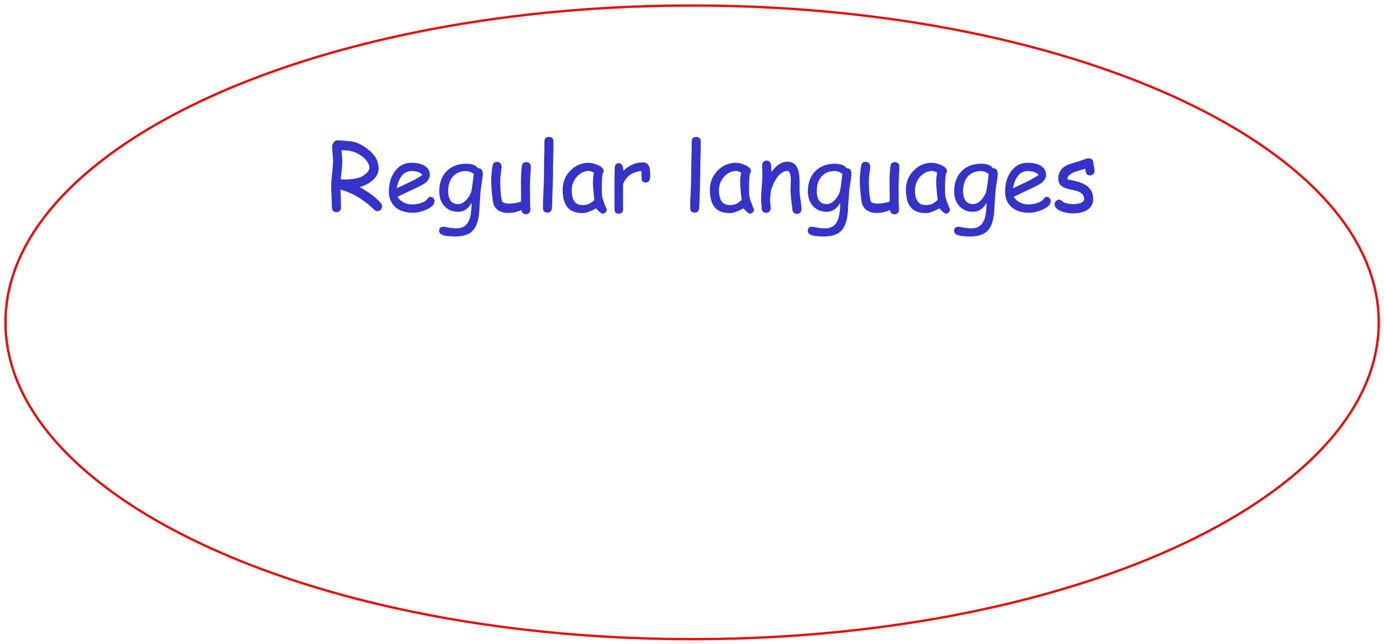
Formal Languages
The Pumping Lemma (2)

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
  - with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L \qquad i = 0, 1, 2, \ldots$

Non-regular languages

$$L = \{vv^R : v \in \Sigma^*\}$$



Theorem: The language

$$L = \{vv^R : v \in \Sigma^*\} \qquad \Sigma = \{a, b\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{vv^R : v \in \Sigma^*\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m b^m a^m$

Write $a^m b^m b^m a^m = x\ y\ z$

From the Pumping Lemma
it must be that length $|x\ y| \le m, \quad |y| \ge 1$

$$xyz = \underbrace{a \ldots a}_{x}\underbrace{a \ldots a}_{y}\underbrace{\ldots a b \ldots b b \ldots b a \ldots a}_{z}$$

$$xyz = \overbrace{a \ldots a a \ldots a \ldots a}^{m}\overbrace{b \ldots b}^{m}\overbrace{b \ldots b}^{m}\overbrace{a \ldots a}^{m}$$

Thus: $y = a^k, \quad k \ge 1$

$$x\,y\,z = a^m b^m b^m a^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \;\in\; L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^2\,z \;\in\; L$

$$x\,y\,z = a^m b^m b^m a^m \qquad\qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^2\,z \;\in\; L$

$$xy^2z = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}}^{m\;+\;k}\underbrace{...a\overbrace{b...b}^{m}\overbrace{b...b}^{m}\overbrace{a...a}^{m}}_{z} \;\in L$$

Thus: $a^{m+k} b^m b^m a^m \quad \notin L$

$$a^{m+k} b^m b^m a^m \quad \notin L \qquad\qquad k \geq 1$$

BUT: $L = \{vv^R : v \in \Sigma^*\}$

$$\Downarrow$$

$$a^{m+k} b^m b^m a^m \quad \notin L$$

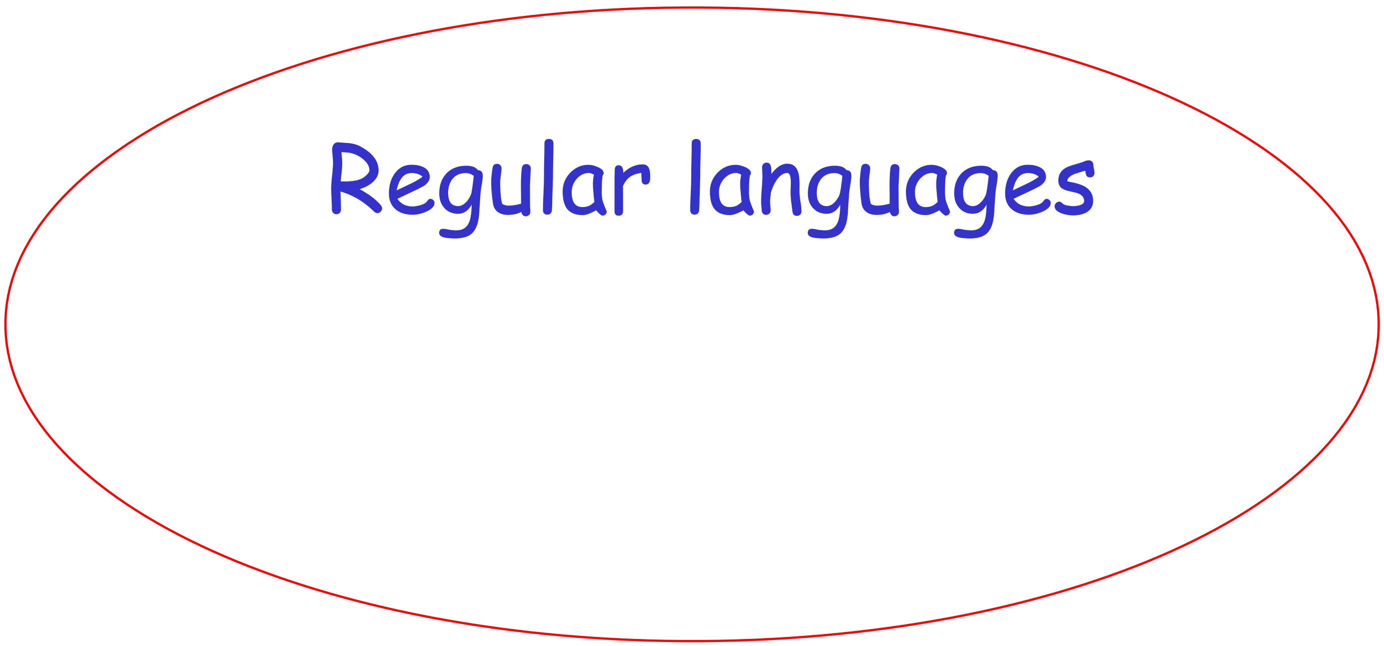
CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$



Theorem: The language

$$L = \{a^n b^l c^{n+l} : \; n,l \geq 0\}$$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$$L = \{a^n b^l c^{n+l} : \ n, l \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$ and

length $|w| \geq m$

We pick $w = a^m b^m c^{2m}$

Write $a^m b^m c^{2m} = x\ y\ z$

From the Pumping Lemma
it must be that length $|x\ y| \le m, \quad |y| \ge 1$

$$xyz = \underbrace{\overbrace{a...a}^{x}}_{x}...$$

$$xyz = \overbrace{\underbrace{a...a}_{x}\underbrace{a...a}_{y}a...a}^{m}\overbrace{b...b}^{m}\overbrace{c...cc...c}^{2m}$$

Thus: $y = a^k, \quad k \ge 1$

$$x\,y\,z = a^m b^m c^{2m} \qquad y = a^k, \quad k \geq 1$$

From the Pumping Lemma: $x\,y^i\,z \in L$

$$i = 0, 1, 2, \ldots$$

Thus: $x\,y^0\,z = xz \in L$

$x\ y\ z = a^m b^m c^{2m}$ $\quad$ $y = a^k, \quad k \geq 1$

From the Pumping Lemma: $\quad xz \in L$

$$xz = \underbrace{\overbrace{a...aa...a}^{m-k}}_{x}\ \underbrace{\overbrace{b...b}^{m}\ \overbrace{c...cc...c}^{2m}}_{z} \in L$$

Thus: $\quad a^{m-k} b^m c^{2m} \notin L$

$$a^{m-k} b^m c^{2m} \quad \notin L \qquad k \geq 1$$

BUT: $L = \{a^n b^l c^{n+l} : \; n, l \geq 0\}$

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