Chapter 12

Coping with the Limitations of Algorithm Power

Tackling Difficult Combinatorial Problems

There are two principal approaches to tackling difficult combinatorial problems (NP-hard problems):

- Use a strategy that guarantees solving the problem exactly but doesn't guarantee to find a solution in polynomial time
- Use an approximation algorithm that can find an approximate (sub-optimal) solution in polynomial time

Exact Solution Strategies

- *exhaustive search* (brute force)
  - useful only for small instances

- *dynamic programming*
  - applicable to some problems (e.g., the knapsack problem)

- *backtracking*
  - eliminates some unnecessary cases from consideration
  - yields solutions in reasonable time for many instances but worst case is still exponential

- *branch-and-bound*
  - further refines the backtracking idea for optimization problems

Backtracking

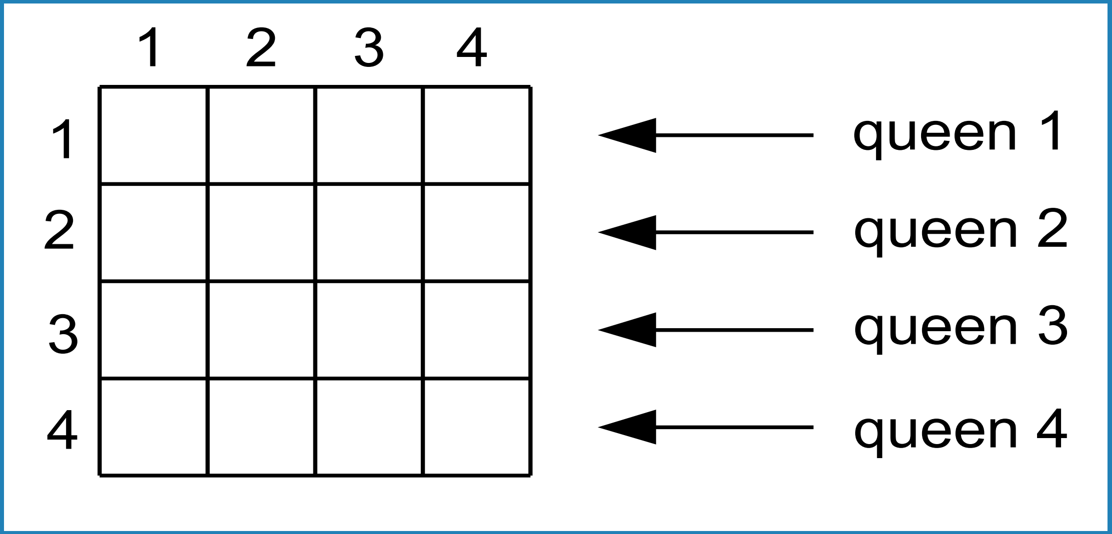
- Construct the ***state-space tree***
  - nodes: partial solutions
  - edges: choices in extending partial solutions

- Explore the state space tree using depth-first search

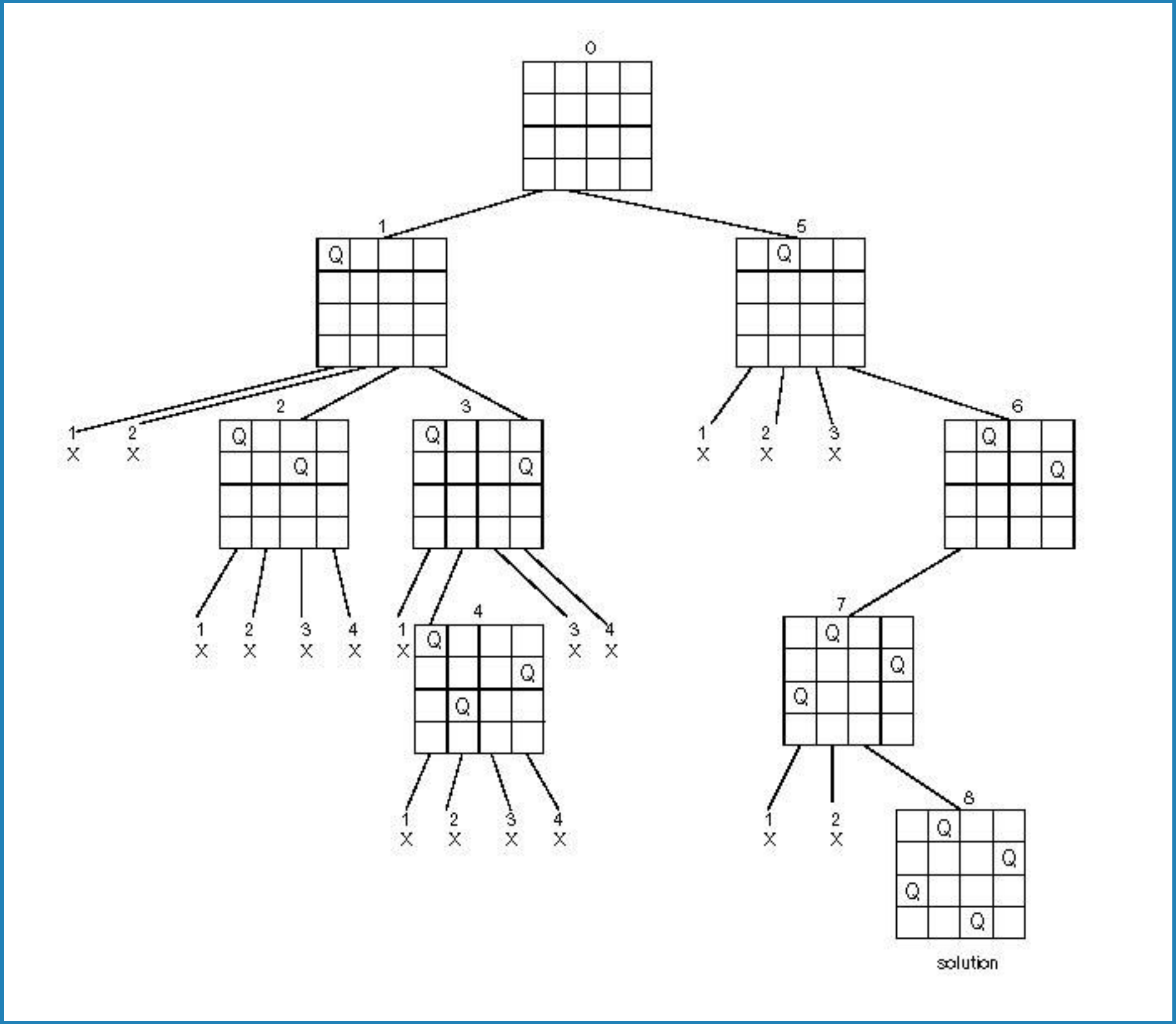
- "Prune" ***nonpromising nodes***
  - dfs stops exploring subtrees rooted at nodes that cannot lead to a solution and backtracks to such a node's parent to continue the search

Example: *n*-Queens Problem

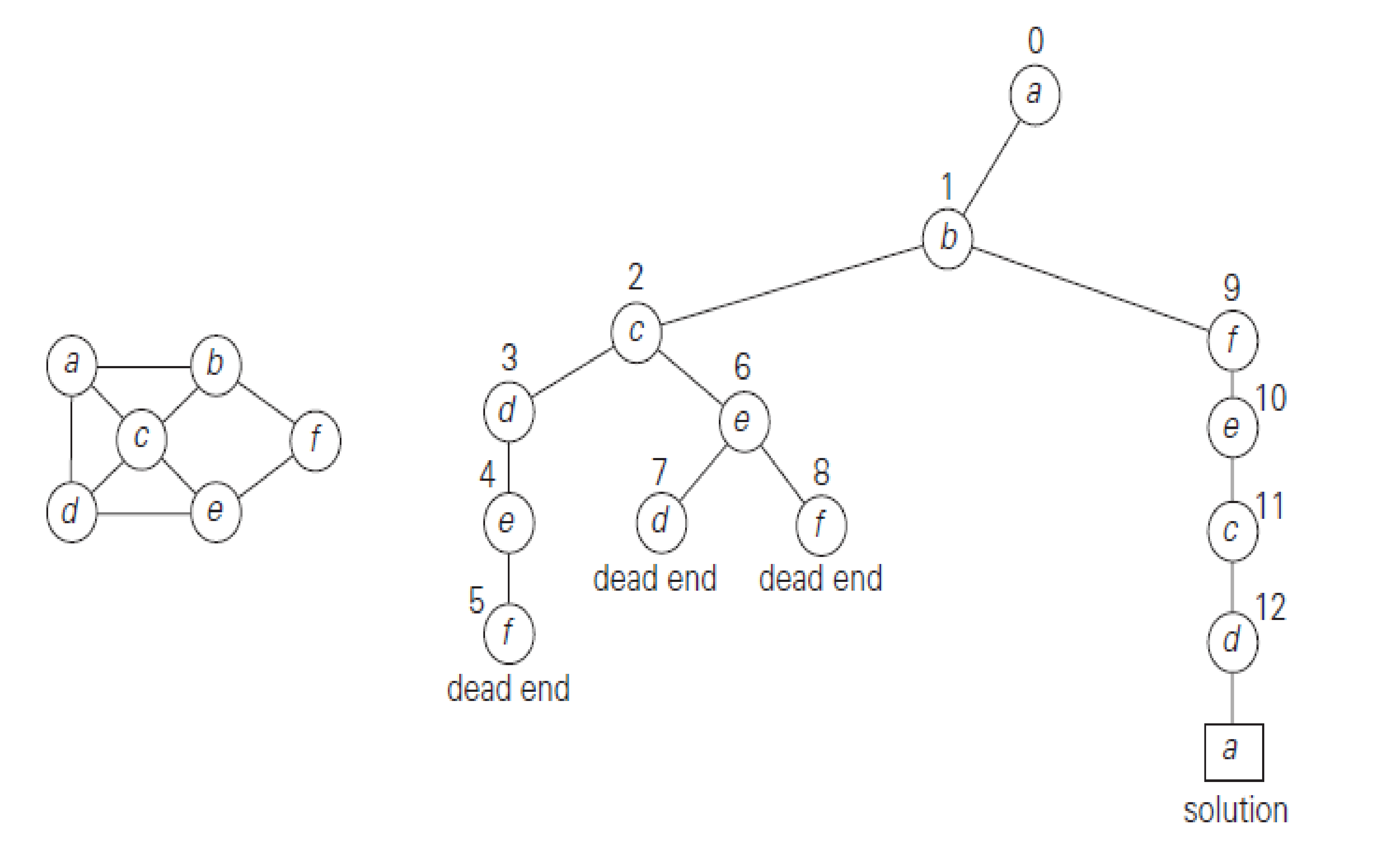
Place *n* queens on an *n*-by-*n* chess board so that no two of them are in the same row, column, or diagonal



State-Space Tree of the 4-Queens Problem

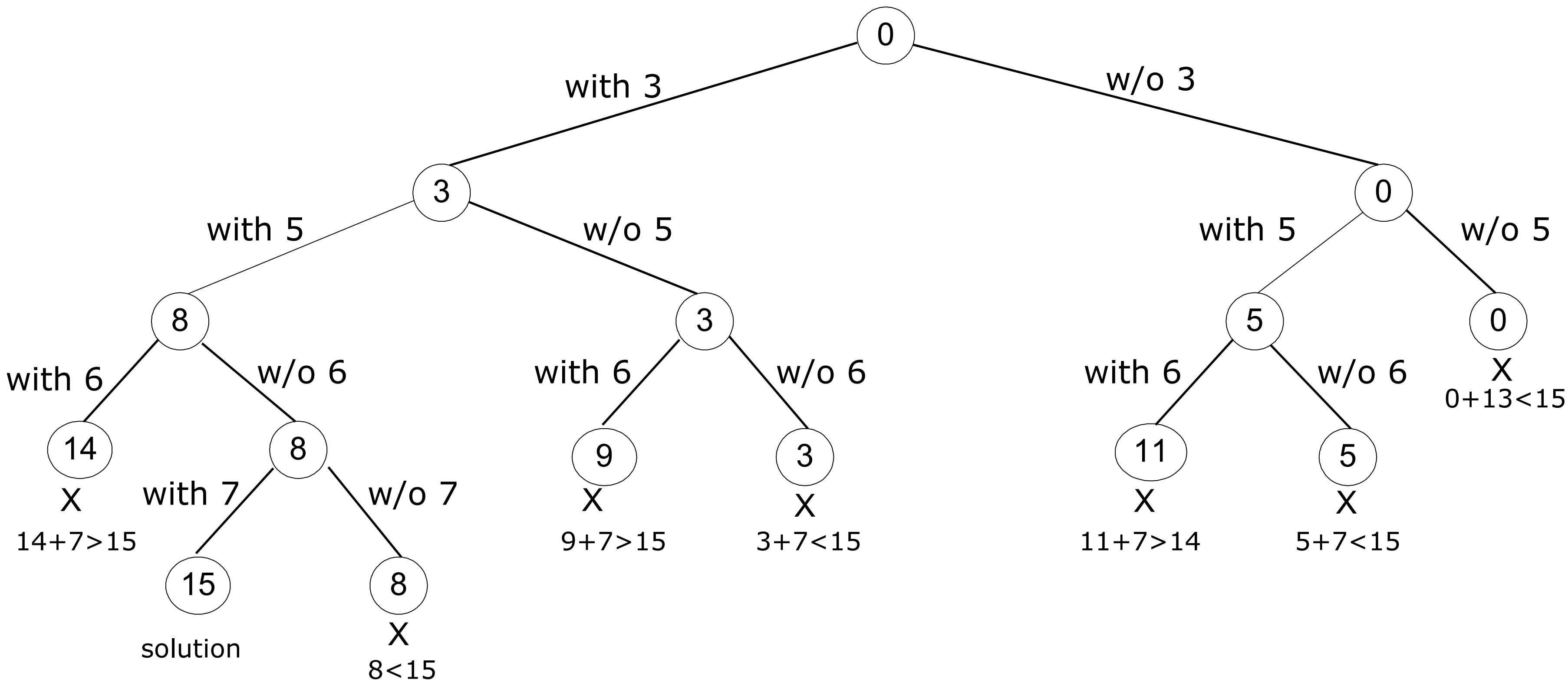


Examples: Hamiltonian Circuit and Subset-Sum Problems



Examples: Hamiltonian Circuit and Subset-Sum Problems

$A = \{3, 5, 6, 7\}$ and $d = 15$.



ALGORITHM *Backtrack*$(X[1..i])$

//Gives a template of a generic backtracking algorithm
//Input: $X[1..i]$ specifies first i promising components of a solution
//Output: All the tuples representing the problem's solutions
if $X[1..i]$ is a solution **write** $X[1..i]$
else
    for each element $x \in S_{i+1}$ consistent with $X[1..i]$ and the constraints **do**
        $X[i+1] \leftarrow x$
        Backtrack$(X[1..i+1])$

Branch-and-Bound

- An enhancement of backtracking
- Applicable to optimization problems
- For each node (partial solution) of a state-space tree, computes a bound on the value of the objective function for all descendants of the node (extensions of the partial solution)
- Uses the bound for:
  - ruling out certain nodes as "nonpromising" to prune the tree – if a node's bound is not better than the best solution seen so far
  - guiding the search through state-space

Example: Assignment Problem

Select one element in each row of the cost matrix *C* so that:
- **no two selected elements are in the same column**
- **the sum is minimized**

Example

	Job 1	Job 2	Job 3	Job 4
Person *a*	9	2	7	8
Person *b*	6	4	3	7
Person *c*	5	8	1	8
Person *d*	7	6	9	4

<u>Lower bound</u>: Any solution to this problem will have total cost at least: 2 + 3 + 1 + 4 (or 5 + 2 + 1 + 4)

Example: First two levels of the state-space tree

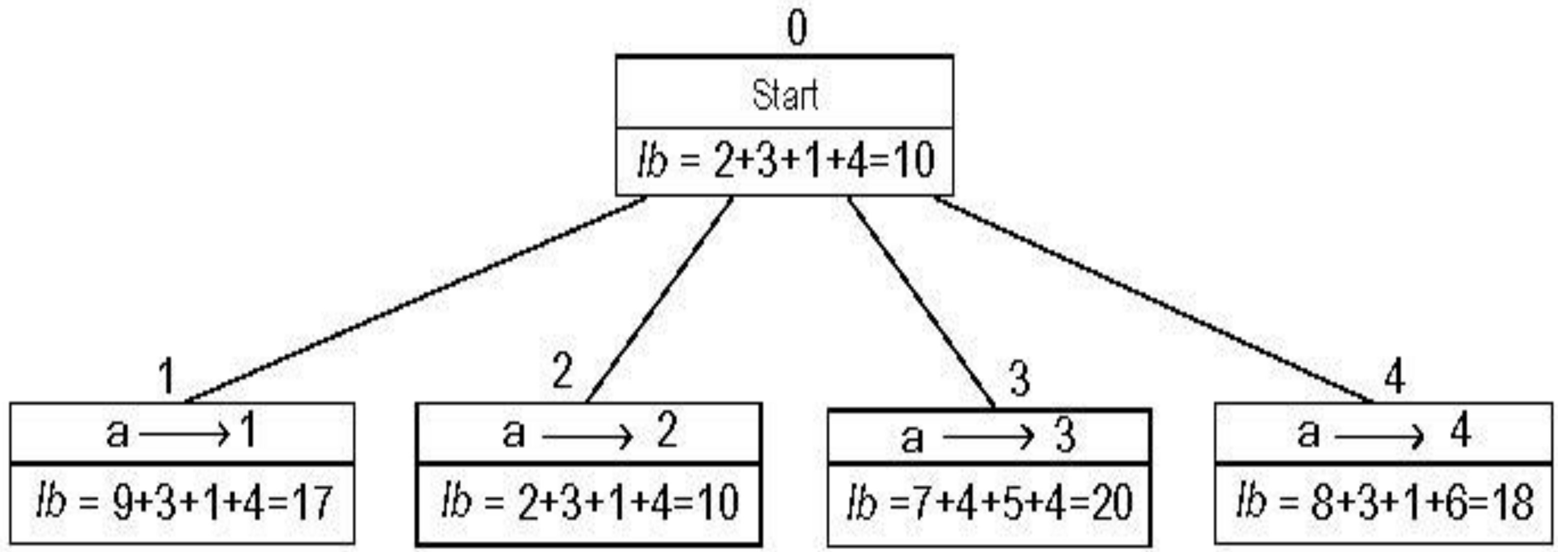


Figure 11.5 Levels 0 and 1 of the state-space tree for the instance of the assignment problem being solved with the best-first branch-and-bound algorithm. The number above a node shows the order in which the node was generated. A node's fields indicate the job number assigned to person *a* and the lower bound value, *lb*, for this node.

Example (cont.)

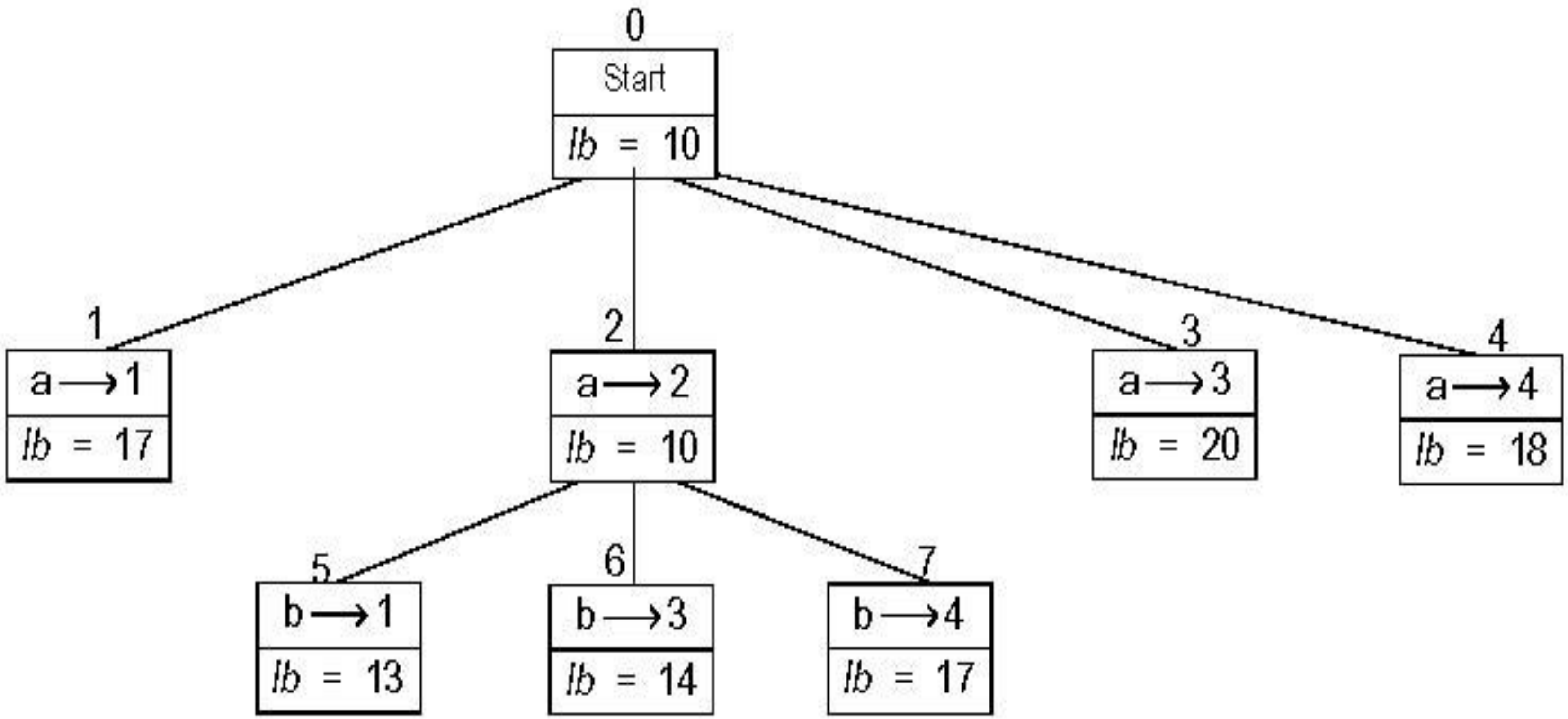


Figure 11.6 Levels 0, 1, and 2 of the state-space tree for the instance of the assignment problem being solved with the best-first branch-and-bound algorithm

Example: Complete state-space tree

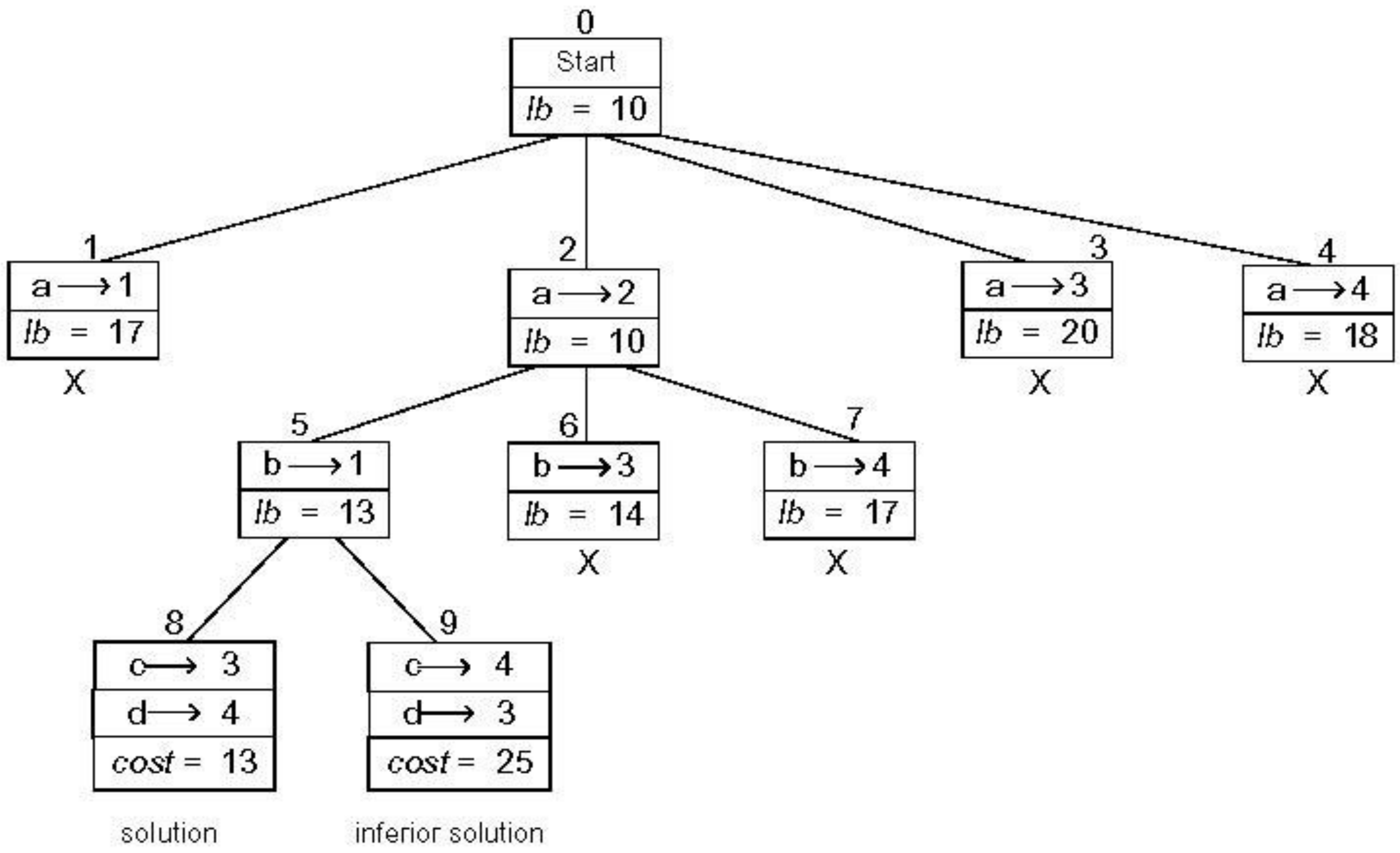
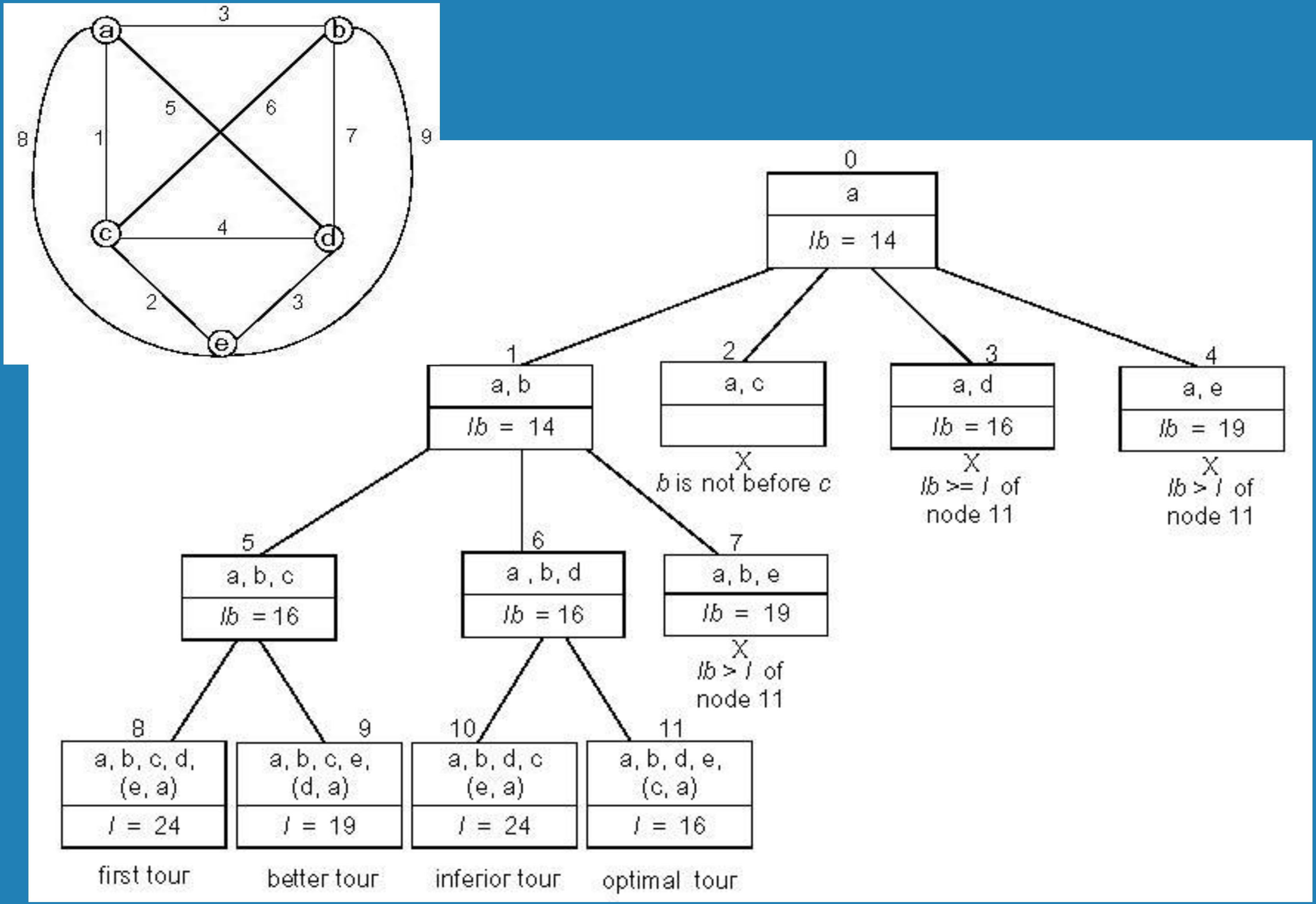


Figure 11.7 Complete state-space tree for the instance of the assignment problem solved with the best-first branch-and-bound algorithm

Example: Traveling Salesman Problem



Knapsack Problem

$$v_1/w_1 \geq v_2/w_2 \geq \cdots \geq v_n/w_n.$$

$$ub = v + (W - w)(v_{i+1}/w_{i+1}).$$

item	weight	value	$\frac{\text{value}}{\text{weight}}$
1	4	$40	10
2	7	$42	6
3	5	$25	5
4	3	$12	4

The knapsack's capacity W is 10.