# Properties of Regular Languages

Chapter – 4

4.1 – Closure Properties of Regular Languages

## Closure Properties of Regular Languages

- If $L1$ and $L2$ are regular languages, then so are

→ $L1 \cup L2$,

→ $L1 \cap L2$,

→ $L1L2$,

→ $\overline{L1}$ ,

→ $L1^*$

- The family of regular languages is closed under **union, intersection, concatenation, complementation, and star-closure**.

**Proof:**

- If *L*1 and *L*2 are regular, then there exist regular expressions *r*1 and *r2* such that *L*1 = *L*(*r*1) and *L*2 = *L*(*r*2).
- By definition, *r*1 + *r*2, *r*1*r*2, and r1* are regular expressions denoting the languages *L*1 ∪ *L*2, *L*1*L*2, and L1*.

- To show closure under complementation,

    let *M* = (*Q*, Σ,δ, *q*0,*F*) be a dfa that accepts *L*1.

Then the dfa $\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$ accepts $\overline{\text{L1}}$

- Assume δ* to be a total function, so that δ* (*qo*,*w*) is defined for all *w* ∈ Σ*.
- Consequently either δ*(*q*0,*w*) is a final state, in which case *w* ∈ *L*, or δ*(*q*0, *w*) ∈ Q – F and *w* ∈ $\overline{L}$.

- To demonstrating **closure under intersection** Let *L*1 = *L* (*M*1) and *L*2 = *L*(*M*2), where *M*1 = (*Q*,Σ,δ1,*q*0,*F*1) and *M*2 = (*P*,Σ, δ2,*p*0,*F*2) are dfa's.

- So the combined automaton $\widehat{M} = \left(\widehat{Q}, \Sigma, \widehat{\delta}, (q_0, p_0), \widehat{F}\right)$ whose state set $\widehat{Q} = Q \times P$ consists of pairs (*qi*, *pj*), and whose transition function $\widehat{\delta}$ is such that $\widehat{M}$ is in state (*qi*, *pj*) whenever *M1* is in state *qi* and *M2* is in state *pj*. So

$\widehat{\delta}((q_i, p_j), a) = (q_k, p_l),$ Where $\delta_1(q_i, a) = q_k$ and $\delta_2(p_j, a) = p_l.$

- Using DeMorgan's Law, taking complement on both sides of $\overline{L1 \cap L2} = \overline{L1} \cup \overline{L2}$

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

- If L1 and L2 are regular then by closure under complementation , Union and again complementation , $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$ is regular.

- Similarly regular languages are closed under difference as well as under reversal.

## Homomorphism

- Definition of homomorphism is given as

Suppose $\Sigma$ and $\Gamma$ are alphabets. Then a function

$$h : \Sigma \rightarrow \Gamma^*$$

is called a **homomorphism**. In words, a homomorphism is a substitution in which a single letter is replaced with a string. The domain of the function $h$ is extended to strings in an obvious fashion; if

$$w = a_1 a_2 \cdots a_n,$$

then

$$h(w) = h(a_1) h(a_2) \cdots h(a_n).$$

If $L$ is a language on $\Sigma$, then its **homomorphic image** is defined as

$$h(L) = \{h(w) : w \in L\}$$

**Example 1:**

Let $\Sigma = \{a, b, c\}$ and $\Gamma = \{a, b, c,\}$ define $h$ by

$$h(a) = ab,$$
$$h(b) = bbc.$$

Then $h(aba) = abbbcab$. The homomorphic image of $L = \{aa, aba\}$ is the language $h(L) = \{abab, abbbcab\}$.

**If $L$ is a regular language, then its homomorphic image $h(L)$ is also regular. The family of regular languages is therefore closed under homomorphisms.**

**Example 2:**

Take $\Sigma = \{a, b\}$ and $\Gamma = \{b, c, d\}$. Define $h$ by

$$h(a) = dbcc,$$
$$h(b) = bdc.$$

If $L$ is the regular language denoted by

$$r = (a + b^*)(aa)^*,$$

then

$$r_1 = \left(dbcc + (bdc)^*\right)(dbccdbcc)^*$$

denotes the regular language $h(L)$.

# Right Quotient

Let $L_1$ and $L_2$ be languages on the same alphabet. Then the **right quotient** of $L_1$ with $L_2$ is defined as

$$L_1/L_2 = \{x : xy \in L_1 \text{ for some } y \in L_2\}.$$

**Example 1:**

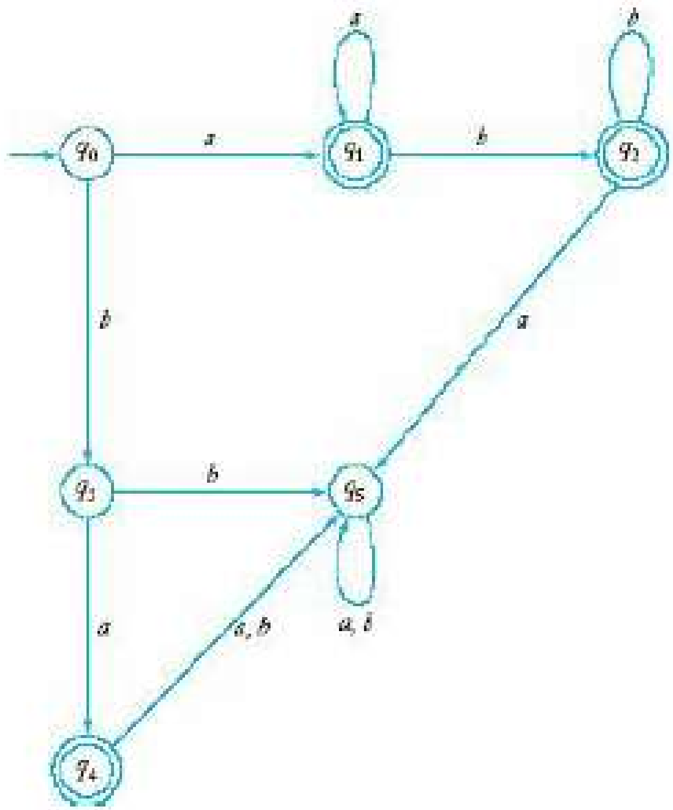
$$L_1 = \{a^n b^m : n \geq 1, m \geq 0\} \cup \{ba\}$$

$$L_2 = \{b^m : m \geq 1\},$$

$$L_1/L_2 = \{a^n b^m : n \geq 1, m \geq 0\}.$$

$$L_1 = \{a^n b^m : n \geq 1, m \geq 0\} \cup \{ba\}$$

$$L_2 = \{b^m : m \geq 1\},$$



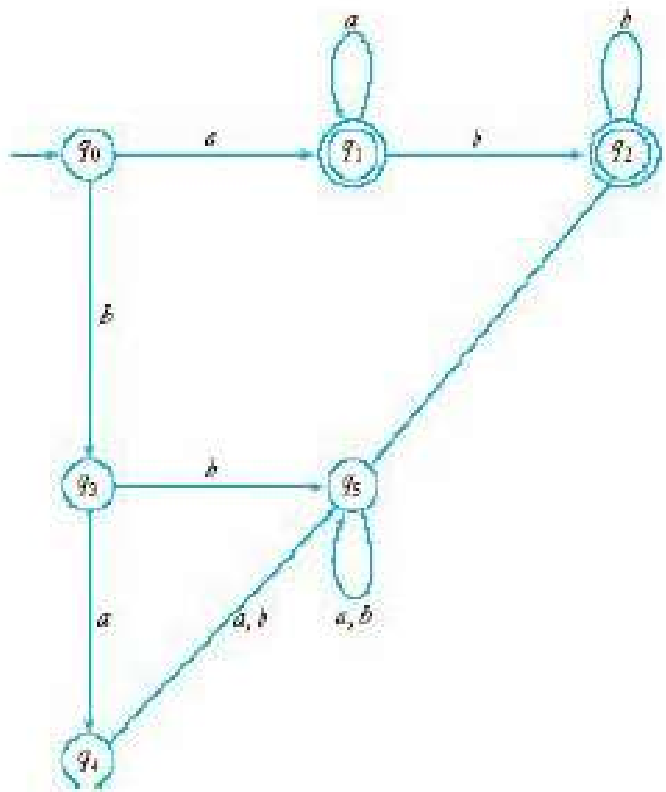

L(M0) = aa* + aa*bb* + ba
L(M1) = a* +a*bb*
L(M2) = b*
L(M3) = a
L(M4)= λ
L(M5) =Φ

Now Find intersection with L2
L(M0):
L(aa* + aa*bb* + ba) ∩ L(bb*) = Φ
**L(M1):**
**L(a* +a*bb*) ∩ L(bb*) ≠ Φ**
**L(M2):**
**L(b*) ∩ L(bb*) ≠ Φ**
L(M3):
L( a) ∩ L(bb*) = Φ
L(M4):
L(λ) ∩ L(bb*) = Φ
L(M5):
L(Φ) ∩ L(bb*) = Φ

Now only make M1 and M2 as acceptor in L1/L2

$$L_1/L_2 = \{a^n b^m : n \geq 1, m \geq 0\}.$$

Output after making q1 and q2 as acceptor state



Try Example 2.....

Find $L_1/L_2$ for

$$L_1 = L(a^*baa^*),$$
$$L_2 = L(ab^*).$$