MATH 452

Metric Space

Def. A metric space (x, d) is a non empty set X together with a function $d: X \times X \to \mathbb{R}$ which satisfies the following axioms:

for all $x, y, z \in X$

(1) $d(x,y) = d(y,x)$ (symmetry)

(2) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

(3) $d(x,x) = 0$

(4) $d(x,y) \neq 0$ with $x \neq y$

d is called a metric on X.

Example : $\mathbb{R}$ is the metric space if $d(x,y) = |x-y|$ standard metric

Example : X is an arbitrary set, let $d(x,y) = \begin{cases} 0, & x=y \\ 1, & x \neq y \end{cases}$ discrete metric

Example : $X = \{a, b\}$ $d(a,b) = d(b,a) = 10$
$d(a,a) = 0$, $d(b,b) = 0$.

Q: Can $d(x,y)$ be negative? Prove your conclusion.

Proof: $z = x$ in (2) & use (1) & (3).

Vector Space.

$\mathbb{R}^n = \{ (x_1, \ldots, x_n) \mid x_i \in \mathbb{R} \}$ (example)

Definition:

A vector space V over $\mathbb{R}$ is a set with a function (addition)

$$(x,y) \longrightarrow x+y \quad \text{from } V \times V \text{ to } V.$$

and a function (scalar multiplication)

$$(a,x) \longrightarrow ax \quad \text{from } \mathbb{R} \times V \text{ to } V$$

such that the following hold:

$\forall\, x, y, z \in V$ and $a, b \in \mathbb{R}$

(V1) $\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$ associative law of addition.

(V2) $\bar{x} + \bar{y} = \bar{y} + \bar{x}$ commutative law of addition.

(V3) there is an element $\vec{0} \in V$ such that $\vec{0} + \bar{x} = \bar{x}$.

(V4) $\bar{x} + (-\bar{x}) = \vec{0}$, here $-\bar{x}$ for $(-1)\bar{x}$

(V5) $(ab)\bar{x} = a(b\bar{x})$ associative law of scalar mult.

(V6) $(a+b)\bar{x} = a\bar{x} + b\bar{x}$ distributive law of scalar mult.

(V7) $a(\bar{x}+\bar{y}) = a\bar{x} + a\bar{y}$ distributive law of scalar mult

(V8) $\vec{1} . \bar{x} = \bar{x}$

Examples: (1) $\mathbb{R}^n$ (2) the set of all real valued functions on $\mathbb{R}$. i.e $\{f : \mathbb{R} \to \mathbb{R}$

(3) In $\mathbb{R}^n$, Given $(a_1, \ldots, a_n) \in \mathbb{R}^n$, $a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$.
all $\{(x_1, \ldots, x_n)\}$ Satisfying the equation form a vector space.

Q2: Verify Example (3).

- If $a_1x_1 + \dots + a_nx_n = b$, $b \neq 0$ is it still a vector space?

"A subspace of the vector space V is a subset W of V that is itself a vector space (with same operations)".

Lec 2 Normed vector space

The notion of a norm on a vector space allows us to speak about length of a vector.

Defn Let V be a vector space over $\mathbb{R}$.

- A function $\phi : V \to \mathbb{R}$ is called a norm if the following holds,

(1) $\phi(x+y) \leq \phi(x) + \phi(y)$ $\forall\, x, y \in V$ (Subadditive property)

(2) $\phi(ax) = |a|\, \phi(x)$ $\forall\, x \in V, a \in \mathbb{R}$ (homogeneous prop)

(3) $\phi(x) \neq 0$ if $x \neq \vec{0}$ & $x \in V$.

Example in $\mathbb{R}^3$: length of $(x, y, z) = \sqrt{x^2+y^2+z^2}$

We often write $\|\cdot\|$ instead of $\phi(\cdot)$.

We call $(V, \|\cdot\|)$ a normed vector space.

- Examples: $\mathbb{R}^n$ $x = (x_1, x_2, \dots, x_n)$

(a) the maximum norm $\|x\|_0 = \max_{j=1,\dots,n} |x_j|$

(b) the sum norm $\|x\|_1 = \sum_{j=1}^{n} |x_j|$

(c) the Euclidean norm $\|x\| = (\sum^{n} |x_i|^2)^{1/2}$

(d) the p-norm $\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$

Q1: Can we say $\|x\| \geq 0, \forall x \in V$? V is a normed vector space.

Inner Product Space

$\mathbb{R}^n$ with the Euclidean norm is an example of inner product space.

Defn: An inner product space on the vector space V is a function $V \times V \to \mathbb{R}$. which associates with each pair (x,y) of vectors in V, a real number $\langle x,y \rangle$, satisfying

(i) $\langle x,y \rangle = \langle y,x \rangle$ (Symmetry)

(ii) $\langle ax,y \rangle = a\langle x,y \rangle$ (associative law)

(iii) $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$ (distributive law)

(iv) $\langle x,x \rangle > 0$ if $x \neq \vec{0}$

It follows that $\langle x,\vec{0} \rangle = \langle \vec{0},x \rangle = 0$.

Defn: Orthogonality

The vectors x and y in a vector space V are orthogonal, iff $\langle x,y \rangle = 0$.

If a set of nonzero vectors $\{V_1, V_2, \ldots, V_k\}$ satisfy $\langle V_j, V_k \rangle = 0$ whenever $j \neq k$, it is called an orthogonal set.

> If in addition $<v_j, v_j> = 1$ for all j, then it is orthonormal.

Remark: The norm associated with the inner product is defined by $||x|| = \sqrt{<x,x>}$

Example: $\mathbb{R}^n$ $\left.\begin{array}{l} x = (x_1, \dots, x_n) \\ y = (y_1, \dots, y_n) \end{array}\right\} <x,y> = x_1y_1 + x_2y_2 + \dots + x_ny_n$

It implies $||x|| = \sqrt{<x,x>} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

Example: Let $I = [a,b]$; $C[a,b]$ is the set of continuous functions $f: I \to \mathbb{R}$. $C[a,b]$ is a vector space.

We define an inner product on $C[a,b]$ by

$$<f,g> = \int_a^b f(t)g(t)\,dt$$

Then the norm $||f|| = \left(\int_a^b |f(t)|^2 dt\right)^{1/2}$

So far, we have studied
1. metric space
2. vector space
3. normed space
4. inner product space

Q2: Use a diagram to demonstrate the relations of the four space.

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Lec 2 Questions from Lecture

Q1 Can we say $||x|| \geq 0, \ \forall \ x \in V$?

Answer: Yes.

Proof: Consider two vectors $\vec{x}, -\vec{x}$

Using (1) $\phi(\vec{x} - \vec{x}) \leq \phi(\vec{x}) + \phi(-\vec{x})$

$\Rightarrow \phi(\vec{0}) \leq \phi(\vec{x}) + \phi(-\vec{x})$ ($\vec{x} + (-\vec{x}) = \vec{0}$ is from (V4) of vector spaces)

Claim 1: $\phi(\vec{0}) = 0$

By definition, $0.(\vec{0}) = \vec{0}$

$\Rightarrow \phi(\vec{0}) = \phi(0.\vec{0}) \underset{(2)}{=} |0|.\phi(\vec{0}) = 0$

Claim 2: $\phi(-\vec{x}) = \phi(\vec{x})$

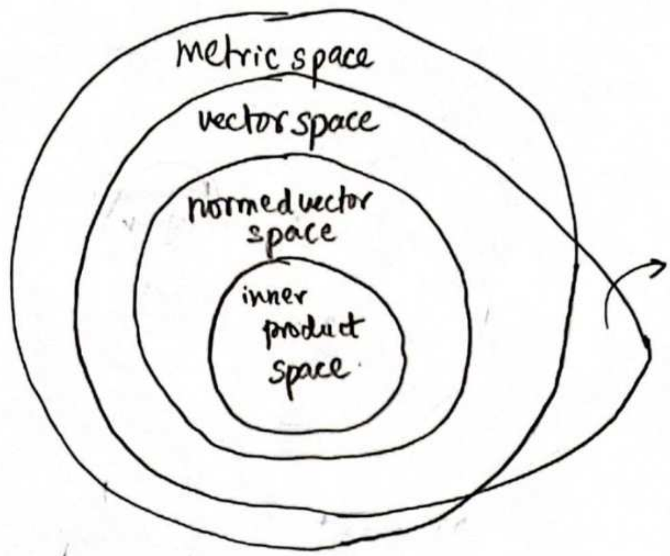
$\phi(-\vec{x}) = \phi(-1.\vec{x}) = |-1|.\phi(\vec{x}) = \phi(\vec{x})$

$\therefore \quad \phi(\vec{0}) \leq \phi(\vec{x}) + \phi(\vec{x})$

$\Rightarrow 2\phi(\vec{x}) \geq 0$

$\Rightarrow \boxed{\phi(\vec{x}) \geq 0}$

Q2 Use a diagram to demonstrate the relations of the four spaces.



Although a vector space can accomodate a metric it does not come preequipped with one. Additional information needs to be specified to derive a metric so we have included here vector spaces with no metric.

> We can always define a discrete metric on a vectorspace.

> normed vector space is by definition a vector space. Also the metric can be derived from the norm but not vice versa.

> norm can be defined from the inner product but not vice versa.

* $(V, \|\cdot\|) \Rightarrow d(x,y) = \|x-y\|$ * $(V, \langle\cdot,\cdot\rangle) \Rightarrow \|x\| = \sqrt{\langle x, x\rangle}$

Lec #3

Convergence

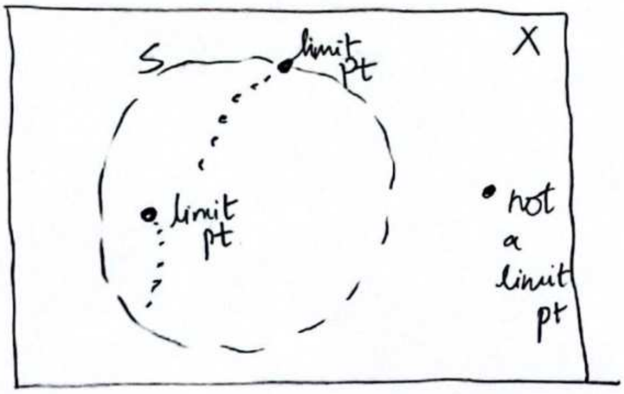
Let (X, d) be a metric space and $\{x_j\}$ a sequence in X. We say that $\{x_j\}$ converges to x if, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $d(x_j, x) < \epsilon$, $\forall j > N$.

notation: $x = \lim_{j \to \infty} x_j$ or $x_j \to x$, $j \to \infty$

limit points

Let S be a subset of (X, d) metric space. A point $x \in X$ is called a limit point of S if there is a sequence $\{x_n\}_{n=1}^{\infty}$ in S such that $x_n \neq x$ for any $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$.

(textbook P.41: The point a is a limit pt. of the set D iff every open ball centered at a contains points of D other that a)



example: $\{x_0\}$ does not have a limit point.

Closure

$S \subseteq X$ is a metric space.

The closure of S, denoted by $\bar{S}$, is the set of all limit points of S. S is called closed if S contains all it's limit points.

a) $S = \{x_0\}$, $\bar{S} = \phi$, $\bar{S} \subseteq S$ so $S = \{x_0\}$ is closed.

b) $\bar{S}$ is closed.

Open and Closed balls.

(X, d) metric space, the open ball with center $x \in X$ and radius $r > 0$ is defined as $B_r(X) = \{y \in X : d(y, x) < r\}$

The closed ball with center $x \in X$ and radius $r > 0$ is defined as $\bar{B}_r(x) = \{y \in X, d(y, x) \leq r\}$

Interior point

$S \subseteq X$, X metric space. A point $x \in S$ is called an interior pt. of S if there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq S$.

The set of all interior points of S is called the interior of S.

$X = \mathbb{R}$, $S = \{x_0\}$ is x_0 an interior point of S? No.

Open set

$S \subseteq X$, X metric space. S is called open if every point in S is an interior point of S.

Q: Prove (1) every open ball is open. } In a metric space.
(2) every closed ball is closed.

Thm: S is open if and only if $X \setminus S$ is closed.

Exercise: Suppose A, B are open subsets of a metric space (X, d)
Prove $A \cap B$ is open.

Proof: Let $x \in A \cap B$ be arbitrary. Goal: prove x is an interior point of $A \cap B$.

Since A is open and $x \in A$, x is an int. pt. of A.
$\exists\ \epsilon_1 > 0$ s.t $B_{\epsilon_1}(x) \subseteq A$.

Similarly $\epsilon_2 > 0$ s.t $B_{\epsilon_2}(x) \subseteq B$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ so $B_\epsilon(x) \subseteq \underset{\subseteq A}{B_{\epsilon_1}(x)}$ & $B_\epsilon(x) \subset \underset{\subseteq B}{B_{\epsilon_2}(x)}$

$B_\epsilon(x) \subseteq A \cap B$

So X is an int pt. of $A \cap B$. ■

→ X is both open & closed. ϕ is both open & closed.

Cauchy sequence.

$\{x_n\}$ in a metric space is called a Cauchy sequence if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $d(x_j, x_k) < \epsilon$ for all $j, k \in \mathbb{N}$ with $j, k > N$.

Thm: If a sequence converges, then it is a Cauchy sequence.

Complete

$S \subseteq X$ metric. S is complete if every Cauchy sequence in S has a limit in S. (Every cauchy sequence converges in a complete set)

Lec 4

Eg: $[0,1] \subseteq \mathbb{R}$ is complete

$S = (0,1) \subseteq \mathbb{R}$ is not complete. $\{\frac{1}{n}\}$ $n = 2, 3, \dots$

$\frac{1}{n} \to 0 \notin S$

→ A normed vector that is complete under the metric $d(x,y) = \|x-y\|$ is called a Banach space.

→ An inner product space that is complete under the metric $\|x\| = \sqrt{\langle x, x\rangle}$, then $d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$ is called a Hilbert space

Let $D \subseteq \mathbb{R}^n$

<u>Interior points</u>: $x_0 \in D$ is an interior pt. iff $\exists\, r>0$ s.t $B_r(x_0) \subseteq D$

<u>Exterior points</u>: $y_0 \notin D$ is an exterior pt. of D iff $\exists\, r>0$ s.t $B_r(y_0) \subseteq \mathbb{R}^n \setminus D$

<u>Boundary points</u>: $z_0 \in \mathbb{R}^n$ is a boundary pt. of D iff $\forall\, r>0$

$B_r(z_0) \cap D \neq \phi$ and $B_r(z_0) \cap (\mathbb{R}^n \setminus D) \neq \phi$.

Boundary of a set: all the boundary points (∂D).

<u>Bounded sets</u>: A set D is called bounded iff $\exists\, r>0$ s.t $D \subseteq B_r(\vec{0})$.

<u>Compact set</u>: A set $D \subseteq \mathbb{R}^n$ is called compact iff every sequence $\{x_n\}$ in D has a subsequence which has a limit in D.

In sequence $\{x_1, x_2, x_3, \ldots x_n, x_{n+1}, \ldots\}$,

$\{x_2, x_3, x_{10}, x_{15}, x_{30}, \ldots\}$ is a subsequence

$\{x_2, x_{10}, x_5, x_{15}, x_3 \ldots\}$ is not a subsequence.

(index must be an increasing set of natural numbers)

Remark: Let $\{x_n\}_{n=1}^{\infty}$ and $\{t_K\}_{K\geq 1}^{\infty}$ be sequences in a set.

$\{t_k\}$ is a subsequence of $\{x_n\}$ if there exists a strictly increasing sequence $\{n_k\}$ of natural numbers such that $t_k = X_{n_k}$ for all $K \in \mathbb{N}$.

Another definition of compact sets.

$D \subseteq \mathbb{R}^n$ is compact iff every infinite subset of D has a limit point which lies in D.

E.g.: (1) $\mathbb{R}$, $\mathbb{R}^n$ is not compact. sequences diverge to ∞.

(2) $(0,1) \subseteq \mathbb{R}$ is not compact. $\{\frac{1}{n}\}$, $n = 2, 3 \ldots$
$0 \notin (0,1)$.

(3) Finite set is automatically compact.

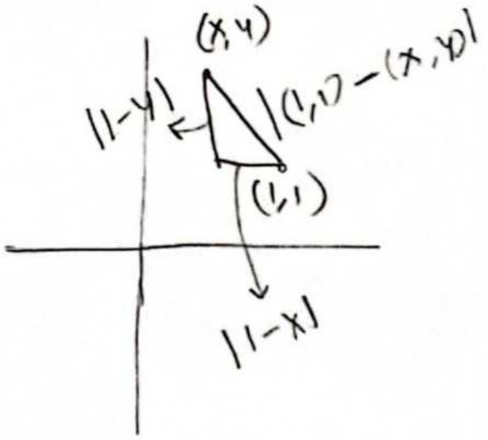
Continuous functions.

A function $f : X_1 \rightarrow X_2$ (X_1, X_2 are metric spaces), f is continuous at a point $\vec{x}_0 \in X_1$ if for each $\epsilon > 0$, $\exists$ $\delta > 0$ s.t $\forall \vec{x} \in X_1$
$d_1(\bar{x}, \bar{x}_0) < \delta \Rightarrow d_2(f(\bar{x}), f(\bar{x}_0)) < \epsilon$. f is continuous if f is continuous at every point in X_1

<u>E.g.:</u> $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ given $f(x,y) = x^2 + xy + y$. Show that f is continuous at $(1,1)$.

<u>Proof</u>: Let $\epsilon > 0$ be arbitrary. We need to find a δ s.t

$d((1,1),(x,y)) < \delta \Rightarrow d(3, (x^2+xy+y)) < \epsilon$. We use Euclidean metric



$d((1,1),(x,y)) = \sqrt{(x-1)^2 + (y-1)^2}$

$d(3, (x^2+xy+y)) = |3 - (x^2+xy+y^2)|$

WLOG, we restrict $\delta < 1$.

We choose $\delta < \min(1, \frac{\epsilon}{7})$

Note that $|1-x| \leq d((1,1),(x,y)) < \delta$
$|1-y| = d((1,1),(x,y)) < \delta$

$-\delta < 1-x < \delta \Rightarrow 1-\delta < x < 1+\delta \Rightarrow |x| < 1+\delta < 2$

$|1+x| < 2+\delta < 3$

$$\begin{aligned}
|3-(x^2+xy+y)| &\leq |1-x^2| + |1-xy| + |1-y| \\
&\leq |1-x||1+x| + |1-x+x-xy| + |1-y| \\
&\leq |1-x||1+x| + |1-x| + |x||1-y| + |1-y| \\
&\leq 3\cdot\delta + \delta + 2\delta + \delta \\
&= 7\delta \\
&< \epsilon
\end{aligned}$$

Lec 5

Sequential Characterization of continuity

<u>Thm</u>: Let $f : \mathbb{R}^n \to \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$. The mapping f is continuous iff $f(\vec{x}_n) \to f(\vec{x})$ as $n \to \infty$ for every sequence $\{\vec{x}_n\}$ in $\mathbb{R}^n$ with $\vec{x}_n \to \vec{x}$ as $n \to \infty$.

<u>Proof</u>: "$\Leftarrow$" Prove the contra positive. Suppose f is not continuous at x. We negate the statement in the definition:

$\exists\ \epsilon > 0$ s.t, $\forall\ \delta > 0$, $\exists\ \vec{y} \in \mathbb{R}^n$ with $d(\vec{y}, \vec{x}) < \delta$ but

$$d(f(\bar{y}), f(\bar{x})) \geq \epsilon.$$

$$\| \quad |f(\bar{y}) - f(\bar{x})|$$

In particular, we can choose $\delta = \frac{1}{n}$, $n = 1, 2, 3 \ldots$

$\exists\ \vec{x}_n \in \mathbb{R}^n$ with $d(\vec{x}_n, \vec{x}) < \delta$ but $|f(\vec{x}_n) - f(\vec{x})| \geq \epsilon$.

So we found a sequence $\vec{x}_n \to \vec{x}$ but $f(\vec{x}_n) \not\to f(\vec{x})$ as $n \to \infty$.

"$\Longrightarrow$" "exercise"

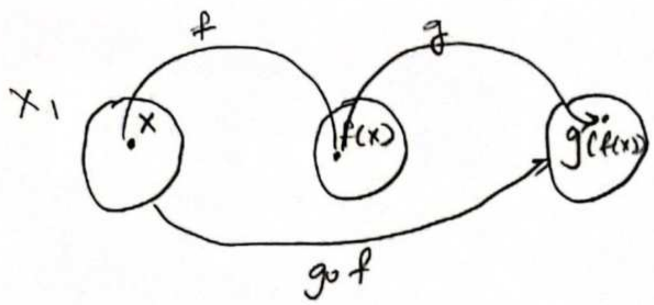
<u>Remark</u>: The mapping $f : D \to \mathbb{R}^n$ is continuous at $x \in D$ iff each coordinate function of f is continuous at x.

Here $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$

<u>Proposition</u>: Let (X_i, d_i) $i = 1, 2, 3$ be metric spaces.

$f : X_1 \to X_2 \qquad g : X_2 \to X_3$

$[g \circ f](x) = g(f(x))$, $x \in X_1$ is a fn from X_1 to X_3.



a) If $x_1 \in X_1$ and f is continuous at x_1 and g is continuous at $x_2 = f(x_1)$, then the composition $g\circ f$ is continuous at x_1.

b) If f and g are continuous, then $g\circ f$ is continuous.

Proof: (a) use d for all the 3 metrics.

Let $\epsilon_3 > 0$. Since g is continuous at $x_2 = f(x_1)$. $\exists\, \epsilon_2 > 0$ s.t $\forall\, y \in X_2$, $d(y, x_2) < \epsilon_2 \Rightarrow d(g(y), g(x_2)) < \epsilon_3$.

Since f is continuous at x_1. For this ϵ_2, $\exists\, \epsilon_1 > 0$ s.t $\forall\, x \in X_1$, $d(x, x_1) < \epsilon_1 \Rightarrow d(f(x), f(x_1)) < \epsilon_2$.

Combine them:

$\forall\, x \in X_1$, $d(x, x_1) < \epsilon_1 \Rightarrow d(g(f(x)), g(f(x_2))) < \epsilon_3$.

so $g\circ f$ is continuous at x by definition.

(b) follows directly from (a).

Definition: Uniformly Continuous.

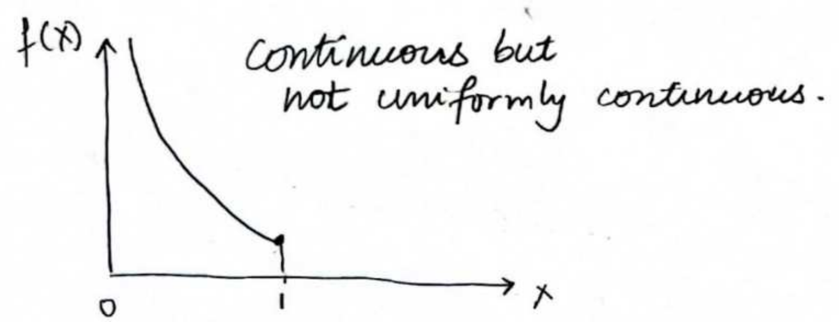
> A function $f: X_1 \to X_2$ $(X_1, d_1), (X_2, d_2)$ metric spaces is uniformly continuous on a subset D of X_1 if for every $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\forall x, y \in D, \quad d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon.$$

> f is uniformly continuous if it is uniformly continuous on X_1.

Remark: The number δ in this definition will not depend upon x, y.

Example: $f: (0,1) \to \mathbb{R}$ with $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous.



Proof of Sequential characterization of continuity

(Aditya Varma Muppala & Ribhu Deb)

$\Longrightarrow$

Let $\vec{x}_n \to \vec{x}$ as $n \to \infty$ and we have $f(\vec{x})$ is continuous

$\Rightarrow$ For any $\epsilon > 0$ $\exists$ $\delta > 0$ s.t

$d(\bar{x}_n, \bar{x}) < \delta \Rightarrow d(f(\bar{x}_n), f(\bar{x})) < \epsilon$.

Choose an $N \in \mathbb{N}$ s.t $d(\bar{x}_n, \bar{x}) < \delta \quad \forall\ n \geq N$

$\Rightarrow$ from continuity $d(f(\bar{x}_n), f(\bar{x})) < \epsilon \quad \forall\ n \geq N$.

$\Rightarrow f(\vec{x}_n) \to f(\vec{x})$

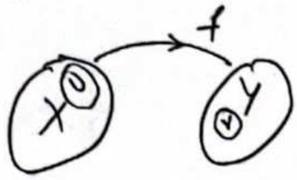
Lec 6

Notation:

X, Y are two metric spaces. d denotes the metric on both.

Let $f: X \to Y$. The image of a set under a function.

for $U \subseteq X$



$f(U) = \{y \in Y : y = f(x) \text{ for some } x \in U\}$

The preimage of a set under a function for $V \subseteq Y$

$f^{-1}(V) = \{x \in D : f(x) \in V\}$

Remark: preimage definition doesn't require f to be invertible

<u>Thm</u>: The following are equivalent.

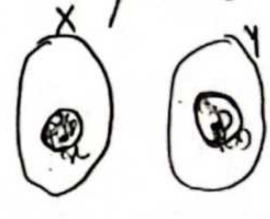
(i) f is continuous.

(ii) If D is any closed subset of Y, then $f^{-1}(D)$ is closed.

(iii) If V is any open subset of Y, then $f^{-1}(V)$ is open.

<u>Proof</u>: <u>(i) $\Rightarrow$ (ii)</u> Assume that f is continuous, and D is a closed subset of Y.

<u>Goal</u>: prove $f^{-1}(D)$ is closed. (pick a limit point, show it is in $f^{-1}(D)$).



Let $x \in X$ be a limit point of $f^{-1}(D)$.

$\exists \{x_n\}$ in $f^{-1}(D)$ such that $x_n \to x$ as $n \to \infty$.

f is cont. so $f(x_n) \to f(x)$. Noticing $f(x_n) \in D$ $\forall n$.

$f(x)$ is a limit point of D.

Since D is closed, $f(x) \in D$. Therefore, $x \in f^{-1}(D)$.

<u>(ii) $\Rightarrow$ (iii)</u> Let $V \subseteq Y$ be open. Then $Y \setminus V$ (complement) is closed. $f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$.

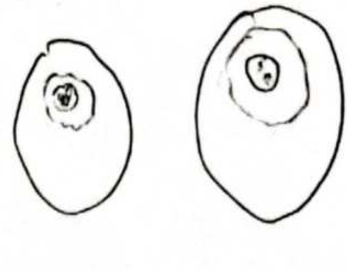
By (ii), $X \setminus f^{-1}(V)$ is closed. Then $f^{-1}(V)$ is open.

(iii) $\Rightarrow$ (i) Let $x \in X$, we aim to show f is continuous at x. Note $f(x) \in Y$. Let $\epsilon > 0$ and $V = B_\epsilon(f(x))$.

By (iii) $f^{-1}(V) = f^{-1}(B_\epsilon(f(x)))$ is open.

Note $x \in f^{-1}(B_\epsilon(f(x)))$.

Since $f^{-1}(B_\epsilon(f(x)))$ is open. $\exists\, \delta > 0$ such that



$B_\delta(x) \subseteq f^{-1}(B_\epsilon(f(x)))$.

Let $z \in X$ and $d(z, x) < \delta$.

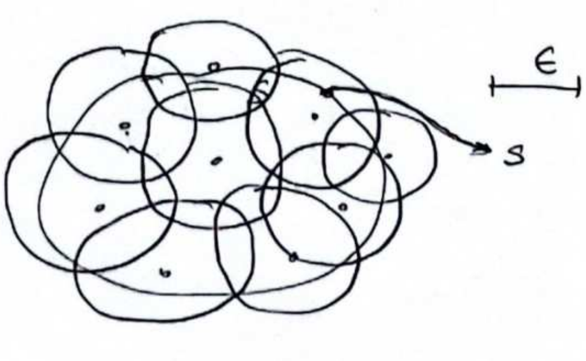
Then $z \in B_\delta(x)$ so $f(z) \in B_\epsilon(f(x))$, ie. $d(f(z), f(x)) < \epsilon$

So f is continuous at x. Since x is arbitrary, f is continuous. ■

Definition: Let S be a subset of a metric space (X, d)

S is called totally bounded if, for each $\epsilon > 0$, there are finitely many points $x_1, \ldots, x_n \in S$ such that

$$S \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$



* In other words, for each $x \in S$, $\exists\ j \in \{1, 2, \ldots n\}$ s.T $d(x, x_j) < \epsilon$. we call $\{x_1, \ldots, x_n\}$ an ϵ-net. *

Lemma 1

If S is totally bounded, then S is bounded.

Proof: Pick $\epsilon = 1$, there are $\{x_1, \ldots, x_n\} \subseteq S$ s.T $\forall\ x \in S$, $\exists j$, $d(x, x_j) < 1$. Set $r = \max\limits_{i=1,\ldots,n} \{d(x_1, x_i)\}$

By triangle inequality

$$d(x, x_1) \le d(x, x_j) + d(x_j, x_1) < 1 + r$$

$$d(x, \vec{0}) \le d(x, x_1) + d(x_1, \vec{0}) < 1 + r + \underbrace{d(x, \vec{0})}_{\text{fixed number}}$$

$$C = 1 + r + d(x, \vec{0})$$

and $S \subseteq B_C(\vec{0})$ so S is bounded.

Thm: If $S \subseteq X$ is compact then S is totally bounded.

Proof: Let S be compact. Suppose it is not totally bounded. $\Rightarrow S \neq \phi$. $\exists$ some $\epsilon > 0$ s.T, $\forall\ n \in \mathbb{N}$ and every collection $\{x_1, \ldots, x_n\}$ there exists $x \in S$ such that $d(x, x_i) \ge \epsilon$ $\forall\ i = 1, \ldots, n$.

Start with some $x_1 \in S$, then there exists $x_2 \in S$ such that $d(x_1, x_2) \ge \epsilon$.

Now we have $\{x_1, x_2\}$. Then $\exists\, x_3 \in S$:

$d(x_3, x_1) \geq \epsilon$. $d(x_3, x_2) \geq \epsilon$. Continue this process

$x_1, x_2, x_3, \ldots, x_m$. $\exists\, x_{m+1} \in S$ such that $d(x_{m+1}, x_i) \geq \epsilon$

$\forall\, i = 1, \ldots, m$. By induction, we find a sequence $\{x_n\}$

and $d(x_j, x_k) \geq \epsilon > 0 \;\forall j, k \; j \neq k$.

This sequence has no subsequence that is a Cauchy sequence, therefore no subsequence that converges. $\divideontimes$

Contradiction to that S is compact.

Compactness $\Rightarrow$ totally bounded $\Rightarrow$ bounded.

Lec #7

<u>Thm:</u> A closed subset D of a compact set S is compact

<u>Proof:</u> Choose an arbitrary sequence $\{x_n\}$ in D.

$\{x_n\}$ is also a sequence in S.

Since S is compact, there exists a subsequence $\{x_{n_j}\}$,

$x_{n_j} \to x \in S$.

$\{x_{n_j}\}$ is a sequence in D and D is closed, so x is a limit point of D, and $x \in D$.

Therefore, D is compact.

<u>Lemma 2</u> : If S is a compact subset of X and V is a compact subset of Y, then $S \times V$ is a compact subset of $X \times Y$. $(x \in S, y \in V)$.

<u>Proof</u> : For an arbitrary sequence in $S \times V$,

$\{z_n\}_{n=1}^{\infty}$, $z_n = (x_n, y_n)$ with $x_n \in S$ and $y_n \in V$.

Since S is compact, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ with $x_{n_j} \to x \in S$ as $j \to \infty$.

Now we consider $\{y_{n_j}\}$ in V.

Since V is compact, $\{y_{n_j}\}$ has a subsequence $\{y_{n_{j_k}}\}_{k=1}^{\infty}$.

$y_{n_{j_k}} \to y \in V$ as $k \to \infty$.

Thus, $z_{n_{j_k}} = (x_{n_{j_k}}, y_{n_{j_k}}) \to (x, y) \in S \times V$.

So $S \times V$ is compact

Thm: (Heine Borel Theorem for $\mathbb{R}^n$)

Consider $\mathbb{R}^n$ is a metric space induced by an arbitrary norm. Let S be a subset of $\mathbb{R}^n$. S is compact iff S is bounded and closed.

Proof: "$\Rightarrow$" Suppose S is compact.

By the theorem and lemma 1 from last time, we have S is bounded.

To show S is closed. Let x be a limit point of S. There is a sequence $\{x_n\}$ in S, $x_n \to x$ as $n \to \infty$.

Since S is compact, there is a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ this has a limit in S, $x_{n_j} \to \hat{x} \in S$.

Simple proof for $\hat{x} = x$, $d(x, \hat{x}) \le d(x, x_{n_k}) + d(x_{n_k}, \hat{x})$,

make k large, $\Rightarrow$ $\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

$\because$ For any $\epsilon > 0$, we can find k large enough to satisfy

$d(x, x_{n_k}) < \frac{\epsilon}{2}$ & $d(x_{n_k}, \hat{x}) < \epsilon/_2$.

$0 \le d(x, \hat{x}) < \epsilon$. Therefore $d(x, \hat{x}) = 0$

Therefore, $x = \hat{x} \in S$. S is closed.

"$\Leftarrow$" Suppose S is closed and bounded.

Choose $r>0$ so large that $S \subseteq B_r(\vec{0}) \subseteq I \times I \times \dots \times I$
$\hookrightarrow$ n dimension cube.

$I = [-r, r]$.

Note, I is a compact subset of $\mathbb{R}$.

(Read textbook appendix P.445-448

Every bounded sequence of real numbers has a convergent subseq.

$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}$.

By lemma 2, I^n is compact.

By theorem (today), S is a closed subset of I^n, thus, S is compact. $\blacksquare$

<u>Thm:</u> Let X, Y be two metric spaces.
$D \subseteq X$ is compact. $f: D \to Y$ is continuous. Then $f(D)$ is a compact subset of Y.

"Continuous function preserves compactness".

<u>Proof:</u> Let $\{y_n\}$ be a sequence in $f(D)$.

There are $x_n \in D$ such that $y_n = f(x_n)$, $\forall n$.

D is compact, hence, $\{x_{n_j}\}$ a subsequence of $\{x_n\}$ converges in D. $x_{n_j} \to x \in D$, $j \to \infty$.

f is continuous, therefore

$$y_{n_j} = f(x_{n_j}) \to f(x) \in f(D)$$

Therefore, $f(D)$ is compact.

<u>Thm:</u> Let D be a subset of a metric space $\mathbb{R}^n$.
If $f: D \to \mathbb{R}$ is continuous and D is compact, then f attains maximum and minimum values at points of D.
That is, there exist points a and b in D such that $f(a) \le f(x) \le f(b)$ for all $x \in D$.

<u>Proof:</u> We only prove the maximum value. By previous theorem, $f(D)$ is a compact subset of $\mathbb{R}$.
$f(D)$ is closed and bounded, by Heine Borel Thm
$\exists$ a least upper bound $u \in \mathbb{R}$ of $f(D)$.

Hence, (i) $f(x) \le u$, for all $x \in D$
(ii) for any $\delta = \frac{1}{n} > 0$, there is $y_n \in f(D)$ such that $u - y_n < \delta = \frac{1}{n}$.

So $y_n \to u$ as $n \to \infty$.

Since $f(D)$ is closed, $u \in f(D)$. There exists $b \in D$ such that $f(b) = u$.

$f(x) \le f(b)$ for all $x \in D$.

For minimum use greatest lower bound & proceed.

<u>Thm:</u> Let D be a compact subset of a metric space X $f: D \to \mathbb{R}$ is continuous. Then f is uniformly continuous.

<u>Proof:</u> Suppose f is not uniformly continuous.

$\exists \epsilon > 0$, $\forall \delta > 0$, $\exists x, y \in D$ s.t $d(x,y) < \delta$ but $|f(x), f(y)| \ge \epsilon$

Let $\delta = \frac{1}{n}$, $n \in \mathbb{N}$

We can find (x_n, y_n) with $d(x_n, y_n) < \frac{1}{n}$, $|f(x_n) - f(y_n)| \ge \epsilon$.

Since D is compact $\{x_n\}$ has a subsequence $\{x_{n_j}\}$

$x_{n_j} \to \hat{x} \in D$ as $j \to \infty$.

$$0 \leq d(y_{n_j}, \hat{x}) \leq d(y_{n_j}, x_{n_j}) + d(x_{n_j}, \hat{x})$$

$$< \frac{1}{n_j} + d(x_{n_j}, \hat{x})$$

As $j \to \infty$, $\frac{1}{n_j} \to 0$, $d(x_{n_j}, \hat{x}) \to 0$

$$0 \leq d(y_{n_j}, \hat{x}) < \frac{1}{n_j} + d(x_{n_j}, \hat{x}) \to 0$$

$\Rightarrow y_{n_j} \to \hat{x}$ as $j \to \infty$.

f is continuous, $f(x_{n_j}) \to f(\hat{x})$ and $f(y_{n_j}) \to f(\hat{x})$

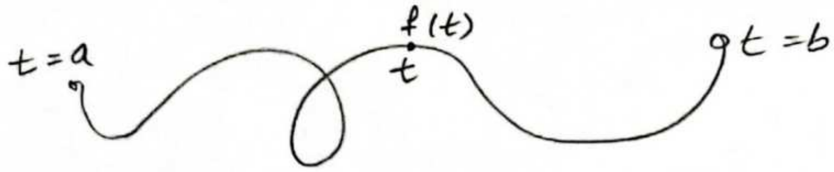
$\#$ ■

Lec 8

Curves in $\mathbb{R}^n$

- Consider $f: \mathbb{R} \to \mathbb{R}^n$. This is a parametric curve in $\mathbb{R}^n$

$t \to f(t)$
time position



each $f_j : \mathbb{R} \to \mathbb{R}$.

Derivative of f_j

- $$\lim_{h \to 0} \frac{f_j(a+h) - f_j(a)}{h} = f_j'(a)$$

Then the derivative of $f'(a) = (f_1'(a), \ldots, f_n'(a))$

$$f' = (f_1', \ldots, f_n')$$

Recall: $f(t)$ position

$f'(t)$ velocity

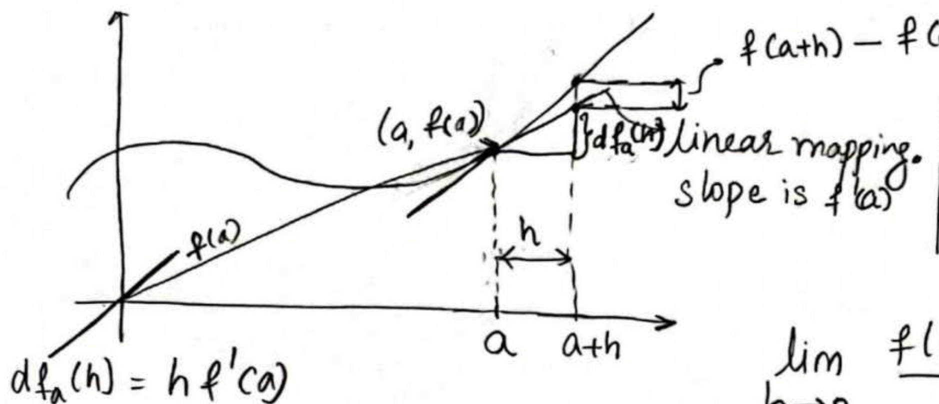
$\|f'(t)\|$ (Euclidean norm) speed.

$f''(t)$ acceleration.

- <u>Definition</u> : <u>differential</u>

The linear mapping $df_a : \mathbb{R} \to \mathbb{R}^n$, defined by $df_a(h) = h f'(a)$, is called the <u>differential of f at a.</u>

Remark: For a point a the derivative is a vector $f'(a)$, the differential is a linear mapping.



$f(a) + d f_a(h) \simeq f(a+h)$

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - d f_a(h)}{h} = 0$$

Thm: The mapping $f: \mathbb{R} \to \mathbb{R}^n$ is differential at $a \in \mathbb{R}$ if and only if there exists a linear mapping $L: \mathbb{R} \to \mathbb{R}^n$ s.t

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{h} = \vec{0}$$ in which case L is defined by $L(h) = df_a(h) = h f'(a)$.

Proof: "$\Leftarrow$" Since L is a linear mapping from $\mathbb{R} \to \mathbb{R}^n$ there exists $n\times 1$ matrix such that $L(x) = Mx$.

$$\vec{0} = \lim_{h\to 0} \frac{f(a+h) - f(a) - Mh}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} - M$$

So $$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = M = (M_1, M_2, \ldots, M_j)$$

$$\lim_{h\to 0} \frac{f_j(a+h) - f(a)}{h} = M_j \quad , j = 1, \ldots, n \quad \Rightarrow M = f'(a)$$

"$\Rightarrow$" f is differentiable at a.

By definition, coordinate-wise

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a).$$

and $df_a(h) = hf'(a)$

$$\vec{0} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} - f'(a)$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)-hf'(a)}{h} = \lim_{h\to 0} \frac{f(a+h)-f(a)-df_a(h)}{h}$$

Denote $L = df_a$, linear mapping from $\mathbb{R}$ to $\mathbb{R}^n$.

Remark: In the above theorem, L is unique.

Proof: Suppose L_1 and L_2 both satisfy

$$\lim_{h\to 0} \frac{f(a+h)-f(a)-L(h)}{h} = \vec{0}$$

We aim to show $L_1 = L_2$

The linear mapping always maps zero to zero

$L_1(0) = L_2(0) = \vec{0}$. ✓

Let $x \in \mathbb{R}$, $x \neq 0$, with $L_1(x) \neq L_2(x)$

$\vec{y} = L_1(x) - L_2(x) \neq \vec{0}$

Let $h = tx$ with $t \neq 0$.

$L_1(h) = L_1(tx) = t L_1(x)$

$L_2(h) = L_2(tx) = t L_2(x)$

However,

$$\lim_{h \to 0} \frac{L_1(h) - L_2(h)}{h} = \lim_{h \to 0} \frac{L_1(h) - f(a+h) + f(a) + f(a+h) - f(a) - L_2(h)}{h}$$

$$= \vec{0}$$

$$\lim_{h \to 0} \frac{L_1(h) - L_2(h)}{h} = \lim_{h \to 0} \frac{t L_1(x) - t L_2(x)}{h} = \lim_{t \to 0} \frac{t\vec{y}}{tx} = \frac{\vec{y}}{x} \neq \vec{0}$$

It's a contradiction.

Q: A particle has a position vector $\gamma(t) = \left(\frac{4}{5}\cos(t),\ 1 - \sin(t),\ -\frac{3}{5}\cos(t)\right)$.

(a) Find velocity vector for time t.

(b) Find speed at time $t = 2\pi$

(c) What is the $d\gamma_{2\pi}$?

(d) Bonus Find a time when the velocity vector is perpendicular to x axis

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(33)

(a) $v(t) = \left(-\frac{4}{5}\sin(t), \quad -\cos(t), \quad \frac{3}{5}\sin(t)\right)$

(b) speed $s(t) = \sqrt{\frac{16}{25}\sin^2 t + \cos^2 t + \frac{9}{25}\sin^2 t}$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$= \sqrt{1}$$

$$= \underline{\underline{1}}$$

(c) $d\gamma_{2\pi} = 2\pi \, v(t)$

$$= 2\pi\left(-\frac{4}{5}\sin(t), -\cos(t), \frac{3}{5}\sin(t)\right)$$

$d\gamma_{2\pi}(h) = h\,(v(2\pi))$

$= h(0, -1, 0)$

(d) $\langle v(t), (1,0,0)\rangle = 0$

$\Rightarrow \quad -\frac{4}{5}\sin(t) = 0 \quad \Rightarrow \underline{\underline{t = \pi}}$

$v(t)$ at $t = \pi$ is $(0, 1, 0)$.

Lec 9

Tangent line to f at a: $T_a(t) = f(a) + df_a(h)$

$t = a$, at the pt. on f $\qquad h = t - a$.

$= f(a) + f'(a)(h)$

$\Rightarrow \boxed{T_a(t) = f(a) + f'(a)(t-a)}$

df_a is a linear mapping $df_a : \mathbb{R} \to \mathbb{R}^n$.

$df \approx \Delta f$; $\boxed{df = \frac{df}{dx} \cdot dx}$ where $dx \approx \Delta x$

identity map

$x : \mathbb{R} \to \mathbb{R}$; $dx_a(h) = x' \cdot h = 1 \cdot h = h$. $\qquad dx_a : \mathbb{R} \to \mathbb{R}$. $\quad h \to h$.

$f : \mathbb{R} \to \mathbb{R}$; $df_a(h) = f'(a) \cdot h = f'(a)\, dx_a(h)$.

$\boxed{df_a(h) = \left.\frac{df}{dx}\right|_{x=a} dx_a(h)}$ $\qquad dx_a(h) = h$

$\left.\frac{df}{dx}\right|_{x=a} = f'(a)$

Today, we study surfaces in $\mathbb{R}^m$.

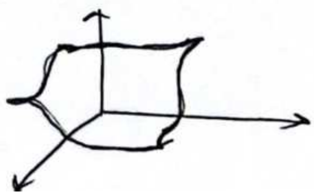
$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$(x_1, \dots, x_n) \longrightarrow (f_1(x_1, \dots x_n), \dots, f_m(x_1, \dots, x_n))$

The image of F is an n-dimensional "surface" in $\mathbb{R}^m$.

$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$.

eg. $(x,y) \longrightarrow (x, y, \underbrace{f(x,y)}_{z = f(x,y)})$



In general,

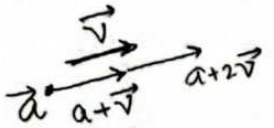
$$(x,y) \to (f_1(x,y), f_2(x,y), f_3(x,y))$$

$$(s,t) \to (x(s,t), y(s,t), z(s,t))$$

<u>Definition</u>: The directional derivative with respect to $\vec{v}$ of $F : \mathbb{R}^n \to \mathbb{R}^m$ at a point $\vec{a}$ is
($\vec{a}, \vec{v} \in \mathbb{R}^n$)

$$\boxed{D_{\vec{v}} F(\vec{a}) = \lim_{h \to 0} \frac{F(\vec{a}+h\vec{v}) - F(\vec{a})}{h}}$$

provided the limit exists.



The directional derivatives of F w.r.t $\underbrace{\vec{e_1}, \ldots, \vec{e_n}}_{}$ (standard unit basis.) are called the partial derivatives of F.

The i^{th} partial derivative of F : $\boxed{D_{\vec{e_i}} F(\vec{a}) = D_i F(\vec{a}) = \frac{\partial F(\vec{a})}{\partial x_i}}$

$$\therefore \quad D_i F(\vec{a}) = \lim_{h \to 0} \frac{F(\vec{a} + h e_i) - F(\vec{a})}{h}$$

$$= \lim_{h \to 0} \frac{F(a_i, \ldots, a_i + h, \ldots, a_n) - F(\vec{a})}{h}.$$

Definition: Let D be an open subset of $\mathbb{R}^n$, and $F: D \to \mathbb{R}^m$. F is differentiable at $\vec{a} \in D$ iff there exists a linear mapping $L: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\vec{h} \to \vec{0}} \frac{F(\vec{a}+\vec{h}) - F(\vec{a}) - L(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

$L(\vec{h}) = dF_a(\vec{h}) = M\vec{h}$

$dF_a(\vec{x}) = M\vec{x}$

$M = F'(\vec{a})$

Recall,

$$\lim_{h \to 0} \frac{f_i(a+h) - f_i(a)}{h} \iff \lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{h} = \vec{0}$$

exists $\Rightarrow$ differentiable $\qquad\qquad \Rightarrow$ differentiable.

L is denoted by dF_a and is called the differential of F at $\vec{a}$. There is a unique $m \times n$ matrix, denoted by M such that $dF_a(\vec{x}) = M\vec{x}$, $\forall \vec{x} \in \mathbb{R}^n$.

M is denoted by $F'(\vec{a})$, and is called the derivative of F at $\vec{a}$.

Example: If F is constant, $F(\vec{x}) = \vec{b}$ $\forall \vec{x} \in \mathbb{R}^n$ then F is differentiable everywhere, with dF_a being the zero mapping. $dF_a(\vec{x}) = \vec{0}$, $\forall \vec{x} \in \mathbb{R}^n$

$$\lim_{\vec{h} \to \vec{0}} \frac{\overbrace{F(\vec{a}+\vec{h})}^{\vec{b}} - \overbrace{F(\vec{a})}^{\vec{b}} - \overbrace{dF_a(\vec{h})}^{\to \vec{0}}}{\|\vec{h}\|} = \vec{0}$$

If F is linear, then F is differentiable everywhere $dF_a = F$

$$\lim_{\vec{h}\to\vec{0}} \frac{F(\vec{a}+\vec{h}) - F(\vec{a}) - dF_a(\vec{h})}{\|\vec{h}\|} = \lim_{\vec{h}\to\vec{0}} \frac{\cancel{F(\vec{a})} + F(\vec{h}) - \cancel{F(\vec{a})} - dF_a(\vec{h})}{\|\vec{h}\|}$$

$\Rightarrow$ Make $F(\vec{h}) = dF_a(\vec{h})$ to get $\vec{0}$.

<u>Example</u>. This is an example which shows that the existence of all directional derivatives is not sufficient for differentiability.

Let $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

f is not continuous at $(0,0)$.

[Choose $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$ as $n \to \infty$

But $\dfrac{2\left(\frac{1}{n}\right)^2 \cdot \left(\frac{1}{n^2}\right)}{\frac{1}{n^4} + \left(\frac{1}{n^2}\right)^2} = \frac{2}{2} = 1 \not\to 0 = f(0,0)$]

(<u>Thm</u>: If F is differentiable at $\vec{a}$, then F is continuous at $\vec{a}$)

HW next week!

f is not differentiable at $(0,0)$.

directional derivatives :

$\vec{v}(a,b)$, $b \neq 0$

$$D_{\vec{v}} f(0,0) = \lim_{h \to 0} \frac{f(ah, bh) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{2a^2h^2 bh}{a^4h^4 + b^2h^2} - 0}{h} = \frac{2a^2b}{b^2} = \frac{2a^2}{b}$$

$\vec{v}(a,0)$,

$$D_{\vec{v}} f(0,0) = \lim_{h \to 0} \frac{\overset{f(ah,0)}{0} - \overset{f(0,0)}{0}}{h} = 0.$$

$\Rightarrow$ directional derivatives exist !

Lec10 . $F: \mathbb{R}^n \to \mathbb{R}^m$ n-dim surface

$dF_{\vec{a}}$ $\vec{a} \in \mathbb{R}^n$ differential $dF_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{\vec{h} \to \vec{0}} \frac{F(\vec{a} + \vec{h}) - F(\vec{a}) - dF_a(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

$F'(\vec{a})$ derivative of F at $\vec{a}$ is $m \times n$ matrix

$dF_{\vec{a}}(\vec{x}) = F'(\vec{a})\,\vec{x}$

directional derivative : $D_{\vec{v}} F(\vec{a}) = \lim_{h \to 0} \frac{F(\vec{a} + h\vec{v}) - F(\vec{a})}{h}$

$\star$ All directional derivatives exist $\not\Rightarrow$ F is differentiable at this pt.

<u>Thm</u>: If F is differentiable at $\vec{a}$, then the directional derivative $D_{\vec{v}} F(\vec{a})$ exists for all (nonzero) $\vec{v} \in \mathbb{R}^n$, and

$$\boxed{D_{\vec{v}} F(\vec{a}) = dF_{\vec{a}}(\vec{v})} \quad \text{with} \quad \vec{v} = (v_1, v_2, \ldots, v_n)$$

Moreover $$\boxed{D_{\vec{v}} F(\vec{a}) = \sum_{j=1}^{n} v_j D_j F(\vec{a})}$$

<u>Proof</u>: Let $\vec{h} = t\vec{v}$, take $t \to 0$. $\|\vec{h}\| = |t| \, \|\vec{v}\| \to 0$,

thus, $\vec{h} \to \vec{0}$

$$\lim_{\vec{h} \to \vec{0}} \frac{F(\vec{a}+\vec{h}) - F(\vec{a}) - dF_a(\vec{h})}{\|\vec{h}\|} = \lim_{t \to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a}) - dF_a(t\vec{v})}{|t| \, \|\vec{v}\|}$$

dF_a is a linear mapping.

$$= \lim_{t \to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a}) - t \, dF_a(\vec{v})}{|t| \, \|\vec{v}\|}$$

$$= \frac{1}{\|\vec{v}\|} \left(\lim_{t \to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a})}{|t|} - \lim_{t \to 0} \frac{t \, dF_a(\vec{v})}{|t|} \right)$$

So $$\lim_{t \to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a})}{|t|} = \lim_{t \to 0} \frac{t \, dF_a(\vec{v})}{|t|} = \begin{cases} dF_a(\vec{v}), & t > 0 \\ -dF_a(\vec{v}), & t < 0 \end{cases}$$

When $t > 0$,

$$\lim_{t\to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a})}{t} = dF_{\vec{a}}(\vec{v})$$

When $t < 0$

$$\lim_{t\to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a})}{-t} = -dF_{\vec{a}}(\vec{v})$$

$$\Longleftrightarrow \lim_{t\to 0} \frac{F(\vec{a}+t\vec{v}) - F(\vec{a})}{t} = dF_{a}(\vec{v})$$

By definition of $D_{\vec{v}}F(\vec{a})$

$$D_{\vec{v}}F(\vec{a}) = dF_{\vec{a}}(\vec{v})$$

$\vec{v} = (v_1, v_2, \dots, v_n) = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$, $v_j \in \mathbb{R}$

$$D_{\vec{v}}F(\vec{a}) = dF_{\vec{a}}(\vec{v}) = \underbrace{dF_{\vec{a}}}_{\text{linear}}\left(\sum_{j=1}^{n} v_j\vec{e}_j\right) = \sum_{j=1}^{n} v_j \, dF_{\vec{a}}(\vec{e}_j)$$

$$= \sum_{j=1}^{n} v_j D_{\vec{e}_j}F(\vec{a}) = \boxed{\sum_{j=1}^{n} v_j D_j F(\vec{a}).}$$

<u>Thm</u>: If F is differentiable at $\vec{a}$, then $F'(\vec{a})$ is

(Recall $dF_{\vec{a}}(\vec{x}) = F'(\vec{a})\,\vec{x}$)

$$F'(\vec{a}) = \left(D_j\, F(\vec{a})\right) = \begin{bmatrix} D_1 F_1(\vec{a}) & \cdots & D_n F_1(\vec{a}) \\ D_1 F_2(\vec{a}) & \cdots & D_n F_2(\vec{a}) \\ \vdots & & \\ D_1 F_m(\vec{a}) & \cdots & D_n F_m(\vec{a}) \end{bmatrix}^{m\times n} \begin{array}{l} j=1,\dots m \\ j=1,\dots n \end{array}$$

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to F(\vec{x}) = \begin{pmatrix} F_1(\vec{x}) \\ F_2(\vec{x}) \\ \vdots \\ F_m(\vec{x}) \end{pmatrix} \qquad F_j : \mathbb{R}^n \to \mathbb{R}$$

<u>Proof</u>: $dF_{\vec{a}}(\vec{x}) = F'(\vec{a})\vec{x}$, $\vec{x} = \sum_{j=1}^{n} x_j \vec{e_j} \in \mathbb{R}^n$

$$\underbrace{dF_{\vec{a}}(\vec{x})}_{\mathbb{R}^m} = \begin{bmatrix} dF_{1\vec{a}}(\vec{x}) \\ dF_{2\vec{a}}(\vec{x}) \\ \vdots \\ dF_{m\vec{a}}(\vec{x}) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} x_j D_j F_1(\vec{a}) \\ \sum_{j=1}^{n} x_j D_j F_2(\vec{a}) \\ \vdots \\ \sum_{j=1}^{n} x_j D_j F_m(\vec{a}) \end{bmatrix}$$

Previous theorem

$$dF_{\vec{a}}(\vec{v}) = \sum_{j=1}^{n} v_j D_j F(\vec{a})$$

$$= \begin{bmatrix} x_1 D_1 F_1(\vec{a}) + x_2 D_2 F_1(\vec{a}) + \dots + x_n D_n F_1(\vec{a}) \\ \vdots \\ x_1 D_1 F_m(\vec{a}) + x_2 D_2 F_m(\vec{a}) + \dots + x_n D_n F_m(\vec{a}) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} D_1 F_1(\vec{a}), & D_2 F_1(\vec{a}), & \dots, & D_n F_1(\vec{a}) \\ D_1 F_2(\vec{a}), & D_2 F_2(\vec{a}), & \dots, & D_n F_2(\vec{a}) \\ \vdots & & & \\ D_1 F_m(\vec{a}), & D_2 F_m(\vec{a}), & \dots, & D_n F_m(\vec{a}) \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = F'(\vec{a})\,\vec{x}$$

↳ Jacobian Matrix

Defn: F is said to be continuously differentiable at $\vec{a}$ if the partial derivatives $D_j F(\vec{a})$, $j=1,\dots,n$ all exist at each point of some open set containing $\vec{a}$ and are continuous at $\vec{a}$.
(all partial derivatives)

Theorem: If F is continuously differentiable then it is differentiable at $\vec{a}$.

We will prove it next time.

Problem

Let $f: B_4(\vec{0}) \to \mathbb{R}^4$ be a differentiable function given by $f(x,y,z) = (x^3y^2,\ ze^{2y},\ z^2,\ 3x-2z)$.

(1) Write the defn. of $D_{(4,-5,-1)} f(2,0,3)$.

(2) Find $f'(2,0,3)$.

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Last time:

$F : \mathbb{R}^n \to \mathbb{R}^m$ differentiable at $\vec{a} \in \mathbb{R}^n$

$$D_{\vec{v}} F(\vec{a}) = dF_{\vec{a}}(\vec{v}) = \sum_{j=1}^{n} v_j D_j F(\vec{a})$$

and $dF_{\vec{a}}(\vec{x}) = F'(\vec{a})\vec{x}$

$$= \underbrace{\begin{bmatrix} D_1 F_1(\vec{a}) & D_2 F_1(\vec{a}) & \dots & D_n F_1(\vec{a}) \\ D_1 F_2(\vec{a}) & D_2 F_2(\vec{a}) & \dots & D_n F_2(\vec{a}) \\ \vdots & & & \\ D_1 F_m(\vec{a}) & D_2 F_m(\vec{a}) & \dots & D_n F_m(\vec{a}) \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

When $m = 1$, $F'(\vec{a}) = (D_1 F(\vec{a}), D_2 F(\vec{a}), \dots, D_n F(\vec{a}))$

$f : \mathbb{R}^n \to \mathbb{R}$ $\quad f'(a) = \nabla f(\vec{a})$ $\quad df_{\vec{a}}(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{x}$

gradient vector

Theorem: If F is continuously differentiable at $\vec{a}$, then F is differentiable at $\vec{a}$.

$$\text{diff} \iff \lim_{\vec{h} \to \vec{0}} \frac{F(\vec{a}+\vec{h}) - F(\vec{a}) - dF_{\vec{a}}(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

$\Uparrow$

cont. diff $\iff$ all partial derivs. exist & continuous at $\vec{a}$.

Proof: $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

We prove it for $f: \mathbb{R}^n \to \mathbb{R}$, then it can be generalized componentwise to F.

f is continuously differentiable at $\vec{a}$.

Recall: $df_{\vec{a}}(\vec{h}) = f'(\vec{a}) \cdot \vec{h} = \nabla f(\vec{a}) \cdot \vec{h} = \sum_{j=1}^{n} D_j f(\vec{a}) h_j$

Let $\vec{h} = (h_1, h_2, \dots, h_n)$

Denote $\vec{h_0} = (0, 0, 0 \dots, 0)$

$\vec{h_1} = (h_1, 0, 0, \dots, 0)$

$\vec{h_2} = (h_1, h_2, 0, \dots, 0)$

$\vdots$

$\vec{h_j} = (h_1, h_2, h_3, \dots, h_j, 0, \dots, 0)$

$\vdots$

$\vec{h_n} = \vec{h}.$

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = f(\vec{a}+\vec{h_n}) - f(\vec{a}+\vec{h}_{n-1}) + f(\vec{a}+\vec{h}_{n-1}) - f(\vec{a}+\vec{h}_{n-2}) + f(\vec{a}+\vec{h}_{n-2}) - \dots - f(\vec{a}+\vec{h_1}) + f(\vec{a}+\vec{h_1}) - f(\vec{a}+\vec{h_0})$$

$$= \sum_{j=1}^{n} \left(f(\vec{a}+\vec{h_j}) - f(\vec{a}+\vec{h}_{j-1}) \right)$$

For each j: $f(\vec{a}+\vec{h_j}) - f(\vec{a}+\vec{h}_{j-1}) = f(a_1+h_1, a_2+h_2, \dots, a_{j-1}+h_{j-1}, a_j+h_j, a_{j+1}, a_{j+2}, \dots, a_n)$

$- f(a_1+h_1, a_2+h_2, \dots, a_{j-1}+h_{j-1}, a_j$

$a_n)$

Denote $s_j(t) = f(a_1+h_1, \dots, a_{j-1}+h_{j-1}, t, a_{j+1}, a_{j+2}, \dots, a_n)$

$s_j : \mathbb{R} \to \mathbb{R}$ single valued single variable function.

$$s_j'(t) = \lim_{\tau \to 0} \frac{s_j(t+\tau) - s_j(t)}{\tau}$$

$$= \lim_{\tau \to 0} \frac{f(a_1+h_1, \dots, a_{j-1}+h_{j-1}, t+\tau, a_{j+1}, \dots a_n) - f(a_1+h_1, a_{j-1}+h_{j-1}, t, a_{j+1}, \dots, a_n)}{\tau}$$

$$\vec{b} = (a_1+h_1, \dots, a_{j-1}+h_{j-1}, t, a_{j+1}, \dots, a_n)$$

$$= \lim_{\tau \to 0} \frac{f(\vec{b}+\tau\vec{e_j}) - f(\vec{b})}{\tau} = D_j f(\vec{b})$$

f is continuously differentiable $\Rightarrow$ $D_j f$ exists near $\vec{a}$ and $D_j f$ is continuous at $\vec{a}$.

We use the mean value thm for $s_j(t)$

$$s_j(a_j+h_j) - s_j(a_j) = s_j'(c_j)\, h_j$$

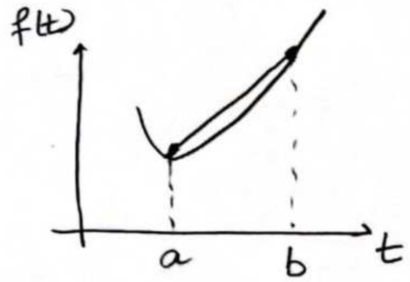
c_j is between a_j and a_j+h_j

Therefore, let $\vec{b_j} = (a_1+h_1, \dots, a_{j-1}+h_{j-1}, c_j, a_{j+1}, \dots, a_n)$

$$s_j'(c_j) = D_j f(\vec{b_j})$$

$$s_j(a_j+h_j) - s_j(a_j) = D_j f(\vec{b_j}) \cdot h_j$$

<u>MVT</u>



f cont in $[a,b]$

f diff in (a,b)

$\Rightarrow f(b)-f(a) = f'(c)(b-a)$

$$LHS = f(\vec{a}+\vec{h_j}) - f(\vec{a}+\vec{h_{j-1}}) = D_j f(\vec{b_j})\cdot h_j$$

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = \sum_{j=1}^{n}\left[f(\vec{a}+\vec{h_j}) - f(\vec{a}+\vec{h_{j-1}})\right]$$

$$= \sum_{j=1}^{n} D_j f(\vec{b_j}) h_j$$

$$\vec{b_j} = (a_1+h_1, \ldots, a_{j-1}+h_{j-1}, \underline{c_j}, a_{j+1}, \ldots, a_n)$$

$\hookrightarrow$ c_j is between a_j and a_j+h_j

When $\vec{h}\to\vec{0}$, each $\vec{b_j}\to\vec{a}$.

Let L be a linear mapping with $L(\vec{h}) = \sum_{j=1}^{n} D_j f(\vec{a})\, h_j$

then

$$0 \leq \lim_{\vec{h}\to\vec{0}} \frac{|f(\vec{a}+\vec{h}) - f(\vec{a}) - L(\vec{h})|}{\|\vec{h}\|} = \lim_{\vec{h}\to\vec{0}} \frac{\left|\sum_{j=1}^{n} D_j f(\vec{b_j})h_j - \sum_{j=1}^{n} D_j f(\vec{a})h_j\right|}{\|\vec{h}\|}$$

$$= \lim_{\vec{h}\to\vec{0}} \frac{\left|\sum_{j=1}^{n} h_j\,(D_j f(\vec{b_j}) - D_j f(\vec{a}))\right|}{\|\vec{h}\|}$$

triangle ineq

$$\leq \lim_{\vec{h}\to\vec{0}} \left[\frac{\sum_{j=1}^{n} |h_j|\,|D_j f(\vec{b_j}) - D_j f(\vec{a})|}{\|\vec{h}\|}\right] \overset{①}{\leq} \lim_{\vec{h}\to\vec{0}} \sum_{j=1}^{n} |D_j f(\vec{b_j}) - D_j f(\vec{a})| \overset{②}{=} 0$$

①

For each j: $\dfrac{|h_j|\,|D_j f(\vec{b_j}) - D_j f(\vec{a})|}{\|\vec{h}\|} \leq \dfrac{\cancel{|h_j|}\,|D_j f(\vec{b_j}) - D_j f(\vec{a})|}{\cancel{|h_j|}}$

$\hookrightarrow \left(\|\vec{h}\| = \sqrt{h_1^2+\ldots+h_n^2} \geq \sqrt{h_j^2} = |h_j|\right)$

② $\vec{b_j} \to \vec{a}$ and $D_j f$ is continuous at $\vec{a}$, so

$D_j f(\vec{b_j}) \to D_j f(\vec{a})$ as $\vec{h} \to \vec{0}$

$$\Rightarrow \lim_{\vec{h}\to\vec{0}} \frac{|f(\vec{a}+\vec{h}) - f(\vec{a}) - L(\vec{h})|}{\|\vec{h}\|} = 0.$$ as desired, with

$$L(\vec{h}) = \sum_{j=1}^{n} h_j D_j f(\vec{a})$$

$\blacksquare$

Summary

all directional derivatives exist (partial derivatives)

differentiable $\Rightarrow$ all directional derivatives exist; all directional derivatives exist $\not\Rightarrow$ differentiable

all directional derivatives exist + all partial derivatives are continuous. $\Rightarrow$ Continuously differentiable

Continuously differentiable $\Rightarrow$ differentiable.

differentiable $\not\Rightarrow$ Continuously differentiable

Eg: $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) ; & x \neq 0 \\ 0 ; & x = 0 \end{cases}$

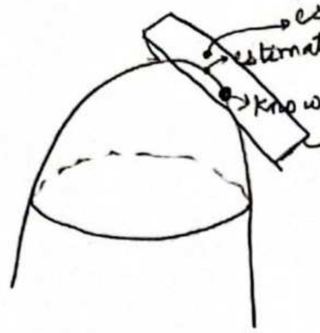
Linear approximation (estimate a function)

$$\lim_{\vec{h}\to\vec{0}} \frac{F(\vec{a}+\vec{h}) - F(\vec{a}) - dF_{\vec{a}}(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

when $\vec{h}$ is small

$$F(\vec{a}+\vec{h}) \approx F(\vec{a}) + dF_{\vec{a}}(\vec{h}) = \underbrace{F(\vec{a}) + F'(\vec{a})\,\vec{h}}_{\text{Tangent Plane.}}$$

$T(\vec{x}) = F(\vec{a}) + F'(\vec{a})(\vec{x} - \vec{a})$



⇈ Till here is Midterm 1

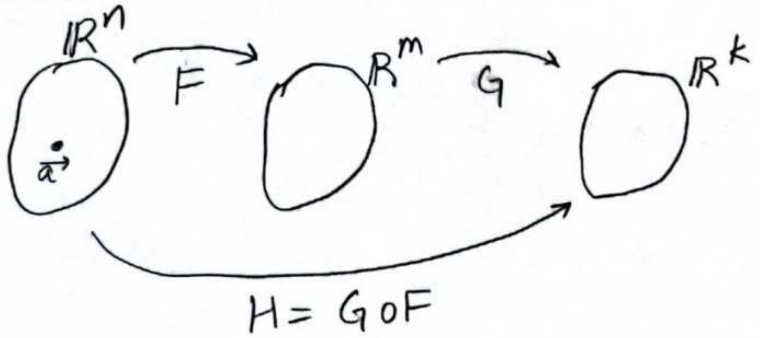
Lec

Chain Rule.

Recall: $f: \mathbb{R} \to \mathbb{R}$ $\quad g: \mathbb{R} \to \mathbb{R}$ $\quad h = g \circ f$ $\quad h(t) = g(f(t))$

$\Rightarrow h'(t) = g'(f(t))\, f'(t).$

In general: $F: \mathbb{R}^n \to \mathbb{R}^m$ $\quad G: \mathbb{R}^m \to \mathbb{R}^k$



$H'(\vec{a}) = ?$ $\quad dH_{\vec{a}} = ?$

Theorem Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open subsets. The mappings $F: U \to \mathbb{R}^m$ and $G: V \to \mathbb{R}^k$ and suppose they are differentiable at $\vec{a} \in \mathbb{R}^n$ and $F(\vec{a}) \in V$ respectively. Their composition $H = G \circ F$ is differentiable at $\vec{a}$, and $dH_{\vec{a}} = dG_{F(\vec{a})} \circ dF_{\vec{a}}$ → (linear mapping)

and $\underset{(k \times n)}{H'(\vec{a})} = \underset{\substack{\text{(matrix)} \\ (k \times m)}}{G'(F(\vec{a}))} \cdot \underset{(m \times n)}{F'(\vec{a})}$

Proof: We aim to show that
→ linear mapping.

$$\lim_{\vec{h} \to \vec{0}} \frac{H(\vec{a}+\vec{h}) - H(\vec{a}) - dG_{F(\vec{a})} \circ dF_{\vec{a}}(\vec{h})}{\|\vec{h}\|} = \vec{0}$$

Since F and G are differentiable at $\vec{a}$, $F(\vec{a})$ respectively.

$$\lim_{\substack{\vec{h} \to \vec{0} \\ \vec{h} \in \mathbb{R}^n}} \frac{F(\vec{a}+\vec{h}) - F(\vec{a}) - dF_{\vec{a}}(\vec{h})}{\|\vec{h}\|} = \vec{0} \quad \&$$

$$\lim_{\substack{\vec{k} \to \vec{0} \\ \vec{k} \in \mathbb{R}^m}} \frac{G(F(\vec{a}) + \vec{k}) - G(F(\vec{a})) - dG_{F(\vec{a})}(\vec{k})}{\|\vec{k}\|} = \vec{0}$$

Define: $R(\vec{h}) = \dfrac{F(\vec{a}+\vec{h}) - F(\vec{a}) - dF_a(\vec{h})}{\|\vec{h}\|}$, $\vec{h} \in \mathbb{R}^n$, $\vec{h} \neq \vec{0}$,

So $R(\vec{h}) \to \vec{0}$ as $\vec{h} \to \vec{0}$. $\vec{b} = F(\vec{a}) \in \mathbb{R}^m$.

$S(\vec{k}) = \dfrac{G(\vec{b}+\vec{k}) - G(\vec{b}) - dG_{\vec{b}}(\vec{k})}{\|\vec{k}\|}$, $\vec{k} \in \mathbb{R}^m$, $\vec{k} \neq \vec{0}$ so

$S(\vec{k}) \to \vec{0}$ as $\vec{k} \to \vec{0}$.

<u>Step 1</u>: $H(\vec{a}+\vec{h}) - H(\vec{a}) = ?$

$$H(\vec{a}+\vec{h}) - H(\vec{a}) = G(F(\vec{a}+\vec{h})) - G(F(\vec{a}))$$

$$= G(F(\vec{a}+\vec{h})) - G(\vec{b})$$

$$= G(\vec{b} + \underbrace{F(\vec{a}+\vec{h}) - \vec{b}}_{\text{call it by } \vec{k}}) - G(\vec{b})$$

(Let $\vec{k} = F(\vec{a}+\vec{h}) - \vec{b} = F(\vec{a}+\vec{h}) - F(\vec{a})$)

$$= G(\vec{b}+\vec{k}) - G(\vec{b})$$

$$= \underbrace{\|\vec{k}\|\, S(\vec{k})}_{\text{step2}} + \underbrace{dG_{\vec{b}}(\vec{k})}_{\text{Step3}}$$

$\vec{k} = \|\vec{h}\|\, R(\vec{h}) + dF_{\vec{a}}(\vec{h})$

<u>Step 2</u>: Because $H(\vec{a}+\vec{h}) - H(\vec{a}) = \|\vec{k}\|\, S(\vec{k}) + dG_{\vec{b}}(\vec{k})$

$\|\vec{k}\|\, S(\vec{k})$?

$$\|\vec{k}\|\, S(\vec{k}) = \big\| \|\vec{h}\| R(\vec{h}) + dF_{\vec{a}}(\vec{h}) \big\|\, S(\vec{k})$$

Let $\vec{v} = \dfrac{\vec{h}}{\|\vec{h}\|}$ unit vector, $dF_{\vec{a}}$ is linear, then $dF_{\vec{a}}(\vec{v}) = \dfrac{1}{\|\vec{h}\|} dF_{\vec{a}}(\vec{h})$

$$\|\vec{k}\|\, S(\vec{k}) = \big\| \|\vec{h}\| R(\vec{h}) + \|\vec{h}\|\, dF_{a}(\vec{v}) \big\|\, S(\vec{k})$$

$$= \|\vec{h}\| \, \big\| R(\vec{h}) + dF_{\vec{a}}(\vec{v}) \big\|\, S(\vec{k})$$

Step 3: $dG_{\vec{b}}(\vec{k})$?

$$dG_{\vec{b}}(\vec{k}) = dG_{\vec{b}}\left(\|\vec{h}\| R(\vec{h}) + dF_{\vec{a}}(\vec{h}) \right)$$

$$\overset{\text{linear}}{=} \|\vec{h}\|\, dG_{\vec{b}}\left(R(\vec{h})\right) + dG_{\vec{b}} \circ dF_{\vec{a}}(\vec{h})$$

Step 4:
$$\frac{H(\vec{a}+\vec{h}) - H(\vec{a}) - dG_{F(\vec{a})} \circ dF_{\vec{a}}(\vec{h})}{\|\vec{h}\|}$$

$$= \frac{\cancel{\|\vec{h}\|} \big\| R(\vec{h}) + dF_{\vec{a}}(\vec{v}) \big\| S(\vec{k}) + \|\vec{h}\|\, dG_{F(\vec{a})}\left(R(\vec{h})\right) + \cancel{dG_{F(\vec{a})} \circ dF_{\vec{a}}(\vec{h})} - \cancel{dG_{F(\vec{a})} \circ dF_{\vec{a}}(\vec{h})}}{\cancel{\|\vec{h}\|}}$$

$$= \underbrace{\big\| R(\vec{h}) + dF_{\vec{a}}(\vec{v}) \big\|\, S(\vec{k}) + dG_{F(\vec{a})}\left(R(\vec{h})\right)}$$

Step 5: $\to 0$? when $\vec{h} \to 0$?

Part ① $\|R(\vec{h}) + dF_{\vec{a}}(\vec{v})\| \, S(\vec{k})$

$S(\vec{k}) = S(F(\vec{a}+\vec{h}) - F(\vec{a}))$ as $\vec{h} \to \vec{0}$, $\vec{a}+\vec{h} \to \vec{a}$

F is diff. at $\vec{a}$ thus continuous at $\vec{a}$.

$$\vec{k} = F(\vec{a}+\vec{h}) - F(\vec{a}) \longrightarrow \vec{0}$$

So $S(\vec{k}) \longrightarrow \vec{0}$

Next, $\|R(\vec{h}) + dF_{\vec{a}}(\vec{v})\|$ bounded?

$\vec{v}$ is a unit vector, so it is on unit sphere in $\mathbb{R}^n$.

This unit sphere is closed and bounded, so it is compact.

And $dF_{\vec{a}}$ is continuous.

$dF_{\vec{a}}(\vec{v})$ is bounded for $\vec{v}$ in the unit sphere.

$R(\vec{h}) \to \vec{0}$ as $\vec{h} \to \vec{0}$.

Therefore, $\|R(\vec{h}) + dF_{\vec{a}}(\vec{v})\|$ is bounded.

$\Rightarrow \|R(\vec{h}) + dF_{\vec{a}}(\vec{v})\| \, S(\vec{k}) \longrightarrow \vec{0}$ as $\vec{h} \to \vec{0}$.

Part ② $dG_{F(\vec{a})}(R(\vec{h}))$. $\vec{h} \to \vec{0}$, $R(\vec{h}) \to \vec{0}$, then

$dG_{F(\vec{a})}$ is continuous & linear. $dG_{F(\vec{a})}(R(\vec{h})) \to dG_{F(\vec{a})}(\vec{0}) \underset{\text{linear}}{=} \vec{0}$

<u>Example</u>: $f: \mathbb{R} \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}$. $\Rightarrow h = g \circ f : \mathbb{R} \to \mathbb{R}$.

$h'(t)$ is scalar, $f'(t) = (f_1'(t), f_2'(t), \ldots, f_m'(t))$

$$g'(\vec{x}) = \nabla g(\vec{x}) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_m} \right)$$

$$\boxed{h'(t) = \nabla g(f(t)) \cdot f'(t)}$$

<u>Example</u>: $\underset{F' \to 3 \times 2}{F}: \mathbb{R}^2 \to \mathbb{R}^3$, $\underset{G' \to 2 \times 3}{G}: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad H = G \circ F : \mathbb{R}^2 \to \mathbb{R}^2$

$$H'(\vec{a}) = \begin{bmatrix} D_1 H_1(\vec{a}) & D_2 H_1(\vec{a}) \\ D_1 H_2(\vec{a}) & D_2 H_2(\vec{a}) \end{bmatrix} = \underbrace{\begin{bmatrix} D_1 G_1(F(\vec{a})) & D_2 G_1 & D_3 G_1 \\ & & \end{bmatrix}}_{2 \times 3} \underbrace{\overset{3 \times 2}{\begin{bmatrix} D_1 F_1(\vec{a}) & D_2 F_1(\vec{a}) \\ \vdots & \vdots \\ & \end{bmatrix}}}_{F'(\vec{a})}$$

$$dH_{\vec{a}}(\vec{x}) = H'(\vec{a}) \cdot \vec{x}, \quad \vec{x} \in \mathbb{R}^2.$$

<u>Lec 16</u>

$f: \mathbb{R} \to \mathbb{R}^m$, $g: \mathbb{R}^m \to \mathbb{R}$; $h = g \circ f : \mathbb{R} \to \mathbb{R}$

$$h'(t) = \nabla g(f(t)) \cdot f'(t).$$

$t \to f(t) = \vec{x} = (x_1, \ldots, x_m) \qquad x_i = f_i(t)$

$u = h(t) = g(f(t)) = g(\vec{x})$

$$\nabla g(\vec{x}) = \left(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \ldots, \frac{\partial g}{\partial x_m}\right) \quad ; \quad f'(t) = \left(\frac{df_1}{dt}, \frac{df_2}{dt}, \ldots, \frac{df_m}{dt}\right)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_m}\frac{dx_m}{dt}$$

$F: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad$ $G: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad$ $H = G \circ F = \mathbb{R}^2 \to \mathbb{R}^2.$

$(s,t) \to (x,y,z)$ $\quad$ $(x,y,z) \to (u,v)$ $\quad$ $(s,t) \to (u,v)$

$$H' = \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix}$$

$$H'(\vec{a}) = G'(F(\vec{a})) \cdot F'(\vec{a})$$

$$G' = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t}$$

$$F' = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}$$

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial s}$$

Similarly for $\frac{\partial u}{\partial s}$ & $\frac{\partial v}{\partial t}$.

$f: \mathbb{R} \to \mathbb{R}$ $\quad f'(t) = 0 \quad \forall t \in \mathbb{R} \Leftrightarrow f(t) = C, \quad \forall t \in \mathbb{R}.$

$f: \mathbb{R}^n \to \mathbb{R}$?

$f: \mathbb{R}^n \to \mathbb{R}^m$?

<u>Definition</u>: The open set $U \subseteq \mathbb{R}^n$ is said to be connected iff given any two points $\vec{a}, \vec{b} \in U$, there is a differentiable mapping $\phi: \mathbb{R} \to U$ s.t $\phi(0) = \vec{a}$ & $\phi(1) = \vec{b}$

In $\mathbb{R}^n$ connected $\Leftrightarrow$ path connected $\underset{U \text{ open}}{\Longleftrightarrow}$ ϕ being differentiable.

<u>theorem</u>: Let U be a connected open subset of $\mathbb{R}^n$. The differentiable mapping $F: U \to \mathbb{R}^m$ is constant on U iff $F'(\vec{x}) = 0$ (zero matrix) for all $\vec{x} \in U$.

<u>Proof</u>: Note F is constant iff each component is constant. We only need to prove it for $f: U \to \mathbb{R}$.

"$\Rightarrow$" $f: U \to \mathbb{R}$

$(x_1, \dots, x_n) \to f(x_1, \dots, x_n)$

Suppose f is constant

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{x}+\vec{h}) - f(\vec{x}) - 0}{\|\vec{h}\|} = 0 \quad \forall \vec{x} \in U.$$

$df_{\vec{x}}$ is zero mapping $f'(\vec{x}) = \vec{0}$

"$\Leftarrow$" For $\vec{a}, \vec{b} \in U$ arbitrary, there is a differentiable function $\phi : \mathbb{R} \to U$, $\phi(0) = \vec{a}$, $\phi(1) = \vec{b}$.

$$\phi : \mathbb{R} \to U, \quad f : U \to \mathbb{R}$$

$$t \to \underset{\substack{\uparrow \\ U}}{\phi(t)} \longrightarrow f(\phi(t)) \longrightarrow \mathbb{R}$$

Let $h = f \circ \phi : \mathbb{R} \to \mathbb{R}$

$$h'(t) = \nabla f(\phi(t)) \cdot \phi'(t)$$

∇f is always zero vector by assumption. So $h'(t) = 0 \ \forall t \in \mathbb{R}$.

Therefore, $h(0) = h(1) = f(\phi(1)) = f(\vec{b})$.

$$h(0) = f(\vec{a}) = f(\phi(0))$$

$$\Rightarrow f(\vec{a}) = f(\vec{b})$$

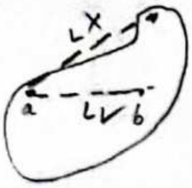
Since $\vec{a}, \vec{b}$ are arbitrary in U, f is constant on U. ∎

<u>Corollary</u>: F, G two differentiable $U \to \mathbb{R}^m$. U is a connected open subset of $\mathbb{R}^n$.

If $F'(\vec{x}) = G'(\vec{x}) \ \forall \vec{x} \in U$, then there exists $\vec{C} \in \mathbb{R}^m$ s.t. $F(\vec{x}) = G(\vec{x}) + \vec{C} \quad \forall \vec{x} \in U$.

Mean Value Theorem for $f: \mathbb{R}^n \to \mathbb{R}$ (not really applicable for $\mathbb{R}^n \to \mathbb{R}^m$)

<u>Theorem</u>: Suppose U is an open subset of $\mathbb{R}^n$ and $\vec{a}, \vec{b}$ are two points in U such that U contains the line segment L from $\vec{a}$ to $\vec{b}$.



If $f: U \to \mathbb{R}$ is differentiable, then

$$f(\vec{b}) - f(\vec{a}) = f'(\vec{c})\,(\vec{b} - \vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$$

for some $\vec{c} \in L$.

<u>Proof</u>: Let $\phi: [0,1] \to L$. $\phi(t) = (1-t)\vec{a} + t\vec{b}$

$$\phi(0) = 1\vec{a} + 0\vec{b} = \vec{a}$$

$$\phi(1) = (1-1)\vec{a} + 1\vec{b} = \vec{b}$$

$$\phi'(t) = \vec{b} - \vec{a}$$

Let $h = f \circ \phi : [0,1] \to \mathbb{R}$ Use mean value thm for h (single variable)

$$h(1) - h(0) = h'(\xi)(1-0) = h'(\xi) \quad , \xi \text{ b/w } 0 \,\&\, 1.$$

$$h(1) = f(\phi(1)) = f(\vec{b}) \;; \quad h(0) = f(\phi(0)) = f(\vec{a}).$$

Chain Rule $h'(\xi) = \nabla f(\phi(\xi)) \cdot \phi'(\xi) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$

(Let $\vec{c} = \phi(\xi) \in L$)

Therefore, $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$ ∎

<u>Definition</u>: The zero set S of a function $g: \mathbb{R}^n \to \mathbb{R}$ is

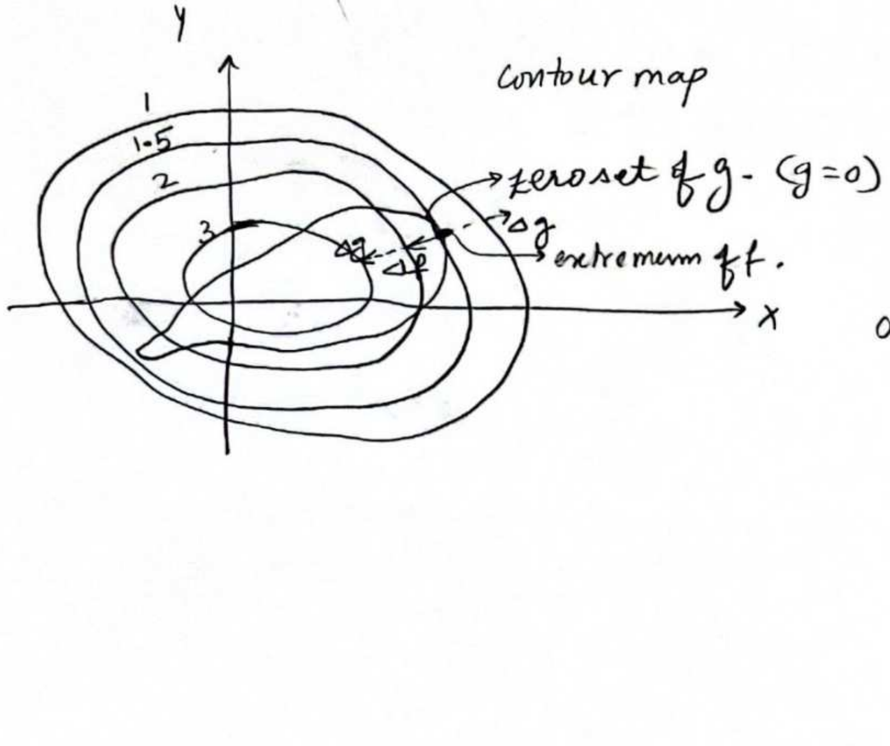
$$S = \{\vec{x} \in \mathbb{R} : g(\vec{x}) = 0\}$$

<u>Theorem</u>: Let f, g be continuously differentiable real-valued functions on $\mathbb{R}^2$; $f, g : \mathbb{R}^2 \to \mathbb{R}$.

Suppose f attains it's maximum or minimum value on the zero set S of g at the point $\vec{p}$ where $\nabla g(\vec{p}) \neq \vec{0}$.

Then $\nabla f(\vec{p}) = \lambda \nabla g(\vec{p})$ for some $\lambda \in \mathbb{R}$.

λ is called a <u>Lagrange Multiplier</u>.



Δg & Δf will line up at the extreme value of f.

Lec 16

Symmetry of Second derivatives

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable

$D_i F : \mathbb{R}^n \to \mathbb{R}^m$ fix i

$$D_j (D_i F)(\vec{a}) = D_j D_i F(\vec{a}) = \frac{\partial^2 F}{\partial x_j \partial x_i}$$

<u>Theorem:</u> Let $f : U \to \mathbb{R}$, U is an open set in $\mathbb{R}^n$.
If the first and second partial derivatives of f exist and are continuous on U, then $D_i D_j f = D_j D_i f$ on U.

<u>Proof:</u> f is continuously differentiable, because of the existence of the second partial derivatives.
$D_i f$ (any i) is continuously differentiable, because we assume 2^{nd} derivatives are continuous.

Thus f and $D_i f$ are differentiable on U.

$$D_{\vec{v}} f(\vec{a}) = \sum_{j=1}^{n} v_j \cdot D_j f(\vec{a})$$

By definition,

$$D_i D_j f(\vec{a}) = \lim_{h \to 0} \frac{D_j f(\vec{a} + h e_i) - D_j f(\vec{a})}{h}$$

$$= \lim_{h \to 0} \left(\frac{\lim_{k \to 0} \frac{f(\vec{a} + h\vec{e_i} + k\vec{e_j}) - f(\vec{a} + h\vec{e_i})}{k} - \lim_{k \to 0} \frac{f(\vec{a} + k e_j) - f(}{k}}{h} \right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\lim_{k\to 0}\frac{f(\vec{a}+h\vec{e_i}+k\vec{e_j})-f(\vec{a}+h\vec{e_i})-f(\vec{a}+k\vec{e_j})+f(\vec{a})}{k}\right)$$

$$D_j D_i f(a) = \lim_{k\to 0}\left(\frac{1}{k}\lim_{h\to 0}\frac{f(\vec{a}+k\vec{e_j}+h\vec{e_i})-f(\vec{a}+k\vec{e_j})-f(\vec{a}+h\vec{e_i})+f(\vec{a})}{h}\right)$$

The only difference is the order of the limits.

Note: the limiting operations are not always interchangeable.

eg: $g(h,k) = e^{\frac{1}{h}-\frac{1}{k}}$ $\quad \lim_{h\to 0}\left(\lim_{k\to 0} g(h,k)\right) = 0$

$\lim_{k\to 0}\left(\lim_{h\to 0} g(h,k)\right) = \infty$

Denote $\Delta^2 f_{\vec{a}}(\vec{u};\vec{v}) = f(\vec{a}+\vec{u}+\vec{v}) - f(\vec{a}+\vec{u}) - f(\vec{a}+\vec{v}) + f(\vec{a})$

$= [f(\vec{a}+\vec{u}+\vec{v}) - f(\vec{a}+\vec{u})] - [f(\vec{a}+\vec{v}) - f(\vec{a})]$

$\Delta^2 f_{\vec{a}}(h\vec{e_i}, k\vec{e_j})$ is what we want.

Define $g(\vec{x}) = f(\vec{x}+\vec{v}) - f(\vec{x}) \Rightarrow \nabla g(\vec{x}) = \nabla f(\vec{x}+\vec{v}) - \nabla f(\vec{x})$

$\Rightarrow \Delta^2 f_{\vec{a}}(\vec{u},\vec{v}) = g(\vec{a}+\vec{u}) - g(\vec{a})$

Apply twice the mean value theorem.

b/w $\vec{a}$, $\vec{a}+\vec{u}$

$\Delta^2 f_{\vec{a}}(\vec{u},\vec{v}) = g(\vec{a}+\vec{u}) - g(\vec{a}) \overset{MVT}{=} \nabla g(\vec{a}+\alpha\vec{u}) \cdot (\vec{a}+\vec{u}-\vec{a})$

where $\alpha \in [0,1]$

$= [\nabla f(\vec{a}+\alpha\vec{u}+\vec{v}) - \nabla f(\vec{a}+\alpha\vec{u})] \cdot \vec{u}$

$\hookrightarrow \vec{u} = (u_1, u_2, \dots, u_n)$

& $\nabla f = (D_1 f, \dots, D_n f)$

$$= \sum_{j=1}^{n} u_j \left(D_j f(\vec{a}+\alpha\vec{u}+\vec{v}) - D_j f(\vec{a}+\alpha\vec{u}) \right)$$

$$= \sum_{j=1}^{n} u_j D_j f(\vec{a}+\alpha\vec{u}+\vec{v}) - \sum_{j=1}^{n} u_j D_j f(\vec{a}+\alpha\vec{u})$$

$$= D_{\vec{u}} f(\vec{a}+\alpha\vec{u}+\vec{v}) - D_{\vec{u}} f(\vec{a}+\alpha\vec{u})$$

$$\overset{MVT}{=} \nabla D_{\vec{u}} f(\vec{a}+\alpha\vec{u}+\beta\vec{v}) \cdot \vec{v}$$

$\beta \in [0,1]$

$$= \sum_{j=1}^{n} v_j D_j (D_{\vec{u}} f)(\vec{a}+\alpha\vec{u}+\beta\vec{v})$$

$$= D_{\vec{v}} D_{\vec{u}} f(\vec{a}+\alpha\vec{u}+\beta\vec{v}) \qquad \exists \alpha, \beta \in [0,1]$$

Let $\vec{u} = h\vec{e_i}$; $\vec{v} = k\vec{e_j}$

$$\Rightarrow \Delta^2 f_{\vec{a}}(h\vec{e_i}, k\vec{e_j}) = D_{k\vec{e_j}} D_{h\vec{e_i}} f(\vec{a}+\alpha h\vec{e_i}+\beta k\vec{e_j})$$

Property : $D_{t\vec{v}} f(\vec{a}) = t D_{\vec{v}} f(\vec{a})$, t is fixed.

$$\lim_{h\to 0} \frac{f(\vec{a}+ht\vec{v}) - f(\vec{a})}{h} \underset{\substack{\text{let} \\ k=th}}{=} \lim_{k\to 0} \frac{f(\vec{a}+k\vec{v}) - f(\vec{a})}{\frac{1}{t}k}$$

$$= t \lim_{k\to 0} \frac{f(\vec{a}+k\vec{v}) - f(\vec{a})}{k}$$

$$= t D_{\vec{v}} f(\vec{a})$$

$$= k\, D_{\vec{e_j}} \left(h\, D_{\vec{e_i}}\, f(\vec{a} + \alpha h \vec{e_i} + \beta k \vec{e_j}) \right)$$

<u>Property</u>: $D_{\vec{v}} (t\, f(\vec{a})) = t\, D_{\vec{v}} f(\vec{a})$, t fixed.

$$\| $$

$$\lim_{h \to 0} \frac{t\, f(\vec{a} + h\vec{v}) - t\, f(\vec{a})}{h} = t \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

$$kh\, D_{\vec{e_j}} D_{\vec{e_i}}\, f(\vec{a} + \alpha h \vec{e_i} + \beta k \vec{e_j}) = kh\, D_j D_i\, f(\vec{a} + \alpha h \vec{e_i} + \beta k \vec{e_j})$$

$$D_i D_j\, f(\vec{a}) = \lim_{h \to 0} \left(\lim_{k \to 0} \frac{\cancel{kh}\, D_j D_i\, f(\vec{a} + \alpha h \vec{e_i} + \beta k \vec{e_j})}{\cancel{hk}} \right)$$

$$= \lim_{h \to 0} \left(\lim_{k \to 0} D_j D_i\, f(\vec{a} + \alpha h \vec{e_i} + \beta k \vec{e_j}) \right)$$

Because $D_j D_i f$ is continuous.

$$\vec{a} + \alpha h \vec{e_i} + \beta k \vec{e_j} \xrightarrow{k \to 0} \vec{a} + \alpha h \vec{e_i}$$

$$\therefore D_j D_i f(\ldots) \xrightarrow{k \to 0} D_j D_i f(\vec{a} + \alpha h \vec{e_i})$$

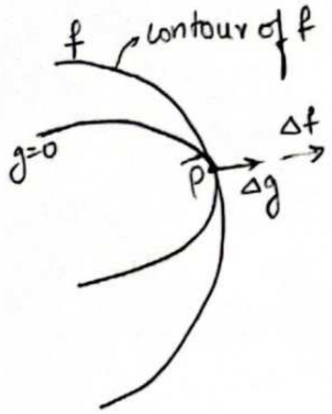
Then $h \to 0$ by the continuity again,

$$D_j D_i\, f(\vec{a} + \alpha h \vec{e_i}) \xrightarrow{h \to 0} D_j D_i f(\vec{a})$$

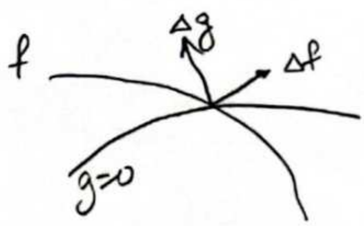
So $D_i D_j f(\vec{a}) = D_j D_i f(\vec{a})$

$\blacksquare$

Lecture #17 (Lagrange Multipliers)



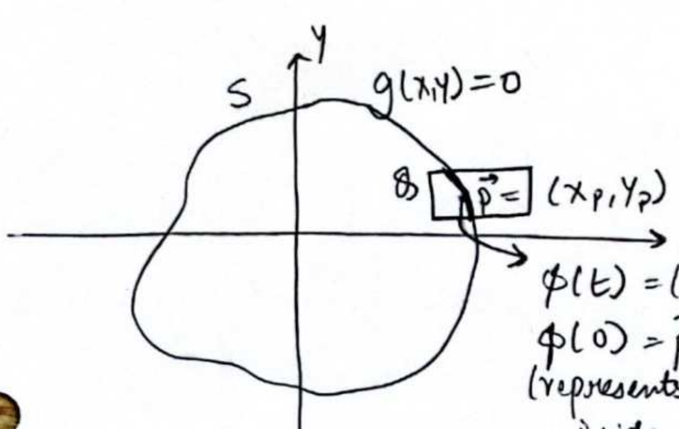
If Δg & Δf are aligned, the contours must be tangent at $\vec{P}$. Therefore when f moves on the contour of g away from $\vec{P}$ it increases in both directions. $\Rightarrow$ Min.



If Δg & Δf are not aligned, the contours cross & f goes ↑ along g in one direction & f goes ↓ in the other direction $\Rightarrow$ not max/min.

Proof: There is a rectangle Q centered at $\vec{P}$ and a differentiable curve $\phi : \mathbb{R} \to \mathbb{R}^2$ with $\phi(0) = \vec{P}$ and $\phi'(0) \neq \vec{0}$ s.t. S and the image of ϕ agree inside Q.

(ϕ is used to parameterize the zero set $S = \{(x,y) \mid g(x,y) = 0\}$)



$\phi(t) = (\phi_1(t), \phi_2(t))$
$\phi(0) = \vec{p}$
(represents curve around p inside rectangle)

This result is from the implicit function theorem (we will prove it later)

$\boxed{G(x,y,z) = 0 \text{ + some condition } \quad z = h(x,y) \text{ near a pt.}}$

$g(\phi(t)) = 0$ for t sufficiently small ($\phi(t)$ is inside Q).

By the chain rule at point $\vec{p}$

$$\underbrace{\nabla g(\phi(t))}_{\hookrightarrow \text{constant}(=0).} \cdot \phi'(t) = 0.$$

$t=0$: $\nabla g(\vec{p}) \cdot \phi'(0) = 0$

Since f attains an extremum on S at $\vec{P}$, $h(t) = f(\phi(t))$

$h(t)$ has an extremum at $t=0$.

$h'(t) = \nabla f(\phi(t)) \cdot \phi'(t)$

$h'(0) = \nabla f(\vec{p}) \cdot \phi'(0) = 0$ because here is the extremum.

Therefore, $\nabla g(\vec{p}) \perp \phi'(0)$ and $\nabla f(\vec{p}) \perp \phi'(0)$.

$\nabla g(\vec{p}) \parallel \nabla f(\vec{p})$ (parallel). $\exists \lambda \in \mathbb{R}$ s.t $\nabla f(\vec{p}) = \lambda \nabla g(\vec{p})$

Example: Find the maximum and minimum of $f(x,y) = \underbrace{ax^2+2bxy+cy^2}_{\text{quadratic form.}}$ on the unit circle $g(x,y) = x^2+y^2-1=0$

Solution: Unit circle is compact $\Rightarrow$ existence of max/min ✓
f is cont.

Set up $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 2ax+2by = \lambda 2x & \Rightarrow ax+by = \lambda x \quad ① \\ \cancel{2}bx+\cancel{2}cy = \cancel{2}y\lambda & \Rightarrow bx+cy = \lambda y \quad ② \end{cases}$

$\nabla f = (2ax+2by, 2bx+2cy)$

$\nabla g = (2x, 2y)$

also $x^2+y^2-1=0$ ③ Solve for x, y, λ.

① $\Leftrightarrow ax - \lambda x + by = 0 \Leftrightarrow (a-\lambda)x + by = 0$

$\Leftrightarrow (a-\lambda, b) \cdot (x,y) = 0$

② $bx + (c-\lambda)y = 0$

$\Leftrightarrow (b, c-\lambda) \cdot (x, y) = 0$

$\Rightarrow (a-\lambda, b) \perp (x, y)$ & $(b, c-\lambda) \perp (x, y)$

$\Rightarrow (a-\lambda, b) \parallel (b, c-\lambda)$

$\Rightarrow (a-\lambda, b) \times (b, c-\lambda) = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a-\lambda & b & 0 \\ b & c-\lambda & 0 \end{vmatrix} = (0, 0, (a-\lambda)(c-\lambda)-b^2) = 0$$

$\Rightarrow (a-\lambda)(c-\lambda) - b^2 = 0 \quad \Rightarrow \quad \lambda^2 - (a+c)\lambda + ac - b^2 = 0$

$\lambda_1 + \lambda_2 = a+c \qquad \lambda_{1,2} = \frac{1}{2}\left(a+c \pm \sqrt{(a-c)^2 + 4b^2}\right)$

$\lambda_1 \lambda_2 = ac - b^2$

For each λ_i, we assume the solution is (x_i, y_i, λ_i)

$$f(x_i, y_i) = ax_i^2 + 2bx_iy_i + cy_i^2 = (ax_i + bx_iy_i) + (bx_iy_i + cy_i^2)$$

$$= (ax_i + by_i)x_i + (bx_i + cy_i)y_i$$

$$= \lambda_i x_i x_i + \lambda_i (y_i y_i)$$

$$= \lambda_i x_i^2 + \lambda_i y_i^2 \quad \text{on } g(x,y) = 0$$

$$= \underline{\underline{\lambda_i}} \qquad i = 1, 2.$$

Therefore λ_1, λ_2 are the max & min values on $x^2 + y^2 = 1$

<u>Remark</u>: The quadratic form $f(x,y) = ax^2 + 2bxy + cy^2$ is

(i) positive definite if $a > 0$ and $ac - b^2 > 0$.

Recall: symmetric matrix A & $\vec{x}^T A \vec{x} > 0 \Rightarrow$ pos. def.

$f(x,y) > 0$ whatever $(x,y) \in \mathbb{R}^2$.

(Similar to $f(\vec{x}) = \vec{x}^T A \vec{x} > 0 \;\; \forall \vec{x}$)

(given) $ac - b^2 > 0 \Rightarrow \lambda_1 \lambda_2 > 0 \Rightarrow$ Same sign of max / min

$ac > b^2 \geq 0 \Rightarrow ac > 0 \Rightarrow a, c$ same sign.

also $\lambda_1 + \lambda_2 = a + c > 0 \iff a > 0$ (given)

$\Rightarrow \lambda_1, \lambda_2$ are positive. $\Rightarrow f$ is positive!

(ii) negative definite if $a < 0$ and $\underbrace{ac - b^2 > 0}_{\lambda_1, \lambda_2 \text{ same sign.}}$

(iii) non semidefinite if $ac - b^2 < 0$. (λ_1, λ_2 different signs)
$\Rightarrow f$ will have both positive & negative values.

Lecture #

Taylor's Formula.

Review: $f: \mathbb{R} \to \mathbb{R}$ (single variable) real valued

Suppose $f^{(K+1)}$ exists at each point of the open interval between a and x,

$f^{(K)}$ is continuous on the closed interval between a and x.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$
$$+ \frac{f^{(K)}(a)}{K!}(x-a)^k + R_K$$

$$R_K = \frac{f^{(K+1)}(\xi)}{(K+1)!}(x-a)^{K+1} \qquad \exists\, \xi \text{ b/w a and } x.$$

Theorem: $f: \mathbb{R} \to \mathbb{R}$, $f^{(K+1)}$ exists in a neighbourhood of a & is continuous at a. Suppose $f^{(j)}(a) = 0$ for $j = 1, 2, \dots, k-1$ but $f^{(K)}(a) \neq 0$, then (1) f has a local minimum at a if K is even and $f^{(K)}(a) > 0$.

(2) f has a local maximum at a if K is even and $f^{(K)}(a) < 0$.

(3) f has neither a maximum nor a minimum at a if K is odd

The above theorem allows us to classify $\underbrace{\text{critical points}}_{\text{ie } f'(a)=0}$.

$f: \mathbb{R}^n \to \mathbb{R}$

<u>Notation</u>: $D_i f(\vec{a})$ $\quad \vec{a} \in \mathbb{R}^n$ $\quad$ 2^{nd} derivative : $D_i D_j f$ $\quad i,j \in \{1 \dots, n\}$

$\quad$ k_{th} derivatives? assume f is <u>smooth enough</u> $\to$ interchange order. , we can arrange D_i's : $D_1^{j_1} D_2^{j_2} D_3^{j_3} \dots D_n^{j_n} f(\vec{a})$; here $j_1+j_2+\dots+j_n = k$,

$j_1, \dots, j_n \in \mathbb{Z}_+ \cup \{0\}$.

$\vec{h} = (h_1, \dots, h_n)$

<u>Notation</u>: $D_{\vec{h}} f \triangleq \sum_{i=1}^{n} h_i D_i f = h_1 D_1 f + h_2 D_2 f + \dots + h_n D_n f$.

<u>2nd derivative</u>:

$$D_{\vec{h}} D_{\vec{h}} f = \sum_{i=1}^{n} h_i D_i (D_{\vec{h}} f) = \sum_{i=1}^{n} h_i D_i \Big(\underbrace{\sum_{j=1}^{n} h_j D_j f}_{\text{independant of } i}\Big)$$

$n=2$: $\quad h_1 D_1 (h_1 D_1 f + h_2 D_2 f) + h_2 D_2 (h_1 D_1 f + h_2 D_2 f)$

$\Rightarrow h_1^2 D_1^2 f + 2 h_1 h_2 D_1 D_2 f + h_2^2 D_2^2 f$ $\quad$ (since smooth)

$$= \sum_{j_1+j_2=2} (\text{coef})\, h_1^{j_1} h_2^{j_2} D_1^{j_1} D_2^{j_2} f$$

$n=3$: $\quad h_1 D_1 (h_1 D_1 f + h_2 D_2 f + h_3 D_3 f) + h_2 D_2 (\text{''}) + h_3 D_3 (\text{''})$

$= (h_1 D_1 + h_2 D_2 + h_3 D_3)^2 f$

$$\Rightarrow \sum_{j_1+j_2+j_3=2} \text{coeff}\; h_1^{j_1} h_2^{j_2} h_3^{j_3} D_1^{j_1} D_2^{j_2} D_3^{j_3} f$$

coeff : $\binom{2}{j_1 j_2 \cdots j_n} = \dfrac{2!}{j_1!\, j_2! \ldots j_n!}$

from multinomial theorem.

$(x_1 + \ldots + x_n)^k =$

k^{th} derivative :

$$\underbrace{D_{\vec h} D_{\vec h} \cdots D_{\vec h}}_{k} f = \boxed{D_{\vec h}^{k} f} = (h_1 D_1 + h_2 D_2 + \ldots + h_n D_n)^k f$$

$$= \sum_{j_1+\ldots+j_n=k} \binom{k}{j_1\, j_2 \ldots j_n} h_1^{j_1} h_2^{j_2} \cdots h_n^{j_n} D_1^{j_1} D_2^{j_2} \cdots D_n^{j_n} f$$

coefficient : $\binom{k}{j_1 \ldots j_n} = \dfrac{k!}{j_1!\, j_2! \ldots j_n!}$

Taylor's Formula

U is open, all iterated partial derivatives of f, of order atmost k+1 exist & continuous on U

$f: \mathbb{R}^n \to \mathbb{R}$ is of class C^{k+1} on an open set containing the line segment L from $\vec a$ to $\vec a + \vec h$, then there exists $\vec\xi \in L$ such that

$$f(\overset{\vec x}{\vec a + \vec h}) = \underbrace{P_k(\overset{\vec x - \vec a}{\vec h})}_{k\text{th Taylor polynomial}} + R_k(\vec h) \longrightarrow k\text{th degree remainder.}$$

$$P_K(\vec{h}) = \sum_{r=0}^{K} \frac{D_{\vec{h}}^{r} f(\vec{a})}{r!} \qquad R_K(\vec{h}) = \frac{D_{\vec{h}}^{(K+1)} f(\vec{\xi})}{(K+1)!}$$

<u>Example</u>: If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $K=2$ 2nd order Taylor's formula

For $\vec{a} = (a_1, a_2)$ $\quad \vec{h} = (h_1, h_2)$ $\quad \vec{x} = (a_1+h_1, a_2+h_2)$

$$f(\vec{x}) = f(\vec{a}+\vec{h}) = f(a_1, a_2) + \frac{D_{\vec{h}}^{1} f(a_1, a_2)}{1!} + \frac{D_{\vec{h}}^{2} f(a_1, a_2)}{2!} + R$$

$$D_{\vec{h}} f = \sum_{j_1+j_2=1} \underbrace{\binom{1}{j_1 \quad j_2}}_{= \frac{1!}{j_1!\, j_2!}} h_1^{j_1} h_2^{j_2} D_1^{j_1} D_2^{j_2} f = h_1 D_1 f + h_2 D_2 f$$

$$D_{\vec{h}}^{2} f = \sum_{j_1+j_2=2} \binom{2}{j_1 \quad j_2} h_1^{j_1} h_2^{j_2} D_1^{j_1} D_2^{j_2} f$$

$$= h_1^2 D_1^2 f + 2 h_1 h_2 D_1 D_2 f + h_2^2 D_2^2 f$$

$$P_2(\vec{h}) = f(a_1, a_2) + h_1 D_1 f(a_1, a_2) + h_2 D_2 f(a_1, a_2)$$
$$+ \frac{1}{2}\Big(h_1^2 D_1^2 f(a_1, a_2) + 2 h_1 h_2 D_1 D_2 f(a_1, a_2)$$
$$+ h_2^2 D_2^2 f(a_1, a_2)\Big)$$

$\vec{x} = (x, y)$

$$\Rightarrow D_1 f = \frac{\partial f}{\partial x}, \quad D_2 f = \frac{\partial f}{\partial y}, \quad D_1 D_2 f = \frac{\partial^2 f}{\partial x \partial y}.$$

Lecture

- **Review:**

$f: \mathbb{R}^n \to \mathbb{R}$, C^{k+1} on open set $U \subseteq \mathbb{R}^n$ containing L from $\vec{a}$ to $\vec{x}$.

$\exists\, \vec{\xi}$ on L : $f(\vec{x}) = f(\vec{a} + \vec{h}) = P_k(\vec{h}) + R_k(\vec{h})$

$\vec{h} = \vec{x} - \vec{a}$

with $$P_k(\vec{h}) = \sum_{r=0}^{k} \frac{D^r_{\vec{h}} f(\vec{a})}{r!}$$

$$R_k(\vec{h}) = \frac{D^{k+1}_{\vec{h}} f(\vec{\xi})}{(k+1)!}$$

- **Remark.** : $$\lim_{\vec{h} \to \vec{0}} \frac{R_k(\vec{h})}{\|\vec{h}\|^k} = 0$$

Proof: (Idea)

$$D^{k+1}_{\vec{h}} f(\vec{\xi}) = \sum_{\substack{j_1 + j_2 + \dots + j_n \\ = k+1}} \binom{k+1}{j_1\ j_2\ \dots\ j_n} \underbrace{h_1^{j_1} h_2^{j_2} \dots h_n^{j_n}}_{k+1 > k.} D_1^{j_1} \dots D_n^{j_n} f(\vec{\xi})$$

$\vec{\xi}$ approaches $\vec{a}$ as $\vec{h} \to \vec{0}$

($\Rightarrow$ as $\vec{h} \to 0$ numerator goes to 0 faster)

Example : If $f: \mathbb{R}^2 \to \mathbb{R}$ 2nd order Taylor's Polynomial about (a_1, a_2).

Last time: $$P_2(\vec{h}) = f(a_1, a_2) + h_1 D_1 f(a_1, a_2) + h_2 D_2 f(a_1, a_2) + \frac{1}{2}\left(h_1^2 D_1^2 f(a_1, a_2) + 2 h_1 h_2 D_1 D_2 f(a_1, a_2) + h_2^2 D_2^2 f(a_1, a_2)\right)$$

$f(x,y)$: $D_1 f = \frac{\partial f}{\partial x}$; $D_2 f = \frac{\partial f}{\partial y}$

$$f' = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$H_f = f'' = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

Hessian

assume f is of class C^3 $\quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

$P_2(\vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h}$ → Similar to $f(a) + f'(a)(x-a) + \ldots$ from single variable Taylor

$$+\frac{1}{2}\left(h_1^2 \frac{\partial^2 f}{\partial x^2}(\vec{a}) + 2 h_1 h_2 \frac{\partial^2 f}{\partial x \partial y}(\vec{a}) + h_2^2 \frac{\partial^2 f}{\partial y^2}(\vec{a})\right)$$

$$= f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \underbrace{\frac{1}{2}(h_1, h_2)\begin{bmatrix} & H_f & \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}_{\text{Similar to } \frac{1}{2} f''(a)(x-a)^2 \,\to\, \text{third term!}}$$

3^{rd} degree Taylor polynomial. f of class C^4.

$$P_3(\vec{h}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{h} + \frac{1}{2} \vec{h}^T f''(\vec{a}) \vec{h} + \frac{1}{6}\left(D^3_{\vec{h}} f(\vec{a})\right)$$

$$D^3_{\vec{h}} f(\vec{a}) = \Big[\underset{1}{?}\, h_1^3 \frac{\partial^3 f}{\partial x^3}(\vec{a}) + \underset{3}{?}\, h_1^2 h_2 \frac{\partial^3 f}{\partial x^2 \partial y}(\vec{a}) + \underset{3}{?}\, h_1 h_2^2 \frac{\partial^3 f}{\partial x \partial y^2}(\vec{a}) + \underset{1}{?}\, h_2^3 \frac{\partial^3 f}{\partial y^3}(\vec{a})$$

$$\binom{3}{j_1, j_2} = \frac{3!}{j_1! \, j_2!}$$

↳ 1 3 3 1

- For critical points we need to classify them as maxima, minima, saddle, none.
- Recall $\nabla f = 0$ at critical points. We need to consider 2^{nd} degree Taylor polynomial.

Quadratic Form

$f: \mathbb{R}^n \to \mathbb{R}$ of at least C^3; $\vec{a} \in \mathbb{R}^n$ is a critical point of f, $\nabla f(\vec{a}) = \vec{0}$; $f(\vec{a}+\vec{h}) = f(\vec{a}) + \nabla f(\vec{a}).\vec{h} + \frac{1}{2}\vec{h}^T f''(\vec{a})\vec{h}$

$$\boxed{f(\vec{a}+\vec{h}) - f(\vec{a}) = \frac{1}{2}\vec{h}^T f''(\vec{a})\vec{h}} \quad \text{since } \nabla f(\vec{a}) = 0$$

$q(\vec{h}) = \frac{1}{2}\vec{h}^T f''(\vec{a})\vec{h}$ is called the quadratic form of f at the critical point $\vec{a}$.

Theorem: Let f be of class C^3 in a neighbourhood of the critical point $\vec{a}$.

(a) f has a local minimum if $q(\vec{h})$ is positive definite

(b) f has a local maximum if $q(\vec{h})$ is negative definite

(c) f has a saddle point at $\vec{a}$ is $q(\vec{h})$ is non semidefinite

(d) otherwise, anything can occur.

[positive def $\Rightarrow$ for any $\vec{h}$, if $q(\vec{h})$ is positive & vice versa.
nonsemidef $\Rightarrow$ for $\vec{h}$ if $q(\vec{h})$ is positive or negative]

$\ulcorner f''(\vec{a}) = H_f \quad q(\vec{h}) = \frac{1}{2}\vec{h}^T H_f \vec{h}$.

H_f is symmetric $\Rightarrow q(\vec{h})$ is positive definite

$\Rightarrow H_f$ is positive def.

Similarly, H_f is neg def $\Rightarrow q(\vec{h})$ is neg def. $\lrcorner$

<u>**Proof**</u>: Let $\vec{h} \neq \vec{0}$. $\vec{v} = \frac{\vec{h}}{\|\vec{h}\|}$ unit vector on unit sphere in $\mathbb{R}^n$

<u>for (a)</u>

$S^{n-1} \to$ n-1 dimension sphere

then $q(\vec{h}) = \frac{1}{2}\vec{h}^T H_f \vec{h}$

$$= \frac{1}{2}\|\vec{h}\|\, \vec{v}^T H_f \|\vec{h}\|\, \vec{v}$$

$$= \|\vec{h}\|^2 \frac{1}{2}\vec{v}^T H_f \vec{v} = \|\vec{h}\|^2 q(\vec{v})$$

$\vec{v}$ is a unit vector $\Rightarrow S^{n-1}$ is compact $\Rightarrow q(\vec{v})$ attains max + min on S^{n-1}

$\vec{b} \in S^{n-1}$, $q(\vec{b})$ is min. $\Rightarrow q(\vec{v}) \geq \underset{\text{positive def}}{q(\vec{b})} > 0 \Rightarrow \underline{\underline{q(\vec{h}) > 0}}$.

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = \underbrace{q(\vec{h})}_{\text{always positive}} + R_2(\vec{h})$$

Since $\lim_{\vec{h}\to\vec{0}} \frac{R_2(\vec{h})}{\|\vec{h}\|^2} = 0$

For $\delta = \frac{1}{2}q(\vec{b})$, there exists $\epsilon > 0$, s.t $\|\vec{h}\| < \epsilon$,

$$\left|\frac{R_2(\vec{h})}{\|\vec{h}\|^2}\right| < \delta$$

$$-\frac{1}{2}q(\vec{b}) \le \frac{R_2(\vec{h})}{\|\vec{h}\|^2} \le \frac{1}{2}q(\vec{b})$$

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = \|\vec{h}\|^2\left[q(\vec{v}) + \frac{R_2(\vec{h})}{\|\vec{h}\|^2}\right] > \|\vec{h}\|^2\left(q(\vec{v}) - \frac{1}{2}q(\vec{b})\right)$$

$$\ge \|\vec{h}\|^2\left(q(\vec{b}) - \frac{1}{2}q(\vec{b})\right)$$

$$= \frac{1}{2}\|\vec{h}\|^2 q(\vec{b})$$

$$> 0$$

$\Rightarrow f(\vec{a}+\vec{h}) - f(\vec{a}) > 0 \Rightarrow f(\vec{a})$ is minimum.

Similar for max & saddle point. ■

Practice q:

$f(x,y) = e^{xy}$ about $(1,0)$ $\quad \vec{h} = (x-1, y-0) = (x-1, y)$

(1) Find first & 2nd Taylor polynomial about $(1,0)$

$P_1(\vec{h}) = P_1(x-1, y)$ $\qquad P_2(\vec{h}) = P_2(x-1, y)$

(2) Use a graphing tool, (MATLAB), show $f(x,y)$, P_1, P_2 (on one figure) & $(1,0,1)$. Set $x \in [0,2]$; $y \in [-1,-1]$.

(ccordor)
(uniqname)

How to classify critical points of $f: \mathbb{R}^2 \to \mathbb{R}$, f is C^3.

① Find critical points by $\nabla f(\vec{x}) = \vec{0}$ solve for $\vec{x} = (x_1, \dots, x_n)$

② $$H_f(\vec{a}) = \left.\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}\right|_{\vec{x} = (x,y) = \vec{a}}$$

③ Determine $H_f(\bar{a})$ pos-def, neg-def, nonsemidef, none of these

use linear algebra.

Find eigenvalues of $H_f(\vec{a}) = A$.

$\overset{\lambda_i}{\vec{v}_i}$ eigenvectors $\to \{\vec{v}_i\}$ basis

$$\vec{h} = \sum_{i=1}^{n} y_i \vec{v}_i$$

$$q(\vec{h}) = \vec{h}^T A \vec{h} = \sum_{i=1}^{n} y_i^2 \lambda_i = (y_1, \dots, y_n) \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$q_{max} = \max\{\lambda_i\}$, $q_{min} = \min\{\lambda_i\}$ for unit $\vec{h}$ on S^{n-1} sphere.

* Find all eigenvalues of $H_f(\vec{a})$

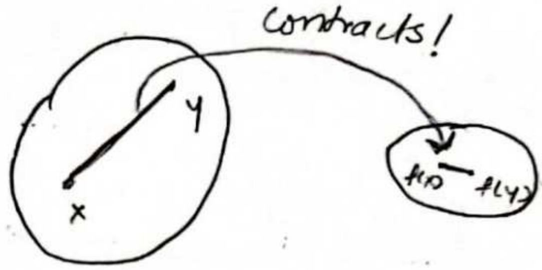
if all $\lambda_i > 0$ $\Rightarrow$ pos def $\Rightarrow$ local min

if all $\lambda_i < 0$ $\Rightarrow$ neg def $\Rightarrow$ local max

if $\max(\lambda_i) > 0$ & $\min(\lambda_i) < 0$ $\Rightarrow$ nonsemidef $\Rightarrow$ saddle!

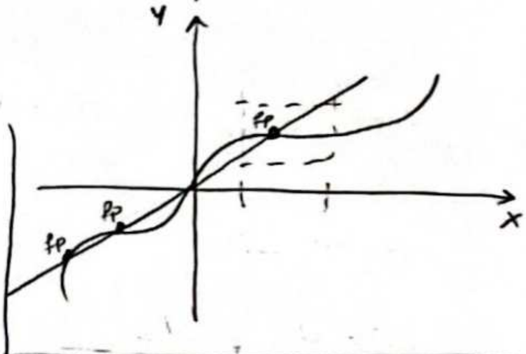
Chapter III Focus on Sec 1.2.3

Definition: The mapping f is called a contraction mapping with contraction constant $K<1$ if $\|f(x)-f(y)\| \leq K\|x-y\|$ for all x, y in the domain.



Remark: A contraction mapping is always uniformly continuous. (I think just pick $\epsilon < K\delta$).

Definition: A point x^* is called a fixed point of a mapping f, if $x^* = f(x^*)$



Theorem: Let $f: [a,b] \to [a,b]$ be a contraction mapping with contraction constant $K<1$. Then f has a unique fixed point x^*.

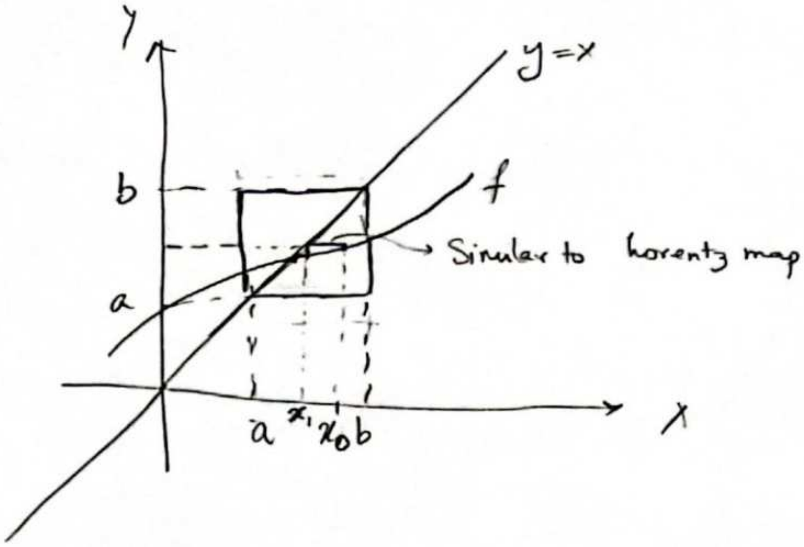
(Converse is not true).

MVT

$|f(y)-f(x)| = |f'(z)||y-x|$

$\Rightarrow$ if $|f'(z)| < 1$ we have a contraction map.

Moreover, given x_0, then the sequence $\{x_n\}$ defined inductively by $x_{n+1} = f(x_n)$ converges to x^*, $x_n \to x^*$ as $n \to \infty$



If f is not a contraction mapping the sequence diverges from the fixed point

Proof: First, we confirm the fixed point is unique.

Given x_1^*, x_2^* are two fixed points.

$$|x_1^* - x_2^*| = |f(x_1^*) - f(x_2^*)| \leq k|x_1^* - x_2^*|$$

→ contraction map!

Since $k<1 \Rightarrow x_1^* = x_2^*$.

Next, we want to show $x_n \to x^*$.

$$|x_2 - x_1| = |f(x_1) - f(x_0)| \leq k|x_1 - x_0|$$

$$\vdots$$

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq k|x_n - x_{n-1}| \leq kk|x_{n-1} - x_{n-2}|$$

$$\leq kkk|x_{n-2} - x_{n-3}|$$

By induction, $|x_{n+1} - x_n| \leq k^n|x_1 - x_0|$

Let $m>n>0$, $m, n \in \mathbb{N}$

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} + x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq k^{m-1}|x_1 - x_0| + k^{m-2}|x_1 - x_0| + \dots + k^n|x_1 - x_0|$$

$$= (k^{m-1} + k^{m-2} + \dots + k^n)|x_1 - x_0|$$

$$= k^n \left(\frac{1 - k^{m-n}}{1-k} \right) |x_1 - x_0|$$

$$< k^n \frac{1}{1-k} |x_1 - x_0|$$

$$\Rightarrow |x_m - x_n| < \frac{k^n}{1-k} |x_1 - x_0| \longrightarrow 0 \text{ as } m, n \to \infty$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in $[a, b]$.
By completeness of $[a, b]$ we know the cauchy sequence is convergent. $\Rightarrow x_n \longrightarrow \hat{x}$ in $[a, b]$.

By uniqueness of fixed point $\hat{x} = x^*$.

Since $x_{n+1} = f(x_n)$ As $n \to \infty$ $\hat{x} = f(\hat{x})$. since f is cont.
$\Rightarrow \hat{x}$ is <u>the</u> fixed pt.

<u>Remark</u>: The error estimation of the fixed point iteration $x_{n+1} = f(x_n)$.

$$|x_n - x^*| \leq \frac{k^n}{1-k} |x_0 - x_1|$$

estimate

true fixed pt.

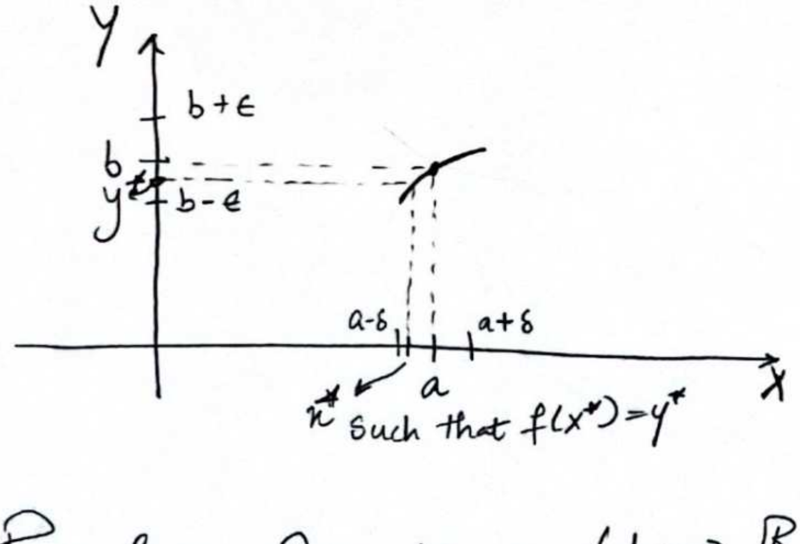
Lec #

Inverse Function Theorem.

Let $f: \mathbb{R} \to \mathbb{R}$ be a C' function such that $f(a) = b$ and $f'(a) \neq 0$. Then there exists neighbourhood $U = [a-\delta, a+\delta]$ and $V = [b-\epsilon, b+\epsilon]$ such that given $y^* \in V$, the sequence $\{x_n\}$ defined by

$$x_0 = a \quad , x_{n+1} = x_n - \frac{f(x_n) - y^*}{f'(a)} \quad \to \text{(Newton's method)}$$

converges to a unique point $x^* \in U$ such that $f(x^*) = y^*$



Proof: Denote $g: U \to \mathbb{R}$ with $g(x) = x - \frac{f(x) - y^*}{f'(a)}$

$$g(x_n) = x_n - \frac{f(x_n) - y^*}{f'(a)} = x_{n+1}$$

We need to prove g has a unique fixed point on $\mathcal{U}$.
(By previous theorem, $g: \mathcal{U} \to \mathcal{U}$ & g is a contraction mapping, if we can show both, we are done).

$$|g(x_1) - g(x_2)| = \left| \left(x_1 - \frac{f(x_1) - y^*}{f'(a)} \right) - \left(x_2 - \frac{f(x_2) - y^*}{f'(a)} \right) \right|$$

(may not work)

Let's use the Mean Value Theorem for g:

$$|g(x_1) - g(x_2)| = |g'(c)||(x_1 - x_2)| \qquad c \text{ is b/w } x_1, x_2.$$

need to show $|g'(c)| = k < 1$

Notice

$$|g'(x)| = \left| 1 - \frac{f'(x)}{f'(a)} \right| = \left| \frac{f'(a) - f'(x)}{f'(a)} \right|$$

$$x \in \mathcal{U} = [a-\delta, a+\delta]$$

Because f' is continuous, we can choose δ small enough such that $|f'(a) - f'(x)| \leq \frac{1}{2}|f'(a)|$

$$|g'(x)| \leq \frac{1}{2} \frac{\cancel{|f'(a)|}}{\cancel{|f'(a)|}} = \frac{1}{2}$$

By MVT: $|g(x_1) - g(x_2)| \leq \frac{1}{2}|x_1 - x_2| \Rightarrow g$ is a contraction mapping //

It remains to show that g maps $[a-\delta, a+\delta]$ to $[a-\delta, a+\delta]$. Let $x \in [a-\delta, a+\delta]$ be arbitrary.

$$|g(x) - a| \leq |g(x) - g(a)| + |g(a) - a|$$

$$\leq \frac{1}{2}|x-a| + \left| \frac{\overbrace{f(a)}^{b} - \overbrace{y^*}^{y^* \in [b-\epsilon, b+\epsilon]}}{f'(a)} \right| \leq \frac{1}{2}\delta + \frac{\epsilon}{|f'(a)|}$$

We want to control $\frac{\epsilon}{|f'(a)|} \leq \frac{1}{2}\delta$

We restrict $\epsilon \leq \frac{1}{2}\delta |f'(a)|$

So $|g(x) - a| \leq \delta$ for all $x \in [a-\delta, a+\delta]$.

Therefore, $g(x)$ has a unique fixed point x^* on $[a-\delta, a+\delta]$

$$x^* = g(x^*) = x^* - \frac{f(x^*) - y^*}{f'(a)}$$

$$0 = 0 - \frac{f(x^*) - y^*}{f'(a)} \Rightarrow f(x^*) - y^* = 0.$$

$$\Rightarrow f(x^*) = y^*$$

$\blacksquare$

Implicit function theorem

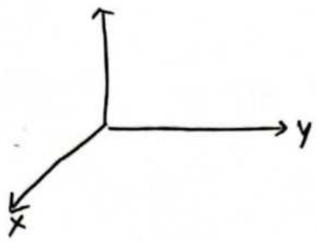
Let $G : \mathbb{R}^2 \to \mathbb{R}$ be C^1. $G(a,b)=0$ and $D_2 G(a,b) \neq 0$

Then there exists a continuous function

$f : [a-\delta, a+\delta] \to \mathbb{R}$ such that $y = f(x)$ solves

the equation $G(x, \overset{f(x)}{y}) = 0$ in a neighbourhood of

(a,b). (watch recording ending)
for intuition on why $D_2 \neq 0$.



In particular, define

$$f_0(x) \equiv b, \quad f_{n+1}(x) = f_n(x) - \frac{G(x, f_n(x))}{D_2 G(a,b)}$$

then the sequence $\{f_n\}$ converges uniformly to f on the interval $U = [a-\delta, a+\delta]$.

Idea of proof : existence of f $\to$ iteration is true.

$\downarrow$

$f_n \xrightarrow{\text{uniformly}} f$

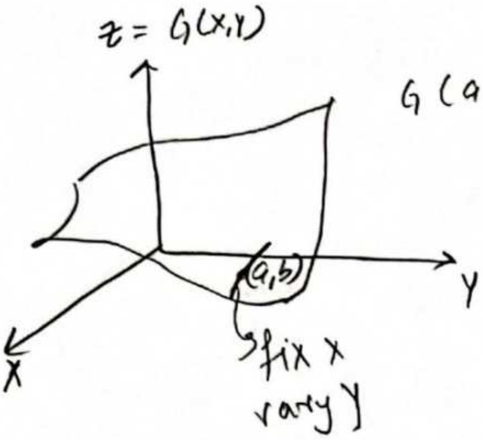
$\downarrow$

f is continuous.

Lecture#

Proof (beautiful): We only prove it for $D_2 G(a,b) > 0$.

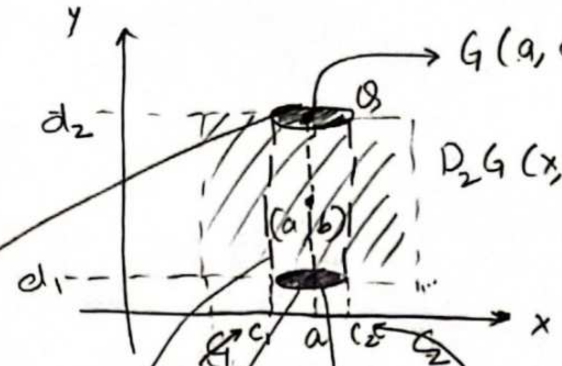
G is C^1 so G and $D_2 G$ are both continuous.



$G(a,b) = 0$

We construct a rectangle

$$Q = [c_1, c_2] \times [d_1, d_2]$$



$\to G(a, d_2) > 0$

$D_2 G(x,y) > 0$ on Q.

In this neighbourhood $G > 0$ since G is continuous

$\to G(a, d_1) < 0$

In this neighbourhood $G < 0$

Shrink Q
So it is sandwiched b/w the two neighbourhoods.

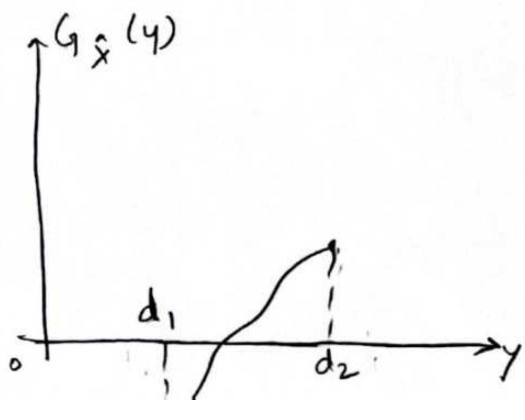
Let us still denote $Q = [c_1, c_2] \times [d_1, d_2]$, so we have

$$\begin{cases} D_2 G(x,y) > 0, & (x,y) \in Q \\ G(x, d_2) > 0, & c_1 \le x \le c_2 \\ G(x, d_1) < 0, & c_1 \le x \le c_2 \end{cases}$$

Fix $\hat{x} \in [c_1, c_2]$. Define $G_{\hat{x}} : [d_1, d_2] \to \mathbb{R}$

$$G_{\hat{x}}(y) = G(\hat{x}, y)$$

$$\begin{cases} G'_{\hat{x}} > 0 \quad \text{on } [d_1, d_2] \\ G_{\hat{x}}(d_2) > 0 \\ G_{\hat{x}}(d_1) < 0 \end{cases}$$



There is a unique $\hat{y} \in [d_1, d_2]$

such that $G_{\hat{x}}(\hat{y}) = 0$, i.e. $G(\hat{x}, \hat{y}) = 0$.

This proves the existence of f.

Next, we study the iteration.

$\hat{x} \in [c_1, c_2]$ fixed. How to compute $\hat{y}$.

$y_0 = b$, $\boxed{y_{n+1} = y_n - \dfrac{G(\hat{x}, y_n)}{D_2 G(a,b)}} \to y_{n+1} = g(y_n)$

($y_n \xrightarrow{?} \hat{y}$ and $\hat{y}$ is unique?)

(Take $n \to \infty$, $\cancel{\hat{y}} = \cancel{\hat{y}} - \dfrac{G(\hat{x}; \hat{y}) = 0}{D_2 G(a,b)}$?)

Define $g : [d_1, d_2] \to \mathbb{R}$

$$g(y) = y - \frac{G(\hat{x}, y)}{D_2 G(a,b)}$$

Then $\hat{y}$ is a fixed point of g. (because $\hat{y} = g(\hat{y})$)

We want to show that $\hat{y}$ is unique.

First, we show that g is a contraction mapping.

$$|g'(y)| = \left| 1 - \frac{D_2 G(\hat{x}, y)}{D_2 G(a,b)} \right| = \left| \frac{D_2 G(a,b) - D_2 G(\hat{x}, y)}{D_2 G(a,b)} \right|$$

Since $D_2 G$ is continuous, we can choose a small enough neighbour of (a,b) such that

$$|D_2 G(a,b) - D_2 G(\hat{x}, y)| \leq \frac{1}{2} D_2 G(a,b).$$

(This may change δ)

Next, we show $g : [b-\epsilon, b+\epsilon] \longrightarrow [b-\epsilon, b+\epsilon]$

for $\epsilon > 0$ small, we restrict $[b-\epsilon, b+\epsilon] \subseteq [d_1, d_2]$

then we replace $[d_1, d_2]$ by $[b-\epsilon, b+\epsilon]$

Suppose $y \in [b-\epsilon, b+\epsilon]$

$$|g(y) - b| \leq \underbrace{|g(y) - g(b)|}_{|g'(\xi)||y-b|} + |g(b) - b| \leq \frac{1}{2}|y-b| + \left|\frac{G(\hat{x}, b)}{D_2 G(a,b)}\right|$$

Since $G(a,b) = 0$ and G is continuous, $D_2 G(a,b)$ is a fixed number, there exists $\delta > 0$ such that

$\forall\ |\hat{x} - a| < \delta$, we can guarantee $|G(\hat{x}, b)| \leq \frac{1}{2} \epsilon |D_2 G(a,b)|$

$$|g(y) - b| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

So $g(y) \in [b-\epsilon, b+\epsilon]$

By previous thm, $\exists\ \hat{y}$ fixed pt of g & $\hat{y}$ is unique.

$\therefore$ The iteration

$$f_0(x) \equiv b\ ; \quad f_{n+1}(x) = f_n(x) - \frac{G(x, f_n(x))}{D_2 G(a,b)}$$

$$\left(f_n(\hat{x}) = \hat{y}_n \text{ then } \hat{y}_n \to \hat{y}\right)$$

Review : $f_n(t)$ $\quad t \in I \quad f_n : I \to \mathbb{R} \quad f : I \to \mathbb{R}$

$$f_n \to f \text{ as } n \to \infty$$

$\forall\, \epsilon > 0,\ \forall\, t \in I,\ \exists N = N(t, \epsilon),\ |f_n(t) - f(t)| < \epsilon\ \forall\, n > N.$

$f_n \to f$ uniformly means

$\forall\, \epsilon > 0,\ \exists N = N(\epsilon),\ |f_n(t) - f(t)| < \epsilon\ \forall\, n > N,\ \forall\, t \in I$

<u>Lec:</u> contd. Let $\hat{x} \in [a-\delta, a+\delta]$ use $y_0 = f_0(\hat{x}) = b$, $y_{n+1} = f_{n+1}(\hat{x})$

$$= \underbrace{f_n(\hat{x})}_{y_n} - \frac{G(\hat{x}, f_n(\hat{x}))}{D_2 G(a,b)}$$

we have proved $y_n \to \hat{y} = f(\hat{x})$ as $n \to \infty$.

It means $f_n \to f$ pointwisely for every $\hat{x} \in [a-\delta, a+\delta]$

Let $\gamma > 0$, $\exists N = N(\hat{x}, \gamma)$ $\quad |f_n(\hat{x}) - f(\hat{x})| < \gamma\ \forall\ n > N$ $\hookrightarrow$ pointwise convergence

Next, we want to show $f_n \to f$ uniformly.

Let $x \in [a-\delta, a+\delta]$ be arbitrary, let $y_n = f_n(x)$.

rom last time

$$g(\cdot) = \cdot - \frac{G(x, \cdot)}{D_2 G(a,b)}$$ is a contraction mapping with contraction constant $\frac{1}{2}$.

$$|g(z_1) - g(z_2)| \leq \frac{1}{2}|z_1 - z_2|$$

$$g: [b-\epsilon, b+\epsilon] \longrightarrow [b-\epsilon, b+\epsilon]$$

By the error estimation of the contraction mapping's fixed point, y is the fixed point of g, $y_{n+1} = g(y_n)$, $y_0 = b$, $y_n \to y$

$$|f_n(x) - f(x)| = |y_n - y| \leq \frac{(\frac{1}{2})^n}{1-(\frac{1}{2})} |y_0 - y_1| = (\tfrac{1}{2})^{n-1} |y_0 - y_1| = \frac{1}{2^{n-1}} |b - g(b)|$$

$$\leq \frac{\epsilon}{2^{n-1}}.$$

Let $\gamma > 0$. Choose N such that $\frac{\epsilon}{2^{n-1}} < \gamma$, $N > 1 + \log_2\left(\frac{\epsilon}{\gamma}\right)$

$\forall x$: $|f_n(x) - f(x)| \leq \frac{\epsilon}{2^{n-1}} < \gamma \quad \forall n > N$.

$f_n \to f$ uniformly.

Last, we prove f is continuous.

$f_0(x) \equiv b$ is continuous

$f_1(x) = f_0(x) - \frac{G(x, f_0(x))}{D_2 G(a,b)}$ is continuous, because $f_0, G, D_2 G$ are continuous.

$\vdots$

$f_{n+1}(x) = f_n(x) - \frac{G(x, f_n(x))}{D_2 G(a,b)}$ is continuous by induction.

Every f_n is continuous.

Let $\gamma > 0$, try to show for some $\tau > 0$ such that

$$|x - \hat{x}| < \tau \Rightarrow |f(x) - f(\hat{x})| < \gamma \ ?$$

Let $\hat{x} \in [a-\delta, a+\delta]$ be arbitrary.

$$|f(x) - f(\hat{x})| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(\hat{x})| + |f_n(\hat{x}) - f(\hat{x})|$$

① $f_n \to f$ uniformly, set n to be very large,

$|f(x) - f_n(x)| < \frac{\gamma}{3}$ and $|f(\hat{x}) - f_n(\hat{x})| < \frac{\gamma}{3}$

② f_n is continuous, $\exists\, \tau > 0$ $|x - \hat{x}| < \tau$

$\Rightarrow |f_n(x) - f_n(\hat{x})| < \frac{\gamma}{3}$.

Combine ① & ②, $|f(x) - f(\hat{x})| < \gamma$

So f is a continuous function. $\blacksquare$

Textbook Section III Theorem 3.4 is a more generalised version of the Implicit function theorem.

$G: \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^n$. C^1

$$\underset{\mathbb{R}^m}{\underset{\uparrow}{G(\vec{a}}}, \underset{\mathbb{R}^n}{\underset{\uparrow}{\vec{b}}}) = \underset{\mathbb{R}^n}{\underset{\uparrow}{\vec{0}}}$$

$D_2 G(\vec{a}, \vec{b})$ is a matrix. <u>not zero</u>

$\uparrow$ w.r.t $\vec{b}$ $\quad D_2 G(\vec{a}, \vec{y}): \mathbb{R}^n \to \mathbb{R}^n$

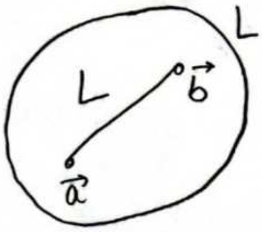
$\vec{y} = h(\vec{x})$ near $(\vec{a}, \vec{b})$

$\vec{b} = h(\vec{a})$ $\qquad G(\vec{x}, h(\vec{x})) = 0$

$h_0(\vec{x}) = \vec{b}$, $\quad h_{n+1}(\vec{x}) = h_n(\vec{x}) - \left(D_2 G(\vec{a}, \vec{b})\right)^{-1} G(\vec{x}, h_n(\vec{x}))$

III.2 Mean Value Theorem for $f: U \to \mathbb{R}^m$, $m > 1$
$U \subseteq \mathbb{R}^n$

$\vec{a}, \vec{b} \in U$.



$L \subseteq U$

If $f: U \to \mathbb{R}$

$$\underbrace{(f(\vec{b}) - f(\vec{a}))}_{\text{Scalar}} = \underbrace{f'(\vec{c})}_{\text{vector } \nabla f} . \underbrace{(\vec{b} - \vec{a})}_{\text{vector}}$$

$\vec{c} \in L$

Scalar

If $f: U \to \mathbb{R}^n$

seem like $\left(\underbrace{f(\vec{b})}_{\text{vector}} - \underbrace{f(\vec{a})}_{\text{vector}}\right) \neq \underbrace{f'(\vec{c})}_{\text{matrix}} . \underbrace{(\vec{b} - \vec{a})}_{\text{vector}}$

but not true!

We next want to define norm of a matrix.

Definition: Let X be a metric space, two metrics, d_1 & d_2 are strongly equivalent if and only if $\exists$ positive constants α & β such that $\forall\, x, y \in X$ $\quad \alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$

Theorem: Let X be a normed vector space.

$\|\cdot\|_A$, $\|\cdot\|_B$ are two norms.

When the two metrics d_1, d_2 are those induced by $\|\cdot\|_A$, $\|\cdot\|_B$

Then, strong equivalence of the metric is equivalent to the condition that $\forall x \in X$, $\alpha \|x\|_A \leq \|x\|_B \leq \beta \|x\|_A$.

Here α, β are inherited from $\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$

We say $\|.\|_A$, $\|.\|_B$ are equivalent.

Consequently, it makes no difference using an equivalent norm in the definitions of limits, continuity, etc.

Proof: homework 7

Theorem: The max-norm and Euclidean norm are equivalent on $\mathbb{R}^n$. (In fact, all p-norms are equivalent on $\mathbb{R}^n$).

Proof: homework.7

Lec:

Definition: Let X, Y be normed vector space.

A linear map $L: X \to Y$ is called a bounded linear map if there exists a constant $c > 0$ such that

$\|Lx\| \leq c \quad \forall x \in X$ with $\|x\| \leq 1$

We then define the norm of L as

$\|L\| := \sup\{\|Lx\| ; x \in X, \|x\| \leq 1\}$

Lemma: If L is a bounded linear map, then

$\|Lx\| \leq \|L\| \|x\|$ for all $x \in X$.

Proof: Let $x \in X$ be arbitrary and nonzero.

Denote $v = \frac{x}{\|x\|}$ so $\|v\| = 1$

Then $\|Lv\| = \|L\|$

Since L is linear,

$\|Lx\| = \|L(\|x\| v)\| = \|\|x\| Lv\| = \|x\| \|Lv\| \leq \|x\| \|L\|$

Proposition: If L is a bounded linear map, then $\|L\|$ is the least number M such that $\|Lx\| \leq M|x|$ for all $x \in X$.

Proof: We need to show $M \geqslant \|L\|$

Since the set $\{x \in X : \|x\| \leq 1\}$ is compact in $\mathbb{R}^n$;

$f(x) = \|Lx\|$ is continuous.

$\|Lx\|$ attains maximum at some $b \in \{x \in \mathbb{R}^n : \|x\| \leq 1\}$

$\|L\| = \sup \{\|Lx\| : \|x\| \leq 1\} = \|Lb\| \leq M\|b\| \leq M$

For convenience, we now use the max-norm $\|\vec{x}\|_0 = \max_{i=1,\ldots,n} |x_i|$.
$\vec{x} \in \mathbb{R}^n$

Definition: Let $A = (a_{ij})$ be a $m \times n$ matrix, it's norm $\|A\|$ is defined.

$$\|A\| = \max_{1 \leq i \leq m} \left(\sum_{j=1}^{n} |a_{ij}| \right)$$

$\|A\|$ is simply the maximum of the 1norms of the row vectors.

Theorem: Let A be the matrix of the linear map $L: \mathbb{R}^n \to \mathbb{R}^m$

$$L(x) = \underset{m\times n}{A} \; \underset{n\times 1}{x} \quad \text{for all } x \in \mathbb{R}^n$$

then $\|A\| = \|L\|$

<u>Proof:</u> " $\|L\| \geq \|A\|$ " : Suppose the k^{th} row of A has the maximal 1-norm.

$$\|A\| = \sum_{j=1}^{n} |a_{kj}|$$

Let $\vec{x} = (\text{sign}(a_{k1}), \text{sign}(a_{k2}), \ldots, \text{sign}(a_{kn}))^T$

$\|\vec{x}\| = 1.$ by max norm

$$\|\underbrace{A\vec{x}}_{\text{Vector}}\| = \max \left|\sum_{j=1}^{n} a_{ij}\, x_j\right| \geq \left|\sum_{j=1}^{n} a_{kj}\, \underset{\nwarrow \text{sign } a_{kj}}{x_j}\right| = \sum_{j=1}^{n} |a_{kj}| = \|A\|$$

$$\left(i_{th} \to (a_{i1}\; a_{i2} \cdots a_{in}) \begin{pmatrix} \vdots \\ \vec{x} \\ \vdots \end{pmatrix} \right)$$

$$\|L\| \geq \|L\vec{x}\| = \|A\vec{x}\| \geq \|A\|$$

sup

" $\|L\| \leq \|A\|$ ": Let k be the index for $\max \left|\sum_{j=1}^{n} a_{ij} \cdot x_j\right|$ $k \in \{1, \ldots, m\}$

$$\|L(\vec{x})\| = \|A\vec{x}\| = \max \left|\sum_{j=1}^{n} a_{ij}\, x_j\right| = \left|\sum_{j=1}^{n} a_{kj}\, x_j\right| \leq \sum_{j=1}^{n} |a_{kj}\, x_j|$$

Same as $L\vec{x}$

$$\|\vec{x}\| = \max_{j=1,\ldots,n} |x_j| \qquad\qquad \leq \sum_{j=1}^{n} |a_{kj}|\, \|\vec{x}\|$$

$$\|\vec{x}\| \sum^{n} |a_{kj}| \leq \|\vec{x}\| \ldots$$

That is, $\|L(\vec{x})\| \leq \|\vec{x}\| \, \|A\|$ ($\|A\|$ is M in the proposition)

So $\|A\| \geqslant \|L\|$

Mean Value Theorem for vector valued functions.

Theorem: Let $U \subseteq \mathbb{R}^n$ be a neighbourhood of the line segment with endpoints $\vec{a}$ and $\vec{b}$.

$f: U \to \mathbb{R}^m$ be C^1:

Then $\|f(\vec{b}) - f(\vec{a})\| \leq \|\vec{b} - \vec{a}\| \max\limits_{x \in S} \|f'(\vec{x})\|$

Remark: $\|f(\vec{x})\|$, $\|\vec{x}\|$ max-norm.

$\|f'(\vec{x})\|$ matrix norm.

Proof: Let $\phi : [0,1] \to S$

$\phi(t) = (1-t)\vec{a} + t\vec{b}$

and $\gamma = f \circ \phi$ $\quad \gamma(0) = f(a)$, $\gamma(1) = f(b)$

By chain rule $\forall\, t \in [0,1]$

$\gamma'(t) = f'(\phi(t)) \cdot \phi'(t) = f'(\phi(t)) \cdot (\vec{b} - \vec{a})$

Then for some $k \in \{1, \dots, m\}$

$\|f(\vec{b}) - f(\vec{a})\| = \max\limits_{1 \leq i \leq m} |f_i(\vec{b}) - f_i(\vec{a})| = |f_k(\vec{b}) - f_k(\vec{a})|$

$= |f_k(\phi(1)) - f_k(\phi(0))| = |\gamma_k(1) - \gamma_k(0)|$

$$= |\gamma_k'(\xi)|\,|1-0| \qquad (\text{M.V.T for } \gamma_k)$$

$$= |\gamma_k'(\xi)|$$

$$\leq \max_{1\leq i\leq m} |\gamma_m'(\xi)|$$

$$= \|\gamma'(\xi)\| \quad \text{chain rule}$$

$$= \|f'(\phi(\xi)).(\vec{b}-\vec{a})\| \leq \|\vec{b}-\vec{a}\|\,\|f'(\phi(\xi))\|$$

$$\leq \|\vec{b}-\vec{a}\| \max_{x\in S} \|f'(\phi(\vec{x}))\|$$

Mean Value Theorem

$f: U \to \mathbb{R}^m$ is C^1 ; $U \subseteq \mathbb{R}^n$

norm of matrices
maximal 1-norm of rows

$$\|f(\vec{b}) - f(\vec{a})\|_0 \leq \|\vec{b} - \vec{a}\| \max_{\vec{x} \in S} \|f'(x)\|$$

max norm.

→ line segment b/w $\vec{a}$ & $\vec{b}$

→ Why not equal?

Example: $f: [0, 2\pi] \longrightarrow \mathbb{R}^2$

$f(t) = (\cos t, \sin t)$ f is C^1

$f'(t) = (-\sin t, \cos t)$ $\|f'(t)\|$ is never zero.

old M.V.T, $f(b) - f(a) = f'(\xi).(b-a)$. Let $b = 2\pi$ & $a = 0$

$\Rightarrow f(b) = (1, 0)$; $f(a) = (1, 0)$

$\Rightarrow f(b) - f(a) = (0, 0)$

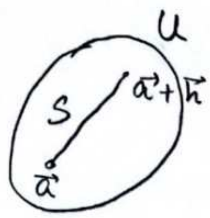
$f'(\xi)(b-a) = (-\sin \xi, \cos \xi)(2\pi)$

RHS is never zero ~~so~~ RHS is never zero. So old MVT doesn't work. New version works.

* Need new M.V.T when $m > 1$ (ie $f: \mathbb{R}^n \to \mathbb{R}^{\overset{>1}{m}}$)

Corollary: Let $f: U \to \mathbb{R}^m$ be C^1; $U \subseteq \mathbb{R}^n$

U is a neighbourhood of the line segment S with end points $\vec{a}$ & $\vec{a} + \vec{h}$.



If $\lambda : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then

$$\|f(\vec{a}+\vec{h}) - f(\vec{a}) - \lambda(\vec{h})\|_0 \le \|\vec{h}\|_0 \max_{\vec{x}\in S} \|df_{\vec{x}} - \lambda\|$$

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + \underbrace{df_{\vec{a}}(\vec{h})}_{} \ldots$$

differential of f at $\vec{a}$ (linear mapping)
linear approximation.

When $\vec{h} \to \vec{0}$, $\vec{a}+\vec{h} \to \vec{a}$ & $\vec{x} \to \vec{a}$

Proof: Let $g(\vec{x}) = f(\vec{x}) - \lambda(\vec{x})$

$$g(\vec{a}+\vec{h}) - g(\vec{a}) = (f(\vec{a}+\vec{h}) - \lambda(\vec{a}+\vec{h}) - (f(\vec{a}) - \lambda(\vec{a}))$$

(λ is linear) $= f(\vec{a}+\vec{h}) - \lambda(\vec{a}) - \lambda(\vec{h}) - f(\vec{a}) + \lambda(\vec{a})$

$$= f(\vec{a}+\vec{h}) - \lambda(\vec{h}) - f(\vec{a})$$

$$\text{LHS} = \|g(\vec{a}+\vec{h}) - g(\vec{a})\|_0 \underset{\text{MVT}}{\le} \|\vec{h}\|_0 \max_{\vec{x}\in S} \|g'(\vec{x})\|$$

We want to show $\|\underset{\text{matrix}}{g'(\vec{x})}\| = \|\underset{\text{linear map}}{df_{\vec{x}}} - \lambda\|$

$$(df_x - \lambda)\vec{h} \underset{\text{want to show}}{=} g'(\vec{x})\vec{h} = g_{\vec{x}}(\vec{h})$$

$$dg_{\vec{x}} = df_{\vec{x}} - \lambda \text{ ??}$$

$g = f - \lambda$ so $dg_{\vec{x}} = df_{\vec{x}} - d\lambda_{\vec{x}}$

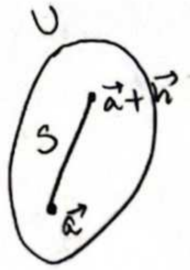
λ is linear, linear map's differential is itself. → since this is just a linear approx.

$\Rightarrow$ $d\lambda_{\vec{x}} = \lambda$ as desired. $\Rightarrow \boxed{dg_x = df_x - \lambda}$

Corollary: Let $f: U \to \mathbb{R}^m$ be C^1 at $\vec{a} \in U \subseteq \mathbb{R}^n$.

If $df_{\vec{a}}: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one (injective),

then f is one to one on some neighbourhood of $\vec{a}$.



Proof: Goal: $\forall \vec{x} \neq \vec{y}$ in a neighbourhood of $\vec{a} \Rightarrow f(\vec{x}) \neq f(\vec{y})$

$$\Updownarrow$$

$$\|f(\vec{x}) - f(\vec{y})\|_0 > 0$$

$df_{\vec{a}}$ is one to one, $df_{\vec{a}}(\vec{0}) = \vec{0}$

So $df_{\vec{a}}(\vec{v}) \neq \vec{0} \ \forall \ \vec{v} \neq \vec{0}$, Let $\vec{v}$ unit $\Rightarrow \|\vec{v}\|_0 = 1$

Let $C = \{\vec{v} \in \mathbb{R}^n, \|\vec{v}\| = 1\}$ unit cube

∂C is a compact set.

$df_{\vec{a}}$ is continuous on ∂C.

$\Rightarrow \exists \vec{b} \in \partial C$ such that $\|df_{\vec{a}}(\vec{v})\|_0 \geq \|df_{\vec{a}}(\vec{b})\| = \beta > 0$ ($\vec{b} \neq \vec{0}$)

$\forall \vec{v} \in \partial C$.

$\forall \vec{x}$ in the neighbourhood of $\vec{a}$ $\|(df_{\vec{x}} - df_{\vec{a}})(\vec{v})\| = ?$

$$\|(df_{\vec{x}} - df_{\vec{a}})\vec{v}\| = \|df_{\vec{x}}(\vec{v}) - df_{\vec{a}}(\vec{v})\|$$

$$\overset{f \text{ is } C^1}{=} \left\|\sum_j v_j D_j f(\vec{x}) - \sum_j v_j D_j f(\vec{v})\right\|_0$$

$$= \max_{1 \leq i \leq m} \left| \sum_{j=1}^{n} v_j \left(D_j f_i(\vec{x}) - D_j f_i(\vec{a})\right)\right|$$

$D_j f_i$ is continuous (f is C')

Let $\epsilon = \beta/2 > 0$

$\exists\ \delta_{ij} > 0$ such that $\forall\ \vec{x} : \|\vec{x} - \vec{a}\| < \delta_{ij}$

$\Rightarrow |D_j f_i(\vec{x}) - D_j f_i(\vec{a})| \leq \frac{\epsilon}{n}$

$\delta = \min_{i,j} \{\delta_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\} > 0$

$\forall\ \vec{x} : \|\vec{x} - \vec{a}\| < \delta \Rightarrow$

$$\underbrace{\|(df_{\vec{x}} - df_{\vec{a}})(\vec{v})\|}_{\text{norm of linear map}} \leq \max_{1 \leq i \leq m} \sum_{j=1}^{n} |v_j| \, |D_j f_i(\vec{x}) - D_j f_i(\vec{a})|$$

$$\leq \max_{\substack{1 \leq i \leq m \\ \text{no } i \text{ left}}} \frac{\epsilon}{n} \sum_{j=1}^{n} |v_j|$$

$$\leq \frac{\epsilon}{n} \sum_{j=1}^{n} \|\vec{v}\|_0$$

$$= \underbrace{\epsilon}_{\text{constant}} \underbrace{\|\vec{v}\|_0}_{\text{norm of vector.}}$$

Recall: $\|L\vec{x}\| \leq M\|\vec{x}\|$ then $\|L\| \leq M$.

So $\|df_{\vec{x}} - df_{\vec{a}}\| \leq \epsilon = \beta/2$

In previous corollary, let $\lambda = df_{\vec{a}}$

Let $\vec{x}, \vec{y}$ be distinct. $\|\vec{x} - \vec{a}\| < \delta$, $\|\vec{y} - \vec{a}\| < \delta$.

$\vec{h} = \vec{y} - \vec{x} \neq \vec{0}$

(101)

$$\| f(\vec{x}+\vec{h}) - f(\vec{x}) - df_{\vec{a}}(\vec{h})\| < \|\vec{h}\| \max_{\vec{z}\in S} \| df_{\vec{z}} - df_{\vec{a}}\|$$

$$\| f(\vec{y}) - f(\vec{x}) - df_{\vec{a}}(\vec{y}-\vec{x})\| \leq \underbrace{\|\vec{h}\|}_{\|\vec{y}-\vec{x}\|} \max_{\vec{z}\in S} \underbrace{\| df_{\vec{z}} - df_{\vec{a}}\|}_{\leq \epsilon.}$$

$$\leq \epsilon \|\vec{y}-\vec{x}\|$$

$\|f(\vec{y}) - f(\vec{x})\|$?

$$\|df_{\vec{a}}(\vec{y}-\vec{x})\| \leq \|df_{\vec{a}}(\vec{y}-\vec{x}) - f(\vec{y}) + f(\vec{x})\| + \|f(\vec{y}) - f(\vec{x})\|$$

$$\leq \epsilon\|\vec{y}-\vec{x}\| + \|f(\vec{y}) - f(\vec{x})\|$$

$\|df_{\vec{a}}(\vec{y}-\vec{x})\| = \|df_{\vec{a}}(\vec{h})\| = \|\vec{h}\|\ df_{\vec{a}}\left(\frac{\vec{h}}{\|\vec{h}\|}\right)$ since linar map.

unit vector.

$$\geq \beta\|\vec{h}\| = \beta\|\vec{y}-\vec{x}\|$$

$$\Rightarrow \beta\|\vec{y}-\vec{x}\| \leq \|df_{\vec{a}}(\vec{y}-\vec{x})\| \leq \frac{\beta}{2}\|\vec{y}-\vec{x}\| + \|f(\vec{y}) - f(\vec{x})\|$$

$$\Rightarrow \|f(\vec{y}) - f(\vec{x})\| \geq \frac{\beta}{2}\|\vec{y}-\vec{x}\| > 0 \quad \forall\ \vec{x}\neq\vec{y}$$

f is one to one

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11 Midterm #2

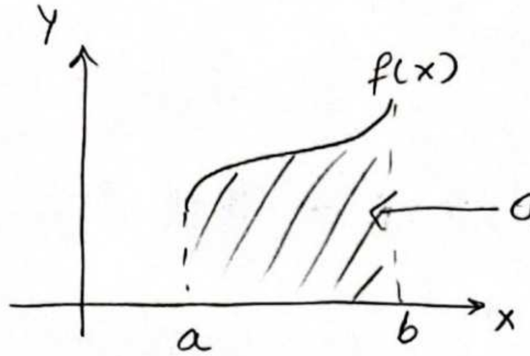
Lec # Integration

Review of Integrals for $f: \mathbb{R} \to \mathbb{R}$.

- If $f: [a,b] \to \mathbb{R}$ continuous and non negative.

$$\mathcal{O}_f = \{ (x,y) \in \mathbb{R}^2 : x \in [a,b],\ 0 \le y \le f(x) \}$$

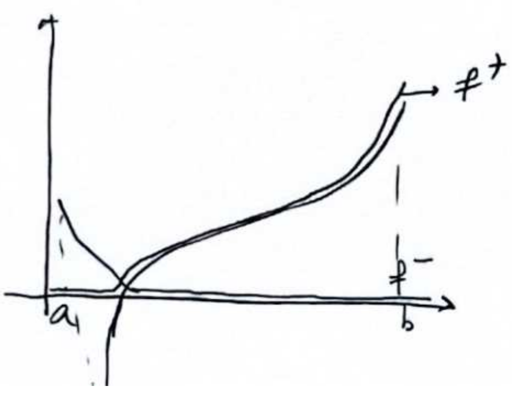
ordinate set of f.



notation $\displaystyle\int_a^b f = v(O_f)$

- If $f: [a,b] \longrightarrow \mathbb{R}$ continuous & arbitrary,

$$\left.\begin{array}{l} f^+ = \max(+f, 0) \\ f^- = \max(-f, 0) \end{array}\right\} \text{non negative}$$



$$\int_a^b f = \int_a^b f^+ - \int_a^b f^-$$

$$= v(O_{f^+}) - v(O_{f^-})$$

- F.T.C

(I) If f continuous, & $F' = f$ then $\int_a^b f = F(b) - F(a)$

(II) If f continuous & $F(x) = \int_a^x f$ then $F' = f$.

- Substitution: $f \in C^1$ & $g \in C^0$

$$\int_a^b g(f(x))\, f'(x)\, dx = \int_{f(a)}^{f(b)} g(u)\, du$$

- Integration by parts

$f \in C^1$ $g \in C^1$

$$\int_a^b f(x)\, g'(x)\, dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)\, g(x)\, dx.$$

Definition of area of a set in $\mathbb{R}^2$

- Intuitively some rules for well defined area.

① $A \subseteq B$. then $V(A) \leq V(B)$.
volume

② A, B nonoverlapping, then $V(A \cup B) = V(A) + V(B)$.

③ A is translation, rotation, reflection of B, then $V(A) = V(B)$.

④ Closed rectangle, $I = [a,b] \times [c,d]$ in $\mathbb{R}^2$

$$V(I) = (b-a) \cdot (d-c)$$

<u>Definition</u>: Let S be a bounded subset of $\mathbb{R}^2$, we say its area/volume is $C \iff$ given $\epsilon > 0$, there exists both i) a finite nonoverlapping collection of closed rectangles $\{I_k \subseteq S ; k = 1, \dots, n\}$

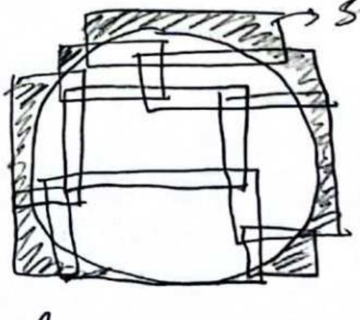
ii) a finite collection of closed rectangles

$\{J_p ; p = 1, \dots, m\}$

with $S \subseteq \bigcup_{p=1}^{m} J_p$

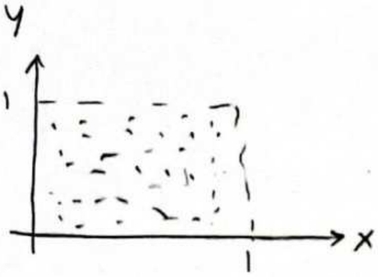
for i) $\sum_{k=1}^{n} V(I_k) > C - \epsilon$

for ii) $\sum_{p=1}^{m} V(J_p) < C + \epsilon$



<u>Example</u>: Not every set has area.

$S = \{(x,y) : x, y$ are rational $0 \le x \le 1$, $0 \le y \le 1\}$



For i) $I_k \subseteq S$

For ii) $\bigcup_{p=1}^{m} J_p \supseteq S$

For i) $v(I_k) = 0$ for the rectangles to be contained in the rational numbers, but they need to be finite whereas rationals are infinite. $\Rightarrow \sum_{k=1}^{n} v(I_k) = 0$ — ①

For ii) S is dense in $[0,1] \times [0,1]$

$$\Rightarrow \bar{S} = [0,1] \times [0,1]$$

$$\Rightarrow \bar{S} \subseteq \bigcup_{p=1}^{m} \overline{J_p} = \bigcup_{p=1}^{m} J_p$$

$$1 = v([0,1] \times [0,1]) \leq \sum_{p=1}^{m} v(J_p) \text{ — ②}$$

① & ② $\Rightarrow$

$\Rightarrow$ There is not a collection of rectangles C such that it is slightly greater than ii) & slightly smaller than i) at the same time. $\Rightarrow$ Area is not defined.

Volume of a set in $\mathbb{R}^n$

Definition

Let $S \subseteq \mathbb{R}^n$ be a bounded set

The volume of S is $C \Leftrightarrow$ given $\epsilon > 0$.

both exist.

i) $\exists$ finite non overlapping I_K closed rectangular boxes,

$$I_K \subseteq S, \quad K = 1, \dots, n$$

with $\sum_{K=1}^{n} v(I_K) > C - \epsilon$.

ii) $\exists$ finite J_p, closed rectangular boxes.
$p = 1, \dots, m$

with $\bigcup_{p=1}^{m} J_p \supseteq S$ & $\sum_{p=1}^{m} v(J_p) < C + \epsilon$.

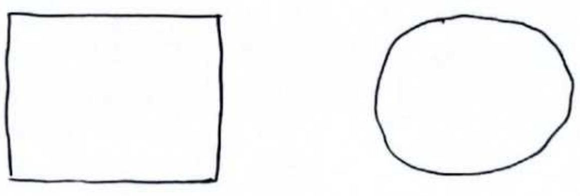
Then we denote $v(S) = C$.

- We say S is $\boxed{\text{contented}}$ if its volume exists.
- The contented set A is called $\boxed{\text{negligible}}$

$\Longleftrightarrow V(A) = 0$

In other words, given any $\epsilon > 0$, $\exists$ closed intervals/rectangular boxes $J_1, \dots, J_m$ such that

$$A \subseteq \bigcup_{p=1}^{m} J_p \quad \& \quad \sum_{p=1}^{m} V(J_p) < \epsilon.$$



→ boundary is negligible.

Theorem

The bounded set A is contended $\Longleftrightarrow$ its boundary ∂A is negligible.

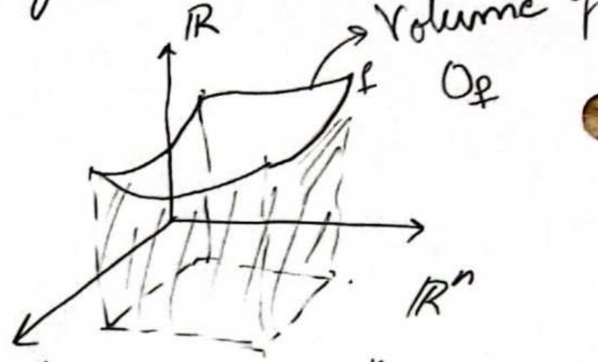
(Rational number's boundary is not negligible).

Exercises

1. Prove by definition. The union of finite negligible sets is still negligible.
2. It is true that an arbitrary union of negligible set $\bigcup_{a \in A} N_\alpha$ is a negligible set? Prove it or give a counter example.

Integration (n dimensional integral)

$f: \mathbb{R}^n \to \mathbb{R}$ non negative



$O_f = \{ (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n, 0 \le x_{n+1} \le f(\vec{x}) \}$

If f is not nonnegative

$f^+ = \max(f, 0)$ intuitively $\Rightarrow$ $\int f = \int f^+ - \int f^-$

$f^- = \max(-f, 0)$

Definitions

(1) $f: \mathbb{R}^n \to \mathbb{R}$, f is bounded if $\exists M > 0$ such that

$|f(\vec{x})| \le M, \quad \forall \vec{x} \in \mathbb{R}^n$.

(2) f has bounded support if $\exists$ a closed interval

$I \subseteq \mathbb{R}^n$

($I_1 \times I_2 \times \cdots \times I_n$ closed rectangular box)

such that $f(\vec{x}) = 0 \quad \forall \vec{x} \notin I$

(3) Suppose f is nonnegative and bounded and has bounded support. f is integrable if & only if O_f is contende

$\int_{\mathbb{R}^n} f = V(O_f)$ simply write $\int f$ for $\int_{\mathbb{R}^n} f$

If f is an arbitrary bounded function and has bounded support.

$$\int f = V(O_{f^+}) - V(O_{f^-})$$

(4) The characteristic function ϕ_A of set $A \subseteq \mathbb{R}^n$

$$\phi_A(\vec{x}) = \begin{cases} 1, & \vec{x} \in A \\ 0, & \vec{x} \notin A \end{cases}$$

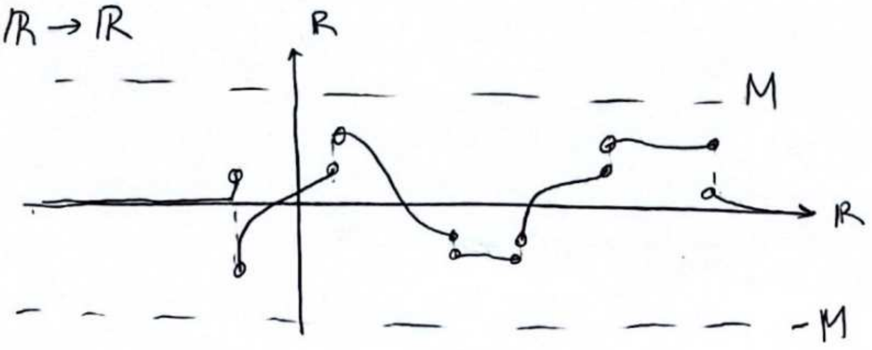
Then the integral of f over A

$$\int_A f = \int_{\mathbb{R}^n} f \phi_A$$

(5) f is called admissible iff

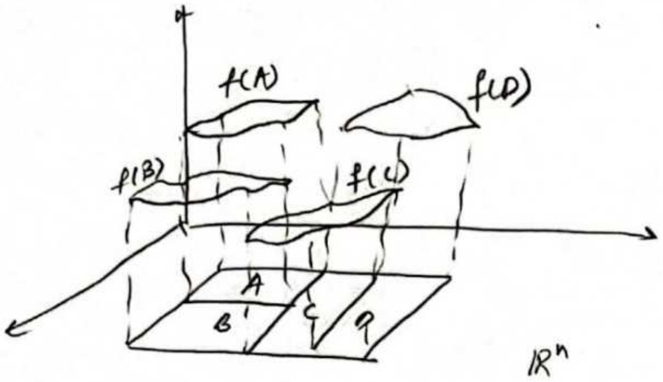
it is $\begin{cases} \text{① bounded} \\ \text{② with bounded support} \\ \text{③ continuous except on a negligible set.} \end{cases}$

$f: \mathbb{R} \to \mathbb{R}$



$f : \mathbb{R}^n \to \mathbb{R}$

Here f is admissible.
since the discontinuity is on a negligible set



<u>Theorem</u> : Every admissible function is integrable.

<u>Proof</u>: We only prove for nonnegative functions.

$f : \mathbb{R}^n \to \mathbb{R}$ is admissible $\begin{cases} \text{bounded} \\ \text{bounded support} \\ \text{continuous except on a negligible set.} \end{cases}$

We aim to prove $v(O_f)$ exists.

Denote D as the points at which f is not continuous.

D is negligible, $v(D) = 0$

For any $\epsilon > 0$, $\exists Q_1, Q_2, \dots, Q_k$ closed intervals in $\mathbb{R}^n$

$$D \subseteq \bigcup_{i=1}^{k} \operatorname{int} Q_i , \quad \sum_{i=1}^{k} v(Q_i) < \frac{\epsilon}{2M}$$

f is continuous on $\Big(\underbrace{S}_{\text{closed}} - \bigcup_{i=1}^{k} \underbrace{\operatorname{int} Q_i}_{\text{open}} \Big)$. f has bounded support. $\exists S$ closed interval $\subseteq \mathbb{R}^n$. $f(\vec{x}) = 0$, $\forall \vec{x} \notin S$.

$(S - \bigcup_{i=1}^{k} \text{int } Q_i)$ is closed and bounded, thus, compact.

$\Rightarrow f$ is uniformly continuous on $(S - \bigcup_{i=1}^{k} \text{int } Q_i)$

$\exists \delta > 0$ such that $\|\vec{x} - \vec{y}\| < \delta \Rightarrow |f(\vec{x}) - f(\vec{y})| < \frac{\epsilon}{2V(S)}$

Divide S into a union of closed intervals $\{S_1, \ldots, S_n\}$

Make the collection fine enough such that the $\underbrace{\text{diameter}}_{\text{max possible}}$

$\sup\{\|\vec{x} - \vec{y}\|, \forall \vec{x}, \vec{y} \in S_j\}$

of S_j is less that δ.

and each Q_i is a union of some S_j's.

not fine enough

$Q_1 = \bigcup (S_{15}, S_{16}, S_{17}, S_{18}, S_{22}, S_{23}, S_{24}, S_{25})$

$\Rightarrow$ fine enough.

<u>Case 1</u>: S_j is contained in some Q_i, let $a_j = 0$, $b_j = M$

<u>Case 2</u>: Otherwise, S_j is compact, f is uniformly cont. on S_j.

f attains max + min on S_j.

max $= b_j$, min $= a_j$

Since $\forall \vec{x}, \vec{y} \in S_j$ satisfies $\|\vec{x} - \vec{y}\| < \delta$,

$|f(\bar{x}) - f(\bar{y})| < \frac{\epsilon}{2V(S)}$. Then $b_j - a_j < \frac{\epsilon}{2V(S)}$.

$D_f \in \mathbb{R}^{n+1}$

construct I_j, $J_j \in \mathbb{R}^{n+1}$

$I_j = S_j \times [0, a_j] \leftarrow$ below surface of f in $\mathbb{R}^{n+1}$

$J_j = S_j \times [0, b_j] \leftarrow$ above surface of f in $\mathbb{R}^{n+1}$

$$\sum_{j=1}^{n} v(J_j) - \sum_{j=1}^{n} v(I_j) = \sum (b_j - a_j) v(S_j)$$

$$= \sum_{S_j \subseteq Q_i} (M - 0) v(S_j) + \sum_{S_j \not\subseteq Q_i} (b_j - a_j) v(S_j)$$

$$\leq M \sum_{i=1}^{k} v(Q_i) + \frac{\epsilon}{2v(S)} \sum_{j=1}^{n} v(S_j)$$

$$< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2v(S)} \cdot v(S)$$

$$= \epsilon$$

as desired.

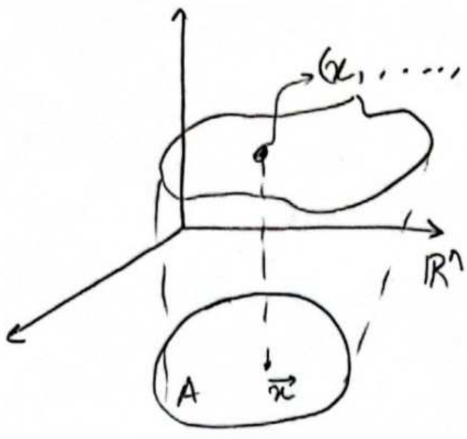
$v(D_f)$ exists ∎

Lec:

Last time: Thm 2.2 text P. 219

admissible function $\Rightarrow$ integrable.

Corollary 2.3: If $f: A \to \mathbb{R}$ is continuous and bounded, and $A \subseteq \mathbb{R}^n$ is contended, then the graph G_f of f is a negligible set in $\mathbb{R}^{n+1}$.



Proof: We extend f to $\mathbb{R}^n$ by setting $f(\vec{x}) = 0 \ \forall \ \vec{x} \notin A$.

① bounded ✓
② with bounded support. (A is bounded)
③ f is continuous except on negligible set. f is not continuous only on ∂A. A is contended $\Leftrightarrow$ ∂A is negligible.

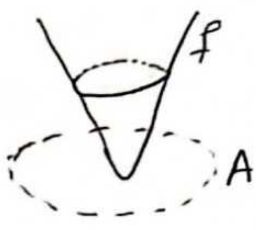
$\Rightarrow$ f is admissible $\Rightarrow$ integrable.

$\Rightarrow$ O_f is contended.
(assume f is nonnegative) $f \geq 0$

Or if f is not nonnegative.
O_{f^+}, O_{f^-} are contended.
∂O_{f^+}, ∂O_{f^-} . . .

$G_f \subseteq \partial O_{f^-} \cup \partial O_{f^+}$. is negligible.

Why bounded? $A = \{(x,y) : x^2+y^2<1\}$



$$f(x,y) = \frac{1}{1-\sqrt{(x^2+y^2)}}$$

text does not include this in the Corollary (which is wrong).

Text: Prop 2.4 – 2.7 & Thm 2.8

(1) If f is admissible and A is contended, then $f \cdot \phi_A$ is admissible.

$$\phi_A = \begin{cases} 1, & \vec{x} \in A \\ 0, & \vec{x} \notin A \end{cases}$$

(2) f, g are admissible and $f \leq g$, A is a contended set, then $\int_A f \leq \int_A g$.

(3) f is admissible and $|f(x)| \leq M$, $\forall \vec{x}$

A is contended,

then $\left|\int_A f\right| \leq M \, \overset{\text{volume.}}{v(A)}$

(4) If A, B are both contended. with $A \cap B$ is negligible, and f is admissible, then $\int_{A \cup B} f = \int_A f + \int_B f$

(5) A is contended, f, g are admissible functions.

If $f \neq g$ only on a negligible set, then

$$\int_A f = \int_A g.$$

<u>Definition</u>: $h: \mathbb{R}^n \to \mathbb{R}$ is called a step function iff

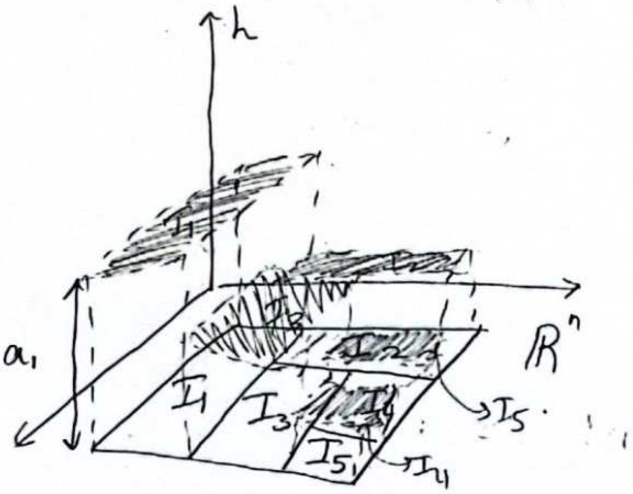
$$h = \sum_{j=1}^{p} a_j \phi_j, \quad a_j \in \mathbb{R}, \ \phi_j \text{ is a characteristic}$$

function of interval $I_j \subseteq \mathbb{R}^n$.

for example

$I_j = [a_1, b_1] \times (a_2, b_2) \times (a_3, b_3] \times [a_4, b_4) \times \ldots \times (a_n, b_n]$.

Also $\{I_1, \ldots, I_p\}$ are mutually disjoint.



<u>Theorem 3.1</u>

$$\int h = \sum_{j=1}^{p} \underbrace{a_j}_{\text{'height'}} \underbrace{v(I_j)}_{\text{Base "area" in } \mathbb{R}^n}$$

<u>Theorem 3.2</u>

h, k are two step functions, then $h+k$ is a step function and $\int h+k = \int h + \int k$.

Proof: $h : \{I_1, I_2, \dots, I_p\}$

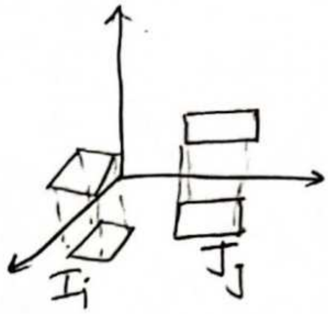
$$h = \sum_{i=1}^{p} a_i \phi_{I_i}$$

$$k : \{J_1, J_2, \dots, J_q\}$$

$$k = \sum_{j=1}^{q} b_j \phi_{J_j}$$

① When I_i, J_j not overlapping

(trivial)

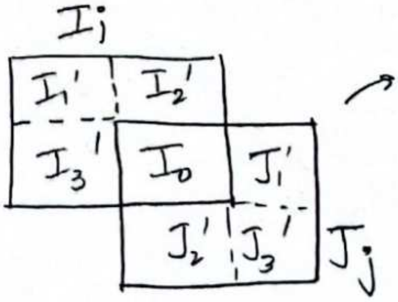


② When $I_i = J_j$

$h + k \Rightarrow (a_i + b_j)\phi_{I_i}$

(trivial)

③ When I_i, J_j overlap but not equal.



→ seperate into cases ① & ②

$\to I_1', I_2', I_3', I_0$

$h : \{I_1, \dots, \cancel{I_i}, I_{i+1}, \dots I_p\}$

$k : \{J_1, \dots, \cancel{J_j}, J_{j+1}, \dots, J_p\}$

$\to I_0, J_1', J_2', J_3'$

So $h+k$ is a step function.

$$\int h+k = \int h + \int k.$$

Lec

Review: $h : U \to \mathbb{R}$

U is subset of $\mathbb{R}^n$

$U = \bigcup_{i=1}^{p} I_i \qquad \{I_1, I_2, \dots, I_p\}$ intervals (mutual disjoint)

$h = \sum_{i=1}^{p} a_i \phi_{I_i} \qquad \phi_{I_i} = \begin{cases} 1 , & \vec{x} \in I_i \\ 0 , & \vec{x} \notin I_i \end{cases}$

$$\int h = \sum_{i=1}^{p} a_i \, v(I_i)$$

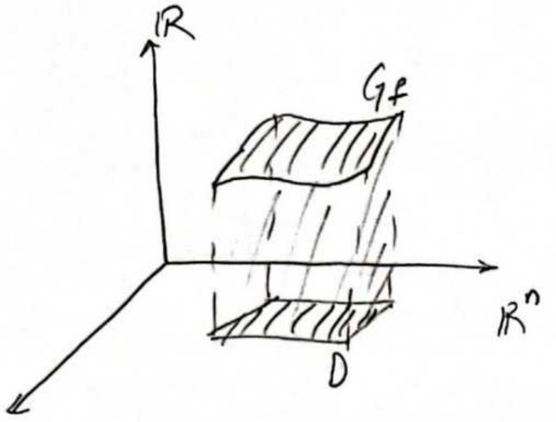
k, h two step functions then so is $k+h$ and $\int k+h = \int k + \int h$.

Theorem

$f : \mathbb{R}^n \to \mathbb{R}$ is bounded and has bounded support, then f is integrable iff given some $\epsilon > 0$, $\exists$ step functions h and k such that $h \le f \le k$ and $\int (k-h) = \int k - \int h < \epsilon$

In this case, $\int h \le \int f \le \int k$

"integrable" means
① If $f \ge 0$, O_f is contented
② otherwise, O_{f^+}, O_{f^-} are contended



$\partial O_f = G_f \cup D \cup \{\text{Connection of } \partial D, \partial G_f\}$

<u>Proof</u>: " $\Longleftarrow$ " Suppose such h, k exist.

We aim to show that $\partial O_{f^+}, \partial O_{f^-}$ are negligible.

f has bounded support, say D. $f=0$ outside of D.

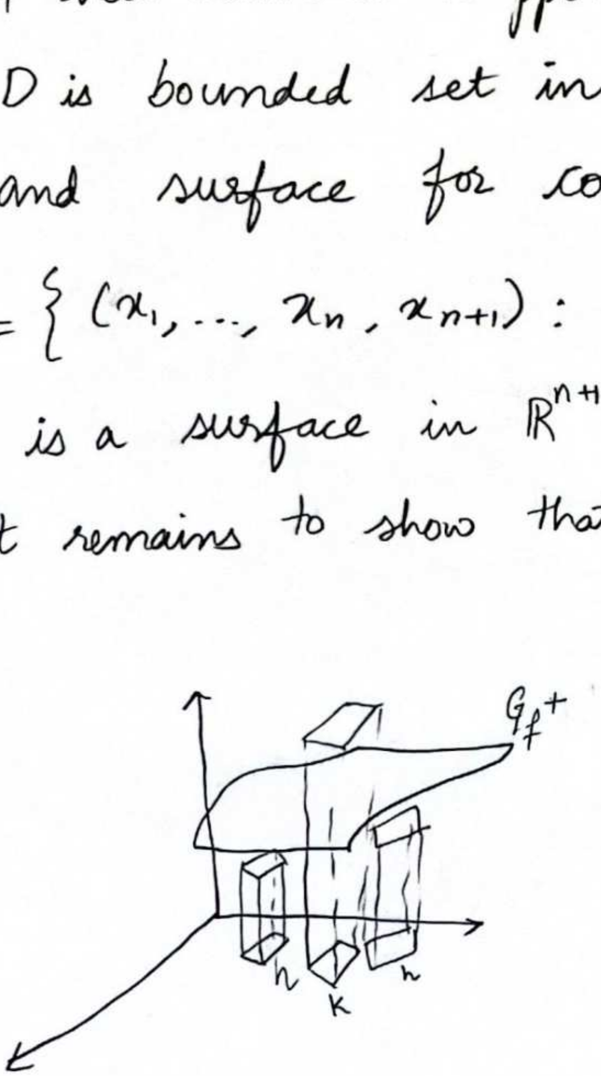
D is bounded set in $\mathbb{R}^n$, so negligible on $\mathbb{R}^{n+1}$.

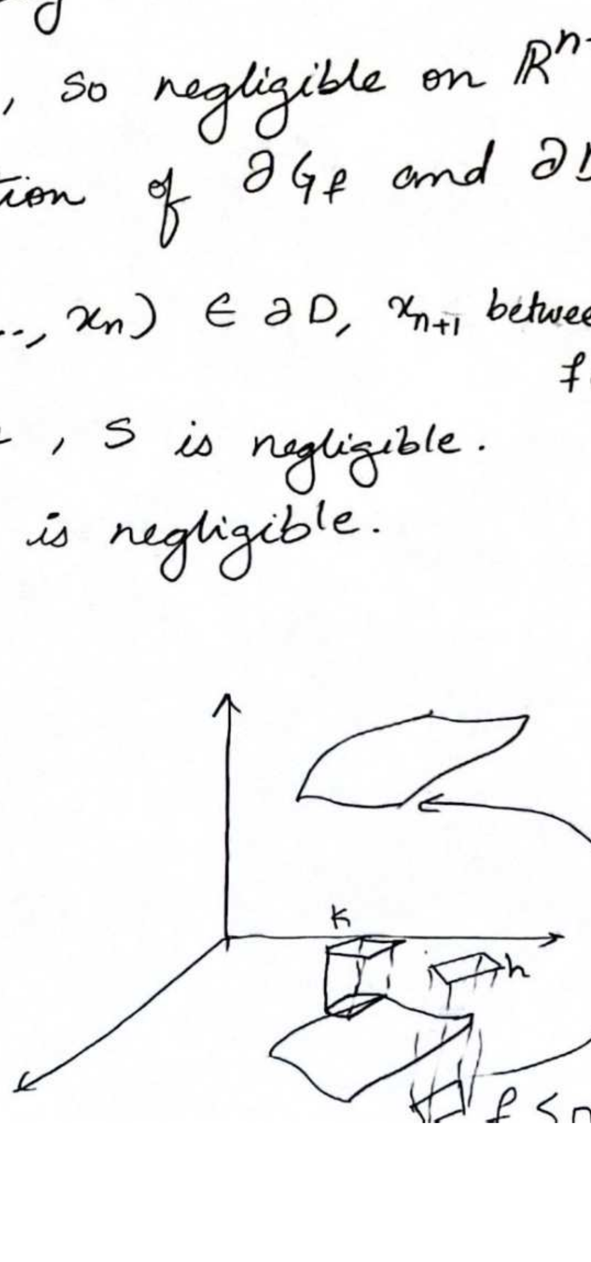
and surface for connection of ∂G_f and ∂D

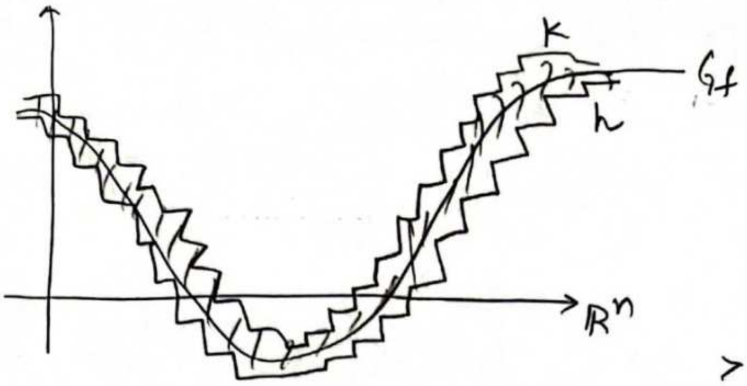
$S = \{(x_1, \dots, x_n, x_{n+1}) : (x_1, \dots, x_n) \in \partial D,\ x_{n+1} \text{ between } 0 \text{ and } f(x_1, \dots, x_n)\}$

S is a surface in $\mathbb{R}^{n+1}$ space, S is negligible.

It remains to show that G_f is negligible.







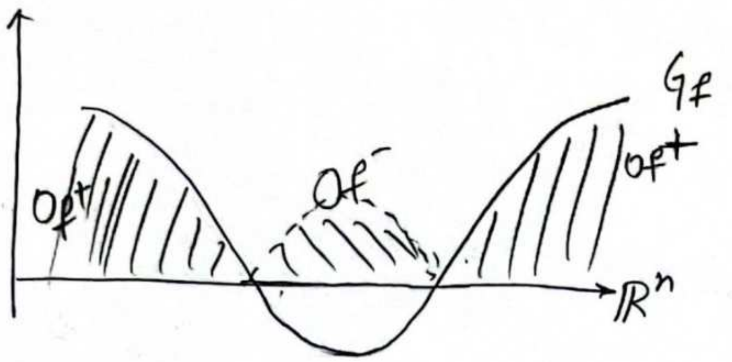
$\int k - \int h < \epsilon$

Volume of shaded part $< \epsilon$

$> V(G_f) <$ vol. of gray $< \epsilon$

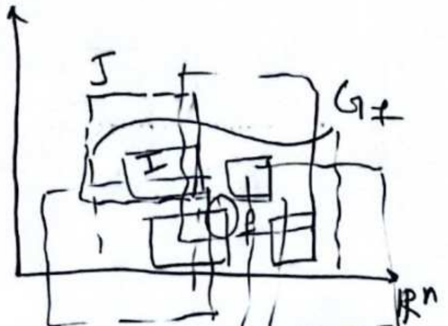
So G_f is negligible.

Therefore, O_f^+, O_f^- are contented, because $\partial O_f^+, \partial O_f^-$ are negligible.



"$\Longrightarrow$" Suppose f is integrable.

We only prove for $f \geq 0$. $\int f = v(O_f)$



O_f is contented

By definition, $\exists I_i$ nonoverlapping

$\bigcup_i I_i \subseteq O_f$

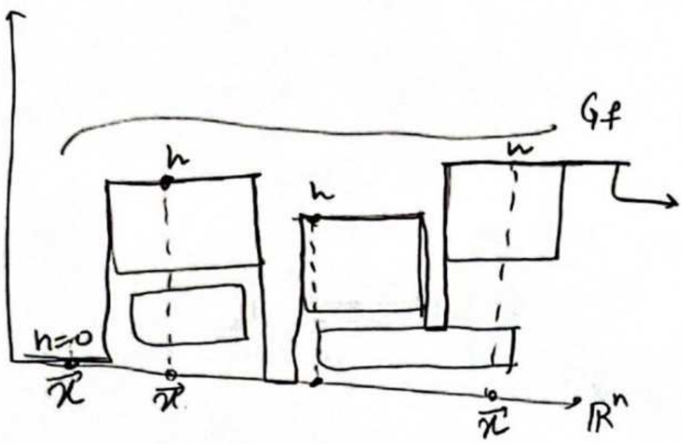
$\exists J_j, \ \bigcup_j J_j \supseteq O_f$

$\sum_j v(J_j) - \sum_i v(I_i) < \epsilon$

Pick h to be top of "purple box" that is right above it. If

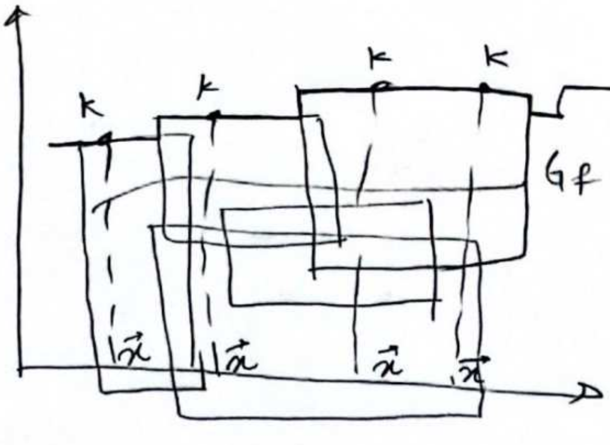
$\hookrightarrow I_{box}$

no box exists set $h=0$



$\int h \geq \sum_i v(I_i)$

Pick k to be highest value on J boxes above f.



$\int k \leq \sum_j v(J_j)$

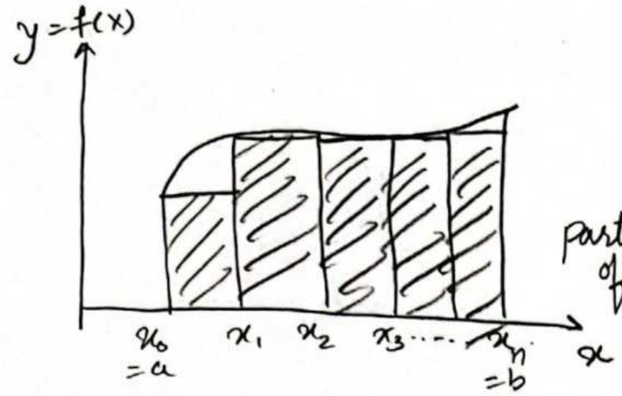
and we know $\sum_i v(J_j) - \sum_i v(I_i) < \epsilon$

So $\int k - \int h < \epsilon$.

Riemann Sums

Review in 1D

$f: [a,b] \to \mathbb{R}$



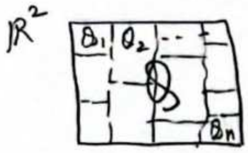
$$\int_a^b f(x)\,dx \cong \sum_{i=0}^{n} f(x_i^*)\,(x_{i+1} - x_i)$$

partition of $[a,b]$ $\{ a = x_0 < x_1 < x_2 < \dots < x_n = b$

$x_i^* \in [x_i, x_{i+1}] \}$ Selection

Generalizing to $\mathbb{R}^n \to \mathbb{R}$

$f: Q \longrightarrow \mathbb{R}$ $\quad Q \subseteq \mathbb{R}^n$ $\quad Q$ is a closed interval.



assume $f = 0$ outside Q.

A partition of Q is a collection of closed intervals with disjoint interiors.

$P = \{ Q_1, \dots, Q_n \}$ * finite sets. $\quad Q = \bigcup_{i=1}^{n} Q_i$

A selection for P is $S = \{ \vec{x}_1, \vec{x}_2, \dots, \vec{x}_n \}$ with $\vec{x}_i \in Q_i$

The mesh of P is the maximum diameter of Q_i.

$\text{mesh } P = \max_{i=1,\dots,n} \{ \text{diam}(Q_i) \}$ $\quad$ Recall $\text{diam}(A) = \sup\{ d(x,y) ; \forall x, y \in A \}$

Riemann Sum with P and S

$$R(f, P, S) = \sum_{i=1}^{n} f(\vec{x_i})\, v(Q_i)$$

*Theorem

Suppose f is bounded, $f = 0$ outside of Q. f is integrable with $\int f = I$ iff $\forall \epsilon > 0, \exists \delta > 0$ such that for any P with mesh $P < \delta \Rightarrow |I - R(f, P, S)| < \epsilon$

S is any selection of P.

Proof: "$\Rightarrow$" Let f be integrable with $I = \int f$

Let $\epsilon > 0$.

There exist step functions h, k on Q,

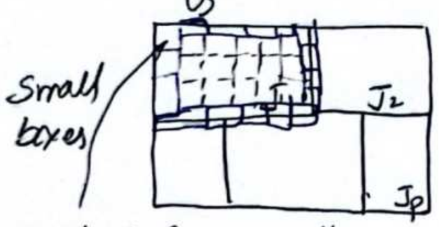
$h \le f \le k$, $\int (k - h) < \frac{\epsilon}{2}$

We can assume that h, k are defined on the same collection

$\{J_1, J_2, \ldots, J_p\}$

(If not we can always redefine J_s to contain both h & k)

$$h = \sum_{j=1}^{p} \alpha_i \phi_{J_i} \qquad k = \sum_{j=1}^{p} \beta_i \phi_{J_i}, \quad \alpha_i \le \beta_i,\ \forall i \quad (\because h \le k)$$



$$Q = \bigcup_{j=1}^{p} J_j$$

$P = \{Q_1, Q_2, \ldots, Q_n\}$ partition of Q (ie the small boxes)

$$Q = \bigcup_{i}^{n} Q_i$$

small boxes inside each J_k

$A = \{ Q_i \in P ; \ Q_i \subseteq \text{int } J_k, \text{ some } k \}$ ⟶ in dashed lines.

small boxes containing boundaries of J_k.

$B = \{ Q_i \in P ; \ Q_i \nsubseteq \text{int } J_k, \ \forall k \in \{1, \ldots, p\} \}$ → in pen

$P = A \cup B$. , $A = \bigcup_{Q_i \in A} Q_i$ $\quad B = \bigcup_{Q_i \in B} Q_i$

Goal: $\left| \int f - R(f, P, s) \right| = \left| \left(\int_A f + \int_B f \right) - R_A - R_B \right|$

$$\leq \left| \int_A f - R_A \right| + \left| \int_B f - R_B \right| \overset{?}{<} \epsilon$$

① on A: $h \leq f \leq k$, $h(\vec{x_j}) \leq f(\vec{x_j}) \leq k(\vec{x_j})$

$$\sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} \alpha_k v(Q_i) = \int_A h \leq \int_A f \leq \int_A k = \sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} \beta_k v(Q_i)$$

Note that R_A = Riemann Sum on $A. = \sum_{Q_i \in A} f(\vec{x_j}) v(Q_i)$

Same

$$\Rightarrow \underbrace{\sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} \alpha_k v(Q_i) - \sum_{Q_i \in A} f(\vec{x_j}) v(Q_i)}_{\geq \sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} (\alpha_k - \beta_k) v(Q_i)} \leq \int_A f - \sum_{Q_i \in A} f(\vec{x_j}) v(Q_j)$$

β
f
α

$$\leq \underbrace{\sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} \beta_k v(Q_i) - \sum_{Q_i \in A} f(\vec{x_j}) v(Q_j)}_{\leq \sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} (\beta_k - \alpha_k) v(Q_i)}$$

So $\left| \int_A f - R_A \right| \leq \sum_{k=1}^{p} \sum_{Q_i \subseteq \text{int} J_k} (\beta_k - \alpha_k) v(Q_i)$

RHS is $\leq \sum_{Q_i \in P} (\beta_k - \alpha_k)\, v(Q_i)$ ↳ includes boxes from B too.

$$= \int (k-h) < \frac{\epsilon}{2}$$

② For B: f is bounded, $\exists\, M>0$, $-M \leq f \leq M$.

We choose δ so small that

$$\sum_{Q_i \in B} v(Q_i) < \frac{\epsilon}{4M}$$

(e.g. Let $\delta < \dfrac{\epsilon/4M}{v(\bigcup_k (\partial J_k)) \text{ in } \mathbb{R}^{n-1} \text{ space}}$)

$$R_B = \sum_{Q_i \in B} f(\vec{x_i})\, v(Q_i)$$

$$-M \sum_{Q_i \in B} v(Q_i) \leq R_B \leq M \sum_{Q_i \in B} v(Q_i)$$

Also,

$$-M \sum_{Q_i \in B} v(Q_i) \leq \int_B f = v(O_{f_{\phi_B}}) \leq M \sum_{Q_i \in B} v(Q_i)$$

$$\left| \int_B f - R_B \right| \leq 2M \sum_{Q_i \in B} v(Q_i) < 2M \cdot \frac{\epsilon}{4M} = \frac{\epsilon}{2}$$

Therefore, $\left|\int_A f - R_A\right| + \left|\int_B f - R_B\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

$$\Rightarrow \left| \int f - R(f, P, s) \right| < \epsilon$$

Proof (part 2): "$\Leftarrow$" We aim at constructing step functions h, k such that $h \le f \le k$ and $\int (k-h) < \epsilon$.

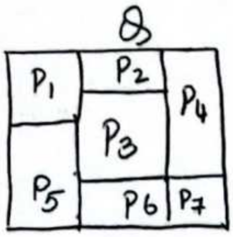
Let $\epsilon < 0$.

There is $\delta > 0$ s.t. P with mesh of $P < \delta$ and any selection S of P

$$\left| I - R(f, P, S) \right| < \frac{\epsilon}{4}$$

Note: I is a known constant, but we haven't confirmed $I = \int f$.

We fix this partition $P = \{P_1, P_2, \ldots, P_n\}$



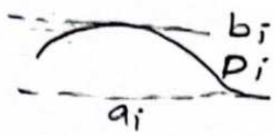
Choose $J_1 = P_1$ Define on J_1 $h = a_1$

$J_5 = P_5$ on J_5 $h = a_5$

What is h on the joint edge?

Define $h = \sum_{i=1}^{n} a_i \phi_{J_i}$ here $\overline{J_i} = P_i$ and $\bigcup_{i=1}^{n} J_i = Q$. J_i must be mutually disjoint.

& $k = \sum_{i=1}^{n} b_i \phi_{J_i}$



On each J_i, $D_i = \{ f(\vec{x}) ; \vec{x} \in J_i \} \subseteq \mathbb{R}$

a_i = greatest lower bound of D_i
b_i = lowest upper bound of D_i
$\Big\}$ possible because D_i is bounded. (it may not be closed so

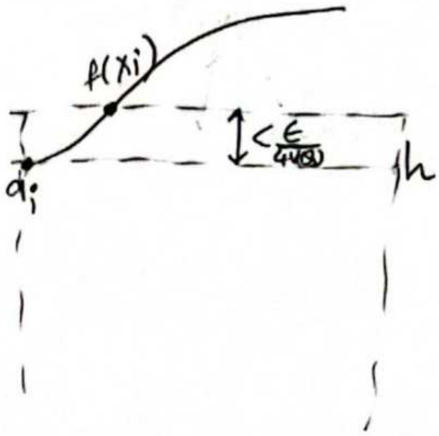
(note: f may not attain a_i, b_i on J_i because J_i may not be closed)

Choose a selection S on P.

$$S = \{ \vec{x_1}, \vec{x_2}, \ldots, \vec{x_n} \ ; \ \vec{x_i} \in P_i \ , \forall i \}$$

one selection for h

$$S_h : \ |a_i - f(\vec{x_i})| < \frac{\epsilon}{4V(Q)}$$



Another selection for k

$$S_k : \ |b_i - f(\vec{x_i})| < \frac{\epsilon}{4V(Q)}$$

$$\left| R(f, P, S_h) - \int h \right| = \left| \sum_{i=1}^{n} f(\vec{x_i}) \, V(P_i) - \sum_{i=1}^{n} a_i \, V(Q_i) \right|$$

equal

$$= \left| \sum_{i=1}^{n} (f(\vec{x_i} - a) \, v(J_i) \right| \leq \sum_{i=1}^{n} |f(\vec{x_i}) - a_i| \, V(J_i)$$

$$< \sum_{i=1}^{n} \frac{\epsilon}{4V(Q)} V(J_i) = \frac{\epsilon}{4V(Q)} \sum_{i=1}^{n} V(J_i) = \frac{\epsilon}{4V(Q)} V(Q) = \frac{\epsilon}{4}$$

Similarly, $\left| R(f, P, S_k) - \int k \right| < \frac{\epsilon}{4}$

Therefore,

$$\int (k-h) = \left|\int k - \int h\right| \leq \left|\int k - R(f,P,S_k)\right| + \left|R(f,P,S_k) - I\right|$$
$$+ \left|I - R(f,P,S_h)\right| + \left|R(f,P,S_h) - \int h\right|$$
$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

So we have done with finding such h & k. f is integrable

Lastly, we want to confirm $I = \int f$.

$$\left|\int f - I\right| \leq \underbrace{\left|\int f - \int h\right|} + \left|\int h - R(f,P,S_h)\right| + \left|R(f,P,S_h) - I\right|$$
$$\left(\int f - \int h \leq \int k - \int h < \epsilon\right)$$
$$< \epsilon + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{3\epsilon}{2} \qquad \forall \epsilon > 0.$$

So $\int f = I$

∎

<u>Remark:</u> The operation of integration is a limit process.

Take $\lim_{k\to\infty} (\text{mesh of } P_k) = 0$

$$\int f = \lim_{k\to\infty} R(f, P_k, S_k)$$

Theorem: A, B contented sets $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$

$f: A \times B \to \mathbb{R}$ f is $\boxed{\text{uniformly continuous.}}$

then $$\lim_{\vec{x} \to \vec{a}} \underbrace{\boxed{\int_B f(\vec{x}, \vec{y}) d\vec{y}}}_{g(\vec{x}) \text{ fn. of } \vec{x}} = \int_B \lim_{\vec{x} \to \vec{a}} f(\vec{x}, \vec{y}) d\vec{y}$$

$\vec{x} \in A$, $\vec{y} \in B$.

Proof (outline): $\boxed{\text{Show } g \text{ is continuous on } A.}$? use this to show

then $\vec{x}_n \to \vec{a}$ $\lim_{n \to \infty} g(\vec{x}_n) = g(\vec{a})$

$$\lim_{n \to \infty} f(\vec{x}_n, \vec{y}) = f(\vec{a}, \vec{y})$$

$$\text{LHS} = \lim_{n \to \infty} g(\vec{x}_n) = g(\vec{a}) = \int_B f(\vec{a}, \vec{y}) d\vec{y} = \int_B \lim_{n \to \infty} f(\vec{a}, \vec{y}) d\vec{y}$$
$$= \text{RHS}.$$

thm: Let f_n be a sequence of functions

Each f_n is integrable on a contented set A.

& $f_n \to f$ $\boxed{\text{uniformly}}$

Then $\lim_{n \to \infty} \int_A f_n = \int_A \lim_{n \to \infty} f_n = \int_A f$.

(see HW)

Thm: Let $f: A \times J \to \mathbb{R}$

$A \subseteq \mathbb{R}^n$ contented

J is an open interval $J \subseteq \mathbb{R}$ (J for time t)

If $\frac{\partial}{\partial t} f(\vec{x},t)$ is uniformly continuous.

then $\frac{\partial}{\partial t} \int_A f(\vec{x},t)\, d\vec{x} = \int_A \frac{\partial}{\partial t} f(\vec{x},t)\, d\vec{x}$

(see HW)

Lec

Fubini's Theorem.

Let $f: \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$ & f is integrable

$(\vec{x},\vec{y}) \longrightarrow f(\vec{x},\vec{y})$

$\vec{x} \in \mathbb{R}^m, \vec{y} \in \mathbb{R}^n$

$\int_{\mathbb{R}^m \times \mathbb{R}^n} f$? How to calculate it?

Let $f_{\vec{x}}(\vec{y}) = f(\vec{x},\vec{y})$ for fixed $\vec{x} \in \mathbb{R}^m$

$f_{\vec{x}} : \mathbb{R}^n \rightarrow \mathbb{R}$ Suppose $f_{\vec{x}}$ is integrable

Given $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^n$ are contended.

$F(\vec{x}) = \int_B f_{\vec{x}}(\vec{y})\, d\vec{y}$ is a function of $\vec{x}$, and is integrable on A.

Then,

$$\int_{A\times B} f = \int_A F(\vec{x})\, d\vec{x} = \int_A \left(\int_B f_{\vec{x}}(\vec{y})\, d\vec{y} \right) d\vec{x} = \int_A \left(\int_B f(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x}$$

If we also denote $f_{\vec{y}}(\vec{x}) = f(\vec{x},\vec{y})$ for fixed $\vec{y} \in \mathbb{R}^n$

$f_{\vec{y}} : \mathbb{R}^m \to \mathbb{R}$ is integrable.

Then $\int_{A\times B} f = \int_B \left(\int_A f(\vec{x},\vec{y})\, d\vec{x} \right) d\vec{y}$

$$\boxed{\int_{A\times B} f = \int_A \left(\int_B f(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x} = \int_B \left(\int_A f(\vec{x},\vec{y})\, d\vec{x} \right) d\vec{y}}$$

Proof (Outline)

$\underset{\substack{\text{interval} \\ \text{on } \mathbb{R}^m}}{I} \times \underset{\substack{\text{Interval} \\ \text{on } \mathbb{R}^n}}{J}$

$\phi_{I\times J} \longrightarrow h$ step $\longrightarrow f$ integrable $\left(\begin{array}{c} * \ h,k \\ h \le f \le k \\ \int (k-h) < \epsilon \end{array} \right)$

(1) $\phi_{I\times J} = \begin{cases} 1 & \text{on } I\times J \\ 0 & \text{otherwise} \end{cases}$

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} \phi_{I\times J} = \int_{I\times J} \phi_{I\times J} = V(I\times J) = V(I) \times V(J)$$

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \phi_{I\times J}(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x} = \int_I \left(\int_J \phi_{I\times J}(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x}$$

$$= \int_I V(J)\, d\vec{x}$$

$$= V(J) \int_I 1 \qquad = V(J) \times V(I)$$

So $\int_{\mathbb{R}^m \times \mathbb{R}^n} \phi_{I \times J} = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \phi_{I \times J}(\vec{x}, \vec{y}) d\vec{y} \right) d\vec{x}$

<u>Next:</u> $h = \sum_{i=1}^{n} a_i \phi_{I_i \times J_j}$

$$\int_{\mathbb{R}^m \times \mathbb{R}^n} h = \int_{\mathbb{R}^m \times \mathbb{R}^n} \sum_{i=1}^{n} a_i \phi_{I_i \times J_i} = \sum_{i=1}^{n} a_i \int_{\mathbb{R}^m \times \mathbb{R}^n} \phi_{I_i \times J_i}$$

$$= \sum_{i=1}^{n} a_i \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \phi_{I_i \times J_i} \right)$$

$$= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} \sum_{i=1}^{n} a_i \phi_{I_i \times J_i} \right)$$

$$= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} h(\vec{x}, \vec{y}) d\vec{y} \right) d\vec{x}$$

<u>Last:</u> f is integrable $\Rightarrow \forall \epsilon > 0$

$\exists h, k$ step functions $h \le f \le k$, $\int (k - h) < \epsilon$

Fix $\vec{x} \in \mathbb{R}^m$

$f_{\vec{x}}(\vec{y})$ is integrable.

$\exists h_{\vec{x}}, k_{\vec{x}}$ such that $h_{\vec{x}} \le f_{\vec{x}} \le k_{\vec{x}}$, $\int_{\mathbb{R}^n} (k_{\vec{x}} - h_{\vec{x}}) < \epsilon$

In fact, $h_{\vec{x}}(\vec{y}) = h(\vec{x}, \vec{y})$, $k_{\vec{x}}(\vec{y}) = k(\vec{x}, \vec{y})$

Since, $\int h \leq \int f \leq \int k$

$$\int h = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} h_{\vec{x}}(\vec{y})\, d\vec{y} \right) d\vec{x}$$ similarly for k.

$$\underbrace{\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} h_{\vec{x}}(\vec{y})\, d\vec{y} \right) d\vec{x}}_{a} \leq \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x} \leq \underbrace{\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} k_{\vec{x}}(\vec{y})\, d\vec{y} \right) d\vec{x}}_{= b}$$

& $b - a < \epsilon$

Therefore,

$$\left| \int f - \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x} \right| < \epsilon$$

both in $[a,b]$ & $|b-a| < \epsilon$

It's true for any $\epsilon > 0$. Thus, $\int f = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(\vec{x},\vec{y})\, d\vec{y} \right) d\vec{x}$ ■

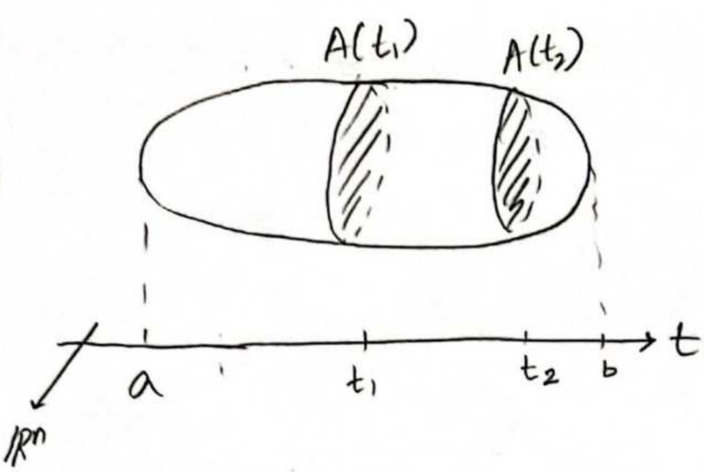
<u>Applications of Fubini's Theorem.</u>

① $S \subseteq \mathbb{R}^{n+1}$ contended $\quad S = A \times [a,b]$

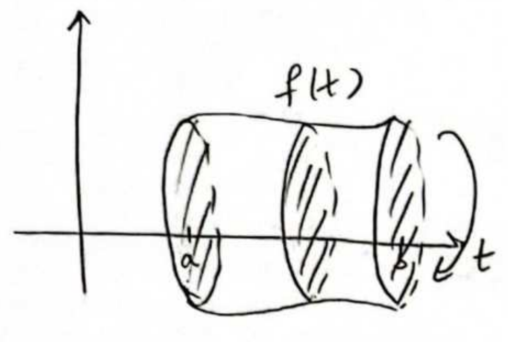
$A \subseteq \mathbb{R}^n$ contented

denote cross sectional region

$A(t) = \{ \vec{x} \in \mathbb{R}^n : (\vec{x}, t) \in S \}$
$t \in [a,b]$



$$V(S) = \int_a^b v(A(t))\,dt$$

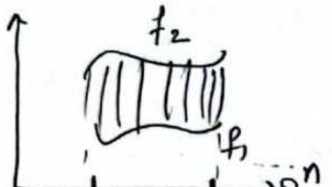


Volume of revolution

$$V = \int_a^b \pi (f(t))^2\,dt$$

- aka Cavalieri's principle

(2) Let $A \subseteq \mathbb{R}^n$ be contended.



$f_1, f_2 : A \to \mathbb{R}$ continuous with $f_1 \leq f_2$

The region bounded by them.

$S \subseteq \mathbb{R}^{n+1}$

$$S = \{(\vec{x}, y) \in \mathbb{R}^{n+1}, \vec{x} \in A, f_1(\vec{x}) \leq y \leq f_2(\vec{x})\}$$

Suppose S is contented

- Define $g : \mathbb{R}^{n+1} \to \mathbb{R}$ we want to compute $\int_S g$

$$\int_S g = \int_A \left(\int_{f_1(\vec{x})}^{f_2(\vec{x})} g(\vec{x}, y)\,dy \right) d\vec{x}$$

) $f: \mathbb{R}^n \to \mathbb{R}$ $\qquad \vec{x} = (x_1, x_2, \dots, x_n)$ integrable.

Apply Fubini's theorem to $Q = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$.

$$\int_Q f = \int_{a_n}^{b_n} \cdots \left(\int_{a_3}^{b_3} \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_2} f(\vec{x})\, dx_1 \right) dx_2 \right) dx_3 \dots dx_n \right.$$

iterated integral.

<u>Lec</u> <u>Fubini's Theorem</u> (Review) : $\int_Q f$ $\qquad$ Q is an interval in $\mathbb{R}^n$

$$\int_{a_n}^{b_n} \cdots \left(\int_{a_1}^{b_1} f(\vec{x})\, dx_1 \right) \dots dx_n \qquad [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \times \dots \times [a_n, b_n]$$

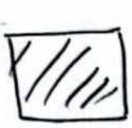
How about non standard (rectangular) shapes?

A:  $\int_A f = ?$

$T \uparrow \downarrow T^{-1}$ transformation

Assume C^1 invertible mapping

$T: \mathbb{R}^n \to \mathbb{R}^n$, T, T^{-1} are both C^1.

Q: 

$T(Q) = A$

$$\int_A f = \int_Q \tilde{f} \qquad \tilde{f} = ? \quad \text{new integrand depends on } T.$$

Let's start with easy T (linear)

- $T(\vec{x}) = L\vec{x}$ $\quad$ L: matrix.

<u>Thm:</u> If L is a linear mapping and $A \subseteq \mathbb{R}^n$ is contented, then $L(A)$ is also contented with

$$v(L(A)) = |\det L| \, v(A)$$

here $L(\vec{x}) = L\vec{x}$
↳ matrix

<u>eg:</u> rotation $\alpha = \frac{\pi}{3}$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = L \qquad |\det L| = \cos^2\alpha + \sin^2\alpha = 1$$

- $L = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad (x,y) \to (2x, y)$ $\quad |\det L| = 2$

<u>Proof:</u> By linear algebra

$$L = L_1 L_2 \dots L_n$$

either $L_i = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$ or $L_i = \begin{array}{c} \begin{array}{ccc} & k & p \end{array} \\ \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 \cdots & 1 & \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \end{array}$

multiply k^{th} column by a $\qquad$ copy k^{th} col add it to p^{th} col.

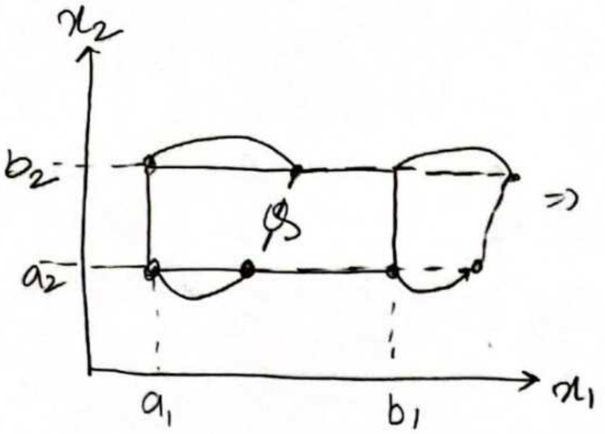
(column operations of matrices) $\qquad$ $(p\text{-th})_{new} = (k\text{-th}) + (p\text{-th})_{old}$

- first, for an interval $Q \subseteq \mathbb{R}^n$: $Q = [a_1, b_1] \times \dots \times [a_n, b_n]$

<u>Type I:</u> $v(L_i(Q)) = |a| \, v(Q)$ and $\det L_i = a$

Type II: Consider $L_i = \begin{bmatrix} 1 & 1 & 0 & & \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & & 1 \end{bmatrix}$

$$L_i \vec{x} = \begin{bmatrix} 1 & 1 & & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$



$\Rightarrow$ $(a_1, a_2) \to (a_1 + a_2, a_2)$

$(b_1, a_2) \to (b_1 + a_2, a_2)$

$(a_1, b_2) \to (a_1 + b_2, b_2)$

$(b_1, b_2) \to (b_1 + b_2, b_2)$

converts to parallelogram.

The area doesn't change under Type II, (also $\det L_i = 1$)

Note: $\det L = (\det L_1)(\det L_2) \dots \det(L_n)$

We confirm it for any interal S.

For any contented $A \subseteq \mathbb{R}^n$, we find intervals I_k, J_k as in the definition of volume. Since the formula is true for I_k, J_k so the formula holds for A. ▨

Consider $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ C^1- invertible

by Taylor's formula, around $\vec{a} \in \mathbb{R}^n$

$$\underset{\uparrow \text{ small}}{T(\vec{a}+\vec{h})} = \underbrace{T(\vec{a})}_{\substack{\uparrow \\ \text{constant} \\ \text{vector}}} + \underbrace{T'(\vec{a})\vec{h}}_{\substack{\uparrow \\ \text{Jacobian} \\ \text{matrix}}} + O(\|\vec{h}\|^2)^{\text{ignore}}$$

Let Q be a small neighbourhood of $\vec{a}$.

$T(\vec{a})$: constant vector gives a rigid shift, it doesn't change the area

$T'(\vec{a})\vec{h}$: linear mapping applied on $\vec{h}$, $v(\text{new } Q) \approx |\det T'(\vec{a})| v(Q$

Thm (Change of variable)

Let $Q \subseteq \mathbb{R}^n$ be an interval and $T: \mathbb{R}^n \to \mathbb{R}^n$ be C^1 invertible on a neighbourhood of Q. If $f: \mathbb{R}^n \to \mathbb{R}$ is integrable (on $\mathbb{R}^n$) such that $f \circ T$ is also integrable.

then $$\boxed{\int_{T(Q)} f = \int_Q (f \circ T)\, |\det T'|}$$

Sketch of proof: Use a partition $\{Q_i\}_{i=1}^{k}$ for Q.

$$\int_{T(Q)} f = \sum_{i=1}^{k} \int_{T(Q_i)} f \simeq \sum_{i=1}^{k} f(T(\vec{a_i}))\, V(T(Q_i))$$

each i $a_i \in Q_i$

$\hookrightarrow$ Riemann sum.

$$\simeq \sum_{i=1}^{k} \underbrace{f(T(\vec{a_i}))\left|\det T'(\vec{a_i})\right|}_{\text{new integrand.}} V(Q_i)$$

$$\cong \int_{Q} f(T(\vec{x})).\ \left|\det T'(\vec{x})\right| d\vec{x}$$

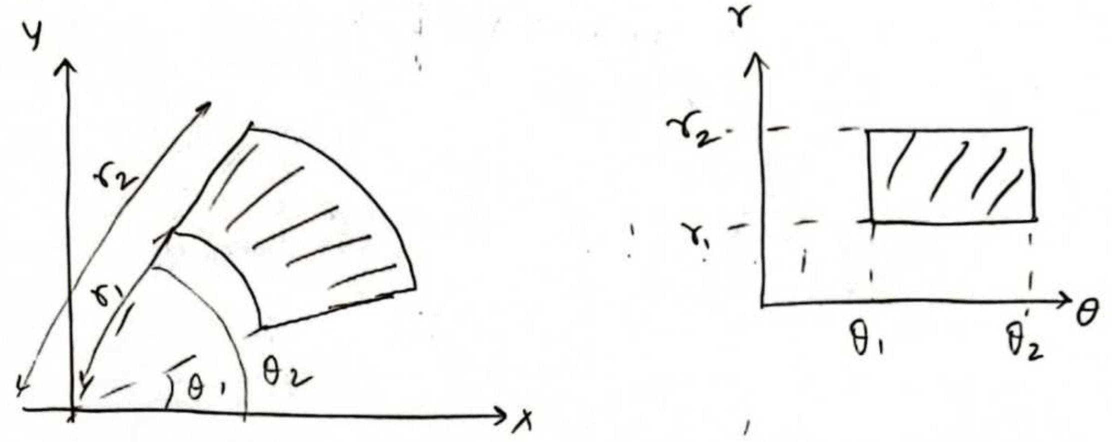
$$= \int_{Q} (f \circ T) \left|\det T'\right|$$

Take $k \longrightarrow \infty$, $\displaystyle\int_{T(Q)} f = \int_{Q} (f \circ T) \left|\det T'\right|$

Lec:

$$\int_{T(Q)} f = \int_Q (f \circ T)\,|\det T'|$$

example (last problem on HW is similar)



$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$f(x,y) = xy^2 \qquad \int_A f = ?$$

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$T(Q) = A$$

$$(r,\theta) \xrightarrow{T} (x,y)$$

$$(x,y) = T(r,\theta)$$

$$= (r\cos\theta, r\sin\theta) \qquad r \geq 0 \qquad \theta \in [0, 2\pi).$$

$$\int_A f = \int_Q (f \circ T)\,|\det T'|$$

$$T'(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det T' = r\cos^2\theta + r\sin^2\theta = r$

$|\det T'| = |r| = r$

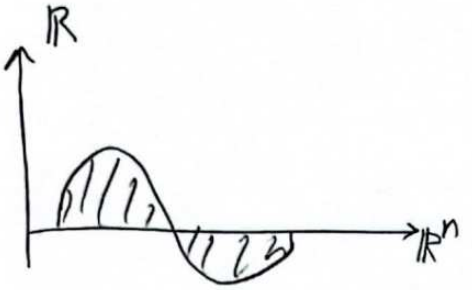
$$\int_A f = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(T(r,\theta))\, r\, dr\, d\theta$$

$f(x,y) = xy^2$

$$= \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} (r\cos\theta)(r\sin\theta)^2\, r\, dr\, d\theta$$

Improper Integrals.

Review: The definition for $\int_{\mathbb{R}^n} f$ → f is bounded; → f has bounded support.



Definition

$f : U \to \mathbb{R}$ $U \subseteq \mathbb{R}^n$

f is said to be locally integrable iff f is integrable on every compact contented subset of U.

<u>example</u> : 1) $f(x) = \frac{1}{x^2}$ $f: [1, \infty) \longrightarrow \mathbb{R}$, U is unbounded.

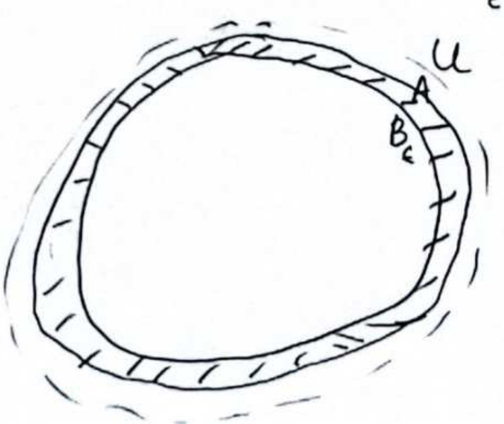
$\forall b < \infty$ on $[1, b]$ $\int_1^b f(x)dx$ exists. $\Rightarrow$ locally integrable

2) $f(x) = \frac{1}{\sqrt{x}}$ on $(0,1)$ f becomes unbounded near 0.

$[a,b] \subseteq (0,1)$ is compact & therefore f is locally integrable $\int_a^b \frac{1}{\sqrt{x}}dx$

<u>Def:</u> Let $f: U \longrightarrow \mathbb{R}$ be a locally integrable function on an open U.
Then f is said to be absolutely integrable on U if and only if given $\epsilon > 0$, $\exists$ a compact contended $B_\epsilon \subset U$ such that $\forall$ compact contented A with $B_\epsilon \subseteq A \subset U$.

$$\left| \int_{A-B_\epsilon} f \right| = \left| \int_A f - \int_{B_\epsilon} f \right| < \epsilon$$



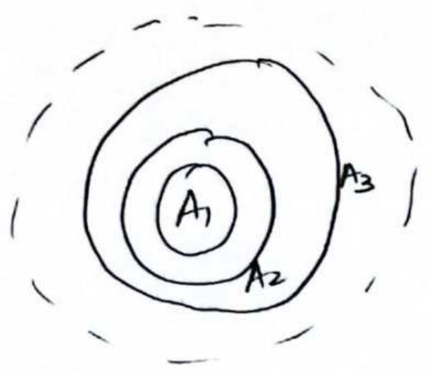
We need shaded part $\rightarrow \epsilon$ for all $A \Rightarrow B_\epsilon$ must be very close to U.

$$\int_{\text{shaded}} f < \epsilon.$$

<u>Theorem</u>: Let f be an absolutely integrable function on open U. Given $\epsilon > 0$, there exists number $I_U f$ $(\in \mathbb{R})$ and a compact contented set C_ϵ $(\subset U)$ such that $\left| I_U f - \int_A f \right| < \epsilon$ for all compact contented A with $C_\epsilon \subseteq A \subset U$.

$I_U f$ is called the improper integral of f on U.

Further, $I_U f = \lim\limits_{k \to \infty} \int_{A_k} f$ for every approximating sequence of compact contented sets $\{A_k\}_{k=1}^{\infty}$ with $A_k \subseteq A_{k+1}$ and $U = \bigcup_{k=1}^{\infty} A_k$.



Outline of proof : ① What is $I_U f$?

② How to find C_ϵ?

③ Confirm $\forall \{A_k\}$ approximating U: $I_U f = \lim_{k\to\infty} \int_{A_k} f$.

Proof: f is abs. int.

$\forall \epsilon > 0$, $\exists B_{\epsilon/2}$ compact contented subset of U, s.t

$\forall A : B_{\epsilon/2} \subseteq A \subset U$ $\left| \int_A f - \int_{B_{\epsilon/2}} f \right| < \epsilon/2$.

Consider a fixed sequence of compact contended

$B_k \subseteq B_{k+1}$ and $U = \bigcup_{k\geq 1}^{\infty} B_k$.

for N sufficiently large: $B_k \supseteq B_{\epsilon/2}$ $\forall k \geq N$

use $A = B_k$ in the defn. of abs. int.

$$\left| \int_{B_k} f - \int_{B_{\epsilon/2}} f \right| < \frac{\epsilon}{2}$$

$\forall k, j \geq N$: $$\left| \int_{B_k} f - \int_{B_j} f \right| \leq \left| \int_{B_k} f - \int_{B_{\epsilon/2}} f \right| + \left| \int_{B_{\epsilon/2}} f - \int_{B_j} f \right|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\left\{ \int_{B_k} f \right\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$.

$\mathbb{R}$ is complete, so $\int_{B_K} f$ has a limit in $\mathbb{R}$.

We denote $I_u f = \lim_{K\to\infty} \int_{B_K} f$. ① ✓

Choose $C_\epsilon = B_{\epsilon/3} \longrightarrow$ in definition of abs. int.

$$\left| \int_A f - \int_{B_{\epsilon/3}} f \right| < \frac{\epsilon}{3} \quad \forall A : B_{\epsilon/3} \subseteq A \subset \mathcal{U}$$

compact contended

Let A be an arbitrary compact contended set with $C_\epsilon \subseteq A \subset \mathcal{U}$.

$$\left| I_u f - \int_A f \right| \le \underbrace{\left| I_u f - \int_{B_K} f \right|}_{K \text{ large: } < \frac{\epsilon}{3}} + \underbrace{\left| \int_{B_K} f - \int_{C_\epsilon} f \right|}_{= \left|\int_{B_K} f - \int_{B_{\epsilon/3}} f\right| < \frac{\epsilon}{3}} + \underbrace{\left| \int_{C_\epsilon} f - \int_A f \right|}_{= \left|\int_{B_{\epsilon/3}} f - \int_A f\right| < \frac{\epsilon}{3}}$$

because $I_u f = \lim_{K\to\infty} \int_{B_K} f$

because K large $B_{\epsilon/3} \subseteq B_K \subset \mathcal{U}$

Last, confirm: for any sequence $\{A_K\}$ approximating $\mathcal{U}$.

Let $\epsilon > 0$, $\exists\, C_\epsilon$ such that $A_K \supseteq C_\epsilon \quad \forall K \geqslant N$ N large.

By ②: $\left| I_u f - \int_{A_K} f \right| < \epsilon, \quad \forall K \geqslant N$. Since ϵ is arbitrary

$I_u f = \lim \int f$

Improper Integrals (Contd.)

The procedure for improper integrals (Assume f is locally integrable)

1) Choose a convenient approximating sequence of compact contented sets $\{A_k\}_{k=1}^{\infty}$ $\to$ $A_1 \subseteq A_2 \subseteq A_3 \dots$, $A_k \subset U$,

$$U = \bigcup_{k=1}^{\infty} A_k$$

example : if $U = \mathbb{R}^n$, $A_k = \bar{B}_k(\vec{0})$ A_k is compact contented.

2) Compute $\int_{A_k} f$

3) Take the limit $\lim_{k\to\infty} \int_{A_k} f$. If this limit is a finite number, then $\int_U f = \lim_{k\to\infty} \int_{A_k} f$. If the limit does not exist (including $\pm\infty$), then f is not absolutely integrable.

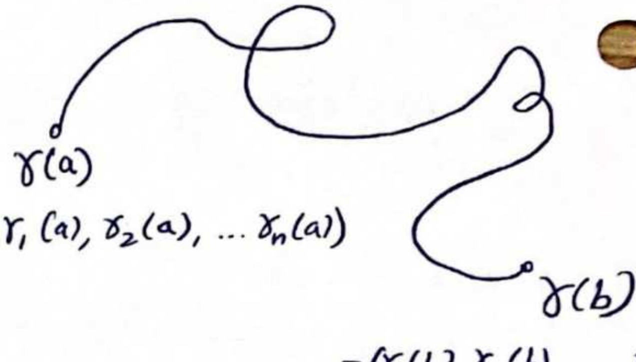
<u>Comparison test</u> : f, g are locally integrable on U and $0 \le f \le g$. If g is absolutely integrable on U, so is f.

Chapter V

Path in $\mathbb{R}^n$ $\quad \gamma : I \to \mathbb{R}^n$.

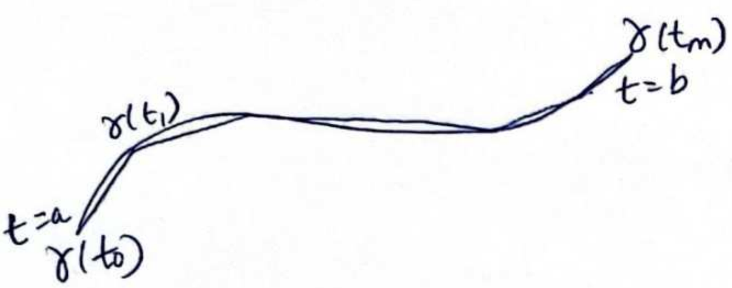
interval $I \subseteq \mathbb{R}$

$I = [a,b]$



If γ is C' (continuously differentiable), we say that the path given by γ is C'.

Question: total length of γ?



Partition P for $[a,b]$

$a = t_0 < t_1 < t_2 < \dots < t_m = b$

mesh $P = \max \{ t_{i+1} - t_i \}$

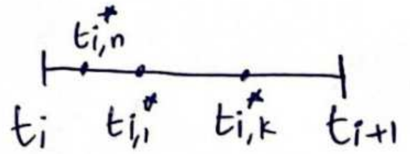
$$s(\gamma, P) = \sum_{j=1}^{m} \| \gamma(t_{j+1}) - \gamma(t_j) \|$$

Idea: take mesh $P \to 0$ $\quad (m \to \infty)$, $s(\gamma, P) \to s(\gamma)$

$$\| \gamma(t_{j+1}) - \gamma(t_j) \| = \sqrt{(\gamma_1(t_{j+1}) - \gamma_1(t_j))^2 + \dots + (\gamma_n(t_{j+1}) - \gamma_n(t_j))^2}$$

γ_k : Mean Value Theorem.

$[a,b] \to \mathbb{R}$ $\quad \gamma_k(t_{i+1}) - \gamma_k(t_i) = \gamma_k'(t_{i,k}^*)(t_{i+1} - t_i)$



$$\gamma_1(t_{i+1}) - \gamma_1(t_i) = \gamma_1'(t_{i,1}^*)(t_{i+1} - t_i)$$

$$\vdots$$

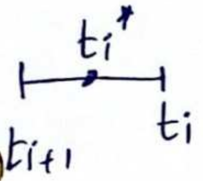
$$\gamma_n(t_{i+1}) - \gamma_n(t_i) = \gamma_n'(t_{i,n}^*)(t_{i+1} - t_i)$$

$$\|\gamma(t_{i+1}) - \gamma(t_i)\| = \sqrt{(\gamma_1'(t_{i,1}^*))^2(t_{i+1} - t_i)^2 + \dots + (\gamma_n'(t_{i,n}^*))^2(t_{i+1} - t_i)^2}$$

$$= \sqrt{(\gamma_1'(t_{i,1}^*))^2 + \dots + (\gamma_n'(t_{i,n}^*))^2}\;(t_{i+1} - t_i)$$

Size of $[t_i, t_{i+1}] \to 0$ all points $t_{i,1}^*, \dots, t_{i,n}^*, t_i, t_{i+1}$ merge together.

we can make $t_{i,1}^*, \dots, t_{i,n}^*$ be the same point t_i^*



$$\sum_{i=1}^{m} \|\gamma(t_{i+1}) - \gamma(t_i)\| \approx \sum_{i=1}^{m} \|\gamma'(t_i^*)\| . (t_{i+1} - t_i)$$

Reiemann Sum for

$$\int_a^b \|\gamma'(t)\|\, dt$$

Theorem: If $\gamma : [a,b] \to \mathbb{R}^n$ is a C^1 path, then $S(\gamma)$ (total length of γ) exists and $S(\gamma) = \int_a^b \|\gamma'(t)\| dt$

Proof: We want to prove that $\forall \epsilon > 0$, $\exists \delta > 0$ s.t

$$\left| \int_a^b \|\gamma'(t)\| dt - S(\gamma, P) \right| < \epsilon \quad \forall \text{ partition } P \text{ with mesh } < \delta.$$

We showed above that

$$S(\gamma, P) = \sum_{i=1}^{m} \sqrt{(\gamma_1'(t_{i,1}^*))^2 + \dots + (\gamma_n'(t_{i,n}^*))^2} \cdot (t_{i+1} - t_i)$$

Define $F : [a,b]^n \longrightarrow \mathbb{R}$

$t_{i,1}^* \in [a,b], \dots, t_{i,n}^* \in [a,b]$

$$F(\vec{x}) = \sqrt{(\gamma_1'(x_1))^2 + \dots + (\gamma_n'(x_n))^2}$$

γ is C^1, so f is continuous. $[a,b]^n$ is compact.

So F is uniformly continuous on $[a,b]^n$.

As long as $\|\vec{x}-\vec{y}\| < \delta$, we have $|F(\vec{x})-F(\vec{y})| < \frac{\epsilon}{2(b-a)}$

Pick a $t_i^* \in [t_{i+1}, t_i]$ if $|t_{i+1}-t_i| < \tilde{\delta}_1 = \frac{1}{\sqrt{n}}\delta_1$

$$\|(t_{i,1}^*, t_{i,2}^*, \dots, t_{i,n}^*) - (t_i^*, t_i^*, \dots, t_i^*)\| < \delta_1$$

$$\Rightarrow \left| F(t_{i,1}^*, \dots, t_{i,n}^*) - \|\gamma'(t_i^*)\| \right| < \frac{\epsilon}{2(b-a)}$$

$$\left| s(\gamma,P) - \sum_{i=1}^{m} \|\gamma'(t_i^*)\| (t_{i+1}-t_i) \right| < \sum_{i=1}^{m} \frac{\epsilon}{2(b-a)} (t_{i+1}-t_i)$$

$$< \frac{\epsilon}{2(b-a)} \cdot (b-a) = \frac{\epsilon}{2}$$

Additionally by the theories of Riemann Sums,

$$\left| \int_a^b \|\gamma'(t)\| dt - \sum_{i=1}^{m} \|\gamma'(t_i^*)\| (t_{i+1}-t_i) \right| < \frac{\epsilon}{2} \text{ for some } \delta_2 > 0$$

$\&\ \max |t_{i+1}-t_i| < \delta_2$

Combine the two inequalities

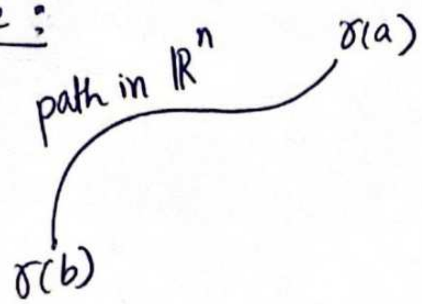
$\delta = \min\{\tilde{\delta}_1, \delta_2\}$ so $\forall P : \max |t_{i+1}-t_i| < \delta$

$$\left| s(\gamma,P) - \int_a^b \|\gamma'(t)\| dt \right| \le \left| s(\gamma,P) - \sum_{i=1}^{m} \|\gamma'(t_i^*)\| (t_{i+1}-t_i) \right|$$

$$+ \left| \sum_{i=1}^{m} \|\gamma'(t_i^*)\| (t_{i+1}-t_i) - \int_a^b \|\gamma'(t)\| \right|$$

$$< \epsilon/2 + \frac{\epsilon}{2} = \epsilon$$

Lec:



$\gamma : [a,b] \longrightarrow \mathbb{R}^n$

$$\underset{\text{arc length}}{S(\gamma)} = \int_a^b \|\gamma'(t)\|\, dt$$

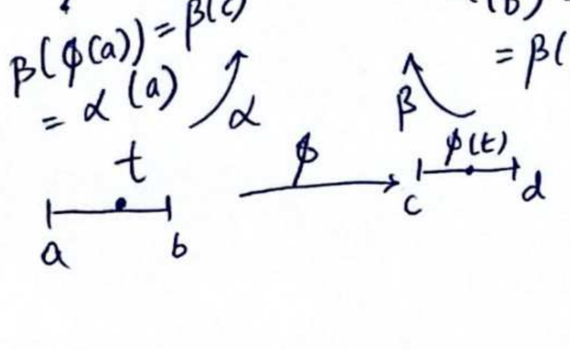
Notation: $ds = \|\gamma'(t)\|\, dt.$

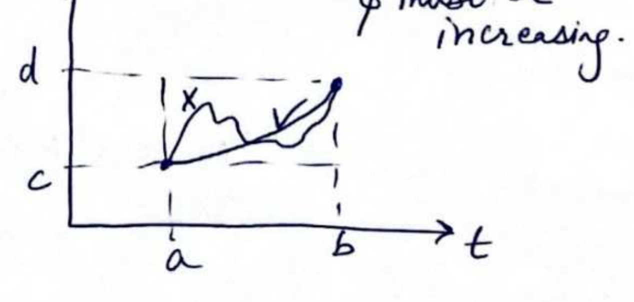
ds ↳ arc length coordinate

$$S(\gamma) = \int_\gamma ds$$

Definition: $\alpha : [a,b] \longrightarrow \mathbb{R}^n$, $\beta : [c,d] \longrightarrow \mathbb{R}^n$ are two paths.

They are equivalent if and only if there exists a C' mapping $\phi : [a,b] \longrightarrow [c,d]$. such that $\phi([a,b]) = [c,d]$ and $\alpha = \beta \circ \phi$ with $\phi'(t) > 0$, $\forall\, t \in [a,b]$





<u>Theorem:</u> Suppose $\alpha : [a,b] \to \mathbb{R}^n$ and $\beta : [c,d] \to \mathbb{R}^n$ are equivalent C' paths. Let f be a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ then $\int_a^b f(\alpha(t)) \|\alpha'(t)\| \, dt$

$$= \int_c^d f(\beta(t)) \|\beta'(t)\| \, dt$$

or simply we write $\int_\alpha f \, ds = \int_\beta f \, ds$.

Remark: If $f = 1$ $\underbrace{\int_\alpha 1 \, ds}_{s(\alpha)} = \underbrace{\int_\beta 1 \, ds}_{s(\beta)}$

<u>Proof:</u> $\alpha = \beta \circ \phi$ $\quad \phi : [a,b] \to [c,d]$, C', $\phi'(t) > 0$,

$\phi([a,b]) = [c,d]$

$$\int_a^b f(\alpha(t)) \|\alpha'(t)\| \, dt = \int_a^b f(\beta(\phi(t))) \|\beta'(\phi(t)) . \phi'(t)\| \, dt$$

chain rule

$$= \int_a^b f(\beta(\phi(t))) \|\beta'(\phi(t))\| \underbrace{\phi'(t)}_{>0 \text{ scalar}} dt$$

Let $u = \phi(t)$, $du = \phi'(t) \, dt$.

$$= \int_{\phi(a)}^{\phi(b)} f(\beta(u)) \|\beta'(u)\| \, du$$

$$= \int_c^d f(\beta(u)) \|\beta'(u)\| \, du \quad \blacksquare$$

Definition: A path is called smooth if

$\gamma : [a,b] \to \mathbb{R}^n$, γ is C^1 and $\underbrace{\gamma'(t) \neq \vec{0}}_{\text{nonzero velocity}}$ $\forall t \in [a,b]$

$\gamma(t)$ is the location of a particle on the path.

$\gamma'(t)$: velocity

$\|\gamma'(t)\|$: speed.

Proposition: Every smooth path $\gamma : [a,b] \to \mathbb{R}^n$ is equivalent to a smooth unit-speed path.

$\|\hat{\gamma}'(t)\| = 1$, $\forall t$ $\quad$ $\hat{\gamma}$ new path.

Proof: We aim to find $\hat{\gamma}$ with $\|\hat{\gamma}'\| = 1$

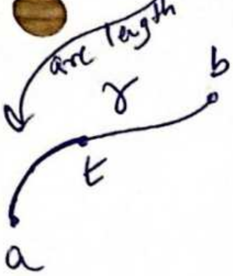
Denote $L = s(\gamma)$

It takes time from 0 to L for $\hat{\gamma}$ to travel from one endpoint to the other.

$\hat{\gamma} : [0, L] \to \mathbb{R}^n$

We define $\phi : [a,b] \to [0, L]$

$$\phi(t) = \int_a^t \|\gamma'(u)\| \, du = \int_{\gamma[a,t]} ds, \quad t \in [a,b]$$



We confirm:

$\gamma, \hat{\gamma}$ are equivalent
$$\begin{cases} ① \ \phi([a,b]) = [0,L] & \text{this is obvious} \\ ② \ \phi'(t) > 0 \\ ③ \ \gamma = \hat{\gamma} \circ \phi \iff \hat{\gamma} = \gamma \circ \phi^{-1} \\ ④ \ \|\hat{\gamma}'\| = 1 \end{cases}$$

For ②: $\phi'(t) = \dfrac{d}{dt} \displaystyle\int_a^t \|\gamma'(u)\| \, du = \|\gamma'(t)\| > 0$

(because γ is smooth $\gamma'(t) \neq \vec{0}$)

For ③: we define $\hat{\gamma}(x) = \gamma(\phi^{-1}(x)) \ \forall x \in [0, L]$

It remains to show $\|\hat{\gamma}'\| = 1$

$$\hat{\gamma}'(x) \overset{\text{chain}}{\underset{\text{rule}}{=}} \gamma'(\phi^{-1}(x)) \underbrace{(\phi^{-1})'(x)}_{} = \gamma'(\phi^{-1}(x)) \cdot \frac{1}{\phi'(\phi^{-1}(x))}$$

$$= \gamma'(t) \cdot \frac{1}{\phi'(t)}$$

$$= \frac{\gamma'(t)}{\|\gamma'(t)\|} \Rightarrow \|\hat{\gamma}'(x)\| = 1 \quad \forall x \in [0, L]$$

∎

{Inverse derivative formula.}

$y = f(x)$

$x = f^{-1}(y) = f^{-1}(f(x))$

$\dfrac{d}{dx}(x) = \dfrac{d}{dx}(f^{-1}(f(x)))$

$1 = (f^{-1})'(f(x)) \cdot f'(x)$

$\Rightarrow (f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow (f^{-1})'(y) = \dfrac{1}{f'(f^{-1}(y))}$ //

Lec: $\int_n \mathbb{R}^n$

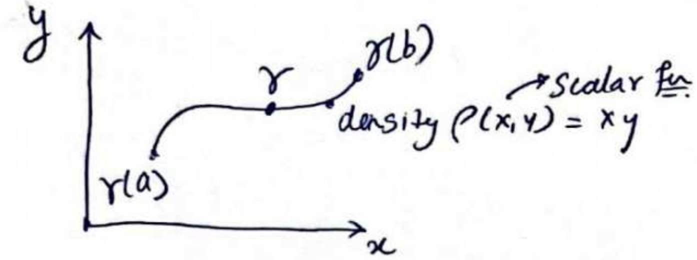
$\gamma : [a,b] \longrightarrow \mathbb{R}^n$

$ds = \|\gamma'(t)\| \, dt$

Define a function on γ, f.

$$\int_\gamma f = \int_a^b f(\gamma(t)) \; \|\gamma'(t)\| \, dt .$$

Example: heavy rope



$\gamma(t) = (x(t), y(t)) = (t^2, t^3 + 3t)$

$t \in [1,2]$

total mass of rope?

$$\int_\gamma \rho \, ds = \int_1^2 \rho(x(t), y(t)) \|\gamma'(t)\| dt$$

$$= \int_1^2 x(t)\, y(t) \sqrt{(2t)^2 + (3t^2+3)^2} \, dt$$

$$= \int_1^2 (t^2)(t^3+3t)\sqrt{(2t)^2+(3t^2+3)^2} \, dt$$

Work done by force

$$W = \int_\gamma \vec{F} \cdot d\vec{D} = \int_a^b \vec{F}(t) \cdot \underbrace{\gamma'(t)}_{\text{velocity}} \underbrace{dt}_{\text{time}} .$$

Instead of $\vec{F}(t)$, in practice we often define the force field according to the location $\vec{x} \in \mathbb{R}^n$, $\vec{F}(\vec{x})$

Change $\vec{F}(t)$ to $\vec{F}(\gamma(t))$

Scalar fn. for γ

$$\text{Work done} = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt \qquad \int_\gamma f.$$

Assume $\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n$

$\vec{F}(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \ldots, F_n(\vec{x})) \in \mathbb{R}^n$

$$\text{Work done} = \int_a^b (F_1(\gamma(t)), \ldots, F_n(\gamma(t))) \cdot (x_1'(t), \ldots, x_n'(t))\, dt$$

$$= \int_a^b \Big(F_1(\vec{x}) \underbrace{x_1'(t)\, dt}_{dx_1} + F_2(\vec{x}) \underbrace{x_2'(t)\, dt}_{dx_2} + \ldots + F_n(\vec{x})\,(x_n'(t)\, dt)\Big)$$

$$= \int_\gamma F_1(\vec{x})\, dx_1 + F_2(\vec{x})\, dx_2 + \ldots + F_n(\vec{x})\, dx_n$$

Definition: A differential 1 form on an open set U of $\mathbb{R}^n$ is an expression.

$$\omega = F_1(\vec{x})\,dx_1 + F_2(\vec{x})\,dx_2 + \dots + F_n(\vec{x})\,dx_n$$

$$= F(\vec{x}) \cdot d\vec{x} \quad \text{(my notation)}$$

$\Rightarrow$ work done $\quad W = \int_\gamma \omega$

Remark: If every F_i is continuous/ differentiable/ C^1/...

we say ω is continuous/ differentiable/ C^1/ ...

Definition: A C^1 differential 1-form $\omega = F_1 dx_1 + F_2 dx_2 + \dots + F_n dx_n$ is said to be exact if there is a C^2 function f with

$df = \omega$, ie. $F_i = \dfrac{df}{dx_i}$, $i = 1, \dots, n$.

total diff. ie. $df = \dfrac{\partial f}{\partial x_1} dx_1 + \dfrac{\partial f}{\partial x_2} dx_2 + \dots + \dfrac{\partial f}{\partial x_n} dx_n$.

$f: U \to \mathbb{R}$ $\quad f = f(x_1, \dots x_n)$ scalar fn.

U open $\subseteq \mathbb{R}^n$

Example : (1) $\omega = y\,dx + x\,dy$ $\qquad F_1 = y \overset{?}{=} \frac{\partial f}{\partial x} \qquad F_2 = x \overset{?}{=} \frac{\partial f}{\partial y}$

~~f = xdx~~

$f = xy$ is exact.

(2) $\omega = \underbrace{3y}_{F_1}\,dx + \underbrace{x}_{F_2}\,dy$

$$\frac{\partial f}{\partial x} = 3y \Rightarrow f = \int 3y\,dx = 3xy + \underbrace{g_1(y)}_{\text{no } x}$$

$$\frac{\partial f}{\partial y} = x \Rightarrow f = \int x\,dy = xy + \underbrace{g_2(x)}_{\text{no } y}$$

$\Rightarrow$ not exact.

Theorem: If $f : U \to \mathbb{R}$ is C^1 on open $U \subseteq \mathbb{R}^n$ containing $\gamma : [a,b] \to U$, γ is a C^1 path, then

$$\omega = df \qquad \int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

Proof: Let $g = f \circ \gamma \quad g : [a,b] \to \mathbb{R}$

$$g'(t) = \nabla f(\gamma(t)) \cdot \gamma'(t) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right)\Bigg|_{\vec{x} = \gamma(t)} \cdot \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt}\right)$$

$$= \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt} \Bigg|_{\vec{x} = \gamma(t)}$$

$$\int_\gamma df = \int_\gamma \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n = \int_a^b g'(t)dt \overset{FTC}{=} g(b) - g(a)$$

$$= f(\gamma(b)) - f(\gamma(a))$$

$\blacksquare$

<u>Remark:</u> If ω is exact, $\int_\gamma \omega$ does not depend on the path, it only depends on the two endpoints of the path.

Field $\vec{F} = (F_1, \dots, F_n)$ if $\omega = F_1 dx_1 + \dots + F_n dx_n$ is exact, we say the field is conservative (path-independent)

Lec: $\omega = F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz$

1-form in $\mathbb{R}^3$

$\exists f: \mathbb{R}^3 \to \mathbb{R}$ $\quad \frac{\partial f}{\partial x} = F_1,\ \frac{\partial f}{\partial y} = F_2,\ \frac{\partial f}{\partial z} = F_3 \longrightarrow \omega$ is exact.

"$\wedge$" Exterior product (wedge product) (not exactly the same as cross product).

[the resulting object $a \wedge b$ is not a vector but a bivector] ↳ lives in a different vector space.

(Cross product is only well defined in 3 dimensions & 7 D I believe)

$\vec{u} = (u_1, u_2)$ $\quad \vec{v} = (v_1, v_2)$

Properties of "$\wedge$":

1) $\vec{u} \wedge \vec{v} = -\vec{v} \wedge \vec{u}$
2) $\vec{u} \wedge \vec{u} = 0$
3) $c(\vec{u} \wedge \vec{v}) = (c\vec{u}) \wedge \vec{v} = \vec{u} \wedge (c\vec{v})$
4) $\vec{u} \wedge \vec{v} + \vec{u} \wedge \vec{w} = \vec{u} \wedge (\vec{w} + \vec{v}) = \vec{u} \wedge (\vec{v} + \vec{w})$

   $(u + \vec{v}) \wedge \vec{w} = \vec{u} \wedge \vec{w} + \vec{v} \wedge \vec{w}$
5) $(\vec{u} \wedge \vec{v}) \wedge \vec{w} = \vec{u} \wedge (\vec{v} \wedge \vec{w}) \longrightarrow$ associative (note that cross product is not associative)

$\vec{u} = u_1 \vec{e}_1 + u_2 \vec{e}_2$; $\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2$

$$\vec{u} \wedge \vec{v} = (u_1 \vec{e}_1 + u_2 \vec{e}_2) \wedge (v_1 \vec{e}_1 + v_2 \vec{e}_2) = u_1 \vec{e}_1 \wedge v_1 \vec{e}_1 + u_1 \vec{e}_1 \wedge v_2 \vec{e}_2 + u_2 \vec{e}_2 \wedge v_1 \vec{e}_1 + u_2 \vec{e}_2 \wedge v_2 \vec{e}_2$$

$$= (u_1 v_2 - u_2 v_1)\, \underbrace{\vec{e}_1 \wedge \vec{e}_2}$$

area of a unit square in $\mathbb{R}^2$ (following right hand rule)

$$[\vec{u},\vec{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \quad \det[\vec{u},\vec{v}] = u_1v_2 - u_2v_1.$$

$$\vec{u} \wedge \vec{v} \wedge \vec{w} = \det(\vec{u},\vec{v},\vec{w}) \underbrace{\vec{e_1} \wedge \vec{e_2} \wedge \vec{e_3}}_{\text{unit volume in } \mathbb{R}^3}$$

$dx \wedge dy = dS$ infinitesimal area.

$dx \wedge dy \wedge dz = dV$ infinitesimal volume.

Differential 2 form ("derivative" of 1-form)

$$d\omega = F(x,y,z)\, dx \wedge dy + G(x,y,z)\, dy \wedge dz + H(x,y,z)\, dz \wedge dx$$

$F, G, H : \mathbb{R}^3 \to \mathbb{R}$.

Question :- $\omega = P(x,y)\,dx + Q(x,y)\,dy$ 1-form in $\mathbb{R}^2$

$$d\omega = d\left(P(x,y)\,dx,\ Q(x,y)\,dy\right) = d(P\,dx) + d(Q\,dy)$$

Rules: α, β C^1 1-forms.

$f : \mathbb{R}^2$ or $\mathbb{R}^3 \to \mathbb{R}$ C^1 function.

1) $d(\alpha+\beta) = d\alpha + d\beta$

2) $d(f\alpha) = df \wedge \alpha + f\,d\alpha$ where $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

3) $d(dx) = 0$, $d(dy) = 0$, $d(dz) = 0$.

$$\Rightarrow d\omega = d(P\,dx + Q\,dy) = d(P\,dx) + d(Q\,dy)$$

$$= dP \wedge dx + P\,d(dx) + dQ \wedge dy + Q\,d(dy).$$

$$= dP \wedge dx + dQ \wedge dy$$

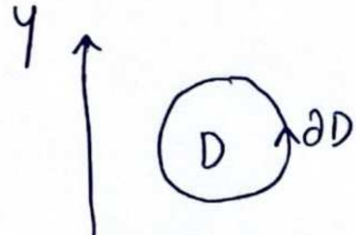
$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)\wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)\wedge dy$$

$$= \frac{\partial P}{\partial y}\,dy \wedge dx + \frac{\partial Q}{\partial x}\,dx \wedge dy$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(dx \wedge dy)$$

Try : $\omega = P\,dx + Q\,dy + R\,dz \Rightarrow d\omega = ?$ (Stokes' Thm)

<u>Recall Green's Theorem</u> (from Calc III)



D region in $\mathbb{R}^2$, ∂D is $\boxed{\text{Counterclockwise}}$

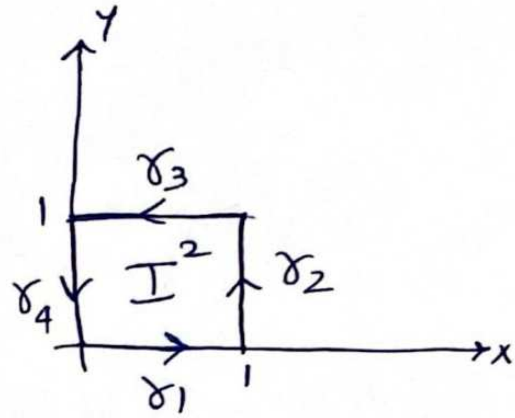
P, Q, C^1 functions $\mathbb{R}^2 \to \mathbb{R}$

$$\downarrow \quad dx \wedge dy \quad \downarrow \quad dxdy$$

$$\int_{\partial D} \underbrace{(P\,dx + Q\,dy)}_{\omega} = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy}_{d\omega}$$

$$\boxed{\int_{\partial D} \omega = \int_{D} \partial \omega}$$ Green's Theorem.

Green's theorem on unit square.



$$\int_{\partial I^2} \omega = \int_{I^2} d\omega$$

Proof: $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$

$$\int_{I^2} d\omega = \int_0^1 \left(\int_0^1 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \right) dy = \int_0^1 \left(\int_0^1 \frac{\partial Q}{\partial x} dx \right) dy - \int_0^1 \left(\int_0^1 \frac{\partial P}{\partial y} dy \right) dx$$

$$\overset{FTC}{=} \int_0^1 \left(Q(1,y) - Q(0,y) \right) dy - \int_0^1 \left(P(x,1) - P(x,0) \right) dx$$

γ_1 : ~~x~~ $x = t, \ y = 0 \qquad t \in [0,1]$

$$\int_{\gamma_1} \omega = \int_{\gamma_1} P dx + Q dy = \int_0^1 P(t,0)\, dt$$

γ_2: $x=1$, $y=t$ $\quad t\in[0,1]$.

$$\int_{\gamma_2}\omega = \int_0^1 Q(1,t)\,dt.$$

γ_3: $x=(1-t)$, $y=1$ $\quad t\in[0,1]$

$$\int_{\gamma_3}\omega = \int_0^1 P(1-t,1)\,(-dt) = -\int_0^1 P(u,1)\,du$$

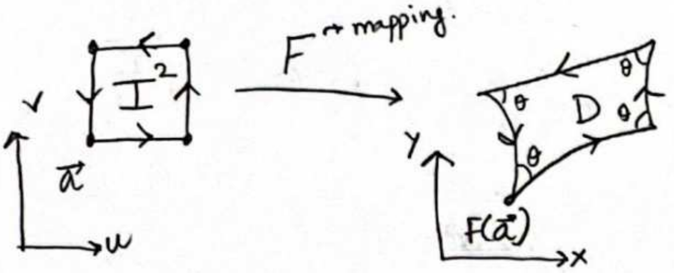
γ_4: $x=0$ $\quad y=1-t$ $\quad t\in[0,1]$

$$\int_{\gamma_3}\omega = -\int_0^1 Q(0,y)\,dy. \quad \blacksquare$$

<u>Lec</u>:

Last time we proved Green's Theorem for a unit square I^2. $I = [0,1]$

$$\int_{dI^2} \omega = \int_{I^2} d\omega$$



4 edges, 4 vertices.
$\theta < \pi$

F: one-to-one ; F is C^1 ; $F(I^2) = D$; $\underbrace{\det(F') > 0}_{\text{preserves orientation of the boundary.}}$ ← Jacobian matrix of F

$$(x,y) = F(u,v) = (\underset{x}{\underset{\shortparallel}{F_1(u,v)}}, \underset{y}{\underset{\shortparallel}{F_2(u,v)}})$$

<u>Goal:</u> $\displaystyle\int_{\partial D} \omega \stackrel{?}{=} \int_D d\omega$

<u>Idea</u>: Pull back from D to I^2

on D: $\omega = P(x,y)\,dx + Q(x,y)\,dy$ $\qquad P, Q : C^1$

$$\begin{array}{ll} D: & \omega \longrightarrow d\omega = \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right) dx \wedge dy \\ & \downarrow \text{pullback using F} \quad \downarrow \\ I^2: & \omega^* \qquad\quad (d\omega)^* \end{array}$$

→ This is the main part.

$$\text{on } I^2 \quad \underset{\underset{\int_{\partial D}\omega}{\shortparallel}}{\int_{dI^2} \omega^*} \stackrel{\text{Green's}}{=} \int_{I^2} d(\omega^*) \stackrel{?}{=} \underset{\underset{\int_D d\omega}{\shortparallel}}{\int_{I^2} (d\omega)^*}$$

Step ① ω^*, $(d\omega)^*$?

Step ② confirm $d(\omega^*) = (d\omega)^*$

$P(x,y) \to F_1(u,v)$, $Q(x,y) \to F_2(u,v)$

ω on (x,y) : $\omega = P dx + Q dy$

ω^* on (u,v) : $\omega^* = (P \circ F)\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right)$

$$+ (Q \circ F)\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right)$$

$d\omega$ on (x,y) : $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \underbrace{dx \wedge dy}_{\to\, du,\, dv?}$

$$dx \wedge dy = \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) \wedge \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right)$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}(du \wedge dv) + \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}(dv \wedge du)$$

$$= \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du \wedge dv$$

$$= \det \underbrace{\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}_{F' \to \text{Jacobian.}} du \wedge dv$$

notation

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

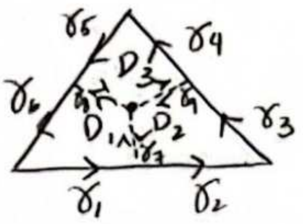
$$\Rightarrow \quad dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}(du \wedge dv)$$

$(d\omega)^*$ on (u,v) : $(d\omega)^* = \left(\frac{\partial Q}{\partial x}\circ F - \frac{\partial P}{\partial y}\circ F\right)\frac{\partial(x,y)}{\partial(u,v)}(du\wedge dv)$

Step 2 : $d(\omega^*) \stackrel{?}{=} (d\omega)^*$ (Seperate notes)

This completes Green's theorem for D. ✓

Triangle?



The line integrals on the dashed lines cancel.

$\partial D_1 = \gamma_1 + \cancel{\gamma_7} + \cancel{\gamma_8} + \gamma_6$

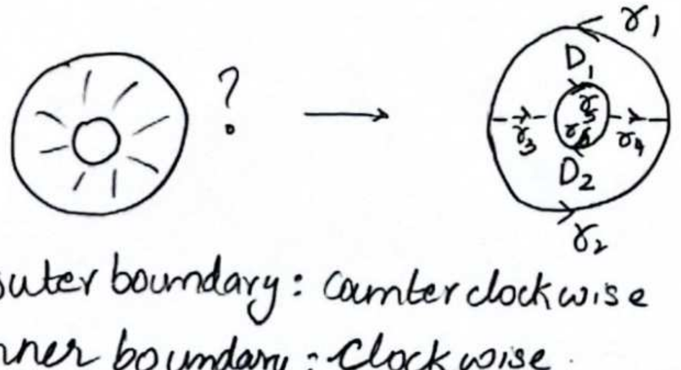
$\partial D_2 = \gamma_2 + \gamma_3 + \cancel{\gamma_9} - \cancel{\gamma_7}$

$\partial D_3 = \gamma_4 + \gamma_5 - \cancel{\gamma_8} - \gamma_9$

$$\int_D d\omega = \int_{D_1} d\omega + \int_{D_2} d\omega + \int_{D_3} d\omega \qquad D = D_1 \cup D_2 \cup D_3$$

$$\stackrel{\text{Green's}}{=} \int_{\partial D_1}\omega + \int_{\partial D_2}\omega + \int_{\partial D_3}\omega = \int_{\partial D}\omega$$

> The same technique applies to any polygon or even a circle.



> D_1 & D_2 are 4 sided so it is covered.

$\partial D_1 = \gamma_1 + \cancel{\gamma_4} + \gamma_5 + \cancel{\gamma_3}$

$\partial D_2 = \gamma_2 - \cancel{\gamma_3} + \gamma_6 - \cancel{\gamma_4}$

outer boundary: counterclockwise

inner boundary: clockwise

> Multiple holes can also be dealt with similarly.

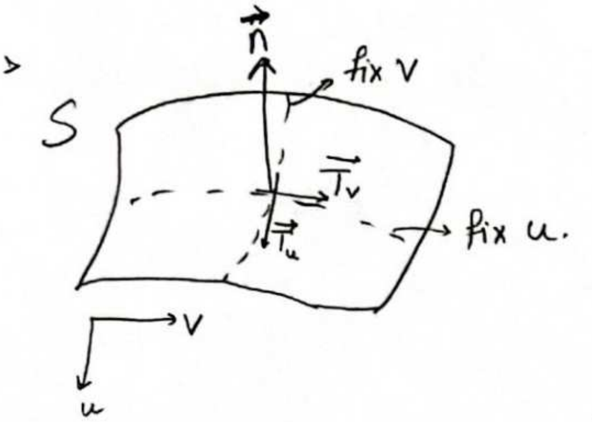
> "nice" region: can be seperated into cells: Green's theorem applies!

Surface Integral & Stoke's theorem.

Surface in $\mathbb{R}^3$ can be represented by 2 parameters (u, v)

on S

$$(x, y, z) = \left(F_1(u,v),\ F_2(u,v),\ F_3(u,v)\right)$$



$\vec{T_u}, \vec{T_v}$ tangent vectors.

$$\vec{n} = \frac{\vec{T_u} \times \vec{T_v}}{\|\vec{T_u} \times \vec{T_v}\|}$$ unit normal.

$\vec{n}$ defines orientation of ∂S from right hand rule.

Stoke's Theorem

$$\int_S d\omega = \int_{\partial S} \omega$$

(Green's theorem is a special case of this on a flat surface)

$$d\omega = F(x,y,z)\, dx \wedge dy + G(x,y,z)\, dy \wedge dz + H(x,y,z)\, dz \wedge dx.$$

where,

$$dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv$$

Exam: Lagrange multiplier (problem & proof?)
See additional notes.
Integrals (>60%) of final.