AML Assignment-1

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1 Question 1

Without loss of generality, assume the x_i are topologically ordered (from ancestors to descendants w.r.t to the DAG). This assumption can be done because the only difference in the formula made by permuting the indices by a permutation ψ is that π_i is replaced by some π_i' obtained by applying the permutation ψ to the set of parents of x_i.
Now, by factorizing the joint distribution as per the DAG, we get:

$$\begin{aligned}
P(x_1, \cdots x_{n-1}) &= \sum_{x_1, \cdots x_{n-1}} P(x_1, \cdots x_{n-1}) \\
&= \prod_{i=1}^{n-1} f_i(x_i, x_{\pi_i}) \cdot \left(\sum_{x_n} f_n(x_n, x_{\pi_n}) \right) \\
&= \prod_{i=1}^{n-1} f_i(x_i, x_{\pi_i})
\end{aligned}$$

Proceeding inductively, we can observe:

$$\begin{aligned}
P(x_1, \cdots x_j) &= \prod_{i=1}^{j} f_i(x_i, x_{\pi_i}) \cdot \prod_{i=j+1}^{n} \left(\sum_{x_i} f_i(x_i, x_{\pi_i}) \right) \\
&= \prod_{i=1}^{j} f_i(x_i, x_{\pi_i}) \cdot \prod_{i=j+1}^{n} 1 \\
&= \prod_{i=1}^{j} f_i(x_i, x_{\pi_i})
\end{aligned}$$

Thus, $P(x_i | \{x_1, \dots, x_{i-1}\}) = \frac{P(x_1, \dots, x_i)}{P(x_1, \dots, x_{i-1})} = f_i(x_i, x_{\pi_i})$
However, by the fact that P factorizes over the given DAG, $P(x_i | \{x_1, \dots, x_{i-1}\}) = P(x_i | x_{\pi_i})$
Thus, $f_i(x_i, x_{\pi_i}) = P(x_i | x_{\pi_i})$

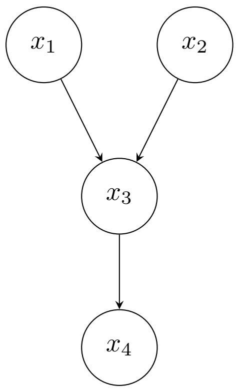
2 Question 2

2.1 Part a

$ZP(1,0,1,1) = e^0 \cdot e^0 = 1$
$ZP(1,1,0,0) = e^0 \cdot e^0 = 1$
$ZP(1,1,0,1) = e^0 \cdot e^1 = e$
$ZP(1,1,1,0) = e^1 \cdot e^1 = e^2$
$ZP(1,1,1,1) = e^1 \cdot e^0 = e$

2.2 Part b

First, x_1 is added to the BN. It has no parents. When adding x_2, observe that $P(x_1, x_2) = \frac{1}{4}$ for all combinations of (x_1, x_2). This equals $P(x_1) \cdot P(x_2)$. Thus, $x_2 \perp\!\!\!\perp x_1$.
When x_3 is added, given both x_1 and x_2, the probability that $x_3 = x_1 \oplus x_2$ is $\frac{1}{e}$ times the probability of $x_3 = \neg(x_1 \oplus x_2)$. Similarly, x_3 is dependent on x_2 given x_1. Thus, we add edges from x_1 and x_2 to x_3.
When x_4 is added, given x_3, the probability that $x_4 = x_3$ is e times the probability that $x_4 = \neg x_3$ and is independent of x_1 and x_2. Thus, we add an edge from x_3 to x_4.



2.3 Part c

Node	**0**	**1**
x_1	$\frac{1}{2}$	$\frac{1}{2}$

Node	**0**	**1**
x_2	$\frac{1}{2}$	$\frac{1}{2}$

x_1, x_2	**(0, 0)**	**(0, 1)**	**(1, 0)**	**(1, 1)**
$x_3 = 0$	$\frac{1}{e+1}$	$\frac{e}{e+1}$	$\frac{e}{e+1}$	$\frac{1}{e+1}$
$x_3 = 1$	$\frac{e}{e+1}$	$\frac{1}{e+1}$	$\frac{1}{e+1}$	$\frac{e}{e+1}$

x_3	**0**	**1**
$x_4 = 0$	$\frac{1}{e+1}$	$\frac{e}{e+1}$
$x_4 = 1$	$\frac{e}{e+1}$	$\frac{1}{e+1}$

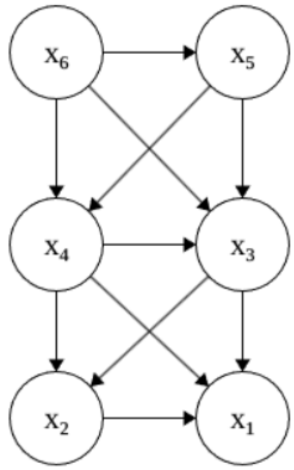
3 Question 3

3.1 Part a

x_1, x_2, x_3, x_4 are all unconditionally independent of x_6. This is because the only possible path from x_6 to any of them uses x_5 as the center of a v-structure, and x_5 is not observed and has no observed descendants when no variables are observed, and hence cannot be the center of a v-structure. x_5 is dependent on x_6 because they are directly connected.

3.2 Part b

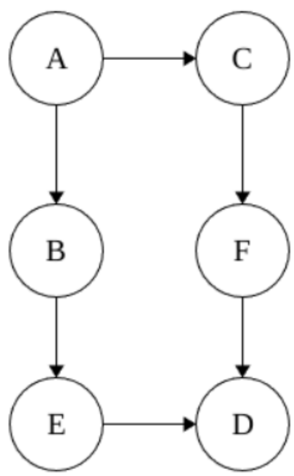
x_5 is dependent on x_6. $x_4 \not\perp\!\!\!\perp x_5|S$ for any S by virtue of a direct edge between them. $x_4 \not\perp\!\!\!\perp x_6|x_5$ by virtue of x_5 being a valid center of a v-structure.
$x_3 \not\perp\!\!\!\perp x_5|S$ and $x_3 \not\perp\!\!\!\perp x_4|S$ for any S. $x_3 \not\perp\!\!\!\perp x_6|\{x_4, x_5\}$ due to x_5 being a valid center of a v-structure.
$x_2 \not\perp\!\!\!\perp x_4|S$ for any S. $x_2 \not\perp\!\!\!\perp x_3|x_4$. Thus, $x_2 \perp\!\!\!\perp x_5, x_6|\{x_4, x_3\}$ is the best we can do.
$x_1 \not\perp\!\!\!\perp x_4|S$ for any S. $x_1 \not\perp\!\!\!\perp x_3|x_4$ and $x_1 \not\perp\!\!\!\perp x_2|x_4$. Thus, $x_1 \perp\!\!\!\perp x_5, x_6|\{x_4, x_3, x_2\}$ is the best we can do.



4 Question 4

4.1 Part 1

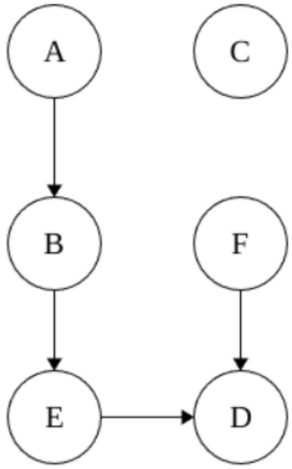
Consider the following BN for the given model:



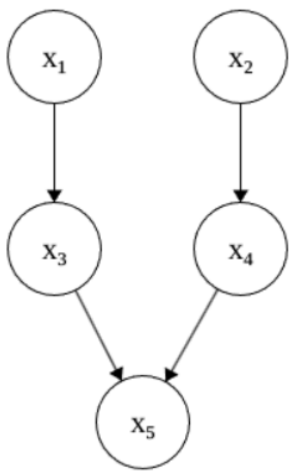
Since P factorizes over G, G is an I-map for P. Since $I(G) \subseteq I(P)$, the conditional independence $F \perp\!\!\!\perp B|A$, which holds in this graph, holds in P as well.

4.2 Part 2

We cannot guarantee that C is independent of E. In this very graph, C-A-B-E is an active path from C to E given no observed variables. We at the same time cannot guarantee that C is dependent on E. Consider the graph below, which satisfies all the conditional independencies satisfied in P and some more.



5 Question 5



6 Question 6

$$x_4 \perp\!\!\!\perp x_3, x_1 | x_2 (\text{given}) \tag{1}$$
$$x_4 \perp\!\!\!\perp x_3 | x_1, x_2 (\text{Weak Union}) \tag{2}$$
$$x_3 \perp\!\!\!\perp x_2 | x_1 (\text{given}) \tag{3}$$
$$x_3 \perp\!\!\!\perp x_4, x_2 | x_1 (\text{Contraction on 2 and 3}) \tag{4}$$
$$x_3 \perp\!\!\!\perp x_4 | x_1 (\text{Decomposition}) \tag{5}$$
$$x_4 \perp\!\!\!\perp x_1 | x_2 (\text{Decomposition on 1}) \tag{6}$$
$$x_1 \perp\!\!\!\perp x_2 (\text{given}) \tag{7}$$
$$x_1 \perp\!\!\!\perp x_4 (\text{Contraction on 6 and 7}) \tag{8}$$
$$x_4 \perp\!\!\!\perp x_3 (\text{Contraction on 5 and 8}) \tag{9}$$

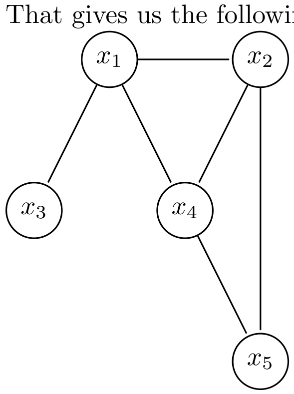
7 Question 7

Given that $Pa(x_i)$ and $Pa(x_j)$ are observed, notice that no path emerging out of x_i in the direction of one of its parents can be active (since the edge is directed towards x_i, the parent is not the center of a v-structure, and is observed, hence the parent is blocked).
Thus, any active path must come out of x_i towards one of its children. If we consider this path to keep going downwards in the graph until some node x_k which it enters from a parent of x_k and exits via another parent of x_k. Since x_k is the center of a v-structure, it has to be an ancestor of $Pa(x_j)$ (since if it was an ancestor of $Pa(x_i)$ and also a descendant of x_i, the graph would no longer be a DAG.)
Thus, x_j is a descendant of x_k and x_k is a descendant of x_i. Thus, x_j is a descendant of x_i.
By the same argument with x_i and x_j swapped, x_i is a descendant of x_j. Thus, x_i and x_j are descendants of each other.
This is not possible in a DAG.

8 Question 8

8.1 Part a

We use Proposition 4.8 from Koller-Friedman: Let G be any Bayesian network graph. The moralized graph M[G] is a minimal I-map for G.
That gives us the following diagram equivalent Markov network.

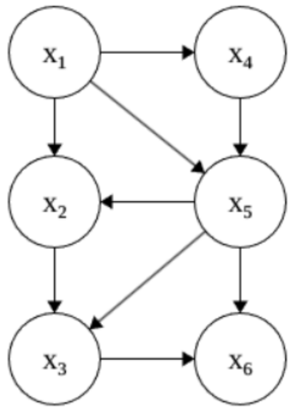


8.2 Part b

$x_1 \perp\!\!\!\perp x_2$ and $x_1 \perp\!\!\!\perp x_2|x_3$ are two CIs that hold in G but not in H.

9 Question 9

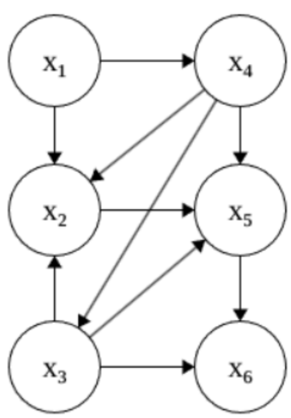
9.1 Part a



Justification: We use the algorithm for constructing an I-map of a given distribution given in Algorithm 3.2 of Friedman-Koller.
When x_5 is inserted, it has of course to be connected with x_4, but x_1 is also not independent of x_5 given x_4 in H. For x_2, x_5 and x_1 together separate it from x_4, and this blanket is minimal since it has to be connected to both x_1 and x_5. For x_3, x_2 is necessarily its parent. x_3 cannot be rendered independent of x_5 by any subset of $\{x_1, x_2, x_4\}$, so x_5 is also a parent. At this point, x_3 is separated from x_1 and x_4. For x_6, it is necessary to add its neighbours in H as its parents in G, and these d-separate it in H from all other vertices, so we are done. In total, this approach created 9 directed edges.

9.2 Part b

A simple change that generates more variables is to consider the order $x_1, x_4, x_3, x_2, x_5, x_6$.



When x_3 is inserted, no subset of $\{x_4, x_1\}$ d-separates it from the other vertex. When x_2 is now added, in the absence of x_5, it cannot be d-separated from x_4 by x_1, x_3 (note that due to monotonicity of d-separation in Markov networks, we do not need to separately consider subsets of $\{x_1, x_3\}$). x_5 needs to be connected with both its neighbours, and also with x_3 since $\{x_2, x_4\}$ does not d-separate it from x_3. x_6 has to be connected to all its neighbours, who form a Markov blanket. In total, this approach created 10 directed edges.

9.3 Part c

$x_3 \perp\!\!\!\perp x_5 | x_2, x_6$ and $x_5 \perp\!\!\!\perp x_1 | x_2, x_4$ are two CIs that hold in H but not in G (since x_3 and x_5, x_5 and x_1 now have edges between them in G, but were d-separated from each other by an appropriate set of vertices in H).