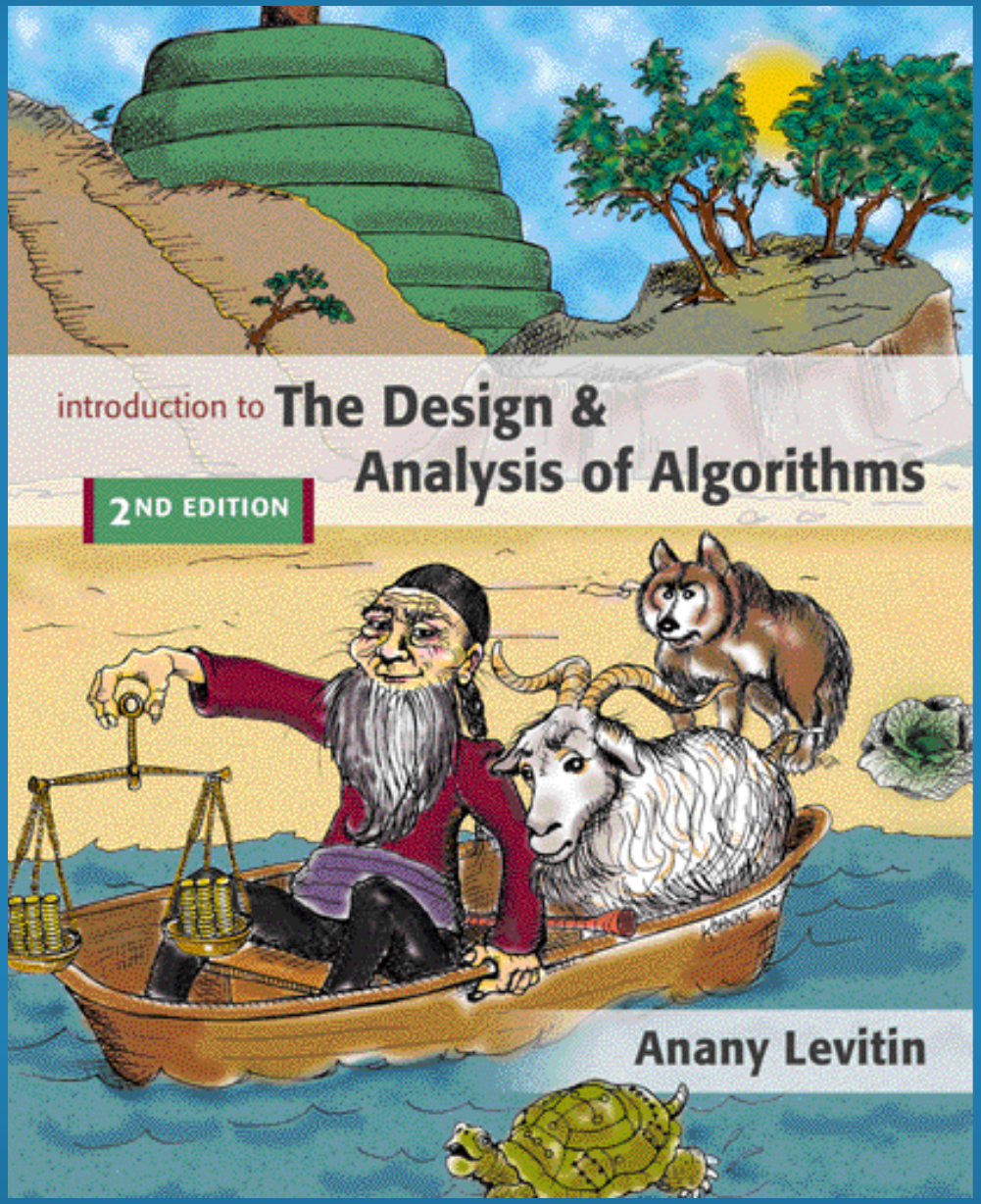
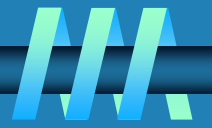
Chapter 3

Brute Force



Brute Force

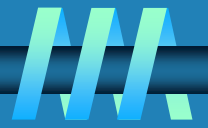


A straightforward approach, usually based directly on the problem's statement and definitions of the concepts involved

Examples:

1. **Computing a^n ($a > 0$, n a nonnegative integer)**
2. **Computing $n!$**
3. **Multiplying two matrices**
4. **Searching for a key of a given value in a list**

Brute-Force Sorting Algorithm



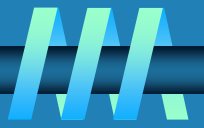
Selection Sort Scan the array to find its smallest element and swap it with the first element. Then, starting with the second element, scan the elements to the right of it to find the smallest among them and swap it with the second elements. Generally, on pass i ($0 \le i \le n-2$), find the smallest element in $A[i..n-1]$ and swap it with $A[i]$:

$$A[0] \le \ldots \le A[i-1] \mid A[i], \ldots, A[min], \ldots, A[n-1]$$

in their final positions

Example: 7 3 2 5

Analysis of Selection Sort



ALGORITHM *SelectionSort*$(A[0..n-1])$

```
//Sorts a given array by selection sort
//Input: An array A[0..n − 1] of orderable elements
//Output: Array A[0..n − 1] sorted in ascending order
for i ← 0 to n − 2 do
    min ← i
    for j ← i + 1 to n − 1 do
        if A[j] < A[min]   min ← j
    swap A[i] and A[min]
```

Time efficiency:

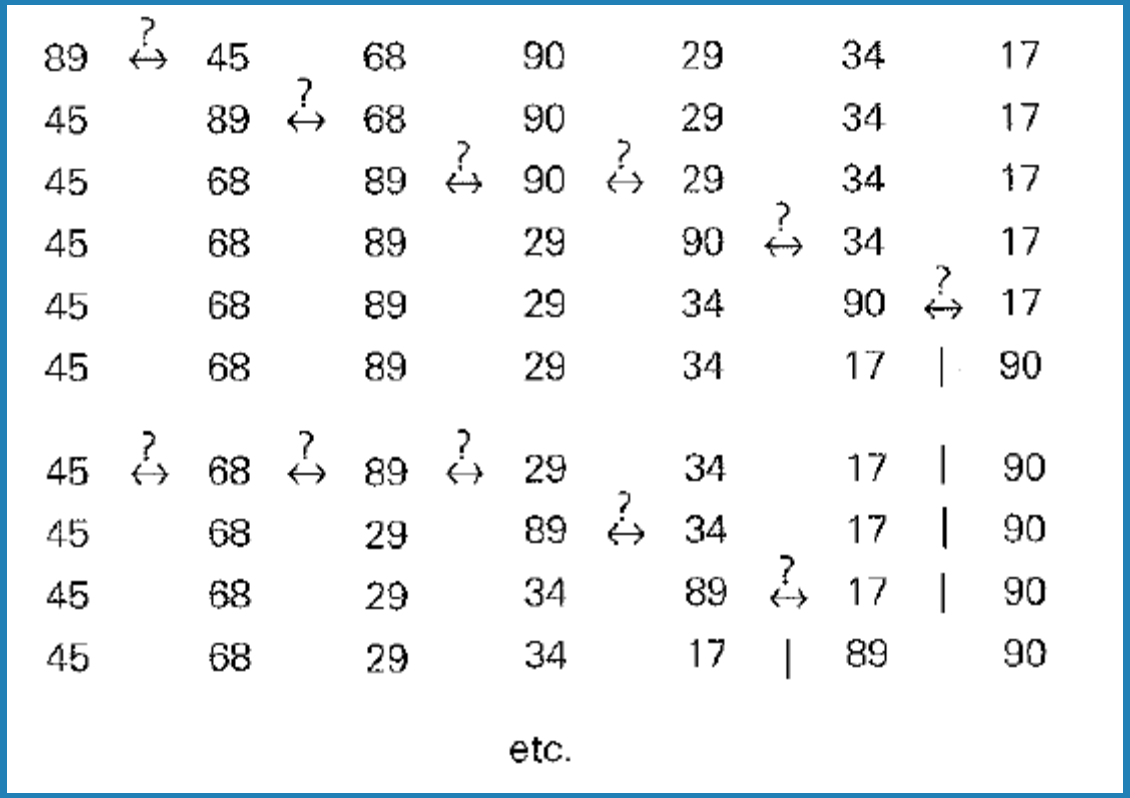
89	45	68	90	29	34	**17**
17	45	68	90	**29**	34	89
17	29	68	90	45	**34**	89
17	29	34	90	**45**	68	89
17	29	34	45	90	**68**	89
17	29	34	45	68	90	**89**
17	29	34	45	68	89	90

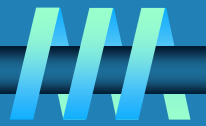
$$A_0, \ldots, A_j \overset{?}{\leftrightarrow} A_{j+1}, \ldots, A_{n-i-1} \mid A_{n-i} \leq \ldots \leq A_{n-1}$$
in their final positions

Here is a pseudocode of this algorithm.

ALGORITHM *BubbleSort*$(A[0..n-1])$

```
//Sorts a given array by bubble sort
//Input: An array A[0..n − 1] of orderable elements
//Output: Array A[0..n − 1] sorted in ascending order
for i ← 0 to n − 2 do
    for j ← 0 to n − 2 − i do
        if A[j + 1] < A[j]  swap A[j] and A[j + 1]
```

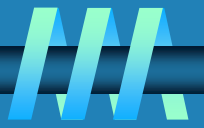




ALGORITHM *SequentialSearch2*$(A[0..n], K)$

//Implements sequential search with a search key as a sentinel
//Input: An array A of n elements and a search key K
//Output: The index of the first element in $A[0..n-1]$ whose value is
// equal to K or -1 if no such element is found
$A[n] \leftarrow K$
$i \leftarrow 0$
while $A[i] \neq K$ **do**
  $i \leftarrow i + 1$
if $i < n$ **return** i
else return -1

Brute-Force String Matching



- *pattern*: a string of *m* characters to search for
- *text*: a (longer) string of *n* characters to search in
- problem: find a substring in the text that matches the pattern

Brute-force algorithm

Step 1 Align pattern at beginning of text

Step 2 Moving from left to right, compare each character of pattern to the corresponding character in text until

- all characters are found to match (successful search); or
- a mismatch is detected

Step 3 While pattern is not found and the text is not yet exhausted, realign pattern one position to the right and repeat Step 2

Examples of Brute-Force String Matching

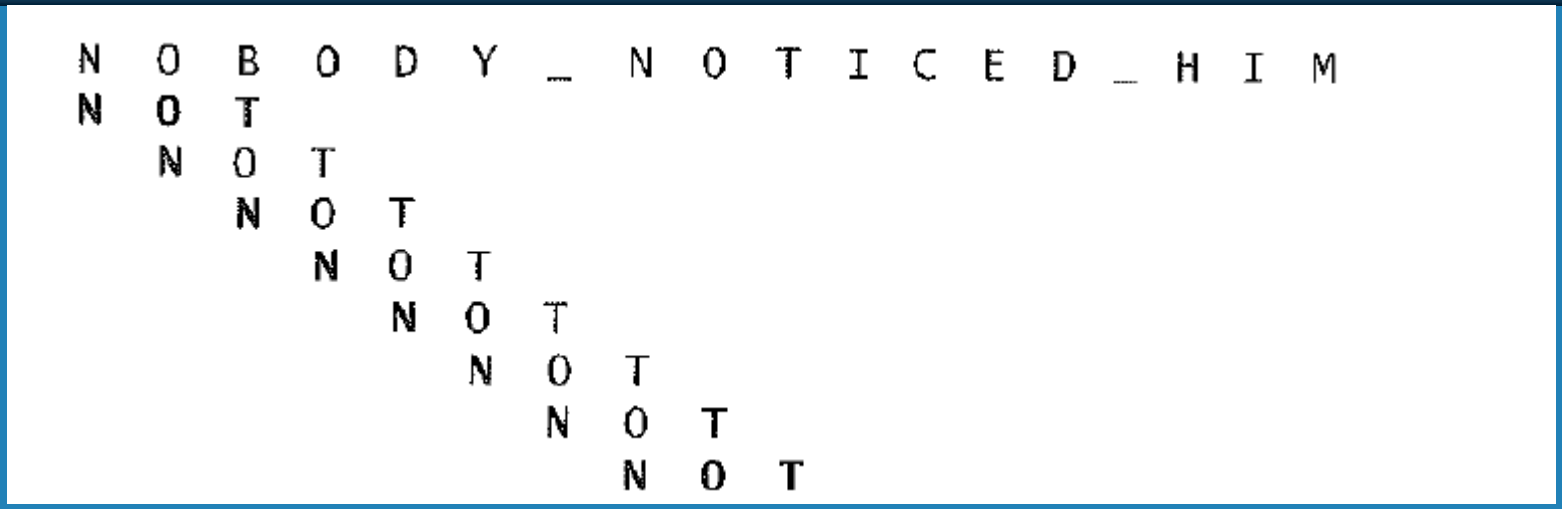
1. **Text:** 10010101101001100101111010
   Pattern: 001011

2. **Text:** It is never too late to have a happy childhood.
   Pattern: happy

Pseudocode and Efficiency

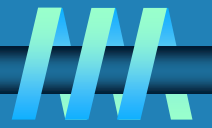
ALGORITHM *BruteForceStringMatch*$(T[0..n-1], P[0..m-1])$

```
//Implements brute-force string matching
//Input: An array T[0..n − 1] of n characters representing a text and
//       an array P[0..m − 1] of m characters representing a pattern
//Output: The index of the first character in the text that starts a
//        matching substring or −1 if the search is unsuccessful
for i ← 0 to n − m do
    j ← 0
    while j < m and P[j] = T[i + j] do
        j ← j + 1
    if j = m return i
return −1
```



Time efficiency: $\Theta(mn)$ **comparisons (in the worst case)**

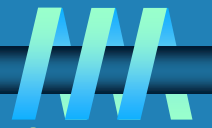
Brute-Force Strengths and Weaknesses



- **Strengths**
  - wide applicability
  - simplicity
  - yields reasonable algorithms for some important problems (e.g., matrix multiplication, sorting, searching, string matching)

- **Weaknesses**
  - rarely yields efficient algorithms
  - some brute-force algorithms are unacceptably slow

Exhaustive Search



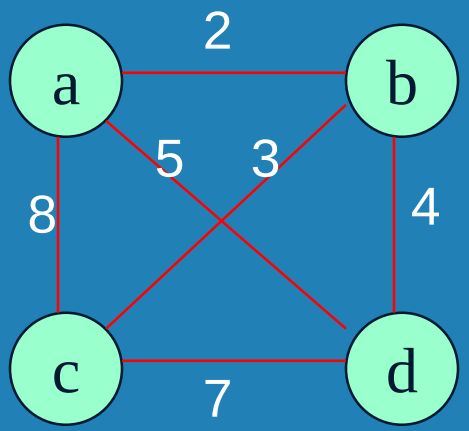
A brute force solution to a problem involving search for an element with a special property, usually among combinatorial objects such as permutations, combinations, or subsets of a set.

Method:

- **generate a list of all potential solutions to the problem in a systematic manner**
- **evaluate potential solutions one by one, disqualifying infeasible ones and, for an optimization problem, keeping track of the best one found so far**
- **when search ends, announce the solution(s) found**

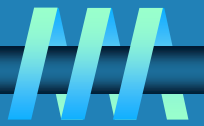
Example 1: Traveling Salesman Problem

- Given *n* cities with known distances between each pair, find the shortest tour that passes through all the cities exactly once before returning to the starting city
- Alternatively: Find shortest *Hamiltonian circuit* in a weighted connected graph
- Example:



How do we represent a solution (Hamiltonian circuit)?

TSP by Exhaustive Search

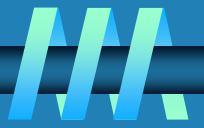


Tour	Cost
a→b→c→d→a	2+3+7+5 = 17
a→b→d→c→a	2+4+7+8 = 21
a→c→b→d→a	8+3+4+5 = 20
a→c→d→b→a	8+7+4+2 = 21
a→d→b→c→a	5+4+3+8 = 20
a→d→c→b→a	5+7+3+2 = 17

Efficiency: $\Theta((n-1)!)$

Chapter 5 discusses how to generate permutations fast.

Example 2: Knapsack Problem



Given *n* items:

- **weights:** w_1 w_2 … w_n
- **values:** v_1 v_2 … v_n
- **a knapsack of capacity *W***

Goal:

Find most valuable subset of the items that fit into the knapsack

Example: Knapsack capacity W=16

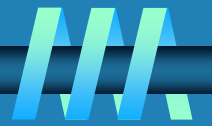
item	weight	value
1	2	$20
2	5	$30
3	10	$50
4	5	$10

Knapsack Problem by Exhaustive Search

Subset	Total weight	Total value
{1}	2	$20
{2}	5	$30
{3}	10	$50
{4}	5	$10
{1,2}	7	$50
{1,3}	12	$70
{1,4}	7	$30
{2,3}	15	$80
{2,4}	10	$40
{3,4}	15	$60
{1,2,3}	17	not feasible
{1,2,4}	12	$60
{1,3,4}	17	not feasible
{2,3,4}	20	not feasible
{1,2,3,4}	22	not feasible

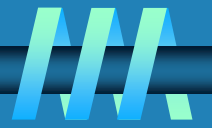
Efficiency: Θ(2^n)

Each subset can be represented by a binary string (bit vector, Ch 5).



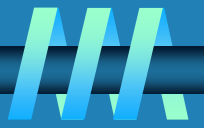
- Given n objects, the total number of subsets obtained will be 2^n .
- Note that Exhaustive search for TSP and knapsack are extremely inefficient on every input.

Example 3: Assignment problem



- Given n jobs and n persons, it is required to assign all n jobs to n persons with the constraint that one job has to be assigned to one person.

Example 3: The Assignment Problem



There are *n* people who need to be assigned to *n* jobs, one person per job. The cost of assigning person *i* to job *j* is C[*i*,*j*]. Find an assignment that minimizes the total cost.

	Job 1	Job2	Job3	Job 4
Person 1	9	2	7	8
Person 2	6	4	3	7
Person 3	5	8	1	8
Person 4	7	6	9	4

Algorithmic Plan: Generate all legitimate assignments, compute their costs, and select the cheapest one.

How many assignments are there? n!

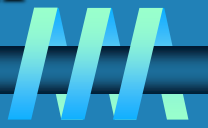
Assignment Problem by Exhaustive Search

$$C = \begin{matrix} 9 & 2 & 7 & 8 \\ 6 & 4 & 3 & 7 \\ 5 & 8 & 1 & 8 \\ 7 & 6 & 9 & 4 \end{matrix}$$

Assignment (col.#s)	Total Cost
1, 2, 3, 4	9+4+1+4=18
1, 2, 4, 3	9+4+8+9=30
1, 3, 2, 4	9+3+8+4=24
1, 3, 4, 2	9+3+8+6=26
1, 4, 2, 3	9+7+8+9=33
1, 4, 3, 2	9+7+1+6=23
	etc.

(For this particular instance, the optimal assignment can be found by exploiting the specific features of the number given. It is: 2,1,3,4)

Final Comments on Exhaustive Search

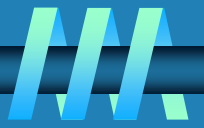


- Exhaustive-search algorithms run in a realistic amount of time only on very small instances

- In some cases, there are much better alternatives!
  - shortest paths
  - minimum spanning tree
  - assignment problem

- In many cases, exhaustive search or its variation is the only known way to get exact solution

Depth-First Search (DFS)

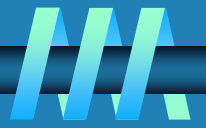


- Visits graph's vertices by always moving away from last visited vertex to an unvisited one, backtracks if no adjacent unvisited vertex is available.

- Recurisve or it uses a stack
  - a vertex is pushed onto the stack when it's reached for the first time
  - a vertex is popped off the stack when it becomes a dead end, i.e., when there is no adjacent unvisited vertex

- "Redraws" graph in tree-like fashion (with tree edges and back edges for undirected graph)

Pseudocode of DFS



ALGORITHM *DFS*(G)

//Implements a depth-first search traversal of a given graph
//Input: Graph $G = \langle V, E \rangle$
//Output: Graph G with its vertices marked with consecutive integers
//in the order they've been first encountered by the DFS traversal
mark each vertex in V with 0 as a mark of being "unvisited"
$count \leftarrow 0$
for each vertex v in V **do**
    if v is marked with 0
        dfs(v)

dfs(v)
//visits recursively all the unvisited vertices connected to vertex v by a path
//and numbers them in the order they are encountered
//via global variable *count*
$count \leftarrow count + 1$; mark v with *count*
for each vertex w in V adjacent to v **do**
    if w is marked with 0
        dfs(w)

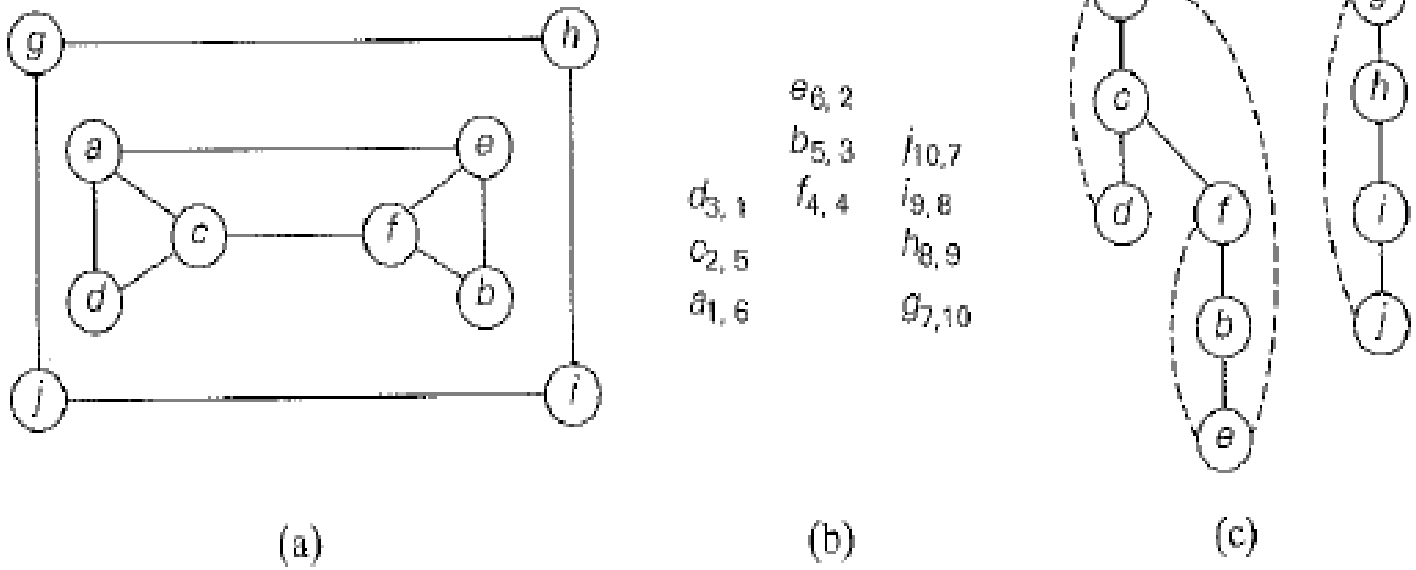
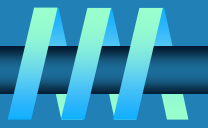


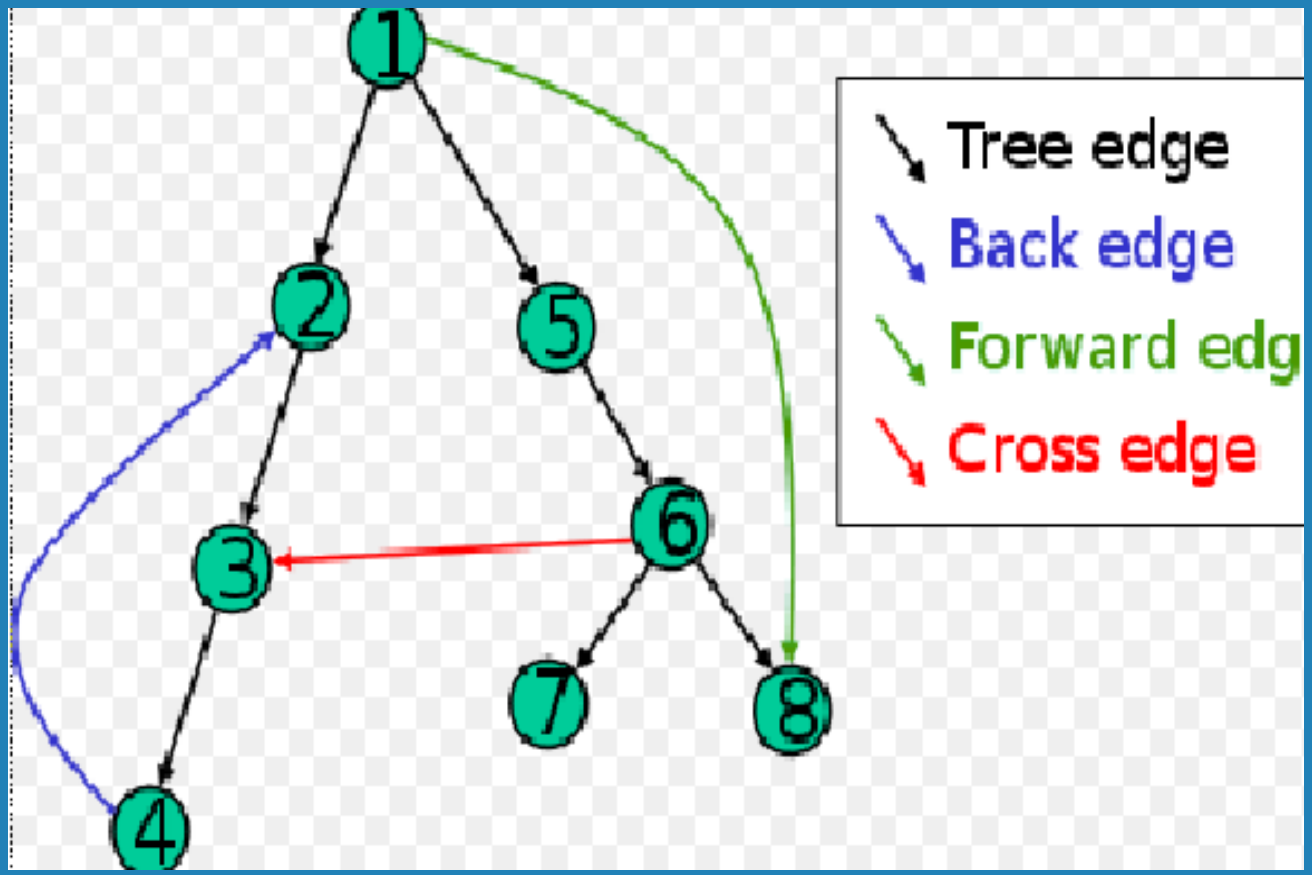
FIGURE 5.5 Example of a DFS traversal. (a) Graph. (b) Traversal's stack (the first subscript number indicates the order in which a vertex was visited, i.e., pushed onto the stack; the second one indicates the order in which it became a dead-end, i.e., popped off the stack). (c) DFS forest (with the tree edges shown with solid lines and the back edges shown with dashed lines).

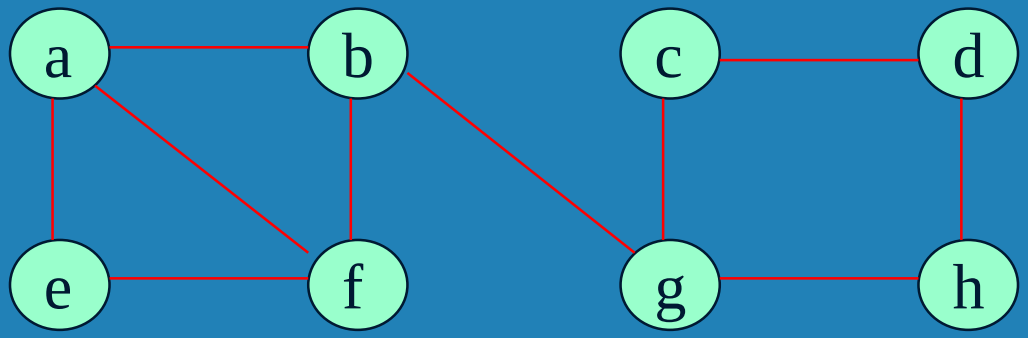
Notes on DFS



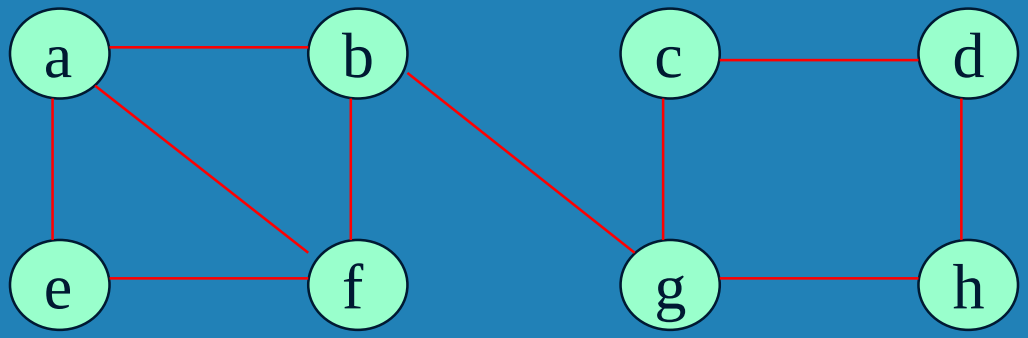
- DFS can be implemented with graphs represented as:
  - adjacency matrices: $\Theta(|V|^2)$.
  - adjacency lists: $\Theta(|V|+|E|)$.
- Yields two distinct ordering of vertices:
  - order in which vertices are first encountered (pushed onto stack)
  - order in which vertices become dead-ends (popped off stack)
- Applications:
  - checking connectivity, finding connected components
  - checking acyclicity (if no back edges)
  - finding articulation points

Contd…





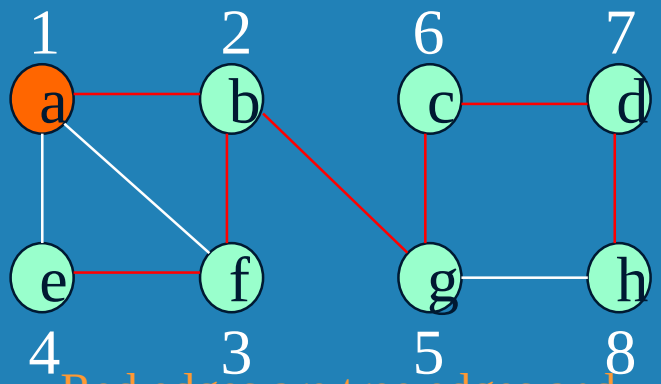
Example: DFS traversal of undirected graph



DFS traversal stack:

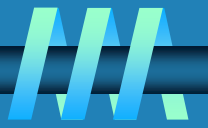
a
ab
abf
abfe
abf
ab
abg
abgc
abgcd
abgcdh
abgcd
:

DFS tree:



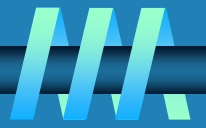
Red edges are tree edges and white edges are back edges.

Breadth-first search (BFS)



- Visits graph vertices by moving across to all the neighbors of the last visited vertex
- Instead of a stack, BFS uses a queue
- Similar to level-by-level tree traversal
- "Redraws" graph in tree-like fashion (with tree edges and cross edges for undirected graph)

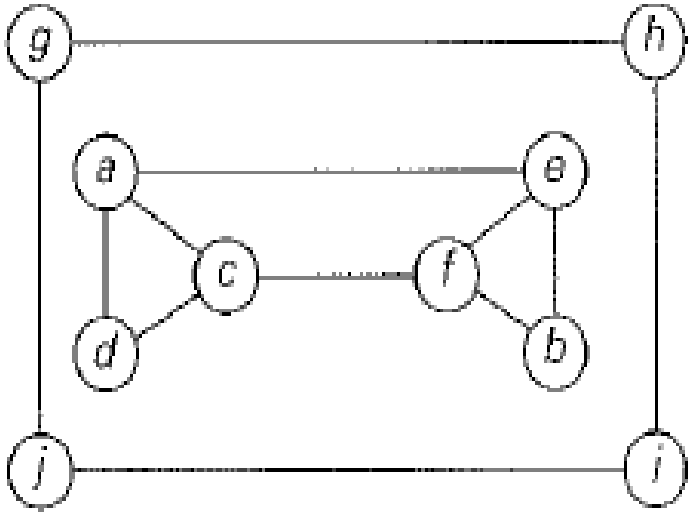
Pseudocode of BFS



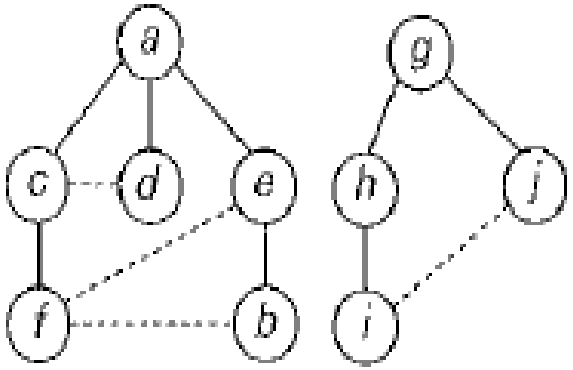
ALGORITHM *BFS*(G)

```
//Implements a breadth-first search traversal of a given graph
//Input: Graph G = ⟨V, E⟩
//Output: Graph G with its vertices marked with consecutive integers
//in the order they have been visited by the BFS traversal
mark each vertex in V with 0 as a mark of being "unvisited"
count ← 0
for each vertex v in V do
    if v is marked with 0
        bfs(v)
```

```
bfs(v)
//visits all the unvisited vertices connected to vertex v by a path
//and assigns them the numbers in the order they are visited
//via global variable count
count ← count + 1;  mark v with count and initialize a queue with v
while the queue is not empty do
    for each vertex w in V adjacent to the front vertex do
        if w is marked with 0
            count ← count + 1;  mark w with count
            add w to the queue
    remove the front vertex from the queue
```



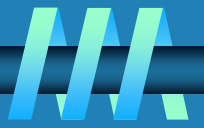
$a_1\ c_2\ d_3\ e_4\ f_5\ b_6$
$g_7\ h_8\ j_9\ i_{10}$



(a) (b) (c)

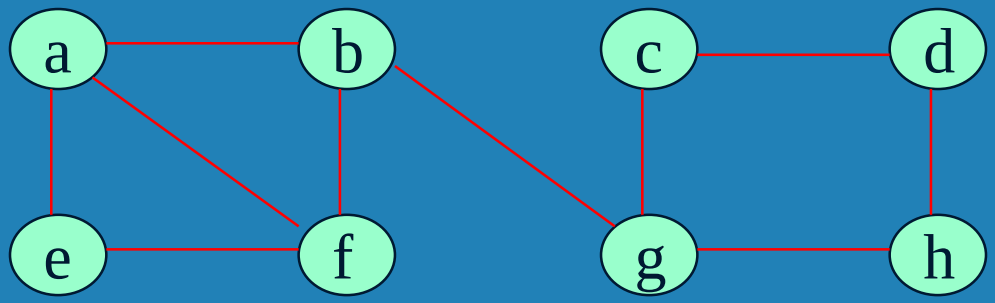
FIGURE 5.6 Example of a BFS traversal. (a) Graph. (b) Traversal's queue, with the numbers indicating the order in which the vertices were visited, i.e., added to (or removed from) the queue. (c) BFS forest (with the tree edges shown with solid lines and the cross edges shown with dotted lines).

Notes on BFS



- BFS has same efficiency as DFS and can be implemented with graphs represented as:
  - adjacency matrices: $\Theta(|V|^2)$.
  - adjacency lists: $\Theta(|V|+|E|)$.
- Yields single ordering of vertices (order added/deleted from queue is the same)
- Applications: same as DFS, but can also find paths from a vertex to all other vertices with the smallest number of edges

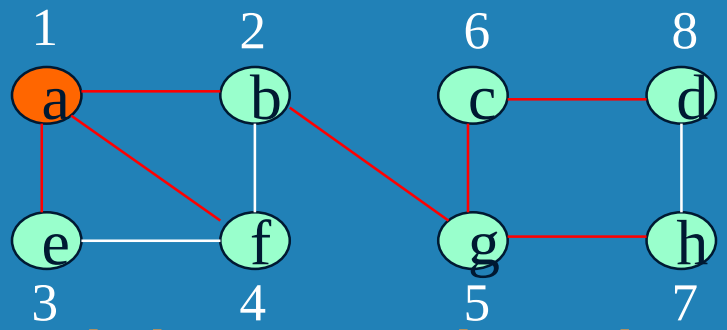
Example of BFS traversal of undirected graph



BFS traversal queue:

a
bef
efg
fg
g
ch
hd
d

BFS tree:



Red edges are tree edges and white edges are cross edges.