Scheme for evaluation: MAT2256(Engineering Mathematics-IV)

1A. Three group of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the 3 selected consist of 1 girl and 2 boys is 13/32.

Solution: The required event of getting 1 girl and 2 'boys among the three selected children can materialise in the following three mutually disjoint cases:

Group No.	I	II	III
(i)	Girl	Boy	Boy
(ii)	Boy	Girl	Boy
(iii)	Boy	Boy	Girl

— 2M

Hence by addition theorem of probability,

Required probability = P (i) + P (ii) + P (iii) = 9/32 + 3 /32 + 1/ 32 = 13/32 — 2M

1B. A bag contains 10 gold coins and 8 silver coins. Two successive drawings of 4 coins are made such that

i) The coins are replaced before the second trial.
ii) The coins are not replaced before the second trial.

Find the probability that the first drawing will give 4 gold coins and second drawing will give 4 silver coins.

Solution:

Let A denote the event of drawing 4 gold coins in the first draw and B denote the event of drawing 4 silver coins in the second draw. Then we have to find the probability of P (A n B).

(i) Draws with replacement. If the coins drawn in the first draw are replaced back in the bag before the second' draw then the events A and B are independent and the required probability is P(AnB)=P(A).P(B)

1st draw. Four coins can be drawn out of 10+8=18 coins in 18C4 ways, which gives the exhaustive number of cases. Drawn 4 out of the 10 gold coins and this can be done in lOC4 ways. Hence, P (A) = lOC4 / 18C4 = 7/102 The probability of drawing 4 silver coins in the 2nd draw is given by P (B) = 8C4 / 18C4 = 7/ 306 — 1.5M

Is this nC_r = ?

P(AnB)= lOC4 X 8C4 / (18C4 X 18C4) = 49/ 31212 = 0.0016

(ii) Draws without replacement, P (B I A) = 8C4 / I4C4 = 10/143 = 0.07
P(AnB)= IOC4 X 8C4 / 18 C4 X14 C4 = 0.004 — 1.5M

1C. Suppose the pdf of a random variable X is given as

$$f(x) = \begin{cases} 4x - 4x^3, & 0 < x < 1 \\ 0, & otherwise \end{cases} .$$ Find

i . V(X)
ii. Mode of the distribution

Solution:

1. $V(X) = \frac{1}{3} - \frac{64}{225} = \frac{11}{225}$ — 1.5 M

2. Mode of the distribution is o.5774 = $\frac{1}{\sqrt{3}}$

Since $\frac{d\,f(x)}{dx}$ = 4 - 12 x^2 = 0 and $\frac{d^2\,f(x)}{dx^2}$ = - 24x < 0 — 1.5 M

2A. If X is normally distributed with mean μ and variance, σ^2 then show that $E(X-\mu)^{2n} = 1.3.5 \ldots . (2n-1)\sigma^{2n}$.

Solution:

Given $X \sim N(\mu, \sigma^2)$, $M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}$ — 1M $1\frac{1}{2}$

Let Y= $(X-\mu)$

$M_Y(t) = E\,(e^{ty}) = E(e^{t(x-\mu)})$

$= E(e^{tx})E(e^{-\mu t})$

$= M_X(t)\,E(e^{-\mu t}) = e^{\frac{t^2\sigma^2}{2}}$ — 1 M — $1\frac{1}{2}$

E(Y^{2n})=The coefficient of $t^{2n}/2n!$ in M_Y(t)

$= \frac{(2n)(2n-1)(2n-2)\ldots(3)(2)(1)}{(2n)(2n-2)(2n-4)\ldots(6)(4)(2)} * \sigma^{2n}$ — 1

$= 1.3.5 \ldots . (2n-1)\sigma^{2n}$. — 2M

2B. An aircraft knows that 5% of the people making reservation on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for the flight that can only hold 50 passengers. What is the probability that there will be a seat available for every passenger who turns up?

Solution: Given n = 52

Let X: number of passengers who won't turn up — 1M

p= passenger will not turn up= 0.05 , q =1-p = 0.95. Using Binomial distribution, P(X= K)= $nC_k\, p^k\, q^{n-k}$ — 0.5 M

Therefore, P($X \geq 2$)= 1- [P(X=0) + P(X=1)] — 1.5 M

= 0.7405

2C. If X_1, X_2, X_3 be uncorrelated random variables having same standard deviation. Find the correlation coefficient between $U = X_1 + X_2$ and $V = X_3 + X_2$.

Solution: : Given that , X_1, X_2, X_3 be uncorrelated random variables and having same standard deviation.

$\Rightarrow$ E((X_1X_2) = E(X_1) (E(X_2) , E((X_3X_2) = E(X_3) E(X_2) , E(X_1) (E(X_3) = (E(X_1X_3)

And V(X_2) = V(X_1) = V(X_3) — 1M

Let U = $X_1 + X_2$, V = $X_2 + X_3$

Consider $E(UV)$ - E(U) E(V) = $V(X_2)$ — 1M

V(U) V(V) = 4 $V(X_2)$

$\rho(U, V) = \frac{\text{Cov}(U, V)}{\sqrt{V(U)\,V(V)}} = \frac{V(X_2)}{2\,V(X_2)} = 0$ — 1 M

3A. Let X has pdf $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$ where $\theta > 0$. To test $H_0\colon \theta = 1$ against $H_1\colon \theta = 2$, a random sample (X_1, X_2) of size 2 is used with critical region $C = \left\{(x_1, x_2) \mid x_1 x_2 \geq \frac{3}{4}\right\}$. Compute the power function and significance level of the test.

Solution: The joint pdf is $g(x_1, x_2; \theta) = \theta^2 (x_1 x_2)^{\theta-1}$, $0 < x_1, x_2 < 1$. – 1M — 1½

The power function is

$$\begin{aligned} K(\theta) &= P\left[X_1 X_2 \geq \frac{3}{4}\right] \\ &= \int_{3/4}^{1} \int_{3/4x_1}^{1} \theta^2 (x_1 x_2)^{\theta-1}\, dx_2\, dx_1 \\ &= 1 - \left(\frac{3}{4}\right)^{\theta} \left(1 - \theta \log \frac{3}{4}\right). \end{aligned}$$

– 2M — 1½

The significance level is

$$\alpha = K(1) = \frac{1}{4} + \frac{3}{4} \log \frac{3}{4} \approx 0.034.$$

– 1M

3B. Four roads A, B, C and D lead away from a jail. A prisoner escaping from the jail selects a road at random. If road A is selected, the probability of escaping is $\frac{1}{8}$. Similarly, for road B it is $\frac{1}{6}$, for road C it is $\frac{1}{4}$ and for road D it is $\frac{4}{5}$. What is the probability that the prisoner will succeed in escaping?

Solution: Let A: A road lead away from jail

B: B road lead away from jail

C: C road lead away from jail

D: D road lead away from jail

– 1M

P(A) = P(B) = P(C) = P(D) = 0.25

X: The event that the prisoner will succeed in escaping

Given that, P(X|A) = 1/8, P(X|B) = 1/6, P(X|C) = 1/4, P(X|D) = 4/5 .

1M

Using Total probability theorem, $P(X) = \frac{1}{4}\{\ 1/8 + 1/6 + 1/4 + 4/5\} = \frac{161}{480}$ – 1M

3C. Suppose that an electronic device has a life length X which satisfies Gamma Distribution with mean 1 and variance 2. Then, determine $P(X \geq 0)$.

Solution : Gamma random variable Mean and variance are 1 and 2 respectively,

$\frac{r}{\alpha} = 1, \frac{r}{\alpha^2} = 2$

$\Rightarrow r = \alpha = \frac{1}{2}.$

So, $f(x) = \frac{1}{\sqrt{2\pi}} x^{\frac{-1}{2}} e^{-\frac{x}{2}}, x > 0$

– 1M — 1½

$P(X > 0) = \int_0^\infty f(x)\, dx \quad \int_0^\infty \frac{1}{\sqrt{2\pi}} x^{\frac{-1}{2}} e^{-\frac{x}{2}}\, dx \Rightarrow \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2})$ – 2M — 1½

$$\Rightarrow \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

4A. Two production lines manufacture a certain type of item. Suppose that the capacity (on any given day) is 5 items foe line I and 3 item for line II. Let (X, Y) represent the two-dimensional random variable yielding the number of items produced by line I and line II, respectively.

Y \ X	0	1	2	3	4	5
0	0	0.01	0.03	0.05	0.07	0.09
1	0.01	0.02	0.04	0.05	0.06	0.08
2	0.01	0.03	0.05	0.05	0.05	0.06
3	0.01	0.02	0.04	0.06	0.06	0.05

Find the probability that more items are produced by line I than line II.

Solution:

Thus $p(2, 3) = P(X = 2, Y = 3) = 0.04$, etc. Hence if B is defined as

$$B = \{\text{More items are produced by line I than by line II}\}$$ —0.5M

we find that

$$P(B) = 0.01 + 0.03 + 0.05 + 0.07 + 0.09 + 0.04 + 0.05 + 0.06 + 0.08 + 0.05 + 0.05 + 0.06 + 0.06 + 0.05$$ } 3M

$$= 0.75.$$ — 0.5M

4B. An insurance company has discovered that only about 0.1% of the population is limited in a certain type of accidents each year. If its 10000 policy holders were randomly selected from the population. What is the probability that not more than 5 of the clients are involved in such accidents each year.

Solution: Let X: number of clients involved in accidents

Given p=0.001, n=10000, } — 1M —— 1½

α = np= 10

Therefore, P($X \geq 5$) = 0.8686 using Poisson distribution. } –2M —— 1½

4C. The monthly income of a group of 10,000 person were found to be normally distributed with mean 750 rupees and SD rupees 50. Show that of this group about 95% had income exceeding rupees 668 and only 5% had income exceeding rupees 832. What was the lowest income among the richest 100?

Solution: Let X: Monthly income of a group

To Show P(X>668) =0.95 and P(X>832) =0.05 —— 1.5M

To find "C" such that P(X> μ+C) = 100/10000 =0.01

C= 116.5 .Therefore, lowest income among the richest 100 = $\mu + C$= Rupees 866.5

1.5M

5A. A die was rolled $n = 120$ times independently and the following data was obtained:

Outcome	1	2	3	4	5	6
Frequency	m	20	20	20	20	20

For what values of m would the hypothesis that the die is unbiased be rejected at 0.025 significance level in a chi-square test?

Solution:

Here, $k = 6, n = 120$, and $np_1 = np_2 = \cdots = np_6 = 20$. Hence

$$Q_5 = \frac{(m-20)^2}{20} + 5 \times \frac{(20-20)^2}{20} = \frac{(m-20)^2}{20}.$$ — 1.5M

To reject the hypothesis, Q_5 should be greater than $\chi_5^2(0.025) = 12.833$. — 1M

$$\frac{(m-20)^2}{20} > 12.833 \Rightarrow m < 3.97 \text{ or } m > 36.02.$$ — 1M

Thus, the hypothesis would be rejected if $m < 3$ or $m > 36$. — 0.5 M

5B. Let $(X_1, X_2, \ldots, X_n)$ be a random sample of size n from a distribution whose pdf is given by

$$f(x;\theta) = \begin{cases} \theta^x(1-\theta)^{1-x} & x = 0,1,2,\ldots \, and \; 0 \le \theta \le 1 \\ 0 & otherwise \end{cases}.$$

Find the maximum likelihood estimator for θ

Solution: Let $L = \prod_{i=1}^{n} f(x_i;\theta) = \theta^{\sum_{i=1}^{n} x_i}(1-\theta)^{n-\sum_{i=1}^{n} x_i}$ — 1M

$$\log L = \sum_{i=1}^{n} x_i \log\theta + \left(n - \sum_{i=1}^{n} x_i\right)\log(1-\theta)$$

$$\frac{\partial \log L}{\partial \theta} = \frac{\sum_{i=1}^{n} x_i}{\theta} + \frac{n}{1-\theta} + \frac{\sum_{i=1}^{n} x_i}{1-\theta} = 0$$

1.5 M

$$Thus, \theta = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}$$ — 0.5 M.

5C. The mean life length of a certain cutting tool is 41.5 hours with a standard deviation of 2.5 hours. What is the probability that a random sample of size 50 drawn from this population will have a sample mean between 40.5 and 42 hours.

Solution: Given $\mu = 41.5, \sigma = 2.5$ and $n = 50$, Using central limit theorem,

$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ — 1M — 1/2

Required probability is $P(40.5 < \bar{X} < 42) = P\left(\frac{40.5-\mu}{\frac{\sigma}{\sqrt{n}}} < \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} < \frac{42-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$

$= P(-2.83 < Z < 1.42)$ — 1M

$= \phi(1.42) - \phi(-2.83)$ (From normal distribution table) — 1/2

$= \mathbf{0.9199}$. — 1M