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Discrete optimization

Dynamic rebalancing optimization for bike-sharing systems: A modeling framework and empirical comparison

Jiaqi Liang [a,b,e], Sanjay Dominik Jena [c,e,*], Andrea Lodi [b,d,e]

[a] Polytechnique Montreal, 2500 Chemin de Polytechnique, Montreal, H3T 1J4, Quebec, Canada
[b] Canada Excellence Research Chair in Data-Science for Real-time Decision-Making (CERC), 2920 Chemin de la Tour, Montreal, H3T 1J4, Quebec, Canada
[c] Universite du Quebec a Montreal, 315 Ste-Catherine Street East, Montreal, H2X 3X2, Quebec, Canada
[d] Cornell Tech and Technion - IIT, 2 West Loop Road, New York, 10044, NY, USA
[e] Centre Interuniversitaire de Recherche sur les Reseaux d'Entreprise, la Logistique et le Transport (CIRRELT), 2920 Chemin de la Tour, Montreal, H3T 1J4, Quebec, Canada

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ABSTRACT

Station-based Bike-sharing systems have been implemented in multiple major cities, offering a low-cost and environmentally friendly transportation alternative. As a remedy to unbalanced stations, operators typically rebalance bikes by trucks. The resulting dynamic planning has received significant attention from the Operations Research community. Due to its modeling flexibility, mixed-integer programming remains a popular choice. However, the complex planning problem requires significant simplifications to obtain a computationally tractable model. As a result, existing models have used a large variety of modeling assumptions and techniques regarding decision variables and constraints. Unfortunately, the impact of such assumptions on the solutions' performance in practice remains generally unexplored.

In this paper, we first systematically survey the literature on rebalancing problems and their modeling assumptions. We then propose a general mixed-integer programming model for multi-period rebalancing problems that can be easily adapted to different assumptions, including trip modeling, time discretization, trip distribution, and event sequences. We develop an instance generator to synthesize realistic station networks and customer trips, as well as a realistic fine-grained simulator to evaluate the operational performance of rebalancing strategies. Finally, extensive numerical experiments are carried out, both on the synthetic and real-world data, to analyze the effectiveness of various modeling assumptions and techniques. Based on our results, we identify the assumptions that empirically provide the most effective rebalancing strategies in practice. Specifically, a set of specific trip distribution constraints and event sequences ignored in the previous literature seem to provide particularly good results.

1. Introduction

Bike-sharing systems (BSSs) are quickly gaining popularity worldwide, as they help to reduce traffic congestion and vehicle CO_2 emissions. Over the past few years, BSSs were implemented in most major cities such as New York, Boston, London, Sydney, Beijing, Paris, Toronto, and Montreal. We focus on station-based BSSs, where stations with specific capacities are installed in the city, holding an inventory of rentable bikes. Users may rent available bikes from these stations and return their bikes to available docks.

Throughout the day, BSS stations are often unbalanced, which leads to demand unsatisfaction, given that rental demand may not be satisfied when a station is empty, and return demand may not be satisfied when the station is full (i.e., no empty docks are available). As a remedy, BSS operators employ trucks to rebalance bikes among stations. However, due to the uncertain rental and return demand, as well as the complexity of the dynamic planning problem, manual planning tends to be sub-optimal.

To provide decision support, the scientific community has provided a large variety of predictive and prescriptive models, aiming at improving station rebalancing. Planning models can roughly be divided into those based on Markov Decision Process (MDP) and those based on Mixed-integer Linear Programming (MILP). The former, more recently applied in the context of BSSs (see, e.g., Brinkmann, Ulmer, & Mattfeld, 2019, 2020; Fricker & Gast, 2016; Legros, 2019; Luo, Li, Zhao, & Lin, 2022; Seo, Kim, Kang, Byon, & Kho, 2022; Xiao, 2018), implicitly considers uncertainty. However, in order to remain computationally

* Corresponding author at: Universite du Quebec a Montreal, 315 Ste-Catherine Street East, Montreal, H2X 3X2, Quebec, Canada.
E-mail address: jena.sanjay-dominik@uqam.ca (S.D. Jena).

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tractable, MDP models rely on state and action spaces limited in size and are therefore constrained to small station networks, small rebalancing fleets, or limited rebalancing decisions. The majority of the literature proposes MILP models to represent the planning problem, where MILP remains the predominant modeling tool due to its flexibility, as well as the availability of a well-established process to integrate and maintain such models and a wide range of solution methods within an industrial decision-making process. For MILP, both deterministic customer demands (see, e.g., Contardo, Morency, & Rousseau, 2012; Ghosh, Varakantham, Adulyasak, & Jaillet, 2017; Kloimüllner, Papazek, Hu, & Raidl, 2014) and simplified variants dealing with uncertain demand Dell'Amico, Iori, Novellani, and Subramanian (see, e.g., 2018), Fu, Zhu, Ma, and Liu (see, e.g., 2022), Ghosh, Trick, and Varakantham (see, e.g., 2016), Lowalekar, Varakantham, Ghosh, Jena, and Jaillet (see, e.g., 2017) have been considered in the literature. In both cases, due to the complexity of the planning problem and the synergies between customer arrivals and rebalancing operations, its modeling requires assumptions that greatly vary within the literature. These assumptions range from the planning objective to the actual decisions and practical constraints used within the models. Further, customer demands may occur continuously in time. To remain computationally feasible, the planning horizon is typically divided into a set of time-periods, which raises questions concerning the sequence of occurring rental and return demands, as well as the moment of the planned rebalancing operations. As a result, both the planning problems and the mathematical optimization models proposed in the literature are highly diverse. Unfortunately, most of those works have been proposed isolated from the remainder of the literature, which therefore lacks consensus on which assumptions and modeling techniques are best to use. Thus, operators are rather uncertain about which modeling assumptions should be used in the context of their specific BSS and which modeling techniques provide solutions that perform best in practice.

Objective, scope, and contributions. The objective of this paper is to provide guidelines to both practitioners and academics on which assumptions and techniques are most appropriate and likely to produce rebalancing solutions that perform well in practice. To this end, we provide a systematic review of the modeling assumptions and techniques presented in the literature, mostly focused on multi-period models, and propose a general MILP modeling framework that encapsulates most of the relevant modeling assumptions. While several modeling assumptions have been used in the literature without further justification or comparison, we explicitly discuss the alternatives and provide intuitive insights on which assumptions may be more appropriate in practice. We then provide extensive numerical results to empirically evaluate the realism of the various assumptions and modeling techniques, such as variable domains, time discretization, the distribution of trip demand, and the assumed sequence of bike rentals, bike returns, and rebalancing operations.

We develop a realistic simulator that emulates customer trips and the given rebalancing strategy on a minute-to-minute basis. This simulator evaluates the quality of a proposed rebalancing planning solution and hence the realism of the modeling assumptions and techniques of the associated optimization model. Throughout our experiments, the most relevant combinations of modeling assumptions and techniques are then empirically tested on a large set of synthetically generated problem instances that have been carefully designed to represent different BSS settings and fit the demand patterns observed in real-world trip data from BIXI Montreal.

Based on our modeling framework and empirical results on both synthetic and real-world data, practitioners can derive an optimization model tailored to their BSS environment. In particular, our empirical results suggest that (i) both variable domain and type strongly impact the realism and tractability of the model, (ii) time-related constraints are particularly important when time-periods are short, (iii) a new set of trip distribution constraints performs better than those previously considered in the literature, and (iv) the sequences of trips and rebalancing operations used in the literature are outperformed by the new event sequences proposed in our paper.

We note again that we are less concerned with the possible uncertainty surrounding the input parameters of the models, but rather with the underlying assumptions required to tractably represent the reality within the mathematical model. As such, the presented modeling framework is deterministic (i.e., it uses a single set of expected trip demand) for several reasons. Stochastic, scenario-based formulations are equally subject to such modeling assumptions. Our conclusions are therefore still likely to hold for such problem variants. While we review and discuss the corresponding literature, for the purpose of our study, it is not necessary to explicitly represent uncertainty. Finally, a stochastic problem variant would be computationally intractable, thus forbidding us to obtain close-to-optimal solutions that are required for our analysis.

Outline. The rest of this paper is organized as follows. Section 2 reviews related literature on BSS rebalancing problems and summarizes the assumptions. Section 3 describes the modeling framework for the multi-period rebalancing problem, including a basic model that can be extended by several constraints according to the various modeling assumptions. Section 4 presents the general framework used to evaluate the practical performance of a given multi-period rebalancing strategy. Numerical tests and analyses on synthetic and real-world problem instances are illustrated in Section 5. This is followed by the conclusions in Section 6.

2. A review of BSS rebalancing modeling assumptions

The literature mainly focuses on two types of rebalancing in BSSs (Vallez, Castro, & Contreras, 2021). User-based rebalancing incentivizes users to rent or return bikes at specific stations (Haider, Nikolaev, Kang, & Kwon, 2018). Such an approach is more common in dockless BSSs. In contrast, operator-based rebalancing involves the active management of a rebalancing fleet (e.g., trucks) that relocates bikes from one station to another. Such an approach is specific to station-based BSSs (Hu, Huang, Zhang, Ge, & Yang, 2021). According to a recent statistical report (*The meddin bike-sharing world map report*, 2022), station-based systems are, by far, more common. Even in the case of rebalancing planning in dockless BSSs, it has been a common practice to divide the studying area into different sub-areas, which are then seen as stations (see, e.g., Du, Deng, & Liao, 2019; Luo et al., 2022; Xu, Ji, & Liu, 2018; Zhang, Yang, Zheng, Jin, & Zhou, 2019). We, therefore, focus on operator-based rebalancing.

Operator-based bike-sharing rebalancing problems can be divided into static bicycle repositioning problems (SBRP) and dynamic bicycle repositioning problems (DBRP) (Pal & Zhang, 2017; Raviv, Tzur, & Forma, 2013). SBRP typically rebalance stations overnight, while the operations during the day are not considered. Static rebalancing, therefore, prepares the station inventories for the next day. However, it cannot explicitly react to the demand fluctuation occurring during the day. In contrast, DBRP focus on intraday rebalancing, where customer trips carried out during the day heavily affect the availability of bikes and docks (Espegren, Kristianslund, Andersson, & Fagerholt, 2016). Indeed, demand satisfaction highly depends on the real-time status of the stations and customers' stochastic rentals and returns, which poses practical challenges (Shui & Szeto, 2020). We here focus on dynamic rebalancing planning, which has a higher impact on practice since it considers continuous rebalancing throughout the day. Given that DBRP consider trip demand and rebalancing operations over the day, the models proposed in the literature either approach this problem using a repeatedly solved single-period model or a multi-period model.

Single-period rolling vs. multi-period planning. A single-period model generally spans a duration between 10 and 60 min. Single-period models are typically solved in a rolling horizon fashion throughout the day, while multi-period models are either solved once at the beginning of the planning horizon or several times throughout the day.

Solving a single-period model is computationally easier than a multi-period one. However, single-period models tend to be myopic, i.e., the decisions made at the current time-period cannot take into consideration the consequences in future time-periods. Such models therefore greedily maximize demand satisfaction at the current time-period, even if this results in station inventories that cannot satisfy demand well in the next time-periods. In contrast, multi-period models benefit from integrated planning over all time-periods and avoid suffering from myopic behavior. On the downside, these models may be more difficult to solve, given that they consider several sets of decision variables and constraints for each time-period.

Since we are concerned with finding the model that performs best in practice, we focus on the multi-period planning model. In the following, we summarize the main assumptions made in multi-period planning models proposed in the literature, review existing alternatives and propose some that might not have been considered yet.

2.1. Time discretization and time constraint

To represent the change in stations' status and vehicles, the planning horizon is discretized into time-periods. One mainly has two possibilities to discretize the planning horizon.

- **Time-periods of equal length**. The planning horizon is discretized into a set of evenly-spaced time points and the length of each period is the same, which is employed in most multi-period models. In the multi-period planning framework, we could obtain the changes of stations for each time-period and gain the look-ahead information. However, it is hard to define the optimal length of the time-period. It should depend on the model complexity and the length of the planning horizon we focus on.
- **Time-periods of different lengths**. Here, the length of each period in the planning horizon can be different. Kloimüllner et al. (2014) split each cumulated demand function into weakly-monotonic segments with extreme values that are regarded as end-of-segment events. Further, the arrival of a vehicle at a station to rebalance bikes is referred to as a station-visit event. These two types of events are sequentially considered to separate the planning horizon.

For multi-period dynamic rebalancing, time-period with equal length is most common in the literature (see, e.g., Chiariotti, Pielli, Zanella, & Zorzi, 2018; Lowalekar et al., 2017; Zheng, Tang, Liu, Xian, & Zhuo, 2021). Typically, it is assumed that each vehicle rebalances at most one station during one time-period. Selecting an appropriate length of time-period is crucial to the rebalancing strategy and its performance. Several works have investigated the effect of such aggregation on the rebalancing performance (see, e.g. Ghosh et al., 2017; Lowalekar et al., 2017; Roshan Zamir, 2020). Generally, aggregations over shorter time-periods allow for more rebalancing operations and, therefore, lower lost demand. However, this typically comes at the cost of increased computing times.

Moreover, when the time-period is short, a time constraint may be required to ensure that the required time for rebalancing and transiting to the next station fits into the length of the time-period. Time constraints, therefore, interdict truck relocations that are unrealistic in practice. Ghosh et al. (2017) and Shu, Chou, Liu, Teo, and Wang (2013) do not use time constraints, while follow-up work (Ghosh, Koh, & Jaillet, 2019; Ghosh et al., 2016) apply time constraints within a single-period rolling planning framework. Contardo et al. (2012) and Lowalekar et al. (2017) use time constraints considering only vehicle traveling time, while the handling time at stations is ignored. Kloimüllner et al. (2014) and Zhang, Meng, Wong, Ieromonachou, and Wang (2021) consider both traveling and handling time, where the latter is computed as an average value regardless of the number of loading/unloading operations. A summary of how time constraints are handled in the literature, along with other model characteristics, is presented in Table 14 in the Online Supplement.

Note that the introduction of time constraints may make the model difficult to solve. It is therefore crucial to select a proper time-period length that can lead to a reasonable solution time while providing high-quality rebalancing strategies.

2.2. Trip modeling and variable domains

Each customer trip contains one bike rental demand and one bike return demand. To model successful trips, mainly two types of variables have been considered:

- **Origin–destination (O–D) variables** $x^{t_1,t_2}_{s_1,s_2}$, contain departure station s_1, arrival station s_2, departure time t_1, and arrival time t_2.
- **Station-based variables** represent the departure of a trip, i.e., satisfied rental demand $x^{+,t_1}_{s_1}$ and the arrival of a trip, i.e., satisfied return demand $x^{-,t_2}_{s_2}$.

Station-based variable models require fewer variables but lack the connection between rentals and returns. The O–D variable models avoid this issue at the expense of a large number of variables, which may complicate the solutions of the model. A general classification of existing models can be found in Table 14 in the Online Supplement.

Next to the type of variables, the variable domains may also impact the performance of the induced solutions. In the rebalancing model, variables represent three main actions: routing, rebalancing, and the above-discussed user trips. The routing variables typically indicate the location of a vehicle along the planning horizon and the route taken. These variables are always binary. Rebalancing variables represent the inventory of vehicles and the number of bikes to be rebalanced. Most models define them as continuous variables (see, e.g., Contardo et al., 2012; Ghosh et al., 2016; Kloimüllner et al., 2014; Lowalekar et al., 2017; Shu et al., 2013, etc.). Zhang et al. (2021) use integer rebalancing variables. User trip variables for rentals and returns also interact with the inventories of stations. Most models in the literature use continuous variables, except for (Zhang, Yu, Desai, Lau, & Srivathsan, 2017), which use integer O–D variables.

2.3. Trip distribution

If the rental/return demand exceeds the current inventory of bikes/docks, a basic optimization model (such as the one in Section 3.1) may select the rentals/returns opportunistically according to the objective function. In reality, however, demand will be satisfied based on a first-come-first-serve rule. Several works therefore aimed at enforcing a more realistic distribution of the trips by adding specific constraints. We review the existing assumptions on demand distribution below:

- **Station-based variables without distribution**. Two variables are created to present rentals and returns for each station and each period (see, e.g., Kloimüllner et al., 2014). Similarly, Contardo et al. (2012) use two variables to represent the shortage and excess of bikes, which is equivalent to the use of station-based variables. Demands will be greedily satisfied in the optimization model without considering the link between rentals and returns. As a result, the lost demand tends to be underestimated.
- **Station-based variables with proportional distribution**. Lowalekar et al. (2017) enforce a trip distribution proportional to the O–D trip demand as $x^{-,t_2}_{s_2} \leq \sum_{t=0}^{t_2-1} \sum_s x^{+,t}_s \frac{F^{t,t_2}_{s,s_2}}{f^{+,t}_s}$. Especially, the proportion is given by the ratio between the number of trips starting at station s in time-period t and ending at station s_2 in time-period t_2 (F^{t,t_2}_{s,s_2}) and the total number of trips starting at station s in time-period t ($f^{+,t}_s$). The authors also assume that

the number of bikes returned during a specific time-period is not higher than the number of bikes rented in the previous periods multiplied by the corresponding proportion. Similarly, You (2019) consider a return ratio on the number of returns at station s divided by the total number of bikes currently used by customers during the time-period t. The return demand at station s in period t is assumed to be the product of the return ratio and the total number of bikes being used.

- **O–D variables without distribution**. In this case, O–D variables will be created to represent the number of trips starting from one station and ending at another station during one particular time-period.
- **O–D variables with proportional distribution**. Ghosh et al. (2017) and Zheng et al. (2021) apply a similar proportional distribution as (Lowalekar et al., 2017). The constraints $x^{t_1,t_2}_{s_1,s_2} \leq ab^{t_1}_{s_1}\frac{F^{t_1,t_2}_{s_1,s_2}}{f^{+,t_1}_{s_1}}$ imply that the trips rentals from a specific station are limited by the distribution ratio multiplied with the number of currently available bikes ($ab^{t_1}_{s_1}$).
- **Poisson distribution**. Chiariotti, Pielli, Zanella, and Zorzi (2020), Legros (2019), and Shu et al. (2013) model the arrival of rentals and returns as a Poisson process, which implicitly models trip uncertainty.

These trip distribution constraints, as well as other alternatives, will be discussed in detail in Section 3.2.

2.4. Sequence of rebalancing, bike rental, and bike return events

In station-based BSSs, where the operator carries out station rebalancing using trucks, both customers and vehicles interact with the station inventories: customers may rent or return bikes, while trucks may drop off or pick up bikes. While in reality, customers and trucks interact with the station inventory at a specific moment in time, an optimization model aggregates these operations within each time-period.

Different assumptions can be made to deal with this issue. Some or all of these events can be assumed to happen simultaneously, allowing rentals and pick-ups to compensate for returns and drop-offs occurring within the same time-period. Such a generous assumption may achieve a high demand satisfaction within the optimization model. In practice, however, this may be overly optimistic and not perform well, i.e., rentals may occur at empty stations before returns and drop-offs, or returns may occur at full stations before rentals and pick-ups. Alternatively, one may assume that these events occur in a pre-defined chronological sequence within each time-period; for example, bike rentals occur first, then bike returns, and finally, the rebalancing operations. While such an assumption is more restrictive concerning demand satisfaction, it assumes that rentals can only be performed if sufficient bikes are available before the returns, and customer demands cannot benefit from the rebalancing operations that are assumed to happen at the end of the time-period.

Let us denote by **(r)** the event of vehicle **rebalancing**, **(a)** the event of customer **arrival** to return bikes, and **(d)** customer **departure**, i.e., bike rental. While models in the literature have assumed different event sequences, theoretically, any combination of these three event types is possible.

Table 1 summarizes the possible combinations, where events within the same parentheses are assumed to happen simultaneously. For example, (r)(a)(d) assumes that rebalancing is performed first, then customer arrivals, and then customer departures. In contrast (r+a+d) assumes that all three events happen at the same time. Note also that all sequences reported within the same row in Table 1 have the same order of events, but not necessarily within the same time-period. For example, both (r)(a)(d) and (a)(d)(r) assume that arrivals occur after rebalancing and departures occur after arrivals if the sequence is observed over several time-periods, e.g., (r)(a)(d)(r)(a)(d)(r)(a)(d), etc. However, such similarity does not guarantee a similar performance of the obtained solutions.

Table 1
Potential sequences of events.

Three separate events	(r)(a)(d) (r)(d)(a)	(a)(d)(r) (d)(r)(a) (d)(a)(r) (a)(r)(d)
Two events simultaneously	(r)(a+d) (a)(d+r) (d)(r+a)	(a+d)(r) (d+r)(a) (r+a)(d)
All events simultaneously	(r+a+d)	

Within the existing literature, Kloimüllner et al. (2014) use a series of station-visit events and extreme values of cumulated demand to discretize the planning horizon. Since the time required to pick up and drop off bikes is neglected, there is no particular order of events. However, their model essentially assumes (a+d)(r), which means that customers first rent and return bikes before vehicles rebalance. Zhang et al. (2021) assume that rebalancing happens first and then customers rent and return bikes. Ghosh et al. (2016) use sequence (a)(r)(d), while (Lowalekar et al., 2017) use sequence (a)(d+r). Several other works (e.g., Calafiore, Bongiorno, & Rizzo, 2019; Contardo et al., 2012; Ghosh et al., 2017; Mellou & Jaillet, 2019; Yuan, Zhang, Wang, Liang, & Zhang, 2019; Zheng et al., 2021) ignore the issue of event sequences, which is equivalent to assuming a simultaneous event sequence (r+a+d). Finally, a different approach is taken by You (2019), who subdivide each time-period into smaller time-segments and associate rental and return demand to such fine-grained time-segments. Bike pick-ups from rebalancing operations are assumed to happen at the first segment of each time-period, while drop-offs occur at the end. Such an assumption does not fit our classification scheme, but the discretization into time segments resembles the operating mode of our simulator.

Unfortunately, no studies are available exploring the degree of realism and effectiveness of the different event sequences. In Section 3.2.3, we will therefore explicitly review the modeling of the various alternatives and empirically evaluate their effectiveness.

Other assumptions and objective functions can be found in Appendix 1 of the Online Supplement.

3. Multi-period rebalancing modeling framework

We now present a general modeling framework that can be adapted to the various assumptions discussed in Section 2. To this end, we first propose a basic multi-period optimization model for dynamic rebalancing. We then formulate the different assumptions, which can be easily incorporated into the basic model.

3.1. Formulation of the basic model

We first consider a basic multi-period model with minimal assumptions. Its input parameters are summarized in Table 2. Namely, S denotes the set of stations, while V denotes the set of available vehicles. Each station $s \in S$ has a total of C_s docks, referred to as its capacity. Each vehicle $v \in V$ has a bike capacity $\hat{C}_v$. Parameters $D_{i,j}$ and $R^t_{s,s'}$ denote respectively the distance and transit time between stations i and j at time-period t. We consider a planning horizon with $|T|$ time-periods, where each time-period $t \in T$ has a duration of L_t minutes.

We formulate the rebalancing problem as a MILP and assume that each vehicle can only visit one station at each period.

The decision variables are summarized in Table 3. Variables $r^{+,t}_{s,v}/r^{-,t}_{s,v}$ represent the number of bikes picked up/dropped off at station s by vehicle v during period t. Variable $z^t_{s,v}$ takes value 1 if station s is visited by vehicle v at period t; 0 otherwise. For each time-period, intermediate variables are used, such as the number of bikes available

Table 2
Input parameters of the optimization model.

Input	Definition
S	The set of stations.
V	The set of vehicles.
T	The set of discretized time-periods.
$D_{i,j}$	The distance between station $i \in S$ and $j \in S$.
C_s	The capacity of station $s \in S$.
$\hat{C}_v$	The capacity of vehicle $v \in V$.
L_t	The length (in minutes) of time-period $t \in T$.
d_s^1	The initial number of bikes in station $s \in S$.
$\hat{d}_v^1$	The initial number of bikes in vehicle $v \in V$.
$z_{s,v}^1$	1, if vehicle $v \in V$ is at station $s \in S$ at the beginning of planning; 0, otherwise.
$f_s^{+,t}$	The expected rental demand at station $s \in S$ in period $t \in T$.
$f_s^{-,t}$	The expected return demand at station $s \in S$ in period $t \in T$.
$R_{s,s'}^t$	Transit time for vehicles from station $s \in S$ to station $s' \in S$ in period $t \in T$.
$F_{s,s'}^{t,t'}$	The number of trips from station $s \in S$ at period $t \in T$ to $s' \in S$ at period $t' \in T$.

Table 3
Decision variables of the optimization model.

Variable	Definition
$\boldsymbol{r_{s,v}^{+,t}}$	The number of bikes picked up at station s by vehicle v in period t
$\boldsymbol{r_{s,v}^{-,t}}$	The number of bikes dropped off at station s by vehicle v in period t
$\boldsymbol{z_{s,v}^t}$	1, if vehicle $v \in V$ is at station $s \in S$ at period $t \in T$; 0, otherwise.
d_s^t	The number of bikes available in station $s \in S$ at the beginning of period t
$\hat{d}_v^t$	The number of bikes in vehicle $v \in V$ at the beginning of period t
$p_{s,s',v}^t$	1, if vehicle v is at station s in period t and at station s' in period $t+1$; 0, otherwise
$x_s^{+,t}$	The number of successful bike trips starting from station s in period t
$x_s^{-,t}$	The number of successful bike trips ending at station s in period t

at stations/in vehicles, successful trips, and the routes of vehicles. We employ rental and return demand without enforcing trip distribution, and use **station-based trip** variables $x_s^{+,t}$ and $x_s^{-,t}$.

Then, the basic MILP model reads as follows:

$$\min \quad \sum_{s\in S}\sum_{t\in T}(f_s^{+,t} - x_s^{+,t}) + \sum_{s\in S}\sum_{t\in T}(f_s^{-,t} - x_s^{-,t}) \tag{1}$$

$$\text{s.t.} \quad \hat{d}_v^{t+1} = \hat{d}_v^t + \sum_{s\in S}(r_{s,v}^{+,t} - r_{s,v}^{-,t}) \qquad \forall\, v \in V,\ t \in T \tag{2}$$

$$d_s^{t+1} = d_s^t - \sum_{v\in V}(r_{s,v}^{+,t} - r_{s,v}^{-,t}) - x_s^{+,t} + x_s^{-,t} \qquad \forall\, s \in S,\ t \in T \tag{3}$$

$$\sum_{s\in S} z_{s,v}^t = 1 \qquad \forall\, v \in V,\ t \in T \tag{4}$$

$$r_{s,v}^{+,t} + r_{s,v}^{-,t} \le \hat{C}_v z_{s,v}^t \qquad \forall\, s \in S,\ v \in V,\ t \in T \tag{5}$$

$$0 \le \hat{d}_v^t \le \hat{C}_v, 0 \le d_s^t \le C_s \qquad \forall\, s \in S,\ v \in V,\ t \in T \tag{6}$$

$$0 \le x_s^{+,t} \le f_s^{+,t}, 0 \le x_s^{-,t} \le f_s^{-,t} \qquad \forall\, s \in S,\ t \in T \tag{7}$$

$$0 \le r_{s,v}^{+,t}, r_{s,v}^{-,t} \le \hat{C}_v \qquad \forall\, s \in S,\ v \in V,\ t \in T \tag{8}$$

$$z_{s,v}^t \in \{0,1\} \qquad \forall\, s \in S,\ v \in V,\ t \in T \tag{9}$$

Objective function (1) minimizes the total lost rental and return demand in the planning horizon over all stations and time-periods. If required, it can be modified according to the preferences of the BSS operators (see Appendix 1.2 of the Online Supplement). Constraints (2) ensure that the number of bikes in each vehicle is synchronized with the vehicles' bike pick-ups and drop-offs. Constraints (3) manage the station inventory along time, considering the rebalancing operations and successful customer trips (i.e., rentals and returns). Note that we use the sequence (r+a+d) in our basic model. The initial inventory of stations d_s^1 is an input of the optimization model and can be obtained from the operators or through static rebalancing. Constraints (4) ensure that each vehicle is at exactly one station at each time-period, which forms the flow of vehicles sequentially. Ghosh et al. (2016) use an alternative constraint: $\sum_{s'} p_{s,s',v}^t - \sum_{s'} p_{s',s,v}^{t-1} = 0$ ($\forall\ s, t, v$), which directly ensures that the flow out of station s for vehicle v at time-period t is equivalent to the flow of v into the station s at time $t-1$. Both of them indicate the relocation of vehicles along time. Note that, in our model, the vehicle can stay at the same station in the next time-period, i.e., $z_{s,v}^t = z_{s,v}^{t+1} = 1$. Constraints (5) ensure that a vehicle only operates at the station where it is currently located. Constraints (6) enforce that the number of bikes in each vehicle is limited by its capacity and the number of bikes at each station is within the station's capacity. Constraints (7) impose that the number of successful trips is bounded by the expected demand for rentals and returns. Constraints (8) force the number of picked-up/dropped-off bikes to respect the vehicle capacity.

The model can easily be reformulated using **O–D** variables $x_{s,s'}^{t,t'}$ instead of station-based variables. In this case, all occurrences of $x_s^{+,t}$ within (1)–(3) simply have to be replaced by $\sum_{s',t'} x_{s,s'}^{t,t'}$, while Constraints (7) have to be replaced by $x_{s,s'}^{t,t'} \le F_{s,s'}^{t,t'}$.

3.2. Formulating different modeling assumptions

We now show how to extend the basic model to account for the additional modeling assumptions discussed in Section 2.

3.2.1. Time constraints

The basic model assumes that vehicles can relocate to any other stations and carry out the rebalancing operations within the duration of a time-period. When the duration of the time-period is short, the resulting planning solution may become infeasible in practice. Since vehicles may not have sufficient time to relocate and rebalance bikes, time constraints (as discussed in Section 2.1) may be added to restrict the vehicle relocation between stations and rebalancing operations to the time available. We formulate time constraints as follows. First, for each pair of stations s and s', vehicle v, and time-period t, Constraints (10) enforce variable $p_{s,s',v}^t$ to take value 1 if both variables $z_{s,v}^t$ and $z_{s',v}^{t+1}$ have value 1. Then, time constraints (11) guarantee that the transit time between stations and the operation time for picking up/dropping off bikes for each period will not surpass the available time L_t.

$$z^t_{s,v} + z^{t+1}_{s',v} - 1 \le p^t_{s,s',v} \qquad \forall\ s, s' \in S,\ v \in V,\ t \in T \tag{10}$$

$$\sum_{s\in S}\sum_{s'\in S} p^t_{s,s',v} R^t_{s,s'} + op \sum_{s\in S}(r^{+,t}_{s,v} + r^{-,t}_{s,v}) \le L_t, \qquad \forall\ v \in V,\ t \in T \tag{11}$$

$$p^t_{s,s',v} \in \{0,1\} \qquad \forall\ s, s' \in S,\ v \in V,\ t \in T, \tag{12}$$

where op is the average operational time to pick up/drop off a single bike. Parameters $|T|$ and L_t can be altered by the decision-maker. In Section 5.3, we will test different lengths of time-periods along with the time constraints to explore their impact on the solution performance.

3.2.2. Trip distribution constraints

We now discuss the proportionality distribution for station-based trip variables. The proportionality distribution constraints for O–D variables can be found in Appendix 2 of the Online Supplement.

Consider a trip starting at station s_1 in period t_1 and ending at station s_2 in period t_2. The station-based trip variables related to this trip are $x^{+,t_1}_{s_1}$ and $x^{-,t_2}_{s_2}$. The *proportional distribution* can be written as:

$$x^{-,t_2}_{s_2} \le \sum_{t=0}^{t_2-1}\sum_{s\in S} x^{+,t}_{s} \frac{F^{t,t_2}_{s,s_2}}{f^{+,t}_{s}} \qquad \forall\ s_2 \in S,\ t_2 \in T \tag{TD1}$$

$$x^{+,t_1}_{s_1} \le \sum_{t=t_1+1}^{|T|}\sum_{s\in S} x^{-,t}_{s} \frac{F^{t_1,t}_{s_1,s}}{f^{-,t}_{s}}, \qquad \forall\ s_1 \in S,\ t_1 \in T \tag{TD2}$$

where, as discussed, $\frac{F^{t,t_2}_{s,s_2}}{f^{+,t}_{s}}$ represents the proportion of all rental demand from s at time-period t that will be returned to s_2 at time-period t_2.

Constraints (TD1) impose that the number of bikes returned to station s_2 during period t_2 is no more than the bikes rented in the previous periods with a proportion of $\frac{F^{t,t_2}_{s,s_2}}{f^{+,t}_{s}}$. Conversely, constraints (TD2) impose that the number of bikes rented in station s_1 during period t_1 is no more than the bikes returned in the later periods with proportion $\frac{F^{t_1,t}_{s_1,s}}{f^{-,t}_{s}}$.

Similarly, instead of considering the number of rented/returned bikes, we can consider the number of available bikes/docks. To this end, we rewrite (TD1) and (TD2) to (TD3) and (TD4) by replacing $x^{+,t}_{s}$ with $ab^{t_1}_{s_1}$ and $x^{-,t}_{s}$ with $ad^{t_1}_{s_1}$.

$$x^{-,t_2}_{s_2} \le \sum_{t_1=0}^{t_2-1}\sum_{s\in S} ab^{t}_{s} \frac{F^{t,t_2}_{s,s_2}}{f^{+,t}_{s}} \qquad \forall\ s_2 \in S,\ t_2 \in T \tag{TD3}$$

$$x^{+,t_1}_{s_1} \le \sum_{t=t_1+1}^{|T|}\sum_{s\in S} ad^{t}_{s} \frac{F^{t_1,t}_{s_1,s}}{f^{-,t}_{s}}, \qquad \forall\ s_1 \in S,\ t_1 \in T \tag{TD4}$$

where ab^t_s and ad^t_s are the number of available bikes and docks respectively at station s in period t.

Given that station-based variables do not maintain the link between bike rental and return, we may use Constraints (TD5) below to enforce that the total number of rentals equals the total number of returns. When all trips are assumed to take no longer than one time-period, one may use Constraints (TD6) below to enforce a stronger relationship. Under the same assumption, Constraints (TD6) can also be derived by summing (TD2) all over s_1.

$$\sum_t \sum_s x^{+,t}_{s} = \sum_t \sum_s x^{-,t}_{s}, \tag{TD5}$$

$$\sum_s x^{+,t}_{s} = \sum_s x^{-,t+1}_{s} \qquad \forall t \in T. \tag{TD6}$$

Practical toy example. To develop an intuition of the impact of the trip distribution constraints, we consider the following toy example. We consider two time-periods with four stations, each of which has a pair of $[ab^t_s, ad^t_s]$ indicating the current number of bikes available for rentals and docks for returns, visualized in Fig. 1.

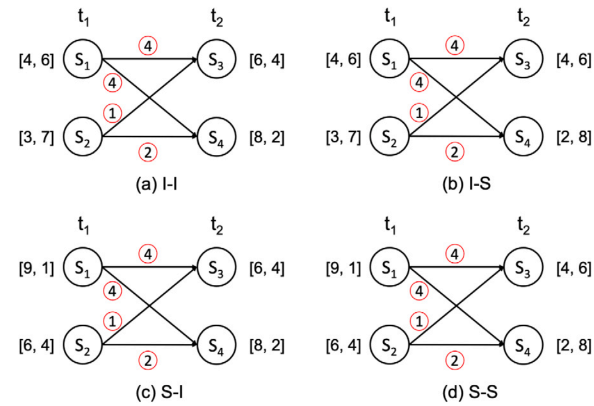

Fig. 1. Toy example with 4 different situations of bike/dock availability.

The value of ab^t_s is equal to $d^t_s - \sum_v (r^{+,t}_{s,v} - r^{-,t}_{s,v}) + x^{-,t}_{s}$ and it is analogous for the value of ad^t_s. The numbers circled in red along the arcs represent the trip demands for each station pair. Stations s_1 and s_2 may either have a sufficient (S) or insufficient (I) number of bikes to satisfy rental demand. Further, station s_3 and s_4 may either have a sufficient (S) or insufficient (I) number of empty docks to satisfy return demand. This leads to 4 different configurations shown in Fig. 1.

Ideally, for scenario I–I, a BSS operator would expect to see 4 trips to station s_3 and 2 trips to station s_4 to fill their empty docks, and will not mind whether the trips come from station s_1 or station s_2 as long as they have sufficient bikes. The ideal distribution is similar for scenario S–I. In contrast, in the case of scenario I–S, the operator would expect to see 4 trips from station s_1 and 3 trips from station s_2 such that all the available bikes can be used. However, there is no preference for the destinations of the 4 trips from station s_1. Clearly, scenario S–S is irrelevant since all trip demand is satisfied.

In order to compute the successful trips under different trip distribution constraints, we solved the basic model with each of them. For station-based trip variables, based on Constraints (3), (6), and (7), we have $x^{+,t_1}_{s_1} \le \min\{f^{+,t_1}_{s_1}, ab^{t_1}_{s_1}\}$ and $x^{-,t_2}_{s_3} \le \min\{f^{-,t_2}_{s_3}, ad^{t_2}_{s_3}\}$.

Minimizing the lost demand for cases I–I, I–S, and S–I in Fig. 1 under the different trip distribution constraints results in the departures and arrivals $[x^{+,t_1}_{s_1}, x^{+,t_1}_{s_2}, x^{-,t_2}_{s_3}, x^{-,t_2}_{s_4}]$ indicated in Table 4. Here, all ideal trip distributions as expected by the BSS operator are indicated in the first row (an '*' refers to any coherent allocation). Note that all trip distribution constraints result in the same solution for case S-S since all trip demands can be satisfied. The solutions that are considered coherent with an ideal solution are indicated in bold.

According to the observed trip distribution, (TD3)+(TD3) and (TD6)+(TD4) have the potential to produce trips that are close to an ideal solution. The combination of (TD1) and (TD2) will be tight for the feasible region, especially for full/empty stations. Constraints (TD1)+(TD6) introduce strict proportional limitations for rentals when the docks are insufficient for returns, which deviates from the ideal solution. Using only (TD1) may result in solutions with more rentals than returns.

Based on this analysis, within our empirical evaluation in Section 5, we will consider the station-based variable model without trip distribution constraints, with (TD1), (TD6), (TD3)+(TD6), and (TD3)+(TD4). The trip distribution of O–D variables, as well as the discussion, is illustrated in Table 16 of the Online Supplement.

3.2.3. Sequences of events

The basic station-based model (1)–(9) implicitly uses event sequence (r+a+d), where the three events are assumed to occur simultaneously. We now show how this model can be modified to take into account the different sequences of rebalancing, rental, and return

Table 4
Trip distribution for 3 different demand scenarios under different trip distribution constraints for station-based variables.

Constraints	I–I	I–S	S–I
Ideal solution	[*, *, 4, 2]	[4, 3, *, *]	[*, *, 4, 2]
without TD	[4, 3, 4, 2]	[4, 3, 5, 6]	[8, 3, 4, 2]
(TD1)	[4, 3, 3, 2]	**[4, 3, 3, 4]**	[8, 3, 4, 2]
(TD2)	[4, $\frac{22}{15}$, 4, 2]	[4, 3, 5, 6]	[$\frac{68}{15}$, $\frac{22}{15}$, **4, 2**]
(TD6)	[$x_{s_1}^{+,t_1} + x_{s_2}^{+,t_1} = 6$, 4, 2]	[4, 3, $x_{s_3}^{-,t_2} + x_{s_4}^{-,t_2} = 7$]	[$x_{s_1}^{+,t_1} + x_{s_2}^{+,t_1} = 6$, **4, 2**]
(TD1)+(TD2)	[$\frac{8}{3}$, 1, $\frac{5}{3}$, 2]	[4, $\frac{3}{2}$, $\frac{5}{2}$, 3]	[$\frac{8}{3}$, 1, $\frac{5}{3}$, 2]
(TD1)+(TD6)	[4, 0, 2, 2]	**[4, 3, 3, 4]**	[4, 0, 2, 2]
(TD2)+(TD6)	[4, $\frac{4}{3}$, $\frac{10}{3}$, 2]	[4, 2, 0, 6]	[$\frac{68}{15}$, $\frac{22}{15}$, **4, 2**]
(TD3)	[4, 3, 3, 2]	**[4, 3, 3, 4]**	[8, 3, 4, 2]
(TD4)	[4, $\frac{22}{15}$, 4, 2]	[4, 3, 5, 6]	[$\frac{68}{15}$, $\frac{22}{15}$, **4, 2**]
(TD3)+(TD4)	[4, $\frac{22}{15}$, 3, 2]	**[4, 3, 3, 4]**	[$\frac{68}{15}$, $\frac{22}{15}$, **4, 2**]
(TD3)+(TD6)	[4, 1, 3, 2]	**[4, 3, 3, 4]**	[$x_{s_1}^{+,t_1} + x_{s_2}^{+,t_1} = 6$, **4, 2**]
(TD1)+(TD4)	[4, $\frac{22}{15}$, $\frac{112}{45}$, 2]	**[4, 3, 3, 4]**	[$\frac{68}{15}$, $\frac{22}{15}$, $\frac{124}{45}$, 2]
(TD2)+(TD3)	[$\frac{56}{15}$, $\frac{19}{15}$, 3, 2]	[4, $\frac{29}{15}$, 3, 4]	[$\frac{68}{15}$, $\frac{22}{15}$, **4, 2**]

events. We give several sequence examples which perform well in the following experiments or are used in the literature.

(r)(a)(d): Since the rebalancing operations performed by the vehicles occur at the beginning of each time-period, Constraints (13)–(16) need to be added, in order to ensure that arrivals consider the rebalancing and departures consider both rebalancing and departures.

$$\sum_v r_{s,v}^{+,t} \le d_s^t \qquad \forall\ s \in S,\ t \in T \qquad (13)$$

$$\sum_v r_{s,v}^{-,t} \le C_s - d_s^t \qquad \forall\ s \in S,\ t \in T \qquad (14)$$

$$x_s^{+,t} \le d_s^t - \sum_v r_{s,v}^{+,t} + \sum_v r_{s,v}^{-,t} + x_s^{-,t} \qquad \forall\ s \in S,\ t \in T \qquad (15)$$

$$x_s^{-,t} \le C_s - d_s^t + \sum_v r_{s,v}^{+,t} - \sum_v r_{s,v}^{-,t} \qquad \forall\ s \in S,\ t \in T \qquad (16)$$

(r)(d)(a): Since rebalancing occurs first, we require Constraints (13) and (14). In order for bike rentals to consider the previous rebalancing, we further require Constraints (17). Finally, bike returns are already correctly implemented due to Constraints (3) and (6).

$$x_s^{+,t} \le d_s^t - \sum_v r_{s,v}^{+,t} + \sum_v r_{s,v}^{-,t} \qquad \forall\ s \in S,\ t \in T \qquad (17)$$

(r)(a+d): Here, we use Constraints (13) and (14) since vehicles operate at the beginning of each period. The restrictions related to the rentals and returns are already satisfied by Constraints (3) and (6). The constraints required for all remaining event sequences can be found in Appendix 3 of the Online Supplement.

4. Dynamic rebalancing evaluation framework

Evaluation framework for rebalancing strategies. The framework used to obtain rebalancing strategies for given problem instances and to evaluate their estimated performance in practice is visualized in Fig. 2. The input set contains the trip demand with exact time-stamps, referring, for example, to historical days with similar demand patterns (e.g., weekdays).

In order to analyze the effect of the various modeling assumptions in a controlled environment, we first generate synthetic problem instances, including a station network along with probability distributions of trip demand over a specified time horizon (in our case one day). For each problem instance, an input set and a test set are sampled, containing a certain amount of days of trip demands. The deterministic optimization model then receives as input the expected trip demand (i.e., a single demand scenario). While this point estimate may be provided by a predictive model, we here refrain from using a predictive model to focus on the impact of the modeling assumptions without potential interference from prediction errors. Instead, we compute the expected demand by averaging over the demand (for each time-period and station) of all days in the input set. Note that, if a stochastic (scenario-based) model was used, each scenario would contain the trip data of a different day.

The rebalancing strategies obtained from the optimization model are then applied to a simulated BSS, which aims at realistically estimating the performance (e.g., the lost demand) of the planning solutions on the various trip demand days within the test data set.

We also define an optimization-simulation-gap, representing the difference between the number of successful trips in the deterministic optimization model and the average number of successful trips simulated on each day in the input set. This gap therefore estimates the deviation from reality, largely explained by the temporal aggregation of the optimization model versus a FIFO policy that applies in reality.

Specifically, $\left(|\sum_{t,s} x_s^{+,t} - \bar{x}_s^{+,t}|\right) / \left(\sum_{t,s} \bar{x}_s^{+,t}\right)$ and $\left(|\sum_{t,s} x_s^{-,t} - \bar{x}_s^{-,t}|\right) / \left(\sum_{t,s} \bar{x}_s^{-,t}\right)$ compute the relative gaps of rentals and returns, respectively, over the entire planning horizon and all stations, where $x_s^{+,t}$/ $x_s^{-,t}$ are rentals/returns as computed by the optimization model and $\bar{x}_s^{+,t}$/ $\bar{x}_s^{-,t}$ are the average number of successful rentals/returns within the simulator.

Fine-grained simulator. Using a simulator to evaluate the performance of the proposed rebalancing strategies has been a common approach in the literature (Ghosh et al., 2019, 2016). Most simulators, however, aggregate the entire demand of each time-period, therefore allowing return demand to cancel out rental demand (and vice-versa). This is overly optimistic and deviates from the first-arrive-first-serve policy in practice. The required operating time for rebalancing is also typically ignored.

Aiming at a more realistic evaluation, we develop a discrete-event simulator taking into account more realistic operational BSS mechanisms. Each time-period used in the optimization model (spanning 30 or 60 min) is further discretized into 1-minute *time-slots* (which is sufficiently fine-grained to be considered real-time in practice). We consider both users' behaviors (rentals and returns) and trucks' operations (pick-ups and drop-offs) as discrete events, each of which is associated with a specific 1-minute time-slot. Rebalancing operations may occur in parallel to rentals and returns and depend on the time the truck arrives at the station. Customer trips and rebalancing operations are therefore considered in chronological order.

For ease of presentation, the operation mode of the simulator is now described verbally. First, the simulator *initializes* the station and vehicle inventories according to the input data. Each rental demand is associated with its respective 1-minute time-slot. Pick-up and drop-off attempts for the first time-period are associated with their respective time-slot, taking into consideration the time to load/unload bikes on/from the truck.

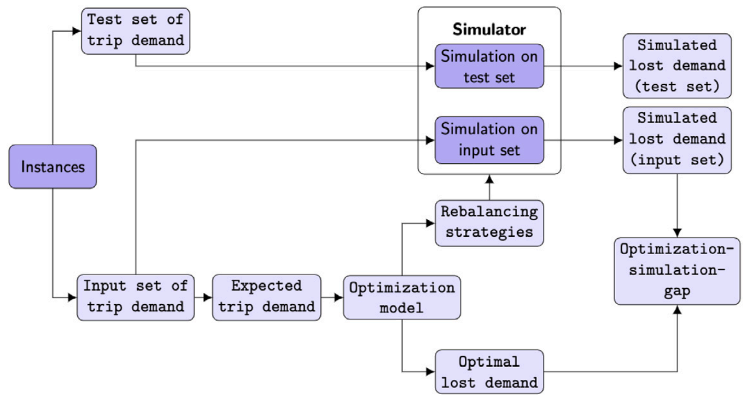

Fig. 2. Evaluation framework for dynamic rebalancing strategies.

The simulator then scrolls through the time-slots minute by minute, attempting to *perform all scheduled events* for that time-slot. The operating rules in this iterative process can be summarized as follows: (i) A rental demand is satisfied if the station holds at least one bike. In this case, a return demand is created for the destination station and associated with a future time-slot based on an estimated travel time. If station inventory is insufficient, the rental demand is counted as lost. (ii) Analogously, a return demand is satisfied if a free dock is available. Otherwise, the bike is returned to the nearest station with an available dock. However, this return demand is then counted as lost. Note that lost returns can only occur if the corresponding rental demand was successful. (iii) Drop-off and pick-up attempts from rebalancing operations are carried out as best as the available inventory and available docks at the stations and the vehicles allow. (iv) Once a truck has finished the rebalancing attempt, it departs to the next station as prescribed by the planning solution for the next time-period. The arrival event is scheduled for a future time-slot based on the estimated travel time of the vehicle. (v) Once a truck arrives at a new station, it is assumed to immediately start the rebalancing operations. However, rebalancing will not start before the first time-slot associated with the current time-period.

A pseudo-code of the simulator, along with a technical description can be found in Appendix 5 of the Online Supplement.

5. Computational experiments

We now report on different sets of computational experiments on both synthetic and real-world data to systematically explore the impact of the various modeling assumptions, based on the evaluation framework we proposed in Section 4.

In Section 5.1, we first elaborate on the synthetic problem instances used throughout the majority of our experiments. In Section 5.2, we compare dynamic and static rebalancing, as well as the usage of different variable types. Section 5.3 focuses on the impact of time discretization and time constraints. Section 5.4 analyzes the importance of the various trip distribution constraints. In Section 5.5, we cross-test whether the previous findings still hold when different variable domains are used. Section 5.6 focuses on the impact of the assumed event sequence. Finally, Section 5.7 empirically validates a variety of such modeling assumptions on real-world data from the Montreal bike-sharing system.

5.1. Generation of synthetic data and computing environment

Even though we have access to real-world trip data from different BSSs, the majority of our experiments are based on synthetically generated instances for a variety of reasons. First, the unobserved demand Albiński, Fontaine, and Minner (2018) (also referred to as censored demand) makes it difficult to obtain accurate rebalancing strategies and evaluate their performance. Second, existing data often contains errors and noise concerning trip and station inventory. Third, rebalancing operations carried out in the BSS alter station inventories, but existing data sets do not provide reliable data on the rebalancing carried out. We therefore develop an instance generator that aims at generating realistic instances with BSS networks of different sizes and characteristics, as well as trip data that is coherent with trips observed in reality (see details in Appendix 4 of the Online Supplement). Note, however, that we validate the most relevant conclusions within a case study based on real-world data in Section 5.7.

For the purpose of our study, we only focus on weekdays, since they have similar demand patterns for work-related trips and demand tends to be much higher than on weekends. We divide the entire daily trip demand into four types. People who live outside city centers and work inside city centers typically use similar origin (outside city centers) and destination stations (within city centers) during peak hours. These trips are denoted as OI trips. Trips of people who live and work outside the centers are referred to as OO trips. In contrast to work-related OI and OO trips, RD trips refer to random trips occurring during the day and RN trips refer to random trips during the night. Such random trips do not have the same origin and destination stations. The departure time for each trip type is characterized by a Beta distribution (see Appendix 4.2 of the Online Supplement). The average demand per 30-minute duration for weekdays of one week in July 2019 at BIXI is visualized in Fig. 3(a). For comparison, the trip demand averaged over 500 days as generated by our instance generator is displayed in Fig. 3(b) and shows a similar pattern as the trip demand observed at BIXI.

We generate 3 ground truths with different station networks. For each ground truth, we generate 5 instances with the proportions for the four trip types. For each instance, we generate 5000 weekdays, from which a single expected demand is computed for each time-period and station as input for the optimization model. We then generate 1000 weekdays of trips as test data, on each of which the planned solution will be simulated. Table 5 shows the characteristics of the 3 ground truths. The percentage of stations within the city centers and those associated with each trip type are indicated in rows 2 to 5. Ground Truth 3 (GT3) has more work-related trips, compared to Ground Truth 1 (GT1). Ground Truth 2 (GT2) has two city centers under the same trip pattern as GT1. A detailed description can be found in Appendix 4.3 of the Online Supplement.

Our optimization models are solved by IBM ILOG CPLEX on 2.70 GHz Intel Xeon Gold 6258R machines with 8 cores. The stopping criterion for the optimization model is a MIP optimality gap of 0.01% and a maximum running time of 24 h. In the following, we will report results for GT1 and GT2. While GT3 has been harder to solve, its results demonstrated the same conclusions. Detailed results can be found in Appendix 6 of the Online Supplement.

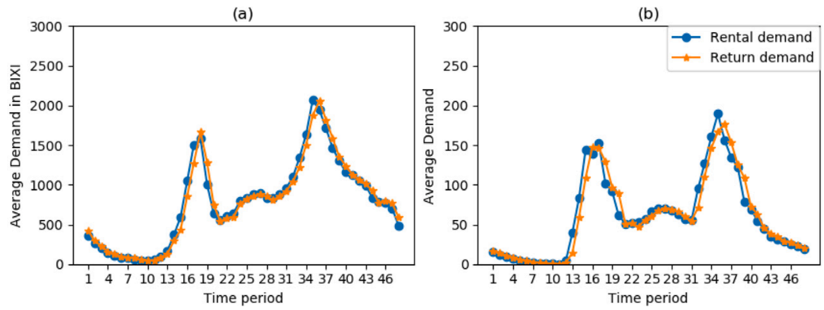

Fig. 3. Total rental and return demand over 24 h (48 time-periods) : (a) Average weekday demand at BIXI ; (b) Average demand in the synthetic instance GT1 (Sample of 500 days).

Table 5
The parameters for the ground truths.

		GT1	GT2	GT3
Station	Number of city centers	1	2	1
Network	City center capacity	26%	35%	26%
	OI	32% - $\beta(3,8)$	32% - $\beta(3,8)$	55% - $\beta(3,8)$
Trip	OO	32% - $\beta(3,7)$	32% - $\beta(3,7)$	25% - $\beta(3,7)$
Pattern	RD	23% - $\beta(3,7)$	23% - $\beta(3,7)$	15% - $\beta(3,7)$
	RN	13% - $\beta(6,8)$	13% - $\beta(6,8)$	5% - $\beta(6,8)$

5.2. Impact of initial station inventory and trip variable types

To quantify the impacts of the initial inventory at stations, we define two baselines as the pre-allocated inventory.

- **Baseline 1: Inventory proportional to rental demands without rebalancing**. The initial inventories of stations at the beginning of a day are set to predefined levels proportional to the rental demands of the first time-period in the planning horizon. We round the values to the closest integer, respecting the station capacities and the total number of bikes available in the system.
- **Baseline 2: Static rebalancing only**. The optimal static rebalancing is obtained by solving the problem given by (30)–(34) in Appendix 6.2 of Online Supplement, where the inventories for the first time-period are decision variables that sum to the total number of available bikes in the system.

In the following, we set the initial inventory of the stations according to these two baselines in the optimization model and run our simulator without dynamic rebalancing for all 3 ground truths. We consider a planning horizon from 6 a.m. to 1 p.m. (i.e., 7 h) and divide the planning horizon into 14 time-periods with a length of 30 min each. We calculate the average rental and return demands for each instance over the input set at each station of each time-period. Rebalancing strategies are obtained through the optimization model and then applied in the simulator to estimate the lost demand on the test set.

Table 6 summarizes the results for GT1 and GT2. The results for GT3, as well as for the O–D variables can be found in Appendix 6.2 of the Online Supplement. We report the optimal value of the objective function as 'O.F. Value' and the running time of the optimization model as 'Opt. Time' in minutes. The 'MIP gap' refers to the optimality gap as reported by CPLEX when the stopping criterion is reached. The lost demand is computed as the relative gap between successful trips and the original demand specified in the problem instances. To be specific, $\frac{\sum_{s,t}(f_s^{+,t}-\hat{x}_s^{+,t})}{\sum_{s,t} f_s^{+,t}}$ defines the lost rental demand over the entire planning horizon, where $\hat{x}_s^{+,t}$ is the number of successful rentals in simulator. Similarly, the lost return demand is computed as $\frac{\sum_{s,t}(\hat{x}_s^{+,t}-\hat{x}_s^{-,t})}{\sum_{s,t}\hat{x}_s^{+,t}}$, where $\hat{x}_s^{-,t}$ is the number of successful returns in simulator. Since, in practice, return demand does not exist when the corresponding rental demand is unsuccessful, the lost returns are only associated with successful rentals $\hat{x}_s^{+,t}$. Lost return demand has to be interpreted critically since the relative lost return may be high when the lost rental is low (which does not indicate a low-quality planning solution). In our result analysis, we, therefore, emphasis on the lost rental.

For each ground truth, the initial inventory observed from Baselines 1 and 2 is directly applied to the simulator without any rebalancing operations, whose average lost demand over 5 instances is reported as 'Baseline 1/2 without rebal.'.

According to the first two rows for each GT in Table 6 (baselines without rebalancing), the initial station inventory seems important to the performance of the BSS. Compared to Baseline 1, static rebalancing (Baseline 2) can significantly improve the lost demand. However, static rebalancing is still insufficient to meet customer demand.

Based on the initial station inventory from Baselines 1 and 2, rows 3-4 for each GT compare the impact of the two different strategies with additional dynamic rebalancing. For the dynamic rebalancing, we use model (1)–(9) without trip distribution and time constraints.

The lost rental is improved when dynamic rebalancing is applied. The performance of dynamic rebalancing varies substantially between the two baselines, which highlights the importance of optimizing the initial station inventory before the dynamic rebalancing. For GT1, the lost rental for dynamic rebalancing with Baseline 1 is higher, leading to a decrease in actual return demands. Given that we only consider lost returns for successful rentals, it is possible that the relative lost return is small when the relative lost rental is high (i.e., only a few successful rentals).

Since strategies based on Baseline 2 outperform those based on Baseline 1, in the following experiments, we will use Baseline 2 to define the initial inventory of each station. Note that we use pre-defined initial locations and inventories for trucks because previous experiments have shown that such assumptions do not affect the performance (see Appendix 6.1 of Online Supplement for details).

5.3. Impact of time discretization and time constraints

We now explore the impact of the length of time-periods and the use of time constraints. We consider two lengths of time-periods: 30-minute and 60-minute, and with or without time constraints. The comparative results for station-based trip variables are summarized in Table 7. The

Table 6
Station-based model with baseline 1 and baseline 2 (60 stations, 4 trucks, 30 min).

	Baselines, configuration, trip modeling	O.F. value	Opt. time (min)	MIP gap (%)	Lost demand (%) Rental	Return
GT1	Baseline 1 without rebal.	–	–	–	26.05	10.27
	Baseline 2 without rebal. (static)	–	–	–	10.40	11.79
	Baseline 1 dyn.rebal. station-based	44.2	1440	0.10	11.11	2.12
	Baseline 2 dyn.rebal. station-based	0.8	<1	0.00	8.78	7.99
GT2	Baseline 1 without rebal.	–	–	–	21.22	6.85
	Baseline 2 without rebal. (static)	–	–	–	11.61	3.18
	Baseline 1 dyn.rebal. station-based	6.4	294	0.01	11.22	1.78
	Baseline 2 dyn.rebal. station-based	0.5	<1	0.00	9.46	1.78

Table 7
Station-based model with/without time constraints in 30/60 min (60 stations, 4 trucks).

	Time period (min)	Time constraints	O.F. value	Time (min)	MIP gap (%)	Lost demand (%) Rental	Return	Opt-sim-gap (%) Rental	Return
GT1	30	No	0.8	<1	0.00	8.78	7.99	9.62	15.88
	30	Yes	0.9	69	0.00	8.13	6.06	8.84	12.69
	60	No	9.5	<1	0.00	8.50	4.39	8.52	11.24
	60	Yes	9.5	2	0.00	8.73	5.85	8.80	13.27
GT2	30	No	0.5	<1	0.00	9.46	1.78	10.42	9.21
	30	Yes	0.5	28	0.00	8.48	1.90	9.24	8.18
	60	No	10.4	<1	0.00	9.72	1.20	9.88	8.90
	60	Yes	10.5	1	0.00	9.63	1.40	9.76	9.02

Table 8
Station-based model with different trip distribution constraints (60 stations, 4 trucks, 30 min).

	Constraints	O.F. value	Time (min)	MIP gap (%)	Lost demand (%) Rental	Return	Opt-sim-gap (%) Rental	Return
GT1	(TD1)	600.3	1440	1.41	3.65	6.34	0.56	40.07
	(TD6)	137.7	<1	0.00	9.94	11.22	5.89	14.28
	(TD3)+(TD6)	240.3	<1	0.00	8.61	6.56	0.01	2.35
	(TD3)+(TD4)	170.5	1440	0.13	6.15	1.57	0.11	1.82
	None	0.8	<1	0.00	8.78	7.99	9.62	15.88
GT2	(TD1)	588.6	1440	1.48	4.78	1.76	1.20	40.70
	(TD6)	101.3	<1	0.00	10.86	3.21	8.40	7.45
	(TD3)+(TD6)	165.0	<1	0.00	8.68	2.08	3.14	0.94
	(TD3)+(TD4)	150.0	1440	0.06	7.34	1.20	2.05	0.56
	None	0.5	<1	0.00	9.46	1.78	10.42	9.21

results of similar experiments using O–D variables can be found in Appendix 6.3 of the Online Supplement.

As shown in Table 7, 30-minute time-periods allow for more rebalancing operations within the optimization model, leading to smaller optimal objective function values than those for cases with 60-minute time-periods. Note again, that in our simulator, we postpone the rebalancing operations if the truck cannot reach the station in time due to long relocation distances. Using 30-minute time-periods, the lost rental without time constraints may therefore be worse than in the case of 60-minute time-periods. Coherently, time constraints with 30-minute time-periods give the best overall performance. Thus, if we have a tolerance for optimization time and the distances between stations tend to be large, a short time-period (30 mins) with time constraints seems beneficial. For longer time-periods, time constraints do not seem necessary. Using the model with O–D variables, we reach similar conclusions (see Appendix 6.3 of the Online Supplement).

5.4. Impact of trip distribution constraints

We implement optimization models with various TD constraints as discussed in Section 3.2 for station-based trip variables with Baseline 2. The results of station-based trip variables are shown in Table 8. The results of O–D variables can be found in Table 28 of the Online Supplement.

According to Table 8, the use of Constraints (TD1) performs best in terms of lost rental for both input and test sets even if optimality has not been proven within 24 h. This suggests that (TD1) reflects the flow of rentals more realistically as also shown by the low rental Opt-sim-gap. Since (TD1) imposes a strict restriction for returns, the lost return of the optimization model is quite high, and lost rental is low. That leads to a small opt-sim-gap for rental but a large one in return. Constraints (TD3)+(TD4) also provide a good performance. Both sets of Constraints (TD1) and (TD3)+(TD4) include the proportionality characteristics of the trip flow. They improve performance, but they require longer computing times.

Based on these experimental results, in the following, we will restrict our analysis to station-based variables with (TD1), (TD3)+(TD4), and (None).

5.5. Impact of variable domains

If the variable domains were selected such that they represent more realistically the BSS operations, trip variables would be binary, while station and vehicle inventory, as well as the rebalancing variables, would be integer. However, using such variable domains may result in models that may be difficult to solve and restricted by certain trip distribution constraints, severely underestimating successful trips. We now explore the impact of using different variable domains. In our models, routing variables $z^t_{s,t}$ are always binary. Let variables d^t_v, $r^{+,t}_{s,v}$, and $r^{-,t}_{s,v}$ be referred to as rebalancing variables and variables d^t_s, $x^{+,t}_s$, and $x^{-,t}_s$ be referred to as station variables. In the previous experiments, both rebalancing and station variables have been continuous, which we

Table 9
Station-based model with different variable domains and trip distribution constraints for GT1 (30 stations, 2 trucks, 30 min).

Variable domains	Constraints	O.F. value	Time (min)	MIP gap (%)	Lost demand (%)		Opt-sim-gap (%)	
					Rental	Return	Rental	Return
All-continuous	(TD1)	262.7	1440	0.18	2.93	1.86	0.82	43.73
	(TD3)+(TD4)	82.7	19	0.00	6.00	0.87	1.62	2.86
	None	0.5	<1	0.00	8.95	4.73	9.83	11.80
Partially-integer	(TD1)	263.6	1440	0.33	2.95	1.29	0.75	44.11
	(TD3)+(TD4)	82.8	50	0.00	5.78	1.47	1.96	2.32
	None	0.5	<1	0.00	6.72	3.56	7.20	7.81
All-integer	(TD1)	656.5	140	0.08	3.89	4.62	30.43	81.39
	(TD3)+(TD4)	408.3	<1	0.00	9.22	1.96	29.56	29.60
	None	380.5	<1	0.00	10.50	7.65	24.88	22.36

denote as an *All-continuous* model. Here, we also consider the other two cases: the *All-integer* model and the *Partially-integer* model. In the All-integer model, both rebalancing and station variables are integers. In the Partially-integer model, the station variables are continuous, while the rebalancing variables are integers.

Since the optimization models with 60 stations and trip distribution constraints (TD1) and (TD3)+(TD4) cannot be solved within the given time limit, we also carry out experiments with the 30-station network to reliably explore the impact of such constraints coupled with different variable domains. Specifically, we consider two ground truths with 30 stations using the same configurations as GT1 (see Table 5).

The results of the corresponding experiments for GT1 under different trip distribution constraints are summarized in Table 9. Surprisingly, the All-integer model is more tractable. Although we have not conducted a complete analysis of this behavior, we observed that more cuts are generated by CPLEX for the All-integer model, which helps in improving the dual bound. More precisely, under Constraints (TD1), three of the five instances of the All-integer model are solved to optimality within 24 h. For the All-continuous and Partially-integer models under Constraints (TD1), none of the instances has been solved to optimality within the time limit. However, the MIP gaps are relatively small. As previously concluded from Table 8, we again observe that Constraints (TD1) provide the lowest lost demand, and this is consistent for all types of variable domains. In terms of the performance for different variable domains, even though it is fast to solve, the All-integer model provides the worst performance, which indicates that restricting station variables to integer values may result in too conservative trips. The Partially-integer model introduces an improvement and performs best in most of the cases, especially with constraints (TD3)+(TD4) and (None). The results for O–D variables can be found in Appendix 6.4 of the Online Supplement.

5.6. Impact of event sequences

Recall that using no particular event sequence, which equals (r+a+d), within the All-continuous station-based model without trip distribution constraints yields an average rental loss of 8.78% on the test set of the 60-station network instances (see Table 8). Based on the model (1)–(9) with continuous variables and the event sequences reviewed in Section 3.2.3, we now analyze the impact of such event sequences for problem instances on the 60-station network in Table 10.

Sequences (d)(r)(a), (r)(d)(a), and (d)(a)(r) perform best for lost rental and return with a small opt-sim-gap. Although GT3 instances are hard to solve, these three sequences still perform well (see Table 30 in Appendix 6.5 of Online Supplement). In contrast, the sequences used in the literature (i.e., (r+a+d), (a)(r)(d), and (a+d)(r)) have performed less well in our experiments. Instead, sequences (d)(r)(a), (r)(d)(a), and (d)(a)(r) may be a better choice, reducing lost rental by an additional 1%–2%.

Table 11 shows the results of the same experiments for the problem instances on a 30-station network and 60-minute time-periods. Here, all instances have been solved to optimality and lead to the same conclusions. Particular sequences help reduce the lost rental: sequences (d)(r)(a) and (r)(d)(a) reduce the lost rental to around 7.45% from 9.12% in the test set.

In Section 5.4, we have concluded that it is beneficial to use trip distribution constraints (TD1) when no particular event sequence is used. The results discussed above suggest that it is beneficial to use a specific event sequence, such as (r)(d)(a) when no trip distribution constraints are used.

We now explore the combination of those two modeling assumptions and further backtest on different variable domains. To this end, Table 12 summarizes the results for GT1 problem instances on the 30-station network using trip distribution constraints (TD1) and various event sequences for All-continuous and Partially-integer variable domains. The models for both variable domains seem to perform similarly well in terms of lost rentals, while the All-continuous models tend to be solved faster.

When comparing with the results in Table 11, the introduction of Constraints (TD1) further decreases the lost rental for any of the event sequences. However, the improvement for (r+a+d) is the highest, indicating that using Constraints (TD1) without any specific event sequence may be the best option if the longer computing time is acceptable. Operators may therefore opt for event sequence (r)(d)(a) with all-continuous variables without trip distribution constraints if a quick solution is required, or Constraints (TD1) without a specific event sequence if higher computing times can be tolerated.

We conclude this section remarking that, in the above experiments, we have scaled a real-world BSS network with over 600 stations, along with its overall demand-level and the size of the rebalancing fleet, to smaller networks with 30 and 60 stations. Such a proportional scaling may have unexpected side effects. In a larger network, conclusions may be attenuated or amplified. Nevertheless, we are confident that the overall conclusions remain valid, as many modeling assumptions are independent of the network size and our conclusions have been coherent for station networks with 30 and 60 stations.

5.7. Case study on BIXI montreal data

We now validate our previous findings by means of a case study based on real-world data from BIXI Montreal (*Bixi open data*, 2023).

Data preprocessing. We focus on the 2019 season. To ensure a coherent association of historical arrivals and rentals to the given station IDs, we first discarded stations that changed locations throughout the 2019 seasons by more than 1 km (which is common given that the operator relocates stations due to events, construction, or holidays). The resulting network contained 606 stations, which, obviously, would result in optimization models that are too large to solve directly. Given that vehicle relocation is time-consuming, efficient rebalancing solutions typically relocate locally rather than over large distances. It is therefore reasonable to assume that the network can be subdivided into sub-clusters (see, e.g., Calafiore et al., 2019; Forma, Raviv, & Tzur, 2015; Ghosh, Varakantham, Adulyasak, & Jaillet, 2015; Huang, Tan, Li,

Table 10
Station-based model with different sequences of events (60 stations, 4 trucks, 30 min).

	Sequences of events	O.F. value	Time (min)	MIP gap (%)	Lost demand (%)		Opt-sim-gap (%)	
					Rental	Return	Rental	Return
GT1	(r)(a)(d)	1.0	288	0.00	7.91	5.22	8.62	1.39
	(a)(d)(r)	3.3	298	0.00	7.64	4.63	8.19	11.83
	(d)(r)(a)	13.4	1399	0.04	6.02	5.81	5.38	9.89
	(r)(d)(a)	8.2	1013	0.03	5.61	3.26	5.36	6.53
	(d)(a)(r)	15.1	879	0.01	6.40	3.48	5.70	7.65
	(a)(r)(d)	2.0	<1	0.00	7.51	2.79	8.12	8.04
	(r)(a+d)	0.8	<1	0.00	8.49	6.86	9.28	14.12
	(a+d)(r)	1.6	<1	0.00	7.42	6.92	7.98	12.87
GT2	(r)(a)(d)	3.2	288	0.00	8.05	3.61	8.71	9.35
	(a)(d)(r)	1.9	864	0.49	8.33	5.11	8.98	11.55
	(d)(r)(a)	17.5	1440	0.16	6.75	5.51	5.88	10.14
	(r)(d)(a)	11.9	1440	0.08	6.39	3.79	5.93	7.77
	(d)(a)(r)	20.0	1341	0.13	6.38	3.77	5.26	7.74
	(a)(r)(d)	3.8	<1	0.00	7.87	1.93	8.50	7.25
	(r)(a+d)	0.7	<1	0.00	8.40	3.29	9.15	9.63
	(a+d)(r)	1.9	576	0.03	7.98	3.66	8.53	9.56

Table 11
Station-based model with different sequences of events (30 stations, 2 trucks, 60 min).

	Sequences of events	O.F. value	Time (min)	MIP gap (%)	Lost demand (%)		Opt-sim-gap (%)	
					Rental	Return	Rental	Return
GT1	(r)(a)(d)	10.4	<1	0.00	8.95	2.09	8.83	7.94
	(a)(d)(r)	15.0	<1	0.00	9.38	1.37	8.47	7.63
	(d)(r)(a)	37.7	<1	0.00	7.43	1.23	1.22	6.01
	(r)(d)(a)	33.1	<1	0.00	7.45	1.10	1.93	6.10
	(d)(a)(r)	42.1	<1	0.00	7.89	1.81	1.04	7.00
	(a)(r)(d)	11.3	<1	0.00	8.62	1.92	8.29	7.34
	(r)(a+d)	5.3	<1	0.00	9.81	2.08	9.89	9.94
	(a+d)(r)	9.8	<1	0.00	9.53	2.48	8.72	10.05
	(r+a+d)	5.3	<1	0.00	9.12	2.55	9.07	9.76

Table 12
Station-based model with (TD1) and sequences of events (30 stations, 2 trucks, 60 min, GT1).

	Sequences of events	O.F. value	Time (min)	MIP gap (%)	Lost demand (%)		Opt-sim-gap (%)	
					Rental	Return	Rental	Return
All-continuous	(r)(d)(a)	499.4	22.9	0.00	6.62	4.09	14.36	65.34
	(d)(a)(r)	515.1	7	0.00	6.78	5.66	16.50	65.35
	(a+d)(r)	503.4	6	0.00	7.10	5.14	14.33	65.12
	(r+a+d)	485.5	237.16	0.00	6.32	3.87	12.37	65.22
Partially-integer	(r)(d)(a)	500.1	18	0.00	6.72	3.22	14.41	65.62
	(d)(a)(r)	515.6	5	0.00	6.73	6.32	16.69	65.08
	(a+d)(r)	504.1	7	0.00	6.97	6.59	14.57	64.65
	(r+a+d)	485.9	440.67	0.00	6.64	3.74	12.15	65.13

Li, & Huang, 2022; Jin, Ruiz, & Liao, 2022; Liu, Sun, Chen, & Xiong, 2016, etc.).

To this end, we first cluster stations via k-means based on demand pattern similarity (see, e.g., Calafiore et al., 2019; Ghosh et al., 2015; Jin et al., 2022), ensuring the inclusion of city center stations and work-related trips. Similar to Forma et al. (2015), we also limit the maximal distance between stations that belong to the cluster, which is motivated by the fact that the vehicles only travel within one cluster. We selected a cluster with 53 stations distributed around the downtown and Plateau areas (see Fig. 4), which has approximately the same number of total rentals and returns (Roshan Zamir, 2020) (a requirement, which is obviously satisfied in closed BSSs systems).

We focus on days with high demand, i.e., weekdays within the summer season of 2019, which have therefore not been affected by the COVID-19 pandemic. Outlier days with extremely low numbers of trips (e.g., due to bad weather or special events) are excluded, resulting in a total of 50 days. Aligned with the experiments on synthetic problem instances, we here consider a planning horizon from 7 a.m. to 2 p.m. (7 h) with 14 time-periods, each of which spans 30 mins.

Empirical results. With the objective of validating the most relevant conclusions drawn from the experiments on synthetic problem

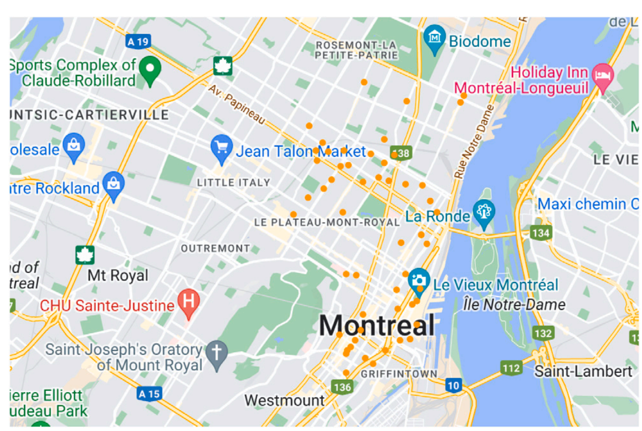

Fig. 4. Considered station cluster from BIXI Montreal (generated through Google Maps (*A set of bixi stations, google maps*, 2023)).

Table 13
Station-based model (53 stations, 4 trucks, 30 min, 50 days high demand, Baseline 2).

	Sequences of events	(TD1)	O.F. value	Time (min)	MIP gap (%)	Lost demand (%)		Opt-sim-gap (%)	
						Rental	Return	Rental	Return
Baseline 2	–	–	–	–	–	8.32	6.12	–	–
All-continuous	(r)(d)(a)		0.0	<1	0.00	7.43	6.15	20.95	18.04
	(d)(a)(r)		0.0	<1	0.00	7.19	6.13	20.93	18.07
	(a+d)(r)		0.0	<1	0.00	8.32	6.11	20.75	18.05
	(r+a+d)		0.0	<1	0.00	9.42	7.65	20.82	18.30
Partially-integer	(r)(d)(a)		0.0	<1	0.00	7.54	7.22	20.58	18.55
	(d)(a)(r)		0.0	<1	0.00	7.36	6.14	20.92	18.05
	(a+d)(r)		0.0	<1	0.00	8.32	6.12	20.75	18.05
	(r+a+d)		0.0	<1	0.00	8.32	6.12	20.75	18.05
All-continuous	(r)(d)(a)	✓	453.4	<1	0.00	5.70	6.67	20.98	25.50
	(d)(a)(r)	✓	453.4	<1	0.00	3.88	7.21	20.83	25.29
	(a+d)(r)	✓	453.4	<1	0.00	5.09	4.99	20.66	25.36
	(r+a+d)	✓	453.4	<1	0.00	6.31	6.23	20.84	25.29
Partially-integer	(r)(d)(a)	✓	453.5	<1	0.00	4.53	6.96	20.91	25.27
	(d)(a)(r)	✓	453.5	<1	0.00	2.82	6.59	20.60	25.42
	(a+d)(r)	✓	453.4	<1	0.00	2.89	6.34	20.61	25.25
	(r+a+d)	✓	453.4	<1	0.00	5.79	6.07	20.70	25.36

instances, we here use the station-based model. As opposed to O–D variables, the use of station-based rental and return variables $x_s^{+,t}$ and $x_s^{-,t}$ also enables us to aim at capturing trips to and from all stations in the original network, not only those included in the considered cluster. We further focus on the impact of variable domains, the event sequences, and the best-performing trip distribution constraints (TD1). Note that the use of time constraints is not necessary, given that all stations within the clusters are located sufficiently close to each other.

Trip constraints (TD1) consider the proportion of successful rentals that depart from any station to a specific station. Given that our model only uses stations within the considered cluster $C \subset S$, this proportion cannot be computed. We may rewrite the right-hand side of (TD1) as $\sum_{t=0}^{t_2-1} \sum_{s\in C} x_s^{+,t} \frac{F_{s,s_2}^{t,t_2}}{f_s^{+,t}} + \sum_{t=0}^{t_2-1} \sum_{s\in S\setminus C} x_s^{+,t} \frac{F_{s,s_2}^{t,t_2}}{f_s^{+,t}}$, where the first term accounts for successful trips from within the cluster, and the second term refers to rentals from stations outside the considered cluster. We replace $x_s^{+,t} (s \in S \setminus C)$ by $f_s^{+,t} (s \in S \setminus C)$ within the second term, therefore optimistically assuming that rentals outside the cluster are all satisfied. The adapted trip constraint then writes as follows:

$$x_{s_2}^{-,t_2} \leq \sum_{t=0}^{t_2-1} \sum_{s\in C} x_s^{+,t} \frac{F_{s,s_2}^{t,t_2}}{f_s^{+,t}} + \sum_{t=0}^{t_2-1} \sum_{s\in S\setminus C} F_{s,s_2}^{t,t_2} \qquad \forall\ s_2 \in C,\ t_2 \in T.$$

Table 13 summarizes the results of the corresponding experiments. Here, Baseline 2 refers to the model that only carries out (overnight) static rebalancing (see Section 5.2). In contrast to the results on synthetic problem instances, the improvement through dynamic rebalancing (as opposed to Baseline 2 only) is not impressive when no trip distribution constraints are used. The small improvement may be explained by the fact that the here-considered real-world data only contains successful trips. All other observations are well aligned with the results for synthetic instances. Using partially-integer variable domains without trip distribution constraints and without a specific event sequence reduces the estimated lost demand (compare Table 9). Further, the use of adapted trip distribution constraints (TD1) positively impacts lost demand (compare Table 9, and Tables 11 and 12). Interestingly, the combination of using partially-integer domains with trip constraints seems particularly effective on real-world data, providing the lowest lost demand among all configurations. Finally, event sequences (r)(d)(a) and (d)(a)(r), which have been found to be among the best-performing sequences on synthetic instances, here also perform well with both all-continuous variables and partially integer variables in lost rental and return without trip distribution constraints. Finally, we note that all models have been solved to optimality within 1 min, which is, of course, sufficiently fast for use in practice.

6. Conclusions

In this paper, we aimed to disentangle and structure the various modeling assumptions and constraints used in the literature on Mixed-Integer Programming models for BSSs rebalancing optimization. To this end, we first surveyed the literature according to modeling techniques and assumptions, with a particular focus on multi-period models. We then introduced a modeling framework, rooted in a basic model, and showed how to adapt it to the various modeling assumptions. Finally, we evaluated the performance of the planned solutions induced by the different model variants as realistically as possible. Specifically, we generated different ground truths that propose BSSs networks and trip patterns matching observed trip patterns at BIXI Montreal. We then developed a fine-grained discrete-event simulator for truck movement, rebalancing operations, as well as bike rentals and returns on a minute-to-minute basis.

6.1. Summary recommendation

Based on the simulation results on a large set of test instances, we focused on two performance measures to analyze the appropriateness of the various modeling assumptions: the lost rental and return demand observed throughout the simulation, and the simulation–optimization-gap that indicates the deviation between the lost demand observed as estimated by the optimization model and by the simulator. Extensive numerical experiments on problem instances with networks including 30 and 60 stations and three different ground truths were carried out. Experiments have also been carried out on real-world data with 53 stations from the BIXI Montreal 2019 summer season. While one is required to be more careful when drawing conclusions from such results, given that the observed trip data refers to trips satisfied in the past, the corresponding conclusions tend to align. Based on these results, our principal conclusions can be summarized as follows:

(i) On the synthetic data sets, adding dynamic rebalancing to static rebalancing reduces the lost rental demand by an additional 2% (e.g., from 10.43% to 8.79% in Table 6).

(ii) Using station-based trip variables instead of more detailed trip variables based on origin–destination pairs generally appears to be competitive and results in faster solution times.

(iii) Shorter time-periods tend to allow for planning more rebalancing operations but may require time constraints to ensure that the resulting rebalancing is time-feasible in practice.

(iv) Trip distribution constraints, especially (TD1), reflect more realistically the trip flow observed in practice and may strongly improve the lost demand on both synthetic and real-world data (e.g., from 8.78%

to 3.65% in Table 8; from 9.42% to 6.31% in Table 13); further, the best performing trip distribution constraint(s) are not necessarily those used in the literature.

(v) Using integer variables exclusively for truck routes, while keeping all other variables continuous (even the pick-up and drop-off decisions in the rebalancing operations) generally approximates reality sufficiently well; in some specific cases, it is beneficial to impose integrality on the rebalancing variables, while it does not seem beneficial to use integer variables for all decisions that, in reality, would also be integer.

(vi) Exploring the various sequences in which bike rentals, bike returns, and rebalancing operations may occur yields interesting conclusions. In particular, event sequences that have not been studied in the literature perform particularly well and tend to reduce lost rental by an additional 2%–3% (Tables 10 and 11). Coupling specific event sequences, in particular (d)(a)(r), with our proposed trip distribution constraints (TD1) provides consistently low lost demand in short computing times on both synthetic and real-world instances.

The optimization model for the real-world problem instance has been solved within 1 min of computing time, while the solution time for synthetic instances may vary strongly with the model configuration. When computing resources are limited and quick decisions are required, using a combination of trip distribution constraints with one of the newly proposed event sequences, in particular the above-mentioned combination of (TD1) with sequence (d)(a)(r), seems to be recommended (compare Table 12).

With unrestricted resources for computing time and memory, we would recommend applying trip distribution constraints with a short time-period for both synthetic data and real-world data. When computing resources are limited and quick decisions are required, we recommend introducing event sequences (r)(d)(a) or (d)(a)(r). Within the experiments on real-world data, partially-integer variable domains, along with constraints (TD1) and event sequence (d)(a)(r) performed particularly well.

Note that we have used a time limit of 24 h in order to be capable of solving the models to optimality, allowing us to draw conclusions on their degree of realism and potential performance in practice. Solving models multiple times throughout the day (albeit over a smaller time horizon) in sufficiently short computing times may require the use of parallel computing, specialized solution methods (e.g., mathematical decomposition), or a combination of both.

6.2. Future work

Having explored the different modeling assumptions and techniques both from methodological and empirical standpoints, this work aimed at shedding light on the modeling jungle and guiding both practitioners and academics in future research on multi-period rebalancing optimization.

While concepts such as the sequences of events cannot be found in most of the related classical optimization problems, such as the Pickup-and-Delivery Problem, such characteristics are not exclusive to BSSs. Indeed, any planning problem in which the interaction between customers and system operator is aggregated in discretized time-periods (e.g., car-sharing, multi-mode transportation planning with synchronization) may exhibit similar event sequences, which may be worth studying.

Certain future research directions may be particularly worthwhile. First, while we have identified the formulations that are likely to provide well-performing planning solutions, solving those models in real-time throughout the day may be challenging; therefore, decomposition algorithms may be employed to speed up the solution time. Second, the proposed models minimize total lost demand. Certain BSS operators consider target intervals, which may be interesting to consider within the objective function.

Finally, while we have intentionally focused on the deterministic planning problem which assumes as input a single set of customer demands, some of our conclusions may also hold for models that consider several demand scenarios simultaneously. Models explicitly considering the underlying uncertainty and demand probability distribution are worthwhile research directions, in particular for the model variants that have shown to be more realistic here.

CRediT authorship contribution statement

Jiaqi Liang: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Resources, Software, Validation, Visualization, Writing – original draft, Writing – review & editing. **Sanjay Dominik Jena:** Conceptualization, Funding acquisition, Methodology, Supervision, Validation, Writing – original draft, Writing – review & editing. **Andrea Lodi:** Conceptualization, Funding acquisition, Methodology, Supervision, Writing – original draft, Writing – review & editing, Validation.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.ejor.2024.04.037.

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