EVALUATION SCHEME – MAT 2226 MID-TERM EXAMINATION

Multiple Choice Questions

1. If the probabilities of hitting a target are 0.5 for A, 0.25 for B, and 0.75 for C, then the probability that at least one of them hits the target, assuming that these events are independent, is ______. ($\frac{1}{2}$M)

Ans. $\dfrac{29}{32}$ — $1/2$

2. A continuous random variable has pdf $f(x) = kx^2e^{-x}$, $x \geq 0$. Then $k =$ ______. ($\frac{1}{2}$M)

Ans. $\dfrac{1}{2}$ — $1/2$

3. Two independent random variables X and Y have variances 0.2 and 0.5 respectively. Let $Z = 5X - 2Y$. The variance of Z is ______. ($\frac{1}{2}$M)

Ans. 7 — $1/2$

4. Six fair coins are tossed. Then the probability of getting exactly 3 heads is ______. ($\frac{1}{2}$M)

Ans. $\dfrac{5}{16}$ — $1/2$

5. If S and T are two independent events with $\mathrm{P}(S) < \mathrm{P}(T)$, $\mathrm{P}(S \cap T) = \dfrac{6}{25}$, and $\mathrm{P}(S \mid T) + \mathrm{P}(T \mid S) = 1$, then $\mathrm{P}(S)$ is ______. ($\frac{1}{2}$M)

Ans. $\dfrac{2}{5}$ — $1/2$

6. The marks obtained were found normally distributed with mean 75 and variance 100. The percentage of students who scored more than 75 marks is ______. ($\frac{1}{2}$M)

Ans. 50 — $1/2$

7. A fair die is rolled 200 times. What is the expected number of rolls giving prime numbers? ($\frac{1}{2}$M)

Ans. 100 — $1/2$

8. The cumulative distribution function of a random variable X is given by ($\frac{1}{2}$M)

$$F(x) = \begin{cases} 1 - 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Ans. $f(x) = 2e^{-2x}$, $x \geq 0$ — $1/2$

9. $\mathrm{Cov}[X+Y, X-Y] =$ ______ ($\frac{1}{2}$M)

Ans. $\mathrm{V}[X] - \mathrm{V}[Y]$ — $1/2$

10. Fifty tickets are serially numbered from 1 to 50. One ticket is drawn from these tickets at random. The probability of its being a multiple of 3 or 4 is ______. ($\frac{1}{2}$M)

Ans. $\dfrac{12}{25}$ — $1/2$

Note. Marks should be awarded suitably to any alternative correct solutions not given here.

Descriptive Questions

11. The probability of Tom hitting a target is $\frac{1}{3}$. (4M)

    (a) If Tom fires 5 times, what is the probability of his hitting the target at least twice?

    (b) How many times must Tom fire so that the probability of his hitting the target at least once is more than 90%?

Ans. The number of times Tom hits the target is $X \sim B\left(n, \frac{1}{3}\right)$. — 1

(a) Given $n = 5$, $P[X \geq 2] = 1 - P[X \leq 1] = 1 - \left(\frac{2}{3}\right)^5 - 5 \times \frac{2^4}{3^5} = \boxed{0.53}$. — 1

(b) $P[X \geq 1] \geq 0.9$, i.e. $P[X = 0] = \left(\frac{2}{3}\right)^n \leq 0.1$. — 1

Hence $n \geq \frac{\log 0.1}{\log\left(\frac{2}{3}\right)} \approx \boxed{6}$. — 1

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12. (a) Let $X \sim N(\mu, \sigma^2)$. Construct a random variable from X with mean 0 and variance 1. (4M)

    (b) A Wall Street analyst estimates that the annual return from the stock of Company A can be considered to be observation from a normal distribution with mean $\mu = 8\%$ and standard deviation $\sigma = 1.5\%$. The analyst's investment choices are based upon the considerations that any return greater than 5% is "satisfactory" and a return greater than 10% is "excellent". Find the probability that Company A's stock will prove to be "unsatisfactory". Find the probability that Company A's stock will prove to be excellent.

Ans. (a) Let $Z = \dfrac{X - \mu}{\sigma}$. Then — 1/2

$$\mathrm{E}[Z] = \frac{\mathrm{E}[X] - \mu}{\sigma} = \frac{\mu - \mu}{\sigma} = 0, \quad \mathrm{V}[Z] = \frac{\mathrm{V}[X]}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1.$$ — 1/2

(b) Let X be the percentage of annual returns. Then $X \sim N(8, 1.5^2)$.
The probability that the stock will prove to be unsatisfactory is

$$\mathrm{P}[X < 5] = \mathrm{P}\left[Z < \frac{5-8}{1.5}\right]$$ — 1/2

$$= \mathrm{P}[Z < -2]$$

$$= \Phi(-2)$$ — 1/2

$$= 1 - \Phi(2)$$

$$= \boxed{0.0228}.$$ — 1/2

The probability that the stock will prove to be excellent is

$$\mathrm{P}[X > 10] = \mathrm{P}\left[Z > \frac{10-8}{1.5}\right]$$ — 1/2

$$= \mathrm{P}[Z > 1.33]$$ — 1/2

$$= 1 - \Phi(1.33)$$

$$= \boxed{0.0918}.$$ — 1/2

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13. Consider a family of n children. Let A be the event that the family has children of both sexes. Let B be the even that there is at most one girl in the family. Find the value of n for which the events A and B are independent. Assume that each child has probability $\frac{1}{2}$ of being a boy. (3M)

Ans. $\mathrm{P}(A) = 1 - \dfrac{1}{2^n} - \dfrac{1}{2^n} = 1 - \dfrac{1}{2^{n-1}}$ — 1

$\mathrm{P}(B) = \dfrac{1}{2^n} + \dfrac{n}{2^n} = \dfrac{n+1}{2^n}$ — $1/2$

$\mathrm{P}(A \cap B) = \dfrac{n}{2^n}$ — $1/2$

If A and B are independent, then

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) \implies 2^{n-1} = n+1$ — $1/2$

Hence $\boxed{n = 3}$. — $1/2$

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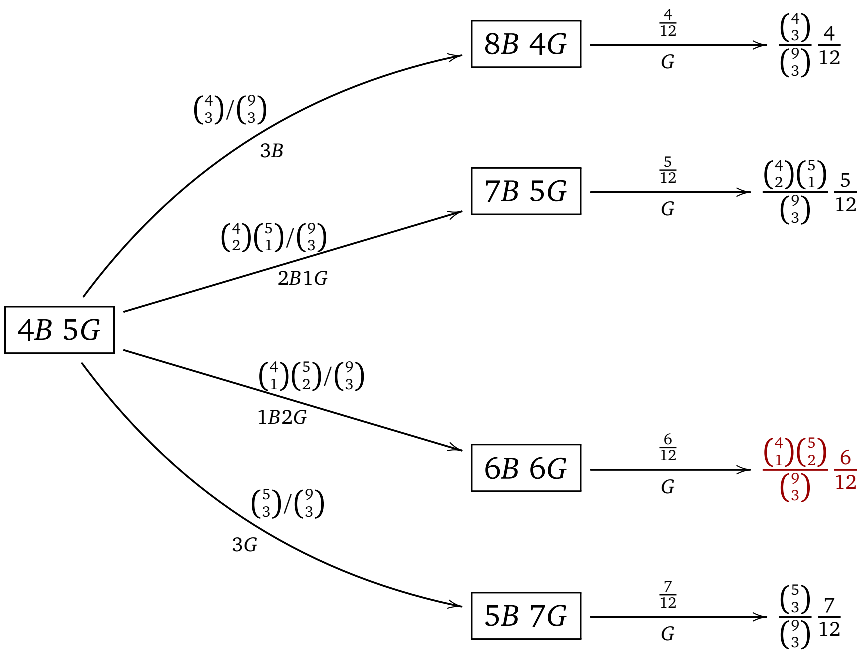
14. Box 1 contains 4 black and 5 green balls and Box 2 contains 5 black and 4 green balls. Three balls are chosen at random from Box 1 without replacement and transferred to Box 2, and then a ball is drawn from Box 2 and is found to be green. What is the probability that 2 green balls and 1 black ball were transferred from Box 1? (3M)

Ans.



— 3

Alternatively, events $A_1, \dots, A_4, B$ **are defined and** $\mathrm{P}(A_1), \dots, \mathrm{P}(A_4)$ **and** $P(B \mid A_1), \dots, P(B \mid A_4)$ **are computed (without drawing the tree diagram).** — 3

$$P(1B2G \mid G) = \frac{\binom{4}{1}\binom{5}{2} \times 6}{\binom{4}{3} \times 4 + \binom{4}{2}\binom{5}{1} \times 5 + \binom{4}{1}\binom{5}{2} \times 6 + \binom{5}{3} \times 7} = \boxed{\frac{60}{119}} \approx \boxed{0.504}$$ — 1

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15. At a telephone centre, the time X (in minutes) for which an agent speaks on a telephone is found to be random, for which the probability of distribution function is given by (3M)

$$f(x) = \begin{cases} kx, & 0 \le x \le 2 \\ 2k, & 2 \le x \le 4 \\ k(6-x), & 4 \le x \le 6. \end{cases}$$

(a) Find the value of k for which $f(x)$ is valid.

(b) Find $\mathrm{P}(4 \le X < 5 \mid X > 3)$.

Ans. (a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ — 1/2

$\implies 8k = 1$ — 1/2

$\implies \boxed{k = \frac{1}{8}}$ — 1/2

(b) $\mathrm{P}(4 \le X < 5 \mid X > 3) = \dfrac{\mathrm{P}(4 \le X < 5)}{\mathrm{P}(X > 3)}$ — 1/2

$= \dfrac{\int_4^5 f(x)\,dx}{\int_3^{\infty} f(x)\,dx}$ — 1/2

$= \dfrac{3/16}{1/2} = \boxed{\frac{3}{8}}$ — 1/2

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16. Let (X, Y) be a two-dimensional random variable with joint pdf $f(x, y) = kxy$, $0 \le x \le 1$, $0 \le y \le \sqrt{x}$. Find the constant k and the marginal pdfs of X and Y. (3M)

Ans. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$ — 1/2

$\implies \boxed{k = 6}$ — 1/2

$$f_1(x) = \int_0^{\sqrt{x}} 6xy\,dy$$ — 1/2

$$= \boxed{3x^2,\ 0 \le x \le 1}$$ — 1/2

$$f_2(y) = \int_{y^2}^{1} 6xy\,dy$$ — 1/2

$$= \boxed{3y(1 - y^4),\ 0 \le y \le 1}$$ — 1/2

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17. If X_1, X_2, and X_3 are pairwise uncorrelated random variables having the same standard deviation. Find the coefficient of correlation between $U = X_1 - X_2$ and $V = X_3 + X_2$. (3M)

Ans. Let σ be the common value of the standard deviation. The covariance between U and V is

$$\mathrm{Cov}[U, V] = \mathrm{E}[UV] - \mathrm{E}[U]\mathrm{E}[V]$$ — 1/2

$$= \mathrm{E}[X_1X_3 + X_1X_2 - X_2X_3 - X_2^2] - (\mathrm{E}[X_1] - \mathrm{E}[X_2])(\mathrm{E}[X_3] + \mathrm{E}[X_2])$$

$$= \mathrm{Cov}[X_1, X_3] + \mathrm{Cov}[X_1, X_2] - \mathrm{Cov}[X_2, X_3] - \mathrm{V}[X_2]$$ — 1/2

$$= -\sigma^2.$$ — 1/2

Their variances are $\mathrm{V}[U] = \mathrm{V}[X_1] + \mathrm{V}[X_2] = 2\sigma^2 = \mathrm{V}[X_3] + \mathrm{V}[X_2] = \mathrm{V}[V]$. — 1/2
Hence the correlation between U and V is

$$\begin{aligned}\rho_{UV} &= \frac{\mathrm{Cov}[U,V]}{\sqrt{\mathrm{V}[U]\mathrm{V}[V]}} \\ &= \frac{-\sigma^2}{\sqrt{(2\sigma^2)(2\sigma^2)}} && \text{— 1/2} \\ &= \boxed{-\frac{1}{2}}. && \text{— 1/2}\end{aligned}$$

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18. Assuming that the year has 365 days, what is the probability that in a random group of k people, at least two people share the same birthday? (2M)

Ans. The probability that no two people in the group have the same birthday is $\frac{1}{365^k}\binom{365}{k} \times k!$. — 1
Hence, the probability that at least two of them share the same birthday is

$$\boxed{1 - \frac{k!}{365^k}\binom{365}{k}}. \qquad \text{— 1}$$
