Q1. i) Find Eign values and Eigen vector for $A = \begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$

ii) Also, check wheather A is diagonalizable. if yes then diagonalize A.

iii) also find eign values of a) $A^4 - 2A^3 + 3I - 5A^{-1}$.

b) A^T.

c) adj(A).

$\Rightarrow$ i) Given $A = \begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$

$\therefore$ The characteristic equation is $[A - \lambda I] = 0$.

$$\therefore \begin{bmatrix} -9-\lambda & 2 & 6 \\ 5 & 0-\lambda & -3 \\ -16 & 4 & 11-\lambda \end{bmatrix} = 0.$$

We know, $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$.

where S_1 = sum of diagonal elements and

S_2 = sum of minor of diagonal elements.

$\therefore S_1 = -9 + 0 + 11 = 2$

$$S_2 = \begin{vmatrix} 0 & -3 \\ 4 & 11 \end{vmatrix} + \begin{vmatrix} -9 & 6 \\ -16 & 11 \end{vmatrix} + \begin{vmatrix} -9 & 2 \\ 5 & 0 \end{vmatrix} = 0+12-99+96+0-10 = -1$$

$\therefore$ Equation becomes, $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$.

also $|A| = \begin{vmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{vmatrix} = 2$.

Solving equation, we get. eign values as,

$\lambda = -1, 2, 1$.

Now, for eigen vectors.

for $\lambda = -1$,

$\therefore [A - (-1)I]X = 0$.

$$\begin{bmatrix} -9+1 & 2 & 6 \\ 5 & 1 & -3 \\ -16 & 4 & 11+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 2 & 6 \\ 5 & 1 & -3 \\ -16 & 4 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Consider, $-8x_1 + 2x_2 + 6x_3 = 0$.

$5x_1 + x_2 - 3x_3 = 0$.

By Cramer's rule, $\dfrac{x_1}{\begin{vmatrix} 2 & 6 \\ 1 & -3 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} -8 & 6 \\ 5 & -3 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} -8 & 2 \\ 5 & 1 \end{vmatrix}} = t$.

$\therefore \dfrac{x_1}{-12} = \dfrac{-x_2}{-6} = \dfrac{x_3}{-18} = t$.

$\therefore X_1 = \begin{bmatrix} -12t \\ 6t \\ -18t \end{bmatrix}$ $\quad \therefore X_1 = t\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ — (dividing by -6) — ①.

for $\lambda = 2$,

$\therefore [A - 2I]X = 0$.

$$\begin{bmatrix} -9-2 & 2 & 6 \\ 5 & -2 & -3 \\ -16 & 4 & 11-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} -11 & 2 & 6 \\ 5 & -2 & -3 \\ -16 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$-11x_1 + 2x_2 + 6x_3 = 0$.

$-16x_1 + 4x_2 + 9x_3 = 0$.

By Cramer's Rule,

$$\frac{x_1}{\begin{vmatrix} 2 & 6 \\ 4 & 9 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -11 & 6 \\ -16 & 9 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -11 & 2 \\ -16 & 4 \end{vmatrix}} = t.$$

$$\therefore \frac{x_1}{-6} = \frac{-x_2}{-3} = \frac{x_3}{-12} = t.$$

$$\therefore X_2 = \begin{bmatrix} -6t \\ -3t \\ -12t \end{bmatrix} \qquad \therefore X_2 = t\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - (\text{dividing by } -3) \text{ — } ②.$$

for $\lambda = 1$,

$$\therefore [A - 1I]X = 0.$$

$$\begin{bmatrix} -9-1 & 2 & 6 \\ 5 & -1 & -3 \\ -16 & 4 & 11-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad 5x_1 - x_2 - 3x_3 = 0$$

$$-16x_1 + 4x_2 + 10x_3 = 0.$$

By Cramer's Rule,

$$\frac{x_1}{\begin{vmatrix} -1 & -3 \\ 4 & 10 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & -3 \\ -16 & 10 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -1 \\ -16 & 4 \end{vmatrix}} = t.$$

$$\therefore \frac{x_1}{2} = \frac{-x_2}{2} = \frac{x_3}{4} = t.$$

$$X_3 = \begin{bmatrix} 2t \\ -2t \\ 4t \end{bmatrix} \quad \therefore X_3 = t\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} - (\text{dividing by } 2) \text{ — } ③.$$

$\therefore$ Eigen values are $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 1$. and Eigen Vectors are $X_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

ii) $\because$ all the eigen values are distinct the matrix A is diagonlizable.

also, The algebric multiplicity and geometrig multipliaity are 1.

for $\lambda = -1$,

algebric multiplicity = geometrig multiplicity = 1.

for $\lambda = 2$,

algebric multiplicity = geometric multiplicity = 1

for $\lambda = 1$,

algebric multiplicity = geometric multiplicity = 1.

Hence, A.M and G.M for each eigen value of A matrix are same.

Therefore, A is digonsiable matrix.

modal Matrix (M) = $[X_1 \; X_2 \; X_3] = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$

$\therefore$ diagonal Matrix,

$$D = M^{-1} A M$$

$$D = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & -1 \\ 3 & 4 & 2 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

iii) a) $A^4 - 2A^3 + 3I - 5A^{-1} = 0$. put $A = \lambda$

for $\lambda = -1$,

$\lambda^4 - 2\lambda^3 + 3I - 5\lambda^{-1} = 0$.

$(-1)^4 - 2(-1)^3 + 3 - 5\left(\frac{1}{-1}\right) = 1 - 2(-1) + 3 - 5(-1)$
$= 1 + 2 + 3 + 5$
$= 11$.

for $\lambda = 2$,

$(2)^4 - 2(2)^3 + 3I - 5\left(\frac{1}{2}\right) = 16 - 2(8) + 3 - \frac{5}{2}$
$= \frac{1}{2}$

for $\lambda = 1$,

$(1)^4 - 2(1)^3 + 3I - 5(1) = 1 - 2 + 3 - 5$
$= -3$

$\therefore$ Eigen values for $A^4 - 2A^3 + 3I - 5A^{-1}$ is $11, \frac{1}{2}, -3$.

b) $A^T = \begin{bmatrix} -9 & 5 & -16 \\ 2 & 0 & 4 \\ 6 & -3 & 11 \end{bmatrix}$

$\therefore$ The characteristic equation is $[A^T - \lambda I] = 0$.

$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A^T| = 0$.

where S_1 is sum of diagonal elements and S_2 is sum of minors of diagonal elements.

$\therefore S_1 = -9 + 11 = 2$

$S_2 = \begin{vmatrix} 0 & 4 \\ -3 & 11 \end{vmatrix} + \begin{vmatrix} -9 & -16 \\ 6 & 11 \end{vmatrix} + \begin{vmatrix} -9 & 5 \\ 2 & 0 \end{vmatrix}$

$= 0 + 12 - 99 + 96 + 0 - 10$ and $|A^T| = -2$

$= -1$

$\therefore \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$.

$\therefore \lambda = -1, 2, 1$

$\therefore$ Eigen values of A^T is $-1, 2, 1$.

c) Adj(A).

we know, $adj(A) = \frac{|A|}{\lambda}$

we know, $A = \begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$ $\therefore |A| = -2$

and $\lambda_1 = -1$, $\lambda_2 = 2$, $\lambda_3 = 1$.

$adj(A) = \frac{|A|}{\lambda} = \frac{-2}{-1}, \frac{-2}{2}, \frac{-2}{1} = 2, -1, -2$

$\therefore$ The ~~as~~ eigen values of adj(A) is $2, -1, -2$.

Q2. Find $A_{2\times2}$ if eigen values are 4 and 1 & their Correspon-ding eigen vectors are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\Rightarrow$ Given that, $\lambda = 4, 1$ and vectors are

$X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ for eigen values $\lambda_1 = 4$ and $\lambda_2 = 1$.

we know, modal Matrix $= M = [X_1 \ X_2] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

and Diagonal Matrix $= D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

We know, $D = M^{-1} A M$

multiplying by M^{-1} and M on Both side,

$\therefore A = M^{-1} D M = M^{-1} M A M M^{-1} = A$

$\therefore A = M^{-1} D M$.

$\therefore$ by putting the matrix,

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 0.2 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 4 & -2 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.4 \\ -0.4 & 0.2 \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 1.6 & 1.2 \\ 1.2 & 3.4 \end{bmatrix} \quad \text{i.e} \quad A = \begin{bmatrix} 8/5 & 6/5 \\ 6/5 & 17/5 \end{bmatrix}$$

$\therefore A = \frac{1}{5}\begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix}$ and determinants $|A| = 4$.

Q3. Using Cayley hamilton find $A^4 - 3A^3 + 5A^2 - 6A + 7I$ for $A = \begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}$

=> by Cayley hamilton,

put we know characteristic equation,

$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$.

put $\lambda = A$, we get —(by Cayley hamilton).

$A^3 - 2A^2 - A + 2I = 0$

$\therefore$ Here, $(I = 1)$.

$$\begin{array}{r|l} & A - I \\ \hline A^3 - 2A^2 - A + 2I & A^4 - 3A^3 + 5A^2 - 6A + 7I \\ & \underset{-}{A^4} \underset{+}{-2A^3} \underset{+}{-A^2} \underset{-}{+2A} \\ \hline & -A^3 + 6A^2 - 8A + 7I \\ & \mp A^3 \pm 2A^2 \pm A \mp 2I \\ \hline & 4A^2 - 9A + 9I. \end{array}$$

$\therefore A^3 - 2A^2 - A + 2I = 4A^2 - 9A + 9I$.

$$= 4\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix}\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix} - 9\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 4\begin{bmatrix} -5 & 6 & 6 \\ 3 & -2 & -3 \\ -12 & 12 & 13 \end{bmatrix} - 9\begin{bmatrix} -9 & 2 & 6 \\ 5 & 0 & -3 \\ -16 & 4 & 11 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 70 & 6 & -30 \\ -33 & 1 & 15 \\ 96 & 12 & -38 \end{bmatrix}.$$

$$\therefore A^4 - 3A^3 + 5A^2 - 6A + 7I = \begin{bmatrix} 70 & 6 & -30 \\ -33 & 1 & 15 \\ 96 & 12 & -38 \end{bmatrix}$$