Dijkstra Algorithm

DESIGN & ANALYSIS OF ALGORITHMS

CSE – AIML

6TH SEMESTER

Dijkstra Algorithm

- Dijkstra Algorithm is a very famous greedy algorithm.
- It is used for solving the single source shortest path problem.
- It computes the shortest path from one particular source node to all other remaining nodes of the graph.

Dijkstra Algorithm Conditions

It is important to note the following points regarding Dijkstra Algorithm-

- Dijkstra’s algorithm works only for connected graphs.
- Dijkstra algorithm works only for those graphs that do not contain any negative weight edge.
- The actual Dijkstra algorithm does not output the shortest paths.
- It only provides the value or cost of the shortest paths.
- By making minor modifications in the actual algorithm, the shortest paths can be easily obtained.
- Dijkstra’s algorithm works for directed as well as undirected graphs.

Dijkstra Algorithm-

```
Dijkstra(G, S)
Create vertex set Q
    for each vertex v in graph G
        dist[v] = ∞
        add v to Q
        dist[S] = 0
    While Q is not empty
        u = Extract_min[Q]
        for each neighbor v of u
            Relax(u, v)
```

Dijkstra Algorithm-

Implementation-The implementation of the above Dijkstra Algorithm is explained in the following steps:

Step-01:

In the first step. two sets are defined-

- One set contains all those vertices which have been included in the shortest path tree. In the beginning, this set is empty.
- Other set contains all those vertices which are still left to be included in the shortest path tree. In the beginning, this set contains all the vertices of the given graph.

Dijkstra Algorithm-

- Step-02:
- For each vertex of the given graph, two variables are defined as-

  Π[v] which denotes the predecessor of vertex ‘v’
- d[v] which denotes the shortest path estimate of vertex ‘v’ from the source vertex.

Initially, the value of these variables is set as-

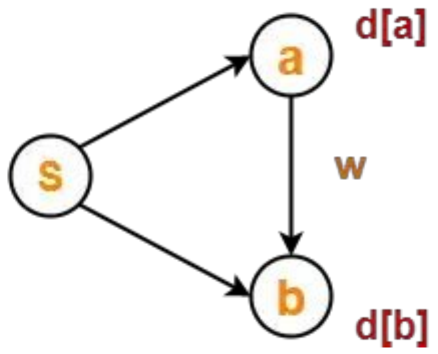
- The value of variable ‘Π’ for each vertex is set to NIL i.e. Π[v] = NIL
- The value of variable ‘d’ for source vertex is set to 0 i.e. d[S] = 0
- The value of variable ‘d’ for remaining vertices is set to ∞ i.e. d[v] = ∞

Dijkstra Algorithm-

- Step-03:The following procedure is repeated until all the vertices of the graph are processed-
- Among unprocessed vertices, a vertex with minimum value of variable ‘d’ is chosen.
- Its outgoing edges are relaxed.
- After relaxing the edges for that vertex, the sets created in step-01 are updated.

What is Edge Relaxation?

- Consider the edge (a,b) in the following graph-



Here, d[a] and d[b] denote the shortest path estimate for vertices a and b respectively from the source vertex ‘S’.

Now, If $d[a] + w < d[b]$ then $d[b] = d[a] + w$ and $\Pi[b] = a$

This is called as edge relaxation.

Time Complexity Analysis-

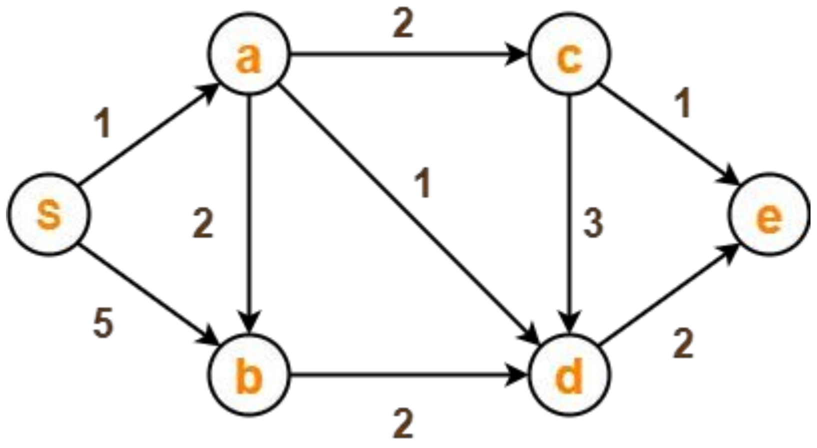
- Case-01: This case is valid when-
- The given graph G is represented as an adjacency matrix.
- Priority queue Q is represented as an unordered list.
- Here,
- A[i,j] stores the information about edge (i,j).
- Time taken for selecting i with the smallest dist is O(V).
- For each neighbor of i, time taken for updating dist[j] is O(1) and there will be maximum V neighbors.
- Time taken for each iteration of the loop is O(V) and one vertex is deleted from Q.
- Thus, total time complexity becomes O(V2).

Time Complexity Analysis-

- Case-02:This case is valid when-
- The given graph G is represented as an adjacency list.
- Priority queue Q is represented as a binary heap.
- Here,
- With adjacency list representation, all vertices of the graph can be traversed using BFS in O(V+E) time.
- In min heap, operations like extract-min and decrease-key value takes O(logV) time.
- So, overall time complexity becomes O(E+V) x O(logV) which is O((E + V) x logV) = O(ElogV)
- This time complexity can be reduced to O(E+VlogV) using Fibonacci heap.

PRACTICE PROBLEM BASED ON THE DIJKSTRA ALGORITHM-

- Problem-Using Dijkstra’s Algorithm, find the shortest distance from source vertex ‘S’ to the remaining vertices in the following graph-



- Also, write the order in which the vertices are visited.

Solution

Step-01:The following two sets are created-

Unvisited set : {S , a , b , c , d , e}

Visited set   : { }

Step-02:

The two variables Π and d are created for each vertex and initialized as-

Π[S] = Π[a] = Π[b] = Π[c] = Π[d] = Π[e] = NIL

d[S] = 0

d[a] = d[b] = d[c] = d[d] = d[e] = ∞

Solution

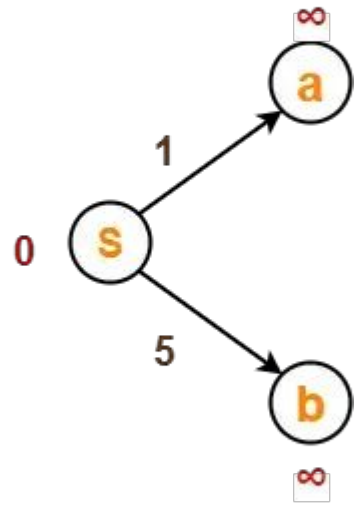
- Step-03:
- Vertex ‘S’ is chosen.
- This is because shortest path estimate for vertex ‘S’ is least.
- The outgoing edges of vertex ‘S’ are relaxed.

Before Edge Relaxation-

- Now,

- d[S] + 1 = 0 + 1 = 1 < ∞
- ∴ d[a] = 1 and Π[a] = S

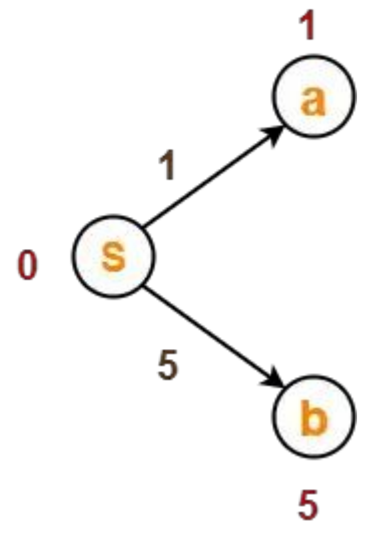
- d[S] + 5 = 0 + 5 = 5 < ∞
- ∴ d[b] = 5 and Π[b] = S



After edge relaxation, our shortest path tree is-

- Now, the sets are updated as-

- Unvisited set : {a , b , c , d , e}
- Visited set : {S}



Step-04:

- Vertex ‘a’ is chosen.
- This is because shortest path estimate for vertex ‘a’ is least.
- The outgoing edges of vertex ‘a’ are relaxed.

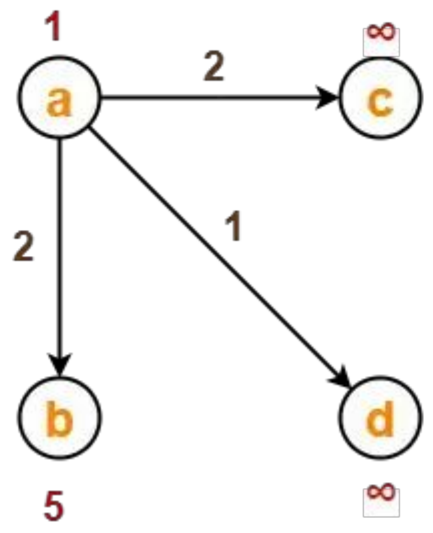
Before Edge Relaxation-

- Now,

- $d[a] + 2 = 1 + 2 = 3 < \infty$
- $\therefore$ $d[c] = 3$ and $\Pi[c] = a$

- $d[a] + 1 = 1 + 1 = 2 < \infty$
- $\therefore$ $d[d] = 2$ and $\Pi[d] = a$

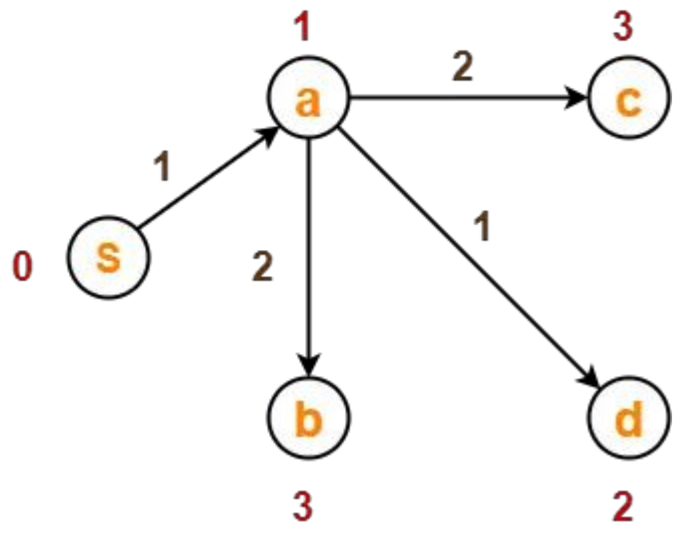
- $d[b] + 2 = 1 + 2 = 3 < 5$
- $\therefore$ $d[b] = 3$ and $\Pi[b] = a$



After edge relaxation, our shortest path tree is-

- Now, the sets are updated as-

- Unvisited set : {b , c , d , e}
- Visited set : {S , a}
-



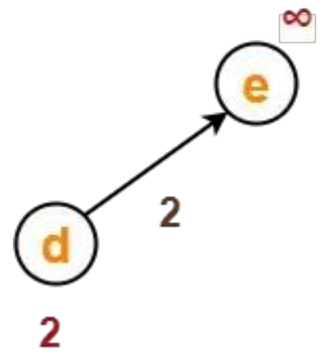
Step-05:

- Vertex ‘d’ is chosen.
- This is because shortest path estimate for vertex ‘d’ is least.
- The outgoing edges of vertex ‘d’ are relaxed.

Before Edge Relaxation-

- Now,

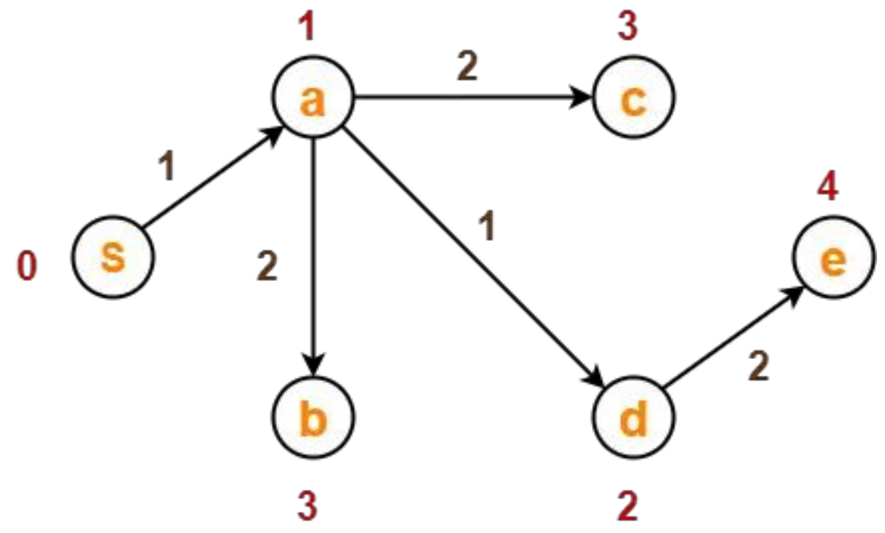
- $d[d] + 2 = 2 + 2 = 4 < \infty$
- $\therefore$ $d[e] = 4$ and $\Pi[e] = d$



After edge relaxation, our shortest path tree is-

- Now, the sets are updated as-

- Unvisited set : {b , c , e}
- Visited set : {S , a , d}



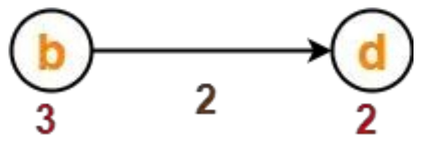
Step-06:

- Vertex ‘b’ is chosen.
- This is because shortest path estimate for vertex ‘b’ is least.
- Vertex ‘c’ may also be chosen since for both the vertices, shortest path estimate is least.
- The outgoing edges of vertex ‘b’ are relaxed.

Before Edge Relaxation-

- Now,

- $d[b] + 2 = 3 + 2 = 5 > 2$
- $\therefore$ No change



After edge relaxation, our shortest path tree remains the same as in Step-05.

- Now, the sets are updated as-

- Unvisited set : {c , e}
- Visited set : {S , a , d , b}

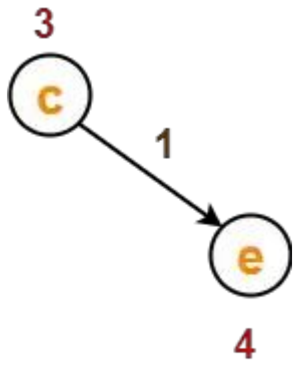
Step-07:

- Vertex ‘c’ is chosen.
- This is because shortest path estimate for vertex ‘c’ is least.
- The outgoing edges of vertex ‘c’ are relaxed.
-

Before Edge Relaxation-

- Now,

- d[c] + 1 = 3 + 1 = 4 = 4
- ∴ No change



After edge relaxation, our shortest path tree remains the same as in Step-05.

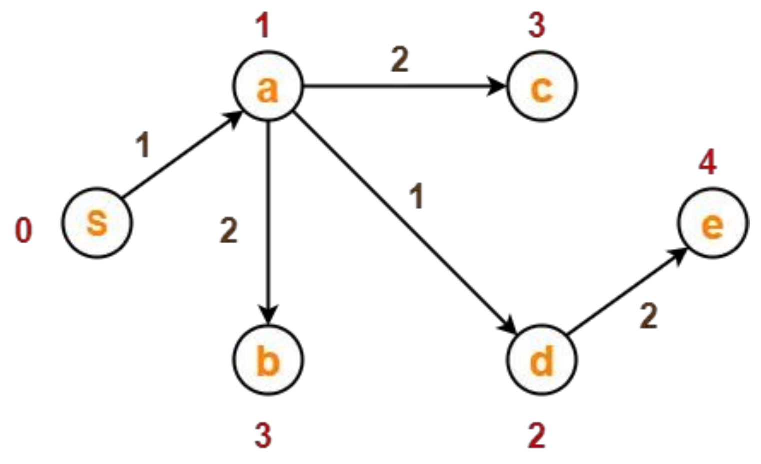
- Now, the sets are updated as-

- Unvisited set : {e}
- Visited set : {S , a , d , b , c}

Step-08:

- Vertex ‘e’ is chosen.
- This is because shortest path estimate for vertex ‘e’ is least.
- The outgoing edges of vertex ‘e’ are relaxed.
- There are no outgoing edges for vertex ‘e’.
- So, our shortest path tree remains the same as in Step-05.
- Now, the sets are updated as-
- Unvisited set : { }
- Visited set : {S , a , d , b , c , e}
- Now,
- All vertices of the graph are processed.
- Our final shortest path tree is as shown below.
- It represents the shortest path from source vertex ‘S’ to all other remaining vertices.

RESULT



Shortest Path Tree

- The order in which all the vertices are processed is :
- S , a , d , b , c , e.