6 Lattice Theory

6.1 INTRODUCTION

In this chapter we shall first discuss what is meant by a partial ordered set and how partial ordered set is represented through a directed graph which is commonly known as Hasse diagram by giving suitable examples. The role of partial ordered is significant while we study the algebraic systems. In section 6.3 we will discuss the properties of partial ordered set. Section 6.4 covers the important concept of partial ordered set that has additional characteristics called lattices. In latter sections we will discuss the properties of lattices and the classifications of the lattices where we study some special type of lattices like sublattices, distributed lattices, complemented lattices, product of lattices and the lattice homomorphism.

6.2 PARTIAL ORDERED SET

Let us start our discussion with partial ordered relation. If a binary relation R defined over set X is (1) reflexive, (2) antisymmetric, and (3) transitive then relation R is a partial ordered relation. Conventionally, partial ordered relation is denoted by symbol "≤" (the symbol "≤" doesn't mean of 'less than or equal to'), i.e.,

- A binary relation R over set X is ***reflexive*** *iff*, $\forall\, x \in X$, i.e., $(x, x) \in R$.
- A binary relation R over set X is ***antisymmetric*** *iff*, $\forall\, x, y \in X$, whenever $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.
- A binary relation R over set X is ***transitive*** *iff*, $\forall\, x, y$, and $z \in X$ whenever, $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Since, symbol "≤" is a partial ordered relation defined over set X, so the ordered pair (X, ≤) is called a *partial ordered set*. Partial ordered set is also known as *poset*. The partial ordered set (X, ≤) will be called *linearly ordered set* if, $\forall\, x$ and $y \in X$, we have $x \le y$ or $y \le x$. A linearly ordered set is a special case of partial ordered set and it is also called a *chain*.

If (X, ≤) is partial ordered set defined by relation R, then (X, ≥) will also be a partial ordered set, and it is defined for inverse of relation R. Hence, poset (X, ≥) is dual of poset (X, ≤). Some of the examples of posets are given below.

1. The relation "less than or equal to" defined over set of real numbers (R) is a partial ordered relation i.e., (R, ≤).
2. The relation "greater than or equal to" defined over set of real numbers is a partial ordered relation i.e., (R, ≥).

3. Similarly the relation "less than" or relation "greater than" are also partial ordered relation over set of real numbers i.e., (R, <) or (R, >).
4. The relation "$\subseteq$" (inclusion) defined over power set of X is a partial ordered relation i.e., (P(X), $\subseteq$). Let X = {a, b}; then power set of X is P(X) = {Ø, {a}, {b}, {a, b}}. Since, every element of P(X) is subset of itself so relation "$\subseteq$" over P(X) is reflexive. It is also antisymmetric and transitive i.e. for the element Ø $\subseteq$ {b} and {b} $\subseteq$ {a, b} then, Ø $\subseteq$ {a, b} and similarly true for all elements of P(X).
5. The relation "perfect division" or "integral multiple" defined over set of positive integer (I^+) are partial ordered relation. For example let X = {2, 3, 4, 6} $\in$ I^+; then partial ordered relation '$\leq$' is "perfect division" (i.e., for any x and $y \in$ X the relation "x divides y") over X will be given as,

$$`\leq' = \{(2, 2), (3, 3), (4, 4), (6, 6), (2, 4), (2, 6), (3, 6)\}$$

Similarly, partial ordered relation "integer multiple" (i.e., for any x and $y \in$ X, and any integer k; $y = x\,k$; 'y is an integer multiple of x') will be given as,

$$\geq = \{(2, 2), (3, 3), (4, 4), (6, 6), (4, 2), (6, 2), (6, 3)\}$$

Here, we used partial ordered relation symbol '$\geq$' because; last poset is dual of previous poset.

Comparability and Noncomparability

Let (X, $\leq$) be a poset, then elements x and $y \in$ X are said to be comparable if $x = y$ or $y = x$. If x and y are not related i.e., $x \not\leq y$ or $y \not\leq x$ then they are called noncomparable. For example, the elements of power set of X say P(X) are noncomparable with respect to partial ordered relation "$\subseteq$".

6.3 REPRESENTATION OF A POSET (HASSE DIAGRAM)

A poset (X, $\leq$) is represented by a diagram called Hasse diagram. *Hasse diagram is a directed graph, where ordered between the elements are preserved*. Since, it is a directed graph which consists of vertices and edges where, all elements of X are in set of vertices that are represented by a circle or dot and the connections between vertices (elements of X) called edges that will be drawn as follows,

- For the element $x, y \in$ X if $x > y$ and there is no element $z \in$ X i.e., $x \geq z \geq y$ then circle for y is putted below the circle for x and are connected by a direct edge.
- Otherwise, if $x > y$ and there exist at least an element $z \in$ X i.e., $x \geq z \geq y$ then they are not connected by a direct edge, however they are connected by other elements of set X.

Example 6.1. *Consider set X = {2, 3, 4, 6, 8, 24} and the partial ordered relation '$\leq$' be i.e.,*

$$x \leq y \Rightarrow \text{"}x \text{ divides } y\text{" (perfect division)}$$

then, poset (X, $\leq$) will be given as,

$$\leq = \{(2, 2), (2, 4), (2, 6), (2, 8), (2, 24), (3, 3), (3, 6), (3, 24), (4, 4), (4, 8), (4, 24), (6, 6), (6, 24), (8, 8), (8, 24)\}$$

Since, relation is understood to be reflexive, so we can leave the pairs of similar elements. Since, relation is understood to be transitive, so we can leave the pairs that come in sequence of pairs i.e. it need not to write the pair (2, 24), if the sequence of pairs (2, 6) and (6, 24) are there.

Thus, a simplified poset will be,

$$\leq = \{(2, 4), (2, 6), (3, 6), (4, 8), (6, 24), (8, 24)\}$$

and there graphical representation is shown below in Fig. 6.1.

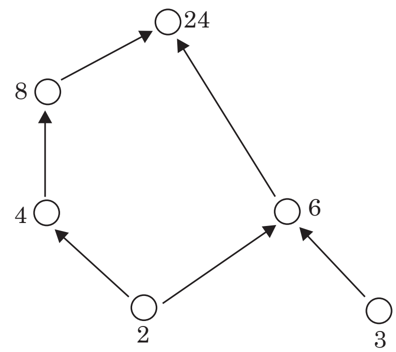

Fig. 6.1

To simplify further, since all arrows pointed in one direction (upward) so, we can omit the arrows. Such a graphical representation of a partial ordered relation in which all arrow heads are understood (to be pointed upward) is called Hasse diagram of the relation shown in Fig. 6.2.

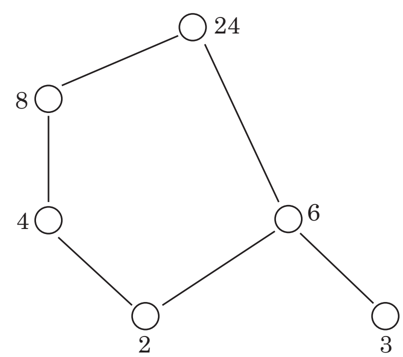

Fig. 6.2 Hasse diagram.

Example 6.2. *Let set X = {*α, β, γ*} then draw the Hasse diagram for the poset (P(X),* $\subseteq$*).*

Sol. The P(x) is the power set of X. So we have,

$$P(X) = \{\emptyset, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\beta, \gamma\}, \{\alpha, \gamma\}, \{\alpha, \beta, \gamma\}\}$$

Therefore, the vertices in the Hasse diagram will corresponds to all elements of P(X) and the edges are constructed according to the partial ordered relation inclusion ($\subseteq$) which are given as,

$$\subseteq = \{(\emptyset, \{\alpha\}), (\emptyset, \{\beta\}), (\emptyset, \{\gamma\}), (\{\alpha\}, \{\alpha, \beta\}), (\{\alpha\}, \{\alpha, \gamma\}), (\{\beta\}, \{\alpha, \beta\}), (\{\beta\}, \{\beta, \gamma\}), (\{\gamma\}, \{\alpha, \gamma\}),$$
$$(\{\gamma\}, \{\beta, \gamma\}), (\{\alpha, \beta\},\{\alpha, \beta, ?\}), (\{\beta, \gamma\}, \{\alpha, \beta, \gamma\}), (\{\alpha, \gamma\}, \{\alpha, \beta, \gamma\})\}.$$

Thus, the Hasse diagram is shown in Fig. 6.3.

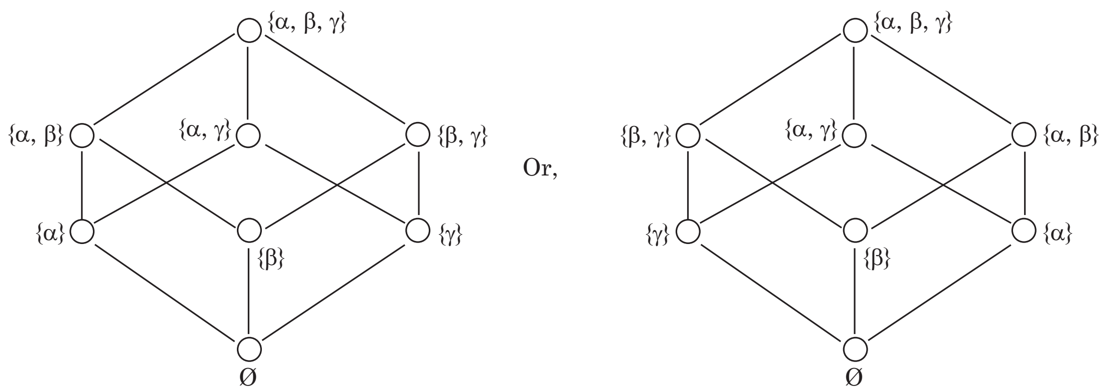

Fig. 6.3 Hasse diagrams.

Note. *For a given poset, Hasse diagram is not unique (e.g. in Fig. 6.3 the vertices order may different on the same level). Conversely, Hasse diagram may be same for different poset. For example, the Hasse diagram for the poset (X, ≤) i.e., for the set X = {1, 2, 3, 4, 6, 8, 12, 24} and the relation ≤ be such that x = y if x divides y; will be same as shown in Fig. 6.3.*

The Hasse diagram of linearly ordered set (X, ≤) consisting of circles one above other is called a ***chain***. For example, if we define the partial ordered relation ≤ is "less than or equal to" over the set X = {1, 2, 3, 4, 5}; then Hasse diagram for poset (X, ≤) is shown in Fig. 6.4.

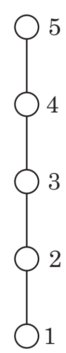

Fig. 6.4

We obtain the same Hasse diagram as above for the poset (X, ≤) where the relation ≤ is "divisibility" i.e., $\forall\, x, y \in$ X we have $x = y \Leftrightarrow$ 'x divides y'; over the set X = {1, 2, 4, 8, 16}.

Example 6.3. *Draw the Hasse diagram for the poset (S, ≤), where set S = {1, 2, 3, 4, 6, 8, 9, 12, 18, 24} and the relation "≤" is "divisibility".*

Sol. The vertices of the Hasse diagram will corresponds to the elements of X and the edges will be formed as,

$$\leq\ = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{2, 6\}, \{3, 6\}, \{3, 9\}, \{4, 8\}, \{4, 12\}, \{6, 12\},$$
$$\{6, 18\}, \{8, 24\}, \{9, 18\}, \{12, 24\}\}$$

that represent the poset.

Hence the Fig. 6.5 shows the Hasse diagram for poset (S, ≤)

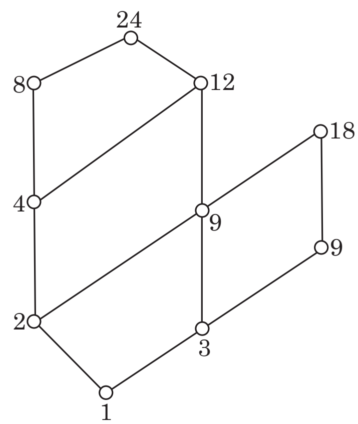

Fig. 6.5

Example 6.4 *1. Draw the Hasse diagram for factors of 6 under relation divisibility.*

2. Draw the Hasse diagram for factors of 8 under relation divisibility.

Sol. 1. Let D_6 is the set that contains possible elements that are factors of 6, i.e., D_6 = {1, 2, 3, 6} then poset will be {{1, 2}, {1, 3}, {2, 6}, {3, 6} whose Hasse diagram is shown in Fig. 6.6.

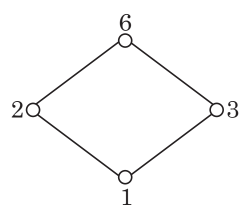

Fig. 6.6

2. Similarly for D_8 = {1, 2, 4, 8} under the relation divisibility the poset will be {1, 2}, {2, 4}, {4, 8}. Hence, its Hasse diagram will be a chain that is shown in Fig. 6.7. Here all elements are comparable to each other such poset is called ***toset.*** Remember, all tosets must be posets but all posets are not necessarily tosets.

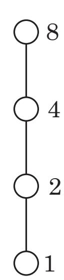

Fig. 6.7

Upper Bound

Let (X, ≤) be a poset and Y ⊆ X, then an element $x \in$ X be the upper bound for Y if and only if, $\forall y \in$ Y s.t. $y \leq x$.

Lower Bound

Let (X, $\leq$) be a poset and Y $\subseteq$ X, then an element $x \in$ X be the lower bound for Y if and only if, $\forall y \in$ Y s.t. $y \geq x$.

Least Upper Bound (LUB)

Let (X, $\leq$) be a poset and Y $\subseteq$ X, then an element $x \in$ X be a least upper bound for Y if and only if, x is an upper bound for Y and $x \leq z$ for all upper bounds z for Y.

Greatest Lower Bound (GLB)

Let (X, $\leq$) be a poset and Y $\subseteq$ X, then an element $x \in$ X be a greatest lower bound for Y if and only if, x is an lower bound for Y and $z \geq x$ for all lower bounds z for Y.

Reader must note that for every subset of poset has a unique LUB and a unique GLB if exists. For example, the GLB and LUB for the poset whose Hasse diagram shown in Fig. 6.3 will be Ø and {α, β, γ} respectively.

Well ordered set

A poset (X, $\leq$) is called well ordered set if every nonempty subset of X has a least element. This definition of well ordered set follows that poset is totally ordered. Conversely, a totally ordered set need not to be always well ordered.

Meet and Join of elements

Let (X, $\leq$) be a poset and x_1 and x_2 are elements $\in$ X, then greatest lower bound (GLB) of x_1 and x_2 is called meet of elements x_1 and x_2 where meet of x_1 and x_2 is represented by $(x_1 \wedge x_2)$ i.e.,

$$(x_1 \wedge x_2) = \text{GLB}\ (x_1, x_2)$$

Similarly, the join of x_1 and x_2 is the least upper bound (LUB) of x_1 and x_2 where join of x_1 and x_2 is represented by $(x_1 \vee x_2)$ i.e.,

$$(x_1 \vee x_2) = \text{LUB}\ (x_1, x_2)$$

6.4 LATTICES

The purpose of the study of the previous sections was to understand the concept of ordered relations. Partial ordered relations plays a significant role in the study of algebraic systems that we shall discuss in the latter sections. Lattice is the partial ordered set that possesses additional characteristics. The feature of lattice as an algebraic system is also significant. The importance of lattice theory associated with Boolean algebra is not only to understand the theoretical aspects and design of computers but many other fields of engineering and sciences.

Definition

A poset (X, $\leq$) is called a lattice if every pair of elements has a unique LUB and a unique GLB. Let x_1 and x_2 are two elements $\in$ X, then for a poset (X, $\leq$) we have,

$$\text{GLB}(x_1, x_2) = x_1 \wedge x_2 \quad \text{and} \quad \text{LUB}(x_1, x_2) = x_1 \vee x_2;$$

- Where the symbols $\wedge$ and $\vee$ are binary operations 'AND' and 'OR' respectively over X.
- In certain cases, symbols $\wedge$ and $\vee$ are also used as the abstraction of '$\cap$' and '$\cup$' respectively.

For example, assume P(X) is the power set for a known set X then over the relation 'inclusion' poset (P(X), $\subseteq$) is a lattice. To show it assume set X = {α, β}; so P(X) = {Ø, {α}, {β}, {α, β}}. Then, poset (P(X), $\subseteq$) in which each pair have a unique LUB & a unique GLB e.g. for the pair ({α, β}, {β}) meet and join will be,

$$\text{GLB}(\{\alpha, \beta\}, \{\beta\}) = \{\alpha, \beta\} \wedge \{\beta\} \Rightarrow \{\alpha, \beta\} \cap \{\beta\} = \{\beta\}$$

$$\text{LUB}(\{\alpha, \beta\}, \{\beta\}) = \{\alpha, \beta\} \vee \{\beta\} \Rightarrow \{\alpha, \beta\} \cup \{\beta\} = \{\alpha, \beta\}$$

(Reader can also verify these results from the Hasse diagram shown in Fig. 6.3)

Similarly we can determine the LUB & GLB for every pair of the poset (P(X), $\subseteq$) that are listed in table shown in Fig. 6.5.

GLB	**Ø**	**{α}**	**{β}**	**{α, β}**
Ø	Ø	Ø	Ø	Ø
{α}	Ø	{α}	Ø	{α}
{β}	Ø	Ø	{β}	{β}
{α, β}	Ø	{α}	{β}	{α, β}

LUB	**Ø**	**{α}**	**{β}**	**{α, β}**
Ø	Ø	{α}	{β}	{α, β}
{α}	{α}	{α}	{α, β}	{α, β}
{β}	{β}	{α, β}	{β}	{α, β}
{α, β}	{α, β}	{α, β}	{α, β}	{α, β}

Fig. 6.5

Example 6.5. Consider the poset (I^+, ≤) where I^+ is the set of positive integers and the partial ordered relation ≤ is defined as i.e. if x and $y \in I^+$ then $x \le y$ means 'x divides y'. Then poset (I^+, ≤) is a lattice. Because,

$$\text{GLB}(x, y) = x \wedge y = lcm(x, y) \qquad [lcm\text{: least common divisor}]$$

and $$\text{LUB}(x, y) = x \vee y = gcd(x, y) \qquad [gcd\text{: greatest common divisor}]$$

Example 6.6. Let R be the set of real numbers in [0, 1] and the relation '≤' be defined as 'less than or equal to' over the set R, then poset (R, ≤) is a lattice. Assume r_1 and r_2 are the elements $\in$ R s.t. $0 \le r_1, r_2 \le 1$ then meet and join are given by,

$$\text{GLB}(r_1, r_2) = r_1 \wedge r_2 = \text{Min}(r_1, r_2)$$

and $$\text{LUB}(r_1, r_2) = r_1 \vee r_2 = \text{Max}(r_1, r_2)$$

In the next section we will discuss the theorems that shows the relationship between the partial ordered relation '≤' and the binary operations GLB & LUB in a lattice (X, ≤).

Theorem 6.4.1. *Let (X, ≤) be a lattice, then for any x, y ∈ X,*

(i) $x \le y \Leftrightarrow GLB(x, y) = x$

and, *(ii)* $x \le y \Leftrightarrow LUB(x, y) = y$

Proof. (Immediately follows from the definition of GLB and LUB)

Assume $x \le y$, since we know that $x \le x \Rightarrow x \le \text{GLB}(x, y)$.

From the definition of GLB(x, y), we have GLB(x, y) $\leq x$.

Hence, $x \leq y \Rightarrow$ GLB(x, y) = x.

Further assume, GLB(x, y) = x; but it is possible only if $x \leq y$.

So GLB(x, y) = $x \Rightarrow x \leq y$. It proved the equivalence (i).

To prove the equivalence (ii) $x \leq y \Leftrightarrow$ LUB(x, y) = y; we proceed similarly.

Alternatively, since GLB(x, y) = x. so we have,

$$\text{LUB}(y, \text{GLB}(x, y)) = \text{LUB}(y, x) = \text{LUB}(x, y)$$

But $\text{LUB}(y, \text{GLB}(x, y)) = y$

Therefore, LUB(x, y) = y follows from GLB(x, y) = x.

Similarly we can show that GLB(x, y) = x follows from LUB(x, y) = y, that proved the equivalence.

Theorem 6.4.2. *Let (X, ≤) be a lattice, then for any x, y and z ∈ X*

$$y \leq z \quad \Rightarrow \begin{cases} GLB(x, y) \leq GLB(x, z) & (i) \\ LUB(x, y) \leq LUB(x, z) & (ii) \end{cases}$$

Proof. From the result of the previous theorem $y \leq z \Leftrightarrow$ GLB(y, z) = y;

To prove GLB(x, y) $\leq$ GLB(x, z), we shall prove

$$\text{GLB}\ (\text{GLB}(x, y), \text{GLB}(x, z)) = \text{GLB}(x, y);$$

Since, $\text{GLB}\ (\text{GLB}(x, y), \text{GLB}(x, z)) = \text{GLB}(x, \text{GLB}(y, z));$

$= \text{GLB}(x, y)$ proved.

Similarly prove the (ii) equality.

Theorem 6.4.3. *Let (X, ≤) be a lattice, then for any x, y and z ∈ X*

(i) $x \leq y \wedge x \leq z \quad \Rightarrow \quad x \leq GLB(y, z)$

(ii) $x \leq y \wedge x \leq z \quad \Rightarrow \quad x \leq LUB(y, z)$

Proof. (i) Inequality can be proved from the definition of GLB and from the fact that both y and z are comparable.

(ii) Inequality is obvious from the definition of LUB.

Since we know that poset (X, ≤) is dual to the poset (X, ≥). So, if A ⊆ X then LUB of A w.r.t. poset (X, ≤) is same as GLB for A w.r.t. poset (X, ≤) and vice-versa. Thus, if the relation interchanges from '≤' to '≥' then GLB and LUB are interchanged. Hence, we say that operation GLB and LUB are duals to each other like as the relation '≤' and '≥'. Therefore, lattice (X, ≤) and (X, ≥) are duals to each other. So, above theorem can be restated as,

(i) $x \geq y \wedge x \geq z \quad \Rightarrow \quad x \geq \text{LUB}(y, z)$

(ii) $x \geq y \wedge x \geq z \quad \Rightarrow \quad x \geq \text{GLB}(y, z)$

Theorem 6.4.4. *Let (X, ≤) be a lattice, then for any x, y and z ∈ X*

(a) LUB(x, GLB(y, z)) = GLB (LUB(x, y), LUB(x, z));

(b) GLB(x, LUB(y, z)) = LUB (GLB(x, y), GLB(x, z));

(These are called distributive properties of a lattice)

Proof. (a) Since, $x \leq \text{LUB}(x, y)$ and $x \leq \text{LUB}(x, z)$;

Then, from (i) equality of theorem 6.4.3

$$x \leq \text{GLB}(y, z) \quad \Rightarrow \quad x \leq \text{GLB}(\text{LUB}(x, y), \text{LUB}(x, z)); \qquad (iii)$$

Further, $\text{GLB}(y, z) \leq y \leq \text{LUB}(x, y)$;

and $\text{GLB}(y, z) \leq z \leq \text{LUB}(x, z)$;

Again using equality (i) we obtain,

$$\text{GLB}(y, z) \leq \text{GLB}(\text{LUB}(x, y), \text{LUB}(x, z)); \qquad (iv)$$

Hence from (iii), (iv) and (i)′ we get the required result that's,

$$\text{LUB}(x, \text{GLB}(y, z)) \leq \text{GLB}(\text{LUB}(x, y), \text{LUB}(x, z)); \quad \text{Proved.}$$

Similarly we can derive the equality (b).

Theorem 6.4.5. *Let (X, ≤) be a lattice, then for any x, y and z ∈ X*

$$x \leq z \iff \text{LUB}(x, \text{GLB}(y, z)) \leq \text{GLB}(\text{LUB}(x, y), z).$$

Proof. Since, $x \leq z \iff \text{LUB}(x, z) = z$ from (ii) inequality of theorem 6.4.1. Put z in place of LUB(x, z) in the inequality (a) of theorem 6.4.4

Thus we have, $\text{LUB}(x, \text{GLB}(y, z)) \leq \text{GLB}(\text{LUB}(x, y), z);$

$$\iff \quad x \leq z.$$

(This inequality is also called *modular* inequality).

Example 6.7. *In a lattice (X, ≤), for any x, y and z ∈ X, if x ≤ y ≤ z then*

(i) LUB(x, y) = GLB(y, z); and

(ii) LUB(GLB(x, y), GLB(y, z)) = y = GLB(LUB(x, y), LUB(x, z));

Sol. (i) Since we know that if $x \leq y \iff \text{LUB}(x, y) = y$; and also if $y = z \iff \text{GLB}(y, z) = y$ (see theorem 6.4.1), therefore

$$x \leq y \leq z \iff \text{LUB}(x, y) = y = \text{GLB}(y, z);$$

(ii) Similarly, $\text{GLB}(x, y) = x$ if $x \leq y$; and $\text{GLB}(y, z) = y$ if $y \leq z$.

So, put these values of x and y in (i), thus we have

$$\underset{LHS}{\text{LUB}(\text{GLB}(x, y), \text{GLB}(y, z)) = y} \qquad (\therefore \quad \text{LUB}(x, y) = y)$$

Further since, $\text{LUB}(x, y) = y$ (if $x \leq y$) and also $\text{LUB}(x, z) = z$ (if $x \leq z$).

From (i) $\quad \text{GLB}(y, z) = y;$

Put the values of y and z in this equation we obtain,

$$\underset{RHS}{\text{GLB}(\text{LUB}(x, y), \text{LUB}(x, z)) = y}$$

Hence, $\quad$ *LHS* = *RHS*; Proved.

6.4.1 Properties of Lattices

Let (X, ≤) be a lattice, than for any x, y and $z \in$ X we have,

(i) $\text{GLB}(x, x) = x$; and $\text{LUB}(x, x) = x$; **[Rule of Idempotent]**

(ii) $\text{GLB}(x, y) = \text{GLB}(y, x)$; and $\text{LUB}(x, y) = \text{LUB}(y, x)$ **[Rule of Commutation]**

(iii) $\text{GLB}(\text{GLB}(x, y), z) = \text{GLB}(x, \text{GLB}(y, z))$; and

$\text{LUB}(\text{LUB}(x, y), z) = \text{LUB}(x, \text{LUB}(y, z))$ **[Rule of Association]**

(iv) $\text{GLB}(x, \text{LUB}(x, y)) = x$; and $\text{LUB}(x, \text{GLB}(x, y)) = x$ **[Rule of Absorption]**

We can prove above identities by using the definition of binary operation GLB and LUB.

(i) Since, we know that,

$$x \leq x \iff \text{GLB}(x, x) = x \qquad \text{(from theorem 6.4.1)}$$

and also $\quad x \leq x \iff \text{LUB}(x, x) = x \qquad$ (from theorem 6.4.1)

Similarly we can prove the other identities (ii) and (iii). To prove identity (iv) we recall the definition of LUB that for any $x \in$ X, $x \leq x$ and $x \leq \text{LUB}(x, y)$.

Therefore, $x \le \text{GLB}(x, \text{LUB}(x, y));$

Conversely, from the definition of GLB, we have GLB(x, LUB(x, y) $\le x$.

Hence, GLB(x, LUB(x, y) $= x$. Proved.

Similarly, LUB(x, GLB(x, y)) $= x$.

6.4.2 Lattices and Algebraic Systems

Let (X, $\le$) be a lattice, then we can define an algebraic system (X, GLB, LUB) where GLB and LUB are two binary operations on X that satisfies (1) commutative, (2) associative, and (3) rule of absorption i.e. for any $x, y \in$ X

$$\text{GLB}(x, y) = x \wedge \mathbf{y};$$

and
$$\text{LUB}(x, y) = x \vee y.$$

6.4.3 Classes of Lattices

In this section we will study the classes of lattices that posses additional properties. To define lattice as an algebraic system, that can anticipate introducing the verity of lattices in a natural way.

6.4.3.1 Distributive Lattice

A lattice (X, GLB, LUB) is said to be distributive lattice if the binary operations GLB and LUB holds distributive property i.e. for any x, y and $z \in$ X,

$$\text{GLB}(x, \text{LUB}(y, z)) = \text{LUB}\ (\text{GLB}(x, y), \text{GLB}(x, z))$$

and
$$\text{LUB}(x, \text{GLB}(y, z)) = \text{GLB}\ (\text{LUB}(x, y), \text{LUB}(x, z))$$

- A lattice is a distributive lattice if distributive equality must be satisfied by all the elements of the lattice.
- Not all lattices are distributive.
- Every chain is a distributive lattice.

For example, lattice shown below in Fig. 6.6 is a distributive lattice. Because if we take the elements {α}, {α, β} and {γ}) then

$$\text{GLB}(x\ \text{LUB}(y, z)) = \text{GLB}\ (\{a\}, \text{LUB}\ (\{a, b\}, \{\gamma\}))$$
$$= \text{GLB}(\{\alpha\}, \{\alpha, \beta, \gamma\}) = \{\alpha\} \quad \text{LHS}$$

Since GLB({α}, {α, β}) = {α} and GLB({α}, {γ}) = Ø

Thus,LUB(GLB ({α}, {α, β}), GLB({α}, {γ})) = GLB({α}, Ø) = {α} RHS.

Similarly it is true for all the elements. Hence, it is an example of distributive lattice.

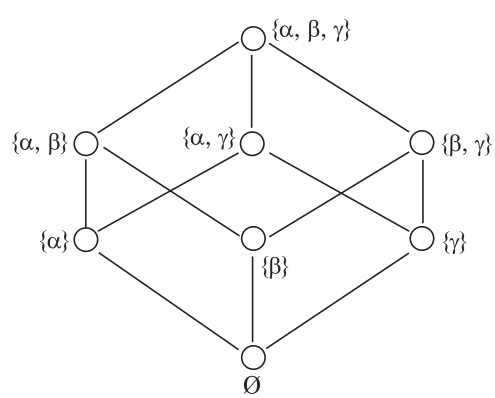

Fig. 6.6

Example 6.8. *Show that lattice of Fig. 6.7 is not a distributive lattice.*

Sol. Since, we can see that all elements of the lattice doesn't satisfies the distributive equalities. For example, between the elements y, z and r

$$\begin{aligned} &\text{GLB}(y, \text{LUB}(z, r)) \\ &= \text{GLB}(y, x) = y; \quad \text{RHS} \end{aligned}$$

and $\quad \text{LUB}(\text{GLB}(y, z), \text{GLB}(y, r) = \text{LUB}(s, s) = s; \quad \text{LHS}$

Therefore RHS ≠ LHS, hence shown lattice is not a distributive lattice.

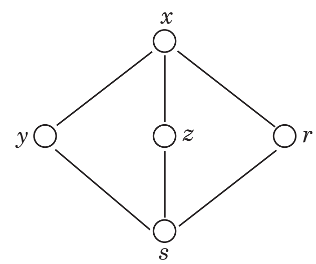

Fig. 6.7

Remember that lattices similar to the lattice of Fig. 6.7 are called 'diamond lattices' and they are not distributive lattices.

Example 6.9. *We can further see that lattice shown in the Fig. 6.8 is not a distributive lattice.*

Sol. A lattice is distributive if all of its elements follow distributive property so let we verify the distributive property between the elements n, l and m.

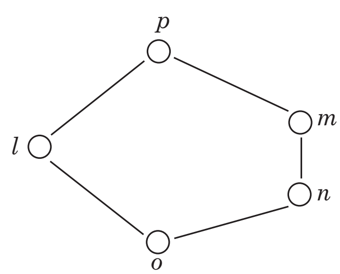

Fig. 6.8

$$\text{GLB}(n, \text{LUB}(l, m)) = \text{GLB}(n, p) \qquad [\because \ \text{LUB}(l, m) = p]$$
$$= n \quad (\text{LHS})$$

also $\quad \text{LUB}(\text{GLB}(n, l), \text{GLB}(n, m)) = \text{LUB}(o, n); \qquad [\because \ \text{GLB}(n, l) = o \text{ and } \text{GLB}(n, m) = n]$
$$= n \quad (\text{RHS})$$

so $\quad \text{LHS} = \text{RHS}.$

But $\quad \text{GLB}(m, \text{LUB}(l, n)) = \text{GLB}(m, p) \qquad [\because \ \text{LUB}(l, n) = p]$
$$= m \quad (\text{LHS})$$

also $\quad \text{LUB}(\text{GLB}(m, l), \text{GLB}(m, n)) = \text{LUB}(o, n); \qquad [\because \ \text{GLB}(m, l) = o \text{ and } \text{GLB}(m, n) = n]$
$$= n \quad (\text{RHS})$$

Thus, LHS ≠ RHS hence distributive property doesn't hold by the lattice so lattice is not distributive.

Example 6.10. *Consider the poset (X, ≤) where X = {1, 2, 3, 5, 30} and the partial ordered relation ≤ is defined as i.e. if x and y ∈ X then x ≤ y means 'x divides y'. Then show that poset (I^+, ≤) is a lattice.*

Sol. Since GLB(x, y) = $x \wedge y$ = $lcm(x, y)$

and LUB(x, y) = $x \vee y$ = $gcd(x, y)$

Now we can construct the operation table I and table II for GLB and LUB respectively and the Hasse diagram is shown in Fig. 6.9.

Table I

LUB	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

Table II

GLB	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

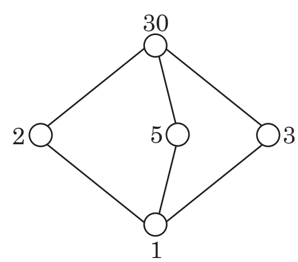

Fig. 6.9 Hasse diagram.

Test for distributive lattice, i.e.,

GLB(x, LUB(y, z)) = LUB(GLB(x, y), GLB(x, z))

Assume $x = 2$, $y = 3$ and $z = 5$, then

RHS: GLB(2, LUB(3, 5)) = GLB(2, 30) = 2

LHS: LUB(GLB(2, 3), GLB(2, 5)) = LUB(1, 1) = 1

Since *RHS* ≠ *LHS*, hence lattice is not a distributive lattice.

6.4.3.2 Bounded Lattice

A lattice (X, GLB, LUB) is said to be bounded if there exist a greatest element 'I' and a least element 'O' in the lattice, i.e.,

(*i*) O ≤ x ≤ I [for any $x \in$ X]

(*ii*) LUB(x, O) = x and GLB(x, O) = O

(*iii*) LUB(x, I) = I and GLB(x, I) = x

(Element I and O are also known as universal upper and universal lower bound)

For example, lattice (P(X), ⊆) is a bounded lattice where I is the set X itself and O is Ø. Consider another example, a Hasse diagram shown in Fig. 6.10 is a bounded lattice. Because its greatest element (I) is 'a' and least element (O) is 'f'.

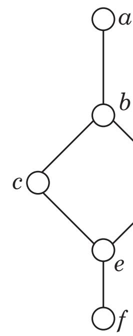

Fig. 6.10

Let $x = c$ then LUB$(c, f) = c$ and GLB$(c, f) = f$

and LUB$(c, a) = a$ and GLB$(c, a) = c$

Similarly, these conditions holds for any $x \in$ X, hence lattice is bounded.

Example 6.12. *Lattice discussed in the example 6.7 is also a bounded lattice.*

Example 6.13. *Show that lattice (X, GLB, LUB) is a bounded lattice, where poset (X = {1, 2, 5, 15, 30}, /) (here partial ordered relation is a 'division' operation).*

Sol. Reader self verify that the Hasse diagram for the given poset is same as the Hasse diagram shown in Fig. 6.10. Since diagram is bounded whose greatest element is 30 and the least element is 1 and the rest of the conditions are also satisfied therefore lattice is a bounded lattice.

6.4.3.3 Complement Lattice

A lattice (X, GLB, LUB) is said to be complemented lattice if, every element in the lattice has a complement. Or,

A bounded lattice with greatest element 1 and least element 0, then for the elements x, $y \in$ X element y is said to be complement of x iff,

$$\text{GLB}(x, y) = 0; \quad \text{and} \quad \text{LUB}(x, y) = 1;$$

- In a complement lattice complements are always unique.
- Since operation GLB and LUB are commutative, so if y is complement of x then, x is also complement of y also 0 is the unique complement of 1 and vice-versa.
- In the lattice *it is possible that* an element has more than one complement and on the other hand it is also possible that an element has no complement.

For example consider the poset (P(X), $\subseteq$) where X = {α, β, γ} and so the lattice (P(X), GLB, LUB) where operation GLB and LUB are $\cap$ and $\cup$ respectively is bounded with greatest element {α, β, γ} and least element Ø. (Fig. 6.11)

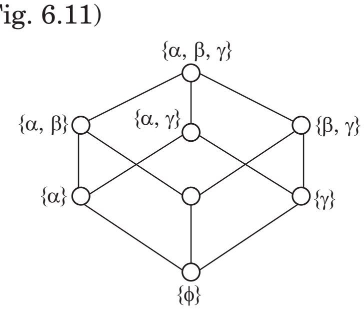

Fig. 6.11

Then we can find the complement of the elements that are listed in the table shown in Fig. 6.12.

No.	Element	Complement	Verification
1	Ø	{α, β, γ}	∴ GLB(Ø,{α, β, γ}) = {α, β, γ}; LUB(Ø,{α, β, γ}) = Ø
2	{α}	{β, γ}	∴ GLB({α}, {β, γ}) = {α, β, γ}; LUB({α}, {β, γ}) = Ø
3	{β}	{α, γ}	∴ GLB({β},{α, γ}) = {α, β, γ}; LUB({β}, {α, γ}) = Ø
4	{γ}	{α, β}	∴ GLB({γ},{α, β}) = {α, β, γ}; LUB({γ}, {α, β}) = Ø
5	{α, β}	{γ}	∴ GLB({α, γ}, {γ}) = {α, β, γ}; LUB({α, β}, {γ}) = Ø
6	{β, γ}	{α}	∴ GLB({β, γ}, {α}) = {α, β, γ}; LUB({β, γ}, {α}) = Ø
7	{α, γ}	{β}	∴ GLB({α, γ}, {β}) = {α, β, γ}; LUB({α, γ}, {β}) = Ø
8	{α, β, γ}	Ø	∴ GLB({α, β, γ}, Ø) = {α, β, γ}; LUB({α, β, γ}, Ø) = Ø

Fig. 6.12

Example 6.14. *Lattices shown in Fig. 6.13 (a), (b) and (c) are complemented lattices.*

Sol.

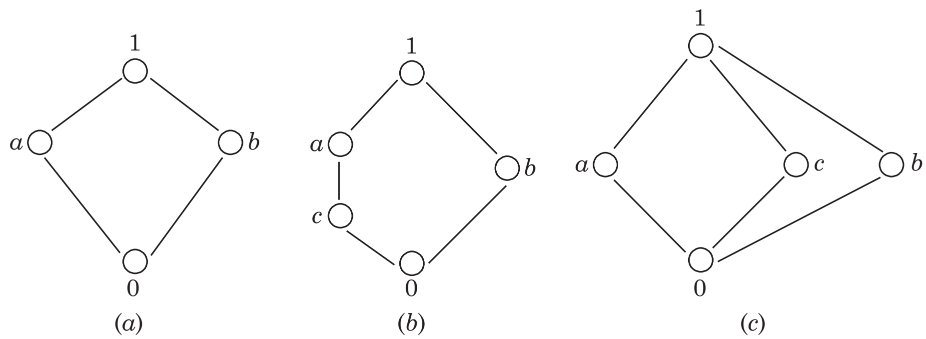

Fig. 6.13

For the lattice (a) GLB(a, b) = 0 and LUB(x, y) = 1. So, the complement a is b and vise versa. Hence, a complement lattice.

For the lattice (b) GLB(a, b) = 0 and GLB(c, b) = 0 and LUB(a, b) = 1 and LUB(c, b) = 1; so both a and c are complement of b. Hence, a complement lattice.

In the lattice (c) GLB(a, c) = 0 and LUB(a, c) = 1; GLB(a, b) = 0 and LUB(a, b) = 1. So, complement of a are b and c. Similarly complement of c are a and b also a and c are complement of b. Hence lattice is a complement lattice.

Example 6.15. *For example lattice (P(X), ⊆) is a complemented lattice. Let us discuss the existence of complement for each elements of the lattice. Since the GLB and LUB operations on P(X) are ∩ and ∪ respectively and so the universal upper bound in the lattice is set X itself (corresponds to symbol 1) and the universal lower bound in the lattice is Ø (corresponds to symbol 0). So, in the lattice (P(X), ⊆) complement of any subset Y of X is the difference of X and Y (i.e. X – Y).*

Example 6.16. *Let (X, ≤) be a distributive lattice, then for any elements x, y ∈ X if, y is complement of x then y is unique.*

Sol. We assume that element x has complement z other than x. So, we have,

$$\text{LUB}(x, y) = 1 \quad \text{and} \quad \text{GLB}(x, y) = 0;$$

and
$$\text{LUB}(x, z) = 1 \quad \text{and} \quad \text{GLB}(x, z) = 0;$$

Further we write,
$$\begin{aligned} z &= \text{GLB}(z, 1) \\ &\Rightarrow \text{GLB}(z, \text{LUB}(x, y)) \\ &\Rightarrow \text{LUB}(\text{GLB}(z, x), \text{GLB}(z, y)) && \text{(distributive property)} \\ &\Rightarrow \text{LUB}(0, \text{GLB}(z, y)) \\ &\Rightarrow \text{LUB}(\text{GLB}(x, y), \text{GLB}(z, y)) \\ &\Rightarrow \text{GLB}(\text{LUB}(x, z), y) && \text{(distributive property)} \\ &\Rightarrow \text{GLB}(1, y) \\ &\Rightarrow y \end{aligned}$$

Hence, complement of x is unique.

6.4.3.4 Sub Lattices

Let (X, ≤) be a lattice and if Y ⊆ X then lattice (Y, ≤) is a sublattice of (X, ≤) if and only if Y is closed under the binary operations GLB and LUB.

[*In the true sense algebraic structure (Y, GLB, LUB) is a sublattice of (X, GLB, LUB). For a lattice (X, ≤) let x, y ∈ X s.t. x ≤ y then, the closed interval [x, y] which contains the entire elements z s.t. x ≤ z ≤ y will be a sublattice of X*]

Example 6.17. *Consider the lattice (I^+, ≤) where I^+ is the set of positive integers and the relation '≤' is "division" in I^+ s.t. for any a, b ∈ I^+; a ≤ b ⇒ 'a divides b'. Let A_k be the set of all divisors of k, for example set A_6 = {1, 2, 3, 6}; which is a subset of I^+. Then lattice (A_k, ≤) is a sublattice of (I^+, ≤).*

Example 6.18. *Let (X, ≤) be a lattice shown in Fig. 6.14, assume X = {x_1, x_2, x_3, x_4, x_5, x_6}. Let subsets of X are X_1 = {x_1, x_2, x_4, x_6}, X_2 = {x_3, x_4} and X_3 = {x_4, x_5} then lattice (X_1, ≤) is a sublattice of (X, ≤) but not lattice (X_2, ≤) and lattice (X_3, ≤).*

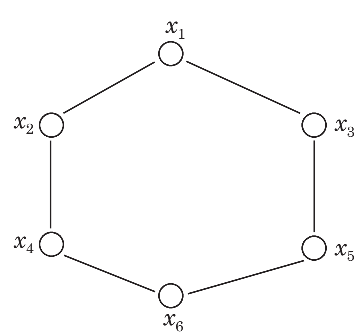

Fig. 6.14

Since,

- For the lattice (X_1, ≤); GLB(x_1, x_2) = x_4 ∈ X_1; GLB(x_2, x_4) = x_6 ∈ X_1; LUB(x_1, x_2) = x_1 ∈ X_1; LUB(x_2, x_4) = x_1 ∈ X_1 and others, we find that subset X_1 is closed under operations GLB and LUB so a sublattice.
- For the lattice (X_2, ≤); GLB(x_3, x_4) = x_6 ∉ X_2; LUB(x_3, x_4) = x_1 ∉ X_2; so subset X_2 is not closed under operations GLB and LUB therefore, (X_2, ≤) is not a sublattice of (X, ≤).

- Also, $\text{GLB}(x_5, x_4) = x_6 \notin X_3$; so subset X_3 is not closed under operations GLB and LUB therefore $(X_3, \leq)$ is not a sublattice of $(X, \leq)$.

6.4.4 Product of Lattices

Let $(X, \text{GLB}_1, \text{LUB}_1)$ and $(Y, \text{GLB}_2, \text{LUB}_2)$ are two lattices, then the algebraic system $(X \times Y, \text{GLB}, \text{LUB})$ in which the binary operations GLB and LUB for any $(x_1, y_1), (x_2, y_2) \in X \times Y$ are such that

$$\text{GLB}((x_1, y_1), (x_2, y_2)) = (\text{GLB}_1(x_1, y_1), \text{GLB}_2(x_2, y_2));$$

and
$$\text{LUB}((x_1, y_1), (x_2, y_2)) = (\text{LUB}_1(x_1, y_1), \text{LUB}_2(x_2, y_2));$$

is defined as the direct product of lattices $(X, \text{GLB}_1, \text{LUB}_1)$ and $(Y, \text{GLB}_2, \text{LUB}_2)$.

- The operations GLB, LUB on $X \times Y$ are (1) commutative, (2) associative and (3) hold absorption law because these operations are defined over operations GLB_1, LUB_1 and GLB_2, LUB_2. Hence direct product $(X \times Y, \text{GLB}, \text{LUB})$ is itself a lattice. In the similar sense we can extend the direct product of more than two lattices and so obtain large lattices from smaller ones. The order of lattice obtain by the direct product is same to the product of the orders of the lattices occurring in the direct product.

6.4.5 Lattice Homomorphism

Let $(X, \text{GLB}_1, \text{LUB}_1)$ and $(Y, \text{GLB}_2, \text{LUB}_2)$ are two lattices, then a mapping $f: X \to Y$ i.e. for any $x_1, x_2 \in X$,

$$f(\text{GLB}_1(x_1, x_2)) = \text{GLB}_2(f(x_1), f(x_2));$$

and
$$f(\text{LUB}_1(x_1, x_2)) = \text{LUB}_2(f(x_1), f(x_2));$$

is called *lattice homomorphism* from lattice (X, GLB1, LUB1) to lattice (Y, GLB2, LUB2).

- From the definition of lattice homomorphism we find that both the operations of GLB and LUB should be preserved.
- A lattice homomorphism $f: X \to Y$ is called a *lattice isomorphism* if, f is one-one onto or bijective.
- A lattice homomorphism s.t. $f: X \to X$ is called a *lattice endomorphism*. Here, image set of f will be sublattice of X.
- And if, $f: X \to X$ is a lattice isomorphism then f is called *lattice automorphism*.

EXERCISES

6.1 Draw the Hasse diagram of lattices $(A_k, \leq)$ for $k = 6, 10, 12, 24$; where A_k be the set of all divisors of k such that for any $a, b \in A_k$; $a \leq b \Rightarrow$ 'a divides b'. Also find the sublattices of the lattice $(A_{12}, \leq)$ and $(A_{24}, \leq)$.

6.2 Let $X = \{\alpha, \beta, \gamma, \delta\}$ then draw the Hasse diagram for poset $(P(X), \subseteq)$.

6.3 Draw the Hasse diagram of $(X, \subseteq)$ where set $X = \{X_1, X_2, X_3, X_4\}$ and the sets are given as,

$$X_1 = \{\alpha, \beta, \gamma, \delta\}; \quad X_2 = \{\alpha, \beta, \gamma\}; \quad X_3 = \{\alpha, \beta\}; \quad X_4 = \{\alpha\};$$

6.4 In a lattice show that $\text{LUB}(x, y) = \text{GLB}(y, z)$ if $x \leq y \leq z$.

6.5 Show that lattice $(X, \leq)$ is distributive *iff* for any x, y and $z \in X$,

$$\text{GLB}(\text{LUB}(x, y), z) = \text{LUB}(x, \text{GLB}(y, z))$$

6.6 For a distributive lattice $(X, \leq)$ show that if,

$$\text{GLB}(\alpha, x) = \text{GLB}(a, y) \quad \text{and} \quad \text{LUB}(\alpha, x) = \text{LUB}(\alpha, y)$$

for some α, then $x = y$.