Vector Calculus

Co-ordinate Systems

1. Cartesian or Rectangular → (x, y, z)

$-\infty \le x \le \infty$
$-\infty \le y \le \infty$
$-\infty \le z \le \infty$

Any vector A in cartesian co-ordinates can be written as → $A_x a_x + A_y a_y + A_z a_z$

(2) Cylindrical → (ρ, ϕ, z)

$0 \le \rho < \infty$
$0 \le \phi < 2\pi$
$-\infty < z < \infty$

Any vector A in cylindrical coordinates can be written as $A_\rho a_\rho + A_\phi a_\phi + A_z a_z$

(3) Spherical → (r, θ, ϕ)

$0 \le r < \infty$
$0 \le \theta \le \pi$
$0 \le \phi < 2\pi$

Any vector A in spherical co-systems $= A_r a_r + A_\theta a_\theta + A_\phi a_\phi$

Conversions.

(1) Cylindrical to Cartesian

$x = \rho \cos\phi$
$y = \rho \sin\phi$
$z = z.$

(2) Cartesian to cylindrical

$\rho = \sqrt{x^2+y^2}$
$\phi = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$

(3) Cartesian to spherical.

$r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$
$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\rho}{z}\right)$
$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

(4) Spherical to Cartesian

$x = \rho\cos\phi = r\sin\theta\cos\phi$
$y = \rho\sin\phi = r\sin\theta\sin\phi$
$z = z = r\cos\theta$

Transformation matrices

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$ (from cartesian to cylinderical)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$ (from cylinderical to cartesian)

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$ From cartesian to spherical.

Differential Elemental Displacement (vector) - length

$$\bar{dl} = dx\,\bar{a}_x + dy\,\bar{a}_y + dz\,\bar{a}_z$$
$$= d\rho\,\bar{a}_\rho + (\rho\, d\phi)\,\bar{a}_\phi + dz\,\bar{a}_z$$
$$= dr\,\bar{a}_r + (r\,d\theta)\,\bar{a}_\theta + (\rho\, d\phi)\,\bar{a}_\phi$$

Differential Elemental area

$$d\bar{S} = dy\,dz\,\bar{a}_x = dx\,dz\,\bar{a}_y = dx\,dy\,\bar{a}_z$$
$$= (\rho\, d\phi)\,dz\,\bar{a}_\rho = (d\rho\, dz)\,\bar{a}_\phi = (d\rho)(\rho\, d\phi)\,\bar{a}_z$$
$$= (r\,d\theta)(\rho\, d\phi)\,\bar{a}_r = (dr)(r\,d\theta)\,\bar{a}_\phi = dr(\rho\, d\phi)\,\bar{a}_\theta$$

Differential volume.

$$dv = dx\,dy\,dz$$
$$= d\rho\, dz\,(\rho\, d\phi)$$
$$= dr\,(r\,d\theta)\,(\rho\, d\phi)$$
$$= r^2 \sin\theta\, dr\, d\phi\, d\theta.$$

is by Aca

Ch

Del operator

$$\nabla = \frac{\partial}{\partial x}\bar{a}_x + \frac{\partial}{\partial y}\bar{a}_y + \frac{\partial}{\partial z}\bar{a}_z \quad \text{(cartesian)}$$

$$= \frac{\partial}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\bar{a}_\phi + \frac{\partial}{\partial z}\bar{a}_z \quad \text{(cylindrical)}$$

$$= \frac{\partial}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\bar{a}_\phi \quad \text{(spherical)}.$$

$\hookrightarrow \rho = r\sin\theta$

Gradient of a Scalar field $\rightarrow$ vector quantity

It is a vector. maximum space rate of increase of V.

$$\nabla V = \frac{\partial V}{\partial x}\bar{a}_x + \frac{\partial V}{\partial y}\bar{a}_y + \frac{\partial V}{\partial z}\bar{a}_z$$

$$= \frac{\partial V}{\partial \rho}\bar{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\bar{a}_\phi + \frac{\partial V}{\partial z}a_z$$

$$= \frac{\partial V}{\partial r}\bar{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\bar{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}a_\phi$$

Divergence of a vector field $\rightarrow$ Scalar Quantity

outward flux per unit volume

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Dot is imp here

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl of vector $\rightarrow$ Vector quantity (Defination on next Sheet)

$$\nabla \times A = \begin{bmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}, \frac{1}{\rho}\begin{bmatrix} \bar{a}_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{bmatrix}, \frac{1}{r^2\sin\theta}\begin{bmatrix} \bar{a}_r & r\bar{a}_\theta & r\sin\theta a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{bmatrix}$$

Divergence Theorem or Gauss's theorem

The Divergence Theorem states that the total outward flux of vector field A through the closed surface S is the same as the volume Integral of the divergence of A.

$$\oint_S A \cdot ds = \int_V \nabla \cdot A \, \partial v$$

Stoke's Theorem

Stoke's Theorem states that the circulation of a vector field A around a closed path L is equal to the Surface Integral of the Curl of A over the open surface S bounded by L, provided A and $\nabla \times A$ are continuous on S.

$$\oint_L A \cdot dl = \int_S (\nabla \times A) \cdot ds$$

Curl of a vector → A Curl of 'A' is an axial vector whose magnitude is the maximum circulation of A per unit area tends to zero and whose direction is the normal direction of Area, when the area is oriented to make the circulation maximum.

$$\text{curl } A = \nabla \times A = \left[\lim_{\Delta S \to 0} \frac{\oint_L A \cdot dl}{\Delta S} \right]_{max} a_n$$

Line Integral → $\oint_L A \cdot dl$

Surface Integral → $\oint_S A \cdot ds$

volume Integral → $\oint_V \rho_v \, dv$

Imp

For solenoidal $\vec{\nabla} \cdot \vec{A} = 0$

For irrotational $\vec{\nabla} \times \vec{A} = 0$

If $\vec{\nabla} \times \vec{E} = 0$ $\vec{E}$ field is conservative