Exam Date & Time: 23-Mar-2024 (10:45 AM - 12:45 PM)

MANIPAL ACADEMY OF HIGHER EDUCATION

PROBABILITY AND OPTIMIZATION [MAT 2233]

Marks: 30 **Duration: 120 mins.**

MCQ

Answer all the questions. Section Duration: 20 mins

1) *A man has 7 different pets and wishes to photograph them 3 at a time arranged in a line. How many different arrangements are possible?*

1)	21	2)	35	3)	210	4)	840

(0.5)

Correct option is: 3

2) *In how many ways a committee of 3 people consisting of atleast one boy can be formed from a class of 12 boys and 10 girls?*

1)	120	2)	540	3)	1420	4)	1540

(0.5)

Correct option is: 3

3) *How many integers from 1 to 520 that are neither divisible by 4 nor by 5?*

1)	187	2)	312	3)	234	4)	208

(0.5)

Correct option is: 2

4) *The coefficient of* x^{16} *in* $(x^2+x^3+x^4+\ldots)^5$ *is*

1)	210	2)	462	3)	120	4)	252

(0.5)

Correct option is: 1

5) *A bag contains 8 red beads, 14 green beads and x blue beads. If a bead is picked at random from the bag, the probability of picking a green bead is 1/3. Find the value of x.*

1)	20	2)	22	3)	23	4)	25

(0.5)

Correct option is: 1

6) *Consider the event that two dice are thrown and the numbers on their top faces are recorded. Let X be a random variable representing the sum of the recorded numbers. What is* $P(X=5)$ *equal to?*

1)	$\frac{1}{9}$	2)	$\frac{1}{5}$	3)	$\frac{1}{7}$	4)	$\frac{1}{11}$

(0.5)

Correct option is: 1

7) *A continuous random variable X has probability density function*

$$f(x) = \begin{cases} 3x^2, & 0 \le x \le 1 \\ 0 & elsewhere \end{cases} \text{. Find } a \text{ such that } P(X \le a) = P(X > a).$$ (0.5)

1)	$\sqrt[3]{\frac{1}{2}}$		2)	$\frac{1}{2}$		3)	$\sqrt{\frac{1}{2}}$		4)	$\sqrt{2}$	

Correct option is: 1

8) *(X,Y) is a two − dimensional random variable with joint pdf f(x,y) and g(x) and h(y) are the marginal pdf's of X and Y respectively. If X and Y are independent then* $f(x,y) =$

1)	0		2)	$g(x).h(y)$		3)	$\frac{g(x)}{h(y)}$		4)	$g(x)+h(y)$	

(0.5)

Correct option is: 2

9) *Given* $P(A) = \frac{1}{3}$ *and* $P(A \cup B) = \frac{1}{2}$. *If A and B are independent then* $P(A^c / B^c)$ *is*

1)	$\frac{1}{3}$		2)	$\frac{1}{6}$		3)	$\frac{3}{4}$		4)	$\frac{2}{3}$	

(0.5)

Correct option is: 4

10) *In a class, 25% of the students have failed in Mathematics, 15% of students failed in Chemistry and 10% failed in both Mathematics and Chemistry. A student is selected at random. If he failed in Chemistry then the probability that he failed in Mathematics is*

1)	$\frac{2}{5}$		2)	$\frac{2}{3}$		3)	$\frac{1}{4}$		4)	$\frac{3}{10}$	

(0.5)

Correct option is: 2

11) *A bag contains 10 White and 5 blue balls. A ball is drawn from the bag at random and its colour is noted. This ball is put back in the bag along with the three more balls of the same colour. Then a ball is drawn again from the bag and is found to be blue. What is the probability that the first ball drawn is blue?* (4)

(11) B_1: Picking a blue ball from the bag

B_2: Picking a white ball from the bag

A : Picking a blue ball second time after putting 4 balls to the bag.

$P(B_1) = 1/3, \quad P(B_2) = 2/3 \; ; \; P(A|B_1) = 8/18, \; P(A|B_2) = 5/18$ — 02

$\therefore P(B_1|A) = \dfrac{P(A|B_1)P(B_1)}{\sum_{i=1}^{2} P(A|B_i)P(B_i)} = 4/9.$ — 01

12) *Suppose that the two dimensional random variable(X,Y) has a joint pdf given by*

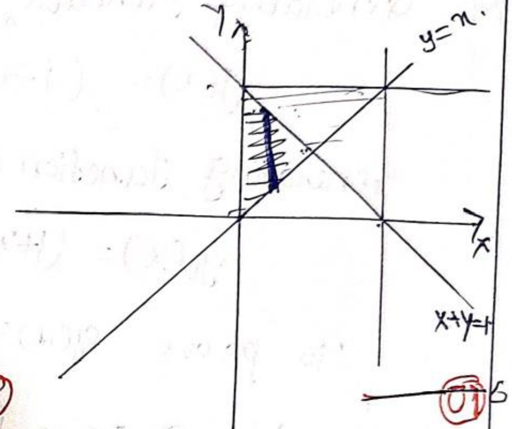
$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1 \\ 0 & \text{elsewhere} \end{cases}$. *Find* $P(x + Y \le 1)$. (4)

(12)

$P(X+Y \le 1)$

$= P(Y \le 1-X)$

$= \int_{x=0}^{1/2} \int_{y=x}^{1-x} 10xy^2 \, dy \, dx$ — 01



— 01

$= 10 \int_0^{1/2} \frac{x}{3} \left[(1-x)^3 - x^3 \right] dx$ — 1

$= \frac{10}{3} \times \frac{17}{320} = 11/96$ — 1

13) *In a bakery 4 types of cookies are available. Using generating function find the number of ways a person can buy 10 cookies if he decides to take at least one cookie of each variety?* (3)

(13.) ~~Nos~~ Generating function is $(x+x^2+\cdots+x^7)^4 = x^4(1+x+x^2+\cdots+x^6)^4$ — 1

No. of ways of buying 10 cookies with atleast one cookie of each variety = coefficient of x^6 in $(1+x+x^2+\cdots+x^6)^4$ — 1

$$= {}^9C_3 = \underline{84}. \quad — 1$$

14)

Six distinct symbols are transmitted through a communication channel. A total of twelve blanks are to be inserted between the symbols with atleast two blanks between every pair of symbols. In how many ways can we arrange the symbols and blanks? (3)

(14) No. of ways to arrange 6 distinct symbols $= 6! = 720$ — 1

Place 2 blanks between each symbol, remaining blanks $= 2$.

No. of ways to place 2 blanks in any of 5 places are — 1

$$ {}^{5+2-1}C_2 = 15.$$ — 1

$\therefore$ The possible arrangements $= 720 \times 15 = \underline{10800}$

15)

Show that the number of partitions of 'n' in which every part is odd is equal to the number of partitions of n with unequal parts. (3)

Generating function of n with odd no. of partition is

$$g(x) = (1-x)^{-1}(1-x^3)^{-1}(1-x^5)^{-1}\cdots$$ — 01

Generating function of n with unequal parts is

$$g_1(x) = (1+x)(1+x^2)(1+x^3)\cdots$$ — 01

To prove $g(x) = g_1(x)$

consider $g_1(x) = (1+x)(1+x^2)(1+x^3)\cdots$

$$= \frac{(1+x)(1-x)}{(1-x)} \frac{(1+x^2)(1-x^2)}{(1-x^2)} \cdots$$

$$= \frac{(1-x^2)(1-x^4)(1-x^6)}{(1-x)(1-x^2)(1-x^3)} \cdots$$

$$= (1-x)^{-1}(1-x^3)^{-1}\cdots = g(x)$$ 01

16) (3)

A coin is known to come up heads three times as often as tail. This coin is tossed three times. Let X be the number of heads that appear. Find the probability distribution of X and also the cumulative distribution function F(x).

$P\{\text{Head}\} = 3/4$; $P\{\text{tail}\} = 1/4$. $0\frac{1}{2}$

Let X : No. of heads.

$R_X = \{0, 1, 2, 3\}$.

$P(X=0) = P\{TTT\} = 1/64$

$P(X=1) = P\{HTT\} + P\{THT\} + P\{TTH\} = 9/64$

$P(X=2) = P\{HHT\} + P\{THH\} + P\{HTH\} = 27/64$ $1\frac{1}{2}$

$P(X=3) = P\{HHH\} = 27/64$

cdf $F(x) = \begin{cases} 0, & x < 0 \\ 1/64, & 0 \le x < 1 \\ 10/64, & 1 \le x < 2 \\ 37/64, & 2 \le x < 3 \\ 1, & x \geq 3 \end{cases}$ — (1)

17) Let (X,Y) be a two dimensional continuous random variable with joint pdf $f(x, y) = \dfrac{1}{\text{Area of } R}$ where $R = \{(x,y) / x^2 + y^2 \le 1, y \ge 0\}$. Find the marginal probability distributions of X and Y. (3)

$f(x) = \frac{2}{\pi}, \ (x, y) \in R.$ — (1)

$g(x) = \int_0^{\sqrt{1-x^2}} \frac{2}{\pi} dy = \frac{4}{\pi} \cdot \sqrt{1-x^2}, \ -1 \le x \le 1.$ — (1)

$h(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi} dx = \frac{4}{\pi} \sqrt{1-y^2}, \ 0 \le y \le 1.$ — (1)

18) Given $f(x) = \begin{cases} kx(1-x), & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$.

Find (a) the value of k. (2)

(b) Find the cumulative distribution function $F(x)$.

18.

(a) $k\int_0^1 x(1-x)\,dx = 1$.

$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow k = 6$.

$\therefore f(x) = \begin{cases} 6x(1-x), & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$

$1\frac{1}{2}$

(b) $F(x) = \begin{cases} 0, & x<0 \\ 3x^2-2x^3 & 0<x<1 \\ 1 & x\geq 1. \end{cases}$

$1\frac{1}{2}$