EVALUATION SCHEME – MAT 2226 END-SEMESTER EXAMINATION (2024)

1A. In a shooting test, the probability of hitting the target is 0.5 for A_1, $\frac{1}{3}$ for A_2, and 0.25 for A_3. If all of them fire at the target, find the probability that (4M)

(i) at least two of them hit the target.

(ii) at most one of them hits the target.

Ans. (i) $\mathrm{P}\left[A_1\left(A_2\overline{A_3} \cup A_3\right) \cup \overline{A_1}A_2A_3\right] = \frac{1}{2}\left(\frac{1}{3} \times \frac{3}{4} + \frac{1}{4}\right) + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \boxed{\frac{7}{24}}$ — 3

(ii) $\mathrm{P}\left[\text{At most one of them hits the target}\right] = 1 - \frac{7}{24} = \boxed{\frac{17}{24}}$ — 1

1B. Two absent-minded room-mates, A and B, forget their umbrellas in one way or another. A always takes his umbrella when he goes out, while B forgets to take his umbrella with probability 0.5. Each of them forgets his umbrella at a shop with probability 0.25. After visiting 3 shops, they return home. Find the probability that (3M)

(i) they have only one umbrella.

(ii) B lost his umbrella, given that there is only one umbrella after their return.

Ans. Let X and Y be the events that A and B, respectively, have umbrellas after returning home.
$\mathrm{P}[X] = \left(\frac{3}{4}\right)^3$ (or $\frac{27}{64} \approx 0.42$). — 1/2
$\mathrm{P}[Y] = \frac{1}{2} + \frac{1}{2} \times \left(\frac{3}{4}\right)^3$ (or $\frac{91}{128} \approx 0.71$) . — 1/2

(i) $\mathrm{P}[\text{Only one umbrella after return}] = P[X\overline{Y} + \overline{X}Y] = \boxed{\frac{2183}{4096}}$ (or 0.53) — 1

(ii) $\mathrm{P}[\text{B lost umbrella} \mid \text{Only one umbrella}] = \frac{\mathrm{P}[X\overline{Y}]}{\mathrm{P}[\text{Only one umbrella}]} = \boxed{\frac{27}{118}}$ (or 0.22) — 1

1C. A random variable has the CDF $F(x) = \begin{cases} 1 - 2e^{-2x}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$ (3M)

Find

(i) the probability density function (pdf)

(ii) $\mathrm{P}[X > 2]$

(iii) $\mathrm{P}[-3 < X < 4]$

Ans. (a) $f(x) = F'(x) = \boxed{2e^{-2x},\ x \geq 0}$. — 1

(b) $\mathrm{P}[X > 2] = 1 - F(2) = \boxed{e^{-4}}$ (or 0.018). — 1

(c) $\mathrm{P}[-3 < X < 4] = F(4) - F(-3) = \boxed{1 - e^{-8}}$ (or 0.99). — 1

2A. (a) The observed values of a random sample from a population having distribution $N(8, \sigma^2)$ are 8.6, 7.9, 8.3, 6.4, 8.4, 9.8, 7.2, 7.8, and 7.5. Construct a 95 percent confidence interval for σ^2. (4M)

(b) A random sample of size 9 from the distribution $N(\mu, \sigma^2)$ yields $s^2 = 7.63$. Determine a 95 percent confidence interval for μ.

Ans. (a) Here, $n = 9$, $\mu = 8$, and $p = 0.95$. — 1/2

$\mathrm{P}[Z < a] = \mathrm{P}[Z > b] = \frac{1-p}{2} = 0.025$, $Z \sim \chi^2_9$. — 1/2

Then $a = 2.7$ and $b = 19.023$. — 1/2

From the given data, $\sum(x_i - \mu)^2 = 7.35$. — 1/2

Thus, the confidence interval for σ^2 is $\boxed{(0.38, 2.72)}$. — 1/2

(b) Here, $n = 9$, $s^2 = 7.63$, and $p = 0.95$. — 1/2

With $T \sim t_8$, $\mathrm{P}[-b < T < b] = 0.95 \implies \mathrm{P}[T < b] = 0.975$. — 1/2

Then $b = \boxed{2.306}$, and hence $\frac{bs}{\sqrt{n-1}} = \boxed{2.25}$. — 1/2

Thus, the confidence interval for μ is $\boxed{(\overline{x} - 2.25, \overline{x} + 2.25)}$.

Up to full marks can be given for other correct steps even if a specific value of $\overline{x}$ is used.

2B. In playing with an opponent of equal ability, which is more probable in each of the following? (3M)

(i) Winning 2 games out of 4 OR 5 games out of 8

(ii) Winning at least 2 out of 4 OR at least 5 out of 8

Ans. The probability of winning exactly k games out of n is $\binom{n}{k}(\frac{1}{2})^n$. — 1

(a) Since $\binom{4}{2}(\frac{1}{2})^4 = 0.37 > 0.21 = \binom{8}{5}(\frac{1}{2})^8$, winning $\boxed{\text{2 games out of 4}}$ is more likely. — 1

(b) Since $[\binom{4}{2} + \binom{4}{3} + \binom{4}{4}](\frac{1}{2})^4 = 0.68 > 0.36 = [\binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}](\frac{1}{2})^8$, winning $\boxed{\text{at least 2 games out of 4}}$ is more likely. — 1

2C. Let X have a pdf of the form $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, where $\theta = 1, 2$. To test the simple hypothesis $H_0: \theta = 1$ against the alternative hypothesis $H_1: \theta = 2$, use a random sample (X_1, X_2) of size $n = 2$ and define the critical region to be $C = \left\{(x_1, x_2) \mid \frac{3}{4} \leq x_1 x_2\right\}$. Find the power function and the significance level of the test. (3M)

Ans. The power function is

$$K(\theta) = \mathrm{P}\left(X_1 X_2 \geq \frac{3}{4}\right) \quad — 1$$

$$= \int_{3/4}^{1} \int_{3/4x_1}^{1} \theta^2 (x_1 x_2)^{\theta-1}\, dx_2\, dx_1 \quad — 1/2$$

$$= \boxed{1 - \left(\frac{3}{4}\right)^{\theta}\left[1 - \theta \log \frac{3}{4}\right]}. \quad — 1/2$$

The significance level is

$$\alpha = K(1) \approx \boxed{0.034}. \quad — 1$$

3A. Let X and Y have the joint pdf $f(x, y) = 15x^2y$, $0 < x < y < 1$. Find the marginal pdfs of X and Y, and compute $\mathrm{P}[X + Y \le 1]$. (4M)

Ans. The marginal pdfs are

$$\begin{aligned} f_1(x) &= \int_x^1 15x^2y\,dy & \text{— } 1/2 \\ &= \boxed{\frac{15}{2}x^2(1-x^2), \quad 0 < x < 1}, & \text{— } 1 \\ f_2(y) &= \int_0^y 15x^2y\,dx & \text{— } 1/2 \\ &= \boxed{5y^4, \quad 0 < y < 1}. & \text{— } 1 \end{aligned}$$

$$\begin{aligned} \mathrm{P}[X + Y \le 1] &= \int_0^{1/2}\int_x^{1-x} 15x^2y\,dy\,dx & \text{— } 1/2 \\ &= \boxed{\frac{5}{64}}. & \text{— } 1/2 \end{aligned}$$

3B. In a courier company, three office assistants are assigned to process incoming mails. The first assistant, A_1, processes 40%, the second assistant, A_2, process 35%, and the third assistant, A_3, processes 25% of the mails. The first assistant has an error rate of 0.04, the second has an error rate of 0.06, and the third has an error rate of 0.03. A mail selected at random is found to have an error. The manager of the company wishes to know the probability that the mail was processed by the first assistant. (3M)

Ans. $\mathrm{P}(A_1) = 0.4$, $\mathrm{P}(A_2) = 0.35$, $\mathrm{P}(A_3) = 0.25$. — 1/2
$\mathrm{P}(E \mid A_1) = 0.04$, $\mathrm{P}(E \mid A_2) = 0.06$, $\mathrm{P}(E \mid A_3) = 0.03$. — 1

$$\begin{aligned} \mathrm{P}(A_1 \mid E) &= \frac{0.4 \times 0.04}{0.4 \times 0.04 + 0.35 \times 0.06 + 0.25 \times 0.03} & \text{— } 1 \\ &= \boxed{0.36}. & \text{— } 1/2 \end{aligned}$$

3C. If the random variable X has a Gamma distribution with parameters r and α, then derive the mean and variance of X. (3M)

Ans. The pdf of X is $f(x) = \frac{\alpha}{\Gamma(r)}(\alpha x)^{r-1}e^{-\alpha x}$, $x \ge 0$. — 1/2

$$\begin{aligned} \mathbb{E}[X] &= \frac{\alpha^r}{\Gamma(r)}\int_0^\infty x^r e^{-\alpha x}\,dx & \text{— } 1/2 \\ &= \frac{\alpha^r}{\Gamma(r)} \times \frac{\Gamma(r)}{\alpha^{r+1}} \\ &= \boxed{\frac{r}{\alpha}} & \text{— } 1/2 \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[X^2] &= \frac{\alpha^r}{\Gamma(r)} \int_0^\infty x^{r+1} e^{-\alpha x}\, dx && \text{— } 1/2 \\
&= \frac{\alpha^r}{\Gamma(r)} \times \frac{\Gamma(r+1)}{\alpha^{r+1}} \\
&= \frac{r^2+r}{\alpha^2} && \text{— } 1/2 \\
\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 &= \boxed{\frac{r}{\alpha^2}}. && \text{— } 1/2
\end{aligned}
$$

Alternatively, these can be obtained from the MGF after first deriving the MGF.

4A. A fair coin is tossed 3 times. Let X be 0 or 1 according to whether H or T is obtained on the first toss. Let Y be the number of heads. Find the joint probability distribution of X and Y, expectation of X, and expectation of Y. (4M)

Ans. The joint pmf of (X, Y) is

X \ Y	0	1	2	3
0	0	1/8	1/4	1/8
1	1/8	1/4	1/8	0

— 2

The expectations are

$$E[X] = \sum_x \sum_y x f(x,y) = \boxed{\frac{1}{2}}. \quad \text{— } 1$$

$$E[Y] = \sum_x \sum_y y f(x,y) = \boxed{\frac{3}{2}}. \quad \text{— } 1$$

The expectations can also be found using the marginal pmfs.

4B. A pot has 10% defective items. What should be the number of items selected from the pot, such that the probability of finding at least 1 defective item in the selection is at least 0.95? (3M)

Let n items be selected from the pot. Then
P[At least one defective] $= 1 - (0.9)^n$. — $1^1/2$
The least n such that $1 - (0.9)^n \geq 0.95$ is $\left\lceil \frac{\log 0.05}{\log 0.9} \right\rceil = \boxed{29}$. — $1^1/2$

4C. The heights of 500 soldiers are found to have a normal distribution. Of them, 258 are found to be within 2 cm of the mean height of 170 cm. Find the standard deviation of the distribution. (3M)

Ans. Let X be the height of a soldier selected at random. Then $X \sim N(170, \sigma^2)$. — 1/2

$$P[|X-170| < 2] = \frac{258}{500} \quad — 1/2$$

$$P\left[-\frac{2}{\sigma} < Z < \frac{2}{\sigma}\right] = 0.516 \quad — 1/2$$

$$\Phi\left(\frac{2}{\sigma}\right) = 0.758 \quad — 1/2$$

$$\frac{2}{\sigma} = 0.7 \quad — 1/2$$

Hence the standard deviation is $\boxed{\sigma = 2.85}$. — 1/2

5A. The Mendelian theory of genetics of crossing two types of peas states that the probabilities of classification of the four resulting types are $\frac{1}{16}$, $\frac{3}{16}$, $\frac{3}{16}$, and $\frac{9}{16}$. From 160 independent observations, the observed frequencies of these classifications are $14, 35, 26, 85$ respectively. Test the consistency of the data with the theory at 0.01 level of significance. (4M)

Ans. Here, $k=4$, $n=160$, and $np_1 = 10$, $np_2 = 30$, $np_3 = 30$, $np_4 = 90$. — 1

$$Q_3 = \frac{(14-10)^2}{10} + \frac{(35-30)^2}{30} + \frac{(26-30)^2}{30} + \frac{(85-90)^2}{90} \quad — 1$$

$$= \boxed{3.24} < 11.345 = \chi_3^2(0.01) \quad — 1$$

Thus, we accept the hypothesis (or, the data is consistent). — 1

5B. If X and Y both follow exponential distributions with parameter 1, find the pdf of $U = X - Y$.

Ans. The joint pdf of (X,Y) is $f(x,y) = e^{-(x+y)}$, $x, y \geq 0$. — 1/2
Since $U = X - Y$, take $V = X + Y$. — 1/2
Then $X = \frac{V-U}{2}$, $Y = \frac{U+V}{2}$, and $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$. — 1/2
$x \geq 0 \implies v - u \geq 0 \implies v \geq u$, and $y \geq 0 \implies v + u \geq 0 \implies v \geq -u$. — 1/2
Thus, the joint pdf of (U,V) is $g(u,v) = \frac{1}{2}e^{-v}$, $v \geq |u|$. — 1/2
Hence, $g_1(u) = \int_{|u|}^{\infty} \frac{1}{2}e^{-v}\,dv = \boxed{\frac{1}{2}e^{-|u|}}$. — 1/2

5C. The mean life length of a certain cutting tool is 41.5, hours with a standard deviation of 2.5 hours. What is the probability that a random sample of size 50 drawn from this population will have a sample mean between 40.5 and 42 hours? (3M)

Ans. The life length $X \sim N(41.5, 2.5^2)$, and $n = 50$. — 1/2
Hence $\overline{X} \sim N(41.5, \frac{1}{8})$ $(\frac{\sigma}{\sqrt{n}} = \frac{1}{2\sqrt{2}})$. — 1/2

$$P\left[40.5 < \overline{X} < 42\right] = P[-2.82 < Z < 1.41] \quad — 1$$

$$= \Phi(1.41) + \Phi(2.82) - 1 \quad — 1/2$$

$$= \boxed{0.92}. \quad — 1/2$$