\theta r \cos\theta} d\cos\theta \cdot V(r) \\ &= -\frac{m\lambda}{\hbar^2} \int_0^\infty dr r^2 \frac{+2r \sin\theta r}{+r \theta r} V(r) \\ &= -\frac{2m\lambda}{\hbar^2 \theta} \int_0^\infty dr r V(r) \sin\theta r \end{aligned}$$

$$\lambda V(r) = \frac{\lambda e^{-\gamma/\xi}}{r}$$

$$\Rightarrow f_B(k, \theta) = -\frac{2m\lambda}{\hbar^2 \theta} \int_0^\infty dr r \sin(\theta r) e^{-\gamma/\xi}$$

$$= -\frac{2m\lambda}{\hbar^2 \theta} \cdot \frac{\theta}{\theta^2 + \gamma/\xi^2} = \frac{-2m\lambda}{\hbar^2} \cdot \frac{1}{\theta^2 + \gamma/\xi^2}$$

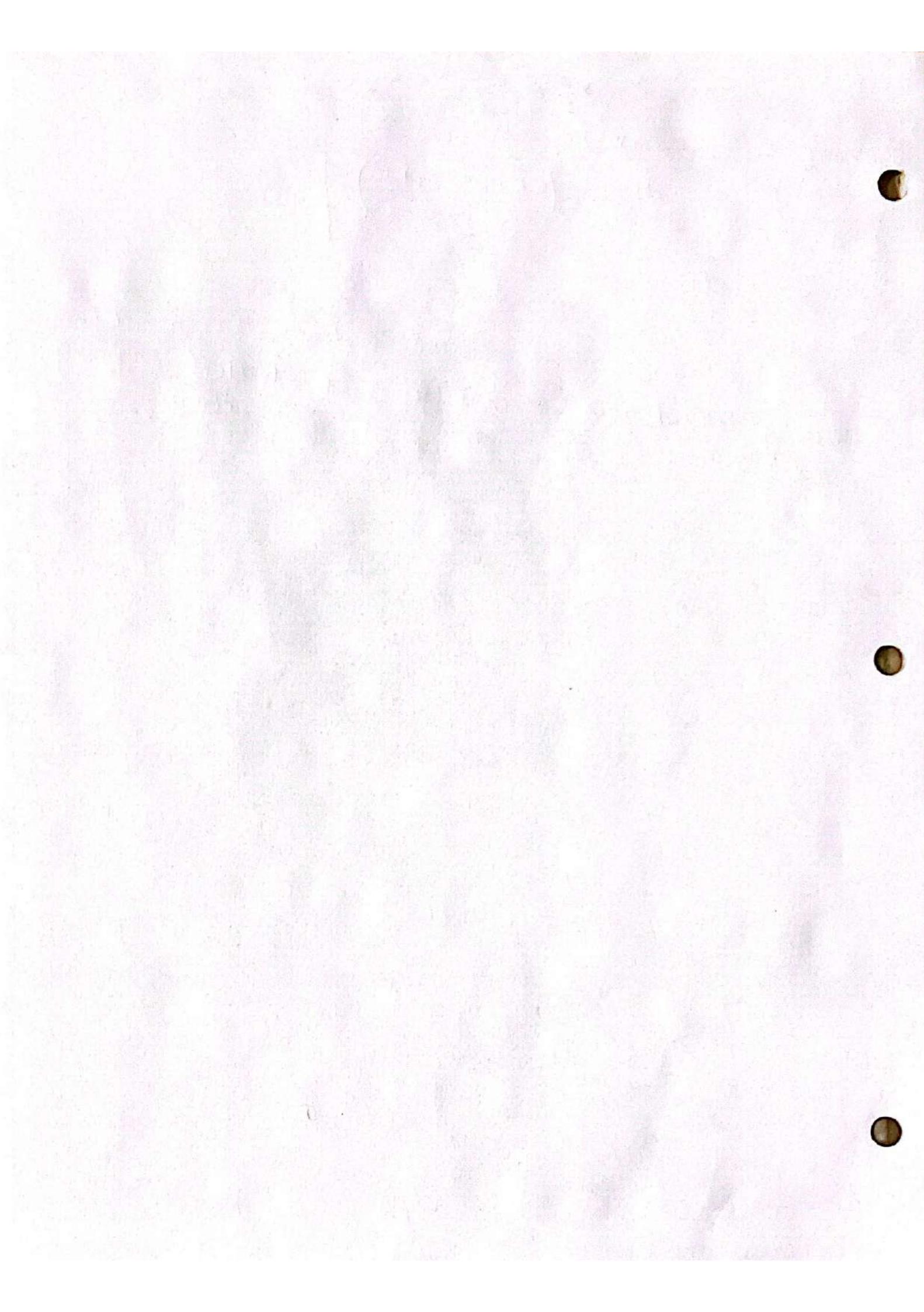
$$\int_0^\infty dx e^{-ax} \left\{ \begin{array}{l} \sin bx \\ \cos bx \end{array} \right\} \rightarrow \frac{b}{a^2 + b^2}$$

$$\begin{aligned} &\int_0^\infty dx e^{-ax} e^{ibx} \\ &= \int_0^\infty dx e^{-(a-ib)x} \\ &= \left[\frac{-e^{-(a-ib)x}}{a-ib} \right]_0^\infty = \left[\frac{1}{a-ib} \right] \\ &= \frac{b}{a^2 + b^2} \end{aligned}$$

$$\cdot \boxed{f_B(k, \theta) = -\frac{2m\lambda}{\hbar^2} \cdot \frac{1}{(2k^2(1-\cos\theta) + \frac{1}{\xi^2})}}$$

Calculate $\frac{d\sigma}{d\Omega}$ & σ

$\xi \rightarrow \infty$ gives the Coulomb potential solution. This cannot be directly derived since $\int_0^\infty \sin x dx$ is not solvable (oscillatory) could solve as $\lim_{a \rightarrow \infty} \int_0^a \sin x e^{-ax} dx \rightarrow \lim_{a \rightarrow \infty} \text{Im} \int_0^\infty e^{x(-a+i)} dx$.



The Wave Equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) f(\vec{r}, t) = g(\vec{r}, t)$$

Green's function is chosen s.t there is no f before g turns on.
AKA principle of causality.

$$f(\vec{r}, t) = \int d^d r' \int dt' \underbrace{G(\vec{r}, t; \vec{r}', t')}_{\text{Green Function.}} g(\vec{r}', t')$$

$$G(\vec{r}, t, \vec{r}', t') \rightarrow \begin{cases} \theta(t-t') \\ (\text{step fn}) \end{cases} K(\vec{r}-\vec{r}', t-t') \quad \hookrightarrow \text{invariance}$$

$$\Rightarrow \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) G(\vec{r}, t, \vec{r}', t') = \delta^{(d)}(\vec{r}-\vec{r}') \delta(t-t')$$

Also $G \rightarrow 0$ as $|\vec{r}-\vec{r}'| \rightarrow \infty$

Also $G=0$ for $t < t'$. (causality)

Let $\vec{R} = \vec{r}-\vec{r}'$, $\tau = t-t'$

$$\Rightarrow \left(\frac{1}{c^2} \frac{\partial^2}{\partial \tau^2} - \nabla_{\vec{R}}^2 \right) G(\vec{R}, \tau) = \delta^{(d)}(\vec{R}) \delta(\tau)$$

> F.T w.r.t space & time.

$$G(\vec{R}, \tau) = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{2\pi} \int dw e^{i(\vec{k} \cdot \vec{R} - \omega \tau)} \tilde{G}(\vec{k}, \omega)$$

deliberately inverted in time
since it looks like a wave.

$$\delta^{(d)}(\vec{R}) \delta(\tau) = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{2\pi} \int dw e^{i(\vec{k} \cdot \vec{R} - \omega \tau)}$$

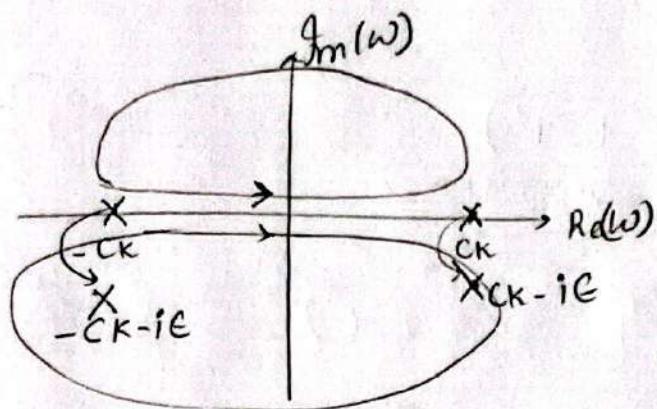
Subs. in wave eqn & equate coefficients. of $e^{i(\vec{k} \cdot \vec{R} - \omega \tau)}$

$$\left(\frac{-\omega^2}{c^2} + k^2 \right) \tilde{G}(\bar{R}, \omega) = 1$$

$$\Rightarrow \tilde{G}(\bar{R}, \omega) = \frac{-c^2}{\omega^2 - k^2 c^2}$$

$$\Rightarrow G(\bar{R}, \tau) = \frac{-c^2}{(2\pi)^d} \int d^d k e^{i\bar{k}\cdot\bar{R}} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega^2 - c^2 k^2}$$

Poles on real axis \Rightarrow use $i\epsilon$ prescription to satisfy the causal B.C.



$\tau < 0$ we want $G \rightarrow 0 \Rightarrow$ need to close in UHP since $e^{-i\omega\tau} \rightarrow e^{i\omega\tau}$
But we want no poles in UHP \Rightarrow move both poles down by $\epsilon \Rightarrow -CK - i\epsilon$ & $CK - i\epsilon$.

$$\Rightarrow \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{(\omega + i\epsilon)^2 - c^2 k^2}$$

\Rightarrow poles now at $CK - i\epsilon$ & $-CK - i\epsilon$.

$$\Rightarrow G(\bar{R}, \tau) = -\frac{c^2 \theta(\tau)}{(2\pi)^d} \int d^d k e^{i\bar{k}\cdot\bar{R}} \cdot \frac{i\pi i}{2\pi} \left[\frac{e^{-iCK\tau}}{2CK} + \frac{e^{iCK\tau}}{-2CK} \right]$$

$$= \frac{c \theta(\tau)}{(2\pi)^d} \int d^d k e^{i\bar{k}\cdot\bar{R}} \frac{\sin CK\tau}{k}$$

for causality.

In reality $t > t'$ is not enough, we need $t \geq t' + \frac{|\bar{R} - \bar{r}|}{c} \rightarrow$ Can we get this?

1 dimension.

$$K = |K|$$

only a few of R due to symmetry.

$$X = x - x'$$

$$G^{(1)}(x, \tau) = \frac{C\theta(\tau)}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \sin\left(\frac{C|k|\tau}{|k|}\right)$$

$$\frac{\sin(C|k|\tau)}{|k|} \rightarrow \frac{\sin(C\tau)}{k}$$

sin part of e^{ikx} vanishes since it is an odd fn.

since it is an even fn.

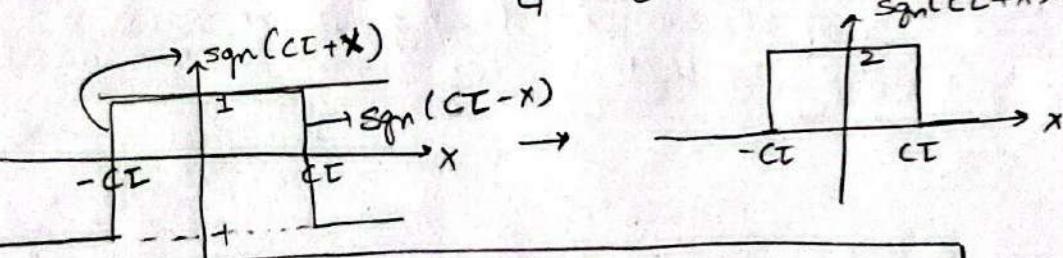
$$\Rightarrow G^{(1)}(x, \tau) = \frac{C\theta(\tau)}{2(2\pi)} \int_{-\infty}^{\infty} dk 2 \frac{\sin(Ck\tau)}{k} \cos kx .$$

$$= \frac{2(C\theta(\tau))}{4\pi} \int_0^{\infty} dk \left[\frac{\sin((C\tau+x)k)}{k} + \frac{\sin((C\tau-x)k)}{k} \right]$$

Symmetric for (even).

$$\Rightarrow G^{(1)}(x, \tau) = \frac{C\theta(\tau)}{2\pi} \cdot \frac{\pi}{2} \left[\operatorname{sgn}(C\tau+x) + \operatorname{sgn}(C\tau-x) \right]$$

$$= \frac{C\theta(\tau)}{4} \left[\operatorname{sgn}(C\tau+x) + \operatorname{sgn}(C\tau-x) \right]$$



$$\Rightarrow G^{(1)}(x, \tau) = \frac{C\theta(\tau)}{2} \theta(C\tau - |x|)$$

= signal only exists when

$$|x| < C\tau$$

$$\Rightarrow \frac{|x|}{C} < \tau - \tau'$$

$$\Rightarrow \tau > \frac{|x-x'|}{C} + \tau'$$

= Causality.
Did not impose, it is inherent.

$C\tau > |x| \Rightarrow$ after the response reaches a point the value of G is constant \Rightarrow the signal received remains for all time.

Weird & only exists in 1D.

2-dimensions

$$G^{(2)}(R, \tau) = \frac{C \theta(\tau)}{(2\pi)^2} \int d^2k e^{ik \cdot R} \sin \frac{c\tau k}{k}. \text{ choose } R_z \text{ along } R.$$

$$= \frac{C \theta(\tau)}{(2\pi)^2} \int_0^\infty k dk \sin \frac{c\tau k}{k} \int_0^{2\pi} d\phi e^{ikR \cos \phi}$$

not an elementary integral.
 $\rightarrow 2\pi J_0(kR)$
 \hookrightarrow Bessel of First kind order 0.

$$= \frac{C \theta(\tau)}{(2\pi)} \int_0^\infty dk \sin(c\tau k) J_0(kR)$$

Not easy but can be done.

$$= \frac{C \theta(\tau)}{\dots} \begin{cases} \frac{1}{\sqrt{c^2 \tau^2 - R^2}} & c^2 \tau^2 > R^2 \rightarrow \text{again ensures retarded.} \\ 0 & c^2 \tau^2 \leq R^2 \end{cases}$$

Here note that the G decreases with time at any point due to $\frac{1}{\tau^2}$

The signal also decays with R as $\frac{1}{R}$ since $c\tau$ always $> R$.

Delta pulse does not travel as a delta but as some decaying fun

In 3D we will see that a δ at the source indeed propagates as a δ fun. & therefore does not undergo distortion. This has made communication possible.

$$\int_0^\infty dk \sin(c\tau k) J_0(kR) = \underbrace{\operatorname{Im} \int_0^\infty dk e^{ic\tau k} J_0(kR)}_{\text{Regard this as the Laplace transform of } s = -ic\tau \text{ of } J_0(kR)}.$$

$$\int_0^\infty dk e^{-sk} J_0(kR) = \frac{1}{\sqrt{s^2 + R^2}}$$

Recall wave equation Green's function.

$$G^{(d)}(R, t) = \frac{c \theta(t)}{(2\pi)^d} \int d^d k e^{i \vec{k} \cdot \vec{R}} \frac{\sin ctk}{k}.$$

$\hookrightarrow \vec{k}_z \text{ along } \vec{R}$

3 dimensions

$$\begin{aligned} G^{(3)}(R, t) &= \frac{c \theta(t)}{(2\pi)^3} \int_0^\infty k^2 dk \sin ctk \cdot \int_{-1}^1 d(\cos \theta) e^{ikR \cos \theta} \\ &= \frac{c \theta(t)}{(2\pi)^2} \int_0^\infty k^2 dk \frac{\sin ctk}{k} \frac{2i \sin kR}{ikR} \\ &= \frac{c \theta(t)}{4\pi^2 R} \int_0^\infty dk 2 \sin ctk \sin kR. \\ &= \frac{1}{2} \frac{c \theta(t)}{4\pi^2 R} \int_{-\infty}^\infty dk [\cos k(ct-R) - \cos k(ct+R)] \end{aligned}$$

Since \cos is even.

Add since since it is an odd fn.

$$= \frac{c \theta(t)}{8\pi^2 R} \underbrace{\int_{-\infty}^\infty dk [e^{ik(ct-R)} - e^{-ik(ct+R)}]}_{2\pi \times \text{delta fns}}$$

$$= \frac{c \theta(t)}{4\pi R} [\delta(ct-R) - \delta(ct+R)]$$

$$= \frac{c \theta(t)}{4\pi R} \left[\frac{\delta(t-\frac{R}{c})}{c} - \frac{\delta(t+\frac{R}{c})}{c} \right]$$

$$= \frac{\theta(t)}{4\pi R} \left[\delta(t-\frac{R}{c}) - \delta(t+\frac{R}{c}) \right]$$

$\Rightarrow 0$ since $t > 0$ &
 $R > 0$

$$\boxed{\Rightarrow G^{(3)}(R, t) = \frac{\theta(t)}{4\pi R} \delta(t-t' - \frac{|R-R'|}{c})}$$

> In the wave equation as $c \rightarrow \infty$ we have $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \Rightarrow -\nabla^2$.
 The Green's fn. of $-\nabla^2$ is $\frac{1}{4\pi r}$ which is what^o the result here also gives. This happens (only?) in 3D.

Solutions of the wave equation.

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) f(\vec{r}, t) = g(\vec{r}, t)$$

$$\Rightarrow f(\vec{r}, t) = \int \frac{d^3 r'}{4\pi |\vec{r} - \vec{r}'|} g(\vec{r}', t')$$

the $\int dt' g(\vec{r}', t')$ gets evaluated only at t' where

$$t' = t + \frac{|\vec{r} - \vec{r}'|}{c}$$

Therefore source at time t' produces signal at time t . Causality!

Note that the wave equation applied to time harmonic quantities becomes the Helmholtz equation in phasor domain -

phasors $\checkmark \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) A(\vec{r}, t)$

$$\hookrightarrow = \left(\frac{1}{c^2} (i\omega)^2 - \nabla^2 \right) A(\vec{r}, \omega)$$

$$= \left(-\nabla^2 - \frac{\omega^2}{c^2} \right) A(\vec{r}, \omega)$$

$$= -(\nabla^2 + k^2) A(\vec{r}, \omega)$$

phasors Let $A(\vec{r}, t) = A_0 \cos(\omega t - kr)$
 $A(\vec{r}, t) = A_0 e^{i(kr - \omega t)}$

$$\Rightarrow \frac{\partial^2}{\partial t^2} \rightarrow -(-i\omega)^2$$