POSETS & LATTICES.

* Q. verify $(Z^+, \backslash)$ is a poset ?

for reflexive

$2 \backslash 2$ is possible

for anti-symmetric

Yes.

$\therefore$ $(Z^+, \backslash)$ is a poset

for transitive

satisfies

Q. Verify $(Z, /)$.

Reflexivity

Yes

Anti-Symmetric

$1/1 = 1$ ✓ $-1/1 = -1$ ✓

$-1/-1 = 1$ — $\therefore$ Not antisymmetric

$(Z, /)$ is not a poset becoz its violating anti-symm. property.

* Similarly.

Verify $(P(S), \subseteq)$ is a Poset ?

where $P(S)$ is power set of S ?

$S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$.

It is a POSET.

★ Comparability.

Q) in $(Z^+, /)$; consider $(2,4) \in Z^+$ are they compara

Yes, because $4/2$ is possible.

consider $(3,5) \in Z^+$,

they aren't comparable because

$3/5 \rightarrow$ not poss - -   incomparable element

$5/3 \rightarrow$ " "

★ Total Ordered Set.

Consider the POSET $(S, \leq)$, if every 2 elements of S are comparable, then S is called as total ordered set or CHAIN.

eg $\rightarrow (Z, <=)$ is a POSET as well as Totally Ordered set.

consider $(Z^+, /) \longrightarrow$ POSET.

Not a Totally Ordered set because

$3/5 \longrightarrow$ not divisible

$5/3 \longrightarrow$

NOTE

Sometimes, the subset of an Ordered Set, may be a totally Ordered Set.

$A = \{2, 6, 12, 36\}$

Now, for $(A, /) \rightarrow$ it is a total orderd set.

★ Every Total Ordered Set is a Poset but the converse may not be True

* Lexicographic Ordering.

* HASSE DIAGRAM.

A Poset can be represented by means of Hasse Diagram. In this diagram, the vertices represent elements of the set.

Let $(x, y) \in S$,

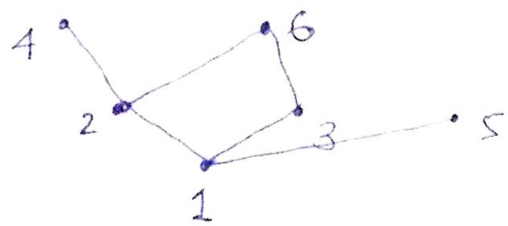
then y is immediate successor of x if there doesn't exist any element z in betⁿ x & y $\ni$ $x \leq z$ & $z \leq y$

$\ni$ → such that

Also, x is called immediate predecessor of 'y'.

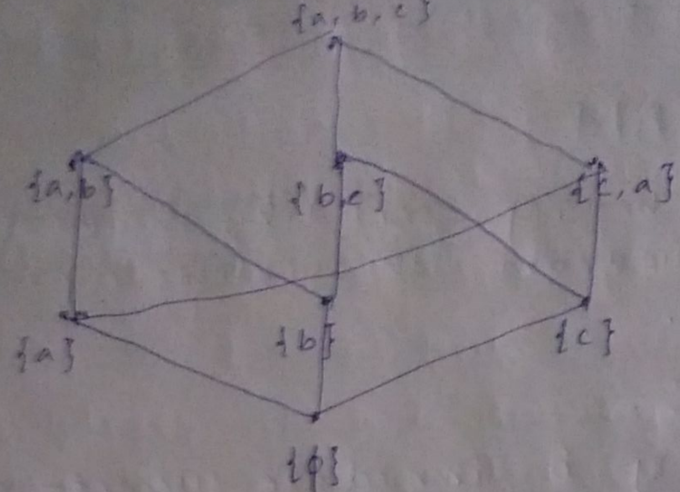
Q. Let $x = \{1, 2, 3, 4, 5, 6\}$ & '/' is the po relatⁿ on x. Draw Hasse diagram.

Clearly $(x, /)$ is a POSET.



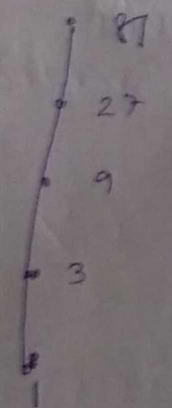
HW.

Consider $S = \{a, b, c\}$. Draw Hasse Diagram for $(P(S), \subseteq)$.



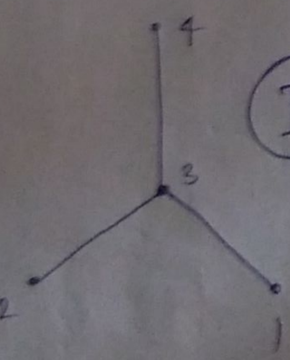
Q. Draw the Hasse Diagram of $(A, /)$

$A = \{1, 3, 9, 27, 81\}$.



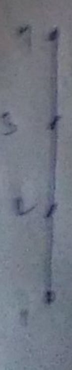
III

Q. Describe the ordered pairs in the relatⁿ determined by the foll. Hasse Diagrams :-



II

{ (1,1) (2,2) (3,3) (4,4) (1,3) (2,3) (3,4) (1,4) (2,4) }



(1,1) (2,2) (3,3) (4,4) (1,2) (1,3) (1,4)
(2,3) (2,4) (3,4).

* SPECIAL ELEMENTS OF POSETS :—

• **Maximal Element :—**

An element 'a' is called maximal element of POSET 'P' if NO element of P strictly succeeds 'a'.

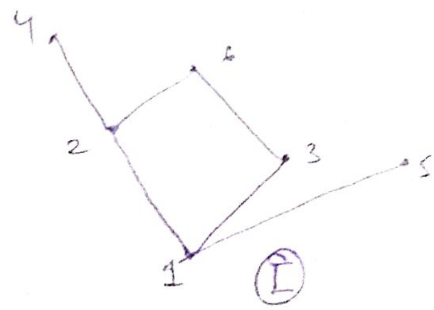
similarly,

• **Minimal Element :—**

An element 'a' is called minimal element if there is no predecessor to 'a'.

Diagram ②

Maximal	Minimal
4	1, 2



I

Dig. III

Max	Min
81	1.

Diag ①

Max	Min
4, 6, 5	1.

• **Greatest Element :—**

An element 'a' belongs to P is greatest element if ~~ekery~~ every element in P preceeds 'a'.

• **Least Element :—**

An element 'b' belongs to P is called Least element if every element in P succeeds 'b'.

★ The greatest & least elements if exists, they are Unique.

Q. Find the least & greatest elements of $(Z^+, /)$.

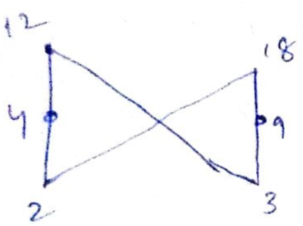
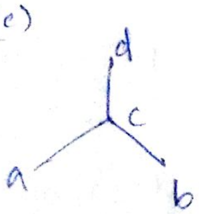
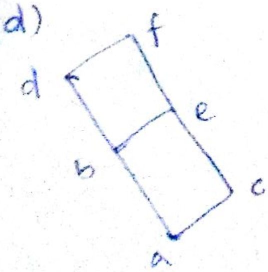
Least no. → 1.

Greatest element → NA.

Clearly $1/n \ \forall \ n \in Z^+$

∴ 1 is the least element. but we cant find a +ve integer which is div. by all +ve integers & hence there is no greatest element.

Q. Identify maximal, minimal, greatest & least from the foll. Hasse diagram :-

	Maxi	Mini	Greatest	Least
(a) 	27	1	27	1
(b) 	12, 18	2, 3	Nil	Nil
(c) 	d	a, b	d	Nil
(d) 	f	a	f	a

WELL ORDERED SET :-

In a POSET $(P, \leq)$, if every subset of P has a least element, then it is called Well Ordered Set.

eg $(N, <=)$

→ $(Z, <=)$ is not a well ordered set because its subset Z^- doesn't contain least element

UPPER & LOWER BOUND :-

Let 'B' be a subset of $(A, \leq)$. An element $u \in A$ is called Upper bound of B if u succeeds every element of 'B' i.e $x \leq u \; \forall \; x \in B$.

Similarly, an element 'l' $\in A$ is called Lower bound of B if 'l' preceeds every element of B i.e $l \leq x \; \forall \; x \in B$.

* There may be more than 1 lower & upper bounds.

Least Upper Bound :-
(LUB) or SUPREMUM.

An element $a \in A$ is called least UB of B if

(i) a is an upper bound.

(ii) $a \leq a'$ where a' is other upper bounds of B.

Greatest Lower Bound :-

An element $a \in A$ is called GLB of B if

(i) a is a LB & all other LBs 'a'' preceeds 'a'. i.e

$$a' \leq a$$

INFIMUM

Q. In a POSET, $A = (\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, /)$.
The subset $\{2, 7\}$ has no UB bcoz there is no integer in A which is div. by both $\{2, 7\}$.

LB $\rightarrow$ 1. as 1 is the element which divides 2 & 7.

Hence, GLB is also 1.

★ Consider $B = \{1, 2, 3\}$

UB $\rightarrow$ 6 LUB $\rightarrow$ 6

LB $\rightarrow$ 1. GLB $\rightarrow$ 1.

★ consider $\{1, 2, 4\}$.

UB $\rightarrow$ 4, 8 LUB $\rightarrow$ 4.

LB $\rightarrow$ 1 GLB $\rightarrow$ 1.

Q. Find the GLB & LUB of set $B1 = \{3, 9, 12\}$ & $B2 = \{1, 2, 4, 5, 10\}$ if they exist in the poset $(Z^+, /)$.

B1 LUB $\rightarrow$ 36

GLB $\rightarrow$ 3.

B2 LUB $\rightarrow$ 20

GLB $\rightarrow$ 1

Q. Consider the POSET defined by the foll. Hasse diagram

$B1 = \{1, 2\}$ find

$B2 = \{3, 4, 5\}$

(i) LB & UB of B1 & B2

(ii) GLB & LUB of B1 & B2

B1 $\longrightarrow$

UB	LB	
3, 4, 5, 6, 7, 8	Not exists	LUB $\longrightarrow$ 3. GLB $\longrightarrow$

We observe that 1 & 2 are related to 3, 4, 5, 6, 7, 8 which are UBs of B1.

& LUB is 3

B2 $\{3, 4, 5\}$.

UB	LB	
6, 7, 8	1, 2, 3.	no LUB, GLB $\downarrow$ 3

LATTICE.

A POSET $(P, \leq)$ is called as a lattice if every 2 elements subset of P has both LUB & GLB. i.e if LUB & GLB exist for every $\{x, y\}$ in P.

We denote

$$x \vee y = lub(x, y) \qquad \vee \rightarrow \text{join}$$

$$x \wedge y = glb(x, y) \qquad \wedge \rightarrow \text{meet}$$

NOTE

every chain is a lattice since any 2 elements x, y are comparable. We can observe than

$$x \vee y = lub(x, y) = y$$

$$x \wedge y = glb(x, y) = x$$

A: Consider the powerset $(P(S), \subseteq)$.

It is a Lattice bcoz if $(A, B) \subseteq S$ then UB of $\{A, B\}$ is subset of S which contains both A & B & the LUB is $A \cup B$ belonging to $P(S)$.

Similarly we can show that the GLB is the intersection of A & B.

Eg. Consider POSET $(\mathbb{Z}^+, /)$, it is Lattice bcoz any UB of $\{a, b\}$ is nothing but an element which is divisible by both A & B which is a common multiple of A & B. & the LUB will be LCM of A & B.

Similarly, the greatest LB will be GCD or HCF.

$6 \vee 4 = 12$

$6 \wedge 4 = 2$

Q. Determine whether the POSETS determined by the foll. Hasse Diagram is Lattice or not?

We construct the closure tables for LUB ($\vee$) & GLB ($\wedge$) as shown below.

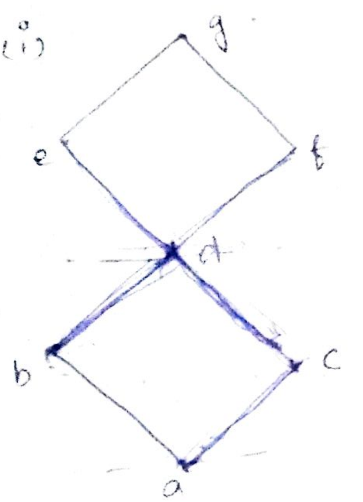
$\vee$	a	b	c	d
a	a	b	c	d.
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d.

✓

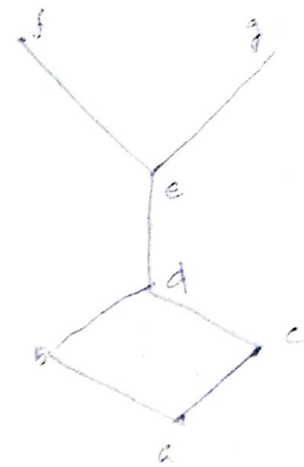
$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

Every pair has GLB & LUB
So the given diagram
is a Lattice

(i)



(ii)



$\vee$	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	b	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	g	g
f	f	f	f	f	g	f	g
g	g	g	g	g	g	g	g

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	b				
c							
d							
e	a	b	c	d			
f	a	b	c	d	f	f	f