UNIVERSITY OF PETROLEUM & ENERGY STUDIES, DEHRADUN

Program	**B. Tech SCS**	**Semester**	**II**
Course	**Mathematics II**	**Course Code**	**MATH 1005**
Session	**Jan-May 2018**	**Topic**	**Differential Equations**

CO1:

1. Let e^x be a solution the differential equation $(D^2 - 2D + 1)y = 0$ then how to find the second linearly independent solution.
2. How to find an integrating factor for the differential equation $\frac{dy}{dx} + P(x)y = R(x)$?
3. Given that $y_1 = e^{\frac{2x}{3}}$ is a solution of $9y'' - 12y' + 4y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.
4. Consider the boundary-value problem $y'' + \lambda y = 0, y(0) = 0 = y\left(\frac{\pi}{2}\right)$. Is it possible to find real values of λ so that the problem possesses (a) trivial solutions? (b) nontrivial solutions?
5. Find the general solution of $x^4 y'' + x^3 y' - 4x^2 y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous equation.
6. Find the transform $u = g(x)$, such that the differential equation $y'' + tanx\, y' + \cos^2 x\ y = 0$ will be transferred into a constant coefficient differential equation.
7. Find the transform $u = g(x)$, such that the differential equation $2xy'' + (5x^2 - 2) + 2x^3\, y = 0$ will be transferred into a constant coefficient differential equation and hence obtain the constant coefficient differential equation.
8. Find complementary function of $(2x + 5)^2 y'' - 6(2x + 5)y' + 8y = 0$
9. The indicated functions are known linearly independent solutions of the associated homogeneous differential equations on $(0, \infty)$. Find the general solution of the given nonhomogeneous equation.

   (a) $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = x^{\frac{3}{2}}$; $\quad y_1 = x^{\frac{-1}{2}} cosx,\ y_2 = x^{\frac{-1}{2}} sinx$

   (b) $x^2 y'' + xy' + y = \sec(lnx)$; $\quad y_1 = \cos(lnx),\ y_2 = \sin(lnx)$

CO2:

10. If the following equations are exact then explain the procedure to obtain the functions $M(x, y)$ and $N(x, y)$.

    (a) $M(x, y)dx + \left(xe^{xy} + 2xy + \frac{1}{x}\right) dy = 0$ (b) $\left(x^{\frac{-1}{2}} y^{\frac{1}{2}} + \frac{x}{x^2+y}\right) dx + N(x, y)dy = 0$
11. Differential equations are sometimes solved by having a clever idea. Here is a little exercise in cleverness: Although the differential equation $\left(x - \sqrt{x^2 + y^2}\right)dx + ydy = 0$ is not exact, show how the rearrangement $\frac{(xdx+ydy)}{\sqrt{x^2+y^2}} = dx$ and the observation $\frac{1}{2}\, d(x^2 + y^2) = xdx + ydy$ can lead to a solution.
12. Show that every separable first-order equation $\frac{dy}{dx} = f(x)g(y)$ is exact.
13. Show that $(x^3 + y)dx + (y^3 + x)dy = 0$ is exact. More generally, is $M(x, y)dx + M(y, x)dy = 0$ exact? Explain.

14. If $M(x,y)dx + N(x,y)dy = 0$ and $P(x,y)dx + Q(x,y)dy = 0$ are exact, is $(M+P)dx + (N+Q)dy = 0$ exact? Explain.
15. (a) Give a convincing demonstration that the second order equation $ay'' + by' + cy = 0$, a, b and c are constants, always possesses at least one solution of the form $y = e^{m_1 x}$, m_1 is a constant.
    (b) Explain why the differential equation in part (a) must then have a second solution either of the form $y_2 = e^{m_2 x}$ or of the form $y_2 = xe^{m_1 x}, m_1$ and m_2 are constants.
16. Suppose that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of a homogeneous differential equation. Explain why $y_3 = coshx$ and $y_4 = sinhx$ are also solutions of the equation.
17. Given that $y = c_1 + c_2 x^2$ is a two-parameter family of solutions of $xy'' - y' = 0$ on the interval $(-\infty, \infty)$, show that constants c_1 and c_2 cannot be found so that a member of the family satisfies the initial conditions $y(0) = 0, y'(0) = 1$.
18. Show that the given functions form a fundamental set of solutions of the differential equation on the indicated interval.
    (a) $y'' - y' - 12y = 0;$ $\quad e^{-3x},\ e^{4x};\quad (-\infty, \infty)$
    (b) $y'' - 2y' + 5y = 0;$ $\quad e^x cos2x, e^x sin2x;\quad (-\infty, \infty)$
    (c) $x^2 y'' + xy' + y = 0;$ $\quad \cos(lnx), \sin(lnx);\quad (0, \infty)$
19. Show that the variable $u = lnx$ transforms the equation $x^2 y'' + axy' + by = 0$ into constant coefficient differential equation $y'' + (a-1)y' + by = 0$.
20. Show that the solution of the differential equation $x^2 y'' - 3xy' + y = \frac{lnx \sin(lnx)+1}{x}$ is
    $y = x^2\left(c_1 x^{\sqrt{3}} + c_2 x^{-\sqrt{3}}\right) + \frac{1}{x}\Big[\frac{lnx}{61}\{5\sin(lnx) + 6\cos(lnx)\} + \frac{2}{3721}\{27\sin(lnx) + 191\cos(lnx)\} + \frac{1}{6}\Big]$
21. Explain the linear dependence and independence of the functions $y_1 = |x|x$ and $y_2 = x^2$ over the indicated intervals.
    (a) $(-1, 1)$ (b) $(0, 1)$ (c) $(-1, 0)$
22. Show that the Wronskian $w(y_1, y_2)$ of any two independent solutions $y_1(x)$ and $y_2(x)$ of the differential equation $y'' + p(x)y' + q(x)y = 0$ satisfies the identity $w = c\, e^{-\int p(x)dx}$ and hence find the Wronskian of $y'' + xy' + xy = 0$.
23. If α is any constant and $F(D) = (D-\alpha)^r\, \phi(D)$, such that $\phi(\alpha) \neq 0$ then show that
    $\frac{1}{F(D)}\ e^{\alpha x} = \frac{e^{\alpha x}}{\phi(\alpha)} \frac{x^r}{r!}$
24. Show that the transformation $x = sinhz$ transforms $(1 + x^2)y'' + xy' = 4y$ to $\frac{d^2 y}{dz^2} = 4y$ and hence solve it.
25. Show that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are respectively particular solutions of
    $y'' - 6y' + 5y = -9e^{2x}$ and $y'' - 6y' + 5y = 5x^2 + 3x - 16$

CO3:

26. Use the substitution $t = -x$ to solve the given initial-value problem on the interval $(-\infty, 0)$
    (a) $4x^2 y'' + y = 0, y(-1) = 2, y'(-1) = 4$; (b) $x^2 y'' - 4xy' + 6y = 0, y(-2) = 8, y'(-2) = 0$
27. Solve the following differential equations:
    (a) $(D^3 - 5D^2 + 7D - 3)y = e^{2x} cosh2x$
    (b) $(D^4 + 3D^2 - 4)y = \cos^2 x - coshx$

(c) $(D^4 - 2D^3 + D^2)y = x^3$
(d) $y'' - 4y' + y = cosx\, cos2x$
(e) $y^{iv} + 3y''' - 3y'' = x^2 + 3e^{2x} + 4sinx$
(f) $(D^4 - 6D^3 + 14D^2 - 14D + 5)y = e^x + \sin(x+3) + x$
(g) $y'' - y = R(x)$ where $R(x) = xsinx + (1+x^2)e^x;\ e^{2x}sinx + e^{\frac{x}{2}}\sin\left(\frac{x\sqrt{3}}{2}\right);\ \left(1+\frac{1}{e^x}\right)^{-2}$
(h) $y'' - 2y' + y = x^2 lnx$
(i) $y'' + y = \sec^3 x\ tanx$
(j) $\frac{1}{5}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 5cos4t$, given that $x(0) = \frac{1}{2}$ and $\left(\frac{dx}{dt}\right)_{t=0} = 0$

28. Apply the mentioned method to obtain the general solution of the following differential equations:
(a) $(1-x)y_2 + xy_1 - y = (1-x)^2$; variation of parameters
(b) $(x^3 - 2x^2)y'' + 2x^2y' - 12(x-2)y = 0$; removal of first derivative
(c) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$; when a part of C.F $y = x^3$ is known
(d) $(1+x)^2y'' + (1+x)y' + y = 2\sin\{\ln(1+x)\}$; change the independent variable
(e) $x^2y_2 + xy_1 - y = x^2e^x$; variation of parameters
(f) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = \frac{e^{2x}}{x^2}$; (i) when a part of C.F is known and (ii) variation of parameters

29. Solve the following differential equations:
(a) $y_2 + (1 - cotx)y_1 - cotx\ y = \sin^2 x$
(b) $x^3y''' + 2x^2y'' + 2y = 10\left(x + \frac{1}{x}\right)$
(c) $x^2(\log_e x)^2 y_2 - 2x\log_e x\ y_1 + \{2 + \log_e x - 2(\log_e x)^2\}y = (\log_e x)^3x^2$
(d) $y_2 + \left(1 + \frac{2}{x}cotx - \frac{2}{x^2}\right)y = x\ cosx$ given that $\frac{sinx}{x}$ is a part of C.F.
(e) $(x\ sinx + cosx)\frac{d^2y}{dx^2} - xcosx\frac{dy}{dx} + cosx\ y = sinsx\ (xsinx + cosx)^2$
(f) $(y'' + y)cotx + 2(y' + tanx\ y) = secx$
(g) $y'' + (tanx - 3cosx)y' + 2\,y\cos^2 x = \cos^4 x$
(h) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$
(i) $(D^2 + 2D + 2)y = e^{-x}\sec^3 x$
(j) $16(x+1)^4\frac{d^4y}{dx^4} + 96(x+1)^3\frac{d^3y}{dx^3} + 104(x+1)^2\frac{d^2y}{dx^2} + 8(x+1)\frac{dy}{dx} + y = x^2 + 4x + 3$
(k) $(3x+2)^2y_2 + 5(3x+2)y_1 - 3y = x^2 + x + 1$
(l) $x^2y'' - 2x(3x-2)y' + 3x(3x-4)y = 2^{3x}$
(m) $(D^4 + m^2D^2 + n^2D^2 + m^2n^2)y = \cos\left\{\left(\frac{m+n}{2}\right)x\right\}\cos\left\{\left(\frac{m-n}{2}\right)x\right\}$
(n) $(D^2 + 2ncos\alpha\ D + n^2)y = L\ cosnx$ given that $y = 0, \frac{dy}{dx} = 0$ when $x = 0$
(o) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$
(p) $(x+1)y'' + xy' - y = (x+1)^2$