CLASSICAL MECHANICS

[For M.Sc. (Physics), B.Sc. (Honours), B.E., Net, GATE and Other Competitive Examinations]

Dr. J.C. Upadhyaya
MSc., PhD., F. Inst. P. (London)
Formerly Director/Professor Incharge,
Dau Dayal Institute, Dr. B.R. Ambedkar University, Agra (India),
Senior Reader in Physics, Agra College, Agra (India)

Himalaya Publishing House
MUMBAI • NEW DELHI • NAGPUR • BENGALURU • HYDERABAD • CHENNAI • PUNE
LUCKNOW • AHMEDABAD • ERNAKULAM • BHUBANESWAR • INDORE • KOLKATA • GUWAHATI

© Author

No part of this book shall be reproduced, reprinted or translated for any purpose, whatsoever, without prior permission of the Author and Publisher in writing.

First Edition	1999	Reprint	2007
Reprint	2000	Reprint	2008
Reprint	2003	Reprint July	2009
Reprint	2004	Reprint	2010
Reprint	Jan. 2005	Reprint	2012
Second Thorougly		Reprint	2013
Reprint	July 2005	Reprint	2014

Published by : Mrs. Meena Pandey
for **HIMALAYA PUBLISHING HOUSE PVT. LTD.,**
"Ramdoot", Dr. Bhalerao Marg, Girgaon, **Mumbai - 400 004.**
Phones: 2386 01 70/2386 38 63, Fax: 022-2387 71 78
Email: himpub@vsnl.com
Website: www.himpub.com

Branch Offices:

New Delhi : "Pooja Apartments", 4-B, Murari Lal Street, Ansari Road, Darya Ganj, **New Delhi - 110 002.** Phone: 23270392, 23278631 Fax: 011-23256286

Nagpur : Kundanlal Chandak Industrial Estate, Ghat Road, **Nagpur - 440 018.** Phone: 2738731, 3296733 Telefax : 0712-2721215

Bengaluru : No. 16/1 (Old 12/1), 1st Floor, Next to Hotel Highlands, Madhava Nagar, Race Course Road, **Bengaluru - 560 001.** Phone : 22281541, 22385461, Telefax: 080-22286611

Hyderabad : No. 3-4-184, Lingampally, Besides Raghavendra Swamy Matham, Kachiguda, **Hyderabad - 500 027.** Phone: 040-27560041, 27550139, Mobile:- 09848130433 Telefax: 040-27560041

Chennai : No. 8/2, Madley 2nd Street, Ground Floor, T. Nagar, **Chennai - 600 017.** Mobile: 09345345055

Pune : No. 527, "Laksha" Apartment, First Floor, Mehunpura, Shaniwarpeth, (Near Prabhat Theatre), **Pune - 411 030.** Phone: 020-24496323/24496333/32326733

Lucknow : House No. 731, Sehkhupura Colony, Near B.D. Convent School, **Lucknow - 226 024.** Mobile : 09307501549

Ahmedabad : No. 114, "SHAIL", 1st Floor, Opp Madhu Sudan House, C.G.Road, Navrang Pura, **Ahmedabad – 380 009.** Phone: 079-26560126, Mobiles: 09327324149,09314679413

Ernakulam : No. 39/104 A, Lakshmi Apartment, Karikkamuri Cross Rd., Ernakulam, **Cochin – 682011**, Kerala. Phone: 0484-2378012, 2378016, Mob.: 09344199799

Bhubaneswar : 5 Station Square, **Bhubaneswar (Orissa) - 751 001.** Mobile: 9861046007, E-mail:- orissa@himpub.com

Indore : Kesardeep Avenue Extension, 73, Narayan Bagh.Flat No. 302, IIIrd Floor, Near Humpty Dumpty School, Narayan Bagh, **Indore (M.P.) 452 007** Mobile: 09301386468

Kolkata : 108/4, Beliaghata, Main Road, Near ID Hospital, Opp. SBI Bank, **Kolkata - 700 010.** Mobile: 09910440956, E-mail:- nkmitra@himpub.com

Guwahati : House No. 15, Behind Prajjyotish College, Near Sharma Printing Press, P.O. Bharalumukh, **Guwahati - 781009** (Assam) Mobile: 07439040301; 07439040302

Printed by : Shri Krishna Offset Press, Delhi-93

Preface to the Second Edition

The author feels pleasure that the present book, entitled **'Classical Mechanics'** has been warmly welcomed and liked by the readers. That is why the first edition of the book was finished soon and it was reprinted four times. Now he is presenting thoroughly revised and enlarged edition of the book in view of the latest syllabi of various Indian Universities and Model Syllabus, approved by the U.G.C., New Delhi.

In the last few decades, a lot of work has been done in the field of non-linear dynamics and chaos. Looking at the importance and wide applications of this subject, several universities in India and abroad have introduced this topic in the syllabi of Classical Mechanics. In the present edition a chapter on 'Non-linear Dynamics and Chaos' has been added to meet this requirement of the students.

The author shall be grateful to the readers who would be kind enough to send their useful and constructive criticisms for the improvement of the subject matter.

July 12, 2005 **J.C. Upadhyaya**

Preface to the First Edition

Present book deals with an advanced course on mechanics, namely classical mechanics, for the students of B.Sc. (Honours), M.Sc. (Physics) and B.E. classes. In addition to a course book, it has been written for the candidates, struggling to qulify competitive examinations at national and state levels such as NET, GATE, SLET, I.A.S. etc. The concepts and formulatations involved in classical mechanics form the base to construct the entire building of physics. Of course, quantum mechanics plays the key role to study the phenomena at atomic scale, specially in the fields of atomic and nuclear physics. The role of classical mechanics is of extreme importance on one hand in modern calculations involved in launching of satellites, motion of rockets and relativily massive bodies, and on the other hand it makes essential background to switch over and move with couriosity and enthusiasm in the various branches of modern physics. In fact classical mechanics provides an opportunity to a student to be familiar with and command many of the mathematical techniques needed in quantum mechanics.

Classical mechanics had been developed over several centuries in particular by Newton, Lagrange, Hamilton and others. At lower level, the Newtonian mechanics and at higher level, the Lagrangian and Hamiltonian dynamics, involving advanced topics, are taught. Here we mean by classical mechanics the mechanics of Lagrange and Hamilton. Classical mechanics was developed over long time on the basis of observations on moving bodies at relatively low speeds. Of course, the relativistic theory of Einstein deals with all particle-speeds, but it does not modify the classical ideas regarding the basic nature of matter and radiation and hence the relativistic theory is generally studied in classical mechanics. Therefore, in the present book, we also include the special theory of relativity and relevant advanced formulations, *e.g.*, four dimensional Minkowski space and convariant formulation of electrodynamics. This course is conceptual in nature and involves intricate formulations. The course has been partly drifted to B.Sc. (Honours) classes in some universities and conventionally taught at M.Sc. level in the different universities. During teaching, the author had a feeling that the students need a textbook which deals the subject matter of classical mechanics with simplified treatments and good number of illustrations. Keeping this idea in mind, the author has made an effort to write a book on the subject in a simplified way with proper explanations so that an average student may not feel difficulty in following the text.

In the universities, a course on Newtonian mechanics and conservation principles is given at lower level. Generally, the students feel much difficulty when at higher level they are taught Lagrangian and Hamiltonian dynamics. This is also the purpose of the book that a student moves smoothly from the Newtonian mechanics to the Lagrangian, Hamiltonian and relativistic mechanics.

In order to be ligible for Lecturership and to obtian Research Fellowship, one has to qualify the competitive tests at state and national levels such as NET, GATE, SLET etc. In these competitive examinations, problem oriented and objective type questions are asked. In order to fulfil the need of such candidates, a good number of problems and objective type questions have been set at the end of each chapter.

For a clear grasp of the physical concepts and to clarify the implications of the theory, the students must have a good practice of solving the relevant problems. Therefore we have included a good number of selected, instructive and modern problems in each chapter of the book. In a chapter, some solved problems have been given as examples and several unsolved problems have been systematically and methodically arranged generally in two sets — Set I and Set II. When an average student solves the problems of Set I, he is encouraged to tackle relatively difficult problems. However a good student feels pleasure and intellectually satisfied by solving the most difficult problems, contained in Set II.

I wish to dedicate this book to my wife, Raj Kumari, and my children, Dr. Sharad, Ram and Tanuja, for their patience and assistance in several ways throughout the writing and preparation of the manuscript. The author is especially thankful to Miss Anita for carefully typing the manuscript on the computer.

I shall feel highly satisfied and amply rewarded in case the student community is benefitted to any substantial extent.

Any suggestion for the improvement of subject matter will be gratefully received.

Contents

Chapter *Page No.*

1. **Introductory Ideas** 1—26
(*Newtonian Mechanics*)
1.1. Introduction 1
1.2. Space and Time (*Frame of Reference*) 1
1.3. Newton's Laws of Motion 2
1.4. Inertial Frames 4
1.5. Gravitational Mass 5
1.6. Mechanics of a Particle : Conservation Laws 6
1.6.1. Conservation of linear momentum 6
1.6.2. Conservation of angular momentum 6
1.6.3. Conservation of energy 7
1.6.3.(a) Work 7
1.6.3.(b) Kinetic energy and work-energy theorem 7
1.6.3.(c) Conservative force and potential energy 7
1.6.3.(d) Conservation theorem 8
1.6.3.(e) First integrals of motion 8
1.7. Mechanics of a System of Particles 9
1.7.1. External and internal forces 9
1.7.2. Centre of mass 10
1.7.3. Conservation of linear momentum 10
1.7.4. Centre of mass-frame of reference 11
1.7.5. Conservation of angular momentum 11
1.7.6. Note on consevation throrems of linear and angular momentum for a system of particles 12
1.7.7. Relation between angular momentum (J) and angular momentum about center of mass (J_{cm}) 12
1.7.8. Conservation of energy 13
1.7.8.(a) Kinetic energy 13
1.7.8.(b) Potential energy 14
1.7.8.(c) Conservation theorem 15
Questions 19
Problems — SET-I and SET-II 20
Objective Type Questions 25
Short Answer Questions 26

2. **Langrangian Dynamics** 27—74
2.1. Introduction 27
2.2. Basic Concepts 27

(1) Coordinate systems 27
(2) Degrees of freedom — Configuration space 28
2.3. Constraints 29
2.3.1. Holonomic constraints 29
2.3.2. Nonholonomic constraints 30
2.3.3. Some more examples of holonomic and non-holonomic constraints 30
2.3.4. Forces of constraints 31
2.3.5. Difficulties introduced by the constraints and their removal 32
2.4. Generalized coordinates 34
2.5. Principle of Virtual Work 35
2.6. D' Alembert's principle 36
2.7. Langrange's Equations from D'Alembert's principle 38
2.8. Procedure for formation of Langrange's Equations 41
2.9. Langrange's Equations in presence of Non-conservative forces 47
2.10. Generalized Potential — Lagrangian for a Charged Particle Moving in an Electromagnetic field (Gyroscopic Forces) 49
2.11. Hamilton's Principle and Langrange's Equations 51
2.12. Superiority of Lagrangian Mechanics over Newtonian Approach 53
2.13. Guage Invariance of the Lagrangian 53
2.14. Symmetry Properties of Space and Time and Conservation laws 55
2.14. Invariance under Galilean Transformation 57
Questions 64
Problems — SET-I and SET-II 66
Objective Type Questions 73
Short Answer Questions 74

3. Hamiltonian Dynamics **75—102**
3.1. Introduction 75
3.2. Generalized momentum and cyclic coordinates 75
3.3. Conservation Theorems 77
3.3.1. Conservation of linear momentum 77
3.3.2. Conservation of angular momentum 78
3.3.3. Significance of translation and rotation cyclic coordinates — symmetry properties 80
3.4. Hamiltonian Function H and Conservation of Energy : Jacobi's Integral 80
3.5. Hamilton's Equations 82
3.6. Hamilton's Equations in Different Coordinate Systems 84
3.7. Examples in Hamiltonian Dynamics 86
(1) Harmonic oscillator 86
(2) Motion of a particle in a central force field 87
(3) Charged particle moving in an electomagnetic field 88
(4) Compound pendulam 89
(5) Two dimensional harmonic oscillator 89
3.8. Routhain 94
Questions 96
Problems — SET-I and SET-I 97

Objective Type Questions 100
Short Answer Questions 101

4. Two-Body Central Force Problem **103—137**

4.1. Reduction of Two - Body Central Force Problem to the Equivalent One-Body Problem 103
4.2. Central Force and Motion in a Plane 106
4.3. Equations of Motion under Central Force and First Integrals 107
4.4. Differential Equation for an Orbit 108
4.5. Inverse Square Law of Force 109
4.6. Kepler's Laws of Plantery Motion and their Deduction 110
4.6.1. Deduction of the Kepler's first law 110
4.6.2. Deduction of the Kepler's second law 112
4.6.3. Deduction of the Kepler's third law (Period of Motiom in an Elliptical Orbit) 112
4.7. Stability of Orbit under Central Force 114
4.8. Artificial Statellites 121
4.9. Virial Theorem 124
4.10. Scattering in a Central Force Field — Scattering cross-section, Scattering angle, Impact parameter 125
4.11. Rutherford Scattering Cross-section 127
Questions
Problems — SET-I and SET-II 132
Objective Type Questions 136
Short Answer Questions 137

5. Variational Principles **138—162**

5.1. Introduction 138
5.2. The Calculus of Variations and Euler-Lagrange's Equations 138
5.3. Deduction of Hamilton's principle from D'Alembert's principle 146
5.4. Modified Hamilton's principle 147
5.5. Deduction of Hamiltion's Equations from Modified Hamiltion's Principle (or Variational Principle) 147
5.6. Deduction of Lagrange's Equations from Variational Principle for Non-conservative Systems (Holonomic Constraints) 148
5.7. Langrange's Equations of Motion for Non-holonomic Systems (Lagrange's Method of Undetermined Multipliers) 149
5.8. Physical Significance of Langrange's Multipliers λ_l 151
5.9. Examples of Lagrange's Method of Undetermined Multipliers 151
(1) Rolling hoop on an inlined plane 151
(2) Simple pendulum 152
5.10. Δ-Variation 153
5.11. Principle of Least Action 154
5.12. Other Forms of Principle of least Action 156
(1) For a conservation system 156
(2) Jacobi's form of the principle of least action 157
(3) Principle of least action in terms of arc length of the particle trajectory 157

Questions 158
Problems — SET-I and SET-II 159
Objective Type Questions 161
Short Answer Question 162

6. Canonical Transformations **163—178**
6.1. Canonical Transformations 163
6.2. Legendre Transformations 163
6.3. Generating Functions 164
6.4. Procedure for Application of Canonical Transformations 167
6.5. Condition for Canonical Transformations 167
6.6. Bilinear Invariant Condition 170
6.7. Intergal Invariant of Poincare 171
6.8. Infinitesimal Contact Transformations 173
Questions 174
Problems — SET-I and SET-II 174
Objective Type Questions 177
Short Answer Questions 178

7. Brackets and Liouville's Theorem **179—196**
7.1. Introduction 179
7.2. Poission's Brackets 179
7.3. Lagrange Brackets 182
7.4. Relation Between Lagrange and Poisson Brackets 182
7.5. Angular Momentum and Poisson Brackets 182
7.6. Invariance of Poisson Bracket with respect to Canonical Transformations 183
7.7. Phase Space 189
7.8. Liouville's Theorem 190
Questions 174
Problems — SET-I and SET-II 194
Objective Type Questions 195
Short Answer Questions 196

8. Hamilton-Jacobi Theory and Transition to Quantum Mechanics **197—231**
8.1. Introduction 197
8.2. The Hamilton-Jacobi Equation 197
8.3. Solution of Harmonic Oscillatior Problem by Hamilton-Jacobi Method 199
8.4. Hamilton-Jacobi Equation: Hamilton's Characteristic Function—Conservative Systems 201
8.5. Kepler's Problem : Solution by Hamilton's-Jacobi Method 204
8.6. Action and Angle Variables 207
8.7. Problem of Harmonic Oscillator using Action-Angle Variables (Deduction of Frequency of Motion) 209
8.8. Action-Angle Variables in General Case 210
8.9. Hamilton-Jacobi Equation-Geometrical Optics and Wave Mechanics (Transition from Classical to Quantum Mechanics) 214

Questions 226
Problems — SET-I and SET-II 226
Objective Type Questions 230
Short Answer Questions 231

9. Small Oscillations and Normal Modes (*Coupled-Oscillators*) **232—272**
9.1. Introduction 232
9.2. Potential Energy and Equilibrium—One Dimensional Oscillator 232
9.2.1. Stable, Unstable and Neutral Equilibrium 233
9.2.2. One-dimensional oscillator 233
9.3. Two Coupled Oscillators 235
9.3.1. Solution of the differential equations 236
9.3.2. Normal coordinates and normal modes 237
9.3.3. Kinetic and potential energies in normal coordinates 239
9.4. General Theory of Small Oscillations 240
9.4.1. Secular equation and eigenvalue equation 241
9.4.2. Solution of the eigenvalue equation 242
9.4.3. Small oscillations in normal coordinates 243
9.5. Examples of Two Coupled Oscillators 246
(1) Two coupled pendulums 246
(2) Double pendulum 250
9.6. Vibrations of a Linear Triatomic Molecule
9.7. Transverse Oscillations of N-coupled Masses on an Elastic String : Many Coupled Oscillators 256
9.8. Transition from Discrete to a Continuous System : Waves on a String 264
Questions 266
Problems — SET-I and SET-II 267
Objective Type Questions 270
Short Answer Questions 272

10. Dynamics of a Rigid Body **273—319**
10.1. Generalized Coordinates of a Rigid Body 273
10.2. Body and Space Reference Systems 274
10.3. Euler's Angles 276
10.4. Infinitesimal Rotations as Vectors — Angular Velocity 280
10.5. Components of Angular Velocity 280
10.6. Angular Momentum and Inertia Tensor 282
10.7. Principle Axes-Principle Moments of Inertia 284
10.8. Rotational Kinetic Energy of a Rigid Body 285
10.9. Symmetric Bodies 287
10.10. Moments of Interia for Different Body Systems 287
10.11. Euler's Equations of Motion for a Rigid Body 289
10.12. Torque-Free Motion of a Rigid Body 291
10.13. Force-free Motion of a Symmetrical Top 295
10.14. Motion of a Heavy Symmetrical Top 298
10.15. Fast Top 303

10.16. Sleeping Top 306
10.17. Gyroscope 307
Questions 314
Problems — SET-I and SET-II 315
Objective Type Questions 318
Short Answer Question 319

11. Noninertial and Rotating Coordinate Systems **320—333**
11.1. Noninertial Frames of Reference 320
11.2. Fictitious or Pseudo Force 320
11.3. Centrifugal Force 322
11.4. Uniformly Rotating Frames 323
11.5. Free Fall of a Body on Earth's Surface 325
11.6. Foucault's Pendulum 327
Questions 329
Problems — SET-I and SET-II 330
Objective Type Questions 332
Short Answer Questions 333

12. Special Theory of Relativity-Lorentz Transformations **334—366**
12.1. Introduction 334
12.2. Galilean Transformations 334
12.3. Principle of Relativity 336
12.4. Transformation of Force from One Inertial System to Another 336
12.5. Covariance of the physical Laws 337
12.6. Principle of Relativity and Speed of Light 337
12.7. The Michelson-Morley Experiments 339
12.8. Ether Hypothesis 341
12.9. Postulates of Special Theory of Relativity 342
12.10. Lorentz Transformations 342
12.11. Consequences of Lorentz Transformations 345
(1) Length contraction 345
(2) Simultaneity 345
(3) Time dilation 346
(4) Addition of velocities 349
12.12. Aberration of Light from Stars 353
12.13 Relativistic Doppler's Effect 355
Questions 359
Problems — SET-I and SET-II 360
Objective Type Questions 365
Short Answer Questions 365

13. Relativistic Mechanics **367—388**
13.1. Introduction 367
13.2. Conservation of Momentum at Relativistic Speeds — Variation of Mass with Velocity 367

13.3. Relativistic Energy — Mass-Energy Relation ($E = mc^2$) 370
13.4. Examples of Mass — Energy Conversion 371
13.5. Relation between Momentum and Energy and Conversation Laws 372
13.6. Transformation of Momentum and Energy 373
13.7. Praticles with Zero Rest Mass 374
13.8. Force in Relativistic Mechanics 374
13.9. Lorentz Transformation for Force 375
13.10. Equilibrium of Right-angled Lever 375
13.11. The Lagrangian and Hamiltonian of a Particle in Relativistic Mechanics 380
13.12. Relativistic Lagrangian and Hamiltonian of a Charged Particle in an Electromagnetic Field-Velocity Dependent Potential 382
Questions 383
Problems — SET-I and SET-II 383
Objective Type Questions 387
Short Answer Questions 387

14. Four Dimensional Formulation — Minkowski Space **389—419**
14.1. Introduction 389
14.2. Minkowski Space and Lorentz Transformations 389
14.3. World Point and World Line 392
14.4. Space-time Invertvals 392
14.5. Four-vectors 396
14.6. Examples of Four-vectors 398
(1) Position four-vector 398
(2) Four velocity or velocity four-vector 398
(3) Momentum four-vector 399
(4) Acceleration four-vector 399
(5) Four-force Minkowski force F_μ 400
14.7. Consevation of Four-momentum — application of Four-vectors 403
14.8. Covariant Formulation of Lagrangian and Hamiltonian 407
14.9. Geometrial Interpretation of Lorentz Transformations: Minkowski Diagrams 411
14.10. Geometrical Representation of Simultaneity, Length Contraction and Time Dilation 414
Questions 416
Problems - SET-I and SET-II 417
Objective Type Questions 418
Short Answer Questions 418

15. Convariant Formulation of Electrodynamics **420—437**
15.1. D'Alembertian Operator $\Box^2$ 420
15.2. Maxwell's Feild Equations 421
15.3. Maxwell's Equations in terms of Electromagnetic Potentials A and ϕ 423
15.4. Current Four-Vector 425
15.5. Transformation of Electromagnetic Potentials A and ϕ (Four-vector Potential) 425

15.6. Covariance of Maxwell's Field Equations in Terms of Four-Vectors 426
15.7. The Electromagnetic Field Tensor 427
15.8. Lorentz Transformations of Electric and Magnetic Fields
15.9. Covariant Form of Maxwell's Fields Equations in terms of Electromagnetic Field Tensor
15.10. Lorentz Force on Charged Particle 432
15.11. Lorentz Force in Covariant Form 433
Questions 434
Problems — SET-I and SET-II 435
Objective Type Questions 435
Short Answer Questions 436

16. Nonlinear Dynamics and Chaos **438—475**
16.1. Indroduction 438
16.2. Nonlinear Differential Equations 438
16.3. Phase Trajectories-Singular Points (Topological Methods) 439
16.4. Phase Trajectories of Linear Systems 440
16.5. Phase Trajectories of Non linear Systems 444
16.6. Limit Cycles-Attractors 451
16.7. N-Torus 453
16.8. Chaos 455
16.9. Logistic Map 455
16.10. Strange Attractor 462
16.11. Sensitivity to Initial Conditions and Parameters — Lyapunov Exponent 463
16.12. Poincare Sections 464
16.13. Driven Damped Harmonic Oscillator 464
16.14. Fractals 466
16.15. Integrable Hamiltonian and Invariant Tori 469
16.16. KAM Theorem 470
Questions 471
Problems 472
Objective Type Questions 473
Short Answer Questions 474

Index **477—480**

CHAPTER 1

Introductory Ideas
(Newtonian Mechanics)

1.1. INTRODUCTION

Mechanics is a branch of physics which deals with physical objects in motion and at rest under the influence of external and internal interactions. Mechanics had developed since ancient times on the basis of observations on the motion of material particles. Although efforts were made earlier to propose theoretical hypotheses regarding the relationship between force and motion, but it was not until Newton announced his famous laws of motion in 1687. *The mechanics based on Newton's laws of motion and alternatively developed by Lagrange, Hamilton and others is called classical mechanics.* When this mechanics deals with the Newton's laws and their consequences, it may be called as *Newtonian* or *vectorial mechanics*, because in this scheme, the quantities such as force, acceleration, momentum etc. are used which are essentially vectors. The alternative and superior schemes in classical mechanics, developed by D'Alembert, Lagrange, Hamilton and others constitute what is known as *analytical mechanics*. In the later, the basic quatities are scalars (*e.g.*, energy) rather than vectors and the dynamical relations are obtained by a systematic process of differentiation. This analytical approach has the further advantage that it can be generalized to quantum mechanics where Newton's laws are not applicable. In the present book, we plan to develop the analytical mechanics in detail. Actually classical mechanics was developed over several centuries on the basis of observations on the moving objects, having relatively low speeds. In 1905, Einstein, by including the experimental fact of constancy of speed of light in vacuum, proposed the special theory of relativity which modifies the classical ideas of space and time and deals with all particle-speeds. The relativistic mechanics of Einstein yeilds the results of Newtonian mechanics at relatively low speeds. However, the relativistic theory of Einstein does not modify the classical ideas regarding the basic nature of matter and radiation and hence it is often studied in classical mechanics.

Classical mechanics is found to be inadequate to describe the behaviour of particles of microscopic size such as electrons in atoms, nuclear particles etc. For the description of the small scale phenomena of atomic and nuclear physics, a new theory, known as quantum mechanics, has been formulated which, when applied to large bodies, gives the results obtained by using classical mechanics. Although classical mechanics fails to the two kinds of extremes, discussed above, but this theory is remarkably successful to deal with the motion of relatively massive particles and relatively slow moving objects, which we come across in innumerable situations. Many of the results of classical mechanics, such as the conservation laws of energy, linear momentum and angular momentum, are of universal validity even in relativistic mechanics and quantum mechanics.

Before starting to study the classical mechanics in depth, we discuss this chapter to summarize some basic concepts of interest from introductory mechanics.

1.2. SPACE AND TIME (*Frame of Reference*)

From our experience, we have some idea about the meaning of space and time. It is assumed (*i*) that the space and time are continuous, (*ii*) that the motion of a particle in space can be described by knowing its position at different instants of time, and (*iii*) that there are universal standards of length and time. S.I. units of measurement of length and time are meter and second respectively.

If a physical phenomenon (*e.g.*, passing of a particle through some point P) occurs in space, its position is known as the point; the time of occurrence and the point taken together are called an ***event***. In classical mechanics, we further assume (*i*) that there is a universal time scale, which means that two observers who have synchronized their clocks will always agree about the time of an event, (*ii*) that the geometry of the space is Euclidean, and (*iii*) that there is no limit, in principle, to the accuracy with which we can measure the position and momentum.

In order to describe the motion of a particle in space, we need to know its position at different instants of time. This needs the choice of reference body or coordinate system. *If we imagine a coordinate system attached to a rigid body and we describe the position of any particle relative to it, then such a coordinate system is called frame of reference.* For the location of the objects, the position vectors are drawn from the origin O of the coordinate system (Fig. 1.1). The simplest frame of reference is a *cartesian coordinate system*. In this system, the position of a particle at any point of its path is given by the position vector $\mathbf{r}$, expressed in terms of three coordinates (x, y, z) as

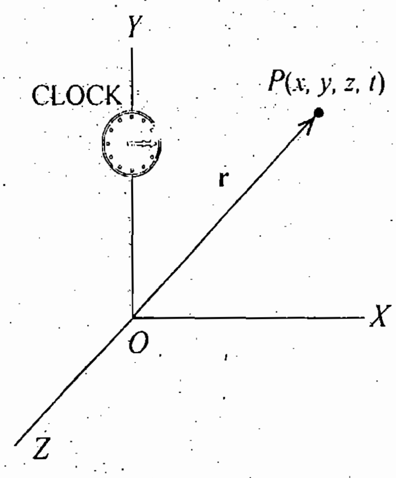


Fig. 1.1. Frame of reference

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad ...(1)$$

In order to know the position of the particle at different instants of time, an observer may be stationed at the origin with a clock to measure the time t. Thus we obtain the position vector $\mathbf{r}$ of the particle as function of time t *i.e.*,

$$\mathbf{r} = \mathbf{r}\,(t) \qquad ...(2)$$

Thus we obtain the velocity and acceleration as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}} \qquad ...(3)$$

$$\text{and } \mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}} \qquad ...(4)$$

The position and time recorded together constitute an event, represented by four coordinates (x, y, z, t) and the reference system, used for this purpose, may be called as *space-time reference system*.

1.3. NEWTON'S LAWS OF MOTION

Sir Isaac Newton expressed his ideas regarding the motion of bodies in the form of three laws which are considered as the basic laws of mechanics. In fact mechanics is a study of certain general relations that describe the interactions of material bodies. One general property of a material body is its *inertial mass*. Another new concept useful in describing interactions is *force*. These two concepts, inertial mass and force, were first defined in a quantitative manner by Isaac Newton. The definition of mass and force are contained in his three laws of motion.

(1) Law of Inertia (*First Law*) : *A body continues in its state of rest or constant velocity, unless not disturbed by some external influence.* The property of a body that it can not change its state of rest or constant velocity is called *inertia* and the influence under which the velocity of a particle changes is called *force*. The quantitative definitions of force and measure of inertia of a body, which we call mass, are contained in second and third laws of motion.

(2) **Law of Force (*Second Law*)** : *The time-rate of change of momentum is proportional to the impressed force, i.e.,*

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad ...(5)$$

Every body possesses the property of inertia or resistance to motion. This inertia is different for different bodies. The measure to this inertia for translation is called the *mass* of a body and is denoted by m. If v be the velocity of a body of mass m, then its momentum is defined by

$$\mathbf{p} = m\mathbf{v} \text{ and thus } \mathbf{F} = \frac{d}{dt}(m\mathbf{v})^* \qquad ...(5a)$$

Newton considered that mass of a body remains constant in motion. Therefore,

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \qquad ...(5b)$$

i.e. Force = mass × acceleration

This is the fundamental law of classical mechanics. Quantitatively, first law is the special case of second law, because if force is not acting on a body, *i.e.*, $\mathbf{F} = 0$, then $\frac{d\mathbf{v}}{dt} = 0$ and therefore $\mathbf{v} = \text{constant}$, including zero.

(3) **Law of Action and Reaction (*Third Law*)** : *To every action there is always equal and opposite reaction.* This means that if 1 and 2 bodies are interacting mutually, then

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \qquad ...(6)$$

i.e., force on 1st body due to 2nd = – force on 2nd body due to 1st.

But $\mathbf{F}_{12}$ = Rate of change of momentun of the 1st body

$$= \frac{d}{dt}(m_1\mathbf{v}_1) = m_1\frac{d\mathbf{v}_1}{dt}$$

and similarly, $\mathbf{F}_{21} = \frac{d}{dt}(m_2\mathbf{v}_2) = m_2\frac{d\mathbf{v}_2}{dt}$

Substituting in (6), we obtain

$$m_1\frac{d\mathbf{v}_1}{dt} = -m_2\frac{d\mathbf{v}_2}{dt}$$

If $a_1 = \left|\frac{d\mathbf{v}_1}{dt}\right|$ and $a_2 = \left|\frac{d\mathbf{v}_2}{dt}\right|$ denote for accelerations, then in magnitude

$$m_1 a_1 = m_2 a_2 \quad \text{or} \quad m_2 = \frac{m_1 a_1}{a_2} \qquad ...(7)$$

---

* Common experience tells us that a greater pull or force is required to change a definite amount of velocity in a certain time for a massive body than a lighter body. So Newton considered that mass also be included in the definition of force by defining the momentum as mass times velocity.

This means that if an isolated system of two bodies is interacting among themselves, then by measuring their accelerations the ratio of their masses can be determined. If one is the standard body of mass 1 kg (say m_1) then the other (m_2) can be determined. Thus Newton's third law defines mass uniquely and hence equation (5) can be used to define the force in a unique way.

1.4. INERTIAL FRAMES

Newton's laws of motion are valid in reference systems, known as *inertial frames. An inertial frame is the one in which the law of inertia holds true** *i.e.*, if a particle, subject to no external force, is found to move in a straight line with constant velocity (or to remain at rest), then the coordinate system used for this purpose is called *inertial frame.* Thus in an inertial frame, a body not experiencing any force ($\mathbf{F} = 0$) appears unaccelerated ($\mathbf{a} = 0$) because from Newton's second law

$$\mathbf{F} = m\mathbf{a} = 0 \text{ or } \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = 0 \quad \ldots(8)$$

All those frames, which are moving with constant velocity relative to an inertial frame, are also inertial. In order to prove this statement, let us consider an inertial frame S and another frame S', which is moving with constant velocity $\mathbf{v}$ relative to S. Initially at $t = 0$, if the positions of the origins of the two frames coincide, then in the two frames, the position vectors of any particle P at any instant t can be related as [Fig. 1.2]

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t \; (\because \overline{OO'} = \mathbf{v}t)$$

$$\text{or } \mathbf{r}' = \mathbf{r} - \mathbf{v}t \quad \ldots(9)$$

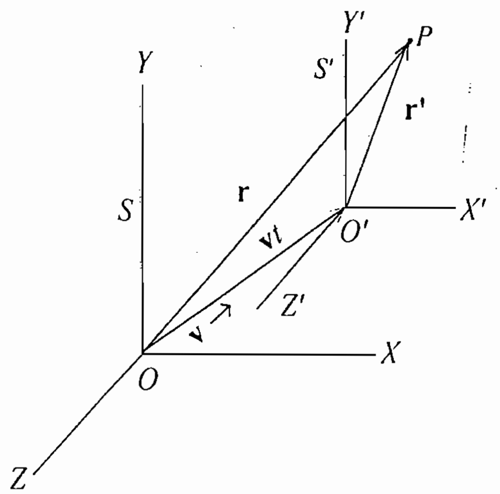


Fig. 1.2 : Frame moving with constant velocity.

In Newtonian mechanics, it is assumed that the time is universal. This implies that the time of an event is the same relative to various observers in different states of motion.

Differentiating eq. (9) with respect to time and writing $\frac{d\mathbf{r}}{dt} = \mathbf{u}$, we obtain

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} \quad \ldots(10)$$

where $\mathbf{u}$ is the velocity of the particle in frame S and $\mathbf{u}'$ in S'. Eqs. (9) and (10) relate the position vectors and velocity vectors of the particle P in S and S' frames.

Differentiating eq. (10) again with respect to time and remembering that $\mathbf{v}$ is constant, we get

$$\frac{d\mathbf{u}'}{dt} = \frac{d\mathbf{u}}{dt} \text{ or } \frac{d^2\mathbf{r}'}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} \text{ or } \mathbf{a}' = \mathbf{a} \quad \ldots(11)$$

Thus a particle experiences the same acceleration in two frames out of which one is inertial and the other is moving with constant velocity relative to the inertial. Now, if the acceleration of the particle in frame S is zero, its acceleration in S' is also zero. But S is an inertial frame, hence S' must also be an inertial

* The law of inertia was first stated by Galileo, therefore the inertial frames are also named as *Galilean frames of reference.*

frame . Thus we conclude that if a frame is inertial, then any frame, moving with constant velocity relative to it, is also an inertial frame.

Eq. (9) represents *Galilean transformation,* connecting the coordinates in two inertial frames (S and S') in constant relative motion.

If the observed acceleration of a body of mass m in an inertial frame is **a**, then the force observed on the body in this frame is

$$\mathbf{F} = m\mathbf{a} \qquad ...(12)$$

Let us think a little in depth about an inertial frame. Obviously *an inertial frame is unaccelerated,* because if the frame is accelerated, an observer stationed in this frame will see an acceleration of a force-free particle which violates the first law. The value of the acceleration **a**, used in the second law $\mathbf{F} = m\mathbf{a}$, is measured with respect to this unaccelerated frame. But the question arises, how is this unaccelerated frame realized in practice. Since fixed stars are at large distances and can be thought as being free from interactions with other bodies.We can think the fixed stars as a standard unaccelerated reference system or inertial frame. The sun with respect to these fixed stars moves with uniform velocity and hence a frame fixed on sun is also an inertial frame. A frame fixed on earth is not inertial because the earth is rotating about its axis and simultaneously it is moving in its orbit around the sun. The centripetal accelerations due to spin and orbital motions of the earth are of the order of 3.4 cm/sec^2 (at the equator) and 0.6 cm/sec^2. Now, *if a frame, fixed on the earth and accounted for these accelerations will work well as an inertial frame.* However, if we are dealing motions negligibly small compared to the earth's motion, a frame fixed on the earth can be approximated as an inertial frame.

Validity of Newton's Laws : Newton's first and second laws do not hold correct in the accelerated and rotating frames. If a particle is experiencing no force in an inertial frame, then the observer of the accelerated frame will see an acceleration and consequently a fictitious force on the particle. *The accelerated frames are called non-inertial frames* and we shall deal them later. Newton's third law of motion is not correct when a force is acting at a distance because the forces and actions can not travel faster than light. In other words, if the first particle produces any change in the second particle at a distance, the reaction reaches on the first particle after a finite interval of time. This means that simultaneously action and reaction are not equal. However, Newton's third law is still correct for the bodies at rest and for contact forces.

1.5. GRAVITATIONAL MASS

The gravitational force exerted on one body by another body, such as earth, is given by

$$F = G\frac{M_E\, m_G}{R^2} \quad \text{or} \quad m_G = \frac{FR^2}{GM_E} \qquad ...(13)$$

The mass of body, determined from the formula (13), is called the *gravitational mass* and is denoted by m_G. In eq. (13) M_E is the mass of the earth and R is its radius.

Let the two bodies of inertial masses m_1 and m_2 be allowed to fall under gravity. Experimentally we know that all bodies on earth fall with the same acceleration g. So that

$$m_1 g = \frac{GM_E\, m_{G_1}}{R^2} \quad \text{and}\, m_2 g = \frac{GM_E\, m_{G_2}}{R^2}$$

Therefore, $$\frac{m_1}{m_{G_1}} = \frac{m_2}{m_{G_2}} \qquad ...(14)$$

This means that the ratio of the inertial mass to the gravitational mass is constant say K, *i.e.*, $\frac{m}{m_G} = K$ or $m = K\, m_G$ *i.e.*, the inertial and gravitational masses are proportional to each other. By a proper choice of units, we can make this raitio (K) equal to unity *i.e.*, $m = m_G$. Thus it is assumed that the gravitational mass and the inertial mass are one and the same. Einstein's general theory of relativity uses this basic postulate as a starting point. This equivalence of inertial mass and gravitational mass is called the *principle of equivalence.* In view of this principle, we shall not distinguish between the inertial mass and gravitational mass.

1.6. MECHANICS OF A PARTICLE : CONSERVATION LAWS

We apply Newtonian mechanics to deduce conservation laws for a particle in motion. These laws tell us under what conditions the mechanical quantities like linear momentum, angular momentum, energy etc. are constant in time.

1.6.1. Conservation of Linear Momentum

If a force $\mathbf{F}$ is acting on a particle of mass m, then according to Newton's second law of motion, we have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) \qquad ...(15)$$

where $\mathbf{p} = m\mathbf{v}$ is the linear momentum of the particle.

If the external force, acting on the particle, is zero, then

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = 0$$

or $$\mathbf{p} = m\mathbf{v} = \text{constant} \qquad ...(16)$$

Thus in absence of external force, the linear momentum of a particle is conserved. This is the conservation theorem for a free particle.

1.6.2. Conservation of Angular Momentum

The angular momentum of a particle P of a mass m about a point O (Fig. 1.3) is defined as

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} \qquad ...(17)$$

where $\mathbf{r}$ is the position vector of the particle P and $\mathbf{p} = m\mathbf{v}$ is its linear momentum.

If the force on the particle is $\mathbf{F}$, then the moment of force or torque about O is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad ...(18)$$

If we differentiate (17) with respect to t, then

$$\frac{d\mathbf{J}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

or $$\frac{d\mathbf{J}}{dt} = \mathbf{r} \times \mathbf{F} \left[\because \frac{d\mathbf{r}}{dt} \times \mathbf{p} = \mathbf{v} \times m\mathbf{v} = 0 \text{ and } \mathbf{F} = \frac{d\mathbf{p}}{dt}\right]$$

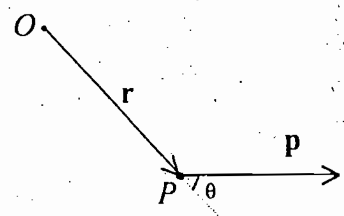


Fig. 1.3. Angular momentum of a particle P along a point O.

Therefore, $$\tau = \frac{d\mathbf{J}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \quad ...(19)$$

Thus *the time rate of change of angular momentum of a particle is equal to the torque acting on it.* This equation (19) is analogous to eq. (15) for linear motion.

Now, if the torque acting on the particle is zero, *i.e.*, $\tau = 0$, then

$$\frac{d\mathbf{J}}{dt} = 0 \quad \text{or} \quad \mathbf{J} = \text{constant} \quad ...(20)$$

Therefore *the angular momentum of a particle is constant of motion in absence of external torque.* This is the *conservation theorem of angular momentum* of a particle.

1.6.3. Conservation of Energy

1.6.3(a). **Work :** Work done by an external force **F** upon a particle in displacing from point 1 to point 2 is defined as

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \quad ...(21)$$

1.6.3(b). Kinetic Energy and Work-Energy Theorem : According to Newton's second law, $\mathbf{F} = m\, d\mathbf{v}/dt$ and hence

$$\mathbf{F} \cdot d\mathbf{r} = m\frac{d\mathbf{v}}{dt} \cdot d\mathbf{r} = m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt \quad \left[\because\ d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = \mathbf{v}\, dt\right]$$

$$= m\frac{d}{dt}\left[\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right] dt$$

$$= d\left[\tfrac{1}{2}mv^2\right]$$

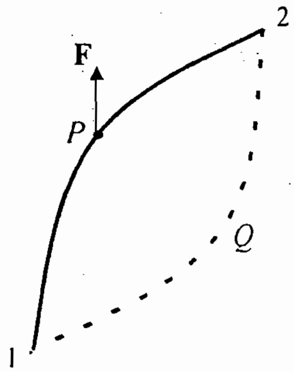


Fig.1.4 : Work done by a force on a particle

Therefore, equation (21) is obtained to be

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 d\left[\tfrac{1}{2}mv^2\right] = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

The scalar quantity ½ mv^2 is defined as the *kinetic energy* and denoted by T. Thus the work done by the force acting on the particle appears equal to the change in the kinetic energy *i.e.*,

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = T_2 - T_1 \quad ...(22)$$

This is known as ***work-energy theorem***.

1.6.3(c). Conservative Force and Potential Energy : If the work done (W_{12}) by the force in moving a particle from point 1 to point 2 is the same for any possible path between the points, then the force (and the system) is said to be conservative. The region in which the particle is experiencing a conservative force is called as conservative force field.

Thus for conservative force [Fig. 1.4]

$$P\int_1^2 \mathbf{F} \cdot d\mathbf{r} = Q\int_1^2 \mathbf{F} \cdot d\mathbf{r} \quad \text{or} \quad P\int_1^2 \mathbf{F} \cdot d\mathbf{r} + Q\int_2^1 \mathbf{F} \cdot d\mathbf{r} = 0 \quad i.e., \oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad ...(23)$$

Thus, *if the force is conservative, the work done on the particle around a closed path in the force field is zero.* In case of a nonconservative force like friction, the amount of work done around different closed paths are different and not zero.

According to Stoke's theorem in vector analysis, we can transform the equation (23) as

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint \text{curl } \mathbf{F} \cdot d\mathbf{s}$$

Since the work done is zero around any closed path in the conservative force field and does not depend on the length of the path, we may carry out the integration over the perimeter of the area $d\mathbf{s}$. This gives

$$\oint \mathbf{F} \cdot d\mathbf{r} = \text{curl } \mathbf{F} \cdot d\mathbf{s} = 0$$

But, $d\mathbf{s} \neq 0$ and hence in general

$$\text{curl } \mathbf{F} = 0 \text{ or } \nabla \times \mathbf{F} = 0 \qquad \text{...(24)}$$

Therefore the force can be expressed as

$$\mathbf{F} = -\nabla V = -\left(\hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\mathbf{j}}\frac{\partial V}{\partial y} + \hat{\mathbf{k}}\frac{\partial V}{\partial z}\right) \qquad \text{...(25)}$$

$$\text{because } \nabla \times \nabla V = \hat{\mathbf{i}}\left(\frac{\partial^2 V}{\partial y\,\partial z} - \frac{\partial^2 V}{\partial z\,\partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial^2 V}{\partial z\,\partial x} - \frac{\partial^2 V}{\partial x\,\partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2 V}{\partial x\,\partial y} - \frac{\partial^2 V}{\partial y\,\partial x}\right) = 0$$

This scalar function V is called the *potential* or *potential energy* and depends on position. In case, if we add any constant quantity to V, equation (25) does not change and hence the *zero* or *reference level* of the potential function V is arbitrary and can be chosen at convenience.

If we take scalar product of $d\mathbf{r}$ with (25) and integrate from position 1 to position 2, we obtain

$$\int_1^2 \mathbf{F} \cdot d\mathbf{r} = -\int_1^2 \nabla V \cdot d\mathbf{r} = -\int_1^2 dV = V_1 - V_2 \qquad \text{...(26)}$$

Now, if we assume the position 1 as ∞ and the potential energy to be zero there, then the potential energy at a point $\mathbf{r}$ (*position* 2) is given by

$$V(\mathbf{r}) = -\int_\infty^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} \qquad \text{...(27)}$$

From eq. (26) we see that the work done by the conservative force is

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = V_1 - V_2 \qquad \text{...(28)}$$

which is the change in potential energy when the particle moves from position 1 to 2.

1.6.3 (d). Conservation Theorem : According to equation (22), the amount of work done by a force in moving a particle from position '1' to '2' is given by the change in kinetic energy *i.e.*,

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = T_2 - T_1 \qquad \text{...(29)}$$

Therefore, from (28) and (29), we obtain

$$V_1 - V_2 = T_2 - T_1 \quad \text{or} \quad T_1 + V_1 = T_2 + V_2 = \text{Constant} \qquad \text{...(30)}$$

Thus *the sum of kinetic and potential energies (i.e., total mechanical energy) of a particle remains constant in a conservative force field.* This is known as *the law of conservation of energy.*

Remember that the law of conservation of energy gives us no new information, not contained in Newton's second law of motion. If we multiply by $\mathbf{v} = d\mathbf{r}/dt$ to both sides of $\mathbf{F} = m \cdot d\mathbf{v}/dt$ and integrate with respect to t, we obtain

$$\int m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v}dt = \int \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt + \text{Constant (say } E)$$

or

$$\int \frac{d}{dt}[\frac{1}{2}m\mathbf{v} \cdot \mathbf{v}]\, dt = \int \mathbf{F} \cdot d\mathbf{r} + E$$

or

$$\int d\,[\tfrac{1}{2}mv^2] - \int \mathbf{F} \cdot d\mathbf{r} = E \text{ or } \tfrac{1}{2}mv^2 - \int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = E$$

i.e.,

$$T + V = E \qquad ...(31)$$

where the constant E is the total energy of the particle. Equation (31) represents the conservation energy theorem.

Conservation laws, obtained above, are the constants of motion and referred as the first integrals of the motion. They are very useful because we get some important information physically about the system just at a glance from these integrals. In fact once integration of the equation of motion under certain condition on the system yields the first integral. Since Newton's equation is a second order differential equation, these first integrals of motion are in fact first order differential equations. We shall discuss further regarding these integrals later on.

1.7. MECHANICS OF A SYSTEM OF PARTICLES

1.7.1. External and Internal Forces

In the last section, we arrived at some results, specially conservation theorems, for the mechanics of a particle. These results can be easily generalized to the case of a system of particles by taking care of mutual interactions. Now, if a mechanical system consists of two or more particles, then the force on the i^{th} particle is given by

$$\mathbf{F}_i = \mathbf{F}_i^e + \sum_{j=1}^{N} \mathbf{F}_{ij} \qquad ...(32)$$

where $\mathbf{F}_i^e$ is the *external force,* acting on the i^{th} particle due to sources outside the system. $\mathbf{F}_{ij}$ is the *internal force* on the i^{th} particle due to the j^{th} particle and the total internal force due to all other particles (j =1 to N) on the i^{th} particle is represented by the sum in equation (32), where N is the number of particles in the system and $\mathbf{F}_{ii}$, the force of i^{th} particle on itself, is naturally zero.

According to Newton's second law

$$\mathbf{F}_i = \dot{\mathbf{p}}_i = m_i\frac{d\mathbf{v}_i}{dt} = m_i\frac{d^2\mathbf{r}_i}{dt^2}$$

Now, when the sum is taken over all the particles of the system, equation (32) takes the form

$$\frac{d^2}{dt^2}\sum_i m_i\mathbf{r}_i = \sum_i \mathbf{F}_i^e + \sum_i\sum_{\substack{j \\ i\neq j}} \mathbf{F}_{ij} \qquad ...(33)$$

On the right hand side of equation (33) first sum represents the total external force $\mathbf{F}^e$. According to Newton's third law, any two particles of the system exert equal and opposite forces on each other, *i.e.*,

$$\mathbf{F}_{ij} + \mathbf{F}_{ji} = 0 \qquad ...(34)$$

Since the second sum in equation (33) represents the internal forces in pairs and for each pair the resultant force is zero, consequently this sum vanishes.

Thus, equation (33) is

$$\mathbf{F}^e = \frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i \qquad ...(35)$$

1.7.2. Centre of Mass

We define the centre of mass $\mathbf{R}$ of the system by

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M} \qquad ...(36)$$

where $\sum_i m_i = M$ is the total mass of the system. In view of eq. (36), eq. (35) assumes the form

$$\mathbf{F}^e = M\frac{d^2\mathbf{R}}{dt^2} = M\,\mathbf{a}_{cm} \qquad ...(37)$$

Thus the acceleration of the centre of mass is due to only the external forces and is given by Newton's second law of motion. Thus *the centre of mass of a system of particles moves as if it were a particle of mass equal to the total mass of the system subjected to the external forces applied on the system.*

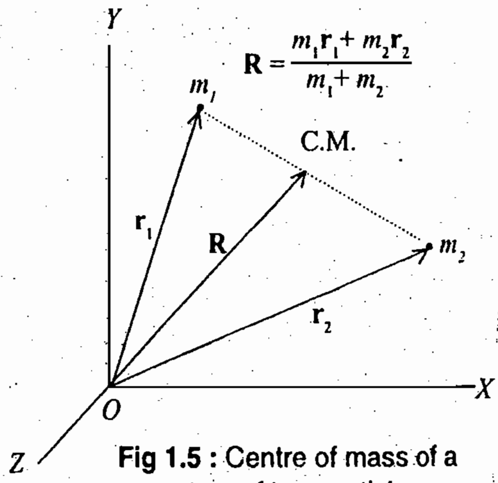


Fig 1.5 : Centre of mass of a system of two particles.

1.7.3. Conservation of Linear Momentum

If we differentiate eq. (36) with respect to t, we obtain

$$M\frac{d\mathbf{R}}{dt} = m_1\frac{d\mathbf{r}_1}{dt} + m_2\frac{d\mathbf{r}_2}{dt} + \ldots + m_N\frac{d\mathbf{r}_N}{dt}$$

or

$$M\mathbf{V} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots\ldots\ldots + m_N\mathbf{v}_N = \sum_{i=1}^{N} m_i \mathbf{v}_i \qquad ...(38)$$

which gives the velocity ($\mathbf{V}$) of centre of mass. The sum $\sum m_i \mathbf{v}_i = \mathbf{P}$ is the total linear momentum of all the particles of the system.

Thus $\qquad \mathbf{P} = M\mathbf{V} \qquad ...(39)$

Thus *the total linear momentum of the system is equal to the product of total mass of the system and the velocity of centre of mass.*

Differentiating eq. (39) with respect to t, we get

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}(M\mathbf{V}) = M\frac{d\mathbf{V}}{dt} = M\frac{d^2\mathbf{R}}{dt^2} \quad ...(40)$$

Hence by using eq. (37), the total external force on the system is

$$\mathbf{F}^e = \frac{d\mathbf{P}}{dt} = \frac{d}{dt}(M\mathbf{V}) \quad ...(41)$$

When $\mathbf{F}^e = 0$,

$$\mathbf{P} = M\mathbf{V} = \sum_i m_i \mathbf{v}_i = \text{constant} \quad ...(42)$$

Thus *if the total external force* $\mathbf{F}^e$ *on the system is zero, its total linear momentum is the constant of motion*. This is the *law of conservation of linear momentum for a system.*

1.7.4. Centre of Mass-Frame of Reference

An inertial frame attached with the centre of mass of an isolated system (*i.e.*, a system free from external forces) of particles is called the *centre of mass-frame of reference* or *C-frame of reference*. In this, C-frame of reference, the centre of mass remains at rest *i.e.* $\mathbf{V} = 0$. So that in view of eq. (39), the total linear momentum of the system in C-frame of reference is always zero, *i.e.*

$$\mathbf{P} = M\mathbf{V} = \sum_i m_i \mathbf{v}_i = 0 \text{ (in C-frame of reference)}$$

This is why the C-frame is called the zero-momentum frame. This C-frame is important because several experiments which we perform in the laboratory (or L-frame) can be more simply analyzed in the centre of mass frame of referene.

1.7.5. Conservation of Angular Momentum

If $\mathbf{J}_1$, $\mathbf{J}_2$, ... are the angular momenta of various particles of a system about a given point O, the total angular momentum about the point O is given by

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + ... + \mathbf{J}_N = (\mathbf{r}_1 \times \mathbf{p}_1) + (\mathbf{r}_2 \times \mathbf{p}_2) + ...(\mathbf{r}_N \times \mathbf{p}_N)$$

or

$$\mathbf{J} = \sum_{i=1}^{N} (\mathbf{r}_i \times \mathbf{p}_i) \quad ...(43)$$

Also

$$\frac{d\mathbf{J}}{dt} = \sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i (\mathbf{r}_i \times \mathbf{F}_i) \qquad (\because \dot{\mathbf{r}}_i \times \mathbf{p}_i = \mathbf{v}_i \times m\mathbf{v}_i = 0) \quad ...(44)$$

If we take product with $\mathbf{r}_i$ in eq. (32) and sum over all the particles of the system, then

$$\sum_i (\mathbf{r}_i \times \mathbf{F}_i) = \sum_i (\mathbf{r}_i \times \mathbf{F}_i^e) + \sum_i \sum_j (\mathbf{r}_i \times \mathbf{F}_{ij}) \quad ...(45)$$

The last term contains the double sum for $i, j = 1$ to N and hence it is a sum of the pairs of the form, given by

$$\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{r}_{ij} \times \mathbf{F}_{ij}$$

because $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$ according to Newton's third law of motion.

Now, if the internal forces between any two particles of the system in addition to being equal and opposite be central *i.e.*, lie along the line joining them, then from the property of cross product $\mathbf{r}_{ij} \times \mathbf{F}_{ij} = 0$.

Thus the last term of eq. (45) vanishes and hence

$$\sum_i (\mathbf{r}_i \times \mathbf{F}_i) = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i^e\right) = \tau^e$$

But from equation (44), we have

$$\sum_i (\mathbf{r}_i \times \mathbf{F}_i) = \frac{d\mathbf{J}}{dt}$$

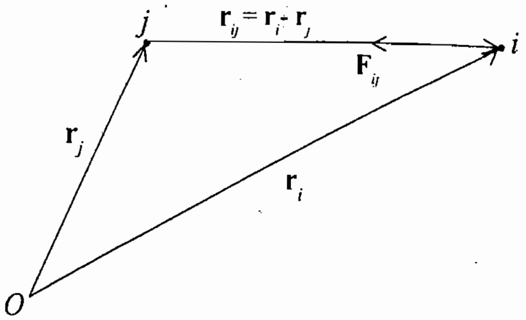


Fig 1.6 : Assumption of the force $\mathbf{F}_{ij}$ parallel to $\mathbf{r}_{ij}$

Thus, $$\tau^e = \frac{d\mathbf{J}}{dt} \qquad ...(46)$$

This means that the time rate of change of total angular momentum of a system of particles is equal to the applied external torque on the system about the same point.

If $\tau^e = 0$,

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + ... + \mathbf{J}_N = \text{constant} \qquad ...(47)$$

In absence of the external torque, the total angular momentum of a system of particles is conserved. This is the *conservation theorem for total angular momentum.*

1.7.6. Note on Conservation Theorems of Linear and Angular Momentum for a System of Particles

We have stated the conservation theorems of linear and angular momentum of a system of particles by assuming the validity of Newton's third law for internal forces in the former case and in the later case additionally the central character of internal forces. Both of these conditions are satisfied for some physical forces, for example gravitational forces in a system, action reaction forces in a rotating mass attached to a string etc. However, there are action and reaction forces which do not obey the third law and also do not lie along the line joining the two particles. For example, if we consider two charges, moving with uniform velocities parallel to each other (which are not perpendicular to the line joining the two charges), then according to Bio-Savart law, the forces on the two charges due to each other are of course equal and opposite, but they do not lie along the line joining them. Further let us consider two charges so that instantaneously one charge is moving directly towards the other but the other is moving at right angles to the direction of the motion of the first. Consequently the other charge exerts a definite force on the first charge, but it does not experience any reaction force at all. In such cases, the conservation theorems of linear and angular momentum appear not to be correct. However, the laws of linear and angular momentum are known as the fundamental laws of nature and therefore, one has to investigate for finding the way for the validity of the conservation theorems. For examples, the sum of mechanical angular momentum and electromagnetic angular momentum of a system of moving charges remains constant in time.

1.7.7. Relation between Angular Momentum (J) and Angular Momentum about Centre of Mass ($\mathbf{J}_{cm}$)

The total angular momentum $\mathbf{J}$ of a system of particles can be expressed in a convenient and important form by using the velocity of the centre of mass and velocities of the particles relative to the centre of mass. If $\mathbf{R}$ and $\mathbf{V}$ are the position vector and velocity of the centre of mass and $\mathbf{r}_{ic}$ and $\mathbf{v}_{ic}$ those of a particle of mass

m_i relative to the centre of mass, then the position vector and velocity of the particle with respect to the reference point (O) are $\mathbf{r}_i = \mathbf{R} + \mathbf{r}_{ic}$ and $\mathbf{v}_i = \mathbf{V} + \mathbf{v}_{ic}$. Hence the total angular momentum is

$$\begin{aligned}\mathbf{J} &= \sum m_i(\mathbf{R} + \mathbf{r}_{ic}) \times (\mathbf{V} + \mathbf{v}_{ic})\\ &= \sum m_i(\mathbf{R} \times \mathbf{V}) + \sum m_i(\mathbf{R} \times \mathbf{v}_{ic}) + \sum m_i(\mathbf{r}_{ic} \times \mathbf{V}) + \sum m_i(\mathbf{r}_{ic} \times \mathbf{v}_{ic})\\ &= \mathbf{R} \times (\sum m_i)\mathbf{V} + \mathbf{R} \times (\sum m_i \mathbf{v}_{ic}) + (\sum m_i \mathbf{r}_{ic}) \times \mathbf{V} + \sum (\mathbf{r}_{ic} \times \mathbf{v}_{ic})\end{aligned} \quad ...(48)$$

But $\quad \mathbf{r}_{ic} = \mathbf{r}_i - \mathbf{R}$ or $m_i\mathbf{r}_{ic} = m_i\mathbf{r}_i - m_i\mathbf{R}$

Hence, $\quad \sum m_i\mathbf{r}_{ic} = \sum m_i\mathbf{r}_i - \sum m_i\mathbf{R} = \sum m_i\mathbf{r}_i - M\mathbf{R}$

where $\quad \sum m_i = M$, mass of all the particles of the system.

But according to the property of centre of mass, we know that

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + ... = \sum m_i\mathbf{r}_i$$

$$\therefore \quad \sum m_i\mathbf{r}_{ic} = 0; \text{ similarly } \sum m_i\mathbf{v}_{ic} = 0 \quad ...(49)$$

Hence, from eq. (48), we have

$$\mathbf{J} = \mathbf{R} \times M\mathbf{V} + \sum (\mathbf{r}_{ic} \times m_i\mathbf{v}_{ic}) \quad ...(50a)$$

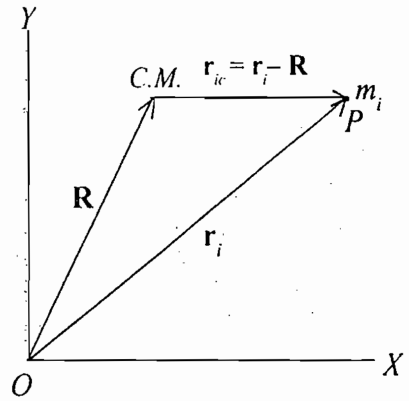


Fig. 1.7 : Coordinates of a particle relative to the centre of mass of the system.

In this equation $\sum(\mathbf{r}_{ic} \times m_i\mathbf{v}_{ic})$ represents the angular momentum of the system about the centre of mass, say $\mathbf{J}_{cm}$, and $M\mathbf{V} = \mathbf{P}$ is the total linear momentum. The quantity $\mathbf{R} \times \mathbf{P}$ or $\mathbf{R} \times M\mathbf{V}$ is the angular momentum of the centre of mass about the origin O. Thus

$$\mathbf{J} = \mathbf{J}_{cm} + \mathbf{R} \times \mathbf{P} \quad ...(50b)$$

In other words, *the total angular momentum of a system of particles about a point is the angular momentum of the system about the centre of mass, plus the angular momentum about the reference point of the system mass, concentrated at the centre of the mass.*

1.7.8. Conservation of Energy

Similar to a single particle, the total amount of work done by the forces acting on various particles of the system from an initial configuration 1 to final configuration 2 is given by

$$W_{12} = \sum_{i=1}^{N} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_i \int_1^2 \mathbf{F}_i^e \cdot d\mathbf{r}_i + \sum_i \sum_j \int_1^2 \mathbf{F}_{ij} \cdot d\mathbf{r}_i \quad ...(51)$$

1.7.8(a). Kinetic Energy : But according to second law, $\mathbf{F}_i = m_i \dfrac{d\mathbf{v}_i}{dt}$

$$W_{12} = \sum_{i=1}^{N} \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \sum_i \int_1^2 m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i \, dt$$

$$= \sum_{i=1}^{N} \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right) = \left[\sum_i \frac{1}{2} m_i v_i^2\right]_1^2$$

$$= T_2 - T_1 \quad ...(52)$$

Thus the work done is again equal to the change in kinetic energy (**work-energy theorem**), where

$$T = \sum_i \frac{1}{2} m_i v_i^2 \quad ...(53)$$

denotes the kinetic energy of the system.

If $\mathbf{v}_{ic} = \mathbf{v}_i - \mathbf{V}$ is the velocity of the i^{th} particle relative to the velocity of the centre of mass, then

$$T = \sum_i \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_i \frac{1}{2} m_i (\mathbf{v}_{ic} + \mathbf{V}) \cdot (\mathbf{v}_{ic} + \mathbf{V})$$

$$= \sum_i \frac{1}{2} m_i v_{ic}^2 + \sum_i \frac{1}{2} m_i V^2 + \sum_i m_i \mathbf{v}_{ic} \cdot \mathbf{V} = \sum_i \frac{1}{2} m_i v_{ic}^2 + \frac{V^2}{2} \sum_i m_i + \mathbf{V} \cdot \sum_i m_i \mathbf{v}_{ic}$$

But $\sum_i m_i = M$, total mass of the system and $\sum_i m_i \mathbf{v}_{ic} = 0$ [from eq. (49)]

Thus $$T = \sum_i \frac{1}{2} m_i v_{ic}^2 + \frac{1}{2} M V^2 \qquad ...(54)$$

Thus the total kinetic energy of a system of particle is the sum of kinetic energy of motion abbut the centre of mass plus the kinetic energy of motion of the total mass of the system, as if it were concentrated at the centre of mass.

1.7.8 (b). Potential Energy : In eq. (51), if the external and internal forces both are conservative and derivable from scalar potential, then

$$\mathbf{F}_i^e = -\nabla_i V_i = -\left[\hat{\mathbf{i}} \frac{\partial V_i}{\partial x_i} + \hat{\mathbf{j}} \frac{\partial V_i}{\partial y_i} + \hat{\mathbf{k}} \frac{\partial V_i}{\partial z_i}\right] \qquad ...(55)$$

and $$\mathbf{F}_{ij} = -\nabla_i V_{ij} = -\left[\hat{\mathbf{i}} \frac{\partial V_{ij}}{\partial x_i} + \hat{\mathbf{j}} \frac{\partial V_{ij}}{\partial y_i} + \hat{\mathbf{k}} \frac{\partial V_{ij}}{\partial z_i}\right] \qquad ...(56)$$

If the internal forces are central in nature, the potential energy V_{ij} will be a function of scalar distance $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ only. Then

$$V_{ij} = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \qquad ...(57)$$

So that $$\frac{\partial V_{ij}}{\partial x_i} = \frac{\partial V_{ij}}{\partial r_{ij}} \frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \qquad ...(58)$$

because $$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}$$

and hence $$\frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{r_{ij}}$$

Similarly, $$\frac{\partial V_{ij}}{\partial y_i} = \frac{(y_i - y_j)}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \text{ or } \frac{\partial V_{ij}}{\partial z_i} = \frac{(z_i - z_j)}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}}$$

Therefore, from equation (56), we have

$$\mathbf{F}_{ij} = -\frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \left[(x_i - x_j)\hat{\mathbf{i}} + (y_i - y_j)\hat{\mathbf{j}} + (z_i - z_j)\hat{\mathbf{k}}\right]$$

$$= -(\mathbf{r}_i - \mathbf{r}_j) \frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}}$$

Also similarly, $$\mathbf{F}_{ji} = -\nabla_j V_{ij} = -\left[\hat{\mathbf{i}} \frac{\partial V_{ij}}{\partial x_j} + \hat{\mathbf{j}} \frac{\partial V_{ij}}{\partial y_j} + \hat{\mathbf{k}} \frac{\partial V_{ij}}{\partial z_j}\right]$$

$$= (\mathbf{r}_i - \mathbf{r}_j)\frac{1}{r_{ij}}\frac{\partial V_{ij}}{\partial r_{ij}} \quad \left[\text{Here, } \frac{\partial r_{ij}}{\partial x_j} = -\frac{(x_i - x_j)}{r_{ij}} \text{ etc.}\right] \qquad ...(60)$$

Thus the internal forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$ between the i^{th} and j^{th} particles are equal and opposite and automatically satisfy third law and lie along the line $(\mathbf{r}_i - \mathbf{r}_j)$ joining the two particles.

Now, if we consider the last term of equation (51), then it can be written as

$$\sum_{\substack{i\ j \\ i \neq j}}\sum \int_1^2 \mathbf{F}_{ij} \bullet d\mathbf{r}_i = \frac{1}{2}\sum_{\substack{i\ j \\ i \neq j}}\sum \int_1^2 (\mathbf{F}_{ij} \bullet d\mathbf{r}_i + \mathbf{F}_{ji} \bullet d\mathbf{r}_j) = \frac{1}{2}\sum_{\substack{i\ j \\ i \neq j}}\sum \int_1^2 -(\nabla_i V_{ij} \bullet d\mathbf{r}_i + \nabla_j V_{ij} \bullet d\mathbf{r}_j)$$

$$= -\frac{1}{2}\sum_{\substack{i\ j \\ i \neq j}}\sum \int_1^2 \nabla_{ij} V_{ij} \bullet d\mathbf{r}_{ij} \qquad ...(61)$$

because $\qquad \nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij} \quad \left[\because \text{From (57)} \frac{\partial V_{ij}}{\partial x_i} = \frac{\partial V_{ij}}{\partial x_{ij}}\right]$

and $\qquad d\mathbf{r}_i - d\mathbf{r}_j = d\mathbf{r}_{ij}$

Thus eq. (51) in view of eqs. (55) and (61) is

$$W_{12} = -\sum_i \int_1^2 \nabla V_i \bullet d\mathbf{r}_i - \frac{1}{2}\sum_{\substack{i\ j \\ i \neq j}}\sum \int_1^2 \nabla_{ij} V_{ij} \bullet d\mathbf{r}_{ij} = -\sum_i \int_1^2 dV_i - \frac{1}{2}\sum_{\substack{i\ j \\ i \neq j}}\sum \int_1^2 dV_{ij}$$

$$= -\left[\sum_i [V_i]_1^2 + \sum_{\substack{i\ j \\ i \neq j}}\sum [V_{ij}]_1^2\right] = V_1 - V_2 \qquad ...(62)$$

where V the total potential energy of the system is defined as

$$V = \sum_i V_i + \frac{1}{2}\sum_{\substack{i\ j \\ i \neq j}}\sum V_{ij} \qquad ...(63)$$

1.7.8(c). Conservation Theorem : Now, we obtain from eqs. (52) and (62)

$$T_2 - T_1 = V_1 - V_2 \text{ or } T_1 + V_1 = T_2 + V_2 \qquad ...(64)$$

which is the law of conservation of energy for a system of particles.

It is to be noted that in eq. (63) the total potential energy V has been defined, provided the external and internal forces are both derivable from scalar potentials. We may call the second term in eq. (63) as the internal potential energy which may not be zero and vary with time. However, for a rigid body, the internal potential energy will remain constant. In fact, *a rigid body is a system of particles with fixed interparticle distances and therefore, the internal forces in a rigid body do not do any work, when the body moves from one configuration to another*. Thus the internal potential energy of a rigid body is constant and can be taken as zero to discuss its motion.

Ex. 1. *Box Train* : *A box train is shown in Fig. 1.8 . Each box has a mass M and the engine applies a net force F. Find the force on each box.*

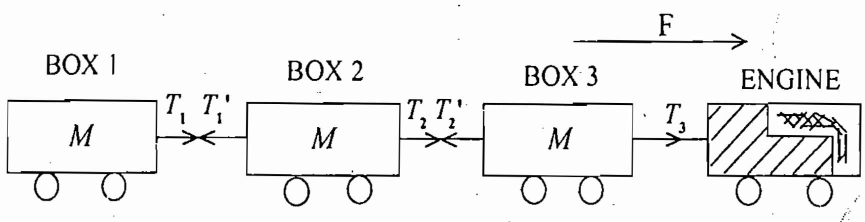


Fig. 1.8 : Box train

Solution : The mass of the box train excluding the engine is $3M$ and a net force F is acting on it. Hence the net acceleration of the box train is

$$a = \frac{F}{3M}$$

For each box, the weight is balanced by the normal reaction. For the box 1, the force T_1 is given by

$$T_1 = Ma = M \times \frac{F}{3M} = \frac{F}{3}$$

Since $T_1 = T_1'$ due to equality of action and reaction in the rod or rope between the box 1 and box 2, the net force on box 2 is

$$T_2 - T_1' = Ma \quad \text{or} \quad T_2 = \frac{F}{3} + \frac{F}{3} = \frac{2F}{3}$$

Similarly, the force on box 3 is

$$T_3 = T_2' + Ma = \frac{2F}{3} + \frac{F}{3} = F$$

Therefore, if there are N boxes, the acceleration will be

$$a = \frac{F}{NM}$$

On each box the net force will be F/N and the forward force on n^{th} box will be $n\ F/N$.

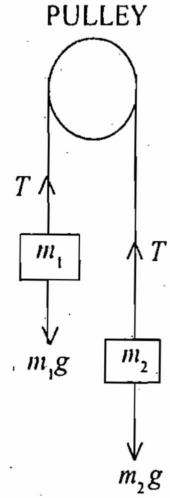


Fig. 1.9 : Atwood machine

Ex.2 : ***Atwood Machine :*** *Consider a massless string going over a frictionless pulley. Blocks of masses m_1 and m_2 are attached at the ends of the string . Such a system is called Atwood machine. Find the acceleration of the blocks and tension in the string.*

Solution : The pulley is frictionless and hence it will not rotate. If m_2 is greater than m_1 and T is the tension in the string, then according to Newton's second law, equations of motion of the two masses are

$$T - m_1 g = m_1 a \qquad ...(i)$$

$$\text{and} \quad m_2 g - T = m_2 a \qquad ...(ii)$$

where a is the acceleration which will be the same for the two masses because the string is continuous.

Adding eqs. (i) and (ii), we obtain

$$(m_2 - m_1)\, g = (m_1 + m_2)\, a \quad \text{or} \quad a = \frac{m_2 - m_1}{m_1 + m_2} g \qquad ...(iii)$$

Therefore,

$$T = m_1 \left(g + \frac{m_2 - m_1}{m_1 + m_2} g \right) = \frac{2 m_1 m_2}{m_1 + m_2} g \qquad ...(iv)$$

Ex. 3. *If* $\mathbf{F} = (2xy + z^2)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 2xz\hat{\mathbf{k}}$ *newton, then show that it is conservative. Calculate the amount of work done by this force in moving a particle from* (0, 1, 2) *to* (5, 2, 7) *m.*

Solution : For a conservative force, curl $\mathbf{F} = 0$

Here, $$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^2 & x^2 & 2xz \end{vmatrix}$$

$$= \hat{\mathbf{i}}\left[\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(x^2)\right] + \hat{\mathbf{j}}\left[\frac{\partial}{\partial z}(2xy + z^2) - \frac{\partial}{\partial x}(2xz)\right] + \hat{\mathbf{k}}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^2)\right]$$

$$= 0 + \hat{\mathbf{j}}\,(2z - 2z) + \hat{\mathbf{k}}\,(2x - 2x) = 0$$

Thus, $\mathbf{F} = (2xy + z^2)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 2xz\hat{\mathbf{k}}$ is a conservative force.

Work done $$= \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \left(F_x dx + F_y dy + F_z dz\right) = \int_1^2 \left[\left(2xy + z^2\right) dx + x^2 dy + 2xz\, dz\right]$$

$$= \int_1^2 \left[\left(2xy\, dx + x^2 dy\right) + \left(z^2 dx + 2xz\, dz\right)\right] = \int_1^2 \left[d\left(x^2 y\right) + d\left(z^2 x\right)\right] = \int_1^2 d\left(x^2 y + z^2 y\right)$$

$$= \left[x^2 y + z^2 x\right]_{1(0,1,2)}^{2(5,2,7)} = [5 \times 5 \times 2 + 7 \times 7 \times 5 - 0 - 0]$$

$= 50 + 245 =$ **295 joule.**

Ex. 4. *Two objects of masses* $m_1 = 200$ *gm and* $m_2 = 500$ *gm possess velocities* $\mathbf{v}_1 = 10\hat{\mathbf{i}}$ *m /sec and* $\mathbf{v}_2 = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ *m/sec just prior to a collision during which they become permanently attached to each other. Calculate (a) the velocity of the centre of mass, (b) the final momentum of the combination in the laboratory frame, and (c) the initial and final momenta in the centre of mass frame. (d) What fraction of the initial total kinetic energy is associated with the motion after collision ?*

Solution : (*a*) $$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \frac{0.2(10\hat{\mathbf{i}}) + 0.5(3\hat{\mathbf{i}} + 5\hat{\mathbf{j}})}{0.2 + 0.5} = 5\hat{\mathbf{i}} + \left(\frac{25}{7}\right)\hat{\mathbf{j}}$$

(*b*) Final momentum $= (m_1 + m_2)\mathbf{V} = 0.7\,\mathbf{V} = 3.5\hat{\mathbf{i}} + 2.5\hat{\mathbf{j}}$

(*c*) In the centre of mass-frame of reference, the centre of mass of the system remains at rest. It means that in the centre of mass-frame, the initial momentum of the system is zero and hence final momentum of the system will also be zero.

(*d*) $$T_i = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 = \tfrac{1}{2}(0.2)(10)^2 + \tfrac{1}{2}(0.5)(3^2 + 5^2) = 10 + 8.5 = 18.5 \text{ joule}$$

Now applying the law of conservation of momentum

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\,\mathbf{v}$$

where $\mathbf{v}$ is the final velocity of the combined particle in the laboratory system. Note that $\mathbf{v}$ is identical to $\mathbf{V}$,

i.e., $\mathbf{v} = 5\,\hat{\mathbf{i}} + (25/7)\,\hat{\mathbf{j}}$

Therefore $T_f = \frac{1}{2}(m_1 + m_2)\,v^2 = \frac{1}{2} \times 0.7\,[5^2 + (\frac{25}{7})^2] = 13.2$ joule

Thus $\dfrac{T_f}{T_i} = \dfrac{13.2}{18.5} = \mathbf{0.72}.$

Note : Definition of Central Force : If a force acts on a particle in such a way that it is always directed towards or away from a fixed point and its magnitude depends only upon the distance (r) from the point, then this force is called a central force. Thus, a central force is represented by

$$\mathbf{F} = F(r)\,\hat{\mathbf{r}} = F(r)\,\mathbf{r}/r$$

where $F(r)$ is any function of distance r and $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector along $\mathbf{r}$ from the fixed centre.

The force $\mathbf{F}$ is attractive or repulsive, if $F(r) < 0$ or $F(r) > 0$ respectively.

Examples of central force are gravitational $\left[F(r) = -\dfrac{Gm_1m_2}{r^2}\right]$ and coulomb $\left[F(r) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1q_2}{r^2}\right]$ forces.

Ex. 5. *When a particle moves under a central force, show that (a) its angular momentum is conserved, (b) the motion takes place in a plane and (c) the areal velocity remains constant.*

Solution : If a particle is moving under the influence of a central force $\mathbf{F}= F(r)\mathbf{r}/r$, so that the torque acting on it is given by

$$\tau = \frac{d\mathbf{J}}{dt} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times F(r)\frac{\mathbf{r}}{r} = 0 \quad [\because\ \mathbf{r} \times \mathbf{r} = 0]$$

where $\mathbf{J}$ is the angular momentum about the origin.

Therefore $\mathbf{J}$ = constant (vector). ...(*i*)

Thus, when a particle moves under the action of a central force, its angular momentum ($\mathbf{J}$) is conserved, *i.e.*, $\mathbf{J}$ remains the same in the magnitude and direction.

But $\mathbf{J} = \mathbf{r} \times \mathbf{p}$,

where $\mathbf{p}$ is the linear momentum.

Taking dot product with $\mathbf{r}$ of the both sides of this equation, we have

$$\mathbf{r} \bullet \mathbf{J} = \mathbf{r} \bullet (\mathbf{r} \times \mathbf{p}) = (\mathbf{r} \times \mathbf{r}) \bullet \mathbf{p} = 0 \qquad ...(ii)$$

(since in a scalar triple product the positions of dot and cross may be interchanged and $\mathbf{r} \times \mathbf{r} = 0$).

Therefore $\mathbf{r}$ is perpenducular to the constant vector $\mathbf{J}$ *i.e.*, the motion takes place in a plane for a central force.

Now, let O be the centre of force. When the vector $\mathbf{r}$ changes to $\mathbf{r} + \Delta\mathbf{r}$, the vector area $\Delta\mathbf{A}$ swept by the radius vector in this time is

$$\Delta\mathbf{A} = \tfrac{1}{2}\mathbf{r} \times \Delta\mathbf{r} \qquad ...(iii)$$

This area is swept in Δt time, therefore dividing both the sides of this equation by Δt and taking the limit as $\Delta t \to 0$, then

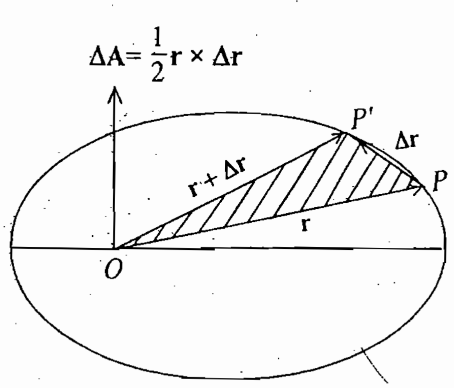


Fig. 1.10 : Vector area swept by the radial vector

$$\frac{d\mathbf{A}}{dt} = \frac{1}{2}\mathbf{r} \times \mathbf{v} = \frac{\mathbf{J}}{2m} \qquad ...(iv)$$

which gives the **areal velocity** of the particle. But $\mathbf{J}$ is constant for the motion under a central force [eq. (*i*)], we mean that the areal velocity remains constant.

Ex. 6. *Harmonic Oscillator* : *The differential equations of a simple harmonic oscillator of mass m is given by*

$$m\frac{d^2x}{dt^2} + Cx = 0$$

where x is the displacement at the time t and C is the force constant. Solve the equation to find the expressions for the displacement, velocity and acceleration.

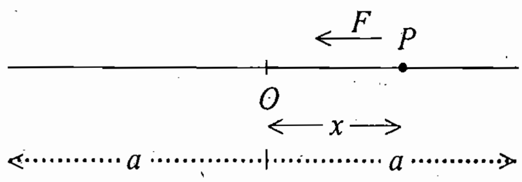


Fig. 1.11: Simple harmonic motion

($F = m\ddot{x} = -Cx$ or $m\ddot{x} + Cx = 0$)

Solution : The differential equation of the harmonic oscillator can be written as

$$\ddot{x} + \omega^2 x = 0 \qquad ...(i)$$

where $\omega = \sqrt{\frac{C}{m}}$ and $\ddot{x} = \frac{d^2x}{dt^2}$.

Acceleration $\ddot{x} = -\omega^2 x$ ...(*ii*)

Multiply eq. (*i*) by $2\dot{x}$, we get

$$2\dot{x}\,\ddot{x} + \omega^2\, 2x\,\dot{x} = 0$$

Integrating it, we obtain

$$\dot{x}^2 + \omega^2 x^2 = A \qquad ...(iii)$$

where A is a constant of integration.

When the displacement is maximum *i.e.*, (equal to the amplitude of motion) $x = a$, velocity $\dot{x} = 0$. Therefore

$$0 + \omega^2 a^2 = A \quad \text{or} \quad A = \omega^2 a^2$$

Therefore $\dot{x}^2 + \omega^2 x^2 = \omega^2 a^2$

Hence velocity $\dot{x} = \frac{dx}{dt} = \omega\sqrt{a^2 - x^2}$ ...(*iv*)

Equation (*iv*) can be written as

$$\frac{dx}{\sqrt{a^2 - x^2}} = \omega\, dt$$

Integrating it, we get

$$\sin^{-1}\frac{x}{a} = \omega t + \phi \quad \text{or} \quad x = a\sin(\omega t + \phi)$$

where ϕ is constant of integration, called the initial phase. The term $(\omega t + \phi)$ is called phase angle or phase of vibration.

Thus displacement $x = a\sin(\omega t + \phi)$ ...(*v*)

Questions

1. What are inertial and non-inertial frames ? Show that if a frame is an inertial frame, then a frame, moving with constant velocity relative to it, is also inertial. How will you realize an inertial frame in practice ?

2. Using Newton's laws of motion, deduce the conservation theorems of linear momentum, angular momentum and energy for the motion of a particle. What are first integrals of motion ?
3. State and prove work-energy theorem. What are conservative forces ? Show that for a conservative force

(*a*) $\oint \mathbf{F} \cdot d\mathbf{r} = 0$, (*b*) curl $\mathbf{F} = 0$, (*c*) $\mathbf{F} = -\nabla V$

Further show that in a conservative force field, the sum of kinetic and potential energies of a particle remains constant.

4. (*a*) What is centre of mass ? Show that in absence of external forces, the velocity of centre mass remains constant.

(*b*) If $\mathbf{v}_c$ is the velocity of any particle of mass m relative to the centre of mass of a number of particles, then show that $\Sigma m\mathbf{v}_c$ is zero whether any force acts on the particles or not.

(*c*) Show that the total linear momentum of a system of particles about the centre of mass is zero. **(Agra 1977)**

(*d*) Show that the acceleration of the centre of mass of a system of particles is only due to external forces. **(Rajasthan 1984)**

5. Using Newton's laws of motion, deduce the conservation laws for a system of particles. Discuss the assumptions involved and failure of Newton's third law.
6. Show that the angular momentum $\mathbf{J}$ of a system of particles can be expressed in the form

$$\mathbf{J} = \mathbf{J}_{cm} + \mathbf{R} \times \mathbf{P}$$

where $\mathbf{J}_{cm}$ = angular momentum about the centre of mass,

$\mathbf{R}$ = position vector of centre of mass,

and $\mathbf{P}$ = total linear momentum. **(Allahabad 1979; Kanpur 80)**

7. Show that the kinetic energy of a system of particles can be expressed as

$$T = \sum_i \frac{1}{2} m_i \mathbf{v}_{ic}^2 + \frac{1}{2} MV^2$$

where $\mathbf{v}_{ic}$ is the velocity of the i^{th} particle in centre of mass system.

8. (*a*) Show that if the internal forces in a system of particles are central in nature, then the forces $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji}$ between i^{th} and j^{th} particles satisfy Newton's third law.

(*b*) Define the total potential energy of a system, when the internal and external forces are conservative in nature. Deduce the law of conservation of energy for the system of particles.

9. (*i*) When a horse pulls a cart, which force helps the horse to move forward the ground on the horse or the horse on the ground ? [**Ans.** Ground on the horse]

(*ii*) A body is kept moving with uniform speed on a circle of radius r by a centripetal force F acting on it. How much is the work done is one rotation? [**Ans.** Zero]

Problems

[SET-I]

1. Consider a system of two masses m_1 and m_2 and a pulley of mass M. Determine the acceleration of the system and the tensions in the strings. Assume $m_2 > m_1$ and pulley to be a solid disc of radius R.

Ans. $$a = \frac{m_2 - m_1}{m_1 + m_2 + \dfrac{M}{2}} g,$$

$$T_1 = m_1 g \frac{\left(2m_2 + \frac{M}{2}\right)}{m_1 + m_2 + \frac{M}{2}},$$

$$T_2 = m_2 g \frac{\left(2m_1 + \frac{M}{2}\right)}{m_1 + m_2 + \frac{M}{2}}$$

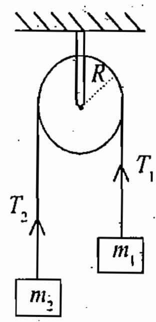


Fig 1.12 : Atwood machine (Pulley having mass M)

[**Hint :** $(T_2 - T_1)R = I\ddot{\theta}$, $R\ddot{\theta} = a$ (acc.) and $I = MR^2/2$.]

2. (*a*) The position vector of a particle of mass *m*, moving in *xy*-plane, is given by

$$\mathbf{r} = a \cos \omega t \,\hat{\mathbf{i}} + b \sin \omega t \,\hat{\mathbf{j}}$$

where *a, b* and ω are positive constants and $a > b$. Show that the path of the particle is an ellipse, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Further show that the force acting on the particle is always directed towards the origin with magnitude $m\omega^2 r$. Is the force conservative ?

(*b*) Find out the expressions for the potential and kinetic energies of the particle in the above problem and show that the total energy of the motion, given by

$E = \frac{1}{2} m\omega^2 (a^2 + b^2)$, remains constant.

3. (*a*) A particle moves under a force $F = -(K/r^2)\,\hat{\mathbf{r}}$. Prove that the angular momentum of the particle is conserved.

(*b*) Find out the nature the force, conservative or non conservative, given by

(*i*) $\mathbf{F} = (2xy + yz^2)\,\hat{\mathbf{i}} + (x^2 + xz^2)\,\hat{\mathbf{j}} + 2xyz\,\hat{\mathbf{k}}$

(*ii*) $\mathbf{F} = yz\,\hat{\mathbf{i}} + zx\,\hat{\mathbf{j}} + xy\,\hat{\mathbf{k}}$ **(Allahabad 1978)**

(*iii*) $\mathbf{F} = (y^2z^3 - 6xz^2)\,\hat{\mathbf{i}} + 2xyz^3\,\hat{\mathbf{j}} + (3xy^2z^2 - 6x^2z)\,\hat{\mathbf{k}}$

(*iv*) $\mathbf{F} = x^2yz\,\hat{\mathbf{i}} - xyz^2\,\hat{\mathbf{k}}$

Ans. (*b*) Conservative : (*i*) , (*ii*) , (*iii*) ; non-conservative : (*iv*).

4. If the magnitude of the force of attraction between the particles of masses m_1 and m_2 is given by

$$F = K\frac{m_1 m_2}{r^2}$$

where *K* is a constant and *r* is the distance between the particles. Determine (*a*) the potential energy function, and, (*b*) the work required to increase the separation of masses from $r = r_0$ to $r = r_0 + R$.

Ans. (*a*) $V(r) = \frac{Km_1m_2}{r^2}$, if $V(\infty) = 0$ (*b*) $\frac{Km_1m_2R}{r_0(r_0 + R)}$.

5. The Yukawa potential is given by

$$U(r) = -\frac{r_0}{r} U_0 \, e^{-r/r_0}$$

This gives a fairly accurate description of the interaction between the nucleons (*i.e.*, neutrons, protons). The constant $r_0 = 1.5 \times 10^{-15}$ m (nearly) and $U_0 = 50$ MeV (nearly). Now, (*a*) calculate the corresponding expression for the force of attraction, and (*b*) for showing the short range nature of this force, find the ratio at $r = 4r_0$ to the force at $r = r_0$.

Ans : (*a*) $F = -\frac{r_0 U_0}{r} e^{-r/r_0} \left[\frac{1}{r_0} + \frac{1}{r}\right]$ (*b*) $\frac{F(4r_0)}{F(r_0)} = \frac{1}{126}$ nearly.

6. A nucleus, initially at rest, decays radioactively by emitting an electron of momentum 1.73 MeV/c and at right angles to the direction of electron a neutrino with momentum 1 MeV/c. In what direction, does the nucleus recoil ? What is its momentum in MeV/c and in S.I. units. If the mass of the residual nucleus is 3.9×10^{-25} kg, calculate its kinetic energy in joules and electron volts ?
Ans. Momentum = 2 MeV/c or 10.66×10^{-22} kg.m /sec; Direction = 210° with the direction of electron; K.E.= 1.456×10^{-18} joule = 9.1 eV.

7. An α-particle is emitted from a uranium nucleus (U^{238}) in a radioactive decay. The speed and kinetic energy of α-particle are 1.404×10^7 m/sec and 4.212 MeV respectively. Calculate the recoil speed and kinetic energy of the residual nucleus.
Ans. -2.4×10^5 m/sec.; 0.072 MeV.

8. In a radioactive decay of a nucleus, an electron and a neutrino with momen tum 1.28×10^{-22} and 6.4×10^{-29} kg-m /sec are emitted at right angles to each other. What is the momentum of the reecoiling nucleus ? If the mass of the residual nucleus is 5.8×10^{-26} kg, calculate its recoil kinetic energy.

Ans. 1.4×10^{-22} kg-m/sec, in a direction $\tan^{-1}(\frac{1}{2})$ with the electron velocity; 1.1 eV.

9. A stationary body of mass 3 kg explodes into three equal pieces. Two of the pieces fly off at right angles to each other, one with $2\hat{\mathbf{j}}$ m/sec and the other with $3\hat{\mathbf{j}}$ m/sec. If the explosion takes in 10^{-5} sec, find the average force acting on each piece during the explosion.
Ans. $2 \times 10^5 \hat{\mathbf{i}}$ newton; $3 \times 10^5 \hat{\mathbf{j}}$ newton ; $-(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) \times 10^5$ newton.

10. A bomb in flight explodes into two fragments when its velocity is $10\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$ m/sec. If the smaller mass M flies with velocity $20\hat{\mathbf{i}} + 50\hat{\mathbf{j}}$ m/sec, deduce the velocity of the larger mass $3M$. Deduce also the velocities of the fragments in the centre of mass-reference frame, and show these in a diagram.

Ans : $\frac{1}{3}(20\hat{\mathbf{i}} + 42\hat{\mathbf{j}})$ m/sec; $10\hat{\mathbf{i}} + 48\hat{\mathbf{j}}$ m/sec and $\frac{1}{3}(10\hat{\mathbf{i}} - 48\hat{\mathbf{j}})$ m/sec.

11. If m_1, m_2 and m_3 be the masses of three particles and $\mathbf{v}_{12}$, $\mathbf{v}_{23}$ and $\mathbf{v}_{13}$ be their relative velocities, prove that the total kinetic energy (E) of the system about the centre of mass is given by

$$E = \frac{m_1 m_2 v_{12}^2 + m_2 m_3 v_{23}^2 + m_1 m_3 v_{13}^2}{m_1 + m_2 + m_3}$$

[SET-II]

1. Two swimmers leave point P on one bank of the river to reach point Q lying right across the other bank. One of them crosses the river along the straight line PQ while the other swims at right angle to

the stream and then walks the distance that he has been carried away by the stream to get the point Q. Show that the velocity v of his walking if both swimmers reaches the destination simultaneously, is given by

$$v = \frac{u}{[(1-u^2/V^2)^{-1/2} - 1]}$$

where u is the stream velocity and V is the velocity of each swimmer with respect to water.

2. Three particles are located at the vertices of an equilateral triangle of side l. Each of the particles starts to move with constant speed v, with the first particle heading continuously for the second, the second for the third, and the third for the first. When will the particles meet each other ?

   Ans : $t = 2l/3v$.

3. A man moves relative to water with a velocity one half than the river flow velocity. At what angle to the stream direction must the man move to minimise drifting ?

   Ans : 120°.

4. A small block Q is placed on another block P of mass 5 kg and length 0.2 m. Initially the block Q is near the right end of the block P [Fig 1.13]. All the surfaces are assumed frictionless. A constant horizontal force of 10 N is applied to the block P. Determine the time elapsed before the block Q separates from P.

   Ans : 0.45 sec.

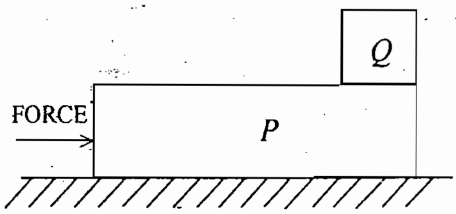


Fig. 1.13

5. Prism of mass M ~~with angle~~ θ rests on a horizontal surface. A small block of mass m is placed on the prism. Find the accelerations of the prism and the block. Assume all the surface to be frictionless.

   Ans : $a_1 = \dfrac{mg\sin\theta\cos\theta}{M + m\sin^2\theta}$, $a_2 = \dfrac{(M+m)g\sin\theta}{M + m\sin^2\theta}$

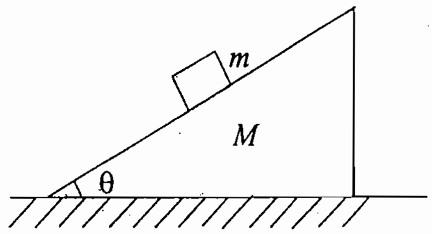


Fig. 1.14

6. A 2 kg block is placed over a 4 kg block and both are placed on a smooth horizontal surface. The coefficient of friction between the blocks is 0.20. Determine the accelerations of the two blocks if a horizontal force of 12 N is applied to (*i*) the upper block, (*ii*) the lower block. Assume g = 10 m/s^2.

   Ans : (*i*) Upper block 4 m/sec^2, lower block 1 m/sec^2 (*ii*) both blocks 2 m/sec^2.

7. A small block starts sliding on an inclined plane forming an angle θ with the horizontal. The friction coefficient depends on the distance x covered as $\mu = \alpha x$, where α is constant. Find the distance covered by the block till it stops, and its maximum speed over this distance.

   Ans : $(2\tan\theta)/\alpha$; $(g\sin\theta\tan\theta/\alpha)^{1/2}$

8. Two blocks of masses m_1 and m_2 connected by a light spring rest on a horizontal plane. If the coefficient of friction between the blocks and the surface is equal to μ, determine the minimum constant force which has to be applied in the horizontal direction to the block of mass m_1 in order to shift the other block.

   Ans : $(m_1 + m_2/2)\,\mu g$

9. A bullet of mass m, moving horizontally with speed v, passes through a pendulum bob of mass M and emerges with a velocity $u/2$. If the length of the pendulum is l, show that the minimum value of v, which will make the pendulum bob to swing through a complete circle is $(2\,M/m)\sqrt{5\,gl}$.

10. A small body of mass m has a horizontal speed v_0 at a point O. Find the maximum instantaneous power developed by the friction force, if the coefficient of friction varies as $\mu = \alpha x$, where α is a constant and x is the distance from O.

Ans : $P_{max} = -\frac{1}{2} m v_0^2 \sqrt{\alpha g}$.

11. A chain hangs from a thread and touches the surface of a table by its lower end. Show that after the thread has been burned through, the force exerted on the table by the falling part of the chain at any moment is twice as great as the force of pressure exerted by the part already resting on the table.

12. The potential energy function for the force between two atoms in a diatomic molecule can approximately be expressed as

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}.$$

where a and b are positive constants, and x is the distance between the atoms.

(*i*) For what values of x, $U(x)$ is equal to zero ?

(*ii*) For what values of x, $U(x)$ is minimum ?

(*iii*) Calculate the force between the two atoms and plot $F(x)$. Show that the two atoms repel each other for x less than x_0 and attract each other for x greater than x_0. What is the value of x_0 ?

(*iv*) Assuming that one of the atoms remains stationary and the other moves along X-axis, describe the possible motions.

(*v*) Calculate the dissociation energy (the energy required to break the molecule into atoms) of the molecule.

Ans. (*i*) $x = (a/b)^{\frac{1}{6}}$ and $x = \infty$,

(*ii*) $(2a/b)^{\frac{1}{2}}$,

(*iii*) $F(x) = \frac{12a}{x^{13}} - \frac{6b}{x^7}$; $x_0 = (2a/b)^{\frac{1}{6}}$,

(*iv*) Oscillatory motion about the point $x = (2a/b)^{1/2}$, (*v*) $b^2/4a$ or more.

13. A smooth sphere of radius R is made to translate in a straight line with aconstant acceleration a. A particle kept on the top of the sphere is released from there at zero velocity with respect to the sphere. Find the speed of the particle with respect to the sphere as a function of the angle θ which it slides.

Ans. $v = [2R(a \sin\theta + g - g\cos\theta)]^{1/2}$

14. A spaceship of mass M_0 moves in the absence of external forces with a constant velocity V_0. In order to change the direction of motion, a jet engine is switched on. It starts ejecting a gas with velocity u which is constant relative to the spaceship motion. The engine is shut down when the mass of the spaceship decreases to M. Through what angle did the motion direction of the spaceship deviate due to the jet engine operation ?

Ans. $F(x) = \frac{u}{V_0} \text{In} \frac{M}{M_0}$.

15. Two identical bodies of mass m lie on a smooth horizontal table. They areinterconnected by a light spring of length l_0 and force constant k. At a certain instant one of the body is set into motion in a horizontal direction perpendicular to the spring with velocity u. Find the maximum extension per unit length of the spring in the process of motion, if it is considerably less than unity.

(*a*) $M\omega^2 L/2$ (*b*) $M\omega^2 L$ (*c*) $M\omega^2 L/4$ (*d*) $M\omega^2L^3/2$

Ans. (*a*)

5. Mutual interaction forces between two particles can change

(*a*) the linear momentum but not the kinetic energy.

(*b*) the kinetic energy but not the linear momentum.

(*c*) the linear momentum as well as kinetic energy.

(*d*) neither the linear momentum nor the kinetic energy.

Ans. (*b*)

6. A particle of mass m moving with speed v collides with a stationary particle of equal mass. After the collision, both the particles move. Let θ be the angle between the two velocity vectors.

(*i*) If the collision is elastic, then

(*a*) θ is always less than 90°
(*b*) θ is always equal to 90°
(*c*) θ is always greater than 90°
(*d*) θ cannot be deduced from the given data

Ans : (*b*) **(Gate 2003)**

(*ii*) If the collision is inelastic, then

(*a*) θ is always less than 90°
(*b*) θ is always equal to 90°
(*c*) θ is always greater than 90°
(*d*) θ could assume any value in the range 0° to 180°

Ans : (*a*) **(Gate 2003)**

Short Answer Questions

1. According to Newton's third law every force is accompanied by an equal and opposite force. How can a movement ever takes place ?

Ans. Action and reaction act always on different bodies. Hence motion can take place.

2. A uniform string, having mass, is suspended from ceiiling with a load at the lower end. Will the tension be uniform in the sting ? Explain. Where will the tension be maximum ?

Ans. No; The tension will be maximum at the upper end of the string.

3. A light body and a heavy body have the same kinetic energy, which one will have the greater momentum.

Ans. Heavy body

4. A coin in left free to fall on the ground from a moving train with constant velocity. Explain the path as seen by an observer on the ground and on the train.

Ans. Parabola; straight line

5. Fill in the blanks :

(*i*) The angular momentum of a panticle in defined as the moment .... **(Agra 2004)**

(*ii*) The work-energy theorem states that the work done is equal to the change in .... **(Agra 2004)**

(*iii*) Rate of change of angular momentum is called ....

Ans. (*i*) of linear momentum, (*ii*) kinetic energy, (*iii*) Torque

Ans : mu^2/kl_0^2.

16. A chain of length l is located in asmooth horizontal tube so that its portion of length h hangs freelyand touches the surface of the table with its end Q [Fig. 1.15]. At a certain instant, the end P of the chain is set free. Determine the speed of this end of the chain , when it slips out of the tube.

Ans : $v = \sqrt{[2gh \ln(l/h)]}$

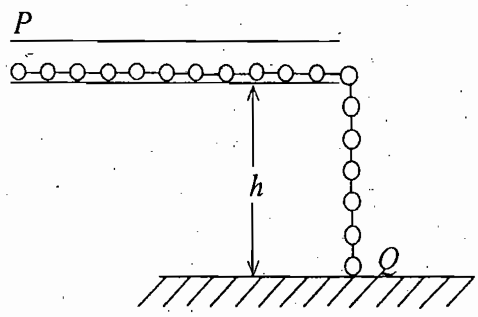


Fig. 1.15

Objective Type Questions

1. A body is moved along a straight line by a machine delivering constant power. The d istance moved by the body in time t is proportional to

(*a*) $t^{1/2}$ (*b*) $t^{3/4}$ (*c*) $t^{3/2}$ (*d*) t^2

Ans : (*c*)

[**Hint :** $P = \mathbf{F} \bullet \mathbf{v} = m\frac{d\mathbf{v}}{dt} \bullet \mathbf{v} = \frac{m}{2}\frac{d}{dt}(v^2) = \text{constant}$

Hence, $v^2 = \frac{2Pt}{m}$ or $v = \frac{ds}{dt} = \sqrt{\frac{2Pt}{m}}$ *i.e.* $s \propto t^{3/2}$

2. A 4 kg slab rests on frictionless floor as shown in Fig.1.16. A 1 kg. block rests on the top of the slab. The static coefficient of friction between the block and the slab is 0.6 while the kinetic coefficient of friction is 0.4. A horizontal force 10 N acts upon 1Kg block. The acceleration of the slab is

(*a*) 0.98 m/s^2 (*b*) 1.47 m/s^2

(*c*) 1.52 m/s^2 (*d*) 6.10 m/s^2

Ans : (*a*)

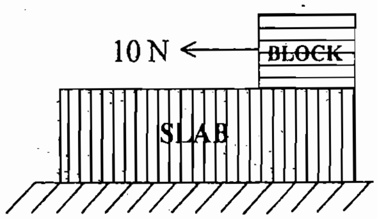


Fig 1.16

3. Two blocks A and B each of mass m are connected by a massless spring of natural length L and spring constant k. The blocks are initially resting on a smooth horizontal floor with the spring at its natural length, as shown in Fig. 1.17. A third identical block C, also of mass m, moves on the floor with a speed v along the line joining A and B, and collides with A. Then

(*a*) the kinetic energy of the A-B system at the maximum compression of the spring is zero.

(*b*) the kinetic energy of the A-B system at the maximum compression of the spring is $mv^2/4$.

(*c*) the maximum compression of the spring is $v\sqrt{(m/k)}$.

(*d*) the maximum compression of the spring is $v\sqrt{(m/2k)}$.

Ans. (*b*) and (*d*)

4. A tube of length L is filled completely with an incompressible liquid of mass M and closed at both ends. The tube is then rotates in a horizontal plane about one of its ends with a uniform angular velocity ω. The force exerted by the liquid at the other end is

CHAPTER 2 Lagrangian Dynamics

2.1. INTRODUCTION

In the previous chapter, we have discussed Newtonian approach to deal with the motion of particles. Application of Newton's laws of motion needs the specification of all forces acting on the body at all instants of time. In practical situations, when the constraint forces are present, the application of the Newtonian approach may be a difficult task. The greatest disadvantage with the Newtonian procedure is that the mechanical problems are always tried to resolve geometrically rather than analytically. In case of constrained motion, the determination of all the unimportant reaction forces is a great nuisance in the Newtonian approach. In order to overcome the difficulties, posed by the Newtonian scheme in solving the problems of constrained motion, methods have been developed by D'Alembert, Lagrange, Hamilton and others. The techniques of Lagrange and Hamilton use generalized coordinates which have been discussed and used in the present and next chapters. In the Lagrangian formulation, the generalized coordinates used are position and velocity, resulting in the second order linear differential equations, while in the Hamiltonian formulation, the generalized coordinates being position and momentum result in the first order linear differential equations.

2.2. BASIC CONCEPTS

We discuss some basic concepts regarding the motion of particles.

(1) Coordinate Systems : The fundamental concept involved in the motion of a particle (or system) is the position coordinate and how it is changing with time. The position of a particle is represented by choosing a coordinate system. In the cartesian or rectangular coordinate system, the position vector **r** of a particle is defined in terms x, y and z coordinates. In a two dimensional motion, rectangular coordinates (x, y) or polar coordinates (r, θ) can represent the position of the particle [Fig. 2.1(a)]. They are related as

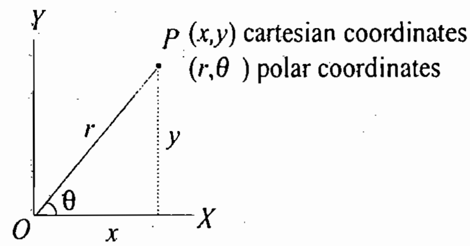


Fig. 2.1(a) : Rectangular and polar coordinates

$$x = r\cos\theta \text{ and } y = r\sin\theta;\ r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\frac{y}{x}.$$

In three dimensions, the cylindrical coordinates (ρ, θ, z) and the spherical coordinates (r, θ, ϕ) of the position of a particle are related to the cartesian coordinates (x, y, z) as follows :

For cylindrical and cartisian coordinates [Fig. 2.1(b)] :

$$x = \rho\cos\theta,\ y = \rho\sin\theta,\ z = z;\quad \rho = \sqrt{x^2 + y^2},\quad \theta = \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{\rho}$$

For spherical and cartesian coordinates [Fig. 2.1(c)]:

$x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$;

$$r = (x^2 + y^2 + z^2)^{1/2}, \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad \phi = \tan^{-1} \frac{y}{x}$$

We may represent, for example, the relationships for spherical and cartesian coordinates as follows :

$$x = x\,(r,\theta,\phi\,),\; y = y\,(r,\theta,\phi\,),\;\; z = z\,(r,\theta\,)$$

or $\qquad \mathbf{r} = \mathbf{r}\,(r,\ \theta,\ \phi\,)$

If we include the time variable also, then

$$\mathbf{r} = \mathbf{r}\,(r,\ \theta,\ \phi,\ t),$$

In general , we may represent the coordinates by q_1, q_2, q_3, having the relationships with the cartesian coordinates as

$$x = x\,(q_1, q_2, q_3, t\,),\quad y = y\,(q_1, q_2, q_3, t\,),$$

$$z = z\,(q_1, q_2, q_3, t\,)$$

or $\qquad \mathbf{r} = \mathbf{r}\,(q_1, q_2, q_3, t\,) \qquad$ ...(1)

In fact, these are the transformation equations from a general system to the cartesian coordinate system.

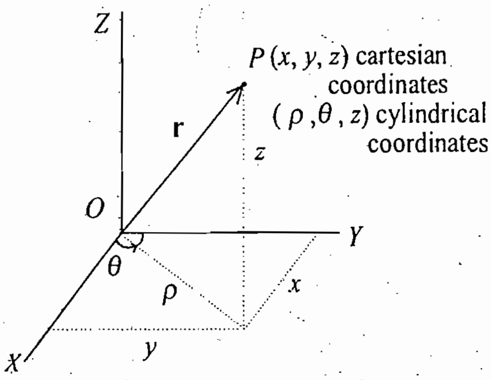


Fig. 2.1(b) : Cartesian and cylindrical coordinates

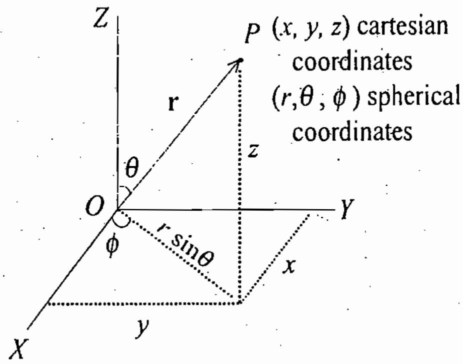


Fig. 2.1(c) : Cartesian and spherical coordinates

(2) Degrees of Freedom — ***Configuration Space*** **:** The minimum number of independent variables or coordinates required to specify the position of a dynamical system, consisting of one or more particles, is called the number of **degrees of freedom** of the system. For example, the motion of a particle, moving freely in space, can be described by a set of three coordinates *e.g.*, (x, y, z) and hence the number of degrees of freedom, possessed by the particle, is three. A system of two particles, moving freely in space, requires two sets of three coordinates [*e.g.*, (x_1, y_1, z_1) and (x_2, y_2, z_2)] *i.e.*, six coordinates to specify its position. Thus the system has six degrees of freedom. If a system consists of N particles, moving freely in space, we need $3N$ independent coordinates to describe its position. Hence the number of degrees of freedom of the system is $3N$.

The configuration of the system of N particles, moving freely in space, may be represented by the position of a single point in $3N$ dimensional space, which is called **configuration space** of the system. The configuration space for a system of one freely moving particle is 3-dimensional and for a system of two freely moving particles, it is six dimensional. In the later case, the configuration of the system of the two particles can be represented by the position of a single point with six coordinates in six dimensional space. This system has six degrees of freedom and its configuration space is six dimensional.

The number of coordinates, needed to specify a dynamical system, becomes smaller, when the constraints (which we describe below) are present in the system. Hence ***the degrees of freedom of a dynamical system is defined as the minimum number of independent coordinates (or variables) required to specify the system compatible with the constraints.*** If there are n independent variables, say $q_1, q_2, \ldots, q_n$ and n constants $C_1, C_2, \ldots, C_n$ such that

$$\sum_{i=1}^{n} C_i\, dq_i = 0 \qquad \text{...(2)}$$

at any position of the system, then we must have

$$C_1 = C_2 = ... = C_n = 0$$

2.3. CONSTRAINTS

Often the motion of a particle or system of particles is restricted by one or more conditions. ***The limitations on the motion of a system are called constraints and the motion is said to be constrained motion*** .

2.3.1. Holonomic constraints

Constraints limit the motion of a system and the number of independent coordinates, needed to describe the motion, is reduced. For example, if a particle is allowed to move on the circumference of a circle, then only one coordinate $q_1 = \theta$ is sufficient to describe the motion, because the radius (a) of the circle remains the same. If **r** is the position vector of the particle at any angular coordinate θ relative to the centre of the circle, then

$$|\mathbf{r}| = a \quad \text{or} \quad r - a = 0 \qquad ...(3)$$

Eq. (3) expresses the constraint for a particle in circular motion. Similarly in the case of a particle, moving on the surface of a sphere, the correct coordinates are spherical coordinates r, θ and ϕ, where θ and ϕ only vary. Therefore $q_1 = \theta$ and $q_2 = \phi$ are the two independent coordinates for the problem, because the constraint is that the radius of the sphere (a) is constant (*i.e.*, $|\mathbf{r}| = a$). Since in the circular motion of the particle, one independent coordinate θ is needed, the number of degrees of freedom of the system is 1. For the particle, constrained to move on the surface of the sphere, two independent coordinates specify its motion and hence the degrees of freedom is 2.

Suppose the constraints are present in the system of N particles. If the constraints are expressed in the form of equations of the form

$$f(\mathbf{r}_1, \mathbf{r}_2, ..., t) = 0 \qquad ...(4)$$

then they are called ***holonomic constraints***. Let there be m number of such equations to describe the constraints in the N particle system. Now, we may use these equations to eliminate m of the $3N$ co-ordinates and we need only n independent coordinates to describe the motion, given by

$$n = 3N - m$$

The system is said to have n or $3N - m$ degrees of freedom. The elimination of the dependent coordinates can be expressed by introducing $n = 3N - m$ independent variables $q_1, q_2, ..., q_n$. These are referred as ***generalized coordinates***.

Superfluous Coordinates : The idea of degrees of freedom makes it clear that when we are using, say rectangular cartesian coordinates, we have several ***redundant*** or ***superfluous*** coordinates, if there are holonomic constraints. This redundance and non-independence of the coordinates makes the problem complicated and this difficulty is resolved by using the generalized coordinates. For example, let us consider a body be thrown vertically upward with an initial velocity v_0. The body will move in a straight line. In cartesian coordinates, the motion will be represented as

$$x = 0, \quad y = v_0 t - \frac{1}{2} g t^2, \quad z = 0$$

where X and Z axes are horizontal and Y-axis is in vertical direction. At different values of the time t, only y coordinate varies and x and z coordinates remain the same. Therefore x and z coordinates are ***superfluous coordinates***. In conclusion, we need only one coordinate $q = y$ to describe the vertical motion.

2.3.2. Nonholonomic constraints

The constraints which are not expressible in the form of eq. (4) are called ***nonholonomic***. For example, the motion of a particle, placed on the surface of a sphere of radius a, will be described by

$$|\mathbf{r}| \geq a \quad \text{or} \quad r - a \geq 0$$

in a gravitational field, where **r** is the position vector of the particle relative to the centre of the sphere. The particle will first slide down the surface and then fall off. The gas molecules in a container are constrained to move inside it and the related constraint is another example of nonholonomic constraints. If the gas container is in spherical shape with radius a and **r** is the position vector of a molecule, then the condition of constraint for the motion of molecules can be expressed as

$$|\mathbf{r}| \leq a \quad \text{or} \quad r - a \leq 0 .$$

It is to be mentioned that in holonomic constraints, each coordinate can vary independently of the other. In a nonholonomic system, all the coordinates cannot vary independently and hence the number of degrees of freedom of the system is less than the minimum number of coordinates needed to specify the configuration of the system. We shall in general consider the holonomic systems.

Constraints are further described as (i) **rheonomous** and (ii) **scleronomous**. In the former, the equations of constraint contain the time as an explicit variable, while in the later they are not explicitly dependent on time. Constraints may also be classified as (i) **conservative** and (ii) **dissipative**. In case of conservative constraints, total mechanical energy of the system is conserved during the constrained motion and the constraint forces do not do any work. In dissipative constraints, the constraint forces do work and the total mechanical energy is not conserved. Time-dependent or rheonomic constraints are generally dissipative.

2.3.3. Some more Examples of Holonomic and Non-holonomic constraints

Besides the above examples of holonomic and nonholonomic constraints, we are giving below some important examples :

(1) **Rigid body** (***Holonomic constraint***) : In case of the motion of a rigid body, the distance between any two particles of the body remains fixed and do not change with time. If $\mathbf{r}_i$ and $\mathbf{r}_j$ are the position vectors of the ith and jth particles, then the distance between them can be expressed by the condition

$$|\mathbf{r}_i - \mathbf{r}_j| = C_{ij} \text{ (constant)}$$

If (x_i, y_i, z_i) and (x_j, y_j, z_j) are the cartesian coordinates (components of position vector) of the two particles, then the constraints will be expressed as

$$(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = C_{ij}^2$$

The constraint is *holonomic* and *scleronomic.*

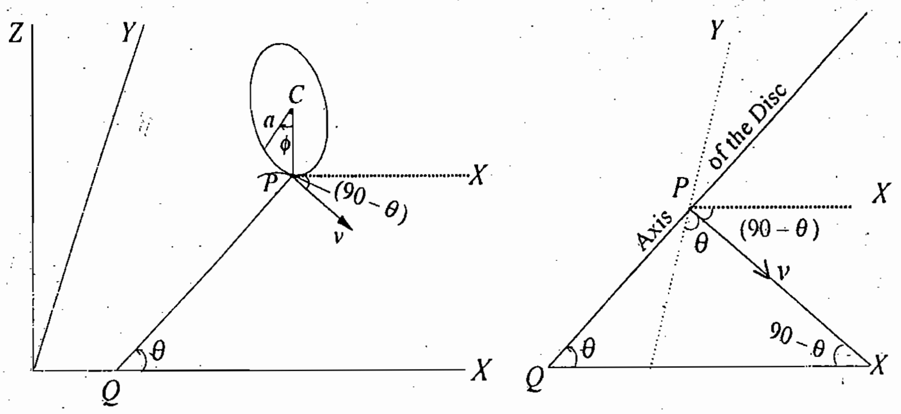
(2) **Simple pendulum with rigid support** (***Holonomic constraint***) : In case of a simple pendulum with rigid support, the constraint is that during the motion, the distance (l) of the bob (particle) from the point of suspension remains constant with time. Thus if **r** is the position vector of the particle relative to the point of suspension, then the condition of constraint can be expressed as

$$|\mathbf{r}| = l \text{ (constant)}$$

This is also an example of *holonomic* and *scleronomic* constraint. If the motion is confined to move in a vertical plane, only one coordinate θ, the angular displacement, is sufficient to describe the motion.

(3) **Rolling disc** (***Non-holonomic constraint***) : The nomenclature '*holonomic*' constraints comes from the word '*holos*' which means '*integer*' in Greek and '*whole*' or '*integrable*' in Latin languages. A system is said to be non-holonomic if it corresponds to non-integrable differential equations of constraints. Such constraints cannot be expressed in the form of eq.(4). Obviously holonomic system has integrable differential

equations of constraints, expressible in the form (4). In order to explain this, let us consider a disc rolling on a rough horizontal X-Y plane with the condition of constraint is that the plane of the disc is always vertical. We choose the coordinates x, y for the centre of the disc, ϕ for the angle of rotation about the axis of the disc and θ for the angle between the axis of the disc and X-axis [Fig. 2.2].



(a) (b)

Fig. 2.2 : Vertical disc rolling on a horizontal XY-plane

If a is the radius of the disc, the constraint that the axis of the disc is perpendicular to the vertical Z-direction, gives the velocity v of the disc with magnitude

$$v = a\dot{\phi} = a\frac{d\phi}{dt}$$

As the direction of the velocity is perpendicular to the axis of the disc, the components of the velocity along X-axis and Y-axis are

$$v_x = \frac{dx}{dt} = v\sin\theta\,,\quad v_y = \frac{dy}{dt} = -\,v\cos\theta$$

Therefore, $$\frac{dx}{dt} = a\frac{d\phi}{dt}\sin\theta \quad \text{and} \quad \frac{dy}{dt} = -\,a\frac{d\phi}{dt}\cos\theta$$

or $$dx - a\sin\theta\, d\phi = 0 \quad \text{and} \quad dy + a\cos\theta\, d\phi = 0 \qquad ...(5)$$

None of the equations, given by (5), can be integrated without solving the entire problem. Thus the constraint cannot be put in the form (4) and hence the constraint is nonholonomic.

2.3.4. Forces of Constraint

Constraints are always related to forces which restrict the motion of the system. These forces are called ***forces of constraint.*** For example, the reaction force on a sliding particle on the surface of a sphere is the force of constraint. In case of a rigid body , the inertial forces of action and reaction between any two particles are the forces of constraint. Constraint force in a simple pendulum is the tension in the string. When a bead slides on a wire, the reaction force exerted by the wire on the bead is the force of constraint. These forces of constraint are elastic in nature and usually appear at the surface of contact because the motion due to external applied forces' is hindered by the contact. However, Newton has not given any prescription to calculate these forces of constraint.

Usually the constraint forces act in a direction perpendicular to the surface of constraints while the motion of the object is parallel to the surface. In such cases, the work done by the forces of constraint is

These constraints are termed as *workless* and may be called as ***ideal constraints***. For example, when a particle slides on a frictionless horizontal surface, the force of constraint is normal to the surface. There are examples, where the constraint force does work. When a body slides on a frictional surface, the work is done by the force of constraint (frictional force) for real displacements.

By definition, the external or applied forces are all known forces. In the solution of dynamical problems either we have to determine all the forces of constraints or we should eliminate them from final equations. If we want to use Newton's form of equations, the forces of constraints are to be determined. We discuss below the difficulties, introduced by such an approach and how to remove them.

2.3.5. Difficulties introduced by the Constraints and their Removal

Two types of difficulties are introduced by constraints in the solution of mechanical problems :

(1) Let us consider a system of N interacting particles. The force on the ith particle is given by

$$\mathbf{F}_i = \mathbf{F}_i^e + \sum_{j=1}^{N} \mathbf{F}_{ij}$$

where $\mathbf{F}_i^e$ stands for an external force and $\mathbf{F}_{ij}$ is the internal (constraint) force on the ith particle due to jth particle. The equation of motion of the ith particle, in view of Newton's second law, is

$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{F}_i^e + \sum_{j=1}^{N} \mathbf{F}_{ij} \qquad \text{...(6)}$$

where $i = 1, 2, \ldots, N$. Thus eq. (6) represents a set of N equations. The coordinates $\mathbf{r}_i$ are connected by equations of constraints of the form :

$$f(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n, t) = 0$$

Hence the coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ of various particles are no longer all independent. This means that N equations represented by (6) are not all independent and therefore, the equations of motion are to be written again taking into consideration the equations of the constraints.

(2) The second difficulty introduced by the constraints is that several times the constraint forces are not known initially and they are among the unknowns of the problem. In fact, these unknown constraint forces are to be obtained from the solution of the problem which we are seeking and thus introduce complications in obtaining the solution. For example, if a bead is moving on a wire, the force (of constraint) which the wire exerts on the bead is not known in the beginning of the problem.

In case of holonomic constraints, as discussed already, the first difficulty is solved by introducing $n = 3N - m$ generalized coordinates, where m is the number of equations of constraints in N particle system. In order to remove the second difficulty, namely the forces of constraint are not known initially, we formulate the mechanics in such a way that the forces of constraint disappear. We require then only the known applied forces. Such an approach is due to D'Alembert which uses the ideas of virtual displacement and virtual work.

Ex. 1. *Determine the number of degrees of freedom in the following cases* : (1) *A particle moving on the circumference of a circle.* (2) *Five particles moving freely in a plane.* (3) *Two particles connected by a rigid rod moving freely in a plane.* (4) *A rigid body moving freely in three dimensional space.* (5) *A rigid body moving in space with one point fixed.* (6) *A rigid body rotating about a fixed axis in space.* (7) *The bob of simple pendulum oscillating in a plane.* (8) *The bob of a conical pendulum.* (9) *Dumbbell moving in space.*

Solution : (1) The constraint is $x^2 + y^2 = a^2$ or $r = a$ (constant). Hence in cartesian coordinates one variable x or y and in polar coordinates one variable θ could suffice to describe the motion on the circle. Hence the degree of freedom is 1.

(2) Each free particle needs two coordinates to specify its position in a plane. Hence 5 free particles will need 10 coordinates and therefore the number of degrees of freedom of the system is 10.

(3) Degrees of freedom = $2 \times 2 - 1 = 3$, because two particles in a plane will need $[(x_1, y_1)]$ and $(x_2, y_2)]$ 4 coordinates and there is one constraint equation for the distance l and between the two particles:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2.$$

(4) A rigid body is a system of particles in which the distance between any two particles remain fixed throughout the motion. Let us consider three non-collinear particles P_1, P_2, P_3 of a rigid body [Fig. 10.1]. As each particle has 3 degrees of freedom and there are three constraints of the form

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = r_{12}^2 \text{ (constraint)}$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = r_{23}^2$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = r_{13}^2,$$

Hence the degrees of freedom for these particles are $3 \times 3 - 3 = 6$. The consideration of any other particle in the body needs three coordinates, say $P_i(x_i, y_i, z_i)$, and obviously there are three equations of constraints because the distances of P_i from P_1, P_2, P_3 are fixed. Hence any other particle will not add any new degree of freedom to the six degrees of freedom of the three-particle system of the body. Thus a rigid body moving freely in a three dimensional space has 6 degrees of freedom.

(5) One point of the rigid body is fixed, say the particle at the origin. Hence for this particle, $x_1 = 0$, $y_1 = 0$, $z_1 = 0$. The constraint equations for other particles are

$$x_2^2 + y_2^2 + z_2^2 = r_2^2, \quad x_3^2 + y_3^2 + z_3^2 = r_3^2$$

and $$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = r_{23}^2 \text{ (constant)}.$$

Hence the degrees of freedom for the system are $3 \times 3 - 6 = 3$.

(6) A rigid body, rotating about a fixed axis, has one degree of freedom, because relative to the origin, say on the fixed axis, taken as Z-axis, z = constant and $x^2 + y^2 = r_0^2$ for a particle, where r_0 is the radius of the circle about the fixed axis.

(7) The bob of an oscillating simple pendulum has one degree of freedom, because the motion is described by one θ coordinate with the constraint $|r| = l$ or in cartesian coordinates by x or y with the constraint $x^2 + y^2 = l^2$.

(8) The bob of a conical pendulum has 2 degrees of freedom as the constraint is $x^2 + y^2 + z^2 = l^2$ relative to the centre of suspension.

(9) In a dumbbel two heavy particles are connected by a rigid massless rod. The system has $3 \times 2 - 1 = 5$ degrees of freedom, because the distance $[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$ = constant] between the two particles is fixed.

Ex. 2. *In the following cases, discuss whether the constraint is holonomic or nonholonomic. Specify the constraint force also* :

(1) *Motion of a body on an inclined plane under gravity* .

(2) *A bead on a circular wire* .

(3) *A particle moving on an ellipsoid under the influence of gravity* .

(4) *A pendulum with variable length* .

Solution : (1) When a body moves on an inclined plane, it is constrained to move on the inclined plane surface. Hence the constraint is holonomic. The force of constraint is the reaction of the plane, acting normal to the inclined surface.

(2) The constraint is that the bead remains at a constant distance a, the radius of the circular wire and can be expressed as $r = a$. Hence the constraint is holonomic. The force of constraint is the reaction of the wire, acting on the bead.

(3) The constraint is nonholonomic, because the particle after reaching a certain point will leave the ellipsoid. Force of constraint is the reaction force of the ellipsoid surface on the particle.

(4) The constraint is holonomic and rheonomic, because the constraint equation is $|\mathbf{r}| = l\,(t)$. The constraint force is the tension in the string.

Ex. 3. *Show that the constraints in a rigid body are conservative .*

Solution : The distance between any two particles i and j of a rigid body can be expressed as

$$r_{ij}^2 = |\mathbf{r}_i - \mathbf{r}_j|^2 = \text{constant.}$$

Therefore,
$$d\,(r_{ij}^2) = 0 = d\,[(\mathbf{r}_i - \mathbf{r}_j)\cdot(\mathbf{r}_i - \mathbf{r}_j)]$$
$$= (\mathbf{r}_i - \mathbf{r}_j)\cdot d\,(\mathbf{r}_i - \mathbf{r}_j) + d\,(\mathbf{r}_i - \mathbf{r}_j)\cdot(r_i - r_j)$$
$$= 2\,(\mathbf{r}_i - \mathbf{r}_j)\cdot d\,(\mathbf{r}_i - \mathbf{r}_j) = 0$$

i.e., $(\mathbf{r}_i - \mathbf{r}_j)\cdot d\,(\mathbf{r}_i - \mathbf{r}_j) = 0$

According to Newton's third law, the force $\mathbf{F}_{ij}$ on the ith particle due to jth particle is equal and opposite to the force on the jth particle due to ith particle *i.e.*,

$$\mathbf{F}_{ij} = -\,\mathbf{F}_{ji}$$

Now the work done

$$W = \sum_{\substack{i,j\\ i\neq j}} \int \mathbf{F}_{ij}\cdot d\mathbf{r}_i = \sum_{\substack{i,j\\ i>j}} \int \left(\mathbf{F}_{ij}\cdot d\mathbf{r}_i + \mathbf{F}_{ji}\cdot d\mathbf{r}_j\right)$$

$$= \sum_{\substack{i,j\\ i\neq j}} \int \mathbf{F}_{ij}\cdot d\left(\mathbf{r}_i - \mathbf{r}_j\right) = \sum_{\substack{i,j\\ i>j}} \int C_{ij}\left(\mathbf{r}_i - \mathbf{r}_j\right)\cdot d\left(\mathbf{r}_i - \mathbf{r}_j\right) = 0$$

because the internal force $\mathbf{F}_{ij}$ is considered parallel to the line joining ith and jth particles of the form : $\mathbf{F}_{ij} = C_{ij}(\mathbf{r}_i - \mathbf{r}_j)$, where C_{ij} are constants.

Hence the work done by constraint forces in a rigid body is zero and consequently the constraint is conservative in nature.

2.4. GENERALIZED COORDINATES

The name generalized coordinates is given to a set of independent coordinates sufficient in number to describe completely the state of configuration of a dynamical system. These coordinates are denoted as

$$q_1, q_2, q_3, \ldots, q_k, \ldots, q_n \qquad \ldots(7)$$

where n is the total number of generalized coordinates. In fact, these are the minimum number of coordinates, needed to describe the motion of the system. For example, for a particle constrained to move on the circumference of a circle only one generalized coordinate $q_1 = \theta$ is sufficient and two generalized coordinates

$q_1 = \theta$, and $q_2 = \phi$ for a particle moving on the surface of a sphere. The generalized coordinates for a system of N particles, constrained by m equations, are $n = 3N - m$. It is not necessary that these coordinates should be rectangular, spherical or cylindrical. In fact, the quantities like length, (length)^2, angle, energy or a dimensionless quantity may be used as generalized coordinates but they should completely describe the state of the system. Further these n generalized coordinates are not restricted by any constraint.

For a system of N particles, if x_i, y_i, z_i are the cartesian coordinates of the ith particle, then these coordinates in terms of the generalized coordinates q_k can be expressed as

$$\begin{aligned} x_i &= x_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \\ y_i &= y_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \\ z_i &= z_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \end{aligned} \quad \text{...(8 a)}$$

or in general the position vector $\mathbf{r}_i(x_i, y_i, z_i)$ of the ith particle is

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \quad \text{...(8 b)}$$

Eqs. (8a) or (8b) give the transformation equations. It may be mentioned that the generalized coordinates may be the cartesian coordinates.

One should note that the system is said to be rheonomic, when there is an explicit time dependence in some or all of the functions defined by eq. (8). If there is not the explicit time dependence, the system is called scleronomic and t is not written in the functional dependence, *i.e.*,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_k, \ldots, q_n) \quad \text{...(9)}$$

2.5. PRINCIPLE OF VIRTUAL WORK

In order to investigate the properties of a system, we can imagine arbitrary instantaneous change in the position vectors of the particles of the system *e.g.*, virtual displacements. An infinitesimal virtual displacement of ith particle of a system of N particles is denoted by $\delta\mathbf{r}_i$. This is the displacement of position coordinates only and does not involve *variation of time i.e.*,

$$\delta\mathbf{r}_i = \delta\mathbf{r}_i(q_1, q_2, \ldots, q_n) \quad \text{...(10)}$$

Suppose the system is in equilibrium, then the total force on any particle is zero *i.e.*,

$$\mathbf{F}_i = 0, \qquad i = 1, 2, \ldots, N$$

The virtual work of the force $\mathbf{F}_i$ in the virtual displacement $\delta\mathbf{r}_i$ will also be zero *i.e.*,

$$\delta W_i = \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0$$

Similarly, the sum of virtual work for all the particles must vanish *i.e.*,

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0 \quad \text{...(11)}$$

This result represents the ***principle of virtual work*** which states that ***the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium***.

The total force $\mathbf{F}_i$ on the ith particle can be expressed as

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i$$

where $\mathbf{F}_i^a$ is the applied force and $\mathbf{f}_i$ is the force of constraint.

Hence eq. (11) assumes the form

$$\sum_{i=1}^{N} \mathbf{F}_i^a \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

We restrict ourselves to the systems where the virtual work of the forces of constraints is zero, *e.g.*, in case of a rigid body. Then

$$\sum_{i=1}^{N} \mathbf{F}_i^a \cdot \delta \mathbf{r}_i = 0 \qquad \text{...(12)}$$

i.e., for equilibrium of a system, the virtual work of applied forces is zero. We see that the principle of virtual work deals with the statics of a system of particles. However, we want a principle to deal with the general motion of the system and such a principle was developed by D'Alembert.

2.6. D'ALEMBERT'S PRINCIPLE

According to Newton's second law of motion, the force acting on the ith particle is given by

$$\mathbf{F}_i = \frac{d\mathbf{p}_i}{dt} = \dot{\mathbf{p}}_i$$

This can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0, \qquad i = 1, 2, ..., N$$

These equations mean that any particle in the system is in equilibrium under a force, which is equal to the actual force $\mathbf{F}_i$ plus a reversed effective force $\dot{\mathbf{p}}_i$. Therefore, for virtual displacements $\delta \mathbf{r}_i$,

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0$$

But $\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{f}_i$, then

$$\sum_{i=1}^{N} \left(\mathbf{F}_i^a - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i + \sum_{i=1}^{N} \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$$

Again, we restrict ourselves to the systems for which the virtual work of the constraints is zero, *i.e.*, $\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0$. Then

$$\sum_{i=1}^{N} \left(\mathbf{F}_i^a - \dot{\mathbf{p}}_i\right) \cdot \delta \mathbf{r}_i = 0 \qquad \text{...(13)}$$

This is known as ***D'Alembert's principle***. Since the forces of constraints do not appear in the equation and hence now we can drop the superscript. Therefore, the D'Alembert's principle may be written as

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \qquad \text{...(14)}$$

Ex. 1. *Two heavy particles of weights W_1 and W_2 are connected by a light inextensible string and hang over a fixed smooth circular cylinder of radius R, the axis of which is horizontal* [*Fig.* 2.3]. *Find the condition of equilibrium of the system by applying the principle of virtual work.*

Solution : According to the principle of virtual work

$$\sum_{i=1}^{N} \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$$

Here, $i = 1, 2$ and therefore

$$W_1 \sin\theta\, \delta r_1 + W_2 \sin\phi\, \delta r_2 = 0$$

But $\delta r_1 = R\, d\theta$, $\delta r_2 = R\, d\phi$

$$\therefore \quad W_1 \sin\theta\, \delta\theta + W_2 \sin\phi\, \delta\phi = 0$$

But $\theta + \phi =$ constant

Therefore $\quad \delta\theta + \delta\phi = 0$ or $\delta\phi = -\delta\theta$

Thus $\quad (W_1 \sin\theta - W_2 \sin\phi)\, \delta\theta = 0$

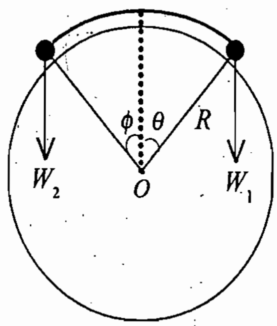


Fig. 2.3

The system is in equilibrium, hence the following condition is to be satisfied ($\delta\theta \neq 0$) :

$$W_1 \sin\theta - W_2 \sin\phi = 0 \quad \text{or} \quad \frac{W_1}{W_2} = \frac{\sin\phi}{\sin\theta}.$$

Ex. 2. *An inextensible string of negligible mass hanging over a smooth peg B [Fig. 2.4] connects one mass m_1 on a frictionless inclined plane of angle θ to another mass m_2. Using D'Alembert's principle, prove that the masses will be in equilibrium, if* $\sin\theta = \dfrac{m_2}{m_1}$.

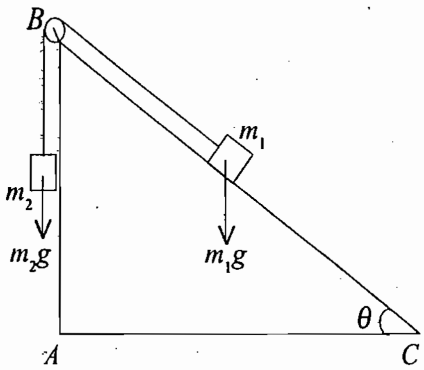


Fig. 2.4

Solution : According to D'Alembert's principle

$$\sum_{i=1}^{2} (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0$$

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of m_1 and m_2 relative to B.

$$(m_1\mathbf{g} - m_1 \ddot{\mathbf{r}}_1) \cdot \delta\mathbf{r}_1 + (m_2\mathbf{g} - m_2 \ddot{\mathbf{r}}_2) \cdot \delta\mathbf{r}_2 = 0$$

or $\quad (m_1 g \sin\theta - m_1 \ddot{r}_1)\, \delta r_1 + (m_2 g - m_2 \ddot{r}_2)\, \delta r_2 = 0 \quad \ldots(i)$

But the string is inextensible,

$r_1 + r_2 =$ a constant or $\delta r_1 + \delta r_2 = 0$, *i.e.*, $\delta r_2 = -\delta r_1$.

Also $\quad \ddot{r}_1 + \ddot{r}_2 = 0$ or $\ddot{r}_2 = -\ddot{r}_1$

Hence eq. (i) takes the form

$$(m_1 g \sin\theta - m_2 g)\, \delta r_1 - (m_1 + m_2)\, \ddot{r}_1\, \delta r_1 = 0$$

The system is in equilibrium, hence $\ddot{r}_1 = 0$. Further dividing by $\delta r_1 \neq 0$, we obtain

$$m_1 g \sin\theta - m_2 g = 0 \quad \text{or} \quad \sin\theta = \frac{m_2}{m_1}$$

Note : *Work Ex. 2 for the incline having the coefficient of friction μ and prove that the equilibrium condition is*

$$\sin\theta - \mu\cos\theta = \frac{m_2}{m_1}.$$

2.7. LAGRANGE'S EQUATIONS FROM D'ALEMBERT'S PRINCIPLE

Consider a system of N particles. The transformation equations for the position vectors of the particles are

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_k, ..., q_n, t) \qquad ...(15)$$

where t is the time and q_k ($k = 1, 2, ..., n$) are the generalized coordinates.

Differentiating eq. (15) with respect to t, we obtain the velocity of the ith particle, *i.e.*,

$$\frac{d\mathbf{r}_i}{dt} = \frac{\partial \mathbf{r}_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial \mathbf{r}_i}{\partial q_2}\frac{dq_2}{dt} + ... + \frac{\partial \mathbf{r}_i}{\partial q_k}\frac{dq_k}{dt} + ... + \frac{\partial \mathbf{r}_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial \mathbf{r}_i}{\partial t}$$

or

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \qquad ...(16)$$

where $\dot{q}_k$ are the **generalized velocities.**

The virtual displacement is given by

$$\delta\mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_1}\delta q_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\delta q_2 + ... + \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k + ... + \frac{\partial \mathbf{r}_i}{\partial q_n}\delta q_n$$

or

$$\delta\mathbf{r}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k \qquad ...(17)$$

since by definition the virtual displacements do not depend on time.

According to D'Alembert's principle,

$$\sum_{i=1}^{N} (\mathbf{F}_i - \dot{\mathbf{p}}_i)\cdot\delta\mathbf{r}_i = 0 \qquad ...(18)$$

Here

$$\sum_{i=1}^{N} \mathbf{F}_i\cdot\delta\mathbf{r}_i = \sum_{i=1}^{N} \mathbf{F}_i\cdot\sum_{k=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\right]\delta q_k = \sum_{k=1}^{n} G_k\,\delta q_k \qquad ...(19)$$

where

$$G_k = \sum_{i=1}^{N} \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[F_{x_i}\frac{\partial x_i}{\partial q_k} + F_{y_i}\frac{\partial y_i}{\partial q_k} + F_{z_i}\frac{\partial z_i}{\partial q_k}\right] \qquad ...(20)$$

are called the components of **generalized force** associated with the generalized coordinates q_k. This may be mentioned that as the dimensions of the generalized coordinates need not be those of length, similarly the generalized force components G_k may have dimensions different than those of force. However, the dimensions of $G_k\,\delta q_k$ are those of work. For example, if δq_k has the dimensions of length, G_k will have the dimensions of force; if δq_k has the dimensions of angle (θ), G_k will have the dimensions of torque (τ).

Further

$$\sum_{i=1}^{N} \dot{\mathbf{p}}_i\cdot\delta\mathbf{r}_i = \sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i\cdot\sum_{k=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_{k=1}^{n}\left[\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\right]\delta q_k \qquad ...(21)$$

Now

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\right) - m_i\dot{\mathbf{r}}_i\cdot\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right)\right] \qquad ...(22)$$

It is easy to prove that

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{d\mathbf{r}_i}{dt}\right) = \frac{\partial \mathbf{v}_i}{\partial q_k} \qquad \text{...(23 a)*}$$

and

$$\frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k} \qquad \text{...(23 b)**}$$

Therefore, eq. (22) is

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_{i=1}^{N}\left[\frac{d}{dt}\left[m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k}\right] - m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial q_k}\right] \qquad \text{...(24)}$$

Substituting in (21) , we get

$$\sum_{i=1}^{N} \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_{k=1}^{n}\sum_{i=1}^{N}\left[\frac{d}{dt}\left(m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial \dot{q}_k}\right) - m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{v}_i}{\partial q_k}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{N}\frac{1}{2}m_i(\mathbf{v}_i \bullet \mathbf{v}_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N}\frac{1}{2}m_i(\mathbf{v}_i \bullet \mathbf{v}_i)\right\}\right]\delta q_k$$

$$= \sum_{k=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right]\delta q_k\,; \qquad \text{...(25)}$$

---

* Here $$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \sum_{j=1}^{n}\frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k}\dot{q}_j + \frac{\partial^2 \mathbf{r}_i}{\partial t\, \partial q_k} \qquad \text{...}(i)$$

which has been obtained by treating $\partial \mathbf{r}_i/\partial q_k$ as a single quantity being the function of the generalized coordinates q_j and time t.

But $$\mathbf{v}_i = \frac{d\mathbf{r}_i}{dt} = \sum_{j=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}$$

and its partial derivative with respect to q_k is

$$\frac{\partial \mathbf{v}_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{d\mathbf{r}_i}{dt}\right) = \sum_{j=1}^{n}\frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial q_j}\dot{q}_j + \frac{\partial^2 \mathbf{r}_i}{\partial q_k \partial t} \qquad \text{...}(ii)$$

From eqs.(i) and (ii)

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k}\right) = \frac{\partial \mathbf{v}_i}{\partial q_k} \qquad \text{...(23}a\text{)}$$

** $$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\sum_{j=1}^{n}\frac{\partial \mathbf{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t}\right] = \frac{\partial \mathbf{r}_i}{\partial q_j}\delta_{jk} = \frac{\partial \mathbf{r}_i}{\partial q_k} \qquad \text{...(23 }b\text{)}$$

as the constraints are holonomic and $\dfrac{\partial \dot{q}_j}{\partial \dot{q}_k} = \delta_{jk}$ is kronecker delta which is 1 for $j = k$ and zero for $j \neq k$.

where $\sum_i \frac{1}{2} m_i(\mathbf{v}_i \bullet \mathbf{v}_i) = \sum_i \frac{1}{2} m_i v_i^2 = T$ is the **kinetic energy** of the system.

Substituting for $\sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i$ from (19) and $\sum_i \dot{\mathbf{p}}_i \bullet \delta\mathbf{r}_i$ from (25) in eq.(18), the D'Alembert's principle becomes

$$\sum_{k=1}^{n}\left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}\right\}-G_k\right]\delta q_k = 0 \qquad ...(26)$$

As the constraints are holonomic, it means that any virtual displacement δq_k is independent of δq_j. Therefore, the coefficient in the square bracket for each δq_k must be zero, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}-G_k = 0 \text{ or } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k} = G_k \qquad ...(27)$$

This represents the ***general form of Lagrange's equations.***

For a **conservative system**, the force is derivable from a scalar potential V:

$$\mathbf{F}_i = \nabla_i V = -\hat{\mathbf{i}}\frac{\partial V}{\partial x_i}-\hat{\mathbf{j}}\frac{\partial V}{\partial y_i}-\hat{\mathbf{k}}\frac{\partial V}{\partial z_i} \qquad ...(28)$$

Hence from eq. (20), the generalized force components are

$$G_k = -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k}+\frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k}+\frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \qquad ...(29)$$

Clearly the right hand side of the above equation is the partial derivative of $-V$ with respect to q_k, *i.e.*,

$$G_k = -\frac{\partial V}{\partial q_k} \qquad ...(30)$$

Thus eq.(27) assumes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} \qquad ...(31)$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial (T-V)}{\partial q_k} = 0 \qquad ...(32)$$

Since the scalar potential V is the function of generalized coordinates q_k only not depending on generalized velocities, we can write eq. (32) as

$$\frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q}_k}\right]-\frac{\partial (T-V)}{\partial q_k} = 0 \qquad ...(33)$$

We define a new function given by

$$L = T - V \qquad ...(34)$$

which is called the ***Lagrangian*** of the system. Thus, eq. (33) takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k} = 0 \qquad ...(35)$$

for $k = 1, 2, ..., n$.

These equations are known as **Lagrange's equations** for conservative system. They are n in number and there is one equation for each generalized coordinate. In order to solve these equations, we must know the Lagrangian function $L = T - V$ in the appropriate generalized coordinates.

2.8. PROCEDURE FOR FORMATION OF LAGRANGE'S EQUATIONS

The Lagrangian function L of a system is given by

$$L = T - V \qquad \text{...(36)}$$

In order to form L, kinetic energy T and potential energy V are to be written in generalized coordinates. This is then substituted in the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{...(37)}$$

to obtain the equations of motion of the system. This involves first to find the partial derivatives of L, *i.e.*, $\partial L/\partial q_k$ and $\partial L/\partial \dot{q}_k$ and then to put their values in eq. (37).

Kinetic Energy in Generalized Coordinates : The transformation equations (15) and (16) are used to transform T from cartesian coordinates to generalized coordinates. Therefore

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \sum_i \frac{1}{2}\left(\sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}\right)^2$$

or

$$T = M_0 + \sum_k M_k \dot{q}_k + \frac{1}{2}\sum_{k,l} M_{kl}\dot{q}_k \dot{q}_l \qquad \text{...(38)}$$

where

$$M_0 = \sum_k \frac{1}{2} m_i \left(\frac{\partial \mathbf{r}_i}{\partial t}\right)^2, \quad M_k = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial t} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

and

$$M_{kl} = \sum_i m_i \frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l}$$

Thus we see from (38) that in the expression for kinetic energy, first term is independent of generalized velocities, while second and third terms are linear and quadratic in generalized velocities respectively.

For scleronomic systems, the transformation equations do not contain time explicitly. So that

$$\mathbf{v}_i = \dot{\mathbf{r}}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k$$

Therefore,

$$T = \sum_i \frac{1}{2} m v_i^2 = \frac{1}{2}\sum_{k,l} M_{kl}\dot{q}_k \dot{q}_l \qquad \text{...(39)}$$

In such a case, the expression for T is a homogeneous quadratic form in generalized velocities.

Ex. 1. ***Newton's equation of motion from Lagrange's equations*** **:** *Consider the motion of a particle of mass m. Using cartesian coordinates as generalized coordinates, deduce Newton's equation of motion from Lagrange's equations.*

Solution : The general form of the Lagrange's equations is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad \text{...}(i)$$

Here, $q_1 = x$, $q_2 = y$, $q_3 = z$ and generalized force components are $G_1 = F_x$, $G_2 = F_y$, $G_3 = F_z$.
The kinetic energy T is

$$T = \frac{1}{2} m \left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right]$$

For x-coordinate, eq.(i) takes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x \qquad ...(ii)$$

But $$\frac{\partial T}{\partial x} = 0 \quad \text{and} \quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}$$

Substituting in eq. (*ii*), we get

$$\frac{d}{dt}(m\dot{x}) = F_x \quad \text{or} \quad F_x = \frac{dp_x}{dt}$$

where $p_x = m\dot{x}$ is the x-component of momentum. Similarly, we can obtain

$$F_y = \frac{dp_y}{dt} \quad \text{and} \quad F_z = \frac{dp_z}{dt}$$

Thus $$\mathbf{F} = \frac{d\mathbf{p}}{dt} \qquad ...(iii)$$

which is Newton's equation of motion.

Ex. 2. *Simple Pendulum* : *Obtain the equation of motion of a simple pendulum by using Lagrangian method and hence deduce the formula for its time period for small amplitude oscillations.*

(Agra 1999, 91; Garwal 98, 97; Kanpur 2003)

Solution : Let θ be the angular displacement of the simple pendulum from the equilibrium position. If l be the effective length of the pendulum and m be the mass of the bob, then the displacement along arc $OA = s$ is given by

$$s = l\theta \qquad \left[\text{because } \theta = \frac{\text{Arc}}{\text{Radius}} = \frac{s}{l}\right]$$

Kinetic energy $$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad \left[\because v = \frac{ds}{dt} = \frac{d(l\theta)}{dt} = \frac{l\,d\theta}{dt} = l\dot{\theta}\right]$$

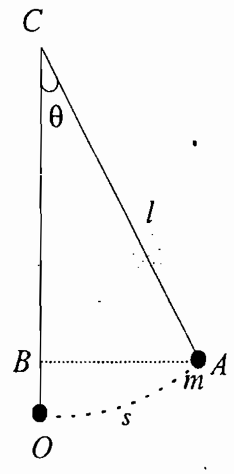


Fig. 2.5 : Simple pendulum

If the potential energy of the system, when the bob is at O, is zero, then the potential energy, when the bob is at A, is given by

$$V = mg\,(OB) = mg\,(OC - BC) = mg\,(l - l\cos\theta) = mgl\,(1 - \cos\theta)$$

Hence $$L = T - V, \quad \text{or} \quad L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1 - \cos\theta)$$

Now, $$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \quad \text{and} \quad \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

Substituting these values in the Lagrange's equation (here there is only one generalized coordinate $q_1 = \theta$)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0,$$

we get

$$\frac{d}{dt}\left[ml^2\dot{\theta}\right] + mgl\sin\theta = 0 \text{ or } ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This represents the equation of motion of a simples pendulum.

For small amplitude oscillations, $\sin\theta \cong \theta$, and therefore the equation of motion of a simple pendulum is

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

This represents a *simple harmonic motion* of period, given by

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Ex. 3. ***Atwood's Machine*** **:** *Obtain the equation of motion of a system of two masses, connected by an inextensible string passing over a small smooth pulley.* **(Mumbai 2002; Agra 1989, 96)**

Solution : The Atwood's machine is an example of a conservative system with holonomic constraint. The pulley is small, massless and frictionless. Let the two masses be m_1 and m_2 which are connected by an inextensible string of length l. Suppose x be the variable vertical distance from the pulley to the mass m_1. Then mass m_2 will be at a distance $l - x$ from the pulley [Fig. 2.6]

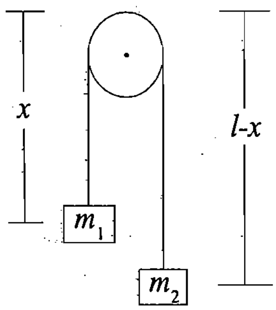


Fig. 2.6 : Atwood's machine

Thus there is only one independent coordinate x. The velocities of the two masses are $v_1 = \dfrac{dx}{dt} = \dot{x}$ and $v_2 = \dfrac{d(l-x)}{dt} = -\dot{x}$.

Therefore, $$T = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}m_2\dot{x}^2 = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2$$

Potential energy of the system with reference to the pulley is

$$V = -m_1gx - m_2g(l - x)$$

Thus the Lagrangian is

$$L = T - V = \tfrac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + m_2g(l - x)$$

Now, $$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} \quad \text{and} \quad \frac{\partial L}{\partial x} = (m_1 - m_2)g$$

Here the generalized coordinate is $q = x$. Now Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0 \quad \text{or} \quad (m_1+m_2)\,\ddot{x}-(m_1-m_2)\,g=0$$

or
$$\ddot{x}=\frac{(m_1-m_2)}{m_1+m_2}\,g$$

which is the desired equation of motion.

If $m_1 > m_2$, the mass m_1 descends with constant acceleration and if $m_1 < m_2$, the mass m_1 ascends with constant acceleration. It is to be noted that the tension in the rope, the force of constraint, is not seen anywhere in the Lagrangian formulation.

Ex. 4 . *In Ex. 3, calculate the acceleration of the system, if the pulley is a disc of radius R and moment of inertia I about an axis through its centre and perpendicular to its plane.*

Solution : Angular velocity of the pulley $\omega=\frac{v}{R}=\frac{\dot{x}}{R}$

Rotational kinetic energy of the pulley $=\frac{1}{2}I\omega^2=\frac{1}{2}I\frac{\dot{x}^2}{R^2}$

where $\quad v=\dot{x}=|v_1|=|v_2|$

$\therefore \quad T=\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}I\frac{\dot{x}^2}{R^2}$

Also, $\quad V=-m_1 gx-m_2 g\,(l-x)$.

Therefore, $\quad L=\frac{1}{2}\left(m_1+m_2+\frac{I}{R^2}\right)\dot{x}^2+(m_1-m_2)\,gx+m_2\,gl$

The Lagrange's equation is

$$\left(m_1+m_2+\frac{I}{R^2}\right)\ddot{x}-[m_1-m_2]\,g=0$$

whence
$$\ddot{x}=\frac{[m_1-m_2]\,g}{m_1+m_2+\frac{I}{R^2}}$$

Equation of motion of Ex. 3 will be obtained for $I=0$.

Ex. 5. *Compound Pendulum* : *Use Lagrange's equations to find the equation of motion of a compound pendulum in a vertical plane about a fixed horizontal axis. Hence find the period of small amplitude oscillations of the compound pendulum.* **(Agra 1999, 97, 93)**

Solution : Let the compound pendulum be suspended from S with C as centre of mass. It is oscillating in the vertical plane which is the plane of the paper.

Moment of inertia of the pendulum about the axis of rotation through S is given by

$$I=I_c+Ml^2=M\,(K^2+l^2)$$

where M is the mass of the pendulum, $I_c=MK^2$ (K = radius of gyration) about a parallel axis through C and l the distance between centre of suspension and centre of mass.

If θ is the instantaneous angle which SC makes with the vertical axis through O, then the kinetic energy of the oscillating system is

$$T = \frac{1}{2} I \dot{\theta}^2 = \frac{1}{2} M\left(K^2 + l^2\right)\dot{\theta}^2$$

Potential energy with respect to horizontal plane through S is

$$V = - Mgl \cos \theta$$

Lagrangian $\quad L = T - V$

or $\quad L = \frac{1}{2} M\left(K^2 + l^2\right)\dot{\theta}^2 + Mgl \cos\theta$

Now, $\quad \dfrac{\partial L}{\partial \theta} = - Mgl \sin \theta$ and $\dfrac{\partial L}{\partial \dot{\theta}} = M(K^2 + l^2)\,\dot{\theta}$

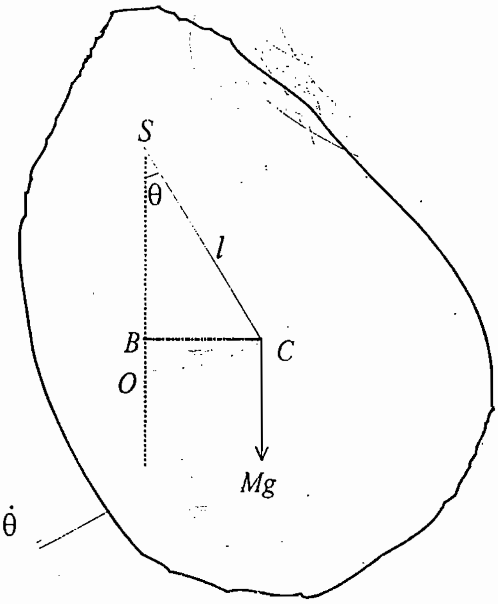


Fig. 2.7 : Compound pendulum

Lagrange's equation in θ coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Therefore, $\quad \frac{1}{2} M\left(K^2 + l^2\right)\ddot{\theta} + Mgl \sin\theta = 0$ or $\ddot{\theta} + \dfrac{gl}{K^2 + l^2}\sin\theta = 0$

This is the equation of motion of the compound pendulum. If θ is small, $\sin\theta \cong \theta$ and then

$$\ddot{\theta} + \frac{gl}{K^2 + l^2}\theta = 0$$

This equation represents a simple harmonic motion whose period is given by

$$T = 2\pi \sqrt{\frac{K^2 + l^2}{lg}} = 2\pi \sqrt{\frac{\frac{K^2}{l} + l}{g}}.$$

Ex. 6. ***Radial and Tangential Components of a Force :*** *Consider the motion of a particle of mass m moving in a plane. Using the plane polar coordinates (r, θ) as generalized coordinates, deduce expressions for the components of generalized force. What are radial and tangential components of the force ?*

Solution : For the motion of a particle in a plane, the cartesian and polar coordinates are related as

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

Hence $\quad \dot{x} = \dot{r} \cos \theta - r\dot{\theta} \sin \theta$ and $\dot{y} = \dot{r} \sin \theta + r\dot{\theta} \cos \theta$

Therefore, $\quad T = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2)$

Here, there are two generalized coordinates, *i.e.*, $q_1 = r$ and $q_2 = \theta$.

Now, $\quad \dfrac{\partial T}{\partial r} = mr\dot{\theta}^2,\ \dfrac{\partial T}{\partial \dot{r}} = m\dot{r},\ \dfrac{\partial T}{\partial \theta} = 0$ and $\dfrac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$

Corresponding to two generalized coordinates, there are two Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = G_r \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = G_\theta$$

or $$m\ddot{r} - mr\dot{\theta}^2 = G_r \quad \text{and} \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) = G_\theta$$

We can express the components of the generalized force in terms of the radial and tangential components of the force. From the definition of the generalized force, we have

$$G_r = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial r} \quad \text{and} \quad G_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} \quad \left[\text{as } G_k = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_k}\right]$$

But $$\mathbf{r} = r\cos\theta\,\hat{\mathbf{i}} + r\sin\theta\,\hat{\mathbf{j}}$$

Hence $$\frac{\partial \mathbf{r}}{\partial r} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$$

and $$\frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\,\hat{\mathbf{i}} + r\cos\theta\,\hat{\mathbf{j}} = r\hat{\boldsymbol{\theta}}$$

where $\hat{\theta}$ = unit vector perpendicular to $\hat{\mathbf{r}}$.

Therefore, $$G_r = \mathbf{F} \cdot \hat{\mathbf{r}} = F_r \quad \text{or} \quad F_r = m\ddot{r} - mr\dot{\theta}^2 \qquad ...(i)$$

and $$G_\theta = \mathbf{F} \cdot r\hat{\boldsymbol{\theta}} = r\,\mathbf{F} \cdot \hat{\boldsymbol{\theta}} = r F_\theta \quad \text{or} \quad r F_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right) \qquad ...(ii)$$

Note that in eq. (ii), $mr^2\dot{\theta} = mvr = J$, the angular momentum and its time derivative is just the applied torque (rF_θ). This is the torque equation, *i.e.*, rate of change of angular momentum is equal to the applied torque.

Thus the radial and tangential components of the force are

$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \quad \text{and} \quad F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) \qquad ...(iii)$$

Ex. 7. *Langrange's equation for L-C circuit* : *Find Lagrange's equation of motion for an electrical circuit comprising an inductance L and capacitance C. The capacitor is charged to q coulombs and current flowing in the circuit is i amperes.* **(Agra 2003, 1999, 1992)**

Solution : Let us consider an electrical circuit, containing inductance L and capacitance C. We want to find Lagrange's equation of motion for the L-C circuit, when the charge on the condenser is q and the current flowing in the circuit is i.

The magnetic energy $\frac{1}{2}Li^2$ in an electrical circuit is analogous to the kinetic energy $\frac{1}{2}mv^2$ in a mechanical system, where we can think inductance L as charge inertia similar to mass inertia and $i = \frac{dq}{dt}$ as $v = \frac{dx}{dt}$; charge q is playing the role of displacement. The electrical potential energy of the circuit is $V = q^2/2C$. Hence the Lagrangian of the L-C circuit is

$$L = T - V = \frac{1}{2}Li^2 - \frac{q^2}{2C} \quad \text{or} \quad L = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C}$$

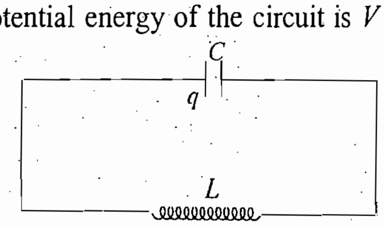


Fig. 2.8 : L-C Circuit

Taking q as the generalized coordinate, the Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$$

Here $$\frac{\partial L}{\partial \dot{q}}=L\dot{q} \text{ and } \frac{\partial L}{\partial q}=-\frac{q}{C}.$$

Therefore, $$\frac{d}{dt}[L\dot{q}]+\frac{q}{C}=0 \text{ or } L\frac{d^2q}{dt^2}+\frac{q}{C}=0 \text{ or } \frac{d^2q}{dt^2}+\frac{q}{LC}=0$$

This is the differential equation for L-C circuit, having frequency $\nu=\dfrac{1}{2\pi\sqrt{LC}}$.

Ex. 8. *Motion under Central Force* : *Derive equations of motion for a particle moving under central force. What is the form of the equations, when the particle is moving under an attractive inverse square law force* $(F=-k/r^2)$. **(Rohilkhand 1998; Agra 1991)**

Solution : When a particle is moving under central force, then the force is conservative and the motion is in a plane.

Let (r, θ) be the plane coordinates of the particle of mass m.

Kinetic energy $\quad T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)$

Lagrangian $\quad L=T-V=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)-V(r)$

where $V(r)$ is the potential energy in the central force field.

Now, $$\frac{\partial L}{\partial r}=mr\dot{\theta}^2-\frac{\partial V}{\partial r},\ \frac{\partial L}{\partial \dot{r}}=m\dot{r} \text{ and } \frac{\partial L}{\partial \theta}=0,\ \frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$$

Hence equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=0$$

or $$m\ddot{r}-mr\dot{\theta}^2+\frac{\partial V}{\partial r}=0 \text{ and } \frac{d}{dt}\left(mr^2\dot{\theta}\right)=0 \qquad ...(i)$$

For attractive inverse square law force $F=-\partial V/\partial r=-k/r^2$, we have equations of motion as

$$m\ddot{r}-mr\dot{\theta}^2+\frac{k}{r^2}=0 \qquad ...(ii\ a)$$

and $$\frac{d}{dt}\left(mr^2\dot{\theta}\right)=0 \text{ or } r\ddot{\theta}+2\dot{r}\dot{\theta}=0 \qquad ...(ii\ b)$$

2.9. LAGRANGE'S EQUATIONS IN PRESENCE OF NON-CONSERVATIVE FORCES

When the forces acting on the system consist of non-conservative forces ($\mathbf{f}_i$) in addition to the conservative forces ($\mathbf{F}_i$), then the components of generalized force can be written as [using eq. (20)];

$$G_k = \sum_{i=1}^{N}[\mathbf{F}_i + \mathbf{f}_i] \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \text{or} \quad G_k = -\frac{\partial V}{\partial q_k} + G_k' \qquad ...(40)$$

where $G_k' = \sum \mathbf{f}_i \cdot \dfrac{\partial \mathbf{r}_i}{\partial q_k}$ are the components of generalized non-potential force resulting from non-conservative forces and $\sum \mathbf{f}_i \cdot \dfrac{\partial \bar{\mathbf{r}}_i}{\partial q_k} = -\dfrac{\partial V}{\partial q_k}$ for conservative part [eq. (30)].

Here V is the scalar potential for conservative forces. In such a case, eq. (35) assumes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = G_k' \qquad ...(41)$$

where $L = T - V$.

Eqs. (41) represent the Lagrange's equations in the presence of non-conservative forces.

An example of the non-conservative force is the presence of frictional force, acting on the system. If the frictional force is proportional to the velocity of a particle, then

$$\mathbf{f}_i = -k_i \mathbf{v}_i \qquad ...(42)$$

where k_i is the constant of proportionality for the movement of the ith particle.

We may derive such frictional forces from a function of the form

$$R = \tfrac{1}{2}\sum_i k_i \mathbf{v}_i^2 = \tfrac{1}{2}\sum_i k_i\left(v_{xi}^2 + v_{yi}^2 + v_{zi}^2\right) \qquad ...(43)$$

This is known as ***Rayleigh's dissipation function.*** Obviously

$$f_{xi} = -\frac{\partial R}{\partial v_{xi}} = -k_i v_{xi}.$$

Thus $\qquad \mathbf{f}_i = -k_i \mathbf{v}_i = -\nabla_v R \qquad ...(44)$

Hence the component of the generalized force due to the force of friction is given by

$$G_k' = \sum_i \mathbf{f}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i \nabla_v R \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i \nabla_v R \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}$$

$$= -\sum_i \left[\frac{\partial R}{\partial v_{xi}}\frac{\partial v_{xi}}{\partial \dot{q}_k} + \frac{\partial R}{\partial v_{yi}}\frac{\partial v_{yi}}{\partial \dot{q}_k} + \frac{\partial R}{\partial v_{zi}}\frac{\partial v_{zi}}{\partial \dot{q}_k}\right] = -\frac{\partial R}{\partial \dot{q}_k} \qquad ...(45)$$

Thus Lagrange's equation (41) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = -\frac{\partial R}{\partial \dot{q}_k}$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0 \qquad ...(46)$$

It can be proved that the Rayleigh's dissipation function R is equal to one half of the rate of dissipation energy against friction. The work done against friction is

$$dW = -\sum_i \mathbf{f}_i \cdot d\mathbf{r}_i = -\sum_i \mathbf{f}_i \cdot \mathbf{v}_i\, dt = [\sum_i k_i v_i^2]\, dt$$

whence
$$\frac{dW}{dt} = \sum_i k_i v_i^2 = 2R \qquad ...(47)$$

This gives the physical interpretation of the Rayleigh's dissipation function.

2.10. GENERALIZED POTENTIAL — *Lagrangian for a Charged Particle Moving in an Electromagnetic Field (Gyroscopic Forces)*

In general, the Lagrange's equations can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad ...(48)$$

For a conservative system, $G_k = -\dfrac{\partial V}{\partial q_k}$ and then the Lagrange's equations in the usual form are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{with} \quad L = T - V \qquad ...(49)$$

However, Lagrange's equations can be put in the form (49), provided the generalized forces are obtained from a function $U(q_k, \dot{q}_k)$, given by

$$G_k = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_k}\right) \qquad ...(50)$$

In such a case, $L = T - U$ ...(51)

where $U(q_k, \dot{q}_k)$ is called ***velocity dependent potential*** or ***generalized potential.*** This type of case occurs in case of a charge moving in an electromagnetic field.

In S.I. system, two of the Maxwell's field equations are

$$\text{div}\,\mathbf{B} = 0 \quad \text{and} \quad \text{curl}\,\mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

or
$$\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad ...(52)$$

where **E** and **B** are electric field and magnetic field vectors respectively.

The force acting on a charge q, moving with velocity **v** in an electric field **E** and magnetic induction **B** is given by

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad ...(53)$$

Since $\nabla \cdot \mathbf{B} = 0$ in eq. (52) and hence **B** can be expressed as curl of a vector *i.e.*,

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad ...(54)$$

where **A** is called the ***magnetic vector potential.*** Substituting for **B** from (54) into the second equation of (52), we get

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t} \nabla \times \mathbf{A} = 0 \quad \text{or} \quad \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \qquad ...(55)$$

Hence we can express the vector quantity $\left(\mathbf{E}+\dfrac{\partial \mathbf{A}}{\partial t}\right)$ as the gradient of a scalar function ϕ, *i.e.*,

$$\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}=-\nabla\phi \quad \text{or} \quad \mathbf{E}=-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t} \qquad ...(56)$$

Substituting for **E** from (56) in (53), we obtain

$$\mathbf{F}=q\left(-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t}+\mathbf{v}\times\nabla\times\mathbf{A}\right) \qquad ...(57)$$

The terms in eq. (57) can be written in a more convenient form.

Let us consider the x-component. Since $\nabla\phi=\hat{\mathbf{i}}\dfrac{\partial\phi}{\partial x}+\hat{\mathbf{j}}\dfrac{\partial\phi}{\partial y}+\hat{\mathbf{z}}\dfrac{\partial\phi}{\partial z}$, x-component of $\nabla\phi$ is $\dfrac{\partial\phi}{\partial x}$. Also,

$$(\mathbf{v}\times\nabla\times\mathbf{A})_x=v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$

We add and subtract the term $v_x\partial A_x/\partial x$. Then

$$(\mathbf{v}\times\nabla\times\mathbf{A})_x=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}-v_x\frac{\partial A_x}{\partial x}-v_y\frac{\partial A_x}{\partial y}-v_z\frac{\partial A_x}{\partial z} \qquad ...(58)$$

However,

$$\frac{dA_x}{dt}=\frac{\partial A_x}{\partial x}\frac{dx}{dt}+\frac{\partial A_x}{\partial y}\frac{dy}{dt}+\frac{\partial A_x}{\partial z}\frac{dz}{dt}+\frac{\partial A_x}{\partial t}=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}+\frac{\partial A_x}{\partial t}$$

whence

$$v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z}=\frac{dA_x}{dt}-\frac{\partial A_x}{\partial t} \qquad ...(59)$$

Further

$$\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})=\frac{\partial}{\partial x}\left(v_xA_x+v_yA_y+v_zA_z\right)$$

$$=v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x} \qquad ...(60)$$

Substituting from (59) and (60) in (58), we get

$$(\nabla\times\nabla\times\mathbf{A})_x=\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t} \qquad ...(61)$$

Hence from eq. (57), the x-component of the force **F** is

$$F_x=q\left(-\frac{\partial\phi}{\partial x}-\frac{\partial A_x}{\partial t}+\frac{\partial}{\partial x}(\mathbf{v}\cdot\mathbf{A})-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}\right)=q\left(-\frac{\partial}{\partial x}(\phi-\mathbf{v}\cdot\mathbf{A})-\frac{dA_x}{dt}\right) \qquad ...(62)$$

Since

$$\frac{\partial}{\partial v_x}(\mathbf{v}\cdot\mathbf{A})=\frac{\partial}{\partial v_x}\left(v_xA_x+v_yA_y+v_zA_z\right)=A_x$$

and scalar potential ϕ is independent of v_x, we have

$$-\frac{dA_x}{dt}=\frac{d}{dt}\frac{\partial}{\partial v_x}(\phi-\mathbf{v}\cdot\mathbf{A})$$

Therefore $F_x = q\left[-\frac{\partial}{\partial x}(\phi - \mathbf{v}\cdot\mathbf{A}) + \frac{d}{dt}\{\frac{\partial}{\partial v_x}(\phi - \mathbf{v}\cdot\mathbf{A})\}\right]$ ...(63)

We define a ***generalized potential*** U, given by

$$U = q\,(\phi - \mathbf{v}\cdot\mathbf{A}) \qquad ...(64)$$

which is a ***velocity dependent potential*** in the sense of eq. (50). Therefore, eq. (63) takes the form

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \qquad ...(65)$$

The Lagrange's equations (48) in this case take the form

$$(q_k = x,\ \dot{q}_k = \dot{x} = v_x \text{ and } G_k = F_x)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_x}\right) - \frac{\partial T}{\partial x} = F_x \qquad ...(66)$$

Substituting F_x from (66) in (65), we get the Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(T-U)\right) - \frac{\partial}{\partial x}(T-U) = 0$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad ...(67)$$

where

$$L = T - U = T - q\phi + q\,\mathbf{v}\cdot\mathbf{A} \qquad ...(68)$$

Eq. (68) gives the Lagrangian for a charged particle moving in an electromagnetic field.

Note : In Gaussian C.G.S. system **B** is to be replaced by **B**/c in eqs. (52) and (53), where c is the speed of light. Therefore the expression for generalized potential is obtained to be $U = q\phi - \frac{q}{c}(\mathbf{v}\cdot\mathbf{A})$.

Gyroscopic Forces

All the velocity dependent forces, which do not consume power, are called gyroscopic forces. If a charge q is moving with velocity **v** in a magnetic field **B**, then the force acting on the particle

$$\mathbf{F} = q\,(\mathbf{v}\times\mathbf{B})$$

is gyroscopic in nature.

For such a force the power consumed happens to be zero, i.e.,

$$P = \mathbf{F}\cdot\mathbf{v} = q\,(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v} = q\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = q(\mathbf{v}\times\mathbf{v})\cdot\mathbf{B} = 0$$

because for a scalar triple product $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = (\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$.

Thus the velocity dependent magnetic force, given in eq. (53) is an example of gyroscopic force. A gyroscopic force can be incorporated in a generalised potential U similar to the one due to magnetic force, given in eq. (64) with the Lagrangian L [eq. (68)] and Lagrange's equation (67).

2.11. Hamilton's Principle and Lagrange's Equations

In Sec. 2.7, we have used the D'Alembert's principle to deduce Lagrange's equations. This principle uses the idea of virtual work and stems from Newton's second law of motion. These Langrange's equations can be derived by an entirely different way, namely Hamilton's variational principle.

Hamilton's principle : *This principle states that for a conservative holonomic system, its motion from time t_1 to time t_2 is such that the line integral* (known as **action** or **action integral**)

$$S = \int_{t_1}^{t_2} L\, dt \qquad ...(69)$$

with $L = T - V$ has stationary (extremum) value for the correct path of the motion.

The quantity S is called as ***Hamilton's principal function***. The principle may be expressed as

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad ...(70)$$

where δ is the variation symbol.

Lagrange's equation from Hamilton's principle : The Lagrangian L is a function of generalized coordinates q_k's and generalized velocities $\dot{q}_k$'s and time t, *i.e.*,

$$L = L(q_1, q_2, ..., q_k, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., \dot{q}_n, t)$$

If the Lagrangian does not depend on time t explicitly, then the variation δL can be written as

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \qquad ...(71)$$

Integrating both sides from $t = t_1$ to $t = t_2$, we get

$$\int_{t_1}^{t_2} \delta L\, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k dt$$

But in view of the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

Therefore,
$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = 0 \qquad ...(72)$$

where $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$.

Integrating by parts the second term on the left hand side of eq. (72), we get

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\, dt = \sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k\, dt \qquad ...(73)$$

At the end points of the path at the times t_1 and t_2, the coordinates must have definite values $q_k(t_1)$ and $q_k(t_2)$ respectively, *i.e.*, $\delta q_k(t_1) = \delta q_k(t_2) = 0$ (Fig. 2.9) and hence

$$\sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k \right]_{t_1}^{t_2} = 0$$

Therefore, eq. (72) takes the form

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k\, dt - \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k\, dt = 0$$

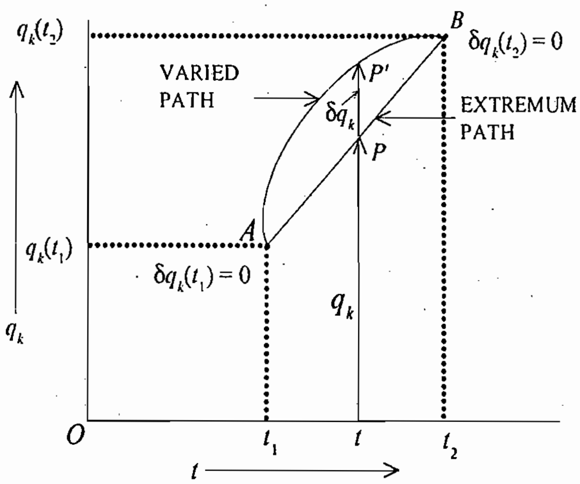


Fig. 2.9 : δ-variation - extremum path

$$\sum_k \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \right] \delta q_k \, dt = 0 \quad ...(74)$$

For holonomic system, the generalized coordinates δq_k are independent of each other. Therefore, the coefficient of each δq_k must vanish, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad ...(75)$$

where $k = 1, 2, ..., n$ are the generalized coordinates.

Eqs. (75) are the *Lagrange's equations of motion.*

2.12. SUPERIORITY OF LAGRANGIAN MECHANICS OVER NEWTONIAN APPROACH

In the Newtonian mechanics, the equations of motion involve vector quantities like force, momentum etc. which increase complexity in solving the problems. This approach also cannot avoid constraints present in a problem. These forces of constraints, if not known, make the solution of the problem difficult and even if they are known, the use of rectangular or other commonly used coordinates may make the solution of the problem to be impossible. These drawbacks are removed in the Lagrangian mechanics, where the technique involves scalars, like potential and kinetic energies, instead of vectors. The use of generalized coordinates in the Lagrangian formulation often allows automatically for the constraints. In this formulation, the difficulty in solving the problems is many times much reduced, when any quantity like momentum or (length)^2 is taken as a generalized coordinate instead of rectangular or commonly used coordinates. Further the form of the Lagrange's equations of motion remains invariant under any generalized coordinate transformation.

2.13. GAUGE INVARIANCE OF THE LAGRANGIAN

If L is a Lagrangian for a system of n degrees of freedom, satisfying Lagrange's equations, it can be shown that

$$L' = L + \frac{dF}{dt} \quad ...(76)$$

also satisfies Lagrange's equations, where F is an arbitrary function :

$$F = F(q_1, q_2, \ldots\ldots, q_n, t) \qquad \ldots(77)$$

Such a function is called **Gauge function** for the Lagrangian.

Proof : Time derivative of the function F is

$$\frac{dF}{dt} = \sum_{k=1}^{n} \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t} \qquad \ldots(78)$$

Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \ldots(79)$$

If $L' = L + \frac{dF}{dt}$ satisfies (79), then

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(L + \frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_k}\left(L + \frac{dF}{dt}\right) = 0 \qquad \ldots(80)$$

Subtracting (79) from (80), we get

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{dF}{dt}\right)\right] - \frac{\partial L}{\partial q_k}\left(\frac{dF}{dt}\right) = 0 \qquad \ldots(81)$$

If we prove L.H.S. of eq. (81) to be equal to zero, then L' will satisfy the Lagrange's equations.

Now, $\text{L.H.S.} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{dF}{dt}\right)\right] - \frac{\partial}{\partial q_k}\left(\frac{dF}{dt}\right)$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial F}{\partial t} + \sum_l \dot{q}_l \frac{\partial F}{\partial q_l}\right)\right] - \frac{\partial}{\partial q_k}\left[\frac{\partial F}{\partial t} + \sum_l \dot{q}_l \frac{\partial F}{\partial q_l}\right]$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{q}_k}\right) + \sum_l \dot{q}_l \frac{\partial}{\partial q_l}\left(\frac{\partial F}{\partial \dot{q}_k}\right) + \sum_l \frac{\partial F}{\partial q_l}\frac{\partial \dot{q}_l}{\partial \dot{q}_k}\right] - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l - \sum_l \frac{\partial F}{\partial q_l}\frac{\partial \dot{q}_l}{\partial q_k}$$

Here, $\frac{\partial F}{\partial \dot{q}_k} = 0$, $\frac{\partial \dot{q}_l}{\partial q_k} = 0$ and $\frac{\partial \dot{q}_l}{\partial \dot{q}_k} = \delta_{lk}$

Hence $\text{L.H.S.} = \frac{d}{dt}\left(\frac{\partial F}{\partial q_k}\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$

$$= \frac{\partial}{\partial q_k}\left(\frac{dF}{dt}\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t} + \sum_l \frac{\partial F}{\partial q_l}\dot{q}_l\right) - \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial t}\right) - \sum_l \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_l}\right)\dot{q}_l$$

$$= 0$$

Thus $L' = L + \frac{dF}{dt}$ satisfies the Langrange's equations.

2.14. SYMMETRY PROPERTIES OF SPACE AND TIME AND CONSERVATION LAWS

When we consider the motion of a free particle or a closed system in an inertial frame, the space is assumed to be **homogeneous** and **isotropic** and the time to be **homogeneous**. By a closed system we mean a system, not acted by any external force.

The space is said to be homogeneous, if the physical properties of a closed system are not affected by an arbitrary displacement of the origin of the frame of reference. This means that in order to describe the state of motion of a closed system, any point in space is equivalent to any other point of the space.

The space is said to be isotropic, if the physical properties of closed system are not changed for arbitrary rotation about the origin of the frame of reference. Therefore, for the description of a closed system, every direction in space is equivalent and any direction for the Cartesian axes can be used.

The time is said to be homogeneous, if the physical properties of a closed system are not affected by an arbitrary displacement of the origin of time. Hence any moment of time can be taken to describe a closed system.

The homogeneity and isotropy of space and homogeneity of time imply the invariance of the physical properties of a closed system under certain operations, known as *symmetry operations*. These operations leave the configuration and states of motion unchanged. The homogeneity of space correspond to an arbitrary translation (symmery operation), isotropy of space to an arbitrary rotation and homogeneity of time to an arbitrary shifting of the time or time-translation.

We can describe a closed system by its Lagrangian. This Lagrangian must be invariant under the operations of translation and rotation in space and time-shifting. These symmetry operations on the Lagrangian have very important consequences. Each symmetry operation results in a conservation law, representing a physical quantity or an integral of motion to be conserved. This physical quantity is additive, *i.e.*, the value of the physical quantity for the entire system is the sum of its values for different parts of the system.

Thus every symmetry in the Lagrangian corresponds to a conservation law. Homogeneity of space results in the conservation law of linear momentum, isotropy of space in the conservation law of angular momentum and homogeneity of time in the conservation law of energy. These conservation laws have been obtained in the following discussion.

(1) Homogeneity of Space and Conservation of Linear Momentum : The homogeneity of space implies that the Lagrangian of a closed system is not changed by an arbitrary translation of all the particles of the system. In Carterian coordinates, a small arbitrary translation of the coordinate of the i^{th} particle can be written as

$$\mathbf{r}_i \to \mathbf{r}_i + \delta\mathbf{r}_i \text{ or } \mathbf{r}_i \to \mathbf{r}_i + \boldsymbol{\varepsilon}$$

where $\delta\mathbf{r}_i = \boldsymbol{\varepsilon}$ is constant small translation for each particle.

Now corresponding to this change in coordinate, the change δL in L is

$$\delta L = \sum_i \frac{\partial L}{\partial \mathbf{r}_i} \cdot \boldsymbol{\varepsilon} = -\boldsymbol{\varepsilon} \cdot \sum_i \frac{\partial L}{\partial \mathbf{r}_i} \tag{82}$$

However for any arbitrary translation $\boldsymbol{\varepsilon}$, $\delta L = 0$. This means

$$\sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0 \tag{83}$$

Langrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) - \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

Hence for all particles of the system

$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) - \sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

Using (83),
$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) = 0 \quad \text{or} \quad \frac{d}{dt}\sum_i \left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) = 0 \qquad \text{...(84)}$$

But
$$L = T - V = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^{\,2} - V(\mathbf{r}_1, \ldots\ldots, \mathbf{r}_N)$$

$\therefore$
$$\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_i = \mathbf{p}_i, \text{ linear momentum of } i^{\text{th}} \text{ particle,} \qquad \text{...(85)}$$

Hence, from (84),

$$\frac{d}{dt}\left(\sum_i \mathbf{p}_i\right) = 0 \quad \text{or} \quad \sum_i \mathbf{p}_i = \text{Constant} \qquad \text{...(86)}$$

where $\sum_i \mathbf{p}_i = \mathbf{P}$ is the total linear momentum of the system.

Thus, *the total linear momentum of a closed system is conserved due to the homogeneity of the space.*

(2) Isotropy of Space and Conservation of Angular Momentum : Due to the isotropy of space, the Lagrangian of a closed system remains unchanged under arbitrary rotation. Let us consider an arbitrary infinitesimal rotation of the system about some direction, say Z-direction. Therefore the position vector $\mathbf{r}_i$ due to rotation $\delta\theta$ will change by (Fig. 2.10)

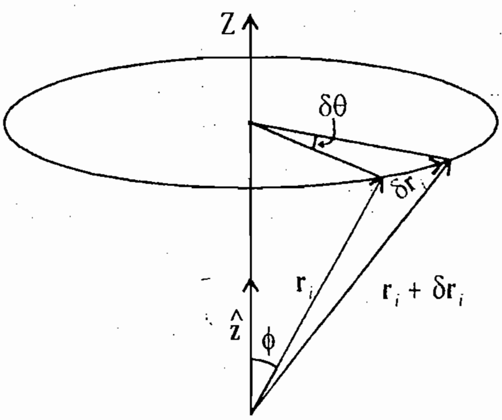


Fig. 2.10

$$\delta \mathbf{r}_i = \delta\theta\, \hat{\mathbf{z}} \times \mathbf{r}_i$$

$$(\because\ |\delta \mathbf{r}_i| = r_i \sin\phi\, \delta\theta = |\hat{\mathbf{z}} \times \mathbf{r}_i|\delta\theta\,) \qquad \text{...(87)}$$

The change in velocity vector $\mathbf{v}_i$ due to arbitrary rotation $\delta\theta$ is

$$\delta \mathbf{v}_i = \delta\theta\, \hat{\mathbf{z}} \times \mathbf{v}_i \qquad \text{...(88)}$$

Now,
$$L = L\,(\mathbf{r}_1, \mathbf{r}_2, \ldots\ldots, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2 \ldots\ldots\ldots)$$

Hence
$$\delta L = \sum_i \left(\frac{\partial L}{\partial \mathbf{r}_i}\cdot \delta\, \mathbf{r}_i + \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot \delta\, \mathbf{r}_1\right)$$

We use
$$\dot{\mathbf{r}}_1 = v_i \text{ and } \frac{\partial L}{\partial \mathbf{r}_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right) = \dot{\mathbf{p}}_i$$

Therefore,
$$\delta L = \sum_i (\dot{\mathbf{p}}_i \cdot \delta\, \mathbf{r}_i + \mathbf{p}_i \cdot \delta\, \mathbf{v}_i) = \delta\theta \sum_i \left[\dot{\mathbf{p}}_i \cdot (\hat{\mathbf{z}} \times \mathbf{r}_i) + \mathbf{p}_i \cdot (\hat{\mathbf{z}} \times \mathbf{v}_i)\right]$$

$$= \delta\theta \sum_i \left[\frac{d}{dt}(\mathbf{r} \times \mathbf{p}_i) \cdot \hat{\mathbf{z}}\right] \quad \text{(using the property of scalar triple product)}$$

$$= \delta\theta\,\hat{\mathbf{z}}\cdot\frac{d}{dt}\left[\sum_i(\mathbf{r}_i\times\mathbf{p}_i)\right] \quad \text{...(89)}$$

As $\delta\theta$ is arbitrary and $\delta L = 0$ for arbitrary rotation, we obtain

$$\frac{d}{dt}\left[\sum_i(\mathbf{r}_i\times\mathbf{p}_i)\right] = 0 \ \text{ or } \ \sum_i(\mathbf{r}_i\times\mathbf{p}_i) = \text{Constant} \quad \text{...(90)}$$

where $\sum_i(\mathbf{r}_i\times\mathbf{p}_i) = \mathbf{J}$ is the total angular momentum of the system.

Thus *the total angular momentum of a closed system is conserved due to the isotropy of space.*

(3) Homogeneity of Time and Conservation of Energy : The homogeneity of time implies that the Lagrangian is invariant under time-translation similar to space-translation.

For arbitrary small time-translation δt, the change in Langragian is

$$\delta L = \frac{\partial L}{\partial t}\delta t \quad \text{...(91)}$$

But for $t \to t + \delta t$, $\delta L = 0$. Hence for arbitrary δt,

$$\frac{\partial L}{\partial t} = 0 \quad \text{...(92)}$$

i.e., L does not depend on time t explicitly.

Thus, $$L = L\,(\mathbf{r}_1, \mathbf{r}_2, \ldots\ldots, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2 \ldots\ldots\ldots) \quad \text{...(93)}$$

Hence, $$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial \mathbf{r}_i}\cdot\dot{\mathbf{r}}_i + \sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\ddot{\mathbf{r}}_i = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right)\cdot\dot{\mathbf{r}}_i + \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\right)\cdot\ddot{\mathbf{r}}_i = \frac{d}{dt}\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right)$$

Thus $$\frac{d}{dt}\left[\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right) - L\right] = 0 \text{ or } \left[\sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right) - L\right] \ \sum_i\left(\frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i\right) - L = \text{Constant} \quad \text{...(94)}$$

But $$L = T - V = \sum_i \frac{1}{2}m_i\dot{\mathbf{r}}_i^2 - V(r_1, r_2, \ldots.)$$

Hence, $$\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = m_i\dot{\mathbf{r}}_i \ \text{ and } \ \sum_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i}\cdot\dot{\mathbf{r}}_i = \sum_i m_i\dot{\mathbf{r}}_i\cdot\dot{\mathbf{r}}_i = \sum_i m_i\dot{\mathbf{r}}_i^2 = 2T \quad \text{...(95)}$$

Therefore, from (94)

$$2T - L = \text{Constant} \quad \text{or} \quad T + V = \text{Constant} \quad \text{...(96)}$$

Thus *the total energy is conserved for a closed system due to homogeneity of time.*

2.15. INVARIANCE UNDER GALILEAN TRANSFORMATION

Consider two inertial frames S and S'. The frame S' is moving with constant velocity $\mathbf{v}_0$ relative to frame S. If $\mathbf{r}_i$ and $\mathbf{r}_i'$ are the position vectors of i^{th} particle in frames S and S' respectively, then the two frames are connected by Galilean transformation with the transformation equations given by

$$\mathbf{r}_i' = \mathbf{r}_i - \mathbf{v}_0\,t \quad \text{[eq. (9), Chapter 1] ...(97)}$$

with the implicit assumption $t' = t$.

If $\mathbf{v}$ and $\mathbf{v}_i'$ be the velocities of the particle in two frames, then

$$\mathbf{v}_i' = \mathbf{v}_i - \mathbf{v}_0 \qquad \text{[eq. (10), Chapter 2] ...(98)}$$

Suppose the particle is moving under the action of external field force due to ordinary potential V. The Lagrangian of the particle in frame S is given by

$$L = \frac{1}{2}mv_i^2 - V \qquad ...(99)$$

where V is the potential function in S. This V is normally a function of difference of position vectors of two particles $\mathbf{r}_2 - \mathbf{r}_1$ and this V will remain the same in S' frame, because $\mathbf{r}_2' - \mathbf{r}_1' = \mathbf{r}_2 - \mathbf{r}_1$ from eq. (97).

The Lagrangian L' in frame S' is given by

$$L' = \frac{1}{2}mv'^2 - V = \frac{1}{2}m|\mathbf{v}_i - \mathbf{v}_0|^2 - V = \frac{1}{2}mv_i^2 - V - m\mathbf{v}_i \cdot \mathbf{v}_0 + \frac{1}{2}mv_0^2$$

Thus

$$L' = L + \frac{d}{dt}\left(\frac{1}{2}mv_0^2 t - m\mathbf{v}_0 \cdot \mathbf{r}_i\right) = L + \frac{dF(\mathbf{r}_i, t)}{dt} \qquad ...(99)$$

where

$$F(\mathbf{r}_i, t) = \frac{1}{2}mv_0^2 t - m\mathbf{v}_0 \cdot \mathbf{r}_i \qquad ...(100)$$

Hence through the gauge function $F(\mathbf{r}_i, t)$ both L and L' must satisfy the same Lagrange equations of motion [see Sec. 2.13]. Thus the from of Lagrange equation retain the same form in S' frame *i.e.*, **the Lagrange equations are invariant under Galilean transformation.**

Further from (97) and (98)

$$\mathbf{r}_i - \mathbf{r}_i' = (\mathbf{v}_i - \mathbf{v}_i')\,t \quad \text{or} \quad \mathbf{r}_i - \mathbf{v}_i t = \mathbf{r}_i' - \mathbf{v}_i' t$$

Hence for the entire system,

$$\sum_i m_i(\mathbf{r}_i - \mathbf{v}_i t) = \sum_i m_i(\mathbf{r}_i' - \mathbf{v}_i' t) \qquad ...(101)$$

But

$$\sum_i m_i \mathbf{r}_i = M\mathbf{R} \quad \text{and} \quad \sum_i m_i \mathbf{v}_i = \mathbf{P} \qquad ...(102)$$

where $M = \sum_i m_i$, total mass of the system, $\mathbf{R}$ the position vector of the centre of mass and $\mathbf{P}$ the total linear momentum of the system.

Thus

$$M\mathbf{R} - \mathbf{P}\,t = M\,\mathbf{R}' - \mathbf{P}'\,t \qquad ...(103)$$

In other words, $M\mathbf{R} - \mathbf{P}\,t$ is a constant of motion which is in fact obtained because of the Galilean invariance of Newton's equations of motion.

Some More Solved Examples

Ex. 1. ***Motion under gravity.*** *Write down the Lagrange's equation of motion for a particle of mass m falling freely under gravity near the surface of earth.* **(Rohilkhand, 1997)**

Solution : If X and Y axes are taken on the surface of the earth and Z-axis vertically upward, then the kinetic energy of the freely falling particle of mass m is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Its potential energy $V \quad = mgz$

Therefore, Lagrangian $L = T - V$

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

The Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

For $q_k = x, y, z$

$$\frac{dL}{d\dot{x}} = m\dot{x}, \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial L}{\partial \dot{z}} = m\dot{z}, \quad \frac{dL}{dx} = 0 = \frac{\partial L}{\partial y} \text{ and } \frac{dL}{dz} = mg$$

Hence Lagrange's equations are

$$\frac{d}{dt}(m\dot{x}) = 0, \ \frac{d}{dt}(m\dot{y}) = 0, \ \frac{d}{dt}(m\dot{z}) + mg = 0$$

or $\qquad \ddot{x} = 0, \quad \ddot{y} = 0, \quad \ddot{z} + g = 0$

Note : The above equations hold good for the case of projectile as well as for a particle falling freely vertically under gravity.

In the later case,

$$T = \frac{1}{2}m\dot{z}^2, \ V = mgz \text{ and } L = \frac{1}{2}m\dot{z}^2 - mgz$$

In such a case, the Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \text{ or } m\ddot{z} + mg = 0 \text{ or } \ddot{z} + g = 0$$

Ex. 2. *A point mass moves in a vertical plane along a given curve in a gravitational field. The equation of motion in parametric form is*

$$x = x(s), \quad z = z(s)$$

Write down the Lagrange's equations. **(Rohilkhand 1996)**

Solution : Here $\dot{x} = \dfrac{dx}{dt} = \dfrac{dx}{ds}\cdot\dfrac{ds}{dt} = x'\dot{s}\left(x' = \dfrac{dx}{ds}, \dot{s} = \dfrac{ds}{dt}\right)$ and $\dot{z} = \dfrac{dz}{dt} = \dfrac{dx}{ds}\dfrac{ds}{dt} = z'\dot{s}\left(z' = \dfrac{dz}{ds}\right)$

Kinetic energy $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{z}^2\right) = \frac{1}{2}\left(x'^2 + z'^2\right)\dot{s}^2$

Potential energy $V = mgz$

where Z-axis is assumed to be vertical upward from the earth.

Therefore, $L = T - V = \frac{1}{2}m\left(x'^2 + z'^2\right)\dot{s}^2 - mgz$

Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0$$

Here $$\frac{\partial L}{\partial \dot{s}} = \frac{1}{2}\left(x'^2 + z'^2\right) 2\dot{s} = \left(x'^2 + z'^2\right)\dot{s}$$

and $$\frac{\partial L}{\partial s} = \left(\frac{1}{2} m\dot{s}^2 (2x'\,x'' + 2z'\,z'')\right) - mg\frac{\partial z}{\partial s} = m\dot{s}^2 (x'\,x'' + z'\,z'') - mgz'$$

Hence $$\frac{d}{dt}\left[m\left(x'^2 + z'^2\right)\dot{s}\right] - m\dot{s}^2 (x'\,x'' + z'\,z'') + mgz' = 0$$

This is the desired Lagrange's equation.

Ex. 3. *Fig. 2.11 (a) shows an inclined plane of mass m_1. It is sliding on a horizontal smooth surface and a body of mass m_2 is sliding on its smooth inclined surface. Derive the equations of motion of the body and the inclined plane.*

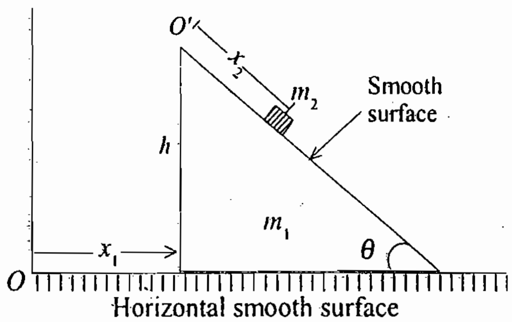


Fig. 2.11 (a)

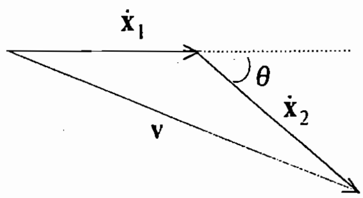


Fig. 2.11 (b)

Solution : Here m_1 slides on the horizontal smooth surface and m_2 slides on the smooth inclined plane of mass m_1. Thus the system has two degrees of freedom and hence we need two generalized coordinates. Let x_1 and x_2 represent the displacements of m_1 and m_2 from O and O' respectively.

Velocity of m_1 with respect to $O = \dot{\mathbf{x}}_1$

Velocity of m_2 with respect to $O' = \dot{\mathbf{x}}_2$

Velocity of m_2 with respect to $O = \mathbf{v} = \dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2$ [Fig. 2.10 (*b*)]

or $$v^2 = \dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\,\dot{x}_2 \cos\theta$$

Kinetic energy of the whole system as observed from O is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2 v^2$$

$$= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\,\dot{x}_2\cos\theta\right)$$

Potential energy of the system is due to the position of the mass m_2 (with respect to horizontal smooth surface) only.

Hence,

$$V = m_2 g\,(h - x_2 \sin\theta)$$

$$\therefore \quad L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left(\dot{x}_1^2 + \dot{x}_2^2 + 2\dot{x}_1\,\dot{x}_2\cos\theta\right) - m_2 g\,(h - x_2\sin\theta) \qquad ...(i)$$

Lagrange's equations for x_1 and x_2 coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

$\therefore$
$$m_1\ddot{x}_1 + m_2(\ddot{x}_1 + \ddot{x}_2 \cos\theta) = 0 \qquad ...(ii)$$

and
$$m_2(\ddot{x}_2 + \ddot{x}_1 \cos\theta) - m_2 g \sin\theta = 0 \qquad ...(iii)$$

Solving eqs. (*ii*) and (*iii*), we get

$$\ddot{x}_1 = \frac{-g\sin\theta\cos\theta}{\dfrac{m_1+m_2}{m_2} - \cos^2\theta} \qquad ...(iv)$$

and
$$\ddot{x}_2 = \frac{g\sin\theta}{1 - \dfrac{m_2\cos^2\theta}{m_1+m_2}} \qquad ...(v)$$

Eqs. (*iv*) and (*v*) are the equations of motion of the inclined plane and sliding body respectively.

Ex. 4. *A particle of mass m moves on a plane in the field of force given by (in polar coordinates)*

$$\bar{F} = -kr\cos\theta\ \hat{\mathbf{r}}$$

where k is constant and $\hat{\mathbf{r}}$ *is the radial unit vector.*

(*a*) *Will the angular momentum of the particle about the origin be conserved ? Justify your statement.*

(*b*) *Obtain the differential equation of the orbit of the particle.* **(Agra 1995)**

Solution : $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

$$\frac{\partial T}{\partial \theta} = 0,\quad \frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta},\quad \frac{\partial T}{\partial r} = mr\dot{\theta}^2 \quad \text{and} \quad \frac{\partial T}{\partial \dot{r}} = m\dot{r}$$

(*a*)
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = G_\theta.$$

Since there is no transverse force, $G_\theta = 0$. Therefore, $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$. Hence the angular momentum about the origin is conserved.

(b)
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = G_r \quad \text{or} \quad m\ddot{r} - mr\dot{\theta}^2 = -kr\cos\theta$$

which is the differential equation of motion of the orbit of the particle.

Ex. 5. *A cylinder of radius a and mass m rolls down an inclined plane making an angle θ with the horizontal. Set up the Lagrangian and find the equation of motion.*

Solution : Let the cylinder start to roll from O so that $x = a\phi$ (Fig. 2.12) and hence $\dot{x} = a\dot{\phi}$.

Now,
$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{ma^2}{2}\frac{\dot{x}^2}{a^2} = \frac{3}{4}m\dot{x}^2$$

(because $I = \dfrac{ma^2}{2}$ for cylinder and $\omega = \dot{\phi} = \dfrac{\dot{x}}{a}$.)

and $$V = mg\,(s - x)\sin\theta + mg\,a\cos\theta$$

$\therefore$ $$L = \frac{3}{4}m\dot{x}^2 - mg\,(s - x)\sin\theta - mg\,a\cos\theta$$

where s is the lenght of the inclined plane.

$\therefore$ Equation of motion is

$$\frac{3}{2}m\ddot{x} - mg\sin\theta = 0.$$

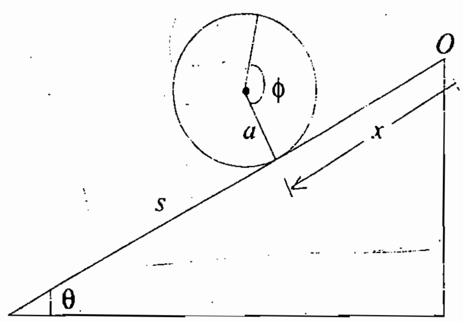


Fig. 2.12

Ex. 6. *A bead slides on a smooth rod which is rotating about one end in a vertical plane with uniform angular velocity* ω [Fig. 2.13]. *Show that the equation of motion is* $m\ddot{r} = mr\omega^2 - mg\sin\omega t$.

Solution : $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, and $V = mgy = mgr\sin\theta$

$\therefore$ $$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - mgr\sin\theta$$

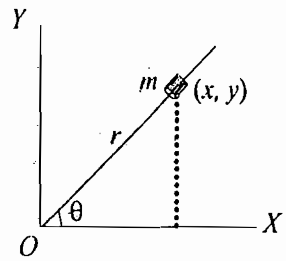


Fig. 2.13

Here, $\dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\sin\theta$ and $\dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$

From Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0,$$

we have $$m\ddot{r} - mr\dot{\theta}^2 + mg\sin\theta = 0 \text{ or } m\ddot{r} - mr\omega^2 + mg\sin\omega t = 0$$

where $$\dot{\theta} = \omega \text{ and } \theta = \omega t.$$

Ex. 7. *A pendulum of mass m is attached to a block of mass M. The block slides on a horizontal frictionless surface* (Fig. 2.14). *Find the Lagrangian and equation of motion of the pendulum. For small amplitude oscillations, derive an expression for periodic time.*

Solution : Let at any time t the coordinates of M and m be $(x_1, 0)$ and (x_2, y_2) respectively.

Here, $$x_2 = x_1 + l\sin\theta \text{ and } y_2 = -l\cos\theta$$

$$T = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$= \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 + l^2\dot{\theta}^2 + 2l\dot{x}_1\dot{\theta}\cos\theta\right)$$

(because $\dot{x}_2 = \dot{x}_1 + l\cos\theta\,\dot{\theta}$ and $\dot{y}_2 = l\sin\theta\,\dot{\theta}$)

$$V = -mgl\cos\theta$$

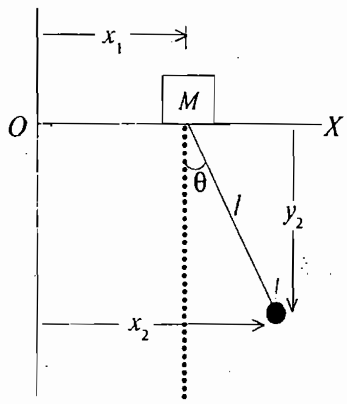


Fig. 2.14

Hence, $$L = T - V = \frac{1}{2}(M + m)\dot{x}_1^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}_1\dot{\theta}\cos\theta + mgl\cos\theta$$

Here, $$\frac{\partial L}{\partial x_1} = 0,\ \frac{\partial L}{\partial \dot{x}_1} = (M + m)\dot{x}_1 + ml\dot{\theta}\cos\theta,$$

$$\frac{\partial L}{\partial \theta} = ml\left(\dot{x}_1\dot{\theta} + g\right)(-\sin\theta) \text{ and } \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + ml\,\dot{x}_1\cos\theta$$

Equation of motion in θ is

$$ml^2\ddot{\theta} + m_1\ddot{x}_1\cos\theta + ml(-\sin\theta)\dot{\theta}\,\dot{x}_1 - ml(-\sin\theta)\,\dot{x}_1\,\dot{\theta} + mgl\sin\theta = 0$$

or $$ml^2\ddot{\theta} + ml\cos\theta\,\ddot{x}_1 + mgl\sin\theta = 0$$

If θ is small, $\sin\theta \cong \theta$ and also $\cos\theta \cong 1$, then

$$ml^2\ddot{\theta} + ml\ddot{x}_1 + mgl\theta = 0$$

or $$\ddot{\theta} + \frac{\ddot{x}_1}{l} + \frac{g}{l}\theta = 0 \qquad ...(i)$$

Equation of motion in x_1 is

$$(M + m)\ddot{x}_1 + ml(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) = 0$$

For small θ, ($\cos\theta \cong 1$, $\sin\theta \cong \theta$ and $\dot{\theta}^2\theta$ is negligible)

$$(M + m)\ddot{x}_1 + ml\ddot{\theta} = 0 \qquad ...(ii)$$

From equations (i) and (ii), we have

$$\ddot{\theta} - \frac{m\ddot{\theta}}{M+m} + \frac{g}{l}\theta = 0$$

Hence $$\ddot{\theta} = -\left[\frac{M+m}{M}\right]\frac{g}{l}\theta \qquad ...(iii)$$

This is the equation of simple harmonic motion whose period is given by

$$T = 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{M}{M+m}}$$

Ex. 8. *In a* ***spherical pendulum*****, *a small bob (particle) of mass m is constrained to move on a smooth spherical surface, say of radius R, R being the length of the pendulum (Fig. 2.15). Set up the Lagrangian for the spherical pendulum and obtain the equation of motion.* **(Rohilkhand 1994)**

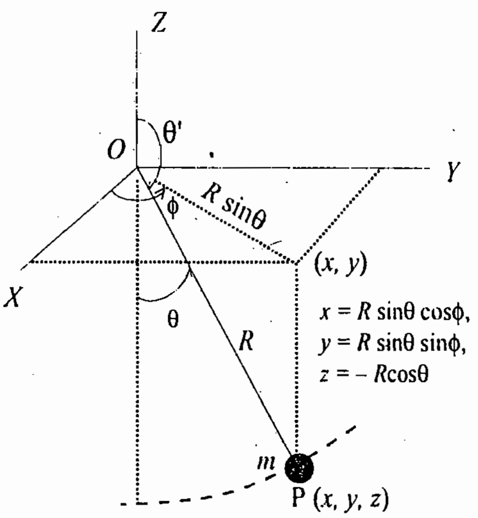


Fig. 2.15

Solution : The constraint of motion is holonomic and the constraint equation is

$$x^2 + y^2 + z^2 - R^2 = 0$$

We take θ and ϕ as the generalized coordinates. The cartesian coordinates of the bob P are

$$x = R\sin\theta'\cos\phi,\ \ y = R\sin\theta'\sin\phi, \text{ and } z = R\cos\theta'$$

or $x = R\sin\theta\cos\phi$, $y = R\sin\theta\sin\phi$, and $z = -R\cos\theta$ (as $\theta' = \pi - \theta$)

Now $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V = \frac{1}{2}mR^2\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + mgR\cos\theta$

* If spherical pendulum moves in a vertical plane, it constitutes a simple pendulum.

Equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \text{ and } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

i.e., $$mR^2\,\ddot{\theta} - mR^2\dot{\phi}^2 \sin\theta\cos\theta + mg\,R\sin\theta = 0 \text{ and } \frac{d}{dt}(mR^2\dot{\phi}\sin^2\theta) = 0$$

or $$\ddot{\theta} - \tfrac{1}{2}\sin 2\theta\dot{\phi} + \frac{g}{R}\sin\theta = 0 \text{ and } mR^2\sin^2\theta\,\dot{\phi} = \text{constant.}$$

These are the equations of motion for spherical pendulum.

Ex. 9. *A particle moves in a plane under the influence of a force, acting towards a centre of force, whose magnitude is*

$$F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}\,r}{c^2}\right)$$

where r is the distance of the particle from the centre of force. Find the generalized potential that will result in such a force and from that the Lagrangian for the motion in a plane. **(Rohilkhand 1986)**

Solution : For velocity dependent potential

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q} = G_k$$

where $G_k = -\dfrac{\partial U}{\partial q_k} + \dfrac{d}{dt}\left(\dfrac{\partial U}{\partial \dot{q}_k}\right)$ is the generalized force and $U(q_k, \dot{q}_k)$ is the generalized potential. The generalized force for $q_k = r$ is

$$G_r = -\frac{\partial U}{\partial r} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{r}}\right)$$

Here $$G_r = F = \frac{1}{r^2}\left(1 - \frac{\dot{r}^2 - 2\ddot{r}\,r}{c^2}\right) = \frac{1}{r^2} - \frac{\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r} = \frac{1}{r^2} + \frac{\dot{r}^2}{c^2r^2} + \frac{2\ddot{r}}{c^2r} - \frac{2\dot{r}^2}{c^2r^2}$$

$$= -\frac{\partial}{\partial r}\left(\frac{1}{r} + \frac{\dot{r}^2}{c^2r}\right) + \frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{1}{r} + \frac{\dot{r}^2}{c^2r}\right) = -\frac{\partial U}{\partial r} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}}$$

where $U = \dfrac{1}{r}\left(1 + \dfrac{\dot{r}^2}{c^2}\right)$. This is the expression for the generalized potential.

Questions

1. (*a*) What are constraints ? Classify the constraints with some examples. **(Agra 2004, 1998, 95; Kanpur 98; Garwal 98, 93)**

   (*b*) What type of difficulties arise due to the constraints in the solution of mechanical problems and how these are removed ? **(Agra 1998, 93)** (*c*) Write a note on 'holonomic and non-holonomic constraints' with two examples of each type. **(Kanpur 1997; Garwal 99; Gorakhpur 95; Agra 2002)**

2. What do you understand by holonomic and nonholonomic constraints ? Obtain differential equations of constraints in case of a disc of radius R, rolling on the horizontal xy plane and constrained to move so that plane of the disc is always vertical. **(Kanpur 1996)**
3. Write down the generalized coordinates for a simple pendulum and exptain why cartesian coordinates are not suitable here. **(Gorakhpur 1995)**
4. What are generalized coordinates and generalized velocities ? Set up the Lagrangian for a spherical pendulum. **(Ruhelkhand 1994)**
5. (*a*) State and prove D 'Alembert's principle. **(Garwal 1996)**

   (*b*) What is D 'Alembert's principle ? Give its one application. **(Kanpur 1997)**

   (*c*) Derive Lagrange's equations from D 'Alemberts principle. **(Kanpur 2001)**
6. What is D 'Alembert's principle ? Derive Lagrange's equations of motion from it for conservation system. How will the result be modified for non-conservative system ? **(Agra 2001, 2000; Meerut 2001; Garwal 1999; Bundelkehand 97)**
7. (*a*) Discuss the superiority of Lagrangian approach over Newtonian approach. **(Rohilkhand 1994)**

   (*b*) Define Lagrangian function for conservative and non-conservative systems.
8. Explain what is meant by generalized coordinates, holonomic constraints and the principle of virtual work. Obtain the D 'Alembert's principle in generalized coordinates and use it to obtain the Lagrange's equations of motion for a holonomic conservative system. **(Agra 1991, 89, 87, 86)**
9. Obtain Lagrange's equations and show that these can be written as $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$, **(Kanpur 1997)**
10. Derive Lagrangian expression for a charged particle in an electromagnetic field. **(Agra 2001, 1999, 95)**
11. Define Raylish's dissipation function for frictional forces, which are proportional to velocities and obtian Lagrange's equations. Also give a physical interpretation of this function. **(Garwal 1995)**
12. What is Hamilton's principle ? Derive Lagrange's equation of motion from it. Find the Lagrangian equation of moton for a L-C circuit and also deduce the time period. **(Agra 1995)**
13. What is Hamilton's principle ? Derive Lagrange's equation with its helps for a conservative system. Derive equation of motion for a particle moving under central force. **(Agra 2002, 1999; Rohilkhand, 96; Merrut, 95)**
14. Set up the Lagrangian and obtain the Lagrange's equation for a simple pendulum. Deduce the formula for its time period. **(Agra 1994, 91)**
15. Prove that if the transformation equations are given by

$$\mathbf{r}_i = \mathbf{r}_i\,(q_1, q_2, ... q_n)$$

which do not involve time explicitly, then the kinetic energy can be written as

$$T = \sum_{\alpha=1}^{n} \sum_{\beta-1}^{n} C_{\alpha\beta}\, \dot{q}_\alpha \dot{q}_\beta$$

where $C_{\alpha\beta}$ are functions of q_α.

16. Write the Lagrangian and equation of motion for the following systems :

(*a*) A mass m is suspended to a spring of force constant k and allowed to swing vertically.

(*b*) A uniform rod of mass m and length a, pivoted at a distance l from the centre of mass, swings in a vertical plane.

Ans. (*a*) $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 ; m\ddot{x} + kx = 0$

(*b*) $L = \frac{1}{2} I\dot{\theta}^2 - mgl\,(1 - \cos\theta);\ I\ddot{\theta} + mgl\sin\theta = 0$

where $I = \dfrac{ma^2}{12} + m\left(\dfrac{l}{2}\right)^2 = \dfrac{m}{4}\left(\dfrac{a^2}{3} + l^2\right)$.

17. The force on a particle of mass m and charge e, moving with a velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, is given by

$\mathbf{F} = \mathbf{e}\left(\mathbf{E} + \dfrac{\mathbf{v} \times \mathbf{B}}{\mathbf{c}}\right)$, where c is the speed of light.

If the fields are expressed by the relations : $\mathbf{E} = -\nabla\phi - \dfrac{1}{c}\dfrac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$, ϕ and $\mathbf{A}$ being the scalar and vector potentials respectively, prove that the Lagrangian for the charged particle is

$$L = \frac{1}{2}mv^2 + \frac{e}{c}(\mathbf{A}\,.\,\mathbf{v}) - e\phi$$ **(Garwal 1996)**

Problems

[SET- I]

1. Determine the number of degrees of freedom in the following cases :

(1) A particle moving on a space curve, (2) 4 particles moving freely in space, (3) 4 particles moving freely in a plane, (4) three particles connected by three rigid massless rods, (5) two particles moving on a space curve and having constant distance between them, (6) a rigid body moving parallel to a fixed plane surface, (7) a rigid body having two points fixed.

Ans : (1) 1, (2) 12, (3) 8, (4) 6, (5) 1, (6) 3, (7) 1.

2. Determine the number of degrees of freedom for a massless rod, moving freely in space with a particle which is constrained to move on the rod.

Ans : 4.

3. Two particles are connected by a rod of variable length $l = f(t)$. What is the nature of the constraint ?

Ans : The constraint is $|\,\mathbf{r}_1 - \mathbf{r}_2\,|^2 = f^2(t)$ which is holonomic and rheonomic.

4. A lever ABC has weights W_1 and W_2 at distances l_1 and l_2 from the fixed support B (Fig. 2.16).

Apply the principle of virtual work to prove that the condition of the equilibrium is $\dfrac{W_1}{W_2} = \dfrac{l_1}{l_2}$

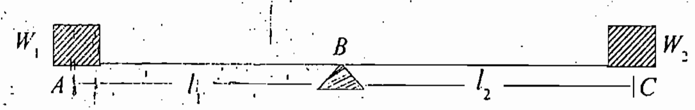


Fig. 2.16

5. Use D'Alembert's principle to determine the equation of motion of a simple pendulum.

Ans : $\ddot{\theta} + (g/l)\,\theta = 0$.

6. An incline that makes an angle α with the horizontal is given a horizontal acceleration a in the vertical plane of the incline so that the sliding of a frictionless block on the incline is prevented. Apply D'Alembert's principle to obtain the value of a.

Ans : $a = g \tan \alpha$.

7. A ladder slides down a smooth wall and smooth floor [Fig. 2.17]. Set up the Lagrangian for the system and deduce the equation of motion.

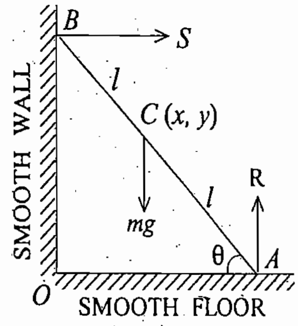


Fig. 2.17

Ans : $L = \frac{1}{2} m\,(l^2 + K^2)\dot{\theta}^2 - mgl \sin\theta$; $\ddot{\theta} = lg \cos\theta / (l^2 + K^2)$,

where K is the radius of gyration.

[Hint : $T = \frac{1}{2} m\,(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I\omega^2$.

Here, $x = a\cos\theta$, $y = a \sin\theta$, $I = mK^2$ and $\omega = \dot{\theta}$]

8. (*a*) Two point masses m are connected by a rod of length $2a$, the centre of which moves on a circle of radius r. Write down kinetic energy in generalized coordinates.

Ans : $m\,(r^2\dot{\theta}^2 + a^2\dot{\phi}^2)$.

(b) Obtain the Lagrangian of a particle moving in a force free field in spherical coordinate and cylindrical coordinate systems.

Ans : $\frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2)$; $\frac{1}{2} m\,(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$.

9. Two particles of masses m_1 and m_2 are located on a frictionless double incline and connected by an inextensible massless string passing over a smooth peg (Fig. 2.18). Use the principle of virtual work to show that for equilibrium, we must have

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{m_2}{m_1}$$

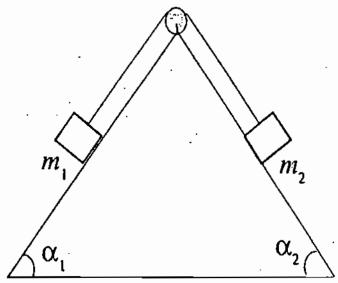


Fig. 2.18

where α_1 and α_2 are the angles of the incline.

Apply D'Alembert's principle to describe the motion. **(Garwal 1992)**

Ans : $a = \dfrac{m_1 g \sin\alpha_1 - m_2 g \sin\alpha_2}{m_1 + m_2}$, where for $m_1 g \sin\alpha_1 > m_2 g \sin\alpha_2$, the particle 1 goes down and particle 2 goes up.

10. A bead is sliding on a uniform rotating rod in a force-free space, find its equation of motion.

Ans : $m\ddot{r} - mr\omega^2 = 0$.

11. A block of mass m is pulled up as the mass M moves down as shown in Fig. 2.19. The coefficient of friction between the incline and m is μ. Find the acceleration of m and M. Assume the pulley P as frictionless.

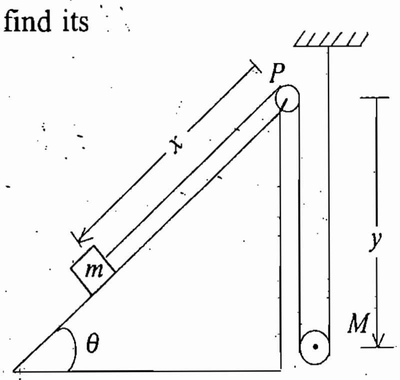


Fig. 2.19

Ans : $\ddot{x} = (-\frac{1}{2} Mg + mg \sin\theta - \mu\, mg \cos\theta)/(m + M/4)$;

$\ddot{y} = \ddot{x}/2$.

12. A bead is constrained to move along a smooth conical spiral [Fig. 2.20] defined by coordinates ρ, ϕ and z, related as

$\rho = az$ and $\phi = -bz$

where a and b are constants. Gravity force is acting in the negative z-direction. Set up the Lagrangian for the system.

Ans : $L = \frac{1}{2} m \dot{z}^2 (a^2 + a^2 b^2 z^2 + 1) - mgz.$

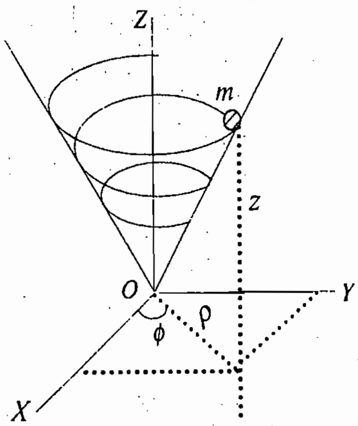


Fig. 2.20

13. Discuss the motion of a particle of mass m moving on the surface of a cone of half angle ϕ and subject to gravitational force only, as shown in Fig. 2.21.

Ans : Equations of motion are

$\ddot{r} - r\dot{\theta}^2 \sin^2 \phi + g \cos \phi \sin \phi = 0;\ J_z = mr^2 \dot{\theta} = \text{constant}.$

[**Hint :** $L = \frac{1}{2} m (\dot{r}^2 \operatorname{cosec}^2 \phi + r^2 \dot{\theta}^2) - mgr \cot \phi$]

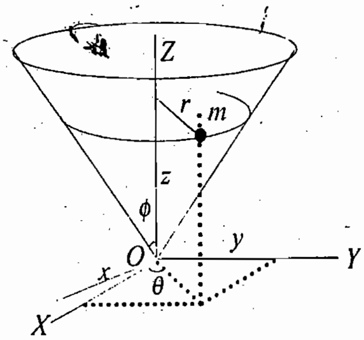


Fig. 2.21

14. In an inverted pendulum, particle of mass m is attached to a rigid massless rod of length l [Fig. 2.22]. If the vertical motion of the point O is represented by the equation $z = a \sin \omega t$, set up the Lagrangian and obtain the equation of motion.

Ans : $L = \frac{1}{2} ml^2 \dot{\theta}^2 - m (g - a\omega^2 \sin \omega t)\, l \cos \theta\,;$

$$\ddot{\theta} - \left(\frac{g}{l} - \frac{\omega^2 a}{l} \sin \omega t\right) \sin \theta = 0.$$

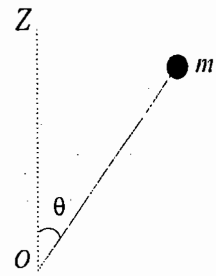


Fig. 2.22

[**Hint :** $T = \frac{1}{2} ml^2 \dot{\theta}^2$; $V = mg'l \cos \theta$, where $g' = g - \ddot{z} = g - \omega^2 a \sin \omega t.$]

15. A particle of mass m is free to slide on a smooth helical wire whose position in cylindrical coordinates is represented as $\rho = a$ and $z = b\phi$. The particle is released from rest at $\rho = a$, $\phi = 0$ and $z = 0$. Discuss the motion of the particle.

Ans : $z = gb^2 t^2 / [2(a^2 + b^2)]$.

[**Hint :** $T = \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m\rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2$

Equations of constraints are $\rho - a = 0$ and $z - b\phi = 0$

Hence there is only one generalized coordinate. Now, $T = m (a^2 + b^2) \dot{z}^2 / 2b^2$

Generalized force $G_z = mg$; $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} = G_z$ or $m (a^2 + b^2) \ddot{z} / b^2 = mg,$

$\therefore \qquad z = gb^2 t^2 / [2(a^2 + b^2)].$

16. A small bead of mass M is initially at rest on a horizontal wire and is attached to a point on the wire by a massless spring of spring constant k and unstretched length a. A mass m is freely suspended from the bead at the end of a wire of length $2b$. For the displacement shown in Fig. 2.23, obtain the Lagrangian.

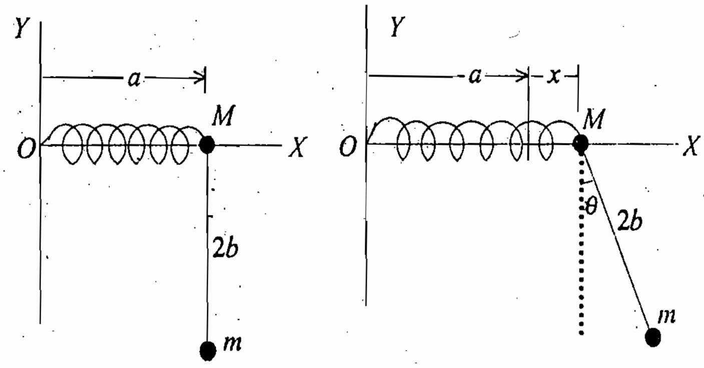


Fig. 2.23

Ans : $L=\frac{1}{2}(m+M)\dot{x}^2 + 2mb\dot{\theta}(b\dot{\theta}+\dot{x}\cos\theta) - \frac{1}{2}kx^2 + 2b\, mg\cos\theta.$

[**Hint :** In the displaced position, coordinate of M, $x' = a + x$ and coordinate of m, $x'' = a + x + 2b\sin\theta$, $y = -2b\cos\theta$].

17. A particle of mass m is projected with initial velocity u at an angle α with the horizontal. Use Lagrange's equations to show that the path of the projectile is parabola. **(Rohilkhand 1999)**

18. A particle of mass m can move in a frictionless thin circular tube of radius r [Fig. 2.24]. If the tube rotates with an angular velocity ω about a vertical diameter, deduce the differential equation of motion of the particle.

Ans : $\ddot{\theta} - \omega^2 \sin\theta\cos\theta - (g/r)\sin\theta = 0.$

[**Hint :** $L=\frac{1}{2}mr^2(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgr\cos\theta$].

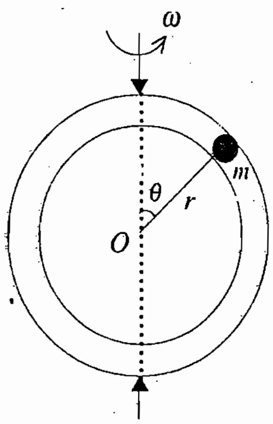


Fig. 2.24

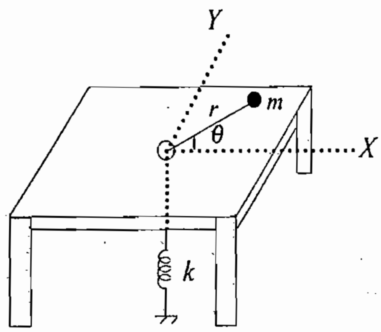


Fig. 2.25

19. Using Lagrangian formulation, find the equation of motion of a particle of mass m, constrained to move on a smooth horizontal table under the action of a spring of force constant k [Fig. 2.25]. In the system, a string attached to the particle passes through a hole in the table and is connected to the spring. Assume the spring is unstretched, when m is at the hole.

Ans : $m\ddot{r} - mr\dot{\theta}^2 + kr = 0$ and $\frac{d}{dt}(mr^2\dot{\theta}) = 0.$

[**Hint :** $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$ and $V = \frac{1}{2}kr^2$]

20. A particle of mass m_2 moves on a vertical axis in the system shown in Fig. 2.26 and the whole system rotates about this axis with a constant angular velocity ω under the action of gravity. Set up the Lagrangian for the system.

Ans : $L = m_1 l^2 (\dot{\theta}^2 + \sin^2\theta\ \omega^2) + 2\, m_2 l^2 \sin^2\theta\ \dot{\theta}^2 + 2\,(m_1 + m_2)\, gl \cos\theta.$

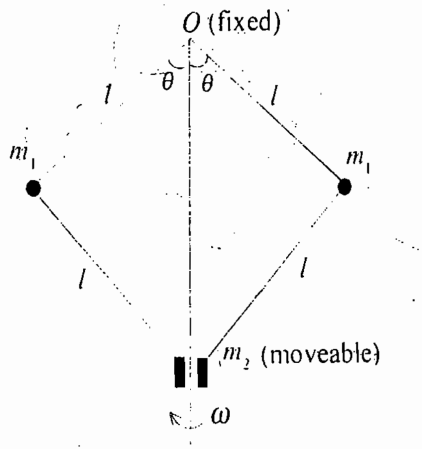


Fig. 2.26

21. Fig. 2.27 shows a mass m resting on a smooth table between two firm supports A and B and controlled by two massless springs of force constants C_1 and C_2. Set up the Lagrangian of the system and deduce the equation of motion.

Ans : $L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2}(C_1 + C_2)\, x^2;\ m\ddot{x} + (C_1 + C_2)\, x = 0$.

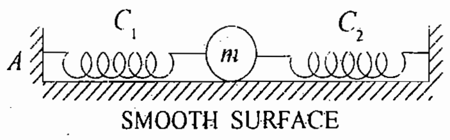


Fig. 2.27

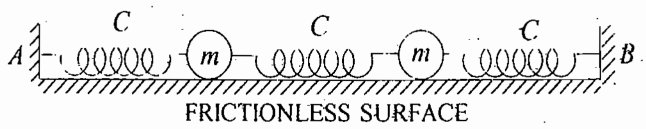


Fig. 2.28

22. Two equal masses are connected by springs having each force constant C [Fig. 2.28]. The masses are free to slide on a frictionless table AB. The walls are at A and B to which the ends of the springs are fixed. Set up the Lagrangian and deduce the equations of motion of the vibrating system.

Ans : $L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} C x_1^2 - \frac{1}{2} C x_2^2 - \frac{1}{2} C (x_1 - x_2)^2;\ m\ddot{x}_1 = C(x_2 - 2x_1),\ \ m\ddot{x}_2 = C(x_1 - 2x_2).$

[**Hint :** Lagrange's equations are $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] - \frac{\partial L}{\partial x_1} = 0$ and $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_2}\right] - \frac{\partial L}{\partial x_2} = 0$].

23. A particle is constrained to move in a plane under the influence of an attraction towards the origin proportional to the distance from it and also of a force perpendicular to the radius vector inversely proportional to the distance of the particle from the origin in anticlockwise direction. Find (*i*) the Lagrangian, and (*ii*) the equations of motion. **(Agra 1999)**

Ans : (i) $L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2} kr^2$ (ii) $m\ddot{r} - mr\dot{\theta}^2 + \frac{k}{r} = 0$; $\frac{d}{dt}(mr^2\dot{\theta}) = k'$;

where $F_r = -kr$ and $F_\theta = \frac{k'}{r}$

[SET- II]

1. A pendulum bob of radius r is rolling on a circular track of radius R [Fig. 2.29]. Set up the Lagrangian, derive the equation of motion and compare its period of small oscillations with that of a simple pendulum of string length $(R - r)$.

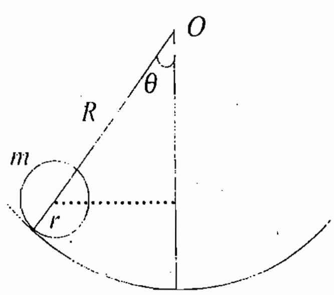


Fig. 2.29

Ans : $L = \frac{1}{2} m\, (R-r)^2 \dot{\theta}^2 + \frac{1}{2}\frac{I\,(R-r)}{r^2}\dot{\theta}^2 - mg\,(R-r)\,(1 - \cos\theta\,)$;

$$\ddot{\theta} + \frac{5}{7}\frac{g}{(R-r)}\theta = 0; \sqrt{1.4} : 1]$$

[**Hint** : $v = r\omega = (R-r)\dot{\theta}$; $T = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}m\frac{(R-r)^2}{r^2}\dot{\theta}^2$; $V = mg(R-r)(1-\cos\theta)$]

2. A solid homogeneous cylinder of radius r rolls without slipping on the inside of a stationary large cylinder of radius R. Find the equation of motion. What is the period of small oscillations about the stable equilibrium position ?

Ans : $\frac{3}{2}(R-r)\ddot{\theta} + g\theta = 0$, $T = 2\pi\sqrt{\frac{3(R-r)}{2g}}$

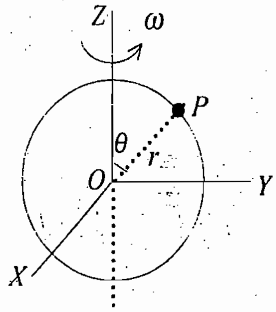


Fig. 2.30

3. (a) A bead of mass m slides on a smooth uniform circular wire of radius r which is rotating with a constant angular velocity ω about a fixed vertical diameter [Fig. 2.30]. Set up the Lagrangian and find the equation of motion of the bead.

Ans : $L = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\omega^2\sin^2\theta - mgr\cos\theta$; $\ddot{\theta} - \frac{1}{2}\omega^2\sin 2\theta - \frac{g}{r}\sin\theta = 0$

(b) In the above problem, if the bead is released with no vertical velocity from a point on the level of the centre of the circular wire, show that it will not reach the lowest point if $\omega > \sqrt{2g/r}$.

4. A bead of mass m can slide freely on a smooth circular wire of radius a. The wire is rotating anticlockwise in a horizontal plane with an angular velocity ω about an axis through O [Fig. 2.31]. Show that the motion of the bead is simple harmonic about the rotating line OA with a period $T = 2\pi/\omega$.

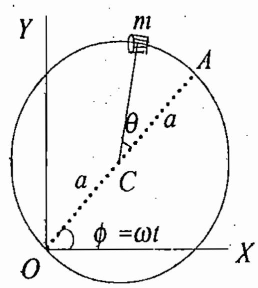


Fig. 2.31

[**Hint** : $x = a\cos\omega t + a\cos(\omega t + \theta)$; $y = a\sin\omega t + a\sin(\omega t + \theta)$;

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ma^2[\omega^2 + (\dot{\theta} + \omega)^2 + 2\omega(\dot{\theta} + \omega)\cos\theta;$$

Lagrange's equation is $ma^2(\ddot{\theta} - \omega\dot{\theta}\sin\theta) + ma^2\omega(\dot{\theta} + \omega)\sin\theta = 0$, whence for small θ, $\ddot{\theta} + \omega^2\theta = 0$].

5. The point of support of a simple pendulum of length l and mass m is moved along a vertical line according to the equation

$$y = y(t)$$

The motion of the pendulum is restricted to a vertical plane. Show that the kinetic energy of the pendulum is given by

$$T = \frac{1}{2}m(l\dot{\theta})^2 + \frac{1}{2}m\dot{y}^2 + ml\dot{y}\dot{\theta}\sin\theta$$

If the potential energy is given by $V = mg\,y - mgl\cos\theta$, derive the equation of motion for the variable θ.

6. A simple pendulum of mass m whose point of support (a) moves uniformly on a vertical circle with constant angular frequency ω [Fig. 2.32], (b) oscillates horizontally in the plane of motion of the pendulum according to the law $x = a\cos\omega t$, (c) oscillates vertically according to the law $y = a\cos\omega t$. Set up the Lagrangian in the three cases.

Ans : (*a*) $L = \frac{1}{2} ml^2 \dot{\theta}^2 + mla\,\omega^2 \sin(\theta - \omega t) + mgl \cos\theta$,

where the terms depending only on time have been omitted together with the total time derivatives of $mla\,\omega \cos(\theta - \omega t)$.

[Hint : $x = a\cos\omega t + l\sin\theta,\ y = -a\sin\omega t + l\cos\theta$]

(*b*) $L = \frac{1}{2} ml^2 \dot{\theta}^2 + mla\omega^2 \cos\omega t \sin\theta + mgl\cos\theta$

[Hint : $x = a\cos\omega t + l\sin\theta,\ y = l\cos\theta$].

(*c*) $L = \frac{1}{2} ml^2 \dot{\theta}^2 + mla\,\omega^2 \cos\omega t \cos\theta + mgl\cos\theta$.]

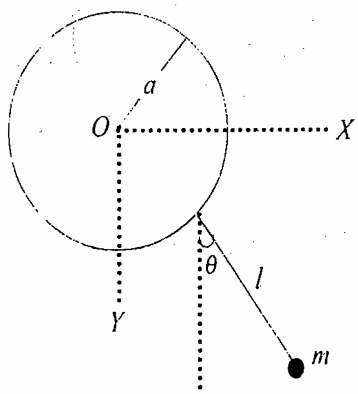


Fig. 2.32

7. A mass m_2 hangs at one end of a string which passes over a fixed frictionless non-rotating pulley. At the other end of the string there is a non-rotating pulley of mass m_1 over which there is a string carrying masses m_1' and m_2' [Fig. 2.33]. Set up the Lagrangian of the system and find the acceleration of the mass m_2.

Ans : $\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_1'(\dot{x}_1 + \dot{y}_1)^2 + \frac{1}{2}m_2'(\dot{x}_1 - \dot{y}_1)^2 + m_1 g x_1$

$+ m_2 g(l_1 - x_1) + m_1' g(x_1 + y_1) + m_2' g(x_1 + l_2 - y_1)$, where l_1 and l_2 are the lengths of upper and lower (in-extensible) strings;

$$a = \frac{(m_2 - m_1)(m_2' + m_1') - 4m_1'm_2'}{(m_1 + m_2)(m_1' + m_2') + 4m_1'm_2'}.$$

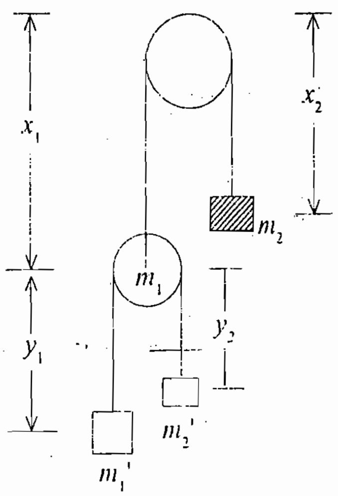


Fig. 2.33

8. A sphere of radius r and mass m rests on the top of a fixed rough sphere of radius R. The first sphere is slightly displaced so that it rolls without slipping down the second sphere. Find out the equation of motion of the rolling sphere.

Ans : $\ddot{\theta} - \frac{5g}{7(r+R)}\sin\theta = 0$, where θ is the angle between vertical and the line joining the centres of two spheres at an instant.

9. A system consists of two equal masses m fixed at the ends of a light rod PQ of length $2l$. The middle point C of this rod is attached to the end of a light rod OC of length a. The rod OC is mounted in such a way that it can move freely in a horizontal plane, while PQ is mounted so that it can rotate freely in a vertical plane through OC [Fig. 2.34]. Set up the Lagrangian and the equation of motion of this syatem, placed in a uniform gravitational field. (This system is called Thompson-Tait pendulum).

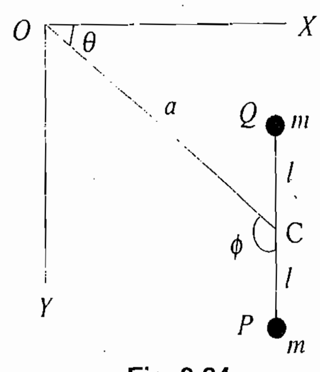


Fig. 2.34

Ans : $L = ml^2\dot{\phi}^2 + m(a^2 + l^2\cos^2\phi)\dot{\theta}^2$; $\ddot{\phi} + \frac{p_\theta^2 \sin\phi\cos\phi}{4m^2(a^2 + l^2\cos^2\phi)^2} = 0$,

where $p_\theta = 2m\dot{\theta}(a^2 + l^2\cos\phi)$.

10. A bead slides on a wire in the shape of a cycloid described by the equations :

$$x = a(\theta - \sin\theta),\ y = a(1 + \cos\theta),\ \text{where } 0 \le \theta \le 2\pi.$$

Find (a) the Lagrangian function, and (b) the equations of motion. Neglect the friction between the bead and wire. **(Agra 1998)**

Ans : $L = ma^2 \dot{\theta}^2 (1 - \cos\theta) - mga\,(1 + \cos\theta)$; $\ddot{\theta}\,(1 - \cos\theta) + \frac{1}{2}\sin\theta\dot{\theta} - \frac{g}{2a}\sin\theta = 0.$

Objective Type Questions

1. A particle is constrained to move along the inner surface of a fixed hemispherical bowl. The number of degrees of freedom of the particle is
   (*a*) one (*b*) two
   (*c*) three (*d*) four **(GATE 1996)**
   Ans : (*b*).
2. A rigid body moving freely in space has degrees of freedom
   (*a*) 3 (*b*) 6
   (*c*) 9 (*d*) 4
   Ans : (*b*)
3. Constraint in a rigid body is
   (*a*) holonomic (*b*) nonholonomic
   (*c*) scleronomic (*d*) rheonomic.
   Ans : (*a*), (*c*).
4. Generalized coordinates
   (*a*) depend on each other. (*b*) are independent of each other.
   (*c*) are necessarily spherical coordinates. (*d*) may be cartesian coordinates.
   Ans : (*b*), (*d*).
5. The constraints on a bead on a uniformly rotating wire in a force free space is
   (*a*) Rheonomous (*b*) Scleronomous
   (*c*) (*a*) and (*b*) both (*d*) None of these **(Kanpur 2003)**
   Ans : (*a*).
6. If the generalized coordinate is angle θ, the corresponding generalized force has the dimensions of
   (*a*) force (*b*) momentum
   (*c*) torque (*d*) energy
   Ans : (*c*).
7. If a generalized coordinate has the dimentions of velocity, generalized velocity has the dimensions of
   (*a*) displacement (*b*) velocity
   (*c*) acceleration (*d*) force
   Ans : (*c*).
8. The Lagrangian for a charged particle in an electromagnetic field is
   (*a*) $L = T + q\,\phi + q\,(\mathbf{v}\,.\,\mathbf{A})$ (*b*) $L = T - q\,\phi - q\,(\mathbf{v}\,.\,\mathbf{A})$
   (*c*) $L = T - q\,\phi + q\,(\mathbf{v}\,.\,\mathbf{A})$ (*d*) $L = T + q\,\phi - q\,(\mathbf{v}\,.\,\mathbf{A})$
   where T is the kinetic energy and ϕ and $\mathbf{A}$ are magnetic scalar and vector potentials.
   Ans. (*c*).

9. A mass m is connected on either side with a spring each of spring constants k_1 and k_2. The free ends of springs are tied to rigid supports. The displacement of the mass is x from equilibrium position. Which one of the following is TRUE ?

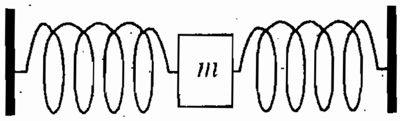


Fig. 2.35

(a) The force acting on the mass is $-(k_1 k_2)^{1/2} x$.

(b) The angular momentum of the mass is zero about the equilibrium point and its Langrangian is $\frac{1}{2}m\dot{x}^2 - \frac{1}{2}(k_1 + k_2) x^2$.

(c) The total energy of the system is $\frac{1}{2}m\dot{x}^2$.

(d) The angluar momentum of the mass is $mx\dot{x}$ and the Lagrangian of the system is $\frac{m}{2}\dot{x}^2 + \frac{1}{2}(k_1 + k_2) x^2$. **(Gate 2004)**

Ans. (b)

10. The homogeneity of time leads to the law of conservation of

(a) linear momentum (b) angular momentum

(c) energy (d) parity. **(Gate 2002)**

Ans. (c)

Short Answer Questions

1. Discuss the D'Alemberts principle. **(Agra 2004, 03)**
2. What do you mean by degrees of freedom ?
3. What are holonomic and non-holonomic constraints ? **(Agra 2002; Kanpur 2002)**
4. Show that the work done by constraint forces in a rigid body is zero. **(Kanpur 2001)**
5. What are generalized coordinates ? What is the advantage of using them ? **(Agra 2004, 02)**
6. Write the Lagrange's equations in presence of non-consecutive forces.
7. For a non-conservative system obtain Lagrange's equations. **(Kanpur 2002)**
8. Write the Lagrangian and equation of motion for a mass M suspended by a spring of force constant k and allowed to swing vertically. **(Kanpur 2003, Rohilkhand 1994)**

   [**Ans.** $L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$; $m\ddot{x} + kx = 0$]
9. Deduce the Lagrange equation of motion for L - C circuit **(Agra 2003)**

   [**Ans.** $L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$]
10. What is Hamilton's's principle ? **(Agra 2004; Kanpur 2001)**
11. Fill in the blanks :

    (i) The number of independent coordinates required to describe a system is called......................... **(Agra 2004)**

    (ii) Generalized coordinates are defined to be any quantities by means of which....................... **(Agra 2004)**

    [**Ans.** (i) Generalized coordinates (ii) we describe the state of configuration of a system]

CHAPTER

3 Hamiltonian Dynamics

3.1. INTRODUCTION

In the earlier chapter, we have seen the use of Lagrangian method, which allows us to find the equations of motion for any system in generalized coordinates $q_1, q_2, ..., q_n$. In the Lagrangian formulation, the equations of motion are in the form of a set of second order differential equations. An alternative formulation, given by Hamilton and known as the Hamiltonian dynamics, makes use of the generalized momenta $p_1, p_2, ..., p_n$ in place of the generalized velocities $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$, used in the Lagrangian formulation. In the Hamiltonian formulation, two sets of first order differential equations are used instead of a set of second order differential equations. Both the formulations are equivalent, but the Hamiltonian formulation is more fundamental to the foundations of statistical and quantum mechanics. This formulation is particularly valuable when some of the generalized momenta are the constants of motion.

3.2. GENERALIZED MOMENTUM AND CYCLIC COORDINATES

In order to define the generalized momentum, we take a simple example of a single particle, moving with velocity $\dot{x}$ along X-axis. The kinetic energy of the particle is

$$T = \frac{1}{2} m\dot{x}^2 \qquad ...(1)$$

The derivative of T with respect to $\dot{x}$ *i.e.*,

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} = p$$

defines the momentum. If V is not a function of the velocity $\dot{x}$, *i.e.*, $V = V(x)$ and $\frac{\partial V}{\partial \dot{x}} = 0$, then the momentum p can be written also as

$$p = \frac{\partial}{\partial \dot{x}}(T - V) \text{ or } p = \frac{\partial L}{\partial \dot{x}} \qquad ...(2)$$

Similarly for a system described by a set of generalized coordinates q_k's and generalized velocities $\dot{q}_k$s, we define the generalized momentum corresponding to the generalized coordinate q_k as

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \qquad ...(3)$$

This is also called ***conjugate momentum*** (conjugate to the coordinate q_k) or ***canonical momentum***. For a conservative system, the Lagrange's equations are given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \qquad ...(4)$$

Substituting for $\frac{\partial L}{\partial \dot{q}_k} = p_k$, we get

$$\frac{dp_k}{dt} - \frac{\partial L}{\partial q_k} = 0 \quad \text{or} \quad \dot{p}_k = \frac{\partial L}{\partial q_k} \qquad ...(5)$$

Now, suppose in the expression for Lagrangian L of a system, a certain coordinate q_k does not appear explicitly. Then

$$\frac{\partial L}{\partial q_k} = 0 \qquad ...(6)$$

This means from eq. (5) that

$$\dot{p}_k = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] = 0 \qquad ...(7)$$

and hence on integration, we get

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \text{ a constant} \qquad ...(8)$$

Thus *whenever the Lagrangian function does not contain a coordinate q_k explicitly, the generalized momentum p_k is a constant of motion. The coordinate q_k is called* ***cyclic*** *or* ***ignorable***. In other words, *the generalized momentum associated with an ignorable coordinate is a constant of motion for the system.*

For example, let us consider the motion of a particle in a central force field. In polar coordinates, the Lagrangian L can be expressed as

$$L = T - V = \frac{1}{2} m [\dot{r}^2 + r^2 \dot{\theta}^2] - V(r) \qquad [\because v^2 = \dot{r}^2 + r^2 \dot{\theta}^2] \qquad ...(9)$$

We see in eq. (9) that L does not contain the coordinate θ. Therefore, θ is the *cyclic* or *ignorable coordinate* and hence the generalized momentum p_θ corresponding to θ is

$$p_\theta = \partial L / \partial \dot{\theta} = mr^2 \dot{\theta} = \text{constant} \qquad ...(10)$$

where the generalized momentum p_θ is the angular momentum and is a constant of motion in time. Thus the angular momentum of the system is conserved in the central force problem. Further the constant of motion is called a *first integral* because $\partial L / \partial \dot{\theta}$ = constant is a first order differential equation and has been obtained by integrating $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = 0$.

It is to be noted that if the generalized coordinate q_k does not have the dimensions of length, the generalized momentum p_k may not have the dimensions of linear momentum. Also, if there is a velocity dependent potential, for example a charge moving in an electromagnetic field, then even with a cartesian coordinate ($q_k = x, y$ or z) the associated generalized momentum will not be identical with the usual mechanical momentum. The Lagrangian L of a particle with charge q in an electromagnetic field is given by [eq. (68), Chapter 2]

$$L = T - q\phi + q\,\mathbf{v} \cdot \mathbf{A} \qquad ...(11)$$

where T is the kinetic energy, ϕ is a scalar function of position and

$$\mathbf{v} \cdot \mathbf{A} = v_x A_x + v_y A_y + v_z A_z$$

Therefore,

$$L = \frac{1}{2} m [v_x^2 + v_y^2 + v_z^2] - q\phi + q [v_x A_x + v_y A_y + v_z A_z] \qquad ...(12)$$

The generalized momentum p_x conjugate to x is given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial v_x} = mv_x + qA_x \quad ...(13)$$

Note that here the generalized momentum is the sum of mechanical momentum (mv_x) and the x component of the electromagnetic linear momentum (qA_x) of the field associated with the charge q.

Suppose the field is such that ϕ and **A** both are independent of x. Thus x does not appear in L and is, therefore, cyclic coordinate. Consequently, the corresponding generalized momentum p_x is conserved *i.e.*,

$$p_x = mv_x + qA_x = \text{constant} \quad ...(14)$$

One sees that here instead of the mechanical linear momentum (mv_x), its sum with qA_x is conserved. Thus the condition for the conservation of generalized momentum is more general than the principle of conservation of linear mechanical momentum, because the former is a conservation theorem for a case where the third law of action and reaction is violated, *e.g.*, when we deal with the motion of a charge in an electromagnetic field.

First Integrals

In eq. (8), the generalized momentum p_k conjugate to a cyclic coordinate q_k is a constant of motion. This is equivalent to integrate the equation of motion once under certain condition and hence this constant of motion is referred as a *first integral.*

By a first integral of the motion, we mean a relation of the form

$$f(q_1, q_2 \,....., \dot{q}_1, \dot{q}_2, ..., t) = \text{Constant}$$

which is a first order differential equation. There may be several first integrals for a particular type of motion of the system. These are very useful because we get some important informations physically about the system just at a glance from these integrals. In fact, conservation laws, eq., (16), (20) and (30) in Chapter 1, are the first integrals of motion.

3.3. CONSERVATION THEOREMS

The theorems of conservation of linear and angular momentum are the special cases of the general principle for cyclic coordinates in the Lagrangian formulation.

3.3.1. Conservation of Linear Momentum

The Lagrange's equation of motion for a generalized coordinate q_k is given by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \quad ...(15)$$

where $L = T - V$.

Suppose dq_k represents a translation of the entire system along a given direction. We consider a conservative system so that V is not a function of velocities and T is not a function of position. Therefore,

$$\frac{\partial V}{\partial \dot{q}_k} = \frac{\partial T}{\partial q_k} = 0$$

Now we can write eq. (15) as

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right] = \dot{p}_k = -\frac{\partial V}{\partial q_k} = G_k \quad ...(16a)$$

or

$$G_k = \dot{p}_k \quad ...(16b)$$

These G_k and p_k are the components of the total force **F** and total linear momentum **P** of the system along the direction of the translation dq_k. For example, if the system is given a translation along X-axis, then $dq_k = dx$ and $G_k = F_x$ and $p_k = P_x$. This can be shown as follows. As we see from Fig. 3.1,

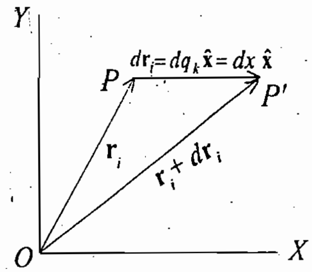


Fig. 3.1 : Change of position vector under translation of the system

$$d\mathbf{r}_i = dq_k\hat{\mathbf{x}} = dx\,\hat{\mathbf{x}}$$

where $\hat{\mathbf{x}}$ is a unit vector along X-axis.

This gives $$\frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \mathbf{r}_i}{\partial x} = \hat{\mathbf{x}}$$

The component of generalized force is given by [eq. (20), Chapter 2] :

$$G_k = \sum_{i=1}^{N} \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_i \mathbf{F}_i \bullet \hat{\mathbf{x}} = \hat{\mathbf{x}} \bullet \sum_i \mathbf{F}_i = \hat{\mathbf{x}} \bullet \mathbf{F} = F_x$$

Also $$T = \sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$$

Therefore, $$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \dot{\mathbf{r}}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \left[\because \dot{\mathbf{r}}_i^2 = \dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i \text{ and } \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k}\right]$$

or $$p_k = \sum_i m_i \dot{\mathbf{r}}_i \bullet \hat{\mathbf{x}} = \hat{\mathbf{x}} \bullet \sum_i m_i \dot{\mathbf{r}}_i = \hat{\mathbf{x}} \bullet \mathbf{P} = P_x$$

Thus eq. (16) represents the equation of motion for the direction of translation.

Suppose that the translation coordinate q_k is cyclic. This means that q_k is not appearing in $L = T - V$. Then

$$\frac{\partial L}{\partial q_k} = -\frac{\partial V}{\partial q_k} = 0$$

and therefore from (16), we get

$$G_k = \dot{p}_k = 0 \text{ or } p_k = \text{constant}$$

For X-direction , $F_x = \dot{P}_x = 0$ or $P_x =$ constant. ...(17)

This is the well known **conservation theorem for linear momentum**. ***Thus in absence of a given component of applied force, the corresponding component of linear momentum is conserved.***

3.3.2. Conservation of Angular Momentum

Let us consider a conservative system as discussed above. Now, if for the generalized coordinate q_k, dq_k represents a rotation $d\theta$, then the Lagrange equation can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = -\frac{\partial V}{\partial q_k}$$

because V is independent of $\dot{q}_k$.

Thus $$\dot{p}_k = G_k \qquad ...(18)$$

Here we want to show that for a rotational coordinate q_k, the generalized force G_k is the component of the total applied torque τ about the axis of rotation and the generalized momentum p_k is the component of the total angular momentum **J** about the same axis.

Now, $$G_k = \sum_{i=1}^{N} \mathbf{F}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_k}$$

Here, $dq_k = d\theta$ is an infinitesimal rotation of the position vector $\mathbf{r}_i$ of the particle of the system about Z-axis such that the magnitude of $\mathbf{r}_i$ remains constant.

From Fig. 3.2., the infinitesimal small distance $|d\mathbf{r}_i|$ is

$$|d\mathbf{r}_i| = r_i \sin\phi \, dq_k = r_i \sin\phi \, d\theta$$

or $$d\mathbf{r}_i = dq_k (\hat{\mathbf{z}} \times \mathbf{r}_i)$$

where $\hat{\mathbf{z}}$ is a unit vector along the axis of rotation.

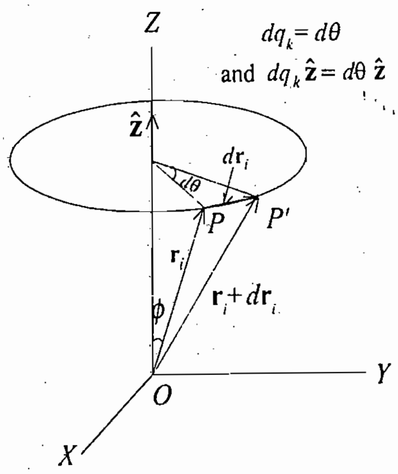


Fig. 3.2 : Change of position vector under rotation of the system.

Therefore , $$\frac{\partial \mathbf{r}_i}{\partial q_k} = \hat{\mathbf{z}} \times \mathbf{r}_i$$

Thus $$G_k = \sum_i \mathbf{F}_i \bullet \hat{\mathbf{z}} \times \mathbf{r}_i$$

Using the property of scalar triple product,

$$G_k = \sum_i \hat{\mathbf{z}} \bullet \mathbf{r}_i \times \mathbf{F}_i = \hat{\mathbf{z}} \bullet \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

$$= \hat{\mathbf{z}} \bullet \sum_i \tau_i = \hat{\mathbf{z}} \bullet \tau = \tau_z$$

where $\sum_i \tau_i = \tau$ is the total applied torque and $\tau_i = \hat{\mathbf{z}} \bullet \tau$ is the component of the total torque τ along Z-axis.

Similarly,

$$p_k = \frac{\partial T}{\partial \dot{q}_k} = \sum_i m_i \mathbf{v}_i \bullet \frac{\partial \mathbf{r}_i}{\partial q_k} = \sum_i m_i \mathbf{v}_i \bullet \hat{\mathbf{z}} \times \mathbf{r}_i$$

$$= \sum_i \hat{\mathbf{z}} \bullet \mathbf{r}_i \times m_i \mathbf{v}_i = \hat{\mathbf{z}} \bullet \sum_i \mathbf{r}_i \times \mathbf{P}_i = \hat{\mathbf{z}} \bullet \sum_i \mathbf{J}_i = \hat{\mathbf{z}} \bullet \mathbf{J} = J_z \qquad ...(19)$$

where $\sum_i \mathbf{J}_i = \mathbf{J}$ is the total angular momentum and $J_z = \hat{\mathbf{z}} \bullet \mathbf{J}$ is the component of the total angular momentum **J** along Z-axis.

Thus eq. (18) represents the equation of motion about the axis of rotation ($\dot{J}_z = \tau_z$).

Now, if the rotation coordinate q_k is cyclic, it will not appear in the Lagrangian L or V and hence

$$G_k = -\partial V / \partial q_k = 0$$

Therefore, from eq. (18), we have

$$G_k = \dot{p}_k = 0 \text{ or } \tau_z = \dot{J}_z = 0$$

or $$J_z = \text{constant} \qquad ...(20)$$

This is the **theorem of conservation of angular momentum** which states that *in absence of a given component of applied torque along an axis, the corresponding component of angular momentum along the same axis is conserved*. Thus we have obtained the two theorems of conservation of momentum from the general conservation theorem of generalized momentum corresponding to cyclic coordinates.

3.3.3. Significance of Translation and Rotation Cyclic Coordinates – Symmetry Properties

If a coordinate corresponding to a translation along a direction is not appearing in the expression of the Lagrangian of a system, this coordinate is cyclic and obviously the translation along the same direction has no effect on the Lagrangian. Thus if the Lagrangian of a system is invariant under translation along a direction, the corresponding linear momentum is conserved. Similarly, if a rotation coordinate about an axis is cyclic, the conjugate angular momentum is conserved and the Lagrangian is invariant under the rotation about the axis.

It is to be mentioned that the conservation principles are the expressions of symmetry properties of the system. For example, we find that in the central force problem, the law of conservation of angular momentum emerges from the fact that the Lagrangian L is independent of the coordinate θ. In fact, this is an expression of the rotational symmetry of the system. In general, for a system, $\partial L/\partial\theta = 0$ means that the Lagrangian of the system does not change on rotation through an angle $\delta\theta$. Consequently the angular momentum is conserved for systems possessing rotational symmetry.

3.4. HAMILTONIAN FUNCTION H AND CONSERVATION OF ENERGY : JACOBI'S INTEGRAL

In the Lagrangian formulation one may expect the deduction of the theorem of conservation of the total energy for a system where the potential energy is a function of position only. In fact we shall see, as discussed below, the theorem of conservation of total energy is a special case of a more general conservation theorem.

Consider a general Lagrangian L of a system given by

$$L = L(q_1, q_2, \ldots, q_k, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k, \ldots, \dot{q}_n, t)$$

We denote it for our convenience by

$$L = L(q_k, \dot{q}_k, t)$$

The total time derivative of L is

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\frac{dq_k}{dt} + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t} \qquad \ldots(21)$$

From Lagrangian equations, we have

$$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$$

Substituting for $\partial L/\partial q_k$ in eq. (21), we get

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\frac{d\dot{q}_k}{dt} + \frac{\partial L}{\partial t}$$

or

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial t}$$

or,

$$\frac{d}{dt}\left(\sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right) = -\frac{\partial L}{\partial t} \qquad \ldots(22)$$

The quantity in the bracket is sometimes called the **energy function** and is denoted by h :

$$h(q_1, q_2, \ldots\ldots\ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots\ldots\ldots \dot{q}_n, t) = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \qquad \ldots(23)$$

Thus from eq. (22) the total time derivative of h is

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t} \qquad ...(24)$$

If the Lagrangian L does not depend on time t explicitly, then $\partial L/\partial t = 0$. So that

$$\frac{dh}{dt} = 0 \text{ i.e., h = Constant} \qquad ...(25)$$

Thus when the lagrangian is not explicit function of time, the energy function is the constant of motion. It is one of the *first integrals of the motion* and is called **Jacobi's integral.**

But from eq. (8) $\partial L/\partial \dot{q}_k = p_k$, hence eq. (22) can be written as

$$\frac{d}{dt}(\sum_k p_k \dot{q}_k - L) = -\frac{\partial L}{\partial t} \qquad ...(26)$$

The quantity in the bracket is called the ***Hamiltonian function*** *H, i.e.*,

$$H = \sum_k p_k \dot{q}_k - L \qquad ...(27)$$

In general, the Hamiltonian function H is the function of generalized momenta p_k, generalized coordinates q_k and time t *i.e.*,

$$H = H(p_1, p_2, ..., p_k, ..., p_n, q_1, q_2, ..., q_k, ..., q_n, t) \qquad ...(28a)$$

or

$$H = H(p_k, q_k, t) \qquad ...(28b)$$

It is to be seen that the energy function h is identical in value with the Hamiltonian H. It is given a different name and symbol because h is a function of q_k, $\dot{q}_k$ and t, while H that of q_k, p_k and t.

If t does not appear in the Lagrangian L explicitly, then $\partial L/\partial t = 0$ and eqs. (26) and (27) give

$$\frac{dH}{dt} = 0 \quad \text{or} \quad H = \sum_k p_k \dot{q}_k - L = \text{ constant} \qquad ...(29)$$

Thus, if the time t does not appear in the Lagrangian L explicitly, we see that the Hamiltonian H is constant in time *i.e.*, conserved. This is a conservation theorem for the Hamiltonian of the system. Under special circumstances, the Hamiltonian H is equal to the total energy E of the system. In-fact, this is the case in most of the physical problems.

Conservation of Energy–Physical Sygnificance

The Hamiltonian takes a special form, if the system is conservative *i.e.*, the potential energy V is independent of velocity coordinates $\dot{q}_k$ and the transformation equations for coordinates do not contain time explicitly *i.e.*,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, ..., q_k, ..., q_n).$$

For a conservative system $\partial V/\partial \dot{q}_k = 0$. From eq. (8), we have

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}(T - V) = \frac{\partial T}{\partial \dot{q}_k}$$

So that eq. (29) is

$$H = \sum_k p_k \dot{q}_k - L = \sum_k \frac{\partial T}{\partial \dot{q}_k}\dot{q}_k - L \qquad ...(30)$$

If $\mathbf{r}_i$ does not depend on time t explicitly, then the kinetic energy T is a homogeneous quadratic function. It is easy to show that

$$\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T \qquad ...(31)^*$$

In fact, for a natural conservative system neither T nor V contains any explicit time dependence (*i.e.*, the Lagrangian does not depend on time explicitly) and T is a homogeneous quadratic function of the time derivatives $\dot{q}_k$. Hence from eq. (30) and eq. (31),

$$H = 2T - L = 2T - (T - V)$$

or $$H = T + V = E, \text{ constant} \qquad ...(31)$$

Thus *the Hamiltonian H represents the total energy of the system E and is conserved, provided the system is conservative and T is a homogeneous quadratic function.*

3.5. HAMILTON'S EQUATIONS

The Hamiltonian, in general, is a function of generalized coordinates q_k, generalized momenta p_k and time t, *i.e.*,

$$H = H(q_1, q_2, ..., q_k, ..., q_n, p_1, p_2, ..., p_k, ..., p_n, t)$$

We may write the differential dH as

$$dH = \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \qquad ...(33)$$

---

* For a system of N particles, when $\mathbf{r}_i$ does not depend on time explicitly,

then $$\mathbf{v}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k$$

Therefore, $$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right)^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right) \cdot \left(\sum_l \frac{\partial \mathbf{r}_i}{\partial q_l} \dot{q}_l \right)$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \sum_k \sum_l \left[\frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l} \right] \dot{q}_k \dot{q}_l$$

$$\therefore \qquad \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} \sum_l m_i \left[\frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l} \right] \dot{q}_l$$

Multiplying by $\dot{q}_k$ and summing over k, we get

$$\sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} = \sum_{i=1}^{N} \sum_k \sum_l m_i \left[\frac{\partial \mathbf{r}_i}{\partial q_k} \cdot \frac{\partial \mathbf{r}_i}{\partial q_l} \right] \dot{q}_k \dot{q}_l = 2T$$

where each k and l run from 1 to n.

But as defined in eq. (27), $H = \sum_k p_k \dot{q}_k - L$ and hence

$$dH = \sum_k \dot{q}_k \, dp_k + \sum_k p_k \, d\dot{q}_k - dL \qquad ...(34)$$

Also, $L = L(q_1, q_2, ..., q_k, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., \dot{q}_n, t)$

Therefore, $$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

But $$\dot{p}_k = \frac{\partial L}{\partial q_k} \text{ [eq. (5)] and } p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ [eq. (3)].}$$

Therefore, $$dL = \sum_k \dot{p}_k \, dq_k + \sum_k p_k \, d\dot{q}_k + \frac{\partial L}{\partial t} dt \qquad ...(35)$$

Substituting for dL from eq. (35) in eq. (34), we get

$$dH = \sum_k \dot{q}_k \, dp_k - \sum_k \dot{p}_k \, dq_k - \frac{\partial L}{\partial t} dt \qquad ...(36)$$

Comparing the coefficients of dp_k, dq_k and dt in eqs. (33) and (36), we obtain

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad ...(37a)$$

$$-\dot{p}_k = \frac{\partial H}{\partial q_k} \qquad ...(37b)$$

and $$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \qquad ...(37c)$$

Eqs. (37a) and (37b) are known as ***Hamilton's equations*** or ***Hamilton's canonical equations of motion***. This procedure of describing the motion of a system by these equations is called **Hamiltonian dynamics.** For $k = 1, 2, ..., n$, in all these are $2n$ first order differential equations which are much easier to solve in comparison to the n second order differential equations in Lagrangian dynamics.

It is clear from eq. (37b) that if any coordinate q_k is cyclic, *i.e.*, not contained in H, then

$$\frac{\partial H}{\partial q_k} = 0 \text{ or } \dot{p}_k = 0 \text{ or } p_k = \text{constant in time} \qquad ...(38)$$

Thus for any cyclic coordinate, corresponding conjugate momentum is a constant of motion. Further from eq. (33), we have

$$\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t} \qquad ...(39)$$

Substituting for $\dot{q}_k$ and $\dot{p}_k$ from eq. (37) in eq. (39), we get

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad ...(40)$$

If the Lagrangian L and hence H does not depend on time t explicitly, then $\partial L/\partial t = -\,\partial H/\partial t = 0$ and hence

$$\frac{dH}{dt} = 0 \text{ or } H = \text{constant}. \qquad ...(41)$$

We are mainly interested in the conservative systems for which $H = T + V = E$ is a constant of motion, as discussed earlier.

3.6. HAMILTON'S EQUATIONS IN DIFFERENT COORDINATE SYSTEMS

(1) In cartesian coordinates :

Kinetic energy of the particle $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$; Potential energy of the particle $V = V(x, y, z)$

Lagrangian $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$

Generalized momentum $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$

Hence, $p_x = \dfrac{\partial L}{\partial \dot{x}} = m\dot{x}$ or $\dot{x} = \dfrac{p_x}{m}$; Similarily $\dot{y} = \dfrac{p_y}{m}$ and $\dot{z} = \dfrac{p_z}{m}$

Hamiltonian $\quad H = \sum_k p_k\,\dot{q}_k - L$

For $k = x, y, z$

$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + V(x, y, z)$$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x, y, z)$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + V(x, y, z) \qquad ...(42)$$

Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

The Hamilton's equations in Cartesian coordinates are

$$\dot{x} = \frac{\partial H}{\partial p_x} \text{ or } \dot{x} = \frac{p_x}{m} \text{ or } \dot{p}_x = -\frac{\partial H}{\partial x} \text{ or } \dot{p}_x = -\frac{\partial V}{\partial x} \qquad ...(43a)$$

Similarly,

$$\dot{y} = \frac{p_y}{m} \text{ and } \dot{p}_y = -\frac{\partial V}{\partial y} \qquad ...(43b)$$

$$\dot{z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial V}{\partial z} \qquad ...(43c)$$

Thus one may express equations of motion as

$$m\ddot{x} = -\frac{\partial V}{\partial x},\quad m\ddot{y} = -\frac{\partial V}{\partial y},\quad m\ddot{z} = -\frac{\partial V}{\partial z} \qquad \text{...(44)}$$

(2) In polar coordinates : If r, θ are the polar coordinates of a particle of mass m, then

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Hence, $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

Now, $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$ and $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r, \theta)$

Generalized momenta $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ or $\dot{r} = \frac{p_r}{m}$ and $\dot{\theta} = \frac{p_\theta}{mr^2}$

Thus, $H = \sum_k p_k\dot{q}_k - L = p_r\dot{r} + p_\theta\dot{\theta} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r,\theta)$

or

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r,\theta) \qquad \text{...(45)}$$

Hamilton's equations are

$$\dot{r} = -\frac{\partial H}{\partial p_r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} \text{ or } \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \qquad \text{...(46a)}$$

and

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_\theta = -\frac{\partial V}{\partial \theta} \qquad \text{...(46b)}$$

(3) In cylindrical coordinates : Here, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Hence

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - V(r, \theta, z)$$

Generalized momentum $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or $\dot{r} = \frac{p_r}{m}$; Similarly, $\dot{\theta} = \frac{p_\theta}{mr^2}$, $\dot{z} = \frac{p_z}{m}$

Now, $H = \sum_k p_k\dot{q}_k - L$, where $k = r, \theta, z$

Thus,

$$H = p_r\dot{r} + p_\theta\dot{\theta} + p_z\dot{z} - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + V(r,\theta,z)$$

$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + V(r,\theta,z) \qquad \text{...(47)}$$

Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Therefore,

$$\dot{r} = \frac{\partial H}{\partial p_r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} \text{ or } \dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \qquad \text{...(48a)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{m} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_\theta = -\frac{\partial V}{\partial \theta} \quad ...(48b)$$

$$\dot{z} = \frac{\partial H}{\partial p_z} \text{ or } \dot{z} = \frac{p_z}{m} \text{ and } \dot{p}_z = -\frac{\partial H}{\partial z} \text{ or } \dot{p}_z = -\frac{\partial V}{\partial z} \quad ...(48c)$$

(4) In spherical coordinates : Here

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

Hence,
$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\sin\theta\dot{\phi}\sin\phi$$

$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\sin\theta\,\dot{\phi}\cos\phi$$

$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

Thus, $L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x, y, z)$ or $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r, \theta, \phi)$

Therefore, $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ or $\dot{r} = \frac{p_r}{m}$; Similarly $\dot{\theta} = \frac{p_\theta}{mr^2}$ and $\dot{\phi} = \frac{p_\phi}{mr^2\sin\theta}$

Now $H = \sum_{k=r,\theta,\phi} p_k\dot{q}_k - L = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - \frac{1}{2}m\left(\dot{r}^2 + \dot{r}^2\dot{\theta}^2 + \dot{r}^2\dot{\phi}^2\sin^2\theta\right) - V(r, \theta, \phi)$

or
$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r, \theta, \phi) \quad ...(49)$$

Hamiltonian equations $\left(\ddot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}\right)$ are

$$\dot{r} = \frac{\partial H}{\partial p_r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } \dot{p}_r = -\frac{\partial H}{\partial r} \text{ or } \dot{p}_r = \frac{p_\theta^2}{mr^3} + \frac{p_\phi^2}{mr^3\sin^2\theta} - \frac{\partial V}{\partial r} \quad ...(50a)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ or } \dot{\theta} = \frac{p_0}{mr^2} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_\theta = \frac{p_\phi^2\cos\theta}{mr^2\sin^3\theta} - \frac{\partial V}{\partial \theta} \quad ...(50b)$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} \text{ or } \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta} \text{ and } \dot{p}_\phi = -\frac{\partial H}{\partial \phi} \text{ or } \dot{p}_\phi = -\frac{\partial V}{\partial \phi} \quad ...(50c)$$

3.7. EXAMPLES IN HAMILTONIAN DYNAMICS

(1) Harmonic oscillator : For a harmonic oscillator, the kinetic energy T and potential energy V are given by

$$T = \frac{1}{2}m\dot{x}^2 \text{ and } V = \frac{1}{2}kx^2 \quad ...(i)$$

Now,
$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad ...(ii)$$

In order to write the Hamiltonian, we must replace $\dot{x}$ by the generalized momentum p_x, *i.e.*,

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or} \quad \dot{x} = \frac{p_x}{m}$$

Hence $$T = \tfrac{1}{2}m\dot{x}^2 = \frac{p_x^2}{2m} \qquad ...(iii)$$

Therefore, $$H = T + V = \frac{p_x^2}{2m} + \tfrac{1}{2}kx^2 \qquad ...(iv)$$

Hence the canonical or Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{or} \quad p_x = m\dot{x} \qquad ...(v)$$

and $$-\dot{p}_x = \frac{\partial H}{\partial x} = kx \quad \text{or} \quad \dot{p}_x = -kx \qquad ...(vi)$$

Substituting for p_x from eq. (v), we get

$$m\ddot{x} = -kx \quad \text{or} \quad m\ddot{x} + kx = 0 \qquad ...(vii)$$

which is the familiar equation of a harmonic oscillator.

(2) Motion of a particle in a central force field : All central forces are conservative in nature and $F(r) = -\partial V/\partial r$. For inverse square law (central) force

$$F = -\frac{K}{r^2} = -\frac{\partial V}{\partial r}; \; V(r) = -\frac{K}{r} \qquad ...(i)$$

If m be the mass of a particle moving in the central force field, then the Lagrangian L in polar coordinates can be expressed as

$$L = T - V(r) = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \qquad ...(ii)$$

In order to write the Hamiltonian, $\dot{r}$ and $\dot{\theta}$ must be replaced by the generalized momenta p_r and p_θ. Now

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

or $$\dot{r} = \frac{p_r}{m} \quad \text{and} \; \dot{\theta} = \frac{p_\theta}{mr^2} \qquad ...(iii)$$

Hence $$H = T + V = \tfrac{1}{2}m\left[\left[\frac{p_r}{m}\right]^2 + r^2\left[\frac{p_\theta}{mr^2}\right]^2\right] + V(r)$$

or $$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + V(r) \qquad ...(iv)$$

The Hamilton's equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{and} \quad -\dot{p}_k = \frac{\partial H}{\partial q_k}.$$

Here $k = r$ and θ; hence in the present case, the equations are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad ...(v)$$

$$-\dot{p}_r = \frac{\partial H}{\partial r} = -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} \quad ...(vi)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad ...(vii)$$

and $$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \quad ...(viii)$$

From eqs. (*vii*) and (*viii*), we get

$$p_\theta = \text{constant} = mr^2\dot{\theta} \quad ...(ix)$$

This is the *familiar equation of conservation of angular momentum of a particle, moving in a central force field*. From eqs. (*v*) and (*vi*), we have

$$-m\ddot{r} = -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} \text{ or } m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} = 0 \quad ...(x)$$

This is an important differentail equation in second order for a particle moving under central force. In case of square law force, $V(r) = -K/r$ and $f(r) = -\partial V/\partial r = -K/r^2$. Then

$$m\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{K}{r^2} = 0 \quad ...(xi)$$

In fact in the next chapter, we shall solve this type of differential equation for a particle moving in a central force field, specifically for planetary motion in a gravitational field.

(3) Charged particle moving in an electomagnetic field : The Lagrangian L for a charged particle in an electromagnetic field [eq. (68), Chapter 2] is given by

$$L = T - U = T - q\phi + q(\mathbf{v} \bullet \mathbf{A})$$

or $$L = \frac{1}{2}\sum_k mv_k^2 + q\sum_k v_k A_k - q\phi$$

The canonical momenta for $k = x, y, z$ are

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial L}{\partial v_k} = mv_k + qA_k \quad ...(iii)$$

Now, $$H = \sum_k p_k \dot{q}_k - L = \sum_k \left(mv_k^2 + qA_k v_k\right) - L$$

$$= mv^2 + q(\mathbf{v} \cdot \mathbf{A}) - \frac{1}{2}mv^2 + q\phi - q(\mathbf{v} \cdot \mathbf{A})$$

$$= \frac{1}{2}mv^2 + q\phi \quad ...(iv)$$

From (*iii*) $v_k = \frac{p_k}{m} - \frac{q}{m}A_k$, then H can be expressed as

$$H = \sum_k \frac{1}{2m}(p_k - qA_k)^2 + q\phi \quad ...(va)$$

or $$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi \quad \text{...}(vb)$$

Hence Hamilton's equations are

$$v_k = \frac{\partial H}{\partial p_k} = \frac{1}{m}(p_k - qA_k) \text{ or } \mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A}) \quad \text{...}(via)$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \text{ or } \dot{\mathbf{p}} = -q\nabla\phi + q\nabla(\mathbf{v} \cdot \mathbf{A}) \quad \text{...}(vib)$$

(4) Compound pendulum : See Fig. 2.7.

Lagrangian $L = T - V = \frac{1}{2} I\dot{\theta}^2 + Mgl\cos\theta$

Generalized momentum $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$ ...(*i*)

Hence $$H = \sum_k p_k \dot{q}_k - L = p_\theta\,\dot{\theta} - \tfrac{1}{2} I\dot{\theta}^2 - Mgl\cos\theta \text{ or } H = \frac{p_\theta^2}{2I} - Mgl\cos\theta$$

Hamilton's equations $\left(\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k} \right)$ are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{I} \text{ and } \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \text{ or } \dot{p}_\theta = -Mgl\sin\theta \quad \text{...}(ii)$$

Using (*i*), eq. (*ii*) assumes the form

$$I\ddot{\theta} = -Mgl\sin\theta \quad \text{...}(iii)$$

For small θ, $\sin\theta \cong \theta$ and the equation of motion is

$$\ddot{\theta} + \frac{Mgl}{I}\theta = 0 \quad \text{...}(iv)$$

Eq. (*iv*) represents simple harmonic motion, whose periodic time is given by

$$T = 2\pi\sqrt{\frac{I}{Mgl}} \quad \text{...}(v)$$

(5) Two dimensional harmonic oscillator :

(*a*) In cartesian coordinates :

$$L = T - V = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - \tfrac{1}{2} K\left(x^2 + y^2\right)$$

$\therefore$ $$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \text{ or } \dot{x} = \frac{p_x}{m} \text{ and } p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \text{ or } \dot{y} = \frac{p_y}{m}$$

Now, $$H = \sum_k p_k\,\dot{q}_k - L \text{ , where, } q_k = x, y$$

$$= p_x\dot{x} + p_y\dot{y} - \tfrac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2} K\left(x^2 + y^2\right)$$

$$= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \tfrac{1}{2} K\left(x^2 + y^2\right)$$

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} \text{ or } \dot{x} = \frac{p_x}{m} \text{ and } \dot{p}_x = -\frac{\partial H}{\partial x} = -Kx \quad ...(i)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} \text{ or } \dot{y} = \frac{p_y}{m} \text{ and } \dot{p}_y = -\frac{\partial H}{\partial y} = -Ky \quad ...(ii)$$

From (*i*) and (*ii*), we obtain the equations of motion of two dimensional harmonic oscillator as

$$m\ddot{x} + Kx = 0 \quad ...(iii)$$

$$m\ddot{y} + Ky = 0 \quad ...(iv)$$

(*b*) **In polar coordinates :** Here, $x = r\cos\theta, y = r\sin\theta$

$$L = T - V = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \tfrac{1}{2}K\left(x^2 + y^2\right) = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \tfrac{1}{2}Kr^2$$

$$\therefore \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \text{ or } \dot{r} = \frac{p_r}{m} \text{ and } p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{mr^2}$$

Now, $$H = \sum_k p_k \dot{q}_k - L = p_r\dot{r} + p_\theta\,\dot{\theta} - \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \tfrac{1}{2}Kr^2$$

or $$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \tfrac{1}{2}Kr^2 \quad ...(i)$$

The Hamilton's eqs. $\left(\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}\right)$ are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad \text{and} \quad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - Kr \quad ...(ii)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \quad \text{and} \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad ...(iii)$$

From (*iii*) $$p_\theta = mr^2\dot{\theta} = \text{constant} \quad ...(iv)$$

From (*ii*) $$\dot{p}_r = m\ddot{r} = \frac{p_\theta^2}{mr^3} - Kr \text{ or } m\ddot{r} - mr\dot{\theta}^2 + Kr = 0 \quad ...(v)$$

Eqs. (*iv*) and (*v*) are the equations of motion for two dimensional harmonic oscillator.

Ex. 1. *Write the Hamiltonian for a simple pendulum and deduce its equations of motion.*

(Kanpur 2002)

Solution : See Fig. 2.5

Kinetic energy $T = \tfrac{1}{2}ml^2\dot{\theta}^2$; Potential energy $V = mgl\,(1 - \cos\theta)$

Lagrangian $$L = T - V = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1 - \cos\theta) \quad ...(i)$$

Hence, $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad ...(ii)$$

Now, Hamiltonian $H = \sum_k p_k \dot{q}_k - L = p_\theta \dot{\theta} - [\frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)]$

$$= ml^2\dot{\theta}^2 - \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) = T + V = \text{Total energy} \qquad ...(iii)$$

$$= \frac{1}{2}ml^2\left[\frac{p_\theta}{ml^2}\right]^2 + mgl(1-\cos\theta) = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta) \qquad ...(iv)$$

where we have used eq. (*ii*).

Hence Hamilton's equations are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \quad \text{and} \quad -\dot{p}_\theta = \frac{\partial H}{\partial \theta} = mgl\sin\theta$$

Thus $$\dot{p}_\theta = ml^2\ddot{\theta} = -mgl\sin\theta \text{ or } l\ddot{\theta} + g\sin\theta = 0$$

or $$\ddot{\theta} + \frac{g}{l}\theta = 0 \text{ for small } \theta \,(\sin\theta \cong \theta) \qquad ...(v)$$

This is the equation of motion of simple pendulum.

Ex. 2. *Describe the Hamiltonian and Hamilton's equations for an ideal spring-mass arrangement.*

Solution : $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$ where k is the force constant of the spring.

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{or} \quad \dot{x} = \frac{p_x}{m}$$

$$H = \sum_k p_k \dot{q}_k - L = p_x\dot{x} - [\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2] = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

or $$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2$$

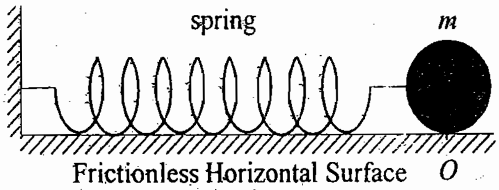


Fig. 3.3 : Spring-mass system

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ and } \dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

Thus the equation of motion of spring-mass system is

$$m\ddot{x} = -kx \text{ or } \ddot{x} + \frac{k}{m}x = 0.$$

Ex. 3. *Using Hamilton's equations of motion, show that the Hamiltonian*

$$H = \frac{p^2}{2m}e^{-rt} + \frac{1}{2}m\omega^2 x^2 e^{rt}$$

leads to the equation of motion of a damped harmonic oscillator

$$\ddot{x} + r\dot{x} + \omega^2 x = 0$$ **(Gorakhpur 1996)**

Solution : Equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} \text{ and } \dot{p} = -\frac{\partial H}{\partial q}$$

For $q = x,$ $\quad \dot{x} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial x}$

Here, $\quad \dot{x} = \frac{p}{m} e^{-rt}$ and $\dot{p} = -m\omega^2 xe^{rt}$ ... (*i*)

whence $\quad p = m\dot{x}e^{rt}$ and $\dot{p} = m\ddot{x}e^{rt} + mr\dot{x}\,e^{rt}$

Substituting for $\dot{p}$ in (*i*), we get

$$m\ddot{x}e^{rt} + mr\dot{x}\,e^{rt} = -m\omega^2 xe^{rt}$$

or $\quad \ddot{x} + r\dot{x} + \omega^2 x = 0$ ... (*ii*)

which is the desired equation of damped harmonic oscillator.

Ex. 4. ***Projectile :*** Obtain the Lagrangian, Hamiltonian and equations of motion for a paojectile near the surface of the earth.

Solution : Let *X* and *Y* axes be fixed on the earth surface and Z-axis be in the upward vertical direction. If the projectile (body if mass *m*) has coordinates (*x, y, z*) at an *t*, then its kinetic energy is given by

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The potential energy at a height *z* is $V = mgz$

Now, Lagragian $\quad L = T - V = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$

Also, $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, whence $\dot{x} = \frac{p_x}{m}$; Similarly $\dot{y} = \frac{p_y}{m}$ and $\dot{z} = \frac{p_z}{m}$

Therefore, $\quad H = \sum_k p_k \dot{q}_k - L$

Here, $\quad q_k = x, y, z$

Therefore, $\quad H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$

$$= \frac{p_x^2}{m} + \frac{p_y^2}{m} + \frac{p_z^2}{m} - \frac{1}{2} m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2}\right) + mgz$$

i.e., $\quad H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz$

Equations of motion are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

Here, $$\frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \frac{\partial H}{\partial p_y} = \frac{p_y}{m}, \frac{\partial H}{\partial z} = \frac{p_z}{m}, \frac{\partial H}{\partial x} = 0 = \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z} = mg$$

Therefore, $$\dot{x} = \frac{p_x}{m}, \dot{y} = \frac{p_y}{m}, \dot{z} = \frac{p_z}{m}, \dot{p}_x = 0 = \dot{p}_y, \dot{p}_z = -mg$$

whence $$\dot{p}_x = m\ddot{x} = 0, \dot{p}_y = m\ddot{y} = 0, \dot{p}_z = m\ddot{z} = -mg$$

or $$\ddot{x} = 0, \ddot{y} = 0 \text{ and } \ddot{z} = -g$$

which are the equations of motion. This shows the acceleration is along negative z direction *i.e.*, vertically downward equal to acceleration due to gravity in magnitude.

Ex. 5. *Describe the motion of a particle of mass m constrained to move on the surface of a cylinder of radius a and attracted towards the origin by a force which is proportional to the distance of the particle from the origin.*

Solution : The particle of mass m is constrained to move on the surface of a cylinder of radius a under an attractive central force F, given by

$$F = -kr \qquad ...(i)$$

where k is the force constant. This force is proportional to the distance of the particle from the origin.

The motion of the particle can be described in terms of the cartesian coordinates x, y, z or cylindrical coordinates ρ, θ, z [Fig. 3.4]. The equation of constraint is

$$\rho^2 = x^2 + y^2 = a^2$$

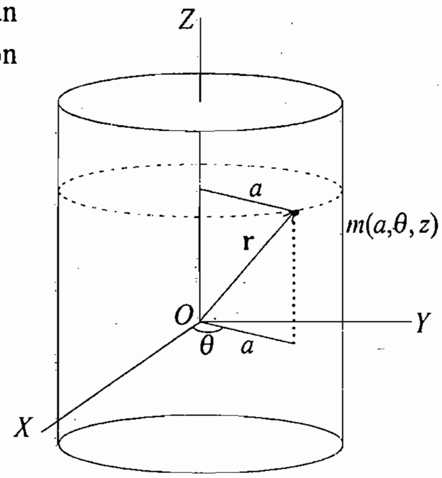


Fig. 3.4 : Motion of a particle on the surface of a cylinder.

In cylindrical coordinates,

$$T = \frac{1}{2}mv^2 = \frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2)$$

Here, $\rho = a$,

$\therefore$ $\dot{\rho} = 0$, and hence

$$T = \frac{1}{2}m\,(a^2\dot{\theta}^2 + \dot{z}^2)$$

and $$V = \frac{1}{2}kr^2 = \frac{1}{2}k\,(x^2 + y^2 + z^2) = \frac{1}{2}k\,(a^2 + z^2)$$

$\therefore$ $$L = T - V = \frac{1}{2}m\,(a^2\dot{\theta}^2 + \dot{z}^2) - \frac{1}{2}k(a^2 + z^2) \qquad ...(ii)$$

Hence $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} \text{ or } \dot{\theta} = \frac{p_\theta}{ma^2}$$

and $$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \text{or} \quad \dot{z} = \frac{p_z}{m}$$

Therefore, $$H = T + V = \frac{1}{2}m\,(a^2\dot{\theta}^2 + \dot{z}^2) + \frac{1}{2}k(a^2 + z^2)$$

or $$H = \frac{p_\theta^2}{2ma^2} + \frac{p_z^2}{m} + \frac{1}{2}k(a^2 + z^2) \qquad ...(iii)$$

Hence the Hamilton's equations are

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \text{ or } p_z = m\dot{z} \quad \text{...}(iv)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ma^2} \text{ or } p_\theta = ma^2\dot{\theta} \quad \text{...}(v)$$

$$-\dot{p}_z = \frac{\partial H}{\partial z} = kz \text{ or } \dot{p}_z = -kz \quad \text{...}(vi)$$

$$-\dot{p}_\theta = \frac{\partial H}{\partial \theta} = 0 \text{ or } p_\theta = \text{constant} \quad \text{...}(vii)$$

From eqs. (*iv*) and (*vi*), we get

$$m\ddot{z} + kz = 0 \quad \text{...}(viii)$$

which shows that the motion of the particle in z direction is simple harmonic with period T, given by

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{...}(ix)$$

From eqs. (*v*) and (*vii*), we get

$$p_\theta = ma^2\dot{\theta} = \text{constant} \quad \text{...}(x)$$

Thus the angular momentum about Z-axis is a constant of motion.

3.8. ROUTHIAN

The Routhian is a potential function constructed out of the Lagrangian and plays a role somewhat in between the Lagrangian and the Hamiltonian. It is a function of mixed variables q_k, $\dot{q}_k$ and p_k, where the number of q_k coordinates is n, the number of degrees of freedom and the rest n velocity like independent ($\dot{q}_k$'s and p_k's) variables are shared in between $\dot{q}_k$'s and p_k's. When there are some cyclic coordinates in the Lagrangian, the construction of Routhian is useful. If the first s out of n coordinates are cyclic in L, then the Routhian for the system under consideration is defined as

$$R = R(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_s; \dot{q}_{s+1}, \ldots, \dot{q}_n; t)$$

$$= \sum_{k=1}^{s} p_k \dot{q}_k - L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t) \quad \text{...}(51)$$

where p_k, $k = 1, 2, \ldots, s$, are constants of motion.

Now,
$$dR = \sum_{k=1}^{n} \frac{\partial R}{\partial q_k} dq_k + \sum_{k=1}^{s} \frac{\partial R}{\partial p_k} dp_k + \sum_{k=s+1}^{n} \frac{\partial R}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial R}{\partial t} dt \quad \text{...}(52)$$

Also,
$$dR = \sum_{k=1}^{s} (p_k\, d\dot{q}_k + \dot{q}_k dp_k) - \sum_{k=1}^{n} \left(\frac{\partial L}{\partial q_k} dq_k + \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k \right) - \frac{\partial L}{\partial t} dt$$

or,
$$dR = -\sum_{k=s+1}^{n} p_k d\dot{q}_k + \sum_{k=1}^{s} \dot{q}_k dp_k - \sum_{k=1}^{n} \dot{p}_k\, dq_k - \frac{\partial L}{\partial t} dt \quad \text{...}(53)$$

Comparing eqs. (52) and (53) for dR, we have for the first s coordinates

$$\dot{q}_k = \frac{\partial R}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial R}{\partial q_k} \text{ for } k = 1,2,\ldots, s \qquad \ldots(54)$$

For the rest $n - s$ coordinates, *i.e.*, from $k = s + 1$ to $k = n$, we get

$$\dot{p}_k = -\frac{\partial R}{\partial q_k} \text{ and } p_k = -\frac{\partial R}{\partial \dot{q}_k} \qquad \ldots(55)$$

Substituting for p_k in the left equation of (55), we obtain

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_k}\right) - \frac{\partial R}{\partial q_k} = 0 \text{ for } k = s+1,\ldots,n \qquad \ldots(56)$$

For t variable, $\dfrac{\partial R}{\partial t} = -\dfrac{\partial L}{\partial t}$ ...(57)

Thus for the first s coordinates, which are supposed to be cyclic, eqs.(54) are similar to the Hamilton's equations of motion, where one has to replace H by R. They would conserve momenta $p_k (k = 1,2,\ldots, s)$ since R is cyclic in the corresponding $q_k (k = 1,2,\ldots,s)$. Therefore one may express R as

$$R = R(q_{s+1},\ldots,q_n ; \alpha_1,\alpha_2,\ldots,\alpha_s ; \dot{q}_{s+1},\ldots,\dot{q}_n ; t) \qquad \ldots(58)$$

where the momenta $p_1, p_2, \ldots, p_s$ are constants $\alpha_1, \alpha_2, \ldots, \alpha_s$ (which are to be determined from the initial conditions).

Rest of the $n - s$ coordinates ($k = s + 1, \ldots, n$) $q_{s+1},\ldots,q_n$ are not cyclic and only the variables in R are the $n - s$ coordinates and the corresponding $n - s$ generalized velocities. These coordinates satisfy in (56) Langrange like equations of motion in R instead of L. Thus the Routhian function R effectively behaves like a Lagrangian of a system, having the number of degrees of freedom $n - s$. In conclusion, in Routh's procedure, a problem with some cyclic and remaining non-cyclic coordinates may be solved in two steps :

(1) Solve Hamilton's equations for cyclic coordinates with the Routhian R as the Hamiltonian of the system, and

(2) Solve Lagrange's equations for non-cyclic coordinates with the Routhian as the Lagrangian of the system.

Ex. 1. *Find the Routhian for the Lagrangian L, given by*

$$L = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}, \quad where \quad \mu = \frac{mM}{m+M}.$$

Solution : $\quad R = \sum_{k=1}^{s} p_k \dot{q}_k - L$

Since θ is the cyclic coordinate,

$$R = p_\theta \dot{\theta} - \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{GMm}{r}$$

But $p_\theta = \dfrac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}, \therefore \dot{\theta} = \dfrac{p_\theta}{\mu r^2}$

Therefore, $\quad R = -\tfrac{1}{2}\mu \dot{r}^2 + \dfrac{p^2_\theta}{2\mu r^2} - \dfrac{GMm}{r}.$

The last two terms in R give the effective potential for r-motion.

Questions

1. Prove that the generalized momentum conjugate to a cyclic coordinate is conserved. Show that the theorems of conservation of linear and angular momentum are contained in this general theorem. **(Meerut 1995)**

2. The Lagrangian for a problems is

$$L = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - V(r)$$

Identify the cycle coordinates and the corresponding conservation law for the problem. **(Meerut 2001)**

3. Define generalized momentum and cycle coordinates. Show that the generalized momentum corresponding to a cyclic coordinate remains conserved. Hence prove the law of conservation of momentum for a system of particles. What is the relation between this law and symmetry properties of the system ? **(Agra 1991; Bundelkhand 1994)**

4. State and prove the conservation theorems for linear momentum, angular momentum and energy for a system of N particles. **(Agra 2003, 01)**

5. (*a*) What is a cyclic coordinate ? Illustrate with examples. **(Kanpur 1998)**

   (*b*) Whenever the Lergrangian function does not contain the coordinate q_k explicitly, the generalized momentum p_k is a constant of motion. Explain. **(Kanpur 2001)**

   (*c*) Prove that the total energy of the system is constant if for a conservative system, the Lagrangian does not depend explicitly on time.

6. (*a*) Define Hamiltonian H. Give its physical significance. **(Kanpur 1999; Rohilkhand 1995)**

   (*b*) Why is the Hamiltonian formulation is preferred over the Lagrangian formulation ? **(Meerut 2001, 1999)**

7. What is the Hamiltonian function ? Derive Hamilton's equations of motion for a system of particles. Hence write down the equations of motion of a particle in a central force field. **(Agra 1991)**

8. Derive Hamiltonian formalism and obtain Hamiltonian equations of motion. **(Garwal 1996)**

9. Derive Hamilton's canonical equations of motion. Obtain Hamilton's equation of motion for a particle moving in a central force field. **(Kanpur 1999; Meerut 2001, 1995)**

10. Show that both H and E are constants of motion but they are not equal to each other. Explain this inequality. Discuss the physical significance of $E = H$. **(Agra 1990)**

11. What is Hamiltonian function ? Explain its physical significance. Prove that the Hamiltonian H of a conservative system is equal to the total energy of the system. **(Agra 2003, 01, 1993)**

12. Establish Hamiltonian function for linear harmonic oscillator and the equation of the motion for it. **(Agra 1997, 92)**

13. Derive Lagrangian expression for a charged particle in an electromagnetic field. Hence obtain the Hamiltonian and an equation of motion for the same particle.. **(Agra 1995, 93)**

14. Derive Hamilton's canonical equations of motion. What is the physical significance of Hamiltonian function ? Obtain the Hamiltonian and Hamilton's equations of a charged particle in an electromagnetic field. **(Rohilkhand 1999; Agra 1991, 89; Meerut 94)**

15. Derive Hamiltonian function and equations of motion for a compound pendulum.

16. Write the Hamiltonian and Hamilton's equations for two dimensional isotropic harmonic oscillator in polar coordinates. **(Agra 1993)**

17. If the Hamiltonian H is independent of time t explicitly, prove that it is, (*a*) constant and (*b*) equal to the total energy of the system.

18. (*a*) Obtain Hamilton's equations in terms of polar coordinates. **(Agra 1998)**

(*b*) Obtain Hamiltonian function, equations of motion and expression for time period of a simple pendulum **(Agra 1998)**

Problems

[SET- I]

1. Write down the Hamiltonian and equation of motion for a simple pendulum.

Ans : $H = \dfrac{p_\theta^2}{2ml^2} + mgl\,(1-\cos\theta);\ \ddot{\theta} + \dfrac{g}{l}\sin\theta = 0$.

2. A particle of mass m is attracted towards a given point by a force of the form k/r^2, where k is a constant. Write down the expression for the Hamiltonian of the system and derive Hamilton's equations of motion.

Ans : $H = \dfrac{1}{2m}\left(p_r^2 + \dfrac{p_\theta^2}{r^2}\right) - \dfrac{k}{r};\ \dot{p}_r = \dfrac{p_\theta^2}{mr^3} - \dfrac{k}{r^2}\ ;\ \dot{p}_\theta = 0,\ \ \dot{r} = \dfrac{p_r}{m},\ \ \dot{\theta} = \dfrac{p_\theta}{mr^2}$.

3. Deduce the Hamiltonian function and equation of motion for a compound pendulum. What is the value of periodic time ? **(Agra 1994)**

Ans : $H = \dfrac{p_\theta^2}{2I} + mgl\,(1-\cos\theta)\,;\ddot{\theta} + \dfrac{mgl}{I}\theta = 0\ ;\ \ T = 2\pi\sqrt{\dfrac{I}{mgl}}$.

4. Consider the spherical pendulum as discussed in Fig. 2.14. Set up the Hamiltonian and show that p is constant of motion.

Ans : $H = \dfrac{1}{2mR^2}\,(p_\theta^{\ 2} + \operatorname{cosec}^2\theta\, p_\phi^{\ 2}) - mgR\cos\theta$.

5. (*a*) A particle is moving near the surface in the earth's gravitational field. Write down the Hamiltonian and equation of motion of the particle. (Neglect the earth's rotation.)

Ans : $H = \dfrac{1}{2m}\,(p_x^{\ 2} + p_y^{\ 2} + p_z^{\ 2}) + mgz\,;\ddot{x} = 0,\ \ddot{y} = 0,\ \ddot{z} = -g$ (assuming Z-direction vertically upward).

(*b*) Whenever the Lagransian function does not contain the coordinate q_k explicitly, the generalized momentum p_k is a constant of motion. Explain. **(Kanpur 2001)**

6. Obtain the Hamiltonian and Hamilton's equations for a projectile. Neglect the earth's rotation.

Ans : $H = \dfrac{1}{2m}\,(p_x^{\ 2} + p_y^{\ 2} + p_z^{\ 2}) + mgz\,;\dot{x} = \dfrac{p_x}{m},\ \dot{y} = \dfrac{p_y}{m},\ \dot{z} = \dfrac{p_z}{m}$

and $\dot{p}_x = 0,\ \dot{p}_y = 0,\ \dot{p}_z = -mg$ or $\ddot{x} = 0,\ \ddot{y} = 0,\ \ddot{z} = -g$.

7. Set up the Hamiltonian for the top whose Lagrangian is

$$L = \tfrac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \tfrac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - V$$

where θ, ϕ, ψ are the generalized coordinates, and I_1, I_3 are constants and V involves the coordinates only.

Ans : $H = \tfrac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \tfrac{1}{2}I_2\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 + V = \dfrac{p_\theta^2}{2I_1} + \dfrac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \dfrac{p_\psi^2}{2I_3} + V$.

8. If the Lagrangian of a particle in a gravitational field is expressed in cylindrical coordinates (r, θ, z) as

$$L = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mgz$$

where r is constrained to have a constant value. Show that (*i*) θ is a cyclic coordinate, (*ii*) the component of angular momentum along Z-direction is conserved, and (*iii*) the Hamiltonian is

$$H = m\dot{z}^2 + mr^2\dot{\theta}^2 + mgz$$

9. A particle of mass m moves in three dimensions under the action of the conservative force with potential energy $V(r)$. Using the spherical coordinates r, θ, ϕ ,obtain the Hamiltonian function for the system. Show that p_ϕ, $\frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r)$ and $p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}$ are the constants of motion.

10. A uniform bar of mass m and length $2l$ is suspended from one end by a spring of force constant K. The spring is constrained to remain vertical and the bar is constrained to remain in a vertical plane. Choose a set of generalized coordinates and obtain Lagrange's equations and Hamilton's equations of motion for the system. Show that they both reduce to the set of equations of motion.

Ans : Choose generalized coordinates (*i*) x, the downward displacement of the end of the spring from the equilibrium position, and (*ii*) θ, the angle the rod makes with the vertical.

$\dot{x} = (4l\,p_x + 3\sin\theta\, p_\theta) / (4ml - 3ml\sin^2\theta)$, $\dot{p}_x = -Kx$, $\dot{\theta} = (3l\sin\theta\, p_x + 3\,p_\theta)/(4ml^2 - 3ml^2\sin^2\theta)$,

$\dot{p}_\theta = ml\cos\dot{x}\dot{\theta} + mgl\sin\theta$.

11. A particle of mass m moves on the inside of a frictionless cone having equation $x^2 + y^2 = z^2\tan^2\alpha$. (*a*) Write the Hamiltonian and (*b*) Hamilton's equations using cylindrical coordinates.

Ans : (*a*) $\frac{p_r^2\sin^2\alpha}{2m} + \frac{p_\theta^2}{2mr^2} + mgr\cot\alpha$ (*b*) $\dot{r} = \frac{p_r\sin^2\alpha}{m}$, $\dot{p}_r = \frac{p_\theta^2}{mr^3} - mg\cot\alpha$

12. A mass m is suspended by means of a string which passes through a small hole in a table. The particle is set in motion in a vertical plane and the string is drawn up through the hole at a constant rate λ.

(*a*) Choosing the angle θ which the string makes with the vertical as the generalized coordinates, find the Hamiltonian function for the system.

(*b*) Is the Hamiltonian equal to the total energy ?

(*c*) Is the Hamiltonian function a constant of the motion ?

Ans : (*a*) $H = \frac{p_\theta^2}{2m(r_0 - \alpha t)^2} - mg(r_0 - \alpha t)\cos\theta - \frac{m\lambda^2}{2}$ (*b*) No, (*c*) No.

[SET- II]

1. The Hamiltonian of a system having two degrees of freedom is

$H = \frac{1}{2}(p_1^2 q_1^4 + p_2^2 q_1^2 - 2\alpha q_1)$

where α is a constant. Show that q_1 varies sinusoidally with q_2.

2. For a system with the Lagrangian $L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_1\dot{q}_2 + \dot{q}_2^2) - V(q)$, show that the Hamiltonian is

$H = \frac{2}{3}(p_1^2 - p_1 p_2 + p_2^2) + V(q)$.

3. The Lagrangian for anharmonic oscillator is given by

$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - \alpha x^3$

Find the Hamiltonian. **(Kanpur 2001)**

Ans : $H(p,x) = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 + \alpha x^3.$

4. If $x_k = \frac{1}{\sqrt{2}}(q_k + ip_k)$ and $\bar{x}_k = \frac{1}{\sqrt{2}}(q_k - ip_k)$, show that the Hamilton's equations of motion can be expressed in the form

$$\frac{dx_k}{dt} + i\frac{\partial H}{\partial \bar{x}_k} = 0, \text{ where } i = \sqrt{-1}.$$

5. Describe the motion of a particle with a Hamiltonian, given by

$$H(p, x) = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2 + \lambda\left(\frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2\right)^2$$

Ans : $x = a\cos(\omega t + \phi)$, $p = -\omega_0 a \sin(\omega t + \phi)$, where $\omega = (1 + 2\lambda E_0)\,\omega_0$, $E_0 = \frac{1}{2}\omega_0^2 a^2$.

6. Prove that the projection of an electron orbit in a uniform magnetic field onto a plane at right angles to **B** in coordinate space can be obtained by rotation and change of scale of the orbit in momentum space.

7. Find the Lagrangian for the case when the Hamiltonian is

$H(p, r) = \frac{p^2}{2m} - (\mathbf{a}\cdot\mathbf{p})$, $\mathbf{a}$ = constant.

Ans : $L = \frac{1}{2}m(\mathbf{v} - \mathbf{a})^2$

8. The Hamiltonian for a three dimensional harmonic oscillator is given by

$$H = \frac{1}{2}\sum_{i=1}^{3}(p_i^2 + Aq_i^2).$$

Show that $F_1 = q_2p_3 - q_3p_2$, $F_2 = q_3p_1 - q_1p_3$, $F_3 = q_1p_2 - q_2p_1$,
$G_1 = Aq_1\cos(At) - p_1\sin(At)$, $G_2 = Aq_2\cos(At) - p_2\sin(At)$,
$G_3 = Aq_3\cos(At) - p_3\sin(At)$ are the constants of motion.

9. Set up the Hamilton's equations for the Lagrangian

$$L(q, \dot{q}, t) = \frac{m}{2}(\dot{q}^2\sin^2\omega t + q\dot{q}\omega\sin 2\omega t + q^2\omega^2).$$

Ans : $\dot{q} = \frac{p}{2\,m\omega\sin^2\omega t}$ and $\dot{p} = p\omega\cot\omega t + m\omega^2 q$.

10. Find the Hamiltonian, if the Lagrangian L is

$L(\theta, z, \dot{\theta}, \dot{z}) = \frac{1}{2}m\left(l^2\dot{\theta}^2 - z\,l\,\dot{\theta}\,\dot{z}\sin\theta\right) + mgl\cos\theta + \frac{1}{2}\dot{z}^2 + mgz$

for a pendulum (l, θ) hung from the ceiling of a moving lift, the instantaneous position of the fulcrum being denoted by $z(t)$.

Ans : $H = \frac{(p + ml\dot{z}\sin\theta)^2}{2ml^2} - mgl\cos\theta - \frac{m\dot{z}^2}{2} - mgz.$

11. Determine the Routhian for the Lagrangian L, given by

$$L = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta\right)^2 - mgl \cos\theta .$$

Ans : $R = R\,(\theta, \dot{\theta}, p_\theta, p_\psi) = \dfrac{p_\psi^2}{2I_3} + \dfrac{\left(p_\phi - p_\psi \cos\theta\right)^2}{2I_1 \sin^2\theta} - \dfrac{1}{2} I_1 \dot{\theta}^2 + mgl \cos\theta$.

Objective Type Questions

1. Whenever the Lagrangian for a system does not contain a coordinate explicitly,
   (*a*) q_k is cyclic coordinate.
   (*b*) p_k is cyclic coordinate.
   (*c*) p_k, the generalized momentum, is a constant of motion.
   (*d*) q_k is always zero.
   Ans : (*a*), (*c*).

2. The dimensions of generalized momentum
   (*a*) are always those of linear momentum.
   (*b*) are always those of angular momentum,
   (*c*) may be those of linear momentum,
   (*d*) may be those of angular momentum.
   Ans : (*c*), (*d*).

3. The generalized momentum p_x of a particle of mass m with velocity v_x in an electromagnetic field is given by
   (*a*) $p_x = mv_x$
   (*b*) $p_x = mv_x + qA_x$
   (*c*) $p_x = mv_x - qA_x$
   (*d*) $p_x = qv_x A_x$
   Ans : (*b*).

4. Choose the correct statements :
   (*a*) The angular momentum is conserved for systems possessing rotational symmetry.
   (*b*) If the Lagrangian of a system is invariant under translation along a direction, the corresponding linear momentum is conserved.
   (*c*) If the Lagrangian of a system is invariant under translation along a direction, we can not say any thing about the corresponding linear momentum.
   (*d*) For a conservative system, the Hamiltonian is equal to the sum of kinetic and potential energies.
   Ans : (*a*), (*b*), (*d*) .

5. If the Lagrangian does not depend on time explicitly,
   (*a*) the Hamiltonian is constant.
   (*b*) the Hamiltonian can not be constant.
   (*c*) the kinetic energy is constant.
   (*d*) the potential energy is constant.
   Ans : (*a*).

6. The product of generalized coordinate and its conjugate momentum has the dimensions of
   (*a*) force
   (*b*) energy
   (*c*) linear momentum
   (*d*) angular momentum
   Ans. (*d*).

7. The Lagrangian of a particle of mass m moving in a plane is given by

$$L = \frac{1}{2} m \left(v_x^2 + v_y^2\right) + a \left(x\, v_y - y\, v_x\right)$$

where v_x and v_y are velocity components and a is a constant. The canonical momenta are given by

(a) $p_x = mv_x$ and $p_y = mv_y$ (b) $p_x = mv_x + ay$ and $p_y = mv_y + ax$

(c) $p_x = mv_x - ay$ and $p_y = mv_y + ax$ (d) $p_x = mv_x - ay$ and $p_y = mv_y - ax$

(Gate 2003)

Ans. (c)

8. Hamilton canonical equations of motion for a conservative system are

(a) $-\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$ and $-\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$ (b) $\frac{dp_i}{dt} = \frac{\partial H}{\partial p_i}$ and $\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$

(c) $-\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$ and $\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$ (d) $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$ and $-\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$ **(Gate 2002)**

Ans. (d)

9. The Lagrangian of a particle moving in a plane under the influence of a central potential is given by $L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$. The generalized momenta corresponding to r and θ are given by

(a) $m\dot{r}$ and $mr^2\dot{\theta}$ (b) $m\dot{r}$ and $mr\dot{\theta}$

(c) $m\dot{r}^2$ and $mr^2\dot{\theta}$ (d) $m\dot{r}^2$ and $mr^2\dot{\theta}^2$ **(Gate 2004)**

Ans. (a)

10. The Hamiltonian corresponding to the Langrangian $L = ax^2 + by^2 - kxy$ is

(a) $\frac{p_x^2}{2a} + \frac{p_y^2}{2b} + kxy$ (b) $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} - kxy$

(c) $\frac{p_x^2}{4a} + \frac{p_y^2}{4b} + kxy$ (d) $\frac{p_x^2 + p_y^2}{4ab} + kxy$ **(Gate 2004)**

Ans. (c)

Short Answer Questions

1. What is generalized momentum ?
2. What is cyclic or ignorable coordinate ?
3. Prove that the generalized momentum conjugate to a cyclic coordinate is conserved. **(Meerut 1995)**
4. What is the Hamiltonian function ?
5. Prove that the Hamiltonian H of a conservative system is equal to the total energy of the system. **(Agra 1999)**
6. Write the Hamilton's equations of motion.
7. Explain physical significance of Hamiltonian. **(Agra 2004, 03, 02)**
8. Whenever the Lagrangian function does not contain coordinate q_k explicitly, the generalized momentum p_k is a constant of motion. Explain. **(Kanpur 2001)**
9. What is Hamiltonian for a simple pendulum ? Obtain its equation of motion. **(Kanpur 2002)**

10. Obtain the Hamiltonian for an anharmonic oscillator, whose Lagrangian is given by

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2 x^2 - ax^3.$$ **(Kanpur 2001)**

Ans. $H(p, x) = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 + ax^3.$

11. Fill in the blanks :

(*i*) In absence of a given component of applied force, the corresponding component of linear momentum is.........

(*ii*) Whenever the Lagrangian function does not contain a coordinate q_k explicitly, the generalized momentum p_k is a ....of motion.

Ans. (*i*) Conserved, (*ii*) Constant.

CHAPTER

4 Two-Body Central Force Problem

4.1. REDUCTION OF TWO-BODY CENTRAL FORCE PROBLEM TO THE EQUIVALENT ONE-BODY PROBLEM

In this chapter, we plan to discuss the motion of two bodies under a mutual central force as an application of Lagrangian formulation. Consider a system of two particles of masses m_1 and m_2 whose instantaneous position vectors in an inertial frame with origin O_i are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively [Fig. 4.1]. Hence the vector distance of m_2 relative to m_1 is

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \qquad ...(1)$$

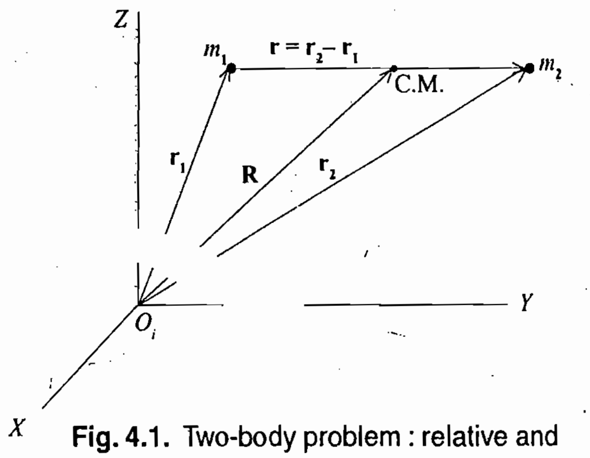


Fig. 4.1. Two-body problem : relative and centre of mass coordinates

The two masses are interacting via central force for which the potential energy for the system $V(r)$ is a function of scalar distance r only. The Lagrangian for the system is

$$L = T - V = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - V(r) \qquad ...(2)$$

This system of two particles has six degrees of freedom and hence six idependent generalized coordinates are required to describe the state of the system. Instead of $\mathbf{r}_1$ and $\mathbf{r}_2$ (six) coordinates, we can choose the three components of the position vector of the centre of mass $\mathbf{R}$, and three components of the relative vector $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$. The position vector of the centre of mass is defined by

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \qquad ...(3)$$

Solving (1) and (3), we get

$$\mathbf{r}_1 = \mathbf{R} - \frac{m_2 \mathbf{r}}{m_1 + m_2} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R} + \frac{m_1 \mathbf{r}}{m_1 + m_2} \qquad ...(4)$$

Therefore $$\dot{\mathbf{r}}_1 = \dot{\mathbf{R}} - \frac{m_2 \dot{\mathbf{r}}}{m_1 + m_2} \quad \text{and} \quad \dot{\mathbf{r}}_2 = \dot{\mathbf{R}} + \frac{m_1 \dot{\mathbf{r}}}{m_1 + m_2} \qquad ...(5)$$

Hence $$L = \frac{1}{2} m_1 \left(\dot{\mathbf{R}} - \frac{m_2 \dot{\mathbf{r}}}{m_1 + m_2} \right)^2 + \frac{1}{2} m_2 \left(\dot{\mathbf{R}} + \frac{m_1 \dot{\mathbf{r}}}{m_1 + m_2} \right)^2 - V(r)$$

or $$L = \frac{1}{2}(m_1 + m_2)\,\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\mathbf{r}}^2 - V(r) \qquad ...(6)$$

We see that the three coordinates **R** (*i.e.*, *X*, *Y*, *Z*) are cyclic and hence corresponding linear momentum $(m_1 + m_2)\,\dot{\mathbf{R}}$ or $\dot{\mathbf{R}}$, the velocity of centre of mass, is constant. This means that the centre of mass is either at rest or moving with constant velocity. Obviously the Lagrange's equations of motion for three generalized coordinates **r** will not contain the terms in **R** and $\dot{\mathbf{R}}$. Hence to discuss the motion of the system, we can drop the first term in the Lagrangian, eq. (6). Thus

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r) \qquad ...(7)$$

where $\mu = (m_1 m_2)/(m_1 + m_2)$ is called the ***reduced mass*** of the two-particle system. The form of the Lagrangian, represented by eq. (7), is exactly the same as that of a particle with mass μ moving at a vector distance **r** from a centre *O*. This centre exerts a central force on the particle and is taken as the origin of the coordinate system [Fig. 4.2] and appears to be fixed. Thus the two-body central force problem has been reduced to the equivalent one-body problem.

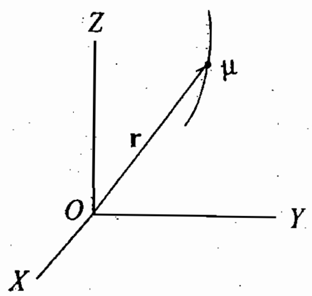


Fig. 4.2 : Reduction of two-body problem to one-body problem

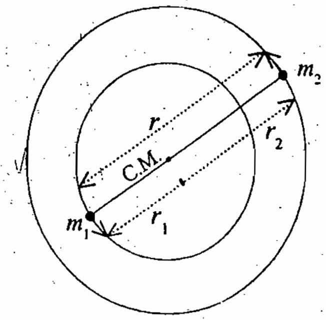


Fig.4.3

This is to be remembered that in fact, **r** is the vector distance of particle 2 relative to the particle 1, which is acting as the origin *O*, and therefore this origin *O* is moving with an acceleration in an inertial system. This means that the coordinate system attached to *O* is a non-inertial frame. However, one may not be aware of this fact just by having a look at the explicit form of the Lagrangian, represented by eq. (7). The actual paths of the two particles will depend upon the law of force, initial positions and initial velocities. One particular case may be of interest to mention. If force $F = \mu\omega^2 r$, the particle of mass μ will move on a circular path of radius r around *O* or m_2 particle will move on a circular path of radius r relative to m_1. However, in the inertial frame, fixed with the centre of mass, the two particles will move in circular orbits around their centre of mass with the same angular velocity. In this frame, r_1 and r_2 are the radii of their orbits and r is the distance between the two particles [Fig. 4.3].

In case, $m_1 >> m_2$, the reduced mass is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_2}{1 + \frac{m_2}{m_1}} \approx m_2 \text{ (say } m)$$

In such a case the problem becomes just one-particle problem. When high accuracy is not required, this approximation is good enough.

In general, if we are dealing with a two-particle central force problem, we need to solve the equivalent one-body problem, where a particle of mass *m* moves about a fixed centre of force with the Lagrangian, given by

$$L = \frac{1}{2} m\dot{r}^2 - V(r) \qquad ...(8)$$

wherever we need we may replace m by μ.

Examples of two-body problem are a planet-sun system, hydrogen atom (consisting of an electron revolving around a proton), positronium, any diatomic molecule like H_2, HCl etc.

Ex. 1. *Calculate the reduced mass of the following systems :*

Hydrogen atom, Positronium and H_2 *molecule.*

Solution : (*i*) In hydrogen atom, an electron of mass m revolves round a proton of much heavier mass M. The reduced mass of the hydrogen atom is given by

$$\mu_H = \frac{mM}{m+M} = m\left(1+\frac{m}{M}\right)^{-1} = m\left(1-\frac{m}{M}\right)$$

(using Bionomial expansion and neglecting higher order terms because $m/M << 1$)

But $\qquad \dfrac{m}{M} = \dfrac{1}{1836}$, hence $\mu_H = m\left(1-\dfrac{1}{1836}\right)$

Hence in case of hydrogen atom, if the energy, period etc. are calculated by assuming that an electron moves round a fixed proton, there will occur some error. To get the correct results m should be replaced by the reduced mass μ_H in the expression of energy, period etc. Actually, in case of hydrogen atom, the error will be very small, because

$$m\left(1-\frac{1}{1836}\right) \cong m \text{ nearly.}$$

(*ii*) ***Positronium*** is a temporary combination of a positron and an electron similar to hydrogen atom. A positron is a particle which has mass equal to the electron mass but it has equal positive charge. The reduced mass of the positronium is given by

$$\mu_p = \frac{mm}{m+m} = \frac{m}{2}$$

(since m = mass of electron = mass of positron).

Thus the reduced mass of the positronium particle is one-half the mass of the electron.

(*iii*) H_2 molecule consists of two hydrogen atoms separated by a distance and bound together by electromagnetic forces. If M is the mass of each hydrogen atom, then the reduced mass of H_2 molecule is given by

$$\mu_{H_2} = \frac{MM}{M+M} = \frac{M}{2}.$$

Ex. 2. *Show that the spectral lines of positronium are arranged in the same pattern as in the case of atomic hydrogen spectrum but have nearly double the wavelengths.*

Solution : According to Bohr's theory, the frequencies of lines in the hydrogen spectrum are given by

$$\nu = \frac{2\pi\mu_H e^4}{h^3}\left(\frac{1}{n^2}-\frac{1}{p^2}\right) \qquad ...(i)$$

where n and p are integers ($p > n$) and μ is the reduced mass of the hydrogen atom.

Now, $\mu_H = m\,[1-(1/1836)] \cong m$, mass of the electron ...(*ii*)

Thus in the case of hydrogen spectrum, the frequencies are directly proportional to the mass of the electron m ($\nu \propto \mu$ or m). Positronium has the structure like hydrogen atom, the expression (*i*) will also provide the spectral lines, rediated by positronium, but the reduced mass of hydrogen atom $\mu_H \equiv m$ is to be replaced by the reduced mass of the positronium $\mu_p = m/2$. Hence in case of positronium, we will get the spectral lines like that of hydrogen but their frequencies will be nearly half, *i.e.*, their wavelengths will be nearly double because

$$\frac{\nu_p}{\nu_H} = \frac{\mu_p}{\mu_H} = \frac{m/2}{m} = \frac{1}{2} \text{ or } \nu_p = \frac{\nu_H}{2} \qquad ...(iii)$$

Therefore,
$$\frac{c}{\lambda_p} = \frac{c}{2\lambda_H} \text{ or } \lambda_p = 2\lambda_H \qquad ...(iv)$$

where ν_H, ν_p and λ_H, λ_p are the frequencies and wavelengths of hydrogen and positronium respectively and c is the speed of light.

Thus we see that the *spectral lines of positronium are arranged in the same pattern as in case of hydrogen but have nearly double wavelengths.*

4.2. CENTRAL FORCE AND MOTION IN A PLANE

If a force acts on a particle in such a way that it is always directed towards or away from a fixed centre and its magnitude depends only upon the distance (r) from the centre, then this force is called ***central force***. Thus a central force is represented by

$$\mathbf{F} = f(r)\,\hat{\mathbf{r}} = f(r)\,\mathbf{r}/r \qquad ...(9)$$

where $f(r)$ is a function of distance r only and $\hat{\mathbf{r}} = \mathbf{r}/r$ is a unit vector along $\mathbf{r}$ from the fixed centre. The force is attractive or repulsive, if $f(r) < 0$ or $f(r) > 0$ respectively.

A central force is always a conservative force and if $V(r)$ is the potential energy, then

$$f(r) = -\frac{\partial V}{\partial r} \text{ or } \mathbf{F} = -\frac{\partial V}{\partial r}\frac{\mathbf{r}}{r} \qquad ...(10)$$

The potential energy for central force depends only on the distance r and hence the system possesses spherical symmetry. Thus any rotation about a fixed axis will not have any effect on the solution and hence an angle coordinate for rotation about a fixed axis must be cyclic. This results in the conservation of angular momentum of the system *i.e.*,

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = \text{constant (vector)} \qquad ...(11)$$

where $\mathbf{p}$ is linear momentum.

Taking dot product with $\mathbf{r}$ in eq. (11), we have

$$\mathbf{r} \cdot \mathbf{J} = \mathbf{r} \cdot (\mathbf{r} \times \mathbf{p}) = (\mathbf{r} \times \mathbf{r}) \cdot \mathbf{p} = 0 \qquad ...(12)$$

since in a scalar triple product the position of dot and cross are interchangeable and $\mathbf{r} \times \mathbf{r} = 0$.

Therefore, position vector $\mathbf{r}$ is always perpendicular to the constant $\mathbf{J}$ vector. This means that the ***motion of the particle under central force takes place in a plane*** and we can describe the instantaneous position of the particle in plane polar coordinates r and θ.

4.3. EQUATIONS OF MOTION UNDER CENTRAL FORCE AND FIRST INTEGRALS

Consider a particle of mass m moving about a fixed centre of force O. [Fig. 4.4]

The particle is moving under a central force

$$\mathbf{F} = f(r)\frac{\mathbf{r}}{r} = -\frac{\partial V}{\partial r}\frac{\mathbf{r}}{r},$$

where $V(r)$ is the potential energy.

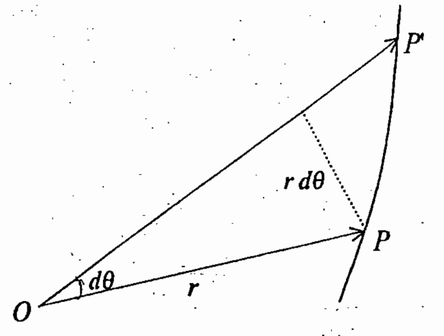


Fig. 4.4 : Area swept out by the radius vector in infinitesimal small time dt

Using polar coordinates (r, θ), the Lagrangian for the system can be written as

$$L = T - V = \frac{1}{2}m\,(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \qquad ...(13)$$

In eq. (13), the Lagrangian L is independent of θ coordinate (*i.e.* $\partial L/\partial\theta = 0$) and hence θ is the cyclic coordinate. The canonical momentum p_θ corresponding to the coordinate θ is given by

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta} \qquad ...(14)$$

which is the angular momentum.

Now, one of the equation of motion (Lagrange's equation for θ coordinate) is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \text{ or } \frac{d}{dt}(mr^2\dot{\theta}) = 0 \qquad ...(15)$$

Integration of this equation gives one of the ***first integral*** of motion. i.e,

$$mr^2\dot{\theta} = J \text{ (constant)} \qquad ...(16)$$

where J is the constant magnitude of the angular momentum and is conserved. Since m is a constant, we obtain from eq. (15)

$$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0 \text{ or } \frac{1}{2}r^2\dot{\theta} = h \text{ (constant)} \qquad ...(17)$$

The factor $\frac{1}{2}$ has been inserted above so that $\frac{1}{2}r^2\dot{\theta}$ represents the ***areal velocity*** (h), *i.e.*, the area swept out by the radius vector per unit time. It can be seen from Fig. 4.4 that the differential area dA swept out in time dt is

$$dA = \frac{1}{2}r\,(r\,d\theta) = \frac{1}{2}r^2 d\theta \text{ and so that } \frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} \text{ is the areal velocity.}$$

Thus from eq. (17) we see that the areal velocity is constant, when the motion is taking place under central force. This is in accordance with the well known *Kepler's second law of planetary motion*. In other words the conservation of angular momentum is equivalent to say that the areal velocity is constant.

Lagrange equation for r coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

From (13), $\quad \dfrac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $\dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 - \dfrac{\partial V}{\partial r}$

Therefore, $$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \qquad ...(18)$$

Putting from eq. (10), $\dfrac{\partial V}{\partial r} = -f(r)$ for central force, eq. (18) takes the form

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \qquad ...(19)$$

Also from (16), $\dot{\theta} = \dfrac{J}{mr^2}$, therefore

$$m\ddot{r} - \frac{J^2}{mr^3} = f(r) \qquad ...(20)$$

This is the second order differential equation in r coordinate only. Eq. (20) can also be written as

$$m\ddot{r} = \frac{J^2}{mr^3} - \frac{\partial V}{\partial r} = -\frac{1}{2}\frac{\partial}{\partial r}\left[\frac{J^2}{mr^2}\right] - \frac{\partial V}{\partial r}$$

or $$m\ddot{r} = -\frac{\partial}{\partial r}\left[\frac{1}{2}\frac{J^2}{mr^2} + V\right] \qquad ...(21)$$

Multiplying both sides of eq. (21) by $\dot{r}$, we get

$$m\ddot{r}\,\dot{r} = -\frac{\partial}{\partial r}\left[\frac{J^2}{2mr^2} + V\right]\dot{r}$$

or $$\frac{d}{dt}\left[\frac{1}{2}m\dot{r}^2\right] = -\frac{d}{dt}\left[\frac{J^2}{2mr^2} + V\right] \text{ or } \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V\right) = 0 \qquad ...(22)$$

Integrating it, we get

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V = E, \text{ constant} \qquad ...(23)$$

Since from (16), $J = mr^2\dot{\theta}$, therefore

$$\tfrac{1}{2}m\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + V(r) = E, \text{ constant or } \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E, \text{ constant} \qquad ...(24)$$

where E represents the total energy.

Thus, the sum of kinetic and potential energy *i.e.*, total energy E, is constant. This is *the statement of conservation of energy.*

Eq. (23) or eq. (24) is known as another ***first integral*** of motion.

4.4. DIFFERENTIAL EQUATION FOR AN ORBIT

In case of a central force, we want to deduce the equation of the orbit whose solution can give us the radial distance (r) as function of θ.

The equation of motion for a particle of reduced mass m, moving under central force, can be written as [from eq. (20)]

$$m\ddot{r} - \frac{J^2}{mr^3} = f(r) \qquad ...(25)$$

Now, $$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{J}{mr^2}\frac{dr}{d\theta} \quad \left[\text{as } \frac{J}{mr^2} = \dot{\theta} \text{ from eq. (16)}\right]$$

and $$\ddot{r} = \frac{d}{dt}\left[\frac{J}{mr^2}\frac{dr}{d\theta}\right] = \frac{d}{d\theta}\left[\frac{J}{mr^2}\frac{dr}{d\theta}\right]\frac{d\theta}{dt} = \frac{J}{mr^2}\frac{d}{d\theta}\left[\frac{J}{mr^2}\frac{dr}{d\theta}\right]$$

Let $$u = \frac{1}{r}, \text{ therefore, } \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}.$$

Then $$\ddot{r} = -\frac{J^2u^2}{m^2}\frac{d}{d\theta}\left[\frac{du}{d\theta}\right] = -\frac{J^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

Hence eq. (25) is $$-\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} - \frac{J^2u^3}{m} = f\left(\frac{1}{u}\right)$$

or $$\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right) \qquad ...(26)$$

This is the differential equation of an orbit, provided the force law $f(r) = f\left(\frac{1}{u}\right) = -\frac{\partial V}{\partial r}$ or the potential V is known.

4.5. INVERSE SQUARE LAW OF FORCE

Gravitational and coulomb force between two particles are the most important examples of central force. The force $f(r)$, as usual, is expressed as

$$f(r) = -\frac{Gm_1m_2}{r^2} \quad \text{(Newton's law of Gravitation)} \qquad ...(27)$$

and $$f(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2} \quad \text{(Coulomb's law)} \qquad ...(28)$$

The general force law, governing eqs. (27) and (28), is the inverse square law of force, given by

$$f(r) = -\frac{K}{r^2} \qquad ...(29)$$

If V is the potential, then

$$f(r) = -\frac{\partial V}{\partial r} = -\frac{K}{r^2}$$

and its integration gives

$$V = -\frac{K}{r} \qquad ...(30)$$

where the constant of integration is taken to be zero by assuming $V(r) = 0$ at infinite separation ($r = \infty$).

4.6. KEPLER'S LAWS OF PLANETARY MOTION AND THEIR DEDUCTION

The motion of planets has been a subject of much interest for astronomers from very early times. Kepler's laws of planetary motion are as follows :

(1) **The law of elliptical orbits :** Every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.

(2) **The law of areas :** The radius vector, drawn from the sun to a planet, sweeps out equal areas in equal times *i.e.*, the areal velocity of the radius vector is constant.

(3) **The harmonic law :** The square of the period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the ellipse.

Kepler's laws have been enunciated purely on the basis of observations, taken for the motion of the planets. These laws give us a simple and accurate description of their motions, but do not offer any explanation.

The planets move around the sun under the influence of gravitational force which is an inverse square law force. Hence we deduce the Kepler's laws of planetary motion around the sun on the basis of inverse square law of force.

4.6.1. Deduction of the Kepler's First Law

For $u = 1/r$, the inverse square law force [$f(r) = -K/r^2$] is given by

$$f\left(\frac{1}{u}\right) = -Ku^2$$

Thus the differential equation (26) of the orbit can be expressed as

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{J^2u^2} Ku^2 \quad [\because f(r) = -Ku^2]$$

or $$\frac{d^2u}{d\theta^2} + u - \frac{mK}{J^2} = 0 \qquad ...(31)$$

Let $$x = u - \frac{mK}{J^2}.$$

Then $$\frac{d^2x}{d\theta^2} + x = 0 \qquad ...(32)$$

which has the solution

$$x = A\cos(\theta - \theta') \qquad ...(33)$$

where A and θ' are the constants of integration.

Since $x = u - \frac{mK}{J^2}$ and $u = \frac{1}{r}$, we can write eq. (33) as

$$\frac{1}{r} - \frac{mK}{J^2} = A\cos(\theta - \theta')$$

or $$\frac{1}{r} = \frac{mK}{J^2} + A\cos(\theta - \theta') \qquad ...(34)$$

or $$\frac{J^2/mK}{r} = 1 + \frac{J^2 A}{mK}\cos(\theta - \theta')$$

or $$\frac{l}{r} = 1 + e\cos(\theta - \theta') \quad ...(35)$$

where $$\frac{J^2}{mK} = l \text{ and } \frac{J^2 A}{mK} = e$$

It is easy to identify $e = \sqrt{1 + \frac{2EJ^2}{mK^2}}$ with the help of the two first integrals of motion*. The constant θ' appearing in eq. (33) is a constant of integration determined by initial conditions.

Thus, when a particle is moving under inverse square law of force, its orbit is represented by eq. (35). This is the general equation of a conic section with one focus at the origin and eccentricity e, given by

$$e = \sqrt{1 + \frac{2EJ^2}{mK^2}} \quad ...(36)$$

The magnitude of e decides the nature of the orbit, represented by eq. (35), as following :

Value of e	*Value of total energy E*	*Conic*
$e > 1$	$E > 0$	Hyperbola
$e = 1$	$E = 0$	Parabola
$e < 1$	$E < 0$	Ellipse
$e = 0$	$E = -\frac{mK^2}{2J^2}$	Circle

* Differentiating eq. (34), we get

$$-\frac{\dot{r}}{r^2} = -A\sin(\theta - \theta')\dot{\theta} = -A\sin(\theta - \theta')\frac{J}{mr^2} \text{ or, } \dot{r} = \frac{AJ}{m}\sin(\theta - \theta')$$

where we have put $\dot{\theta} = J/mr^2$ from one of the first integrals. But from another first integral [eq. (23)]

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{K}{r} = E \qquad \left(\text{Here, } V(r) = -\frac{K}{r}\right)$$

Thus $$\frac{1}{2}m\frac{A^2J^2}{m^2}\sin^2(\theta - \theta') + \frac{J^2}{2m}\left[\frac{mK}{J^2} + A\cos(\theta - \theta')\right]^2$$
$$-K\left[\frac{mK}{J^2} + A\cos(\theta - \theta')\right] = E$$

or $$\frac{A^2J^2}{2m} = E + \frac{mK^2}{2J^2} \quad \text{or} \quad A = \frac{mK}{J^2}\sqrt{1 + \frac{2J^2E}{mK^2}}$$

Therefore, $$e = \frac{J^2 A}{mK} = \sqrt{1 + \frac{2EJ^2}{mK^2}}.$$

Thus the sign of the total energy E tells us about the nature of the orbit. If the total energy E is less than zero or negative, the orbit is an ellipse. This is actually the case for planetary motion, when a planet is moving around the sun under gravitational force. This is known as ***Kepler's first law**, according to which every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.* However, the result does not consider the effect of the presence of the other planets.

4.6.2. Deduction of Kepler's Second Law

According to this law, the radius vector drawn from the sun to a planet, sweeps out equal areas in equal times *i.e., the areal velocity is constant in planetary motion.* A gravitational force is a central force and this law is the same, as given by the statement of conservation of angular momentum in a central force field in eq. (16) or eq. (17) *i.e.,*

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{J}{2m}, \text{ a constant} \qquad ...(37)$$

The inverse square law force or gravitational force in planetary motion is a special case of central force. However, the constancy of areal velocity is a general property taking place under central force.

4.6.3. Deduction of Kepler's Third Law (*Period of motiom in an elliptical Orbit*)

For $e < 1$ or $E < 0$, the orbit is elliptical one, given by [eq. (35)] :

$$\frac{l}{r} = 1 + e \cos(\theta - \theta')$$

where $$l = \frac{J^2}{mK} \text{ and } e = \sqrt{1 + \frac{2EJ^2}{mK^2}}$$

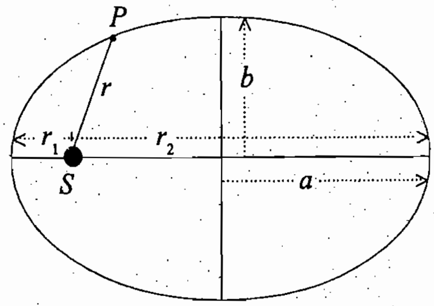


Fig. 4.5 : Motion of a planet around sun

When $\theta - \theta' = 0$ or $\cos(\theta - \theta') = 1$, the value of $r = r_1$ is minimum and when $\theta - \theta' = \pi$ or $\cos(\theta - \theta') = -1$, the value of $r = r_2$ is maximum [Fig. 4.5]. The apsidal distances r_1 and r_2 are known as *perihelion* and *aphelion* and are given by

$$\frac{l}{r_1} = 1 + e \text{ or } r_1 = \frac{l}{1+e} \qquad ...(38)$$

and $$\frac{l}{r_2} = 1 - e \text{ or } r_2 = \frac{l}{1-e} \qquad ...(39)$$

The semimajor axis (a) of the ellipse is one-half the sum of these two apsidal (turning) distances, *i.e.,*

$$a = \frac{r_1 + r_2}{2} = \frac{1}{2}\left[\frac{l}{1+e} + \frac{l}{1-e}\right] = \frac{l}{1-e^2} \qquad ...(40a)$$

or $$a = -\frac{J^2}{mK} \cdot \frac{mK^2}{2EJ^2} = -\frac{K}{2E} \left[\text{from (36), } 1 - e^2 = -\frac{2EJ^2}{mK^2}\right] \qquad ...(40b)$$

or $$E = -\frac{K}{2a} \qquad ...(40c)$$

Thus in case of an elliptical orbit, the total energy depends solely on the major axis.

If T be the periodic time in which the particle or radius vector completes one revolution then the area of the orbit is obtained by using eq. (37) *i.e.*,

$$A = \int_0^A dA = \int_0^T \left(\frac{1}{2} r^2 \dot{\theta}\right) dt = \int_0^T \frac{J}{2m} dt = \frac{JT}{2m} \qquad ...(41)$$

But area of the ellipse $A = \pi ab$, ...(42)

where a and b are the semi-major and semi-minor axes of the ellipse respectively.

Equating (41) and (42), we get

$$T = \frac{2\pi abm}{J} \qquad ...(43)$$

But according to the property of the ellipse

$$b = a\sqrt{1-e^2} = a\sqrt{\frac{J^2}{mKa}} = a^{\frac{1}{2}} \frac{J}{\sqrt{mK}} \qquad ...(44)$$

because from (40 a), $a = \dfrac{l}{1-e^2} = \dfrac{J^2}{mK(1-e^2)}$ or $1-e^2 = \dfrac{J^2}{mKa}$.

Therefore, from eqs. (43) and (44), we obtain

$$T = \frac{2\pi am}{J} \cdot \frac{a^{1/2} J}{\sqrt{mK}} \text{ or } T = 2\pi a^{3/2} \sqrt{\frac{m}{K}} \qquad ...(45)$$

This gives the **periodic time** in an elliptical orbit.

Squaring both sides of eq. (45). we get

$$T^2 = 4\pi^2 a^3 \frac{m}{K} \text{ or } T^2 \propto a^3 \qquad ...(46)$$

Thus the square of period of revolution of a planet around the sun is proportional to the cube of semi-major axis of the elliptical orbit. This is known as ***Kepler's third law of planetary motion.***

It is to be reminded that the motion of a planet around the sun is a two body problem and hence in eq. (45), m is to be replaced by the reduced mass μ of the two-body system, *i.e.*,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad ...(47)$$

where m_1 may be taken to be the mass of the sun and m_2 that of the planet. Also, the force law is

$$f(r) = -\frac{Gm_1 m_2}{r^2} = -\frac{K}{r^2}$$

so that

$$K = Gm_1 m_2 \qquad ...(48)$$

Therefore eq. (45) is

$$T = \frac{2\pi a^{3/2}}{\sqrt{G(m_1 + m_2)}} \text{ or } T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)} \qquad ...(49)$$

Thus we obtain a more correct statement of the third law *i.e.*, the square of the periods of various planets are proportional to the cube of their respective semi-major axes with different proportionality constants, because for each planet, proportionality constant is different. If we neglect the mass (m_2) of a planet compared to the mass of the sun (m_1), we obtain the same proportionality constant for all the planets *i.e.*,

$$T^2 = \frac{4\pi^2}{Gm_1} a^3 \qquad ...(50)$$

or

$$T^2 = 4\pi^2 \frac{m_2}{K} a^3 \; [\because K = Gm_1 m_2] \qquad ...(51)$$

This is an approximate relation and in fact, this is Kepler's third law of planetary motion. For example, $m_2/m_1 = 0.1\%$ for Jupiter and $m_2/m_1 = 3\times10^{-4}\%$ for earth. This means that proportionality constant in (49) is different for different planets. However, in case of the Bohr model of atom, all revolving electrons are identical in mass and hence μ and K are the same for all orbits in an atom. This means that proportionality constant is the same for different electrons in different orbits. Therefore Kepler's third law is very much true for the electron orbits in the Bohr atom.

If an electron of charge $-e$ is moving around a nucleus of charge $+Ze$ (Z = atomic number), the central force acting on the electron is

$$F(r) = -\frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r^2} = -\frac{K}{r^2} \left(K = \frac{Ze^2}{4\pi\epsilon_0}\right)$$

If the total energy $E < 0$, the electron will move in an elliptical orbit with periodic time T, given by

$$T^2 = 4\pi^2 \frac{a^3\mu}{K} = \frac{4\pi^2 a^3 \mu}{Ze^2}(4\pi\epsilon_0)$$

or

$$T = \frac{4\pi}{e}\sqrt{\frac{\pi\epsilon_0 \mu a^3}{Z}}$$

where $\mu = \dfrac{mM}{m+M}$ (m = electronic mass and M = mass of nucleus).

4.7. STABILITY AND CLOSURE OF ORBIT UNDER CENTRAL FORCE

When a particle is moving under central force, total energy E is conserved and is given by [eq. (23)]

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J^2}{mr^2} + V(r)$$

This energy integral can be reduced to a one dimensional motion in r, if an effective potential is defined by

$$V_e(r) = V(r) + \frac{1}{2}\frac{J^2}{mr^2} \qquad ...(52)$$

Hence

$$E = \frac{1}{2}m\dot{r}^2 + V_e(r) \qquad ...(53)$$

This equation suggests that the radial kinetic energy is $\frac{1}{2}m\dot{r}^2$ and the effective potential energy for the radial motion is $V_e(r)$. The later is composed of two parts : (*i*) $V(r)$, the actual potential energy, and (*ii*) $V_{cf}(r) = \frac{1}{2}\frac{J^2}{mr^2} = \frac{1}{2}mr^2\dot{\theta}^2$, the centrifugal energy for radial motion. Obviously the force acting on the particle is composed of a central force and a centrifugal force, because from eq. (19)

$$m\ddot{r} = f(r) + mr\dot{\theta}^2 \qquad ...(54)$$

Suppose the central force is attractive. For example, for gravitational force, $V(r) = -K/r$, which is negative, varying as $-1/r$ and plotted in Fig. 4.6. The centrifugal potential energy $V_{cf}(r)$ increases as $1/r^2$ for $r \to 0$. The plot of the resultant $V_e(r)$ can have a minimum with a finite negative value, thus allowing a range of bounded orbits. In fact, a motion is called bounded in r, if $\dot{r}$ vanishes at the extreme values of r, say $r = r_{max}$ and $r = r_{min}$. For a bounded motion, both of these bounds must exist. Thus for $r = r_{max}$ and $r = r_{min}$, from eq. (53)

$$E = V_e(r)$$

Also from (53), we note that

$$E - V_e(r) = \frac{1}{2}m\dot{r}^2 \geq 0 \text{ for all } r.$$

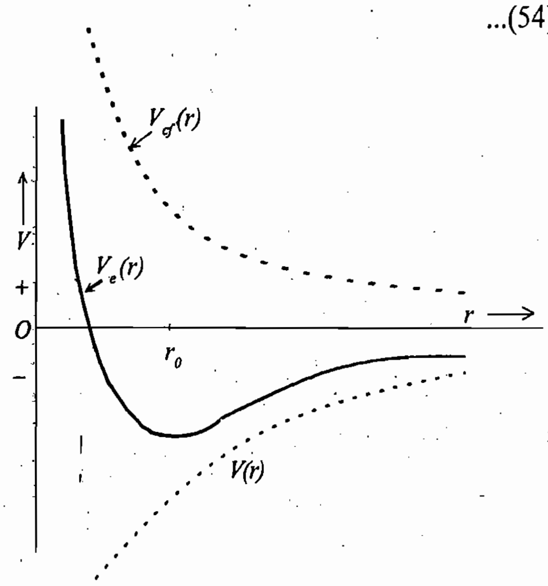


Fig. 4.6 : Effective potential for gravitational force [$V(r) = -K/r$]

Therefore for any possible radial motion, we must have

$$E \geq V_e(r) \qquad ...(55)$$

In other words, some part of the $V_e(r)$ curve must lie below the curve $V_e(r) = E$ for any radial motion.

The path of a particle moving under central force is called an **orbit**. The condition for stability in the radial motion is given by

$$\frac{dV_e}{dr} = 0 \text{ and } \frac{d^2V_e}{dr^2} > 0 \text{ at } r = r_0 \qquad ...(56)$$

If the potential energy function for the central force is of the form $V(r) = a\,r^{n+1}$, a being a constant, and centrifugal energy $V_{cf}(r) = br^{-2}$, ($b > 0$, a constant), then

$$V_e(r) = ar^{n+1} + br^{-2} \qquad ...(57)$$

The condition for stability in the radial motion is

$$\frac{dV_e}{dr} = 0 = (n+1)\,ar^n - 2br^{-3} \text{ at } r = r_0.$$

Thus $\qquad (n+1)\,a = 2br_0^{-(n+3)}$

Therefore $\qquad \left[\frac{d^2V_e}{dr^2}\right]_{r=r_0} = 2br_0^{-4}\,(n+3).$

Thus any **circular orbit** with $r = r_0$ under a central force is stable, if $(n+3) > 0$ or $n > -3$ and the form of the central force is

$$f(r) = -\frac{\partial V}{\partial r} = -(n+1)\,ar^n \text{ or } f(r) = -Kr^n \qquad ...(58)$$

Further an orbit is said to be ***closed*** if the particle eventually retraces its path. The **stable** and **closed orbits** (circular and non-circular) are possible for $n = 1$ and $n = -2$ and the corresponding force laws are as follows :

For $n = 1$, $\quad f(r) = -Kr \quad$ *Hooke's law* ...(59)

For $n = -2$, $\quad f(r) = -K/r^2 \quad$ *Inverse square law* ...(60)

Thus the condition for bounded motion is that there is bounded region of r in which the total energy $E \geq V_e(r)$, effective potential energy. The condition for stability of circular orbit is $n > -3$ with the force law $f(r) = -Kr^n$. Further the orbits are closed only for the inverse square law of force ($n = -2$) of the Coulombian or Newtonian type and for the linear law of force ($n = 1$) of Hooke's type.

From eq. (23), we have

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - V(r) - \frac{J^2}{2mr^2}\right)}$$

Now $\quad \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{J}{mr^2} \quad$ or $\quad \frac{dr}{d\theta} = \frac{mr^2}{J}\dot{r}.$

Thus $\quad \frac{dr}{d\theta} = \sqrt{\frac{2mr^4}{J^2}[E - V(r)] - r^2} \quad$ or $\quad d\theta = \frac{dr}{\sqrt{\frac{2mr^4}{J^2}[E - V(r)] - r^2}}$

Its solution is

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2m}{J^2}\left[E - V(r) - \frac{1}{r^2}\right]}} \quad \text{or} \quad \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\frac{2m}{J^2}\left[E - V\left(\frac{1}{u}\right)\right] - u^2}} \qquad ...(61)$$

where $u = 1/r$.

If E, J and the form of the central force potential $V(r)$ are given, the orbit is fixed. u_0 and θ_0 refer to the starting point on the orbit. For $V(r) = ar^{n+1}$, the above integral can be directly integrated for $n = 1, -2, -3$. For $n = 5, 3, 0, -4, -5$ and -7, the results can be obtained in terms of elliptical integrals. The equation of the orbit is not obtainable in the closed form for other values of n.

Ex. 1. *Use Hamilton's equation to find the differential equation for planetary motion and prove that the areal velocity is constant.* **(Agra 1992)**

Solution : Lagrangian $L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{r}$

Hamiltonian $\quad H = p_r\dot{r} + p_\theta\dot{\theta} - L \quad [\because H = \sum_k p_k\dot{q}_k - L]$

Here, $\quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$ and $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

Therefore, $\quad H = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} - \frac{K}{r}$ or, $H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{K}{r}$

Hamilton's equations are

$$\dot{r} = \frac{\partial H}{\partial p_r}, -\dot{p}_r = \frac{\partial H}{\partial r}, \dot{\theta} = \frac{\partial H}{\partial p_\theta} \text{ and } -\dot{p}_\theta = \frac{\partial H}{\partial \theta}$$

Therefore, $$\dot{r} = \frac{p_r}{m}, -\dot{p}_r = -\frac{p_\theta^2}{mr^3} + \frac{K}{r^2}, \dot{\theta} = \frac{p_\theta}{mr^2} \text{ and } -\dot{p}_\theta = 0.$$

From last two equations, p_θ = constant $= mr^2\dot{\theta}$ or $\frac{1}{2}r^2\dot{\theta}$ = constant. This proves that the areal velocity is constant in planetary motion. From the first two equations

$$-m\ddot{r} = -\frac{p_\theta^2}{mr^3} + \frac{K}{r^2} \text{ or } -m\ddot{r} = -\frac{(mr^2\dot{\theta})^2}{mr^3} + \frac{K}{r^2}, \text{ whence } \ddot{r} = r\dot{\theta}^2 - \frac{K}{mr^2} \quad ...(i)$$

Suppose $$r^2\dot{\theta} = h \text{ and } u = \frac{1}{r}.$$

Then $$\dot{\theta} = \frac{h}{r^2} = hu^2, \ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2} \text{ (as done earlier)}$$

Hence the equation of planetary motion (*i*) is

$$-h^2u^2\frac{d^2u}{d\theta^2} = \frac{1}{u}h^2u^4 - \frac{K}{m}u^2 \text{ or } \frac{d^2u}{d\theta^2} + u = \frac{K}{mh^2}$$

which is the equation of planetary motion with $u = \frac{1}{r}$ and $V = -\frac{K}{r}$.

Ex. 2. *A particle of mass m moves under the action of central force whose potential is* $V(r) = Kmr^3$ $(K > 0)$, *then*

(*i*) *For what kinetic energy and angular momentum will the orbit be a circle of radius R about the origin ?*

(*ii*) *Calculate the period of circular motion.* **(Agra 1992)**

Solution : $$V(r) = Kmr^3$$

Therefore, $$F = -\frac{\partial V}{\partial r} = -3Kmr^2$$

(*i*) For circular motion $F = -\frac{mv^2}{r} = -3Kmr^2$

Kinetic energy $\frac{1}{2}mv^2 = \frac{3}{2}Kmr^3$

Angular momentum = $mvr = mr\,(3Kr^3)^{1/2}$ [because $v^2 = 3Kr^3$]

(*ii*) Since $v = r\omega = \frac{2\pi r}{T}$, therefore from eq. $v^2 = 3Kr^3$, we have

$$\left[\frac{2\pi r}{T}\right]^2 = 3Kr^3 \text{ or } \frac{4\pi^2}{T^2} = 3Kr, \text{ whence } T = \frac{2\pi}{\sqrt{3Kr}}.$$

Ex. 3. *The eccentricity of the earth's orbit is* 0.0167. *Calculate the ratio of maximum and minimum speeds of the earth in its orbit.* **(Agra 1991)**

Solution : From eqs. (38) and (39), the ratio of the minimum value of r, *i.e.*, r_1 (perihelion) and the maximum value of r, *i.e.*, r_2 (aphelion) is given by

$$\frac{r_2}{r_1} = \frac{1+e}{1-e} \qquad ...(i)$$

However, the constancy of angular momentum [mvr] at the two apsidal (turning) points gives

$$mv_1r_1 = mv_2r_2 \text{ or } \frac{v_1}{v_2} = \frac{r_2}{r_1} \qquad ...(ii)$$

Obviously when the radius vector has the minimum value r_1, the speed of the planet (earth) v_1 will be maximum and when it is having maximum value r_2, the speed of the earth v_2 will be minimum. Thus the desired ratio is

$$\frac{v_1}{v_2} = \frac{r_2}{r_1} = \frac{1+e}{1-e} = \frac{1+0.0167}{1-0.0167} = \mathbf{1.03.}$$

Ex. 4. *The maximum and minimum velocities of a satellite are* v_{max} *and* v_{min} *respectively. Prove that the eccentricity of the orbit of the satellite is*

$$e = \frac{v_{max} - v_{min}}{v_{max} + v_{min}}.$$

(Agra 1998, 95, 93)

Solution : As shown in Ex. 3,

$$\frac{v_{max}}{v_{min}} = \frac{1+e}{1-e}, \quad \therefore \quad e = \frac{v_{max} - v_{min}}{v_{max} + v_{min}}.$$

Ex. 5. *A particle of mass m is observed to move in a spiral oribit given by the equation* $r = C\theta$, *where C is a constant. Is it moving in a central force field ? If it is so, find the force law.*

Solution : The differential equation of the orbit [eq.(26)] is given by

$$\frac{d^2u}{d\theta^2} + u = -\frac{mf\left(\frac{1}{u}\right)}{J^2u^2} \qquad ...(i)$$

Since $u = 1/r$ and $r = C\theta$ (given),

hence $$u = \frac{1}{C\theta} \qquad ...(ii)$$

Differentiating it, we get

$$\frac{du}{d\theta} = -\frac{1}{C\theta^2} \text{ and } \frac{d^2u}{d\theta^2} = \frac{2}{C\theta^2} = 2C^2u^3 \qquad ...(iii)$$

Substituting in eq. (*i*) , we get

$$2C^2u^3 + u = -\frac{m}{J^2u^2} f\left(\frac{1}{u}\right)$$

Therefore, $$f\left(\frac{1}{u}\right) = -\frac{J^2}{m}(u^3 + 2C^2u^5) \text{ or } f(r) = -\frac{J^2}{m}\left[\frac{1}{r^3} + \frac{2C^2}{r^5}\right] \qquad ...(iv)$$

Thus the particle is moving under a central force field and the force law is given by eq. (*iv*).

Ex. 6. *A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle. Show that the force varies as the inverse fifth power of the distance.*

(Rohilkhand 1983; Meerut 83)

Solution : Let a be the radius of circle, described by the particle P [Fig. 4.7]. If (r, θ) are the polar coordinates of P, then

$$r = 2a \cos\theta \qquad ...(i)$$

Differential equation of the orbit is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{J^2u^2} f\left(\frac{1}{u}\right)$$

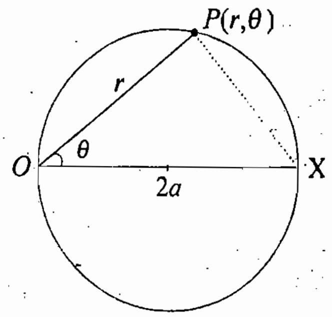


Fig. 4.7

Here, $u = \dfrac{1}{r} = \dfrac{1}{2a\cos\theta}$ or $u = \dfrac{\sec\theta}{2a}$

and hence $\dfrac{du}{d\theta} = \dfrac{1}{2a} \sec\theta \tan\theta$ and $\dfrac{d^2u}{d\theta^2} = \dfrac{1}{2a}[\sec^3\theta + \sec\theta \tan^2\theta]$

Now,
$$f\left(\frac{1}{u}\right) = \frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -\frac{J^2u^2}{2am}[\sec\theta + \sec^3\theta + \sec\theta \tan^2\theta]$$

$$= -\frac{J^2u^2}{2am}[\sec\theta + \sec^3\theta + \sec\theta(\sec^2\theta - 1)]$$

$$= -\frac{J^2u^2}{m} \cdot \frac{2\sec^3\theta}{2a} = -\frac{J^2u^2}{m} \cdot 8a^2u^3 = -\frac{8J^2a^2}{m} \cdot \frac{1}{r^5}$$

Thus
$$f(r) = -\frac{8J^2a^2}{m} \cdot \frac{1}{r^5} \text{ or } f(r) \propto \frac{1}{r^5}$$

Thus the force varies as the inverse fifth power of the distance.

Ex. 7. *A particle, moving in a central force field located at $r = 0$, describes a spiral $r = e^{-\theta}$. Prove that the magnitude of force is inversely proportional* to r^3.

Solution : Here, $r = e^{-\theta}$ and therefore, $u = \dfrac{1}{r} = e^{\theta}$

Differential equation of orbit is

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{J^2u^2} f\left(\frac{1}{u}\right)$$

Substituting $u = e^{\theta}$, we get

$$e^{\theta} + e^{\theta} = -\frac{m}{J^2} e^{-2\theta} f\left(\frac{1}{u}\right) \text{ or } f\left(\frac{1}{u}\right) = -2 \cdot \frac{J^2}{m} e^{3\theta}$$

or
$$f(r) = -\frac{2J^2}{m} \frac{1}{r^3} \text{ i.e., } f(r) \; \alpha \; \frac{1}{r^3}.$$

Ex. 8. *A particle describes a conic* $r = \dfrac{p}{1+e\cos\theta}$, *where p and e involve constant quantities. Show that the force under which the particle is moving is a central force. Deduce the force law.* **(Agra 1992)**

Solution : Since the particle is moving on a curved path in a plane about a centre of force, its angular momentum remains constant *i.e.*,

$$J = mvr = mr^2\dot{\theta} = \text{constant}$$

Therefore,
$$r^2\dot{\theta} = \frac{J}{m} = \text{a constant} \qquad ...(i)$$

The particle is moving on a curved path, hence it is obvious from Newton's first law of motion that a force is acting on it. Consequently an acceleration is acting on the particle. Resolving this acceleration into its two components, along and perpendicular to the radius vector, we have

(1) Component f, along the radius vector, *i.e.*, the radial acceleration of the planet is given by

$$f = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \qquad ...(ii)$$

(2) Component f', perpendicular to the radius vector, *i.e.*, the transverse acceleration of the planet is given by

$$f' = \frac{1}{r}\frac{d}{dt}\left[r^2\frac{d\theta}{dt}\right]$$

But from eq. (*i*), $r^2\dfrac{d\theta}{dt}$ = a constant.

Hence $f' = 0$

Thus the planet has no transverse acceleration and only the radial acceleration is acting on it *i.e.*, the force on the planet is directed towards the centre *i.e.*, it is a central force.

From eq. (*i*) we have

$$\frac{d\theta}{dt} = \frac{J}{mr^2} = \frac{J}{m}u^2 \qquad ...(iii)$$

where $u = 1/r$.

Now, $\dfrac{dr}{dt} = \dfrac{d}{dt}\dfrac{1}{u} = -\dfrac{1}{u^2}\dfrac{du}{dt} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}\dfrac{d\theta}{dt}$ or $\dfrac{dr}{dt} = -\dfrac{1}{u^2}\dfrac{du}{d\theta}\cdot\dfrac{Ju^2}{m}$ or $\dfrac{dr}{dt} = -\dfrac{J}{m}\dfrac{du}{d\theta}$

$$\therefore \qquad \frac{d^2r}{dt^2} = -\frac{J}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -\frac{J^2u^2}{m}\frac{d^2u}{d\theta^2} \qquad ...(iv)$$

Substituting the values of $\dfrac{d\theta}{dt}$ and $\dfrac{d^2r}{d\theta^2}$ from eqs. (*iii*) and (*iv*) in eq.(*ii*), we get

$$f = -\frac{J^2u^2}{m}\left[u + \frac{d^2u}{d\theta^2}\right] \qquad ...(v)$$

The equation of the conic is

$$r = \frac{p}{1+e\cos\theta} \text{ or } \frac{p}{r} = 1 + e\cos\theta \text{ or } pu = 1 + e\cos\theta \qquad ...(vi)$$

Differentiating eq. (*vi*) twice with respect to θ, we have

$$p\frac{d^2u}{d\theta^2} = -e\cos\theta \qquad \text{...(vii)}$$

Adding eq.(*vi*) and (*vii*), we get

$$p\left[u + \frac{d^2u}{d\theta^2}\right] = 1, \text{ hence } u + \frac{d^2u}{d\theta^2} = \frac{1}{p}$$

Substituting this value of $u + \frac{d^2u}{d\theta^2}$ in eq. (*v*), we get

$$f = -\frac{J^2u^2}{mp} = -\frac{J^2}{mp}\cdot\frac{1}{r^2}$$

Let $\frac{J^2}{mp} = k$ (a constant).

Then, $f = -\frac{k}{r^2}$ *i.e.*, $f \propto -\frac{1}{r^2}$ ...(*viii*)

Thus, the acceleration and hence the force acting on the planet is inversely proportional to the square of its distance from the sun. Negative sign indicates that the force is one of attraction.

4.8. ARTIFICIAL SATELLITES

We have studied the motion of a planet and its orbit around the sun. In fact, a body which revolves constantly round a comparatively much larger body is said to be satellite. We know that the earth and other planets revolve round the sun in their specified orbits. The moon revolves round the earth and the planets Jupiter and Saturn have six and nine moons respectively revolving around them. All these are the examples of *natural satellites*. Each one of these satellites is attracted by its primary with a force, given by Newton's law of gravitation.

Scientists have also been able to placed man-made satellites, revolving round the earth or sun. They are called **artificial satellites**. The theory discussed above for the orbits and planetary motion is valid for the discussion of satellites.

An artificial satellite of the earth is a body, place in a stable orbit around the earth with the help of multistage rocket. In order to launch a satellite in a stable orbit, first it is necessary to take the satellite to the altitude h, where at the point P by some mechanism, it is given the necessary orbitting velocity, called the **insertion velocity** v_i (Fig. 4.8).

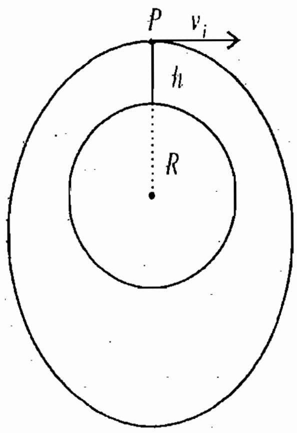


Fig. 4.8 : Elliptical path of a body projected horizontally from a height h above the earth's surface for $v_i^2 < 2GM/(R+h)$

The total energy of the satellite at P relative to the earth is given by

$$E = \frac{1}{2}mv_i^2 - \frac{GMm}{R+h} \qquad \text{...(62)}$$

where m is the mass of the satellite and M that of the earth, having radius R.

The orbit will be an ellipse, a parabola or hyperbola, depending on whether E is negative, zero or positive. In each case, the centre of the earth is at one focus of the path. Therefore, the satellite will be moving in an elliptical orbit (Fig. 4.8), if

$$v_i^2 < \frac{2GM}{R+h} \quad (63)$$

The total energy E determines the size or semi-major axis of the orbit. However the shape or eccentricity e of the orbit is determined by both total energy E and angular momentum J by the relation :

$$e = \sqrt{1+\frac{2EJ^2}{mK^2}} \quad ...(64)$$

with $K = GMm$. For elliptical orbits, layer the angular momentum, the less elongated is the orbit (Fig. 4.9).

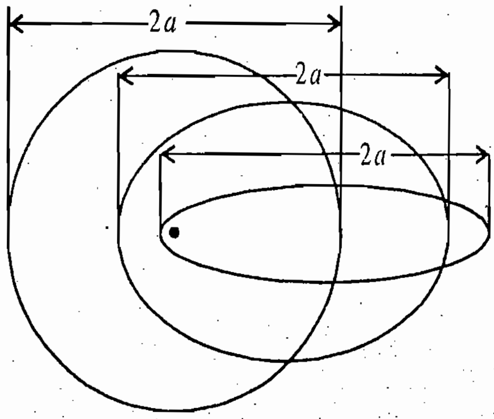


Fig. 4.9 : Elliptical orbits for different values of the angular momentum J with same enegry E; various orbits have the same focus and semi-major axis, but differing in eccentricity.

For circular orbit, the insertion velocity is found by equating the centripetal force mv^2/r to the gravitational force GMm/r^2 *i.e.*,

$$\frac{mv_i^2}{r} = \frac{GMm}{r^2} \text{ or } v_i^2 = \frac{GM}{r} = \frac{GM}{R+h} \quad ...(65)$$

where $\quad r = R + h$

Remember that for circular orbit $e = 0$, so that

$$1+\frac{2EJ^2}{mK^2} = 0,\ 1 + 2 \times \left(-\frac{K}{2a}\right)\times\frac{m^2v_i^2a^2}{mK^2} \text{ or } v_i^2 = \frac{GM}{R+h}$$

where $\quad r = a = R + h$, $K = GMm$ and $J = mv\,(R + h)$.

For the circular obit at the height h above the earth's surface, the period of revolution is

$$T = \frac{2\pi r}{v_i} = \frac{2\pi(R+h)}{v_i} = \frac{2\pi(R+h)^{3/2}}{\sqrt{GM}} \quad ...(66)$$

In Table 1, for some altitudes, we are presenting the values of insertion velocity v_i with corresponding period of revolution T.

TABLE 1

h (km)	v_i (km/s)	T
10	7·894	1 hr 25 min
320	7·714	1 hr 31 min
1600	7·068	1 hr 58 min
35880	3·070	24 hr

We see from the table that as the altitude increases, the insertion velocity decreases and the period of revolution increases.

Geosynchronous orbit : For a satellite, moving around the earth, if the period of revolution is equal to the period of earth's diuranal (one day) rotation, the orbit is said to be geosynchronous orbit. For such an orbit, the period must be 24 hours or more correctly, $T = 23$ hours 56 min. 4·099 sec.

Using Kepler's third law, the semi-major axis of the geosynchronous orbit is

$$a_s = \left(\frac{GM}{4\pi^2}\right)^{1/3} \times T^{2/3} = 42,164{\cdot}2 \text{ km} \quad ...(67)$$

where M is the mass of the earth.

For a geosynchronous orbit, the eccentricity can have any value and the orbit can have any orientation with respect to the equator of the earth.

Geostationary Orbit : If the height of an artificial satellite at equator above the earth's surface is such that its period of revolution is exactly equal to the period of rotation of the earth, then the satellite would appear stationary over a point on earth's equator. Such a satellite is called **geostationary satellite** and its orbit is called **geostationary orbit**. Therefore for a geostationary satellite, we must have the orbit (i) *to be geosynchronous* (ii) *to be circular* and (iii) *to stay over the geographical equator of the earth.*

The height of geostationary satellite is

$$h = a_s - R = 35,786 \text{ km.} \quad ...(68)$$

The geostatinary orbit is often called **parking orbit**. Artificial satellites used for telecasting are put in parking orbits.

Satellites in Space Exploration : The first man made earth satellite was launched by **Russian** scientists on October 4, 1957 and has come to be known as *Sputnik-1* (artificial satellite). This sputnik-1, being of the shape of a ball of diameter 58 cm. and weight 83·6 kgm., was placed in an orbit round the earth and made one full revolution in 96·2 minutes, attaining a speed of 8 km/sec. at a distance of 950 km. from the earth. A three stage rocket was used for this purpose. The rocket was launched vertically and a special device (incorporated in the system) enabled it to gradually curve away from its vertical path. As the fuel of first, second and third stage was exhausted, they dropped away one by one, finally leaving the satellite with a speed of 8 km./sec. to revolve round the earth. Part of the kinetic energy of the satellite was then decreased due to air friction and hence the radius of its path became smaller and smaller. In the denser layers of the atomosphere, it became too much hot and burnt away. Other satellites have been launched by Russian and American scientists. On January 2, 1959, a space rocket was launched by Russians, which became an artificial planet of the sun, having a period of rotation 450 days.

A manned satellite, carrying Major Yuri Gagarin was placed into orbit for the first time by Russians on April 12, 1961. Americans succeded in putting a manned satellite, carrying Col. J. H. Glenn. in orbit in Feb. 1962. The far side of the moon, which was never seen from the earth, was first photographed by a television camera carried by a space rocket Lunik III launched by Russia in Oct. 1959. It revolved round the moon as well as the earth. In November 1969, American scientists launched Appolo-12 with three cosmonauts. Two of them actually landed their plane on the surface of the moon. After collecting some important informations about the surface of the moon, they came back to the earth after a journey of nearly 8 days. Since then a number of satellites have been launched for space exploration.

Uses of Artificial Satellites : Artificial satellites are used in the following :

(1) Distant transmission of radio and TV signals.

(2) To study upper regions of the atmosphere.

(3) High altitude satellites for astronomical observations (as the effects of atmosphere are not present).

(4) Weather forecasting.

(5) Eearth measurements (gravitation and magnetic fields).

Ex. 1. *An artificial satellite is revolving round the earth at a distance of 620 km. Calculate the orbital velocity and the period of revolution. Radius of earth is 6380 km and acceleration due to gravity at the surface of the earth is 9·8 m/sec*^{2}.

Sol. Radius of earth's satellite orbit $r = R + h$

= Radius of the earth + Distance of satellite from earth's surface.

= 6380 + 620 = 7000 km = 7×10^6 m/sec.

∴ Period of revolution $$T = \frac{2\pi r}{R}\sqrt{\frac{r}{g}} = \frac{2\pi \times 7 \times 10^6}{6380 \times 10^3}\sqrt{\frac{7 \times 10^6}{9{\cdot}8}} = \mathbf{5775\ sec.}$$

and orbital velocity $$v = R\sqrt{\frac{g}{r}} = 6380 \times 10^3 \sqrt{\frac{9{\cdot}8}{7 \times 10^6}} = \mathbf{7{\cdot}55 \times 10^3\ m/sec.}$$

Ex. 2. *Calculate the height of an equatorial satellite which is always seen over the same point of earth's surface. ($G = 6{\cdot}66 \times 10^{-11}$ S.I. units; $M = 5{\cdot}98 \times 10^{24}$ kg).*

Sol. Let the height of the equatorial satellite be h. The equatorial satellite is seen over the same point of earth's surface, *i.e.*, the angular velocity of satellite is the same as that the of the earth itself.

Hence, angular velocity of the satellite $$\omega = \frac{2\pi}{24 \times 60 \times 60} = 7{\cdot}27 \times 10^{-5}\ \text{sec}^{-1}.$$

Also, $$GMm/r^2 = m\omega^2 r$$

or $$r^3 = \frac{GM}{\omega^2} = \frac{6{\cdot}66\ 10^{-11} \times 5{\cdot}98 \times 10^{24}}{(7{\cdot}27 \times 10^{-5})^2} = 74{\cdot}74 \times 10^{21}$$

whence, $r = 4{\cdot}21 \times 10^7$ m

Therefore, $h = r - R = 4{\cdot}21 \times 10^7 - 6{\cdot}4 \times 10^6 = 3{\cdot}57 \times 10^7$ **m = $\mathbf{3{\cdot}57 \times 10^4}$ km.**

4.9. VIRIAL THEOREM

This theorem is very useful in a variety of problems in physics. Here, first we shall deduce the virial theorem and then use it to discuss the property of central force as a special case.

Let us consider a system of parcitcles with position vector $\mathbf{r}_i$ and applied force $\mathbf{F}_i$. According to Newton's second law, the equations of motion are

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \qquad ...(69)$$

We introduce a quantity λ , defined by

$$\lambda = \sum_i \mathbf{p}_i \bullet \mathbf{r}_i \qquad ...(70)$$

where the sum is taken for all particles of the system.

The total time derivative of λ is

$$\frac{d\lambda}{dt} = \sum_i \mathbf{p}_i \bullet \dot{\mathbf{r}}_i + \sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i \qquad ...(71)$$

First term of eq. (71) can be written as

$$\sum_i \mathbf{p}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i \dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i = \sum_i m_i v_i^2 = 2T \qquad ...(72)$$

and second term [using eq. (72)] as

$$\sum_i \dot{\mathbf{p}}_i \bullet \mathbf{r}_i = \sum_i \mathbf{F}_i \bullet \mathbf{r}_i \qquad ...(73)$$

Hence, $$\frac{d\lambda}{dt} = \frac{d}{dt}\left[\sum_i \mathbf{p}_i \bullet \mathbf{r}_i\right] = 2T + \sum_i \mathbf{F}_i \bullet \mathbf{r}_i \qquad ...(74)$$

The time average of eq. (74) over a time interval τ is obtained as

$$\frac{1}{\tau}\int_0^\tau \frac{d\lambda}{dt}\,dt = \overline{\frac{d\lambda}{dt}} = \overline{2T} + \overline{\sum_i \mathbf{F}_i.\mathbf{r}_i} \text{ or } \frac{1}{\tau}[\lambda(\tau)-\lambda(0)] = \overline{2T} + \overline{\sum_i \mathbf{F}_i.\mathbf{r}_i} \qquad ...(75)$$

In case of periodic motion, τ is chosen as period and all coordinates repeat after this interval of time. In such a case, $\lambda(\tau) = \lambda(0)$ and then left hand side of eq. (75) vanishes, *i.e.*,

$$2\overline{T} + \overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i} = 0 \text{ or } \overline{T} = -\tfrac{1}{2}\overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i} \qquad ...(76)$$

Eq. (76) is called the ***Virial theorem*** and the quantity $-\frac{1}{2}\overline{\sum_i \mathbf{F}_i \bullet \mathbf{r}_i}$ is known as the ***Virial of Clausius.***

The theorem (76) stands also for non-periodic motions in the condition that the coordinates and velocities of all particles remain finite so there is an upper bound to the quantity λ. By taking τ large enough, left hand side of eq. (75) can be made as small as desired and consequently it may be reduced to zero. Thus in this case also eq. (76) is obtained.

In case of a particle, moving in a central force field,

$$\overline{T} = -\frac{1}{2}\overline{\mathbf{F}_i \bullet \mathbf{r}_i} = -\frac{1}{2}\overline{\left[-\frac{\partial V}{\partial r}\right]\hat{\mathbf{r}} \bullet \mathbf{r}} \text{ or } \overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r} r} \qquad ...(77)$$

where V is the potential energy.

If V is a function of the form

$$V = c\, r^{n+1}, \qquad ...(78)$$

then

$$\overline{T} = \frac{1}{2}\overline{c\,(n+1)r^n r} = \frac{n+1}{2}\overline{cr^{n+1}}$$

or

$$\overline{T} = \frac{n+1}{2}\overline{V} \qquad ...(79)$$

Further in case of ***inverse square law of force*** $n = -2$ and hence

$$\overline{T} = -\frac{\overline{V}}{2} \quad \text{or} \quad 2\overline{T} + \overline{V} = 0 \qquad ...(80)$$

This is a well known form, obtained by Virial theorem for c/r^2 type force.

4.10. SCATTERING IN A CENTRAL FORCE FIELD

The problem of scattering of particles at atomic scale in a central force field is extremely important in modern physics. For example, the scattering of a beam of protons or α-particles by the atomic nuclei of a target is an interesting problem in nuclear physics. Of course, the problem is quantum mechanical, but it can be dealt classically because the procedures for dealing the scattering problem in classical as well as in quantum mechanics are similar. Further a classical study of the scattering problem gives a chance to the beginner to learn the language of the problem.

Let us consider a uniform beam of particles incident on a centre of force. All the particles of the beam have the same mass and energy. The *intensity* of the beam I_0 is the number of particles crossing unit area per unit time normal to the direction of the beam. This I_0 is also called *flux density*. We assume that the force between an incident particle and the particle at the centre of force falls off to zero at large distances.

When a particle approaches the centre of force, it will interact [for example, an α-particle (+ve charge) will experience repulsion from the positively charged nucleus] so that its path will deviate from the

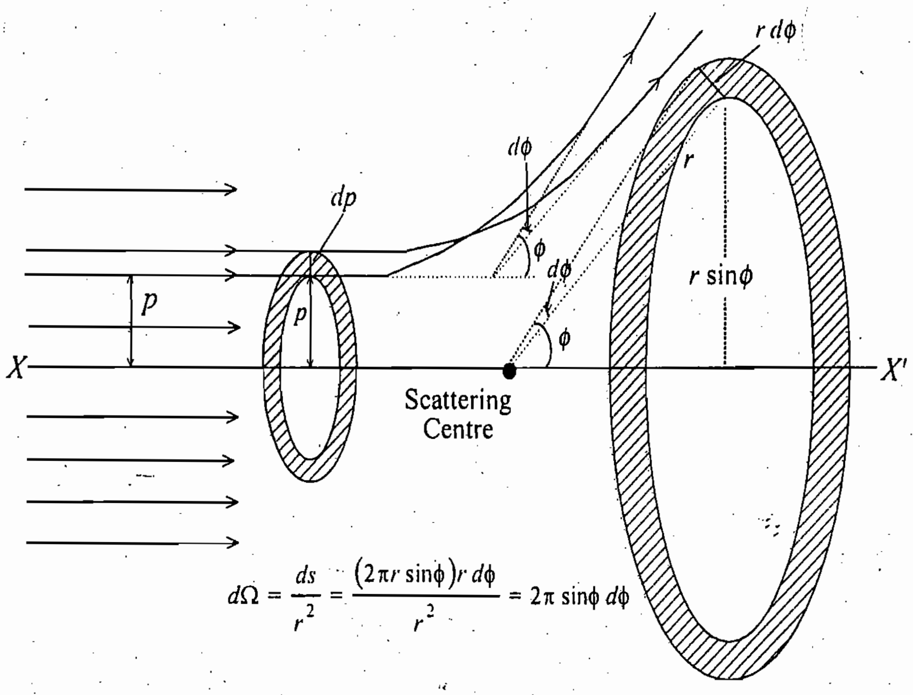


Fig. 4.10 : Scattering–impact parameter (p) and angle of scattering (ϕ)

incident straight line trajectory. After passing the centre of force, as the particle goes away, the force acting on it will decrease and finally at large distances, the force will become zero. This results again in the straight line motion but in general in a different direction and we say that the particle has been ***scattered.***

Scattering cross-section : Consider a uniform beam of particles, moving with a flux of I_0 particles per unit area towards a scattering centre (*e.g.*, an atom of a target). Imagine that the scattering centre presents an area $d\sigma$ perpendicular to the path of the beam such that whatever particles hit $d\sigma$ area are scattered into a solid angle $d\Omega$. Thus the number of particles, scattered into $d\Omega$ solid angle per second are $I_0\, d\sigma$. If $I(\Omega)$ (intensity of the scattered particles) is defined as the number of particles scattered in the direction Ω per unit solid angle per unit time, then the number of scattered particles in the small solid angle $d\Omega$ about Ω direction is given by

$$I_0 d\sigma = I(\Omega)\, d\Omega$$

or
$$\frac{d\sigma}{d\Omega} = \frac{I(\Omega)}{I_0} \qquad \text{...(81)}$$

The quantity $\frac{d\sigma}{d\Omega} = \sigma(\Omega)$ is called ***differential scattering cross-section*** or simply scattering cross-section for scattering in Ω-direction *i.e.*,

$$\sigma(\Omega) = \frac{d\sigma}{d\Omega} = \frac{I(\Omega)}{I_0} \qquad \text{...(82)}$$

Thus the differential scattering cross-section is the ratio of the number of the scattered particles per second per unit solid angle and the flux density of the incident particles.

The total cross-section (σ) is given by

$$\sigma = \int \sigma(\Omega)\, d\Omega = \int \frac{d\sigma}{d\Omega}\, d\Omega \qquad \text{...(83)}$$

Scattering angle (ϕ) : The angle between the incident and scattered directions of the particle is called scattering angle and is denoted by ϕ.

Impact Parameter (p) : If we draw a perpendicular on the direction of the incident particle from the scattering centre, then the length of this perpendicular is known as impact parameter p.

As the force is central, there must be complete symmetry about the axis (XX') of the incident beam. The solid angle $d\Omega$ is given by*

$$d\Omega = 2\pi \sin\phi \, d\phi$$

where ϕ is the scattering angle. The incident particles crossing through area $d\sigma = 2\pi p\, dp$, lying between p and $p + dp$, are given by

$$I_0 \, d\sigma = I_0 \, 2\pi p \, dp \qquad ...(84)$$

These particles are scattered into the solid angle $d\Omega = 2\pi \sin\phi .\, d\phi$. But $I(\Omega) = I_0 \sigma(\Omega)$ is the number of particles scattered in the direction Ω per unit solid angle, hence the number of particles scattered into the solid angle $d\Omega$ are

$$I(\Omega)\, d\Omega = I_0 \, \sigma(\Omega)\, d\Omega$$

or
$$I(\Omega)\, d\Omega = I_0 \, \sigma(\phi)\, 2\pi \sin\phi \, d\phi \qquad ...(85)$$

where $\sigma(\phi)$ represents the differential cross-section for the direction ϕ. Therefore by using eqs. (81) and (84), we get

$$I_0 \, 2\pi \, p dp = -I_0 \, \sigma(\phi)\, 2\pi \sin\phi \, d\phi$$

Negative sign is introduced, because an increase of p will decrease ϕ.

Thus
$$\sigma(\phi) = -\frac{p}{\sin\phi}\left[\frac{dp}{d\phi}\right] \qquad ...(86)$$

This gives the dependence of differential cross-section on the scattering angle ϕ.

4.11. RUTHERFORD SCATTERING CROSS-SECTION

In the Rutherford scattering a positively charged particle of charge ze and mass m is scattered by a heavy nucleus N. The nucleus is assumed to be at rest during the collision. The charge on the nucleus is Ze, where Z is the atomic number. Suppose the positively charged particle is moving towards the heavy nucleus with initial velocity v_0. As the particle approaches the nucleus, the repulsive force $(Zze^2/4\pi\varepsilon_0 r^2)$ increases rapidily and the particle changes from a straight line path to a hyperbola ADB, having one focus at N as shown in Fig. 4.11. The asymptotes AO and BO to the hyperbola give the direction of the incident and scattered particle. The angle COB is the scattering angle ϕ and NM is the impact parameter p.

The charged particle is moving in a central force field, hence the equation of its path is given by eq. (34) *i.e.*,

$$\frac{l}{r} = 1 + e\cos\theta \qquad ...(87)$$

---

* $$d\Omega = \frac{ds}{r^2} = \frac{r\sin\phi \, d\theta \, r\, d\phi}{r^2} = \sin\phi \, d\theta \, d\phi$$

For symmetry about XX' axis, $d\theta$ is to be integrated from 0 to 2π. In such a case, $d\Omega = 2\pi \sin\phi \, d\phi$.

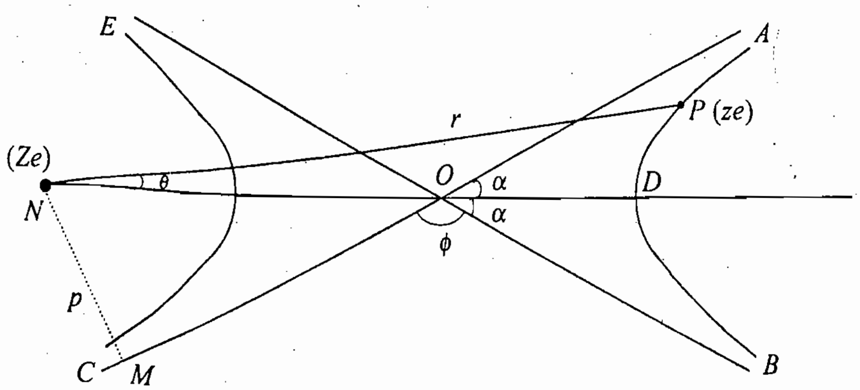


Fig. 4.11 : Scattering in a repulsive force field

where $l = J^2/mK$, $e = \sqrt{1+\dfrac{2EJ^2}{mK^2}}$ and θ', the constant of integration, has been taken to be zero. Here the force F is given by

$$F = \frac{Zze^2}{4\pi\varepsilon_0}\frac{1}{r^2} = -\frac{K}{r^2}$$

Therefore,
$$K = -\frac{Zze^2}{4\pi\varepsilon_0}. \qquad ...(88)$$

Hence
$$l = -\frac{J^2 4\pi\varepsilon_0}{Zze^2 m} \text{ and } e = \sqrt{1+\frac{2EJ^2(4\pi\varepsilon_0)^2}{Z^2z^2e^4m}} \qquad ...(89)$$

As the initial velocity of the particle is v_0, its total energy is given by

$$E = \frac{1}{2}mv_0^2, \text{ whence } mv_0 = \sqrt{2mE} \qquad ...(90)$$

According to the law of conservation of angular momentum,

$$mv_0 p = mr^2\dot{\theta} = J, \text{ whence } mv_0 = \frac{J}{p}$$

Therefore,
$$\frac{J}{p} = \sqrt{2mE} \quad \text{or} \quad J = p\sqrt{2mE} \qquad ...(91)$$

This gives

$$e = \sqrt{1+\frac{2E\left(p\sqrt{2mE}\right)^2(4\pi\varepsilon_0)^2}{mz^2Z^2e^4}} \quad \text{or } e = \sqrt{1+\left(\frac{2Ep\,4\pi\varepsilon_0}{zZe^2}\right)^2} \qquad ...(92)$$

Obviously, $e > 1$, because $(2Ep/zZe^2)^2$ is a positive quantity. Hence eq. (87) represents the path of the charged particle as ***hyperbola***.

Since the hyperbolic path must be symmetric about the direction of the periapsis, the scattering angle ϕ is given by

$$\phi = \pi - 2\alpha \text{ or } \alpha = \frac{\pi}{2} - \frac{\phi}{2} \qquad ...(93)$$

where α is the angle between the direction of the incoming asymptote and the periapsis direction (OD).

Further the asymptotic direction is that for which r is infinite (∞) and then $\theta \to \alpha$. Hence from eq. (87), we have

$$1 + e\cos\alpha = 0 \text{ or } \cos\alpha = -\frac{1}{e} \text{ or } \cos\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = -\frac{1}{e} \text{ or } \sin\frac{\phi}{2} = -\frac{1}{e}$$

Thus
$$\operatorname{cosec}\frac{\phi}{2} = -e$$

Squaring it, we get

$$\operatorname{cosec}^2\frac{\phi}{2} = e^2 \text{ or } 1 + \cot^2\frac{\phi}{2} = 1 + \left[\frac{2Ep\,4\pi\varepsilon_0}{zZe^2}\right]^2 \quad \text{[from (85)]}$$

whence
$$\cot\frac{\phi}{2} = \frac{2Ep\,(4\pi\varepsilon_0)}{zZe^2} \qquad ...(94)$$

From which one can find the ***scattering angle*** ϕ.

Now from eq. (94), we get an expression for ***impact parameter*** :

$$p = \frac{zZe^2 \cot\phi/2}{2E(4\pi\varepsilon_0)} \qquad ...(95)$$

Differentiating it, we get

$$\frac{dp}{d\phi} = -\frac{zZe^2}{4E\,(4\pi\varepsilon_0)}\operatorname{cosec}^2\frac{\phi}{2} \qquad ...(96)$$

Substituting the value of p and $dp/d\phi$ from eqs. (95) and (96) in eq. (86), we get

$$\sigma(\phi) = -\frac{zZe^2\cot\frac{\phi}{2}}{2E\,(4\pi\varepsilon_0)\,2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}\left[-\frac{zZe^2}{4E\,(4\pi\varepsilon_0)}\right]\operatorname{cosec}^2\frac{\phi}{2}$$

or
$$\sigma(\phi) = \frac{1}{4}\left[\frac{zZe^2}{(4\pi\varepsilon_0)\,2E}\right]^2 \operatorname{cosec}^4\frac{\phi}{2} \qquad ...(97)$$

This is the well known expression for the ***Rutherford scattering cross-section***. Thus the scattering cross-section or the number of particles scattered per second along the direction ϕ are proportional to

(1) $\operatorname{cosec}^4\frac{\phi}{2}$,
(2) the square of the charge on the nucleus (Ze),
(3) the square of the charge on the particle (ze), and
(4) inversely proportional to the square of the initial kinetic energy E.

Thus if N_ϕ is the number of particles scattered along the angle ϕ per second, then one can represent

$$N_\phi = C\operatorname{cosec}^4\phi/2 \qquad ...(98)$$

where C is a constant.

Ex. 1. *In Rutherford's experiment 10^5 α-particles are scattered at an angle of 2^0, calculate the number of α-particles scattered at an angle of 20^0* **(Agra 1991)**

Solution : The number of particles N_ϕ scattered per second at an angle ϕ is given by

$$N_\phi = C \operatorname{cosec}^4 \phi/2$$

Therefore, $$\frac{N_{(20°)}}{N_{(2°)}} = \frac{\operatorname{cosec}^4 10°}{\operatorname{cosec}^4 1°} = \left(\frac{\sin 1°}{\sin 10°}\right)^4 = \frac{1}{10^4} \text{ (for small}\theta, \sin\theta \approx \theta)$$

whence $N_{(20°)} = 10^5/10^4 =$ **10** (approximately).

Ex.2. *Show that for any repulsive central force, a formal solution for the angle of scattering can be expressed as*

$$\phi = \pi + \int_0^{u_0} \frac{p\,du}{\sqrt{1 - \frac{V}{E} - p^2u^2}}$$

where V is the potential energy, $u = 1/r$ and u_0 corresponds to the turning point of the orbit. Write down the expression for ϕ for a force K/r^3.

Solution : For a central force,

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{J}{mr^2} + V = E, \therefore \dot{r} = \left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}$$

But $$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\cdot\frac{J}{mr^2}$$

Therefore, $$\frac{dr}{d\theta} = \frac{\dot{r}}{J/mr^2} = \frac{mr^2}{J}\left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}$$

or $$d\theta = \frac{J dr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}} \text{ or } \theta = \int \frac{J dr}{mr^2\left[\frac{2}{m}\left(E - V - \frac{J^2}{2mr^2}\right)\right]^{1/2}}$$

As $r = 1/u$, $dr = -du/u^2$.

Integrating from 0 to u_0 (turning point), we obtain

$$\theta = -\int_0^{u_0} \frac{p\,du}{\left(1 - \frac{V}{E} - p^2u^2\right)^{\frac{1}{2}}}$$

where we have used $J = p\sqrt{2mE}$.

Now, $$\phi = \pi - 2\theta = \pi + 2\int_0^{u_0} \frac{p\,du}{\left(1 - \frac{V}{E} - p^2u^2\right)^{1/2}}$$

For $F = K/r^3$, $V = K/2r^2$ and then

$$\phi = \pi + 2\int_0^{u_0} \frac{p\, du}{\left[1 - \left(\frac{K}{2E} + p^2\right)u^2\right]^{1/2}}$$

Ex. 3. *Determine the differential scattering cross-section and the total scattering cross-section for the scattering of a particle by a rigid elastic sphere.*

Solution : Let the radius of the rigid elastic sphere be a. The impact parameter p for the particle under consideration is

$$p = a \sin\left(\frac{\pi}{2} - \frac{\phi}{2}\right) = a \cos\frac{\phi}{2}$$

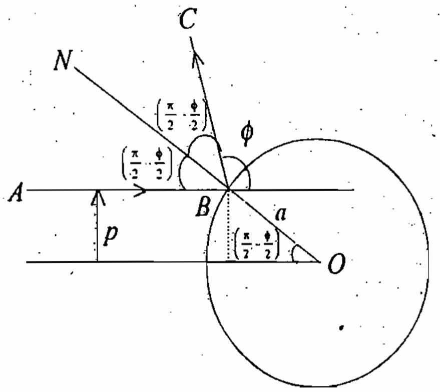


Fig. 4.12

Hence $$\frac{dp}{d\phi} = -\frac{a}{2}\sin\frac{\phi}{2}$$

Differential scattering cross-section

$$\sigma(\phi) = -\frac{p}{\sin\phi}\frac{dp}{d\phi} = \frac{a\cos\frac{\phi}{2} \times \frac{a}{2}\sin\frac{\phi}{2}}{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}} = \frac{a^2}{4}$$

Total scattering cross-section

$$\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{a^2}{4}\int d\Omega = \frac{a^2}{4} 4\pi = \pi a^2.$$

Questions

1. How will you reduce the two-body problem into one-body problem ? Hence explain the concept of reduced mass. Give its two examples. Calculate reduced mass of the hydrogen atom and positronium. **(Agra 1998, 97, 94, 91)**
2. What do you mean by the reduced mass of a two-particle system. If the motion of the hydrogen nucleus is neglected, calculate the percentage errot in the wavelength of any line in hydrogen spectrum determined by Bohr's theory.
3. Discuss a two-body problem reduced into a one-body problem. What is meant by equation of motion and first integrals ? Show that the areal velocity of a planet remains constant. **(Agra 2002)**
4. (*a*) Show that for a particle, moving under central force $f(r)$, the equation of the orbit is given by

$$\frac{d^2u}{d\theta^2} + u = -\frac{m^2}{l^2u^2}\, f\left(\frac{1}{u}\right)$$

where u = $1/r$ and l is the angular momentum. **(Meerut 2001, 1999)**

(*b*) Discuss the orbit if the force law is the attractive inverse square law. **(Meerut 1999)**

5. State and prove the Kepler's laws of planetary motion. **(Agra 2003, 01)**
6. Using the principle of classical mechanics, prove the law of conservauon angular momentum, Kepler's second law and derive the differential equation of the orbit in the central field force. **(Bundelkhand 1996)**

7. What are Kepler's laws of planetary motion ? Give the proof of Kepler's laws of planetary motion and hence deduce that the areal velocity is constant.
(Bundelkhand 1997, 95; Gorakhpur, 95; Agra 2003, 98, 95)

8. What is inverse square law force ? Derive Kepler's laws with its help. **(Meerut 1995; Agra 91)**

9. Derive the differential equation for the orbit of a particle moving under central force.
(Agra 1998, 92)

10. Derive the differential equation of an orbit in polar coordinates under central force. Investigate the motion of a particle under the attractive inverse square law. **(Agra 1990, 85)**

11. Obtain the differential equation for a particle undergoing a central force motion and use it to verify Kepler's laws of planetary motion. **(Agra 1997, 94, 93)**

12. Derive the equation for orbit of a particle moving under the influence of an inverse square central force field. Also calculate the time period of motion in elliptical orbit. **(Meerut 1994 ; Agra, 83)**

13. Show that in an elliptical orbit of a planet around the sun, the major axis solely depends on the total energy. Further prove that the periodic time in an elliptical orbit is given by

$$T = 2\pi a^{3/2} \Big/ \sqrt{G(M+m)}$$

where a is the semi-major axis, M the mass of the sun and m that of the planet.

14. (*a*) State and prove Virial theorem. **(Garwal 1999, 95)**

(*b*) When force is derivable from a potential, $V \propto r^{n+1}$, show that the average kinetic energy equals $\left(\dfrac{n+1}{2}\right)$ times the average polential energy. **(Garwal 1995)**

15. What is differential scattering cross-section ? Discuss the problem of scattering of charged particles by a Coulomb field and obtain Rutherford's formula for the differential scattering cross-section.
(Agra 2004, 1999, 97, 95, 94, 93, 92)

16. Discuss the scattering of α-particles under a central force field and hence obtain the expression for Rutherford scattering cross-section. **(Agra 1998)**

17. Discuss α-particle scattering in coulomb's field. **(Agra 2003, 02)**

18. What is the collision parameter ? Show that the number of particles scattered in the elementary solid angle is inversely proportional to the fourth power of the sign of the deflecting half angle.
(Garwal 1990)

19. If the particles of a system attract each other according to the inverse square law of force, prove that $2\overline{T} + \overline{V} = 0$, where $\overline{T}$ is the average kinetic energy and $\overline{V}$ the average potential energy.

Problems

[SET- I]

1. Calculate the reduced mass of CO and HCl molecules.
(atomic numbers of H, C, O and Cl atoms are 1, 12, 16 and 35.5 respectively; 1 amu $=1.67\times10^{-27}$ kg).
Ans. $\mu(\text{CO}) = 1.15 \times 10^{-26}$ kg; $\mu(\text{HCl}) = 1.62 \times 10^{-27}$ kg.

2. Show that in rectangular coordinates the magnitude of a aeal velocity is

$$\tfrac{1}{2}(x\dot{y} - y\dot{x}).$$

3. Show that the differential equation describing the motion of a particle in a central field can be expressed as

$$\frac{mh^2}{2r^4}\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]-\int f(r)\,dr=E, \text{ where } h=r^2\dot{\theta}.$$

4. A particle moves in a central force field defined by $\mathbf{F}=-Kr^2\hat{\mathbf{r}}$. It starts from rest at a point on the circle $r=a$. Show that when it reaches the circle $r=b$, its speed will be $\sqrt{2K(a^3-b^3)/3m}$.

5. The eccentricity of the earth's orbit is 0.0125. Calculate the ratio of the maximum and minimum speeds of the earth. **(Agra 1998, 95)**
Ans : 1.025

6. A satellite has its largest and smallest orbital speeds v_{max} and v_{min} respectively. If the satellite has a period T, then show that it moves in an elliptical orbit having major axis with length $\frac{T}{2\pi}\sqrt{v_{max}\,v_{min}}$.

7. A particle of mass m describes an elliptical orbit about a centre of attractive force at one of its focus given by $-k/r^2$, where k is a constant. Show that the speed v of the particle at any point of the orbit is given by

$$v^2=\frac{k}{m}\left[\frac{2}{r}-\frac{1}{a}\right]$$

where a is the semimajor axis.

$\left[\textbf{Hint: } E=T(\text{K.E.})+V=\frac{1}{2}mv^2-\frac{k}{r}\right].$

8. If a planet were suddenly stopped in its orbit, supposed to be circular, prove that it will fall into the sun in a time which is $\sqrt{2}/8$ times of its period.

9. A particle of mass m moves in a central force field defined by $F=-\frac{k\mathbf{r}}{r^4}$. Show that if E is the total energy supplied to the particle, then its speed is given by $v=\sqrt{\frac{k}{mr^2}+\frac{2E}{m}}$
Show that for initial condition $E=0$, the equation of orbit is $r=Ae^{\lambda\theta}$ where A and λ are constants.

10. A particle describes a leminiscate, given by $r^2=a^2\cos 2\theta$. Obtain the force law **(Gorakhpur 1991)**
Ans. $F(r)\propto 1/r^7$

11. An election of change $-e$ is moving around a nucleus of atomic number Z. Find the periodic time in case of elliptical orbit.

Ans. $T=\frac{4\pi}{e}\sqrt{\frac{\pi\varepsilon_0\mu a^3}{Z}}$

12. A satellite of radius a revolves in a circular orbit about a planet of radius b with period T. If the *shortest distance between the satellites* is c, show that the mass of the planet is $4\pi^2(a+b+c)^3/GT^2$.

13. A particle moves on a curve $r^n=a^n\cos n\theta$ under the influence of a central force. Find the law of force. **(Gorakhpur 1995)**
Ans : $F(r)\propto r^{-(2n+3)}$.

14. A negatively charged particle is moving under the Coulomb force of the nucleus. Deduce the nature of the orbit of the particle and periodic time. **(Agra 1983)**

Ans : Elliptical orbit, $T = 2\pi a^{3/2}\sqrt{\mu / K}$; $K = Ze^2/4\pi\varepsilon_0$.

[**Hint :** $F = \dfrac{1}{4\pi\varepsilon_0}\dfrac{(Ze)(-e)}{r^2} = -\dfrac{Ze^2}{4\pi\varepsilon_0 r^2}$ (Ze = charge on the nucleus) or $F = -K/r^2$

Eccentricity of the orbit $e = \sqrt{1+\dfrac{2EJ^2}{\mu K^2}} = \sqrt{1-\dfrac{J^2}{\mu K a}}$, $\left(E = -\dfrac{K}{2a}\right)$

Therefore, $e < 1$, hence the orbit is elliptical with period $T = 2\pi a^{3/2}\sqrt{\mu / K}$.]

15. According to Yukawa's theory of nuclear forces, the attractive force between two nucleons inside a nucleus is given by the potential

$$V(r) = k\frac{e^{-\alpha r}}{r}.$$

Find the force law. Calculate the total energy E and angular momentum J, if the particle moves in a circle of radius r_0. Determine also the period of circular motion.

Ans : $f(r) = e^{-\alpha r}\left[\dfrac{k\alpha}{r}+\dfrac{k}{r^2}\right]$; $E = \dfrac{ke^{-\alpha r_0}}{2r_0}(1-\alpha r_0)$; $J = \left[-kmr_0 e^{-\alpha r_0}(1+\alpha r_0)\right]^{1/2}$;

$$T = 2\pi\left[-\frac{ke^{-\alpha r_0}}{mr_0^3}(1+\alpha r_0)\right]^{1/2}$$

16. A body of mass m is moving in a spiral orbit given by $r = r_0 e^{k\theta}$. Show that the force causing such an orbit is a central force and varies as r^{-3}.

17. A particle of mass m moves in an elliptical orbit under the action of an inverse square central force. If α be the ratio of the maximum angular velocity to the minimum angular velocity, show that the eccentricity of the ellipse e is given by $e = (\alpha - 1)/(\alpha + 1)$.

18. Prove that the product of the minimum and maximum speeds of a particle moving in an elliptical orbit is $(2\pi a / T)^2$.

19. If the orbit described under a central force be given by $r = a\,(1+\cos\theta)$ with centre at the origin, find the law of fource. **(Rohilkhand, 1984)**
Ans : $f(r) \propto 1/r^4$.

20. The first artificial satellite was circling round the earth at a distance of 900 km. Taking the radius of the earth equal to 6371 km. and mass $5{\cdot}983 \times 10^{27}$ gm. and G equal to $6{\cdot}66 \times 10^8$ C.G.S. units, find the velocity of satellite and its period of revolution.
Ans. Velocity = 7·40 km./sec.; Time period = 1 hour 42 minutes 52 sec.

21. An artificial satellite is going round the earth close to its surface. Calculate the time taken by it to complete one round. Take the radius of earth 6400 km. and g = 980 cm./sec^2. **(Punjab 1968)**
Ans. 5075 sec.

22. A sputnik revolves round the earth in a circular orbit of radius 13000 km. under the attraction of earth alone. Calculate the time-period of revolution of the sputnik in its orbit, assuming the radius of the earth to be 64000 km. (g = 980 cm./sec.^{2} at the surface of the earth.) **(Ranchi 1966)**
Ans. 6722 sec.

23. In Rutherford experiment, 10^3 α-particles are scattered at an angle of 4^0, calculate the number of particles scattered at an angle of 14^0. **(Agra 1995)**
Ans : 6.7 approx.

24. Determine the differential scattering cross-section for α-particles by Pb ($Z = 82$) nucleus, provided that the initial energy of α-particles is 11×10^{-13} Joule and scattering angle is 30° Calculate the value of impact parameter also.

$$\left(\frac{1}{4\pi \in_0} = 9 \times 10^9 \text{ S.I. unit}, e = 1.6 \times 10^{-19} \text{ coulomb, cosec } 15° = 3.86\right).$$

Ans : 1.8×10^{-26} m^2; 6.4×10^{-14} m.

25. A beam of particles of energy E encounters a spherical potential, given by $V(r) = K$ for $r < a$ and $V(r) = 0$ for $r > a$, where K is a constant. Show that the differential scattering cross-section is given by

$$\sigma(\Omega) = \left[\frac{a^2 f^2}{4 \cos\frac{\phi}{2}}\right]\left[\frac{\left(f\cos\frac{\phi}{2} - 1\right)\left(f - \cos\frac{\phi}{2}\right)}{\left(1 + f^2 - 2f\cos\frac{\phi}{2}\right)^2}\right] + \frac{a^2}{4},$$

for $0 < \phi < 2\cos^{-1} f$ and is zero for $\phi > 2\cos^{-1} f$, where $f = [1-(K/E)]^{1/2}$.

26. For a particle moving under the action of a central force, the effective potential energy is given by

$$U(r) = -\frac{100}{r} + \frac{50}{r^2} \quad (MKS\ units)$$

Sketch roughly U as function of r and find the radius of circular motion. **(Mumbai 2001)**
Ans : $r = 1\ m$.

[SET- II]

1. A particle is describing a parabola about a centre of force which attracts according to the inverse square of the distance. If the speed of the particle is made one half without change of direction of motion when the particle is at one end of the latus rectum, prove that the new path is an ellipse with eccentricity $e = \sqrt{5/8}$.

2. The eccentricity of the earth's orbit is $e = 0.0167$. If the orbit is divided into two by the minor axis, show that the times spent in the two halves of the orbit are $\left(\frac{1}{2} \pm \frac{e}{\pi}\right)$ year. Also evaluate the difference in hours.
Ans : 93.2 hours.

3. Show that the velocity of a planet at any point of its orbit is the same asit would have been if it had fallen that point from rest at a distance from the sun equal to the length of the major axis.

4. Calculate the time in which a particle moving under inverse square law force describes the area $0 \le \theta \le \alpha$ of elliptical orbit.

$$\textbf{Ans :}\ t = \frac{J^3}{\mu K^2 (1-e^2)^{3/2}}\left[2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}\right) - e\sqrt{1-e^2}\,\frac{\sin\alpha}{1+e\cos\alpha}\right]$$

5. A particle moves under the action of a central force and describes a curve $r = a/(1+\theta)^2$, where a is constant. What is the force law ? If when $\phi = 0$, the particle receives an impulse which reduces its radial velocity to zero and doubles its transverse velocity, show that its subsequent path is given by

$$\frac{3a}{2r} = 1 + \frac{1}{2}\cos\left(\frac{\sqrt{3}}{2}\phi\right)$$

6. A particle moves in a bounded orbit under an attractive inverse force. Prove that the time average of the kinetic energy is half the time average of the potential energy.

7. Find the differential scattering cross-section for the scattering of particles by the potential $V(r)$, where $V(r) = \alpha\left(\frac{1}{r} - \frac{1}{R}\right)$ for $r < R$ and $V(r) = 0$ for $r > R$.

Ans : $\dfrac{R^2}{4}\dfrac{(1+\gamma)}{\left(1+\gamma\sin^2\dfrac{\theta}{2}\right)^2}$, where $\gamma = \dfrac{4ER}{\alpha^2}(\alpha + RE)$.

Objective Type Questions

1. A particle is moving under central force about a fixed centre of force. Choose the correct statement :
   (*a*) The motion of the particle is always on a circular path.
   (*b*) Its angular momentum is conserved.
   (*c*) Its kinetic energy remains constant.
   (*d*) Motion of the particle takes place in a plane.
   Ans : (*b*), (*d*).

2. Two particles of masses m and $2m$, interacting via gravitational force are rotating about common centre of mass with angular velocity ω at a fixed distance r. If the particle of mass $2m$ is taken as the origin O,
   (*a*) the force between them can be represented as $F = \mu\omega^2 r$.
   (*b*) in an inertial frame, fixed at the centre of mass, the origin is at rest.
   (*c*) in the inertial frame, the origin O is moving on a circular path of radius $r/3$.
   (*d*) in the inertial frame, the particle of mass m is moving on a circular path of radius $r/3$.
   Ans : (*a*), (*c*).

3. A particle is moving on elliptical path under inverse square law force of the form $F(r) = -K/r^2$. The eccentricity of the orbit is
   (*a*) a function of total energy.
   (*b*) independent of total energy.
   (*c*) a function of angular momentum.
   (*d*) independent of angular momentum.
   Ans : (*a*), (*c*).

4. The maximum and minimum velocities of a satellite are v_1 and v_2 respectively. The eccentricity of the orbit of the satellite is given by

(*a*) $e = \frac{v_1}{v_2}$ (*b*) $e = \frac{v_2}{v_1}$ (*c*) $e = \frac{v_1 - v_2}{v_1 + v_2}$ (*d*) $e = \frac{v_1 + v_2}{v_1 - v_2}$

Ans : (*d*).

5. Rutheford's differential scattering cross-section
   (*a*) has the dimensions of area.
   (*b*) has the dimensions of solid angle.
   (*c*) is proportional to the square of the kinetic energy of the incident particle.
   (*d*) is inversely proportional to $\text{cosec}^4 (\phi/2)$, where ϕ is the scattering angle.
   Ans : (*a*).

6. Consider a comet of mass m moving in a parabolic orbit around the Sun. The closest distance between the comet and the Sun is b, the mass of the Sun is M and the universal gravitation constant is G.
   (*i*) The angular momentum of the comet is
   (*a*) $M\sqrt{Gmb}$ (*b*) $b\sqrt{GmM}$
   (*c*) $G\sqrt{mMb}$ (*d*) $m\sqrt{2GMb}$ **(Gate 2004)**
   Ans. (*d*)
   [**Hint.** Equation of the parabolic orbit ($e = 1$)

$$\frac{l}{r} = 1 + \cos(\theta - \theta')$$

   When the comet is closest to the Sun, $r = b$ and $\cos(\theta - \theta') = 1$. So that $l = 2b$, *i.e.*, $J^2/mK = 2b$ or $J = m\sqrt{2GMb}$ $(K = GMm)$].
   (*ii*) Which one of the following is TRUE for the above system ?
   (*a*) The acceleration of the comet is maximum when it is closest to the Sun.
   (*b*) The linear momentum of the comet is a constant.
   (*c*) The comet will return to the solar system after a specified period.
   (*d*) The kinetic energy of the comet is a constant. **(Gate 2004)**
   Ans. (*a*)

Short Answer Questions

1. Show that the motion of a particle under central force takes place in a plane.
2. What are first integrals ?
3. Calculate the reduced mass of H_2 modecule. Assume the mass of H atom = M.
   Ans. $M/2$.
4. What is differential scattering cross-section ?
5. In Rutherford's experiment, 10^4 particles are scattered at an angle of 2°, calculate the number of α - particles, scattered at an angle of 16°.
   Ans. 2.4 approx.
6. Discuss α-particle scattering in coulomb's field. **(Agra 2003, 02)**
7. Fill in the blanks :
   (*i*) The square of the period of revolution of the planet around the sun is proportional to the cube of the...................
   (*ii*) If e is the eccentricity of the earth's arbit, the ratio of maximum and minimum speeds of the planet is....................
   Ans. (*i*) Semi-major axis of the ellipse, (*ii*) $(1 + e) / (1 - e)$.

CHAPTER 5

Variational Principles

5.1. INTRODUCTION

Many variational principles have been proposed in physics and applied to deal with the problems. The main advantage of such principles lies in their extreme economy of expression. In sec. 2.11, we have made use of such a variational principle, namely Hamilton's principle, to deduce the Lagrange's equations. We discuss here the calculus of variation where the fundamental problem is to find the curve for which a given line integral has a stationary or extremum value. We shall see that the Hamilton's principle is just a special case of the general formulation.

5.2. THE CALCULUS OF VARIATIONS AND EULER-LAGRANGE'S EQUATIONS

Let us have a function $f(y, y', x)$ defined on a curve given by

$$y = y(x) \qquad ...(1)$$

between two points A (x_1, y_1) and B (x_2, y_2). Here, $y' = dy/dx$. We are interested in finding a particular curve $y(x)$ for which the line integral I of the function f between the two points

$$I = \int_{x_1}^{x_2} f(y, y', x)dx \qquad ...(2)$$

has a stationary value.

Suppose that APB be the curve for which I is stationary. Now, consider a neighbouring curve $AP'B$ with the same end points A and B. The point $P(x, y)$ of the curve APB corresponds to the point P' $(x, y+\delta y)$ of the curve $AP'B$, keeping x-coordinate of the points fixed. *This defines a δ-variation of the curve.* The variation is arbitrary but small and may be expressed as

$$\delta y = \frac{\partial y}{\partial \alpha}\delta\alpha = \eta(x)\,\delta\alpha \qquad ...(3)$$

where α is a parameter (independent of x) common to all points of the path and $\eta(x)$ is a function of x with the condition that

$$\delta y_1 = \delta y_2 = \eta(x_1) = \eta(x_2) = 0 \qquad ...(4)$$

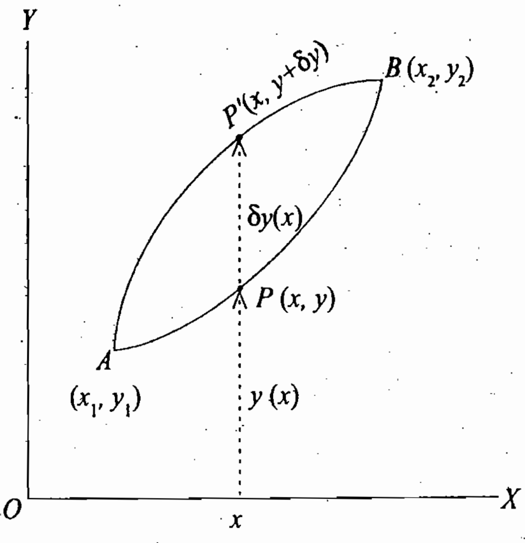


Fig. 5.1 : δ-variation

By choosing different $\eta(x)$, we may construct different varied paths.

The corresponding variation in y' is

$$\delta y' = \eta'(x)\,\delta\alpha \qquad ...(5)$$

Now, the integral on the varied path is

$$I' = \int_{x_1}^{x_2} f(y+\delta y,\ y'+\delta y',\ x)dx$$

or
$$I' = \int_{x_1}^{x_2} f(y+\eta\delta\alpha,\ y'+\eta'\delta\alpha, x)dx \qquad ...(6)$$

Since the variation is small, the integral I' may be obtained by considering only first order terms in the Taylor expansion of the function f *i.e.*,

$$I' = \int_{x_1}^{x_2} [f(y,y',x) + \frac{\partial f}{\partial y}\eta\delta\alpha + \frac{\partial f}{\partial y'}\eta'\delta\alpha]\, dx \qquad ...(7)$$

Hence
$$\delta I = I' - I = \delta\alpha \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right) dx \qquad ...(8)$$

But
$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'\, dx = \frac{\partial f}{\partial y'}\eta \Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\, dx$$

or
$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'\, dx = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\, dx \qquad [\text{as } \eta(x_1) = \eta(x_2) = 0]$$

Therefore,
$$\delta I = \delta\alpha \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\, dx \qquad ...(9)$$

The condition that the integral I is stationary means that $\delta I = 0$, *i.e.*,

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta\, dx = 0 \qquad ...(10)$$

As η is arbitrary, the integrand of (10) must be zero, *i.e.*,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(11)$$

which is known as ***Euler-Lagrange equation.***

The result can easily be generalized to the case where f is a function of many independent variables y_k and their derivatives y'_k. However, y_k and y'_k are function of x. Then

$$\delta I = \delta \int_{x_1}^{x_2} f(y_1, y_2, ..., y_k, ..., y_n, y'_1, y'_2, ..., y'_k, ...y'_n, x)\, dx = 0 \qquad ...(12)$$

leads to the Euler-Lagrange equations

$$\frac{\partial f}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_k}\right) = 0 \qquad ...(13)$$

where, $k = 1, 2, ..., n$.

It is to be pointed out that in most of the problems the stationary value of the integral is seen to be a minimum but occasionally maximum.

Hamilton's principle and Lagrange's equations : If we identify the Lagrangian to

$$L = L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t) \quad ...(14)$$

to the function

$$f = f(y_1, y_2, ..., y_n, y'_1, y'_2, ... y'_n, x)$$

with the transformations

$$x \to t,\ y_k \to q_k,\ y'_k \to \dot{q}_k,$$

then the integral $I \to S$ defines the action integral

$$S = \int_{x_1}^{x_2} L\, dt \quad ...(15)$$

This integral is known as ***Hamilton's principal function.***

The variation

$$\delta S = \delta \int L\, dt = 0 \quad ...(16)$$

corresponding to eq. (12) means that ***the motion of the system from time $t = t_1$ to time $t = t_2$ is such that the line integral (15) has a stationary value for the correct path of the motion.*** This is what is known as **Hamilton's principle.** The necessary condition for the Hamilton's principle ($\delta S = 0$) is given by Lagrange's equations of motion in place of Euler-Lagrange's equations, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad ...(17)$$

where q_k are the generalized coordinates.

These coordinates are independent corresponding to the independent variables y_k and hence the constraints are holonomic.

Ex. 1. *Show that the shortest distance between two points in a plane is a straight line.*

(Garwal 1991; Meerut 99)

Solution : Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points in X-Y plane. An element of length ds of any curve, say $AP'B$, passing through A and B points is given by

$$ds^2 = dx^2 + dy^2$$

or

$$ds = \sqrt{1 + y'^2}\, dx = 0 \quad ...(i)$$

Total length of the curve from point A to the point B is given by

$$s = \int_A^B \sqrt{1 + y'^2} dx = \int_A^B f\, dx \quad ...(ii)$$

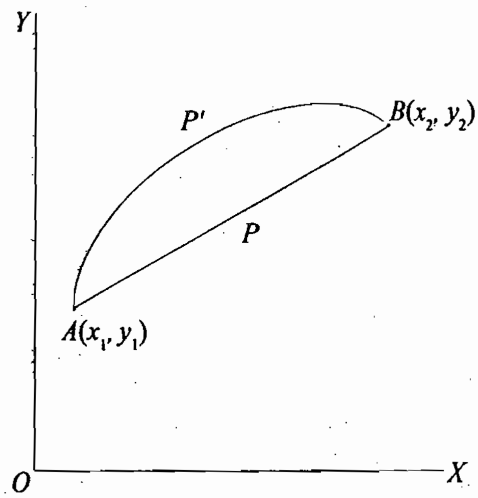


Fig. 5.2 : Shortest distance between two points in a plane.

where $f = \sqrt{1 + y'^2}$. The length of the curve s will be minimum, when $\delta s = 0$. This means that f should satisfy the Euler-Lagrange's equation, *i.e.*,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad ...(iii)$$

Here, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$

Therefore, eq. (*iii*) is

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \text{ or } \frac{y'}{\sqrt{1+y'^2}} = C, \text{ a constant}$$

or $$y'^2 = C^2(1+y'^2) \text{ or } y'^2(1-C^2) = C^2 \text{ or } y' = \frac{C}{\sqrt{1-C^2}} = a \text{ (constant)}$$

or $$\frac{dy}{dx} = a \qquad ...(iv)$$

Integrating it, we get

$$y = ax + b \qquad ...(v)$$

where b is a constant of intrgration. Eq. (v) represents a straight line. Therefore the shortest distance between any two points in a plane is a straight line. The constants of integration a and b can be determined by the condition that the straight line (v) passes through $A(x_1, y_1)$ and $B(x_2, y_2)$.

Ex. 2. *A particle of mass m falls a given distance* z_0 *in time* $t_0 = \sqrt{2z_0/g}$ *and the distance travelled in time t is given by* $z = at + bt^2$, *where constants a and b are such that the time* t_0 *is always the same. Show that the integration* $\int_0^{t_0} L\,dt$ *is an extremum for real values of the coefficients only when* $a = 0$ *and* $b = g/2$.

(Agra 1989)

Solution : Let the particle fall from O ($z = 0$) to P ($OP = z$) in time t.

Kinetic energy of the particle at P,

$$T = \frac{1}{2}m\dot{z}^2.$$

Potential energy of the particle at P, $V = -mgz$

Hence $$L = T - V = \frac{1}{2}m\dot{z}^2 + mgz \qquad ...(i)$$

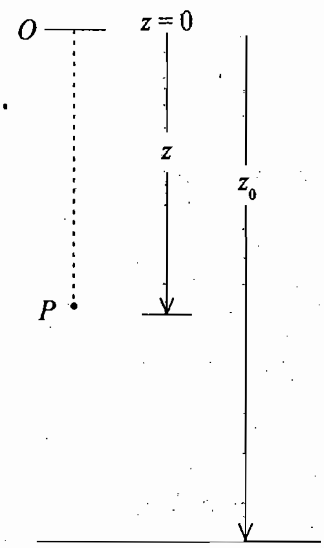


Fig. 5.3

According to the Hamilton's principle

$$\delta\int_0^{t_0} L\,dt = 0$$

or $$\int_0^{t_0} L\,dt = \text{extremum, for which}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}}\right] - \frac{\partial L}{\partial z} = 0 \qquad ...(ii)$$

is to be satisfied.

Here, $\frac{\partial L}{\partial \dot{z}} = m\dot{z}$ and $\frac{\partial L}{\partial z} = mg$. Hence eq. ($ii$) is

$$\frac{d}{dt}(m\dot{z}) - mg = 0 \text{ or } \ddot{z} = g \qquad ...(iii)$$

But $z = at + bt^2$ and therefore $\dot{z} = a + 2bt$ and $\ddot{z} = 2b$ ...(iv)

From (iii) and (iv), we get

$$2b = g \text{ or } b = g/2 \quad ...(v)$$

Also at $t = t_0$, $z = z_0$, we have

$$z_0 = at_0 + bt_0^2 \quad ...(vi)$$

But

$$t_0 = \sqrt{\frac{2z_0}{g}} \text{ or } z_0 = \frac{1}{2}gt_0^2 \quad ...(vii)$$

Comparing (vi) and (vii) and putting $b = g/2$, we get

$$at_0 + \frac{g}{2}t_0^2 = \frac{1}{2}gt_0^2 \text{ or } at_0 = 0$$

Since $t_0 \neq 0$, therefore, $a = 0$.

Thus we find that $\int_0^{t_0} L\, dt$ is extremum, when $a = 0$, $b = g/2$.

Ex. 3. *We take a curve passing through the fixed points (x_1, y_1) and (x_2, y_2) and revolve it about Y-axis to form a surface of revolution. Find the equation of the curve for which the surface area is minimum.*

Solution : Let AB be the curve which passes through the fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$. The curve AB has been revolved about Y-axis to generate a surface. Consider a strip of the surface whose radius is x and breadth is $PP' = ds$, given by

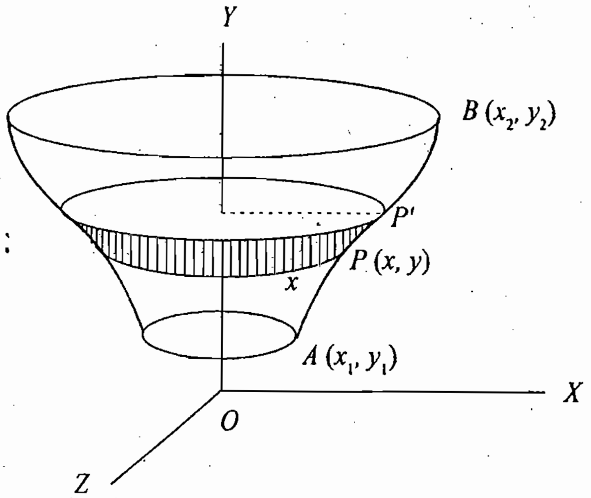


Fig. 5.4 : Minimum surface area of revolution

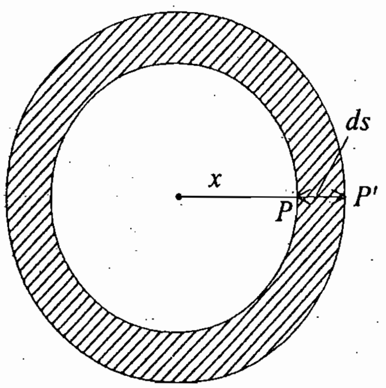


Fig. 5.5 : Circular strip of area $2\pi x\, ds$

$$ds^2 = dx^2 + dy^2 \text{ or } ds = \sqrt{1 + y'^2}\, dx$$

Area of the strip $dS = 2\pi\, x\, ds$ (Fig. 5.5)

$$= 2\pi\, x \sqrt{1 + y'^2}\, dx$$

Total area of revolution $S = 2\pi \int_A^B x\sqrt{1 + y'^2}\, dx$ ...(i)

This area will be minimum, strictly speaking extremum, if $\delta S = 0$, for which Euler-Lagrange equation is to be satisfied, *i.e.*,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(ii)$$

where $f = x\sqrt{1+y'^2}$, when compared to eq. (2),

Here $$\frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial y'} = \frac{xy'}{\sqrt{1+y'^2}}$$

Substituting in eq. (*ii*), we have

$$\frac{d}{dx}\frac{xy'}{\sqrt{1+y'^2}} = 0 \text{ or } \frac{xy'}{\sqrt{1+y'^2}} = a \qquad ...(iii)$$

where a is constant of integration. Squaring (*iii*), we get

$$x^2y'^2 = a^2 + a^2y'^2 \text{ or } y' = \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

Therefore, $$y = \int \frac{a}{\sqrt{x^2-a^2}}\, dx = a\cosh^{-1}\frac{x}{a} + b \qquad ...(iv)$$

where b is another constant of integration.

From (*iv*) we have

$$\cosh^{-1}\frac{x}{a} = \frac{y-b}{a} \text{ or } x = a\cosh\frac{y-b}{a} \qquad ...(v)$$

which is the equation of a catenary.

This is the equation of the curve for which the surface of revolution is minimum. The two constants a and b can be determined by the condition that the curve (v) passes through (x_1, y_1) and (x_2, y_2) points.

Ex. 4. Brachistochrone Problem. *A particle slides from rest at one point on a frictionless wire in a vertical plane to another point under the influence of the earth's gravitational field. If the particle travels in the shortest time, show that the path followed by it is a cycloid.* **(Kanpur 2003)**

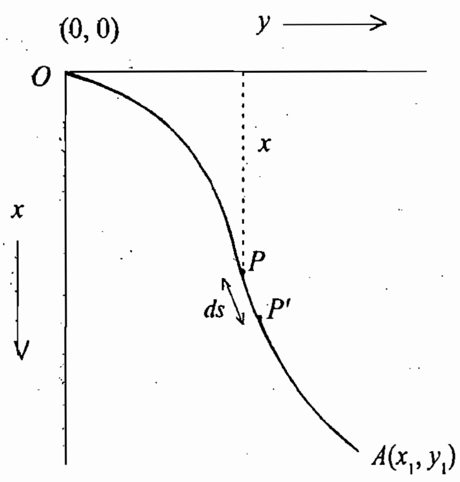


Fig. 5.6 : The brachistochrone problem

Solution : Let the shape of wire be in the form of a curve OA. The particle starts to travel from O (0, 0) from rest and moves to A (x_1, y_1) under the influence of gravity on the frictionless wire.

Let v be the speed at P. Then in moving $PP' = ds$ element, the time taken will be ds/v. Therefore, total time taken by the particle in moving from the higher point O to the lower point A is

$$t = \int_0^A \frac{ds}{v} \qquad ...(i)$$

If the vertical distance of fall from O to P be x, then from the principle of conservation of energy

$\frac{1}{2}mv^2 = mgx$ or $v = \sqrt{2gx}$

Therefore, $$t = \int_0^A \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gx}} \quad \left[ds = \sqrt{dx^2+dy^2} = dx\sqrt{1+y'^2}\right] \quad \text{...(ii)}$$

So that $f = \sqrt{\frac{1+y'^2}{2gx}}$ and for t to be minimum,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \quad \text{...(iii)}$$

Here, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gx}\sqrt{1+y'^2}}$

Subsitituting in (*iii*), we get

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{2gx}\sqrt{1+y'^2}}\right) = 0 \text{ or } \frac{y'}{\sqrt{x}\sqrt{1+y'^2}} = C, \text{ constant}$$

or $$\frac{y'^2}{C^2} = x\,(1+y'^2) \text{ or } y'^2\left(\frac{1}{C^2} - x\right) = x \text{ or } y'^2 = \frac{x}{b-x}$$

(where $b = 1/C^2$, a constant).

or $$\frac{dy}{dx} = \sqrt{\frac{x}{b-x}}, \text{ or } y = \int \sqrt{\frac{x}{b-x}}\,dx + C', \text{ another constant} \quad \text{...(iv)}$$

Let $x = b\sin^2\theta$, then $dx = 2b\sin\theta\cos\theta\,d\theta$.

Therefore, $$y = \int \frac{\sin\theta}{\cos\theta} 2b\sin\theta\cos\theta\,d\theta + C'$$

$$= b\int 2\sin^2\theta\,d\theta + C' = b\int(1-\cos 2\theta)\,d\theta + C'$$

$$= b\left[\theta - \frac{\sin 2\theta}{2}\right] + C' = \frac{b}{2}[2\theta - \sin 2\theta] + C'$$

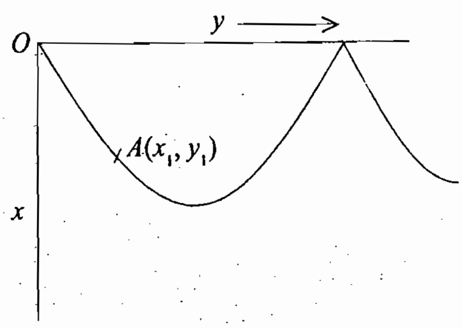


Fig. 5.7 : Cycloid

Thus the parametric equations of the curve are

$$x = b\sin^2\theta = \frac{b}{2}(1-\cos 2\theta) \text{ and } y = \frac{b}{2}(2\theta - \sin 2\theta) + C'$$

Since the curve passes through (0, 0), $C' = 0$. Therefore,

$$x = \frac{b}{2}(1-\cos 2\theta) \text{ and } y = \frac{b}{2}(2\theta - \sin 2\theta) \quad \text{...(v)}$$

Let $2\theta = \phi$ and $b/2 = a$. Then the parametric equations of the curve are

$$x = a\,(1-\cos\phi) \text{ and } y = a\,(\phi - \sin\phi) \qquad ...(vi)$$

This represents a cycloid [Fig. 5.7]. The constant a can be determined because the curve passes through the point $A(x_1, y_1)$.

Ex. 5. *Apply variational principle to find the equation of one dimensional harmonic oscillator.* **(Agra 1988)**

Solution : The Lagrangian L for one dimensional harmonic oscillator is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \text{ or } L = f(x, \dot{x}, t) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

According to Hamilton's principle or variational principle $\int L\,dt$ or $\int f(x,\dot{x},t)\,dt$ is extremum.

Euler-Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) - \frac{\partial f}{\partial x} = 0$$

Here $\qquad \dfrac{\partial f}{\partial x} = -kx\,, \quad \dfrac{\partial f}{\partial \dot{x}} = m\dot{x}$

Therefore, $\qquad m\ddot{x} + kx = 0$

which is the equation of motion for one-dimensional harmonic oscillator.

Ex. 6 . *Show that for a spherical surface, the geodesics are the great circles.* (*For a non-flat surface, the curves of extremal lengths are called geodesics.*) **(Rohilkhand 1999, 78)**

Solution : $ds^2 = a^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$ or $ds = a\,d\theta\,\sqrt{1+\sin^2\theta\,\phi'^2}$

According to the variational principle,

$$\delta s = \delta\int ds = \delta\int a\,d\theta\,\sqrt{1+\sin^2\theta\,\phi'^2} = 0 \text{ or } \delta\int_{\theta_1,\phi_1}^{\theta_2,\phi_2} d\theta\,\sqrt{1+\sin^2\theta\,\phi'^2} = 0$$

Here, $\qquad f = \sqrt{1+\sin^2\theta\,\phi'^2}\;;\; \therefore \dfrac{\partial f}{\partial \phi} = 0 \text{ and } \dfrac{\partial f}{\partial \phi'} = \dfrac{\phi'\sin^2\theta}{\sqrt{1+\sin^2\theta\,\phi'^2}}$

Now, $\qquad \dfrac{\partial f}{\partial \phi} - \dfrac{d}{d\theta}\left(\dfrac{\partial f}{\partial \phi'}\right) = 0 \text{ or } \dfrac{\phi'\sin^2\theta}{\sqrt{1+\sin^2\theta\,\phi'^2}} = C$

or $\qquad \phi' = \dfrac{C\,\mathrm{cosec}^2\theta}{\left(1-C^2-C^2\cot^2\phi\right)^{1/2}} = \dfrac{d\phi}{d\theta}\,, \;\therefore\; \phi = \alpha - \sin^{-1}(C'\cot\theta)$

where α and C' are constants and these may be fixed by limits θ_1, ϕ_1 and θ_2, ϕ_2

$$C'\cot\theta = \sin(\alpha-\phi) \text{ or } C'r\cos\theta = r\sin(\alpha-\phi)\sin\theta$$

or $\qquad C'\,r\cos\theta = \sin\alpha\; r\cos\phi\sin\theta - \cos\alpha\; r\sin\phi\sin\theta$

or $\qquad C'z = x\sin\alpha - y\cos\alpha$

where we have transformed from spherical coordinates to cartesian coordinates.

The above equation represents a plane passing through the origin (0, 0, 0). This plane will cut the surface of the sphere in a great circle (whose centre is at the origin). This indicates that the shortest or longest distance between two points on the surface of the sphere is an arc of the circle with its centre at the origin.

5.3. DEDUCTION OF HAMILTON'S PRINCIPLE FROM D'ALEMBERTS PRINCIPLE

According to D'Alemberts principle [eq.(14), Chapter 2]

$$\sum_{i=1}^{N}(\mathbf{F}_i - \dot{\mathbf{p}}_i)\bullet\delta\mathbf{r}_i = 0$$

or

$$\sum_i \dot{\mathbf{p}}_i \bullet \delta\mathbf{r}_i = \sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i \qquad ...(18)$$

Here

$$\dot{\mathbf{p}}_i \bullet \delta\mathbf{r}_i = m_i\ddot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i = \frac{d}{dt}(m_i\dot{\mathbf{r}}_i)\bullet\delta\mathbf{r}_i$$

$$= \frac{d}{dt}(m_i\dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i) - m_i\dot{\mathbf{r}}_i \bullet \frac{d}{dt}(\delta\mathbf{r}_i) = \frac{d}{dt}(m_i\dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i) - m_i\dot{\mathbf{r}}_i \bullet \delta\left(\frac{d\mathbf{r}_i}{dt}\right)$$

[because the δ-variation in the velocity corresponding to the virtual displacement $\delta\mathbf{r}_i$ is given by

$$\delta\left(\frac{d\mathbf{r}_i}{dt}\right) = \delta(\mathbf{v}_i) = \mathbf{v}_i(\mathbf{r}_i + \delta\mathbf{r}_i) - \mathbf{v}(\mathbf{r}_i) = \frac{d}{dt}(\mathbf{r}_i + \delta\mathbf{r}_i) - \frac{d}{dt}(\mathbf{r}_i) = \frac{d}{dt}(\delta\mathbf{r}_i)]$$

$$= \frac{d}{dt}[m_i\dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i] - \delta\left(\tfrac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)\ \left[\because \delta(\dot{\mathbf{r}}_i^2) = \delta(\dot{\mathbf{r}}_i \bullet \dot{\mathbf{r}}_i) = 2\dot{\mathbf{r}}_i \bullet \delta\dot{\mathbf{r}}_i\right]$$

Thus the eq. (18) is

$$\frac{d}{dt}\left[\sum_i m_i\dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i\right] - \delta\left(\sum_i \frac{1}{2}m_i\mathbf{v}_i^2\right) = \sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i \qquad ...(19)$$

If the forces are conservative,

$$\mathbf{F}_i = -\nabla_i V = -\frac{\partial V}{\partial x_i}\hat{\mathbf{i}} - \frac{\partial V}{\partial y_i}\hat{\mathbf{j}} - \frac{\partial V}{\partial z_i}\hat{\mathbf{k}},$$

and the virtual work done is

$$\sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i = -\sum_i\left[\frac{\partial V}{\partial x_i}\hat{\mathbf{i}} + \frac{\partial V}{\partial y_i}\hat{\mathbf{j}} + \frac{\partial V}{\partial z_i}\hat{\mathbf{k}}\right]\bullet\left[\delta x_i\hat{\mathbf{i}} + \delta y_i\hat{\mathbf{j}} + \delta z_i\hat{\mathbf{k}}\right]$$

$$= -\sum_i\left[\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i\right] = -\delta V \qquad ...(20)$$

Also $\sum_i \frac{1}{2}m_i v_i^2 = T$, represents the kinetic energy of the system. Hence eq. (19) is

$$\frac{d}{dt}\left[\sum_i m_i\dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i\right] = \delta T - \delta V = \delta(T - V) \qquad ...(21)$$

Integrating it from $t = t_1$, to $t = t_2$ with the condition that $\delta\mathbf{r}_i(t_1) = \delta\mathbf{r}_i(t_2) = 0$ at the ends of the path, we get

$$\left[\sum_i m_i \dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \delta(T-V)\, dt \text{ or } \int_{t_1}^{t_2} \delta(T-V)\, dt = 0 \text{ or } \delta\int_{t_1}^{t_2}(T-V)\, dt = 0 \quad ...(22)$$

because $\delta\,[(T-V)\,dt] = \delta\,(T-V)\,dt + (T-V)\,\delta\,(dt) = \delta\,(T-V)\,dt\ [\because\ \delta\,(dt) = 0]$.

Putting $T - V = L$, we get the Hamilton's principle *i.e.*,

$$\delta\int_{t_1}^{t_2} L\, dt = 0 \quad ...(23)$$

5.4. MODIFIED HAMILTON'S PRINCIPLE

According to Hamilton's principle

$$\delta\int_{t_1}^{t_2} L\, dt = 0 \quad ...(24)$$

where $L = T - V = L(q_k, \dot{q}_k, t)$.

Equation (24) can be written in terms of Hamiltonian H by using the relation (25) of Chapter 3 *i.e.*,

$$H(p_k, q_k, t) = \sum_k p_k \dot{q}_k - L(q_k, \dot{q}_k, t)$$

Hence the Hamilton's principle (24) in the new form is obtained as

$$\delta\int_{t_1}^{t_2}\left(\sum_k p_k \dot{q}_k - H\right) dt = 0 \quad ...(25)$$

This is known as ***modified Hamilton's principle.***

In case of Hamilton's principle [eq. (24)] the path refers to configuration space and the variation of path allows for the variations in the generalized coordinates q_k at constant t. In the case of modified Hamilton's principle, the integral is to be evaluated over the path of the representative point of the system in phase space and q_k and p_k coordinates are to be treated as independent coordinates in the phase space. The δ-variation here implies independent variations of both the generalized and momenta coordinates q_k and p_k at constant t.

5.5. DEDUCTION OF HAMILTON'S EQUATIONS FROM MODIFIED HAMILTON'S PRINCIPLE (OR VARIATIONAL PRINCIPLE)

The δ-variation of q_k and p_k coordinates at constant t can be expressed in terms of a parameter α common to all points of the path of integration in phase space [(similar to eq.(3)] as

$$\delta q_k = \frac{\partial q_k}{\partial \alpha}\delta\alpha = \eta_k\,\delta\alpha \text{ and } \delta p_k = \frac{\partial p_k}{\partial \alpha}\delta\alpha = \eta'_k\,\delta\alpha \quad ...(26)$$

where η_k and η'_k are arbitrary subject to the conditions

$$\eta_k(t_1) = \eta_k(t_2) = \eta'_k(t_1) = \eta'_k(t_2) = 0 \quad ...(27)$$

Therefore, the δ-variation of the integral in (25) is

$$\delta\int_{t_1}^{t_2}\left[\sum_k p_k \dot{q}_k - H\right] dt = \int_{t_1}^{t_2} \sum_k \left[\left(\frac{\partial p_k}{\partial \alpha}\dot{q}_k + p_k \frac{\partial \dot{q}_k}{\partial \alpha}\right)\delta\alpha - \frac{\partial H}{\partial q_k}\frac{\partial q_k}{\partial \alpha}\delta\alpha - \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial \alpha}\delta\alpha\right] dt$$

$$= \delta\alpha \int_{t_1}^{t_2} \sum_k \left[\eta'_k \dot{q}_k + p_k \frac{\partial \dot{q}_k}{\partial \alpha} - \eta_k \frac{\partial H}{\partial q_k} - \eta'_k \frac{\partial H}{\partial p_k} \right] dt \qquad ...(28)$$

But
$$\int_{t_1}^{t_2} p_k \frac{\partial \dot{q}_k}{\partial \alpha} dt = \int_{t_1}^{t_2} p_k \frac{d}{dt}\left[\frac{\partial q_k}{\partial \alpha}\right] dt = \int_{t_1}^{t_2} p_k \frac{d\eta_k}{dt} dt$$

$$= \left[p_k \eta_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k \eta_k dt = -\int_{t_1}^{t_2} \dot{p}_k \eta_k dt \quad [\because \ \eta_k(t_1) = \eta_k(t_2) = 0]$$

Also in view the modified Hamilton's principle [eq. (25)], the δ–variation of the integral must be zero. Therefore, we obtain from (28)

$$\delta\alpha \int_{t_1}^{t_2} \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\dot{\eta}'_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right)\eta_k \right] dt = 0$$

or
$$\int_{t_1}^{t_2} \sum_k \left[\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right)\delta q_k \right] dt = 0 \qquad ...(29)$$

Since q_k and p_k are independent variables, the integral will be zero, when

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0 \text{ and } \dot{p}_k + \frac{\partial H}{\partial q_k} = 0 \text{ or } \dot{q}_k = \frac{\partial H}{\partial p_k} \text{ and } \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad ...(30)$$

These are the desired ***Hamilton's canonical equations.***

5.6. DEDUCTION OF LAGRANGE'S EQUATIONS FROM VARIATIONAL PRINCIPLE FOR NON-CONSERVATIVE SYSTEMS (HOLONOMIC CONSTRAINTS)

We deduced Hamilton's principle from D'Alembert's principle for conservative forces. If the forces are not conservative, eq.(19) can be written as

$$\frac{d}{dt}\left[\sum_i m_i \dot{\mathbf{r}}_i \bullet \delta\mathbf{r}_i\right] = \delta T + \delta W \qquad ...(31)$$

where $\delta T = \delta \sum_i \frac{1}{2} m_i v_i^2$ and $\delta W = \sum_i \mathbf{F}_i \bullet \delta\mathbf{r}_i$ = virtual work done.

The integration of (31) from $t = t_1$ to $t = t_2$ with the condition $\delta\mathbf{r}_i(t_1) = \delta\mathbf{r}_i(t_2) = 0$ at the end points, we get

$$\int_{t_1}^{t_2} \delta[T + W]\, dt = 0 \ \text{ or } \ \delta\int_{t_1}^{t_2} [T + W]\, dt = 0 \qquad ...(32)$$

Eq. (32) is known as ***extended Hamilton's principle.*** Here $\mathbf{F}_i$ are the non-conservative forces. We can write as in eq.(19) (Chapter 2)

$$\delta W = \sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = \sum_{i,k} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k = \sum_k G_k \delta q_k \qquad ...(33)$$

where G_k are the components of generalized force.

Thus the extended Hamilton's principle (32) gives

$$\delta\int_{t_1}^{t_2} T\,dt + \int_{t_1}^{t_2} \sum_k G_k\,\delta q_k\,dt = 0 \qquad ...(34)$$

Kinetic energy T in general is function of q_k and $\dot{q}_k$ and hence

$$\delta\int_{t_1}^{t_2} T(q_k,\dot{q}_k)\,dt = \int_{t_1}^{t_2} \delta T(q_k,\dot{q}_k)\,dt = \int_{t_1}^{t_2}\sum_k\left[\frac{\partial T}{\partial q_k}\delta q_k + \frac{\partial T}{\partial \dot{q}_k}\delta\dot{q}_k\right]dt$$

$$= \int_{t_1}^{t_2}\sum_k\frac{\partial T}{\partial q_k}\delta q_k\,dt + \sum_k\left[\frac{\partial T}{\partial \dot{q}_k}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_k}\right]\delta q_k\,dt$$

$$= \int_{t_1}^{t_2}\sum_k\left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)\right]\delta q_k dt \quad [\because \delta q_k(t_1) = \delta q_k(t_2) = 0] \qquad ...(35)$$

Thus eq. (34) is

$$\int_{t_1}^{t_2}\sum_k\left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + G_k\right]\delta q_k\,dt = 0 \qquad ...(36)$$

Since the constraints are holonomic, all δq_k are independent and hence the integral will vanish, if

$$\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + G_k = 0 \text{ or } \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \qquad ...(37)$$

These are the Lagrange's equations for holonomic and non-conservative system.

5.7. LAGRANGE'S EQUATIONS OF MOTION FOR NON-HOLONOMIC SYSTEMS (*LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS*)

In the derivation of Lagrange's equations from D'Alembert's principle or Hamilton's principle, we need the requirement of holonomic constraints at the final step, when the variations δq_k are considered to be independent of each other. In case of non-holonomic systems, the generalized coordinates are not independent of each other. However, we can treat certain types of non-holonomic systems for which the equations of constraint can be put in the form

$$\sum_k a_{lk}\,dq_k + a_{lt}\,dt = 0 \qquad ...(38)$$

These equations of constraints connect the differentials dq_k's by linear relations. For each l, there is one equation and we assume that there are m such equations for $l = 1, 2, ..., m$.

In case of δ-variation, the virtual displacements δq_k are taken at constant times and hence the m equations of constraints, consistent for virtual displacements, are

$$\sum_k a_{lk}\delta q_k = 0 \qquad ...(39)$$

Eq. (39) now can be used to reduce the number of virtual displacements to independent ones. The procedure applied for this purpose is called ***Lagrange's method of undetermined multipliers.***

If eq. (39) is valid, then the multiplication of this equation by λ_l, an undetermined quantity, yields

$$\lambda_l \sum_k a_{lk}\, \delta q_k = 0 \quad \text{or} \quad \sum_k \lambda_l a_{lk}\, \delta q_k = 0 \qquad ...(40)$$

where λ_l (1, 2,.. .,m) are undetermined quantities and they are functions in general of the coordinates and time. Summing eq. (40) over l and then integrating the sum with respect to time from $t = t_1$ to $t = t_2$, we get

$$\int_{t_1}^{t_2} \sum_{k,l} \lambda_l\, a_{lk}\, \delta q_k\, dt = 0 \qquad ...(41)$$

We assume the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \qquad ...(42)$$

to hold for the non-holonomic system. This implies that

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k\, dt = 0 \qquad ...(43)$$

Adding (41) and (43), we obtain

$$\int_{t_1}^{t_2} \sum_{k=1}^{n} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_l \lambda_l a_{lk} \right] \delta q_k\, dt = 0 \qquad ...(44)$$

Still all δq_k's (k =1,2, ..., n) are not independent of each other. First $n - m$ of these δq_k's may be chosen independently and the last m of these δq_k's are then fixed by the eq. (39).

Till now the values of λ_l have not been specified. We choose the λ_l's such that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_k a_{lk} = 0 \qquad ...(45)$$

where $k = n - m + 1, n - m + 2, ..., n$. Thus eqs. (45) will determine m values of λ_l and then eq. (44) can be written as

$$\int_{t_1}^{t_2} \sum_{k=1}^{n-m} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} \right] \delta q_k\, dt = 0 \qquad ...(46)$$

where the δq_k's (k =1, 2, ..., $n - m$), involved, are independent ones. Therefore, for the integrand in (46) to vanish

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0 \qquad ...(47)$$

which are $n - m$ equations for k =1, 2, ..., $n - m$.

Adding eqs. (45) and (47), we get the complete set of the Lagrange's equations for the non-holonomic system, *i.e.*,

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k} \right] - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} \qquad ...(48)$$

where $k = 1, 2, ...n$.

Eq. (48) gives us n equations, but there are $n + m$ unknowns, n coordinates q_k and m Lagrange's multipliers. The remaining m unknown q_k's are determined from m equations of constraints (38), written in the following form of m first-order differential equations :

$$\sum_k a_{lk}\,\dot{q}_k + a_{lt} = 0 \qquad ...(49)$$

5.8. PHYSICAL SIGNIFICANCE OF LAGRANGE'S MULTIPLIERS λ_l

Suppose we remove the constraints on the system, but apply external forces G_k so that the motion of the system remains unchanged. Now, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = G_k \qquad ...(50)$$

Since the applied for... are equal to the forces of constraints, eqs. (48) and (50) must be identical, resulting

$$G_k = \sum_{l=1}^{n} \lambda_l\, a_{lk} \qquad ...(51)$$

Thus the generalized forces of constraints G_k have been identified to $\sum \lambda_l\, a_{lk}$. We observe that in such problems, we need not to eliminate the forces of the constraints but the procedure itself determines these forces by eq. (51).

Eq. (38) does not represent the most general type of nonholonomic constraints because it does not include equations of constraints in the form of inequalities. However, it includes holonomic constraints. Equation representing holonomic constraints is given by

$$f(q_1, q_2, ..., q_n, t) = 0 \qquad ...(52)$$

So that
$$\sum_{k=1}^{n} \frac{\partial f}{\partial q_k} dq_k + \frac{\partial f}{\partial t} dt = 0 \qquad ...(53)$$

This is identical in form to eq. (38) with the coefficients a_{lk} and a_{lt}, given by

$$a_{lk} = \frac{\partial f}{\partial q_k} \text{ and } a_{lt} = \frac{\partial f}{\partial t} \qquad ...(54)$$

Thus one can use Lagrange's method of undetermined multipliers for holonomic constraints when it is not easy to reduce all the q_k's to independent coordinates or we may be interested in knowing the force of constraints.

5.9. EXAMPLES OF LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

(1) ***Rolling hoop on an inclined plane*** : *Discuss the motion of a hoop rolling down an inclined plane without slipping. Find its acceleration and frictional force of constraint by using the method of undetermined multipliers.* **(Agra 1997, 91, 83; Rohilkhand 86)**

Solution : Let ϕ be the inclination of the inclined plane of length l with the horizontal. If a hoop of mass M and radius R is rolling down an inclined plane starting from a point O without slipping, then x and θ are two generalized coordinates and the equation of constraint is

$$x = R\theta \text{ or } dx = R d\theta \text{ or } dx - R d\theta = 0 \qquad ...(i)$$

As there is only one constraint equation, only one Lagrange's multiplier λ will be required. Here

$$a_{1x} dx + a_{1\theta}\, d\theta = 0$$

$$[\because \sum_{k=1}^{n} a_{lk}\, dq_k + a_{lt}\, dt = 0]$$

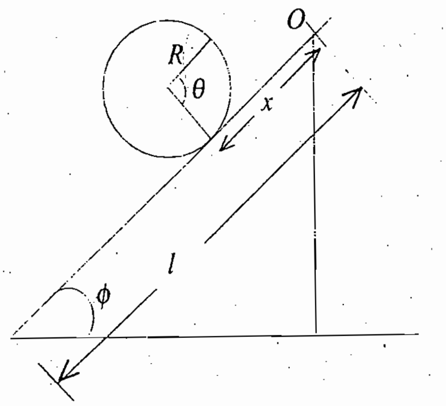


Fig. 5.8 : A hoop rolling down on inclined plane

So that $a_{1x} = 1 = a_x$ (say) and $a_{1\theta} = -R = a_\theta$ (say) ...(*ii*)

Kinetic energy of the hoop $T =$ Kinetic energy of motion of centre of mass + Rotational kinetic energy about the centre of mass

$$= \frac{1}{2} M\dot{x}^2 + \frac{1}{2} MR^2\dot{\theta}^2$$

Potential energy of the hoop $V = Mg(l - x)\sin\phi$

Thus $$L = T - V = \frac{1}{2} M\dot{x}^2 + \frac{1}{2} MR^2\dot{\theta}^2 - Mg(l - x)\sin\phi \qquad ...(iii)$$

Equations of motion for two coordinates x and θ are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \lambda a_x \qquad ...(iv)$$

and $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda a_\theta \qquad ...(v)$$

Here, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \frac{\partial L}{\partial x} = Mg\sin\phi, \frac{\partial L}{\partial \dot{\theta}} = MR^2\dot{\theta}, \frac{\partial L}{\partial \theta} = 0$ and also $a_x = 1$ and $a_\theta = -R$.

Therefore, $$M\ddot{x} - Mg\sin\phi = \lambda \qquad ...(vi)$$

and $$MR^2\ddot{\theta} = -R\lambda \qquad ...(vii)$$

But from (*i*) $\ddot{x} = R\ddot{\theta}$. Hence from eq. (*vii*), we get

$$M\ddot{x} = -\lambda \qquad ...(viii)$$

Substituting for λ in eq. (*vi*), we obtain

$$M\ddot{x} - Mg\sin\phi = -M\ddot{x} \text{ or } \ddot{x} = \frac{1}{2} g\sin\phi \qquad ...(ix)$$

This is the acceleration of the hoop along the inclined plane. Note that it is one half of the acceleration it would have in slipping down a frictionless inclined plane.

The force of constraint λ is [by using eq. (*viii*) and (*ix*)]

$$\lambda = -\frac{1}{2} Mg\sin\phi \qquad ...(x)$$

This gives the frictional force due to constraint which reduces the acceleration $\ddot{x} = g\sin\phi$ (when there is only slipping without friction) to $\ddot{x} = \frac{1}{2} g\sin\phi$ (when the hoop is rolling without slipping).

Note : It is to be remarked that if we take constraint equation as $R\,d\theta - dx = 0$, then the constraint force will be obtained as $\lambda = \frac{1}{2} Mg\sin\phi$ which bears +ve sign. Thus in such problems we obtain only the magnitude of the force of constraint.

(2) *Simple pendulum* : *Find the equation of motion and force of constraint in case of simple pendulum by using Lagrange's method of undetermined multipliers.*

Solution : Referring Fig. 5.9, the Lagrangian L is given by

$$L = \frac{1}{2} m r^2 \dot{\theta}^2 + mgr\cos\theta \qquad ...(i)$$

where $V = -mgr\cos\theta$ with respect to position S.

The equation of constraint is

$$r = l \text{ or } dr = 0 \qquad ...(ii)$$

Here there is only one constraint equation, hence only one Lagrange's multiplier λ will be needed. There are two coordinates r and θ and the general constraint equation will be

$$a_r dr + a_\theta d\theta = 0 \qquad ...(iii)$$

$$\therefore \qquad a_r = 1,\ a_\theta = 0$$

Equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \lambda\, a_r \qquad ...(iv)$$

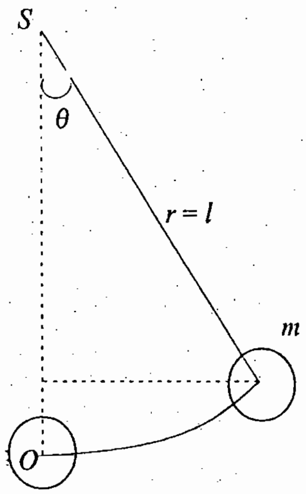


Fig. 5.9 : Simple pendulum

and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \lambda\, a_\theta \qquad ...(v)$$

Here, $\dfrac{\partial L}{\partial \dot{r}} = 0,\ \dfrac{\partial L}{\partial r} = mr\dot{\theta}^2 + mg\cos\theta,\ \dfrac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta},\ \dfrac{\partial L}{\partial \theta} = -mgr\sin\theta$

$$\therefore \qquad -mr\dot{\theta}^2 - mg\cos\theta = \lambda \qquad ...(vi)$$

$$mr^2\ddot{\theta} + mgr\sin\theta = 0 \qquad ...(vii)$$

where $\dot{r} = 0$ from (ii). Using $r = l$, (the constraint equation), equation of motion of simple pendulum is given by eq. (vii), *i.e.*,

$$l\ddot{\theta} + g\sin\theta = 0$$

For small θ, $\sin\theta = \theta$ and hence

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad ...(viii)$$

The force of constraint is

$$\lambda = -ml\dot{\theta}^2 - mg\cos\theta \qquad ...(ix)$$

which gives the force of constraint, *i.e.* tension $F = ml\dot{\theta}^2 + mg\cos\theta$ in magnitude.

5.10. Δ-VARIATION

In the δ-variation, the variation of the path allows for variations in the coordinate q_k at constant t and the varied path and the correct path have the same end points *i.e.*,

$$\delta q_k(t_1) = \delta q_k(t_2) = 0$$

Now, we introduce a new and more general type of variation of the path of a system, known as Δ-variation. In this variation, time as well as position coordinates are allowed to vary. At the end points of the path, the position coordinates are kept fixed, while changes in the time are allowed. The Δ-variation of a coordinate q_k is shown in Fig. 5.10 ; APB is the actual path and $A'P'B'$, the varied path. The end points of the path A and B after time Δt take the position A' and B' so that there is no change in the position coordinates, *i.e.*, $\Delta q_k(1) = \Delta q_k(2) = 0$. A point P on the actual path now goes over to the point P', on the varied path with the correspondence, given by

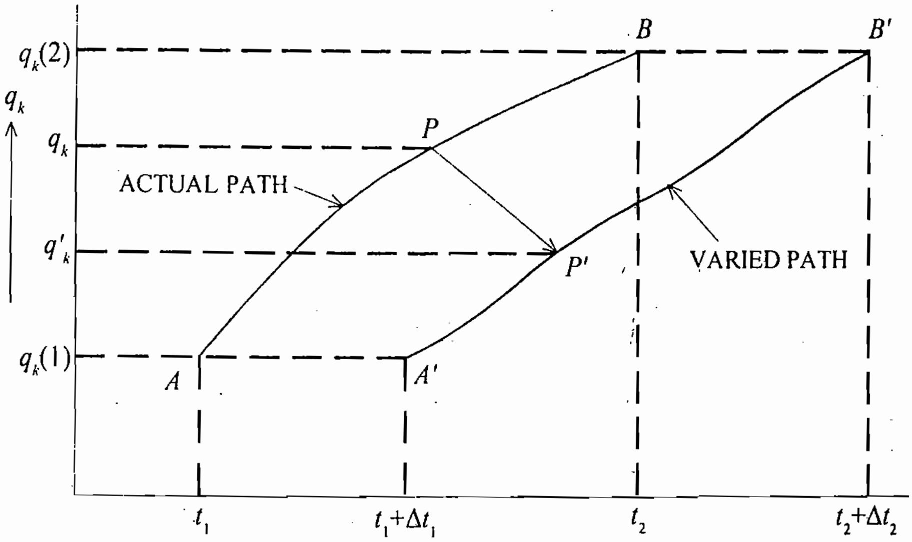


Fig. 5.10 : Δ-variation

$$q_k \to q'_k = q_k + \Delta q_k = q_k + \delta q_k + \dot{q}_k \Delta t \qquad ...(55)$$

where δ-variation has the same meaning as discussed earlier.

The Δ-variation of any function $f = f(q_k, \dot{q}_k, t)$ is given by

$$\Delta f = \sum_k \left[\frac{\partial f}{\partial q_k} \Delta q_k + \frac{\partial f}{\partial \dot{q}_k} \Delta \dot{q}_k \right] + \frac{\partial f}{\partial t} \Delta t = \sum_k \frac{\partial f}{\partial q_k} [\delta q_k + \dot{q}_k \Delta t] + \sum_k \frac{\partial f}{\partial \dot{q}_k} [\delta \dot{q}_k + \ddot{q}_k \Delta t] + \frac{\partial f}{\partial t} \Delta t$$

$$= \sum_k \left[\frac{\partial f}{\partial q_k} \delta q_k + \frac{\partial f}{\partial \dot{q}_k} \delta \dot{q}_k \right] + \left[\sum_k \left(\frac{\partial f}{\partial q_k} \dot{q}_k + \ddot{q}_k \frac{\partial f}{\partial \dot{q}_k} \right) + \frac{\partial f}{\partial t} \right] \Delta t$$

$$= \delta f + \Delta t \frac{df}{dt} \qquad ...(56)$$

Thus the Δ-operation is

$$\Delta = \delta + \Delta t \frac{d}{dt} \qquad ...(57)$$

5.11. PRINCIPLE OF LEAST ACTION

According to the principle of least action for a conservative system

$$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k \, dt = 0 \qquad ...(58)^*$$

---

* In the older literature, the integral in eq. (58) is usually called as action or action integral. The integral in Hamilton's principle is referred as action.

where the quantity $W = \int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt$ is sometimes called ***abbreviated action.***

Eq. (58) was established by Maupertuis (1668-1759) and therefore it is usually referred ***Maupertuis principle of least action.***

Proof : Let us consider Hamilton's principle function (or action integral) S, given by

$$S = \int_{t_1}^{t_2} L\, dt \qquad \text{...(59)}$$

The Δ-variation of S is

$$\Delta S = \Delta \int_{t_1}^{t_2} L\, dt = \left[\delta + \Delta t \frac{d}{dt}\right] \int_{t_1}^{t_2} L\, dt$$

$$= \delta \int_{t_1}^{t_2} L\, dt + \int_{t_1}^{t_2} \Delta t\, d(L) = \delta \int_{t_1}^{t_2} L\, dt + [L\, \Delta t]_{t_1}^{t_2} = \int_{t_1}^{t_2} \delta L\, dt + [L \Delta t]_{t_1}^{t_2} \quad [\because \delta(dt) = 0]$$

$$= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k\right] dt + [L\, \Delta t]_{t_1}^{t_2} \qquad \text{...(60)}$$

In the present case $\delta q_k \neq 0$ at the end points, hence $\delta \int_{t_1}^{t_2} L\, dt$ is not equal to zero. Now, according to Lagrange's equations, we have

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] - \frac{\partial L}{\partial q_k} = 0 \quad \text{or} \quad \frac{\partial L}{\partial q_k} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] \qquad \text{...(61)}$$

Also $$\delta \dot{q}_k = \frac{d}{dt}[\delta q_k] \qquad \text{...(62)}$$

Using (61) and (62), the quantity in the first term of eq. (60) is

$$\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k}\right] \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \frac{d}{dt}[\delta q_k] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right] = \frac{d}{dt}[p_k\, \delta q_k] \qquad \text{...(63)}$$

But in view of eq. (57)

$$\Delta q_k = \delta q_k + \Delta t \frac{dq_k}{dt} \quad \text{or } \delta q_k = \Delta q_k - \Delta t\, \dot{q}_k \quad \text{or } p_k \delta q_k = p_k \Delta q_k - p_k \dot{q}_k \Delta t \qquad \text{...(64)}$$

Hence $$\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k = \frac{d}{dt}[p_k \Delta q_k] - \frac{d}{dt}[p_k \dot{q}_k \Delta t] \qquad \text{...(65)}$$

Thus equation (60) is

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \sum_k \left[\frac{d}{dt}[p_k \Delta q_k] - \frac{d}{dt}[p_k\, \dot{q}_k \Delta t]\right] dt + [L\, \Delta t]_{t_1}^{t_2}$$

$$= \sum_k \int_{t_1}^{t_2} \left[d\,(p_k \Delta q_k) - d\,(p_k \dot{q}_k \Delta t)\right] + [L\, \Delta t]_{t_1}^{t_2}$$

$$= \sum_k [p_k \Delta q_k]_{t_1}^{t_2} - \sum_k [p_k \dot{q}_k \Delta t]_{t_1}^{t_2} + [L\, \Delta t]_{t_1}^{t_2} \qquad \text{...(66)}$$

As $\Delta q_k = 0$ at the end points, $[p_k \Delta q_k]_{t_1}^{t_2} = 0$.

Therefore equation (66) is

$$\Delta\int_{t_1}^{t_2} L dt = \left[\left(L - \sum_k p_k \dot{q}_k\right)\Delta t\right]_{t_1}^{t_2}$$

or $$\Delta\int_{t_1}^{t_2} L\, dt = -[H\Delta t]_{t_1}^{t_2} \qquad \left[\because H = \sum_k p_k \dot{q}_k - L\right] \qquad ...(67)$$

Now, if we restrict to systems for which $\partial H/\partial t = 0$ and to variations for which H remains constant (conservative systems), then

$$\Delta\int_{t_1}^{t_2} H\, dt = \int_{t_1}^{t_2} H d(\Delta t) = [H\Delta t]_{t_1}^{t_2} \qquad ...(68)$$

Substituting for $[H\Delta t]_{t_1}^{t_2}$ in eq. (67), we get

$$\Delta\int_{t_1}^{t_2} L\, dt = -\Delta\int_{t_1}^{t_2} H\, dt \quad \text{or} \quad \Delta\int_{t_1}^{t_2} [H + L]\, dt = 0$$

or $$\Delta\int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \qquad \left[\because H = \sum_k p_k \dot{q}_k - L\right] \qquad ...(69)$$

This is what is known as ***principle of least action.***

The quantity $\int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = W$ is called ***Hamilton's characteristic function***. Hence the principle of least action can be stated as

$$\Delta W = \Delta\int_{t_1}^{t_2} \sum_k p_k \dot{q}_k dt = 0 \qquad ...(70)$$

5.12. OTHER FORMS OF PRINCIPLE OF LEAST ACTION

(1) For a conservative system, the Hamiltonian is constant and the potential energy is independent of time. So that

$$p_k = \frac{\partial L}{\partial \dot{q}_k} = \frac{\partial (T - V)}{\partial \dot{q}_k} = \frac{\partial T}{\partial \dot{q}_k}$$

Therefore, $$\sum_k p_k \dot{q}_k = \sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T \qquad ...(71)$$

because $\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T$ [eq. (30), Chapter 3].

Therefore the principle of least action (70) takes the form

$$\Delta\int_{t_1}^{t_2} 2T\, dt = 0 \quad \text{or} \quad \Delta\int_{t_1}^{t_2} T\, dt = 0 \qquad ...(72)$$

This is another form of principle of least action.

In case, if there is not any external force on the system, its kinetic energy T as well as total energy H will be conserved. Then the principle of least action (72) takes a special form, given by

$$\Delta \int_{t_1}^{t_2} dt = 0 \quad \text{or} \quad \Delta(t_2 - t_1) = 0 \qquad \text{...(73a)}$$

or $\qquad t_2 - t_1 =$ an extremum. ...(73b)

Thus we see that out of all possible paths between two points, the system moves along that particular path for which the time of transit is an extremum, provided that the kinetic energy along with total energy of the system remains constant. This form of principle of least action is the same as Fermat's principle in geometrical optics, which states that ***a ray of light travels between two points along such a path that the time taken is the extremum.***

(2) Jacobi's form of the principle of least action : When transformation equations do not involve time, the kinetic energy of a system can be expressed as a homogeneous quadratic function of the velocities [eq. (39), Chapter 2], *i.e.*,

$$T = \frac{1}{2}\sum_{kl} M_{kl}\, \dot{q}_k\, \dot{q}_l \qquad \text{...(74)}$$

We can construct a configuration space for which M_{kl} coefficients form the metric tensor and the element of path length $d\rho$ in this space is defined as

$$d\rho^2 = \sum_{k,l} M_{kl}\, dq_k\, dq_l \qquad \text{...(75)*}$$

So that $\qquad [d\rho/dt]^2 = \sum_{k,l} M_{kl}\, \dot{q}_k\, \dot{q}_l \qquad$ ...(76)

From eqs. (74) and (76), we get

$$T = \frac{1}{2}\left(\frac{d\rho}{dt}\right)^2 \qquad \text{...(77)}$$

whence $\qquad dt = d\rho / \sqrt{2T} \qquad$ ....(78)

Hence the principle of least action (72) is

$$\Delta \int_{t_1}^{t_2} T dt = \Delta \int_{t_1}^{t_2} \sqrt{2T} d\rho = 0 \qquad \text{...(79)}$$

But $H = T + V(q)$, total energy is constant for conservative system. Thus, the principle of least action takes the form

$$\Delta \int_{t_1}^{t_2} \sqrt{2[H - V(q)]}\, d\rho = 0 \qquad \text{...(80)}$$

This is known as ***Jacobi's form of the least action principle***. This form of principle of least action is related with the path of the system point (in a curvilinear configuration space described by the metric tensor with elements M_{kl}) rather than with its motion in time.

(3) Principle of least action in terms of arc length of the particle trajectory : If the system contains only one particle of mass m, its kinetic energy is given by

* This is similar to Riemannian space in which the path length is given by $ds^2 = \sum_{k,l} g_{k,l}\, dx_k\, dx_l$, where g_{kl} is the element of metric tensor.

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\frac{ds}{dt}\right]^2 \quad ...(81)$$

where ds is the element of arc traversed by the particle in time dt.

From eq. (81), we obtain

$$dt = \sqrt{m/2T}\, ds$$

So that the principle of least action (72) can be written as

$$\Delta\int 2T\sqrt{\frac{m}{2T}}\, ds = 0 \text{ or } \Delta\int \sqrt{2mT}\, ds = 0$$

or

$$\Delta\int \sqrt{2m\,(H-V)}\, ds = 0 \quad ...(82)$$

This equation represents the principle of least action in terms of arc length of the particle trajectory. Eq. (82) is similar to the Jacobi's form of the principle of least action.

Questions

1. What is δ-variation ? Show that the integral $I = \int_{x_1}^{x_2} f(y, y', x)\, dx$ is stationary, when $\frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] - \frac{\partial f}{\partial y} = 0$ where $y' = \frac{dy}{dx}$.
2. Show that for a function $f = f(y_1, y_2, \ldots, y_n, y'_1, y'_2, \ldots, y'_n, x)$, the integral

   $$I = \int_{x_1}^{x_2} f\, dx \text{ will be extremum, if } \frac{d}{dx}\left(\frac{\partial f}{\partial y'_k}\right) - \frac{\partial f}{\partial y_k} = 0,$$

   where $k = 1, 2, \ldots, n$ and $y_k' = dy_k/dx$. **(Meerut 1980)**
3. Obtain the Euler-Lagrange differential equation by variational method. **(Kanpur 2003)**
4. State Hamilton's principle and derive Lagrange's equations of motion from it. Discuss how the result will be modified if the forces are conservative. **(Agra 1992, 74 ; Meerut 81)**
5. Show that the path followed by a particle in sliding from one point to an other in the absence of friction in the shortest time is cycloid. **(Agra 1991, 89)**
6. What is variational principle ? Obtain Hamiltons equations from variational principle. **(Meerut 2001; Kanpur 1998; Rohilkhand 1999)**
7. What is meant by variational principle ? Prove that the equation of curve for which surface area of revolution is minimum, is a catenary $x = a \cosh (y - b)/a$ where a and b are constants. **(Meerut 1981)**
8. Deduce Hamilton's principle from D'Alembert's principle. Derive Lagrange's equations from it. **(Garwal 1999; Agra 99, 90)**
9. Derive the Euler-Lagrange's equations of motion using the calculus of variations and hence obtain Lagrange's equations of motion for a system of particles. **(Meerut 1999, 83; Agra 77)**
10. Derive Hamilton's equations from the variational principle. Deduce Hamilton's principle. How can this principle be used to find the equation of one-dimensional harmonic oscillator ? **(Agra 1988)**
11. Explain the method of Lagrange's undetermined multipliers in deriving the equation of motion for a conservative non-holonomic system from Hamilton's principle. Apply this method to the problem of a hoop rolling down an inclined plane without slipping. **(Agra 1991, 88)**

12. State and explain modified Hamilton's principle. Deduce Hamilton's equations by using this principle. **(Agra 1998, 97)**
13. What is Δ-variation ? Discuss how it differs from δ-variation. State and prove the principle of least action.
14. State and prove the principle of least action. **(Agra 2001, 1991, 90, Kanpur 98; Garwal 93)**
15. State Hamilton's principle of least action. Obtain. Hamilton's equations of motion from this principle. **(Gorakhpur 1996)**
16. (*a*) Deduce the principle of least action in the following form :

$$\Delta \int_{t_1}^{t_2} T\,dt = 0$$

where T is the kinetic energy. **(Kanpur 2002)**

(*b*) If the kinetic energy of the system is conserved, show that out of all the paths between two points, the system moves along that particular path for which the time of transit is an extremum.

17. Describe the principle of least action and deduce the Jacobi's form of the principle of least action. **(Rohilkhand 1999)**

Problems

[SET- I]

1. State and prove the brachistochrone problem. **(Kanpur 2001, 1998; Meerut 1991)**
2. A fixed volume of water is rotating in a cylinder with constant angular velocity. Find the curve of the water surface that will minimize the total potential energy of the water in the combined gravitational centrifugal field.

   Ans : Parabola.
3. Prove that the shortest distance between the points on the surface of a sphere is the arc of the great circle connecting them. **(Rohilkhand 1986)**
4. Find the plane curve of fixed peimeter and maximum area.

   Ans : Circle.
5. Apply variational principle to show that the path of projectile is parabola.
6. Use the variational principle to show that the shortest distance between two points in space is a straight line joining them. **(Meerut 1992)**
7. Apply the variational principles to deduce the equation for stable equilibrium configuration of a uniform heavy flexible string fixed between two points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ in the constant gravity field of the earth [Fig. 5.11].

   Ans : $y = a \cosh (b + x/a)$, catenary; a and b are fixed by the coordinates of the two ends *i.e.*, (x_1, y_1) and (x_2, y_2).

   [**Hint :** The condition of the minimum potential energy can be expressed as $\delta \int_A^B mg\,y\,ds = 0$, where $m\ ds$ is the mass of an element of length ds which is at a vertical height y. But $ds = dx\sqrt{1+y'^2}$, therefore $\delta \int_A^B y\sqrt{1+y'^2}\,dx = 0$.

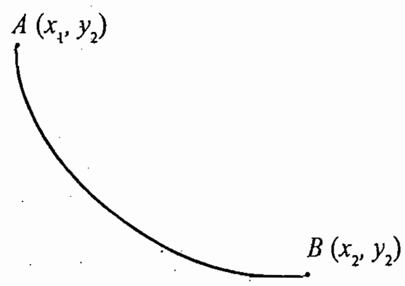


Fig. 5.11

Take $f = y\sqrt{1+y'^2}$ and apply Euler-Lagrange's equation.]

8. A particle moves on the frictionless inner surface of a cone of half angle α under the influence of gravity. Obtain the equations of motion.

9. A curve AB, having end points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$, is revolved about X-axis so that the area of the surface of revolution is a minimum. [Fig. 5.12]. Show that

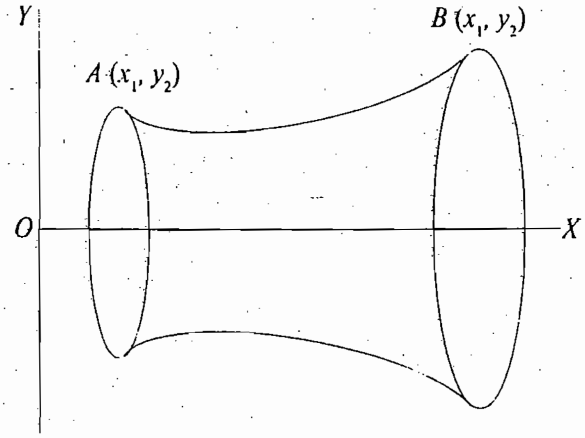


Fig. 5.12.

$$S = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$$

Obtain the differential equation of the curve and prove that the curve represents a catenary.

Ans : $yy'' - y'^2 - 1 = 0$.

10. Two identical circular wires in contact are placed in a soap solution and then they are separated, resulting in the formation of a soap film. Explain why theshape of the surface of the soap film is related to the result of the above problem.

11. Discuss the motion of a disc that is rolling down an inclined plane without slipping. Find the acceleration and the force of constraint by using the method of undetermined multipliers.

Ans : $a = \frac{2}{3} g \sin\theta$; $\lambda = -\frac{1}{3} Mg \sin\theta$.

12. A sphere of radius a and mass m rests on the top of a fixed rough sphere of radius b. The first sphere is slightly displaced so that it rolls without slipping [Fig. 5.13]. Obtain the equation of motion for the rolling sphere.

Ans : $\ddot{\phi} = -\dfrac{5g}{7(a+b)} \sin\phi$

[**Hint :** $bd\phi = ad\psi$ or $bd\phi - ad\psi = 0$

Lagrangian of the rolling sphere is

$$L = \frac{1}{2} m\,(a+b)^2 \dot{\phi}^2 + \frac{1}{2} I\omega^2 - mg(a+b)\cos\phi \quad \text{where } \frac{1}{2} I\omega^2 = \frac{1}{2}\left(\frac{2}{5} ma^2\right)\left(\dot{\phi} + \dot{\psi}\right)^2$$

Now, use $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\phi}}\right) - \dfrac{\partial L}{\partial \phi} = \lambda b$ and $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\psi}}\right) - \dfrac{\partial L}{\partial \psi} = -\lambda a$.]

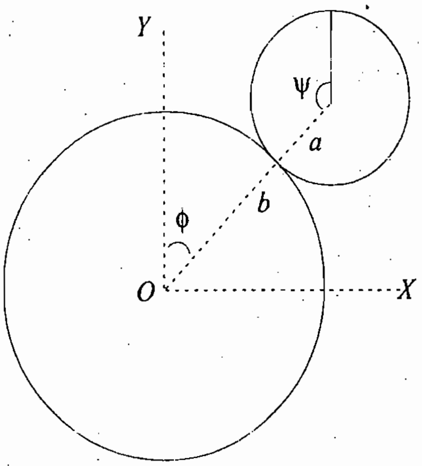


Fig. 5.13.

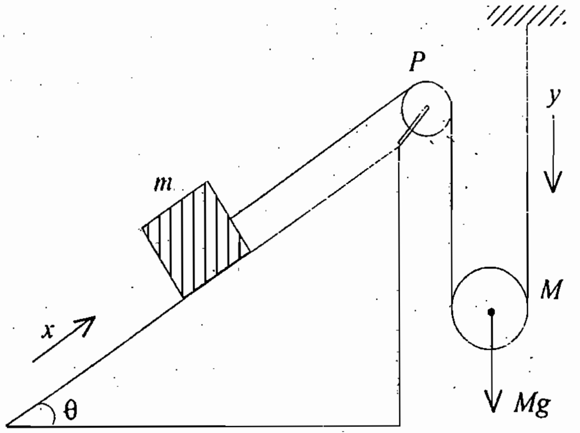


Fig. 5.14.

13. A block of mass m is pulled up as mass M moves down [Fig. 5.14]. The coefficient of friction between m and the incline is μ. Assume the pulley P to be smooth and the string inextensible. Use the Lagrange's method of undetermined multipliers to find the accelerations of m and M.

Ans : $\ddot{x} = 2\ddot{y}\,;\ddot{y} = (Mg - 2\mu\, mg\cos\theta - 2mg\sin\theta)/4(m+M)$

[**Hint :** $2dy = dx$ or $-dx + 2dy = 0$.]

[SET- II]

1. The Lagrangian of a free particle is given in the form

$$L = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}mg_{ik}\left(\frac{dx_i}{dt}\right)\left(\frac{dx_k}{dt}\right).$$

Use Lagrange's equations of motion to show that $\ddot{x}_i = \lambda_{ijk}\,\dot{x}_j\,\dot{x}_k = 0$

become the equations of motion, where $\lambda_{ijk} = \frac{1}{2}g_{il}\left[\frac{\partial g_{kl}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_k} - \frac{\partial g_{jk}}{\partial x_l}\right]$ are the so called Riemann-Christoffel symbols. In the above equations; Einstein's summation convention for indices has been used.

2. Prove that the sphere is the solid figure of revolution which, for a given surface area, has maximum volume.

Objective Type Questions

1. Choose the correct statement/statements :
   (*a*) In δ-variation, time as well as position coordinates are allowed to vary.
   (*b*) In Δ-variation, time as well as position coordinates are allowed to vary.
   (*c*) δ-variation does not involve time.
   (*d*) Δ-variation does not involve time.
   Ans : (*b*), (*c*).
2. In case of modified Hamilton's principle,
   (*a*) the path refers to configuration space. (*b*) the path refers to phase space.
   (*c*) $\int_{t_1}^{t_2} p_j \cdot dq_j - \delta \int_{t_1}^{t_2} H\, dq_j = 0$ (*d*) $\delta\int_{t_2}^{t_1}(\sum_k p_k\dot{q}_k - H)dt = 0$

where the terms have usual meaning.

Ans : (*b*), (*d*).

3. According to the principle of least action

(*a*) $\Delta\int(\sum_k p_k\dot{q}_k - H)dt = 0$

(*b*) $\Delta\int\sum_k p_k\dot{q}_k\, dt = 0$

(*c*) $\Delta\int(H + L)\, dt = 0$

(*d*) $\int\sum_k p_k\dot{q}_k\, dt = 0$

Ans : (*b*), (*c*) .

4. The modified Hamilton's principle is given by

(*a*) $\delta\sum_j\int_{t_1}^{t_2} p_j dq_j - \delta\int_{t_1}^{t_2} Hdt = 0$

(*b*) $\delta\sum_j\int_{t_1}^{t_2} p_j\, dq_j + \int_{t_1}^{t_2} H\, dt = 0$

(*c*) $\delta\sum_j\int_{t_1}^{t_2} p_j\, dq_j - \delta\int_{t_1}^{t_2} H\, dq_j = 0$

(*d*) $\delta\sum_j\int_{t_1}^{t_2} p_j\, dq_j + \delta\int_{t_1}^{t_2} H\, dq_j = 0$

(Kanpur 2003)

Ans : (*a*)

Short Answer Questions

1. What is δ-variation
2. Obtain the Euler-Lagrange differential equation by a variational method. **(Kanpur 2001)**
3. What is Brachistochrone problem. **(Kanpur 2003)**
4. What is extended Hamilton's principle ?
5. Using Hamilton's principle, obtain the modified Hamilton's principle. **(Kanpur 2001)**
6. What is Δ-variation ? Discuss how it differs from δ-variation. **(Kanpur 2002)**
7. Explain the principle of least action. **(Agra 2003, 02)**
8. Fill in the blanks :

   (*i*) Hamilton's principal function is....................

   (*ii*) The Δ - operation is $\Delta = \delta +$....................

Ans. (*i*) $S = \int_{t_1}^{t_2} L\, dt$, (*ii*) $\Delta t\,\frac{d}{dt}$

CHAPTER 6

Canonical Transformations

6.1. CANONICAL TRANSFORMATIONS

In several problems, we may need to change one set of position and momentum coordinates into another set of position and momentum coordinates. Suppose that q_k and p_k are the old position and momentum coordinates and Q_k and P_k are the new ones. Let these coordinates be related by the following transformations :

$$\left.\begin{aligned} P_k &= P_k(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \\ \text{and} \quad Q_k &= Q_k(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \end{aligned}\right\} \qquad \ldots(1)$$

Now, if there exists a Hamiltonian H' in the new coordinates such that

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \quad \text{and} \quad \dot{Q}_k = \frac{\partial H'}{\partial P_k} \qquad \ldots(2)$$

where

$$H' = \sum_{k=1}^{n} P_k \dot{Q}_k - L' \qquad \ldots(3)$$

and L' substituted in the Hamilton's principle

$$\delta \int L' \, dt = 0 \qquad \ldots(4)$$

gives the correct equations of motion in terms of the new coordinates P_k and Q_k, then the transformations (1) are known as *canonical* (or *contact*) *transformations.*

6.2. LEGENDRE TRANSFORMATIONS

This is a mathematical technique used to change the basis from one set of coordinates to another. If $f(x, y)$ is a function of two variables x and y, then the differential of this function can be written as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad \text{or} \quad df = u\,dx + v\,dy \qquad \ldots(5)$$

where

$$u = \partial f / \partial x \quad \text{and} \quad v = \partial f / \partial y \qquad \ldots(6)$$

Now, we want to change the basis from (x, y) to (u, v) so that u is now an independent variable and x is a dependent one. Let f' be a function of u and y such that

$$f' = f - ux \qquad \ldots(7)$$

Then,

$$df' = df - u\,dx - x\,du$$

Substituting for df from (5), we get

$$df' = u\,dx + v\,dy - u\,dx - x\,du$$

or $$df' = v\,dy - x\,du \qquad ...(8)$$

But f' is a function of u and y, therefore

$$df' = \frac{\partial f'}{\partial u} du + \frac{\partial f'}{\partial y} dy \qquad ...(9)$$

Comparing eqs. (8) and (9), we get

$$x = -\frac{\partial f'}{\partial u} \quad \text{and} \quad v = \frac{\partial f'}{\partial y} \qquad ...(10)$$

These are the necessary relations for Legendre transformations.

6.3. GENERATING FUNCTIONS

For canonical transformations, the Lagrangian L in p_k, q_k coordinates and L' in P_k, Q_k coordinates must satisfy the Hamilton's principle, *i.e.*,

$$\delta \int_{t_1}^{t_2} L\,dt = 0 \quad \text{and} \quad \delta \int_{t_1}^{t_2} L'dt = 0 \qquad ...(11)$$

But $$L = \sum_{k=1}^{n} p_k \dot{q}_k - H \quad \text{and} \quad L' = \sum_{k=1}^{n} P_k \dot{Q}_k - H',$$

therefore, $$\delta \int_{t_1}^{t_2} \left[\sum_k p_k \dot{q}_k - H \right] dt = 0 \qquad ...(12)$$

and $$\delta \int_{t_1}^{t_2} \left[\sum_k P_k \dot{Q}_k - H' \right] dt = 0 \qquad ...(13)$$

Subtracting eq. (13) from eq. (12), we get

$$\delta \int_{t_1}^{t_2} \left[\left(\sum_k p_k \dot{q}_k - H \right) - \left(\sum_k P_k \dot{Q}_k - H' \right) \right] dt = 0 \qquad ...(14)$$

In δ-variation process, the condition $\delta \int f\,dt = 0$ is to be satisfied, in general, by $f = dF/dt$, where F is an arbitrary function. Therefore,

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = 0 \qquad ...(15)$$

where $$dF/dt = L - L' \qquad ...(16\text{ a})$$

or $$\frac{dF}{dt} = \left(\sum_k p_k \dot{q}_k - H \right) - \left(P_k \dot{Q}_k - H' \right) \qquad ...(16b)$$

The function F is known as the generating function. The meaning of the name will be clear later on. The first bracket in (16) is a function of p_k, q_k and t and the second as a function of P_k, Q_k and t. F is therefore, in general, a function of $(4n+1)$ variables p_k, q_k, P_k, Q_k and t. It is to be remembered that the variables are subjected to the transformation equations (1) and therefore F may be regarded as the function of $(2n+1)$ variables, comprising t and any $2n$ of the p_k, q_k, P_k, Q_k. Thus we see that F can be written as a function of $(2n+1)$ independent variables in the following four forms :

(*i*) $F_1(q_k, Q_k, t)$, (*ii*) $F_2(q_k, P_k, t)$, ...(17)

(*iii*) $F_3(p_k, Q_k, t)$, and (*iv*) $F_4(p_k, P_k, t)$

The choice of the functional form of the generating function F depends on the problem under consideration.

Case I : If we choose the form (*i*), *i.e.*,

$$F_1 = F_1(q_1, q_2, ..., q_k, ..., q_n, Q_1, Q_2, ..., Q_k, ..., Q_n, t) \quad ...(18)$$

then
$$\frac{dF_1}{dt} = \sum_k \frac{\partial F_1}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k}\dot{Q}_k + \frac{\partial F_1}{\partial t} \quad ...(19)$$

Subtracting (19) from (16 b), we can write

$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right)\dot{q}_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right)\dot{Q}_k + H' - H - \frac{\partial F_1}{\partial t} = 0$$

or
$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right)dq_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right)dQ_k + \left[H' - H - \frac{\partial F_1}{\partial t}\right]dt = 0 \quad ...(20)$$

As q_k, Q_k and t may be regarded as independent variables,

$$p_k = \frac{\partial}{\partial q_k}F_1(q_k, Q_k, t),\ P_k = -\frac{\partial}{\partial Q_k}F_1(q_k, Q_k, t)$$

and
$$H' - H = \frac{\partial}{\partial t}F_1(q_k, Q_k, t) \quad ...(21)$$

In principle, first equation of (21) may be solved to give

$$Q_k = Q_k(q_k, p_k, t) \quad ...(22)$$

Substituting this in the second equation of (21), one gets

$$P_k = P_k(q_k, p_k, t) \quad ...(23)$$

In fact, these are the transformation equations (1). Thus we find that transformation equations can be derived from a knowledge of the function F. This is why F is known as the ***generating function of the transformation***.

Case II : If the generating function is of the type $F_2(q_k, P_k, t)$, then it can be dealt with by affecting a Legendre transformation of $F_1(q_k, Q_k, t)$.

In case of Legandre transformation (7) :

$$f' = f - ux, \text{ where } u = \partial f/\partial x$$

Here, since $P_k = -\partial F_1/\partial Q_k$, we have $u = -P_k$, $x = Q_k$, $f' = F_2$ and $f = F_1$.

Therefore ,
$$F_2(q_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_k \quad ...(24)$$

Evidently, F_2 is independent of Q_k variables, because

$$\frac{\partial F_2}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k = -P_k + P_k = 0 \text{ as } \frac{\partial F_1}{\partial Q_k} = -P_k \text{ in (21).}$$

Using eq.(16)

$$\left(\sum_k p_k\dot{q}_k - H\right) - \left(\sum_k P_k\dot{Q}_k - H'\right) = \frac{dF_1}{dt} = \frac{d}{dt}\left[F_2 - \sum_k P_k Q_k\right].$$

or
$$\frac{dF_2}{dt} = \sum_k p_k\dot{q}_k + \sum_k Q_k\dot{P}_k + H' - H \quad ...(25)$$

Total time derivative of $F_2(q_k, P_k, t)$ is

$$\frac{dF_2}{dt} = \sum_k \frac{\partial F_2}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k} \dot{P}_k + \frac{\partial F_2}{\partial t} \quad \text{...(26)}$$

From (25) and (26), we get

$$p_k = \frac{\partial F_2}{\partial q_k},\ Q_k = \frac{\partial F_2}{\partial P_k} \text{ and } H' - H = \frac{\partial F_2}{\partial t} \quad \text{...(27)}$$

If we look (21) and (27), we find $\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t}$. Further as $\frac{\partial F_1}{\partial q_k} = \frac{\partial F_2}{\partial q_k}$, first equation of (21) and that of (27) are identical. Second equation of (27) appears to be different from the second equation of (21), but in fact it is a rearrangement of it.

Case III : We can again relate the third type of generating function $F_3(p_k, Q_k, t)$ to F_1 by a Legendre transformation in view of the relation $p_k = \partial F_1/\partial q_k$. Here $u = p_k$, $x = q_k$, $f' = F_3$ and $f = F_1$. Therefore,

$$F_3(p_k, Q_k, t) = F_1(q_k, Q_k, t) - \sum_k p_k q_k \quad \text{...(28)}$$

or $$F_1(q_k, Q_k, t) = F_3(p_k, Q_k, t) + \sum_k p_k q_k$$

Using eq. (16), we have

$$\left(\sum_k p_k \dot{q}_k - H\right) - \left(\sum_k P_k \dot{Q}_k - H'\right) = \frac{dF_1}{dt} = \frac{d}{dt}\left(F_3 + \sum p_k q_k\right)$$

or $$\frac{dF_3}{dt} = -\sum_k \dot{p}_k q_k - \sum_k P_k \dot{Q}_k + H' - H$$

Also, $$\frac{dF_3}{dt} = \sum_k \frac{\partial F_3}{\partial p_k} \dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k} \dot{Q}_k + \frac{\partial F_3}{\partial t}$$

Therefore, the new transformation equations are

$$q_k = -\frac{\partial F_3}{\partial p_k},\ P_k = -\frac{\partial F_3}{\partial Q_k} \text{ and } H' - H = \frac{\partial F_3}{\partial t} \quad \text{...(29)}$$

Case IV : Using Legendre transformations, the generating function $F_4(p_k, P_k, t)$ can be connected to $F_1(q_k, Q_k, t)$ as

$$F_4(p_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_k - \sum_k p_k q_k \quad \text{...(30)}$$

Using eq. (16), we have

$$\left(\sum_k p_k \dot{q}_k - H\right) - \left(\sum_k P_k \dot{Q}_k - H'\right) = \frac{d}{dt}\left(F_4 - \sum_k P_k Q_k + \sum_k p_k q_k\right)$$

or $$\frac{dF_4}{dt} = -\sum_k q_k \dot{p}_k + \sum_k Q_k \dot{P}_k + H' - H$$

But $$\frac{dF_4}{dt} = \sum_k \frac{\partial F_4}{\partial p_k} \dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k} \dot{P}_k + \frac{\partial F_4}{\partial t}$$

A comparison of the above two equations gives the fourth set of transformation equations:

$$q_k = -\frac{\partial F_4}{\partial p_k},\; Q_k = \frac{\partial F_4}{\partial P_k},\; H' - H = \frac{\partial F_4}{\partial t} \qquad ...(31)$$

6.4. PROCEDURE FOR APPLICATION OF CANONICAL TRANSFORMATIONS

We note that the relation between H and H' in all the cases has the same form *i.e.*, $H' = H + \partial F/\partial t$. Now, if F has no explicit time dependence, then $\partial F/\partial t = 0$ and hence

$$H' = H \qquad ...(32)$$

Thus, when the generating function has no explicit time dependence, the new Hamiltonian H' is obtained from the old Hamiltonian H by substituting for p_k, q_k in terms of the new variables P_k, Q_k. Further we note that the time t has been treated as an invariant parameter of the motion and we have not made any provision for a transformation of the time coordinate alongwith the other coordinates.

If in the new set of coordinates (P_k, Q_k, t) all coordinates Q_k are cyclic, then

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} = 0 \text{ or } P_k = \text{Constant, say } \alpha_k \qquad ...(33)$$

If the generating function F does not depend on time t explicitly and H is a constant of motion, not depending on time, then from (32) H' is also constant of motion. Thus H' will not involve Q_k and t (explicit time dependence).

Therefore,

$$H(q_k, p_k) = H'(Q_k, P_k) = H'(P_k) = H'(\alpha_1, \alpha_2, ..., \alpha_n)$$

Hamilton's equations for Q_k are

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k} = \frac{\partial H'}{\partial \alpha_k} = \omega_k \qquad ...(34)$$

where ω_k's are functions of the α_k's only and are constant in time.

Eq. (34) has the solution

$$Q_k = \omega_k t + \beta_k \qquad ...(35)$$

where β_k's are the constants of integration, determined by the initial conditions.

6.5. CONDITION FOR CANONICAL TRANSFORMATIONS

Suppose $F = F(q_k, Q_k)$, then obviously $\partial F/\partial t = 0$ and $H = H'$ [from (21)].

Further from (21), we have

$$p_k = \frac{\partial F}{\partial q_k} \text{ and } P_k = -\frac{\partial F}{\partial Q_k}$$

Also
$$dF = \sum_k \frac{\partial F}{\partial q_k} dq_k + \sum_k \frac{\partial F}{\partial Q_k} dQ_k$$

or
$$dF = \sum_k p_k\, dq_k - \sum_k P_k\, dQ_k \qquad ...(36)$$

The left hand side of eq. (36) is an exact differential, hence for a given transformation to be canonical, the right hand side of eq. (36), *i.e.*, $\sum_k p_k\, dq_k - \sum_k P_k\, dQ_k$ must be an exact differential.

Ex. 1. ***Harmonic Oscillator*** **:** *Discuss harmonic oscillator as an example of canonical transformations.* **(Garwal 1997; Kanpur 1999)**

Solution : In case of a harmonic oscillator, the Hamiltonian in terms of q and p coordinates can be expressed as

$$H = \tfrac{1}{2} Cq^2 + \frac{p^2}{2m} \qquad ...(i)$$

Writing $C/m = \omega^2$, H can be written as

$$H = \tfrac{1}{2} m\omega^2 q^2 + \frac{p^2}{2m} \text{ or } H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) \qquad ...(ii)$$

Let us consider the generating function, given by

$$F_1(q, Q, t) = \tfrac{1}{2} m\omega q^2 \cot Q \qquad ...(iii)$$

From (21), we obtain

$$p = m\omega q \cot Q,\ P = \frac{m\omega q^2}{2\sin^2 Q},\ H' = H \qquad ...(iv)$$

Hence $\quad q = \sqrt{2P/m\omega}\, \sin Q;\ p = \sqrt{2m\omega P}\, \cos Q \qquad ...(v)$

Now, the transformation H' is obtained by using (ii) and (v) *i.e.*,

$$H' = H = \frac{1}{2m}\left(2m\omega\, P\cos^2 Q + 2m\omega\, P\sin^2 Q\right)$$

or

$$H' = H = \omega P \qquad ...(vi)$$

Since the Hamiltonian is cyclic in Q, the conjugate momentum P is constant. In fact $H = H' = E$ is the constant energy E so that

$$P = E/\omega \qquad ...(vii)$$

Then the equation of motion for Q reduces to the simple form

$$\dot{Q} = \partial H/\partial P = \omega$$

with the solution

$$Q = \omega t + \phi \qquad ...(viii)$$

where ϕ is a constant of integration .

Thus from (v), (vii) and ($viii$), we get

$$q = \sqrt{2E/m\omega^2}\, \sin(\omega t + \phi) \qquad ...(ix)$$

which is the customary solution of a harmonic oscillator.

Ex. 2. *Prove that the generating function* $F = \sum_i q_i P_i$ *generates the identity transformation.*

Solution : Here, the generating function is $F_2 = \sum_i q_i P_i$ and hence applying eq. (27), we get

$$p_i = \partial F_2/\partial q_i = P_i,\quad Q_i = \partial F_2/\partial P_i = q_i$$

$$H' = H\ (\because\ F_2 \text{ is not } t \text{ dependent})$$

Thus the new and old variables are separately equal and hence F generates an identity transformation.

Ex. 3. *Show that for the function* $F = \sum_k q_k Q_k$, *the transformations are* $p_k = Q_k$, $P_k = -q_k$ *and* $H' = H$.

Solution : Here $F = \sum_k q_k Q_k$ is F_1 and hence applying eqs. (21), we get

$$p_k = \frac{\partial F_1}{\partial q_k} = Q_k, \; P_k = -\frac{\partial F_1}{\partial Q_k} = -q_k \text{ and } H' = H.$$

Ex. 4. *Show that the transformation*

$$P = \tfrac{1}{2}\left(p^2 + q^2\right), \; Q = \tan^{-1}\frac{q}{p}$$

is canonical. **(Kanpur 2003, 01, 1999, 95; Meerut 1995; Agra 99, 97 ; Rohilkhand 79)**

Solution : The transformation will be canonical, if $pdq - PdQ$ is an exact differential. Here

$$dQ = (pdq - qdp)/p^2 + q^2$$

Therefore, $$pdq - PdQ = pdq - \tfrac{1}{2}\left(p^2 + q^2\right)\frac{pdq - qdp}{p^2 + q^2}$$

$$= \tfrac{1}{2}\left(pdq + qdp\right) = d\left(\tfrac{1}{2}pq\right) = \text{an exact differential}$$

This means that the given transformtion is canonical.

Ex. 5. *The transformation equations between two sets of coordinates are*

$$P = 2(1 + q^{1/2}\cos p)\, q^{1/2}\sin p \text{ and } Q = \log\,(1 + q^{1/2}\cos p)$$

Show that (i) the transformation is canonical and (ii) the generating function of this transformation is

$F_3 = -(e^Q - 1)^2\, tan\, p.$ **(Garwal 1993; Rohilkhand 99, 85)**

Solution : Here,

$$(pdq - PdQ) = pdq - 2\left[1 + q^{1/2}\cos p\right] q^{1/2}\cos p \times \frac{\left(-q^{1/2}\sin p\, dp + \frac{1}{2}\cos p\, dq / q^{-1/2}\right)}{(1 + q^{1/2}\cos p)}$$

$$= pdq + 2q\sin^2 p\, dp - \sin p\cos p\, dq$$

$$= (p - \tfrac{1}{2}\sin 2p)\, dq + q\,(1 - \cos 2p)\, dp$$

$$= d\,[q\,(p - \tfrac{1}{2}\sin 2p)]$$

which is an exact differential and hence the transformation is canonical.

Further $\quad Q = \log_e(\,1 + q^{1/2}\cos p)$ or $e^Q = 1 + q^{1/2}\cos p$

or $\quad q^{1/2}\cos p = e^Q - 1$ or $q = (e^Q - 1)^2/\cos^2 p$

For this transformation, we take $F = F_3\,(p, Q)$. So that

$$q = -\frac{\partial F_3}{\partial p} \text{ and } P = -\frac{\partial F_3}{\partial Q}$$

Thus $$-\frac{\partial F_3}{\partial p} = (e^Q - 1)^2\frac{1}{\cos^2 p} \text{ or } F_3 = -\int\frac{\left(e^Q - 1\right)^2}{\cos^2 p}\,dp$$

or $$F_3 = -\left(e^Q - 1\right)^2\tan p + \text{constant}$$

If the constant of integration is zero,

$$F_3 = -(e^Q - 1)^2\tan p.$$

6.6. BILINEAR INVARIANT CONDITION

According to this condition, *if a transformation* (q_k, p_k) *coordinates to* (Q_k, P_k) *coordinates is canonical, then bilinear form*

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) \qquad ...(37)$$

remains invariant. This statement means that

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) = \sum_k (\delta P_k dQ_k - \delta Q_k dP_k) \qquad ...(38)$$

Proof : From Hamilton's canonical equations, we have

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad \text{or} \quad dq_k = \frac{\partial H}{\partial p_k} dt \qquad ...(39a)$$

and
$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad \text{or } dp_k = -\frac{\partial H}{\partial q_k} dt \qquad ...(39b)$$

Similarly,
$$dQ_k = \frac{\partial H}{\partial P_k} dt \quad \text{and } dP_k = -\frac{\partial H}{\partial Q_k} dt \qquad ...(40)$$

Since δp_k and δq_k are arbitrary,

$$\sum_k \delta p_k \left(dq_k - \frac{\partial H}{\partial p_k} dt \right) - \sum_k \delta q_k \left(dp_k + \frac{\partial H}{\partial q_k} dt \right) = 0 \qquad ...(41)$$

Obviously in order to satisfy this equation, the coefficients of δp_k and δq_k must be zero and this gives eqs. (39). Therefore, eq. (41) is correct.

Eq. (41) can be written as

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) - \sum_k \left(\frac{\partial H}{\partial p_k} \delta p_k + \frac{\partial H}{\partial q_k} \delta q_k \right) dt = 0$$

or
$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) - \delta H \, dt = 0 \qquad ...(42)$$

Similarly, for $H' = H$, when F does not depend on time,

$$\sum_k (\delta P_k dQ_k - \delta Q_k dP_k) - \delta H \, dt = 0 \qquad ...(43)$$

Eliminating $\delta H \, dt$ from eqs. (42) and (43), we obtain

$$\sum_k (\delta p_k dq_k - \delta q_k dp_k) = \sum_k (\delta P_k dQ_k - \delta Q_k dP_k) \qquad ...(44)$$

which proves the statement .

Ex. 1. *Show that the transformation* $Q = \frac{1}{p}$ and $P = qp^2$ *is canonical.*

Solution :
$$Q = \frac{1}{p}, \text{ therefore, } dQ = \frac{\partial Q}{\partial p} dp + \frac{\partial Q}{\partial q} dq$$

or
$$dQ = \frac{\partial}{\partial p}\left(\frac{1}{p}\right) dp + \frac{\partial}{\partial q}\left(\frac{1}{p}\right) dq = -\frac{1}{p^2} dp \qquad ...(i)$$

$$\delta Q = \frac{\partial Q}{\partial p}\delta p + \frac{\partial Q}{\partial q}\delta q = -\frac{1}{p^2}\delta p \qquad ...(ii)$$

Similarly, $dP = p^2\, dq + 2qp\, dp \qquad (\because P = qp^2)$ ...(iii)

and $\delta P = p^2\, \delta q + 2qp\, \delta p$ ...(iv)

Therefore,
$$\delta P dQ - \delta Q dP = (p^2 \delta q + 2qp\delta p)\left(-\frac{1}{p^2}dp\right) - \left(-\frac{1}{p^2}\delta p\right)(p^2 dq + 2qp\, dp)$$

$$= -\delta q\, dp - \frac{2q}{p}\delta p\, dp + \delta p\, dq + \frac{2q}{p}\delta p\, dp$$

$$= \delta p\, dq - \delta q\, dp.$$

Therefore, the bilinear form is invariant and hence the transformation is canonical.

6.7. INTEGRAL INVARIANCE OF POINCARE

Phase space is defined as a $2n$ dimensional space formed by the $2n$ coordinates $q_1, q_2, ..., q_n, p_1, p_2, ..., p_n$. In this space a complete dynamical specification of a mechanical system is given by a point.

According to Poincare's theorem, the integral

$$I = \iint_S \sum_k dq_k\, dp_k \qquad ...(45)$$

taken over an arbitrary two dimensional surface S of $2n$ dimensional phase space is invariant under canonical transformation, *i.e.*,

$$\iint_S \sum_k dq_k dp_k = \iint_S \sum_k dQ_k dP_k \qquad ...(46)$$

If S is a 4-dimensional surface in $2n$-dimensional phase space, then according to Poincare's theorem,

$$\iint_S \sum_k \sum_l dq_k\, dq_l\, dp_k\, dp_l = \iint_S \sum_k \sum_l dQ_k\, dQ_l\, dP_k\, dP_l$$

In general, if the surface is $2n$-dimensional in $2n$-dimensional phase space, then the integral invariance of Poincare means

$$\iint..\int dq_1\, dq_2 ... dq_n\, dp_1 ... dp_n = \iint..\int dQ_1\, dQ_2 ... dQ_n\, dP_1 ... dP_n \qquad ...(47)$$

which shows that the volume in phase space is invariant under canonical transformation.

In the advanced calculus, we have the relation

$$\int..\int dQ_1\, dQ_2 ... dQ_n\, dP_1 ... dP_n = \int..\int D\, dq_1\, dq_2 ... dq_n\, dp_1 ... dp_n \qquad ...(48)$$

where D is known as the Jacobian of the transformation, given by

$$D = \frac{\partial(Q_1, Q_2, ..., Q_n, P_1, P_2, ..., P_n)}{\partial(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)} \qquad ...(49)$$

This means that in order to prove the integral invariance (47), we have to show $D = 1$. By using the properties of the Jacobian, it can be written as

$$D = \frac{\dfrac{\partial(Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n)}{\partial(q_1, q_2, \ldots, q_n, P_1, P_2, \ldots, P_n)}}{\dfrac{\partial(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n)}{\partial(q_1, q_2, \ldots, q_n, P_1, P_2, \ldots, P_n)}} \qquad \ldots(50)$$

In the calculus, we know that if the same variables are present in both the partial differentials, the Jacobian is reduced to fewer variables in which the repeated variables are treated as constants in carrying out the differentiation. Thus

$$D = \frac{\left[\dfrac{\partial(Q_1, Q_2, \ldots, Q_n)}{\partial(q_1, q_2, \ldots, q_n)}\right]_{P_1, P_2, \ldots, P_n \text{ as constants}}}{\left[\dfrac{\partial(p_1, p_2, \ldots, p_n)}{\partial(P_1, P_2, \ldots, P_n)}\right]_{q_1, q_2, \ldots, q_n \text{ as constants}}} \qquad \ldots(51)$$

The numerator is a determinant of order n whose element in the ith row and kth column is $\partial Q_k/\partial q_i$, *i.e.*,

$$\frac{\partial(Q_1, Q_2, \ldots, Q_n)}{\partial(q_1, q_2, \ldots, q_n)} = \begin{vmatrix} \dfrac{\partial Q_1}{\partial q_1} & \dfrac{\partial Q_2}{\partial q_1} & \cdots\cdots & \dfrac{\partial Q_n}{\partial q_1} \\ \dfrac{\partial Q_1}{\partial q_2} & \dfrac{\partial Q_2}{\partial q_2} & \cdots\cdots & \dfrac{\partial Q_n}{\partial q_2} \\ \vdots & & & \\ \dfrac{\partial Q_1}{\partial q_i} & \cdots \dfrac{\partial Q_k}{\partial q_i} & \cdots & \dfrac{\partial Q_n}{\partial q_i} \\ \vdots & & & \\ \dfrac{\partial Q_1}{\partial q_n} & \cdots\cdots & \cdots\cdots & \dfrac{\partial Q_n}{\partial q_n} \end{vmatrix} \qquad \ldots(52)$$

Similarly, the denominator is a determinant of the same order n whose element in the ith row and kth column is $\dfrac{\partial p_k}{\partial P_i}$.

If the generating function of the above canonical transformation is written as $F_2(q_k, P_k)$, Then from eq. (27), we obtain

$$Q_k = \frac{\partial F_2}{\partial P_k} \text{ and } p_k = \frac{\partial F_2}{\partial q_k}$$

and hence
$$\frac{\partial Q_k}{\partial q_i} = \frac{\partial^2 F_2}{\partial q_i\, \partial P_k} \text{ and } \frac{\partial p_k}{\partial P_i} = \frac{\partial^2 F_2}{\partial P_i\, \partial q_k} \qquad \ldots(53)$$

Thus, we see that the ik-element of the numerator is the same as the ki-element of the denominator. Since in a determinant, rows and columns can be interchanged and hence the determinant of the numerator is equal to the determinant of the denominator. Therefore, from (51), we get

$$D = 1 \qquad \ldots(54)$$

Thus we see that eq.(47) is true, *i.e.*, the volume in phase space is invariant under canonical transformation.

Also, if we take a two dimensional surface S of $2n$-dimensional phase space, then the invariance of Poincare's integral under canonical transformation means that

$$\iint_S \sum_k dq_k\, dp_k = \iint_S \sum_k dQ_k\, dP_k$$

6.8. INFINITESIMAL CONTACT TRANSFORMATIONS

Those transformations in which the new set of coordinates (Q_k, P_k) differ from the old set (q_k, p_k) by infinitesimals *i.e.*, $Q_k = q_k + \delta q_k$ and $P_k = p_k + \delta p_k$, are called ***infinitesimal contact transformations***.

It has been shown earlier in a solved example that the generating function $F_2 = \sum_k q_k P_k$ generates the identity transformation *i.e.*, $Q_k = q_k$ and $P_k = p_k$. The generating function, giving an infinitesimal change in the variables, can be readily written as

$$F_2 = \sum_k q_k P_k + \in G(q_k, P_k) \qquad ...(55)$$

where $\in$ is an infinitesimal parameter of the transformation and $G\,(q_k, P_k)$, is arbitrary.

Substitution of (55) in eqs. (27) gives

$$p_k = \frac{\partial F_2}{\partial q_k} = P_k + \in \frac{\partial G}{\partial q_k}, Q_k = \frac{\partial F_2}{\partial P_k} = q_k + \in \frac{\partial G}{\partial P_k},\ H' = H \qquad ...(56)$$

Therefore,

$$Q_k - q_k = \delta q_k = \in \frac{\partial G}{\partial P_k} \text{ and } P_k - p_k = \delta p_k = - \in \frac{\partial G}{\partial q_k} \qquad ...(57)$$

Since the difference $(P_k - p_k)$ is infinitesimal. We can replace P_k by p_k in the derivative and also $G(q_k, P_k)$ by $G(q_k, p_k)$. So that eqs. (57) are

$$\delta q_k = \in \frac{\partial G}{\partial p_k} \quad \text{and} \quad \delta p_k = - \in \frac{\partial G}{\partial q_k} \qquad ...(58)$$

In case of infinitesimal contact transformations, the description is transferred to the function G instead of the original generating function F. Thus G is the new generating function which generates the infinitesimal contact transformation.

Let us consider a special case in which $\in\ = dt$ and $G = H$. Eqs. (58) can be written by using Hamilton's equations of motion as

$$\delta q_k = dt \frac{\partial H}{\partial p_k} = dt\, \dot{q}_k = dq_k \text{ and } \delta p_k = - dt \frac{\partial H}{\partial q_k} = dt\, \dot{p}_k = dp_k \qquad ...(59)$$

These changes in the conjugate variables represent an infinitesimal change in coordinates in time dt. Eqs. (59) give thus a transformation from the variables q_k, p_k at time t to $q_k + dq_k$, $p_k + dp_k$ at time $t + dt$. Hence **the motion of the system in a small time *dt* can be described by an infinitesimal canonical transformation generated by the Hamiltonian *H* of the system.** Evidently the motion of the system in a finite interval of time is described by a succession of infinitesimal canonical transformations generated by the same Hamiltonian. In other words, the motion of a system corresponds to the continuous evolution of canonical transformation. Thus we can say that the Hamiltonian of the system is the generator of the motion of the system in phase space with time.

Questions

1. Discuss un detail the canonical transformations ? Solve the problem of harmonic oscillator by using the canonical transformations. **(Agra 2002, 1999, 98, 95, 94)**
2. Define canonical transformations and obtain the transformation equations corresponding to all possible generating functions. Give an example of canonical transformation. **(Agra 2004, 01, 1998, 87)**
3. What are Legendre transformations and the canonical variables ? **(Rohilkhand 1998)**
4. Write down the tests to determine whether a given tansformation is canonical. **(Gorakhpur 1995)**
5. What are canonical transformations ? Explain with examples. **(Garwal 1996)**
6. What are canonical transformations ? Give condition for a transformation to be canonical. **(Agra 1997, 92; Kanpur 2002)**
7. What are cononical transformations? Discuss how the transformation equations can be obtained from generating functions of type F_1 and F_2. **(Meerut 2001)**
8. What is canonical transformations? How is it used to solve the problem of simple harmonic oscillator? **(Kanpur 1999)**
9. What is generating function ? Obtain canonical transformation equations corresponding to first two types of generating functions. **(Agra 1993; Meerut 95)**
10. Let F be a generating function dependent only on Q_α, P_α, t. Prove that

$$P_\alpha = -\frac{\partial F}{\partial Q_\alpha},\ q_\alpha = -\frac{\partial F}{\partial p_\alpha},\ H' = \frac{\partial F}{\partial t} + H.$$

11. Obtain the bilinear invariant condition for the transformation to be canonical. **(Kanpur 1999)**
12. Write a note on 'Infinitesimal Canonical Transformations.' **(Kanpur 2002, 1999)**
13. The motion of the system during at interval of time may be regarded as an infinitesimal contact transformation generated by Hamiltonian. Explain. **(Kanpur 2003)**

Problems

[SET- I]

1. (*i*) Show that the function $F = -\sum_i q_i P_i$ generates the transformation $p_i = -P_i$, $Q_i = -q_i$.

   (*ii*) Show that the function $F = -\sum_i Q_i p_i$ generates the identity transformation.

2. (*i*) Show that the transformation

$$Q_i = \alpha q_i,\ P_i = \beta p_i;\ \alpha \neq 0,\ \beta \neq 0$$

   is canonical and $H' = \alpha\beta H$.

   (*ii*) Show that the transformation

$$Q_i = \alpha p_i,\ P_i = \beta q_i;\ \alpha \neq 0,\ \beta \neq 0$$

   is canonical , and $H' = -\alpha\beta H$.

3. Show that for the transformation $Q = \sqrt{q}\cos 2p$, $P = \sqrt{q}\sin 2p$, the generating function is

$$\tfrac{1}{2} q \cos^{-1}(Q/\sqrt{q}) - \tfrac{1}{2} Q\sqrt{q - Q^2}.$$

4. Determine the values of α and β so that the equations

$$Q = q^{\alpha} \cos \beta p \text{ and } P = q^{2} \sin \beta p$$

is a cononical transformation. Also find the generating function F_3 for this case.

(Rohilkhand 1986, 82)

Ans : $\alpha = \frac{1}{2}$, $\beta = 2$; $F_3 = -\frac{1}{2} Q^2 \tan 2p$.

5. What are canonical transformations ? Using $G_1 = -\frac{1}{2} m\omega q^2 \cot Q$ as a generating function, obtain an expression for the displacement of a harmonic oscillator. **(Meerut 1983, 77)**

Ans : $q = \sqrt{\dfrac{2E}{m\omega^2}} \sin(\omega t + \phi)$.

6. Prove that the transformation

$$P = q \cot p \text{ and } Q = \log \frac{\sin p}{q}$$

is canonical. Show that the generating function $F(q, Q)$ is

$$F = e^{-Q}\left(1 - q^2 e^{2Q}\right)^{\frac{1}{2}} + q \sin^{-1}\left(qe^{Q}\right).$$

(Agra 1998, 93; Garwal 97; 94, Rohilkhand 96, 93; Meerut 82)

7. Show that the generating function for the transformation

$$P = 1/Q, \quad q = PQ^2 \quad \text{is} \quad F = q/Q$$

(Agra 1998)

8. Show that the following transformations are canonical :

(*i*) $Q = p, \; P = -q$

(*ii*) $Q = \sqrt{2q}\, e^{\alpha} \cos p$ and $P = \sqrt{2q}\, e^{-\alpha} \sin p$

(*iii*) $q = \sqrt{2P} \sin Q$ and $p = \sqrt{2P} \cos Q$ **(Garwal 1999)**

(*iv*) $Q = q \tan p$, $P = \log (\sin p)$

(*v*) $Q = p \tan q$, $P = \log (\sin p)$ **(Garwal 1994)**

(*vi*) $Q = \tan^{-1}(\alpha q/p)$, $P = \frac{1}{2}\alpha q^2 (1 + p^2/\alpha q^2)$ for any constant α.

(*vii*) $Q = Aq + Bp$, $P = Cq + Dp$, only if $AD - BC = 1$.

(*viii*) $Q_i = p_i \tan t$, $P_i = q_i \tan t$

(*ix*) $q = P^2 + Q^2$, $p = \frac{1}{2} \tan^{-1}(P/Q)$

(*x*) $p = k\sqrt{2P} \cos Q$, $q = k^{-1}\sqrt{2P} \sin Q$.

9. Show that the transformation

$$q = P/\sqrt{k} \sin Q, \quad p = (mP\sqrt{k})^{1/2} \cos Q$$

is canonical and the generating function is $F = \frac{1}{2}\sqrt{k} q^2 \cot Q$.

10. Show that the tansformation

$$Q = p + i\,a\,q,\; P = \frac{p - i\,a\,q}{2\,i\,a}$$

is canonical and find a generating function. **(Kanpur 1998, 97)**

11. For a canonical transformation, given that

$Q = \sqrt{q^2 + p^2}$ and $F = \frac{1}{2}(q^2 + p^2) + \tan^{-1}(q/p) + \frac{1}{2}qp$, find $P(q, p)$ and $P(q, Q)$.

Ans : $P(q, p) = -\sqrt{p^2 + q^2}\tan^{-1}(q/p)$; $P(q, Q) = -Q\sin^{-1}(q/p)$

12. For the generating function

$$F_1(q, Q, t) = \frac{1}{2}m\omega\left(q - \frac{F(t)}{m\omega^2}\right)^2 \cot Q.$$

find the transformation equations and thus obtain the equations of motion of a simple harmonic oscillator acted by a force $F(t)$ in terms of Q and P.

Ans : $q = \dfrac{F(t)}{m\omega^2} + \sqrt{\dfrac{2P}{m\omega}}\sin Q$, $p = \sqrt{2m\omega P}\cos Q$,

$H' = \omega P - \dfrac{F^2}{2m\omega^2} - \sqrt{\dfrac{2P}{m\omega^3}}\dot{F}\cos Q$, $\dot{P} = \sqrt{\dfrac{2P}{m\omega^2}}\dot{F}\sin Q$,

$\dot{Q} = \omega - \sqrt{\dfrac{1}{2Pm\omega^2}}\dot{F}\cos Q$.

13. Find the canonical transformations defined by the generating functions :

(i) $F_1(q, Q) = qQ - \frac{1}{2}m\omega q^2 - Q^2/4m\omega$

(ii) $F_1(q, Q, t) = \frac{1}{2}m\,\omega(t)\,q^2 \cot Q$

(iii) $F_3(Q, p) = -(e^{\dot{Q}} - 1)^2 \tan p$

Ans : (i) $Q = p + m\omega q$, $P = \dfrac{(p - m\omega q)}{2m\omega}$, $H' = \dfrac{(Q^2 + 4m^2\omega^2 P^2)}{4m}$

(ii) $Q = \tan^{-1}\left(\dfrac{m\omega q}{p}\right)$, $P = \left(p^2 + \dfrac{m^2\omega^2 p^2}{2m\omega}\right)$, $H' = \omega P(1 + \dot{\omega}\sin Q\cos Q/\omega^2)$

(iii) $Q = \log\left(1 + \sqrt{q}\cos p\right)$, $P = 2\left(1 + \sqrt{q}\cos p\right)\sqrt{q}\sin p$,

$$H' = \frac{1}{2m}\left\{\tan^{-1}\left[\frac{P}{2e^Q(e^Q - 1)}\right]\right\}^2 + \frac{1}{2}m\omega^2\left[(e^Q - 1)^2 + \frac{P^2}{4e^{2Q}}\right]^2.$$

[SET- II]

1. Prove that the result of two or more successive canonical transformations is also canonical.
2. Prove that the contact transformation defined by the equations

$q_1 = \lambda_1^{-\frac{1}{2}}(2Q_1)^{\frac{1}{2}}\cos P_1 + \lambda_2^{-\frac{1}{2}}(2Q_2)^{\frac{1}{2}}\cos P_2$,

$$q_2 = -\lambda_1^{-\frac{1}{2}}(2Q_1)^{\frac{1}{2}}\cos P_1 + \lambda_2^{-\frac{1}{2}}(2Q_2)^{\frac{1}{2}}\cos P_2,$$

$$p_1 = \tfrac{1}{2}(2\lambda_1 Q_1)^{\frac{1}{2}}\sin P_1 + \tfrac{1}{2}(2\lambda_2 Q_2)^{\frac{1}{2}}\sin P_2,$$

$$p_2 = -\tfrac{1}{2}(2\lambda_1 Q_1)^{\frac{1}{2}}\sin P_1 + \tfrac{1}{2}(2\lambda_2 Q_2)^{\frac{1}{2}}\sin P_2,$$

changes the system

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i},$$

$$H = p_1^2 + p_2^2 + \tfrac{1}{8}\lambda_1^2(q_1 - q_2)^2 + \tfrac{1}{8}\lambda_2^2(q_1 + q_2)^2$$

into the system

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i},\ \dot{Q}_i = \frac{\partial H'}{\partial P_i},\quad H' = \lambda_1 Q_1 + \lambda_2 Q_2.$$

Intrgrate the equations of motion and express the solution in terms of the original equations.

3. Show that the transformation

$$Q_1 = q_1^2 + \lambda^2 p_1^2,\ Q_2 = \frac{1}{2\lambda^2}(q_1^2 + q_2^2 + \lambda^2 p_1^2 + \lambda^2 p_2^2),$$

$$P_1 = \frac{1}{2\lambda}\left[\tan^{-1}\left(\frac{q_1}{\lambda p_1}\right) - \tan^{-1}\left(\frac{q_2}{\lambda p_2}\right)\right], P_2 = \lambda\tan^{-1}\left(\frac{q_2}{\lambda p_2}\right)$$

is a contact transformation and it reduces the original Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + \frac{q_1^2}{\lambda^2} + \frac{q_2^2}{\lambda^2}\right)$$

to its transformed form $H' = Q_2$.

4. A particle is acted by the force $f = -kq - \alpha/q^3$. Show that the Hamiltonian is $H = \dfrac{p^2}{2m} + \dfrac{kq^2}{2} + \dfrac{cp}{q}$, where c is a constant. Show that the transformation

$$Q = \tan^{-1}\left(\frac{\lambda q}{p}\right),\ P = \frac{p^2 + \lambda^2 q^2}{2\lambda} + R\,(q, p, t).$$

is canonical, where R is a homogeneous function of q, p and t. Find the transformed Hamiltonian and hence solve for the motion of the particle.

Objective Type Questions

1. In case of canonical transformations
   (*a*) Hamilton's principle is satisfied in old as well as in new coordinates.
   (*b*) The form of the Hamilton's equations is preserved.

(c) The form of the Hamilton's equations can not be preserved.
(d) The form of the Hamilton's equations may or may not be preserved.
Ans : (a), (b).

2. If the generating function has the form $F = F(q_k, P_k, t)$, then

(a) $p_k = \frac{\partial F}{\partial q_k},\ Q_k = \frac{\partial F}{\partial P_k},$ (b) $p_k = -\frac{\partial F}{\partial q_k},\ Q_k = \frac{\partial F}{\partial P_k},$

(c) $p_k = \frac{\partial F}{\partial q_k},\ Q_k = -\frac{\partial F}{\partial P_k},$ (d) $p_k = -\frac{\partial F}{\partial q_k},\ Q_k = -\frac{\partial F}{\partial P_k}.$

Ans : (a).

3. Choose the correct statement/statements :

(a) The generating function $F = \sum_k q_k P_k$ generates the identity transfor- mation.

(b) The generating function $F = \sum_k q_k P_k$ cannot generate the identity transformation.

(c) The generating function $F = -\sum_k q_k P_k$ generates the identity transformation.

(d) The generating function $F = -\sum_k q_k P_k$ generates the transformation $p_k = -P_k$ and $Q_k = -q_k$.

Ans : (a), (d).

Short Answer Questions

1. Explain the canonical transformations. **(Kanpur 2003)**
2. Discuss Legendre transformations.
3. Find the condition for a transformation to be canonical. **(Kanpur 2002)**
4. Explain the infinitesimal contact transformation. **(Kanpur 2002)**
5. Show that the transformation

$$P = \frac{1}{2}(p^2 + q^2) \text{ and } Q = \tan^{-1}(q/p)$$

is canonical. **(Kanpur 2003)**

6. The motion of the system during at interval of time may be regarded as an infinitesimal contact transformation generated by Hamiltonian. Explain. **(Kanpur 2003)**
7. Show that the transformation

$$q = \sqrt{2P}\sin Q \text{ and } p = \sqrt{2P}\cos Q$$

is canonical. **(Garwal 1999)**

8. Fill in the blank :

(i) The generating function $F_2 = \sum_i q_i \cdot P_i$ generates the .... transformation.

(ii) Those transformations in which the new set of coordinates (Q_k, P_k) are different from the old set (q_k, p_k) by infinitesimals are called............

Ans : (i) identity, (ii) infinitesimal contact transformations

CHAPTER

7 Brackets and Liouville's Theorem

7.1. INTRODUCTION

In the previous chapter, we have shown that in the case of infinitesimal contact transformations, the changes in the conjugate variables p_k and q_k are given by

$$\delta q_k = \in \frac{\partial G}{\partial p_k} \text{ and } \delta p_k = - \in \frac{\partial G}{\partial q_k} \quad ...(1)$$

where $\in$ is an infinitesimal parameter and the generating function $G(q_k, p_k)$ is arbitrary. Now let us consider some function $F(q_k, p_k)$. The change in the value of $F(q_k, p_k)$ with the changes δq_k and δp_k in the coordinates q_k and p_k respectively can be expressed as

$$\delta F = \sum_k \left(\frac{\partial F}{\partial q_k} \delta q_k + \frac{\partial F}{\partial p_k} \delta p_k \right) \quad ...(2)$$

If the transformation (1), generated by the function G, is applied, we get

$$\delta F = \sum_k \left[\frac{\partial F}{\partial q_k} \left(\in \frac{\partial G}{\partial p_k} \right) + \frac{\partial F}{\partial p_k} \left(- \in \frac{\partial G}{\partial q_k} \right) \right]$$

Since the parameter $\in$ is independent of q_k and p_k, we have

$$\delta F = \in \left[\sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \right] \quad ...(3)$$

The quantity in the big bracket in (3) is called the ***Poisson bracket*** of two functions or dynamical variables $F(q_k, p_k)$ and $G(q_k, p_k)$ and is denoted by $[F, G]$. This definition of Poisson bracket is true for F and G, being functions of time. Thus

$$\delta F = \in [F, G] \quad ...(4)$$

7.2. POISSON'S BRACKETS

If the functions F and G depend upon the position coordinates q_k, momentum coordinates p_k and time t, the Poisson bracket of F and G is defined as

$$[F, G]_{q,p} = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \quad ...(5)$$

For brevity, we may drop the subscripts q, p and write the Poisson bracket as $[F, G]$.

The total time derivative of the function F can be written as

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\dot{q}_k+\frac{\partial F}{\partial p_k}\dot{p}_k\right) \qquad ...(6)$$

Using, Hamilton's equations $\dot{q}_k=\frac{\partial H}{\partial p_k}$ and $-\dot{p}_k=\frac{\partial H}{\partial q_k}$, eq. (6) is obtained to be

$$\frac{dF}{dt}=\dot{F}=\frac{\partial F}{\partial t}+\sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k}-\frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) \qquad ...(7)$$

In view of the definition of Poisson's bracket given by eq. (5), we obtain

$$\frac{dF}{dt}=\frac{\partial F}{\partial t}+[F,H] \qquad ...(8)$$

From this equation we see that the function F is a constant of motion, if

$$\frac{dF}{dt}=0 \quad \text{or} \quad \frac{\partial F}{\partial t}+[F,H]=0 \qquad ...(9)$$

Now, if the function F does not depend on time explicitly, $\frac{\partial F}{\partial t}=0$ and then the condition for F to be constant of motion is obtained to be

$$[F,H]=0 \qquad ...(10)$$

Thus *if a function F does not depend on time explicitly and is a constant of motion, its Poisson bracket with the Hamiltonian vanishes*. In other words, *a function whose Poisson bracket with Hamiltonian vanishes is a constant of motion*. This result does not depend whether H itself is constant of motion.

Equations of motion in Poisson bracket form : Special cases of (8) are

(1) $F=q_k$, $\quad \dot{q}_k=[q_k,H]$ ...(11*a*)

(2) $F=p_k$, $\quad \dot{p}_k=[p_k,H]$ ...(11*b*)

(3) $F=H$, $\quad \dot{H}=\frac{\partial H}{\partial t}$ ...(11*c*)

These equations (11*a*, 11*b*, 11*c*) are identical to Hamilton's equations (37*a*), (37*b*) and (37*c*) or (40) of Chapter 3 and referred as **equations of motion in Poisson bracket form.**

Properties of Poisson brackets and Fundamental Poission brackets : The Poisson bracket has the property of antisymmetry, given by

$$[F,G]=-[G,F] \qquad ...(12)$$

because $$[F,G]=\sum_k\left[\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k}-\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right]=-\sum_k\left[\frac{\partial G}{\partial q_k}\frac{\partial F}{\partial p_k}-\frac{\partial G}{\partial p_k}\frac{\partial F}{\partial q_k}\right]=-[G,F].$$

Thus Poisson bracket does not obey the commutative law of algebra. As an application of the Poisson brackets, we are giving below some of the special cases :

(1) When $G = q_l$,

$$[F, q_l] = \sum_k \left[\frac{\partial F}{\partial q_k} \frac{\partial q_l}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial q_l}{\partial q_k} \right] = -\sum_k \frac{\partial F}{\partial p_k} \delta_{lk}$$

or $$[F, q_l] = -\frac{\partial F}{\partial p_l} \quad \text{...(13)}$$

Also if $F = q_k$, $$[q_k, q_l] = -\frac{\partial q_k}{\partial p_l} = 0 \quad \text{...(14)}$$

and if $F = p_k$, $$[p_k, q_l] = -\frac{\partial p_k}{\partial p_l} = -\delta_{kl} \quad \text{...(15)}$$

(2) When $G = p_l$, $[F, p_l] = \sum_k \frac{\partial F}{\partial q_k} \delta_{kl}$

or $$[F, p_l] = \frac{\partial F}{\partial q_l} \quad \text{...(16)}$$

For $F = p_k$, $$[p_k, p_l] = \frac{\partial p_k}{\partial q_l} = 0 \quad \text{...(17)}$$

and for $F = q_k$, $$[q_k, p_l] = \frac{\partial q_k}{\partial q_l} = \delta_{kl} \quad \text{...(18)}$$

The above results can be summarized as follows :

$$[q_k, q_l] = [p_k, p_l] = 0 \quad \text{...(19)}$$

and $$[q_k, p_l] = \delta_{kl} \quad \text{...(20)}$$

where δ_{kl} is the kronecker delta symbol with the property

$$\delta_{kl} = 0 \text{ for } k \neq 1 \text{ and } \delta_{kl} = 1 \text{ for } k = l$$

Equations (19) and (20) are called the ***fundamental Poisson's brackets***.

Further from the definition of Poisson bracket of any two dynamical variables or functions, one can obtain the following identities :

(*i*) $[F, F] = 0$ ...(21)

(*ii*) $[F, C] = 0$, C = constant ...(22)

(*iii*) $[CF, G] = C[F, G]$ ...(23)

(*iv*) $[F_1 + F_2, G] = [F_1, G] + [F_2, G]$ ...(24)

(*v*) $[F, G_1 G_2] = G_1[F, G_2] + [F, G_1]G_2$ ...(25)

(*vi*) $\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$ ...(26)

(*vii*) $[F, [G, K]] + [G, [K, F]] + [K, [F, G]] = 0$ (Jacobi's identity) ...(27)

7.3. LAGRANGE BRACKETS

The Lagrange bracket of two dynamical variables $F(q_k, p_k)$ and $G(q_k, p_k)$ is defined as

$$\{F,G\} = \sum_k \left[\frac{\partial q_k}{\partial F} \frac{\partial p_k}{\partial G} - \frac{\partial p_k}{\partial F} \frac{\partial q_k}{\partial G} \right] \quad ...(28)$$

The Langrange's bracket does not obey the commutative law of algebra *i.e.*, for Lagrangian bracket

$$\{F, G\} = - \{G, F\} \quad ...(29)$$

because $$\{F,G\} = - \sum_k \left[\frac{\partial q_k}{\partial G} \frac{\partial p_k}{\partial F} - \frac{\partial p_k}{\partial G} \frac{\partial q_k}{\partial F} \right] = - \{G,F\}$$

Further $$\{q_i, q_j\} = \sum_k \left[\frac{\partial q_k}{\partial q_i} \frac{\partial p_k}{\partial q_j} - \frac{\partial p_k}{\partial q_i} \frac{\partial q_k}{\partial q_j} \right] = 0 \quad ...(30)$$

because $$\frac{\partial p_k}{\partial q_j} = \frac{\partial p_k}{\partial q_i} = 0.$$

Similarly, one can prove that for Lagrange brackets

$$\{p_i, p_j\} = 0; \ \{q_i, p_j\} = \delta_{ij} \quad ...(31)$$

7.4. RELATION BETWEEN LAGRANGE AND POISSON BRACKETS

If F_k, $k = 1, 2, ..., 2n$, are $2n$ independent functions such that each F_k is a function of $2n$ coordinates $q_1, q_2, ..., q_n; p_1, p_2, ..., p_n$, then

$$\sum_{k=1}^{2n} \{F_k, F_i\}[F_k, F_j] = \delta_{ij} \quad ...(32)$$

In order to prove the relation (32), we take the left hand side of this equation and use the definitions of Poisson and Lagrange brackets :

$$\sum_{k=1}^{2n} \{F_k, F_i\}[F_k, F_j]$$

$$= \sum_{k=1}^{2n} \left[\sum_{l=1}^{n} \sum_{m=1}^{n} \left(\frac{\partial q_l}{\partial F_k} \frac{\partial p_l}{\partial F_i} - \frac{\partial p_l}{\partial F_k} \frac{\partial q_l}{\partial F_i} \right) \left(\frac{\partial F_k}{\partial q_m} \frac{\partial F_j}{\partial p_m} - \frac{\partial F_k}{\partial p_m} \frac{\partial F_j}{\partial q_m} \right) \right]$$

$$= \sum_{l=1}^{n} \left(\frac{\partial F_j}{\partial p_l} \frac{\partial p_l}{\partial F_i} + \frac{\partial F_j}{\partial q_l} \frac{\partial q_l}{\partial F_i} \right) = \frac{\partial F_j}{\partial F_i} = \delta_{ij}$$

In our work, Poisson bracket is relatively much more useful than Lagrange bracket and therefore we do not discuss further details in relation to Lagrange bracket.

7.5. ANGULAR MOMENTUM AND POISSON BRACKETS

Using the definition of linear and angular momentum, a number of interesting and useful Poisson bracket relations can be obtained.

Poisson brackets relations between the components of p and J : According to the definition of angular momentum,

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \times (p_x\hat{\mathbf{i}} + p_y\hat{\mathbf{j}} + p_z\hat{\mathbf{k}})$$

or $$\mathbf{J} = (yp_z - zp_y)\hat{\mathbf{i}} + (zp_x - xp_z)\hat{\mathbf{j}} + (xp_y - yp_x)\hat{\mathbf{k}}$$

Therefore,

$$J_x = (yp_z - zp_y)\,, J_y = (zp_x - xp_z) \text{ and } J_z = (xp_y - yp_x)$$

From the definition of Poisson bracket (5)

$$[F, G] = \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right),$$

we have,

$$[p_x, p_y] = [p_y, p_z] = [p_z, p_x] = [p_x, p_x] = 0 \qquad \text{...(33)}$$

Next, using the result (16) $[F, p_l] = \dfrac{\partial F}{\partial q_l}$, we have

$$[J_x, p_y] = p_z,\ [J_x, p_z] = -p_y,\ [J_x, p_x] = 0 \qquad \text{...(34 a)}$$

Similarly, $$[J_y, p_x] = -p_z,\ [J_y, p_y] = 0,\ [J_y, p_z] = p_x \qquad \text{...(34 b)}$$

$$[J_z, p_x] = p_y,\ [J_z, p_y] = -p_x,\ [J_z, p_z] = 0 \qquad \text{...(34 c)}$$

Further

$$[J_x, J_y] = \sum_k\left[\frac{\partial J_x}{\partial q_k}\frac{\partial J_y}{\partial p_k} - \frac{\partial J_x}{\partial p_k}\frac{\partial J_y}{\partial q_k}\right]$$

For $q_1 = x$, $q_2 = y$, $q_3 = z$ and $p_1 = p_x$, $p_2 = p_y$, $p_3 = p_z$,

$$[J_x, J_y] = \frac{\partial J_x}{\partial x}\frac{\partial J_y}{\partial p_x} - \frac{\partial J_x}{\partial p_x}\frac{\partial J_y}{\partial x} + \frac{\partial J_x}{\partial y}\frac{\partial J_y}{\partial p_y} - \frac{\partial J_x}{\partial p_y}\frac{\partial J_y}{\partial y} + \frac{\partial J_x}{\partial z}\frac{\partial J_y}{\partial p_z} - \frac{\partial J_x}{\partial p_z}\frac{\partial J_y}{\partial z}$$

$$= 0 - 0 + 0 - 0 + (-p_y)(-x) - (y)(p_x)$$

$$= xp_y - yp_x = J_z \qquad \text{...(35)}$$

Similarly one can prove that

$$[J_y, J_z] = J_x,\ [J_z, J_x] = J_y \qquad \text{...(36)}$$

7.6. INVARIANCE OF POISSON BRACKET WITH RESPECT TO CANONICAL TRANSFOR MATIONS

Poisson brackets are invariant under canonical transformations. First we shall prove this statement for fundamental Poisson brackets and then in general.

Fundamental Poisson brackets under canonical transformation : The fundamental Poisson brackets are invariant under canonical transformation means that if

$$[q_k, q_l] = [p_k, p_l] = 0,\ [q_k, p_l] = \delta_{kl} \qquad \text{...(37)}$$

and the transformation $(q_k, p_k) \to (Q_k, P_k)$ is canonical, then

$$[Q_k, Q_l] = [P_k, P_l] = 0, \quad [Q_k, P_l] = \delta_{kl} \qquad ...(38)$$

According to the definition of Poisson bracket [eq. (5)], we have

$$[F, G]_{q,p} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right). \qquad ...(39)$$

Therefore,

$$[Q_k, Q_l]_{q,p} = \sum_i \left[\frac{\partial Q_k}{\partial q_i} \frac{\partial Q_l}{\partial p_i} - \frac{\partial Q_k}{\partial p_i} \frac{\partial Q_l}{\partial q_i} \right] \qquad ...(40)$$

From eq. (21) of Chapter 6, we get

$$\frac{\partial p_k}{\partial Q_l} = \frac{\partial}{\partial Q_l} \frac{\partial F_1}{\partial q_k} = \frac{\partial}{\partial q_k} \frac{\partial F_1}{\partial Q_l} = -\frac{\partial P_l}{\partial q_k} \qquad ...(41)$$

Similarly eqs. (27), (29) and (31) of Chapter 6 yield

$$\frac{\partial p_k}{\partial P_l} = \frac{\partial}{\partial P_l} \frac{\partial F_2}{\partial q_k} = \frac{\partial}{\partial q_k} \frac{\partial F_2}{\partial P_l} = \frac{\partial Q_l}{\partial q_k} \qquad ...(42)$$

$$\frac{\partial q_k}{\partial Q_l} = -\frac{\partial}{\partial Q_l} \frac{\partial F_3}{\partial p_k} = -\frac{\partial}{\partial p_k} \frac{\partial F_3}{\partial Q_l} = \frac{\partial P_l}{\partial p_k} \qquad ...(43)$$

$$\frac{\partial q_k}{\partial P_l} = -\frac{\partial}{\partial P_l} \frac{\partial F_4}{\partial p_k} = -\frac{\partial}{\partial p_k} \frac{\partial F_4}{\partial P_l} = -\frac{\partial Q_l}{\partial p_k} \qquad ...(44)$$

Hence eq. (40) is [using (41) and (43)]

$$[Q_k, Q_l]_{q,p} = \sum_i \left(-\frac{\partial Q_k}{\partial q_i} \frac{\partial q_i}{\partial P_l} - \frac{\partial Q_k}{\partial p_i} \frac{\partial p_i}{\partial P_l} \right) = -\frac{\partial Q_k}{\partial P_l} = 0 \qquad ...(45)$$

because Q_k and P_k are independent variables. Also we note that

$$[Q_k, Q_l]_{Q,P} = \sum_i \left(-\frac{\partial Q_k}{\partial Q_i} \frac{\partial Q_l}{\partial P_i} - \frac{\partial Q_k}{\partial P_i} \frac{\partial Q_l}{\partial Q_i} \right) = 0$$

Therefore, $[Q_k, Q_l]_{q,p} = [Q_k, Q_l]_{Q,P} = 0$ ...(46)

Similarly we can prove

$$[P_k, P_l]_{q,p} = [P_k, P_l]_{Q,P} = 0 \qquad ...(47)$$

Now, $$[Q_k, P_l]_{q,p} = \sum_i \left(\frac{\partial Q_k}{\partial q_i} \frac{\partial P_l}{\partial p_i} - \frac{\partial Q_k}{\partial p_i} \frac{\partial P_l}{\partial q_i} \right)$$

Using eqs. (41) and (43), we obtain

$$[Q_k, P_l]_{q,p} = \sum_i \left(\frac{\partial Q_k}{\partial q_i} \frac{\partial q_i}{\partial Q_l} + \frac{\partial Q_k}{\partial p_i} \frac{\partial p_i}{\partial Q_l} \right) = \frac{\partial Q_k}{\partial Q_l} = \delta_{kl} \qquad ...(48)$$

By definition $[Q_k, P_l]_{Q,P} = \delta_{kl}$ ...(49)

Thus $$[Q_k, P_l]_{q,p} = [Q_k, P_l]_{Q,P} = \delta_{kl} \qquad ...(50)$$

Eqs. (46), (47) and (50) show the invariance of fundamental Poisson brackets with respect to canonical transformation.

General Poisson brackets under canonical transformation : In general, if Poisson bracket is invariant under canonical transformation (q, p) to (Q, P), we mean that

$$[F, G]_{q,p} = [F, G]_{Q,P} \qquad ...(51)$$

In order to prove this, let us start from the definition of Poisson bracket *i.e.*,

$$[F, G]_{q,p} = \sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \qquad ...(52)$$

As $p_k = P_k\,(Q_1, Q_2, ..., Q_k, ..., P_1, P_2, ..., P_k, ...)$ and

$q_k = q_k\,(Q_1, Q_2, ..., Q_k, ..., P_1, P_2, ..., P_k, ...)$, we can write

$$[F, G]_{q,p} = \sum_k \sum_l \left[\frac{\partial F}{\partial q_k} \left(\frac{\partial G}{\partial Q_l} \frac{\partial Q_l}{\partial p_k} + \frac{\partial G}{\partial P_l} \frac{\partial P_l}{\partial p_k} \right) - \frac{\partial F}{\partial p_k} \left(\frac{\partial G}{\partial Q_l} \frac{\partial Q_l}{\partial q_k} + \frac{\partial G}{\partial P_l} \frac{\partial P_l}{\partial p_k} \right) \right]$$

or $$[F, G]_{q,p} = \sum_l \left(\frac{\partial G}{\partial Q_l} [F, Q_l]_{q,p} + \frac{\partial G}{\partial P_l} [F, P_l]_{q,p} \right) \qquad ...(53)$$

In eq. (53), substituting $F = Q_i$ and $G = F$, we get

$$[Q_i, F]_{q,p} = \sum_l \left(\frac{\partial F}{\partial Q_l} [Q_i, Q_l]_{q,p} + \frac{\partial F}{\partial P_l} [Q_i, P_l]_{q,p} \right) = \frac{\partial F}{\partial P_i} \qquad ...(54)$$

Similarly, substituting $F = P_i$ and $G = F$ in (53), we have

$$[P_i, F]_{q,p} = \sum_l \left(\frac{\partial F}{\partial Q_l} [P_i, Q_l]_{q,p} + \frac{\partial F}{\partial P_l} [P_i, P_l]_{q,p} \right) = -\frac{\partial F}{\partial Q_i} \qquad ...(55)$$

Substituting (54) and (55) in (53), we obtain

$$[F, G]_{q,p} = \sum_l \left(-\frac{\partial G}{\partial Q_l} \frac{\partial F}{\partial P_l} + \frac{\partial G}{\partial P_l} \frac{\partial F}{\partial Q_l} \right) = [F, G]_{Q,P} \qquad ...(56)$$

This proves the statement (51). Thus for the canonical variables, we can drop the subscripts of Poisson brackets.

Ex. 1 : *Prove that the distributive law*

$$[F, G + K] = [F, G] + [F, K]$$

for Poisson brackets holds good.

Solution : $$[F, G + K] = \sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial (G + K)}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial (G + K)}{\partial q_k} \right)$$

$$= \sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} + \frac{\partial F}{\partial q_k} \frac{\partial K}{\partial p_k} \right) - \sum_k \left(\frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} + \frac{\partial F}{\partial p_k} \frac{\partial K}{\partial q_k} \right)$$

$$= \sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) + \sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial K}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial K}{\partial q_k} \right)$$

$$= [F, G] + [F, K].$$

Ex. 2. *If* $[\phi, \psi]$ *be the Poisson bracket, then prove that*

$$\frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \phi}{\partial t}, \psi\right] + \left[\phi, \frac{\partial \psi}{\partial t}\right].$$

Solution : From the definition of Poisson bracket

$$[\phi, \psi] = \sum_k \left(\frac{\partial \phi}{\partial q_k} \frac{\partial \psi}{\partial p_k} - \frac{\partial \phi}{\partial p_k} \frac{\partial \psi}{\partial q_k} \right)$$

Now, $$\frac{\partial}{\partial t}[\phi, \psi] = \sum_k \left[\frac{\partial}{\partial q_k}\left(\frac{\partial \phi}{\partial t}\right) \frac{\partial \psi}{\partial p_k} + \frac{\partial \phi}{\partial q_k} \frac{\partial}{\partial p_k}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial}{\partial p_k}\left(\frac{\partial \phi}{\partial t}\right) \frac{\partial \psi}{\partial q_k} - \frac{\partial \phi}{\partial p_k} \frac{\partial}{\partial q_k}\left(\frac{\partial \psi}{\partial t}\right) \right]$$

$$= \sum_k \left[\frac{\partial}{\partial q_k}\left(\frac{\partial \phi}{\partial t}\right) \frac{\partial \psi}{\partial p_k} - \frac{\partial}{\partial p_k}\left(\frac{\partial \phi}{\partial t}\right) \frac{\partial \psi}{\partial q_k} \right] + \sum_k \left[\frac{\partial \phi}{\partial q_k} \frac{\partial}{\partial p_k}\left(\frac{\partial \psi}{\partial t}\right) - \frac{\partial \phi}{\partial p_k} \frac{\partial}{\partial q_k}\left(\frac{\partial \psi}{\partial t}\right) \right]$$

$$= \left[\frac{\partial \phi}{\partial t}, \psi\right] + \left[\phi, \frac{\partial \psi}{\partial t}\right].$$

Ex. 3. *Show that transformation defined by*

$$q = \sqrt{2P} \sin Q, \quad p = \sqrt{2P} \cos Q$$

is canonical by using Poisson bracket. **(Rohilkhand 1987)**

Solution : The transformation is

$$q = \sqrt{2P} \sin Q, \quad p = \sqrt{2P} \cos Q$$

From these equations, we can write the transformation as

$$\tan Q = \frac{q}{p} \text{ and } P = \tfrac{1}{2}\left(q^2 + p^2\right) \qquad ...(i)$$

In order to show that the given transformation is canonical, the Poisson bracket conditions are

$$[Q, Q] = [P, P] = 0 \text{ and } [Q, P] = 1 \qquad ...(ii)$$

Here, $$[Q, Q] = \frac{\partial Q}{\partial q} \frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial Q}{\partial q} = 0 \qquad ...(iii)$$

Similarly, $$[P, P] = 0 \qquad ...(iv)$$

Also $$[Q, P] = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} \qquad ...(v)$$

But from (*i*),

$$\sec^2 Q\frac{\partial Q}{\partial q}=\frac{1}{p},\ \frac{\partial P}{\partial p}=p,\ \sec^2 Q\frac{\partial Q}{\partial p}=-\frac{q}{p^2},\frac{\partial P}{\partial q}=q$$

Substituting these values in (v), we get

$$[Q,P]=\frac{\cos^2 Q}{p}p+\frac{q\cos^2 Q}{p^2}q=\cos^2 Q+\frac{q^2}{p^2}\cos^2 Q$$

$$=\cos^2 Q\left[1+\frac{q^2}{p^2}\right]=\cos^2 Q\left[1+\tan^2 Q\right]$$

$$=\cos^2 Q\sec^2 Q=1 \qquad \ldots(vi)$$

Thus we prove the conditions (ii) which means that the given transformation is canonical.

Ex. 4. *Jacobi's Identity* : *Prove that for any three functions F, G and K of p_k and q_k, the following relation holds true :*

$$[F,[G,K]]+[G,[K,F]]+[K,[F,G]]=0$$

This relation is known as Jacobi's identity. **(Rohilkhand 1999, 95; Meerut 94; Gorakhpur 96)**

Solution : Let us consider the expression for the following :

$$[F,[G,K]]-[G,[F,K]]$$

$$=\left[F,\sum_k\left(\frac{\partial G}{\partial q_k}\frac{\partial K}{\partial p_k}-\frac{\partial G}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right]-\left[G,\sum_k\left(\frac{\partial F}{\partial q_k}\frac{\partial K}{\partial p_k}-\frac{\partial F}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right]$$

$$=\left[F,\sum_k\left(\frac{\partial G}{\partial q_k}\frac{\partial K}{\partial p_k}\right)\right]-\left[F,\sum_k\left(\frac{\partial G}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right]-\left[G,\sum_k\left(\frac{\partial F}{\partial q_k}\frac{\partial K}{\partial p_k}\right)\right]+\left[G,\sum_k\left(\frac{\partial F}{\partial p_k}\frac{\partial K}{\partial q_k}\right)\right]$$

Now, using the property $[F, GK]=[F,G]K+[F,K]G$, we have

$$[F,[G,K]]-[G,[F,K]]$$

$$=\left[F,\sum_k\frac{\partial G}{\partial q_k}\right]\sum_k\frac{\partial K}{\partial p_k}+\left[F,\sum_k\frac{\partial K}{\partial p_k}\right]\sum_k\frac{\partial G}{\partial q_k}-\left[F,\sum_k\frac{\partial G}{\partial p_k}\right]\sum_k\frac{\partial K}{\partial q_k}$$

$$-\left[F,\sum_k\frac{\partial K}{\partial q_k}\right]\sum_k\frac{\partial G}{\partial p_k}-\left[G,\sum_k\frac{\partial F}{\partial q_k}\right]\sum_k\frac{\partial K}{\partial p_k}-\left[G,\sum_k\frac{\partial K}{\partial p_k}\right]\sum_k\frac{\partial F}{\partial q_k}$$

$$+\left[G,\sum_k\frac{\partial F}{\partial p_k}\right]\sum_k\frac{\partial K}{\partial q_k}+\left[G,\sum_k\frac{\partial K}{\partial q_k}\right]\sum_k\frac{\partial F}{\partial p_k}$$

$$=\sum_k\left\{-\frac{\partial K}{\partial q_k}\left(\left[\frac{\partial F}{\partial p_k},G\right]+\left[F,\frac{\partial G}{\partial p_k}\right]\right)+\frac{\partial K}{\partial p_k}\left(\left[\frac{\partial F}{\partial q_k},G\right]+\left[F,\frac{\partial G}{\partial q_k}\right]\right)\right\}$$

$$+\sum_k\left\{\frac{\partial G}{\partial q_k}\left[F,\frac{\partial K}{\partial p_k}\right]-\frac{\partial G}{\partial p_k}\left[F,\frac{\partial K}{\partial q_k}\right]-\frac{\partial F}{\partial q_k}\left[G,\frac{\partial K}{\partial p_k}\right]+\frac{\partial F}{\partial p_k}\left[G,\frac{\partial K}{\partial q_k}\right]\right\}$$

Using the identity $\frac{\partial}{\partial x}[F,G]=\left[\frac{\partial F}{\partial x},G\right]+\left[F,\frac{\partial G}{\partial x}\right]$, we obtain

$$[F,[G,K]]-[G,[F,K]]=\sum_k\left[-\frac{\partial K}{\partial q_k}\frac{\partial}{\partial p_k}[F,G]+\frac{\partial K}{\partial p_k}\frac{\partial}{\partial q_k}[F,G]\right]+0$$

$$=-[K,[F,G]]$$

Thus, $[F,[G,K]]+[G,[K,F]]+[K,[F,G]]=0$

which proves the *Jacobi's identity.*

Ex. 5. *Show that the Poisson bracket of two constants of motion is itself a constant of motion.*

(Garwal 1995; Agra 81)

Solution : In Jacobi's identity, we put $K = H$, then

$$[F,[G,H]+[G,[H,F]]+[H,[F,G]]=0$$

Now, if F and G are constants of motion, then $[F, H] = 0$ and $[G, H] = 0$

Therefore, $\quad [H,[F,G]]=0$

which means that the dynamic variable $[F, G]$ is constant of motion. Thus the Poisson bracket of two constants of motion is itself a constant of motion.

Ex. 6. *Show that the Lagrange's bracket is invariant under canonical transformation.*

Solution : Invariance of Lagrange's bracket under canonical transformation : According to the Poincare theorem, the integral

$$I_1=\iint_S\sum_k dq_k\,dp_k \qquad ...(i)$$

taken over an arbitrary two dimensional surface S of $2n$ dimensional phase space (q_k, p_k) is invariant under canonical transformation *i.e.*,

$$\iint_S\sum_k dq_k dp_k=\iint_S\sum_k dQ_k dP_k \qquad ...(ii)$$

The position of a point on any two dimensional surface can be completely specified by two parameters, say u and v, so that

$$q_k=q_k(u,v) \quad \text{and} \quad p_k=p_k(u,v) \qquad ...(iii)$$

Transforming the area element in terms of new variables (u, v) by means of Jacobian, we have

$$dq_k\,dp_k=\frac{\partial(q_k,p_k)}{\partial(u,v)}du\,dv \qquad ...(iv)$$

with

$$\frac{\partial(q_k,p_k)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial q_k}{\partial u} & \frac{\partial p_k}{\partial u}\\ \frac{\partial q_k}{\partial v} & \frac{\partial p_k}{\partial v}\end{vmatrix} \qquad ...(v)$$

as the Jacobian.

Eq. (ii) in view of eq. (iv) is obtained to be

$$\iint_S\sum_k\frac{\partial(q_k,p_k)}{\partial(u,v)}du\,dv=\iint_S\sum_k\frac{\partial(Q_k,P_k)}{\partial(u,v)}du\,dv \qquad ...(vi)$$

As the surface S is arbitrary, area $du\,dv$ is arbitrary and therefore the expressions on both sides of eq. (*vi*) will be equal in the condition that the integrals are equal *i.e.*,

$$\sum_k \frac{\partial(q_k, p_k)}{\partial(u,v)} = \sum_k \frac{\partial(Q_k, P_k)}{\partial(u,v)}$$

or
$$\sum_k \begin{vmatrix} \frac{\partial q_k}{\partial u} & \frac{\partial p_k}{\partial u} \\ \frac{\partial q_k}{\partial v} & \frac{\partial p_k}{\partial v} \end{vmatrix} = \sum_k \begin{vmatrix} \frac{\partial Q_k}{\partial u} & \frac{\partial P_k}{\partial u} \\ \frac{\partial Q_k}{\partial v} & \frac{\partial P_k}{\partial v} \end{vmatrix}$$

or
$$\sum_k \left(\frac{\partial q_k}{\partial u}\frac{\partial p_k}{\partial v} - \frac{\partial q_k}{\partial v}\frac{\partial p_k}{\partial u} \right) = \sum_k \left(\frac{\partial Q_k}{\partial u}\frac{\partial P_k}{\partial v} - \frac{\partial Q_k}{\partial v}\frac{\partial P_k}{\partial u} \right)$$

or
$$\{u, v\}_{q,p} = \{u, v\}_{Q,P} \qquad \text{...(vii)}$$

Thus, Lagrange's bracket is invariant under canonical transformation. Therefore it is immaterial which set of canonical coordinates is to be used *i.e.*, that subscripts q, p can be dropped in writing Lagrange's brackets.

Ex. 7. *Prove the following relations :*

(*a*) $\sum_{k=1}^{n} \{p_k, q_i\}[p_k, p_j] + \sum_{k=1}^{n} \{q_k, q_i\}[q_k, p_j] = 0$

(*b*) $\sum_{k=1}^{n} \{q_k, q_i\}[q_k, q_j] + \sum_{k=1}^{n} \{p_k, q_i\}[p_k, q_j] = \delta_{ij}$.

Solution : We know that

$$\{p_k, q_i\} = -\{q_i, p_k\} = -\delta_{ik}, \{q_k, q_i\} = [p_k, p_j] = 0 \text{ and } [q_k, p_j] = \delta_{kj}$$

Therefore
$$\sum_{k=1}^{n} \{p_k, q_i\}[p_k, q_j] + \sum_{k=1}^{n} \{q_k, q_i\}[q_k, p_j] = \sum_{k=1}^{n} -\delta_{ik} \times 0 + \sum_{k=1}^{n} 0 \times \delta_{kj} = 0$$

This proves the relation (*a*).

Also we know that

$$\{q_k, q_i\} = [q_k, q_j] = 0, \{p_k, q_i\} = -\{q_i, p_k\} = -\delta_{ik}$$

and
$$[p_k, q_j] = -[q_j, p_k] = -\delta_{jk}$$

Substituting in (*b*) on left hand side, we get

$$0 + \sum_{k=1}^{n} [-\delta_{ik}][-\delta_{jk}] = \delta_{ij}$$

which proves the relation (*b*).

7.7. PHASE SPACE

In the Hamiltonian formulation, we observe from the equations of motion that the momentum coordinates p_k (k =1, 2,...,n) and position coordinates q_k(k =1,2,...,n) play similar roles. We can imagine a space in $2n$ dimensions, in which a complete dynamical specification of a mechanical system is given by a point, having $2n$ coordinates ($p_1, p_2, ..., p_n, q_1, q_2, ..., q_n$). Such a space is known as $2n$ *-dimensional phase space.*

If we know the state of a mechanical system at time t, *i.e.*, we know all position and momentum coordinates, then this state will be represented by a specific point in the phase space. In other words, a point in phase space specifies the state of a mechanical system.

Symbolically, the representative point for the state of the system in the phase space can be written as

$$\mathbf{r} = (q_1, q_2, ..., q_n, p_1, p_2, ..., p_n) \quad ...(57)$$

As the time advances, the changing state of the system may be described by a curve $\mathbf{r}(t)$ in the phase space. This is called *phase path.*

7.8. LIOUVILLE'S THEOREM

Let us consider a large number of identical mechanical systems. Each system has slightly different values of the coordinates q_k and momenta p_k. This can be represented by a point in the phase space. All these systems can be represented by a swarm of points in the phase space, because each system has slightly different initial coordinates. As the time advances, these systems will move along different paths in the phase space due to different initial conditions. Suppose that the initial points corresponding to all these systems at time $t = t_1$ are contained in a volume Γ_1 of the phase space and after some time at $t = t_2$, these points occupy the region Γ_2. For example , the representative point corresponding to a system moves from a point A $[p_k(t_1), q_k(t_1)]$ to B $[p_k(t_2), q_k(t_2)]$ [Fig. 7.1].

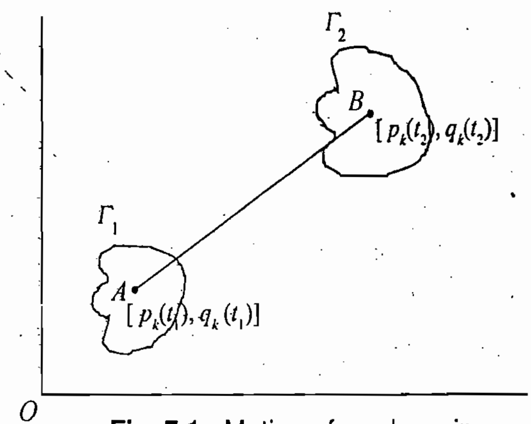


Fig. 7.1 : Motion of a volume in phase space

Clearly the number of points in the volumes Γ_1 and Γ_2 of the phase space are the same. However, it is not obvious how these volumes Γ_1 and Γ_2 are related with the develpoment of the time. Liouville's theorem tells about this relation.

According to Liouville's theorem, the 2n-dimensional volumes Γ_1 and Γ_2 are the same or the 2n-dimensional volume occupied by the swarm of points does not change with time, though its shape may of course change.

In other words, *if we define the number of points per unit volume of the phase space as the density* (ρ), *then according to Liouville's theorem the density of points remains constant with time, i.e.,*

$$\frac{d\rho}{dt} = 0 \text{ or } \rho = \text{constant} \quad ...(58)$$

One can imagine the points in the phase space as particles of an incompressible fluid which move from Γ_1 region to Γ_2 region as time changes from t_1 to t_2.

Proof of Liouvill's Theorem : First we shall prove this theorem for the case of one degree of freedom and then for the general case. In the case of a mechanical system with one degree of freedom, we have two dimensional phase space, described by p and q coordinates. In this case, the volume element reduces to an area element $dp\, dq$ of phase space [Fig. 7.2].

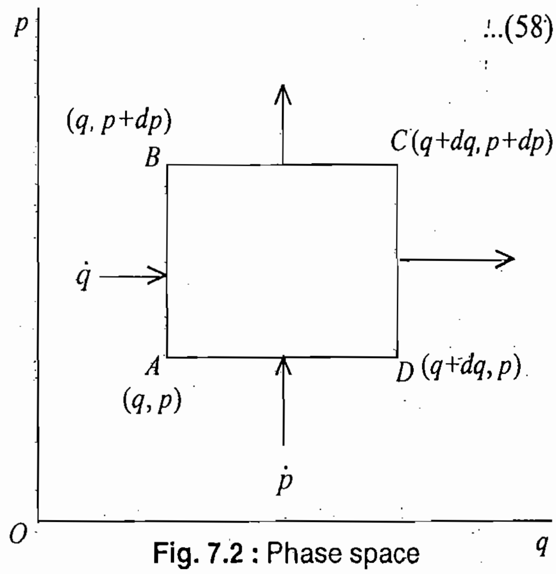


Fig. 7.2 : Phase space

Let $\rho(p, q, t)$ be the density of the representative points in the p-q phase space. Therefore, ρ is the number of points per unit area. As the speed with which representative points enter the element $ABCD$ through AB is $\dot{q}$, the number of representative points, entering through AB per unit time, is $\rho\dot{q}\,dp$. The number of points, leaving through CD, is $\rho\dot{q}\,dp + \frac{\partial}{\partial q}(\rho\dot{q})\,dq\,dp$. Hence the number of points which remain in the element $ABCD$ is

$$\rho\dot{q}dp - [\rho\dot{q}\,dp + \frac{\partial}{\partial q}(\rho\dot{q})dq\,dp] = -\frac{\partial}{\partial q}(\rho\dot{q})dp\,dq \qquad ...(59)$$

Similarly the number of representative points which enter through AD and leave through BC are $\rho\dot{p}\,dq$ and $[\rho\dot{p} + \frac{\partial}{\partial p}(\rho\dot{p})dp]\,dq$. Hence, the number of points remaining in the element $ABCD$ is

$$-\frac{\partial}{\partial p}(\rho\dot{p})\,dp\,dq \qquad ...(60)$$

By adding (59) and (60), we get the increase in the representative points per unit time in the element $ABCD$ *i.e.*,

$$-\left[\frac{\partial(\rho\dot{q})}{\partial q} + \frac{\partial(\rho\dot{p})}{\partial p}\right] dp\,dq \qquad ...(61)$$

But the rate of increase of points per unit time in the element is $\frac{\partial\rho}{\partial t}\,dp\,dq$, therefore we must have

$$\frac{\partial\rho}{\partial t}dp\,dq = -\left[\frac{\partial(\rho\dot{q})}{\partial q} + \frac{\partial(\rho\dot{p})}{\partial p}\right] dp\,dq \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{q})}{\partial q} + \frac{\partial(\rho\dot{p})}{\partial p} = 0$$

or
$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial\dot{q}}{\partial q} + \frac{\partial\rho}{\partial q}\dot{q} + \rho\frac{\partial\dot{p}}{\partial p} + \frac{\partial\rho}{\partial p}\dot{p} = 0 \qquad ...(62)$$

Now Hamilton's equations are

$$\dot{q} = \partial H/\partial p \quad \text{and} \quad \dot{p} = -\partial H/\partial q$$

Hence,
$$\frac{\partial\dot{q}}{\partial q} = \frac{\partial^2 H}{\partial q\,\partial p} \quad \text{and} \quad \frac{\partial\dot{p}}{\partial p} = -\frac{\partial^2 H}{\partial p\,\partial q}$$

We assume that the Hamiltonian H has continuous second order derivatives. Therefore

$$\partial\dot{q}/\partial q = -\partial\dot{p}/\partial p \qquad ...(63)$$

Hence, in view of eq. (63), eq. (62) takes the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q}\dot{q} + \frac{\partial\rho}{\partial p}\dot{p} = 0 \qquad ...(64)$$

But $\rho = \rho\,(p, q, t)$ and hence its total time derivative is

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q}\frac{dq}{dt} + \frac{\partial\rho}{\partial p}\frac{dp}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial q}\dot{q} + \frac{\partial\rho}{\partial p}\dot{p} \qquad ...(65)$$

Therefore, eq. (64) can be written as

$$\frac{d\rho}{dt} = 0 \quad \text{or} \quad \rho = \text{constant} \qquad ...(66)$$

which proves the Liouville's theorem.

In the general case of n degrees of freedom, the volume of element in the phase space is given by

$$d\Gamma = dq_1\, dq_2 \ldots dq_n\, dp_1\, dp_2 \ldots dp_n \qquad \ldots(67)$$

Following exactly the procedure as given above, the increase in the representative points per unit time in the volume $d\Gamma$ is given by

$$\frac{\partial \rho}{\partial t} d\Gamma = -\left[\frac{\partial(\rho \dot{q}_1)}{\partial q_1} + \frac{\partial(\rho \dot{q}_2)}{\partial q_2} + \ldots + \frac{\partial(\rho \dot{q}_n)}{\partial q_n} + \frac{\partial(\rho \dot{p}_1)}{\partial p_1} + \ldots + \frac{\partial(\rho \dot{p}_n)}{\partial p_n}\right] d\Gamma$$

or

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n}\left[\frac{\partial(\rho \dot{q}_k)}{\partial q_k} + \frac{\partial(\rho \dot{p}_k)}{\partial p_k}\right] = 0 \qquad \ldots(68)$$

or

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n}\left[\frac{\partial \rho}{\partial q_k}\dot{q}_k + \rho\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \rho}{\partial p_k}\dot{p}_k + \rho\frac{\partial \dot{p}_k}{\partial p_k}\right] = 0$$

In view of Hamilton's equations, $\partial \dot{q}_k / \partial q_k = -\partial \dot{p}_k / \partial p_k$

Therefore,

$$\frac{\partial \rho}{\partial t} + \sum_{k=1}^{n}\left[\frac{\partial \rho}{\partial q_k}\dot{q}_k + \frac{\partial \rho}{\partial p_k}\dot{p}_k\right] = 0$$

or

$$\frac{d\rho}{dt} = 0 \qquad \ldots(69)$$

Alternative Proof of Liouville's Theorem : Let us consider an infinitesimal volume in phase space surrounding a representative point corresponding to a system. Now, each point in the volume moves in the course of time according to the equations of motion of the corresponding system and hence the region as a whole moves. Clearly the number of representative points within the volume remains constant. Poincare's theorem of integral invariance tells us that a volume element in phase space is invariant under a canonical transformation. Hence the size of the volume element about the representative point cannot vary with time. Thus both the number of representative points in the infinitesimal region, dN, and the volume, $d\Gamma$, are constants and consequently the density, defined by

$$\rho = dN/d\Gamma \qquad \ldots(70)$$

must also be constant in time, *i.e.*,

$$\rho = \text{constant or } d\rho/dt = 0 \qquad \ldots(71)$$

which proves the Liouville's theorem.

We know that for any function F, we can write [eq. (8)].

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] \qquad \ldots(72)$$

For $F = \rho$, Liouville's theorem obtains the form

$$\partial\rho/\partial t + [\rho, H] = 0 \qquad \ldots(73)$$

When the collection or ensemble of systems is in statistical equilibrium, the density of representative points at a given spot does not change with time. Therefore, in statistical equilibrium the partial derivative of ρ with respect to time t must vanish, *i.e.*, $\partial\rho/\partial t = 0$. Then

$$[\rho, H] = 0 \qquad \ldots(74)$$

Thus in statistical equilibrium, the Poisson bracket of density with the Hamiltonian must vanish and hence we can choose the density ρ to be a function of those constants of motion of the system which do not involve time explicitly.

Questions

1. Illustrate Poisson bracket of two dynamical variables. **(Kanpur 1999; Agra 90)**
2. Define Poisson's brakets and discuss their properties. **(Rohilkhand 1998)**
3. Show that the Poisson's bracket of two functions F and G does not obey the commutative law but obeys the distributive law of algebra.
4. If H is the Hamiltonian and f is any function depending on position, momenta and time, show that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$

where [ ] stands for Poisson bracket. **(Rohilkhand 1986, 84; Meerut 99)**

5. Show that if the Hamiltonian and a quantity G are constants of motion, then $\partial G/\partial t$ must also be constant.
6. Derive equations of motion in terms of Poisson's brackets. Prove Jacobi identity. **(Rohilkhand 1999)**
7. What are Lagrange and Poisson's brackets ? Explain and discuss their physical significance. **(Garwal 1999, 96)**
8. What are the Poisson and Lagrange's brackets ? Show that Lagrange's bracket is invariant under canonical transformations. **(Agra 1990; Gorakhpur 96)**
9. Prove that the Poisson bracket of two dynamical variables is invariant under infinitesimal canonical transformation. **(Garwal 1991, 90; Agra 89, 88; Rohilkhand 87)**
10. Write down angular momentum Poisson bracket relations. **(Kanpur 1998)**
11. If $[\phi, \psi]$ be the Poisson bracket, then prove that

$$\frac{\partial}{\partial t}[\phi, \psi] = \left[\frac{\partial \phi}{\partial t}, \psi\right] + \left[\phi, \frac{\partial \psi}{\partial t}\right]$$ **(Agra 1989)**

12. (*a*) Prove that the Lagrangian bracket does not obey the commutative law of algebra. **(Garwal 1992)**
    (*b*) Establish a relation between Lagrange and Poisson's brackets. **(Garwal 1992)**
13. Show that the Poisson bracket of two dynamical variables is invariant under a canonical transformation. Explain Hamilton's equations of motion in Poisson bracket notation. **(Kanpur 1999; Meerut 95, 93; Gorakhpur 96)**
14. Prove the following relation between Lagrange and Poisson brackets :

$$\sum_{i=1}^{2n} \{u_i, u_j\}[u_i, u_k] = \delta_{jk}$$

where u_i are $2n$ independent functions of $q_1, q_2, .., q_n, p_1, p_2, .., p_n$. **(Garwal 1999; Meerut 1994)**

15. Prove that under canonical transformation (q, p) to (Q, P),

$$[F, G]_{q,p} = [F, G]_{Q,P}$$

16. Prove that the Poisson bracket of two constants of motion is itself a constant of motion even when the constants depend on time explicitly. **(Garwal 1995, 91; Agra 81)**
17. What are fundamental Poisson brackets ? Deduce them.
18. Show that the fundamental Poisson brackets are invariant under canonical transformation. **(Agra 1998)**
19. What is phase space ? State and prove Liouville's theorem.
20. Derive Liouville's theorem for change of density of distribution with time and show that the density of the points is conserved. **(Kanpur 1999)**

Problems

[SET- I]

1. Evaluate the Poisson brackets :

   (a) (i) $[J_x, x]$, (ii) $[J_x, y]$, (iii) $[J_y, z]$

   (b) (i) $[J_x, p_x]$, (ii) $[J_x, p_z]$, (iii) $[J_z, p_x]$

   (c) (i) $[J_x, J_y]$. (ii) $[J_y, J_z]$

   Ans : (a) (i) 0, (ii) $-z$, (iii) -1; (b) (i) 0, (ii) $-p_y$, (iii) p_y; (c) (i) J_z, (ii) J_x .

2. If x_1, x_2, x_3 are the cartesian components of **r**, p_1, p_2, p_3 those of **p** and J_1, J_2, J_3 those of **J**, prove that

   $$\left[J_i, x_j\right] = -\sum_k e_{ijk}\, x_k, \left[J_i, p_j\right] = -\sum_k e_{ijk}\, p_k \text{ and } \left[J_i, J_j\right] = -\sum_k e_{ijk}\, J_k$$

   where e_{ijk} is the completely antisymmetric tensor :

   $e_{123} = e_{231} = e_{312} = 1$, $e_{132} = e_{213} = e_{321} = -1$, all other components of e_{ijk} vanish.

3. Prove that [(**a**·**p**), (**b**·**r**)] = **a**·**b**, where **a** and **b** are constant vectors. **(GATE 2004)**

4. Evaluate the Poisson's brackets : (a) [J, (**r** · **p**)] (b) [**p**, r^n]

   Ans : (a) 0 , (b) $n r^{n-2}$ **r**

5. Show that [**p**, (**a** · **r**)^2] = 2**a** (**a** · **r**), where **a** is a constant vector.

6. Prove that (a) [**a** · **J**, **b** · **J**] = (**a** × **b**) · **J** , where **a** and **b** are constant vectors.

   (b) [**f** · **J**, **g** · **J**] = (**g** × **f**) · **J** + $\sum\limits_{i,k} J_i J_k [f_i, g_k]$,

   where **f** = **f** (**r**, **p**) and **g** = **g** (**r**, **p**).

7. Show that [**f**, (**a**·**J**)] = **f** × **a** ,where **a** is a constant vector and **f** = **f** (**r**, **p**).

8. Show that the Poisson Bracket of any three dynamical functions $F(q_i, p_i)$, $G(q_i, p_i)$ and $H(q_i, p_i)$ satisfy the Jacobi's identity $[F, (G, H)] + [G, (H, F)] + [H, (F, G)] = 0$. **(Gorakpur 1996)**

9. Show that $[[[A, B], C], D] + [[[C, D], A], B] + [[[D, A], B], C] = 0$.

10. For a single particle show directly *i.e.*, by direct evaluation of the Poisson brackets that if u is a scatar function only of r^2, p^2 and **r** . **p**, then [u, **J**] = 0. **(Kanpur 1998, 92)**

[SET- II]

1. Show that (i) $[A_i, A_j] = -\sum\limits_k e_{ijk} A_k$, (ii) $[A_j, A_4] = 0$, where i, j and k take on values 1, 2 and 3 and

   $A_1 = \frac{1}{4}(x^2 + p_x^{\,2} - y^2 - p_y^{\,2})$, $A_2 = \frac{1}{2}(xy + p_x p_y)$, $A_3 = \frac{1}{2}(xp_y - yp_x)$, $A_4 = x^2 + y^2 + p_x^{\,2} + p_y^{\,2}$.

2. Evaluate the following :

   (a) $[J_i, \lambda_{jk}]$ (b) $[\lambda_{jk}, \lambda_{il}]$

   Ans : (a) $-\sum\limits_l e_{ijl}\lambda_{lk} - \sum\limits_l e_{ikl}\lambda_{lj}$ (b) $\delta_{ij} J_{lk} + \delta_{ik} J_{ij} + \delta_{jl} J_{ik} + \delta_{kl} J_{ij}$, where $J_{kl} = p_k x_l - p_l x_k$.

3. Evaluate $[[[A, B], C], D] + [[[B, C], D], A] + [[[C, D], A], B] + [[[D, A], B], C]$.

   Ans : $[[A, C], [B, D]]$

4. Prove that $[J_z, \phi] = 0$, where ϕ is an arbitrary function of the coordinates and momenta of a particle. Show also that $[J_z, \mathbf{f}] = \hat{\mathbf{n}} \times \mathbf{f}$, where $\mathbf{f}$ is a vector function of the coordinates and momenta of a particle and $\hat{\mathbf{n}}$ the unit vector along Z-axis.
[**Hint :** If the system as a whole is rotated about the Z-axis over an infinitesimal angle $\in$, the change $\delta\phi$ in any function of the coordinates and momenta is in first order in $\in$ and is given by

$$\delta\phi = \phi\,(x - \in y, y + \in x, z, p_x - \in p_y, p_y + \in p_x, p_z) - \phi\,(x, y, z, p_x, p_y, p_z)$$

$$= \in\left[-\frac{\partial\phi}{\partial x}y + \frac{\partial\phi}{\partial y}x - \frac{\partial\phi}{\partial p_x}p_y + \frac{\partial\phi}{\partial p_y}p_x\right] = \in[J_z, \phi].$$

When ϕ is a scalar, this change under rotation must vanish and thus $[J_z, \phi] = 0$. In case $\phi = f_x$ is the component of a vector function, its change under rotation is $\delta f_x = -\in f_y$, and thus $[J_z, f_x] = -f_y$ or $[J_z, \mathbf{f}] = \hat{\mathbf{n}} \times \mathbf{f}$.

5. A charged particle with charge q is moving in an inhomogeneous magnetic field of induction $\mathbf{B}$. Show that

$$[v_i, v_j] = -q/m^2 \sum_k e_{ijk}\, B_k$$

where v_i are the cartesian components of the velocity of the particle.

6. Prove that the value of any function $f(p(t), q(t))$ of the coordinates and momenta of a system at time t can be expressed in terms of the values of the p and q at $t = 0$ as given below :

$$f(p\,(t), q(t)) = f_0 + \frac{t}{1\,!}[H, f_0] + \frac{t^2}{2\,!}[H,[H, f_0]] + ...$$

where $f_0 = f(p(0), q(0))$ while $H = H(p(0),\ q(0))$ is the Hamiltonian.
Assume that the series converges. Apply this formula to evaluate $p(t)$ and $q(t)$ for (a) a particle moving in uniform field of force, and (b) a harmonic oscillator.

Ans : (a) $\mathbf{p}(t) = \mathbf{p} + \mathbf{F}t$, $\mathbf{r}(t) = \mathbf{r} + \frac{\mathbf{p}}{m}t + \frac{\mathbf{F}}{2m}t^2$ (b) $p\,(t) = p\cos\omega t - m\omega q \sin\omega t$, $q(t) = q\cos\omega t + \frac{p}{m\omega}\sin\omega t$.

Objective Type Questions

1. If the Poisson bracket of a function with the Hamiltonian vanishes,
(a) the function depends upon time.
(b) the function is a constant of motion.
(c) the function does not depend on time explicitly.
(d) the function is not the constant of motion.
Ans : (b), (c).

2. The correct relations for Poisson brackets are :
(a) $[q_k, q_l] = \delta_{kl}$ (b) $[q_k, q_l] = 0$ (c) $[q_k, p_k] = 0$ (d) $[q_k, p_k] = 1$.
Ans : (b), (d).

3. For Lagrange brackets ,

(a) $\{p_i, p_j\} = \delta_{ij}$ (b) $\{p_i, p_j\} = 0$ (c) $\{q_i, p_i\} = 0$ (d) $\{q_i, p_i\} = \delta_{ij}$.

Ans : (b), (d).

4. Poisson brackets for angular momentum components (J_x, J_y, J_z) satisfy the relations,

(a) $[J_x, p_x] = 0$ (b) $[J_x, p_z] = -p_y$ (c) $[J_y, J_z] = J_x$ (d) $[J_y, J_z] = -J_x$.

Ans : (a), (b), (c).

5. If p_k and q_k (k =1, 2, 3) represent the momentum and position coordinates respectively for a particle,
(a) the phase space is six dimensional.
(b) the configuration space is six dimensional.
(c) the phase space is three dimensional.
(d) the configuration space is three dimensional.
Ans : (a), (d).

6. The phase space refers to
(a) position coordinates
(b) momentum coordinates
(c) both position and momentum coordinates
(d) None of these **(Kanpur 2002)**
Ans. (c).

Short Answer Questions

1. Define Poisson bracket.
2. What are simple algebraic properties of Poisson's bracket ? **(Agra 2003)**
3. All functions whose Poisson Brackets with the Hamiltonian vanish will be constants of motion. Explain. **(Kanpur 2002)**
4. If [α, β] is the Poisson bracket, prove that

$$\frac{\partial}{\partial t}[\alpha,\beta] = \left[\frac{\partial\alpha}{\partial t},\beta\right] + \left[\alpha,\frac{\partial\beta}{\partial t}\right].$$ **(Kanpur 2001)**

5. Prove that $[F, G] = -[G, F]$.
6. Show that $[q_k, q_l] = 0$, $[q_k, p_l] = \delta_{kL}$.
7. What is Jacobi's identity.
8. Prove that $[CF, G] = C[F, G]$, C = constant.
9. What is phase space ?
10. Explain Liouville's theorem. **(Kanpur 2003)**
11. Fill in the blanks:
(i) If a function does not depend on time explicitly and is a constant of motion, its poisson bracket with the.....................vanishes.
(ii) $[p_k, p_l]$ = ...
Ans : (i) Hamiltonian, (ii) 0.

CHAPTER 8

Hamilton-Jacobi Theory and Transition to Quantum Mechanics

8.1. INTRODUCTION

In Chapter 6, we have discussed the method of solving mechanical problems by using canonical transformations. The method involves the transformation of old set of coordinates (q_k) to new set of coordinates (Q_k) which are all cyclic and hence all momenta are constants, provided the Hamiltonian is conserved. The equations of motion are then integrated to obtain the solution of the problem. In case, the Hamiltonian involves time, the method is not applied. An alternative approach is to seek a canonical transformation which leads to the new Hamiltonian $H' = 0$, so that the new coordinates and momenta, Q_k and P_k, are constants (because $\dot{P}_k = -\partial H'/\partial Q_k$ and $\dot{Q}_k = \partial H'/\partial P_k$. Using such a transformation, the equations of transformation relating to old and new variables are then exactly the required solution of the mechanical problem. This procedure is due to Jacobi which is a transformation as well as a method itself and applicable for the case, when Hamiltonian involves time.

8.2. THE HAMILTON-JACOBI EQUATION

If we make a canonical transformation from the old set of variables (q_k, p_k) to a new set of variables (Q_k, P_k), then the new equations of motion are,

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \quad \text{and} \quad \dot{Q}_k = \frac{\partial H'}{\partial P_k} \qquad ...(1)$$

Now, if we require that the transformed Hamiltonian H' is identically zero *i.e.*, $H' = 0$, then equations of motion (1) assume the form

$$\dot{P}_k = 0 \quad \text{and} \quad \dot{Q}_k = 0$$

or $$P_k = \text{constant} \quad \text{and} \quad Q_k = \text{constant} \qquad ...(2)$$

Thus the new coordinates and momenta are constants in time and they are cyclic.

The new Hamiltonian H' is related to the old Hamiltonian H by the relation

$$H' = H + \frac{\partial F}{\partial t}$$

which will be zero only when F satisfies the relation

$$H(q_k, p_k, t) + \frac{\partial F}{\partial t} = 0 \qquad ...(3)$$

where $H(q_k, p_k, t)$ is written for $H(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n, t)$.

For convenience, we take the generating function F as a function of the old coordinates q_k, the new constant momenta P_k and time t i.e., $F_2(q_k, P_k, t)$. Then

$$p_k = \frac{\partial F_2}{\partial q_k} \qquad \text{...(4)}$$

Therefore, $$H\left(q_k, \frac{\partial F_2}{\partial q_k}, t\right) + \frac{\partial F_2}{\partial t} = 0 \qquad \text{...(5)}$$

Let us see what is the physical meaning of the generating function $F_2(q_k, P_k, t)$. The total time derivative of F_2 is

$$\frac{\partial F_2}{\partial t} = \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \frac{\partial F_2}{\partial P_k}\dot{P}_k + \frac{\partial F_2}{\partial t}$$

Here, $\dot{P}_k = 0, \frac{\partial F_2}{\partial t} = -H$ from (5) and $\frac{\partial F_2}{\partial q_k} = p_k$ from (4).

Therefore, $$\frac{\partial F_2}{\partial t} = \sum_{k=1}^{n} p_k \dot{q}_k - H = L$$

or $$F_2 = \int L\, dt = S \qquad \text{...(6)}$$

where S is the familiar **action** of the system, known as the **Hamilton's principal function** in relation to the variational principle. Writing $F_2 = S$ in eq. (5), we get

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \qquad \text{...(7)}$$

This is known as the ***Hamilton-Jacobi equation*** which is a partial differential equation of first order in $(n + 1)$ variables $q_1, q_2, \ldots, q_n, t$.

Let the complete solution of equation eq.(7) be of the form

$$S = S(q_1, q_2, \ldots, q_n, \alpha_1, \alpha_2, \ldots, \alpha_n, t) \qquad \text{...(8)}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are n independent constants of integration. Here, we have omitted one arbitrary additive constant which has no importance in a generating function because only partial derivatives of the generating function appear in the transformation equations.

In eq. (8), the solution S is a function of n coordinates q_k, time t and n independent constants. We can take these n constants of integration as the new constant momenta i.e.,

$$P_k = \alpha_k \qquad \text{...(9)}$$

Now, the n transformation equations [eqs. (27) of chapter 6] are

$$p_k = \frac{\partial S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)}{\partial q_k} \qquad \text{...(10)}$$

These are n equations, which at $t = t_0$ (initially) give the n values of α_k in terms of the initial values of q_k and p_k. The other n transformation equations are

$$Q_k = \frac{\partial S}{\partial P_k} = \text{constant, say } \beta_k$$

or
$$\beta_k = \frac{\partial S(q_1,\ldots,q_n,\alpha_1,\ldots,\alpha_n,t)}{\partial \alpha_k} \qquad \ldots(11)$$

Similarly, one can calculate the constants β_k by using initial conditions i.e., at $t = t_0$, the known initial values of q_k, in eq. (11). Thus α_k and β_k constants are known and eq. (11) will give q_k in terms of α_k, β_k and t i.e.,

$$q_k = q_k(\alpha_1, \alpha_2,\ldots,\alpha_n, \beta_1, \beta_2,\ldots,\beta_n, t) \qquad \ldots(12)$$

After performing the differentiation in eq. (10), eq. (12) may be substituted for q_k to obtain momenta p_k. Thus p_k will be obtained as functions of constants α_k, β_k and time t i.e.,

$$p_k = p_k(\alpha_1, \alpha_2,\ldots,\alpha_n, \beta_1, \beta_2,\ldots,\beta_n, t) \qquad \ldots(13)$$

In this way we obtain the desired complete solution of the mechanical problem.

Thus we see that the Hamilton's principal function S is the generator of a canonical transformation to constant coordinates (β_k) and momenta (α_k). Also in solving the Hamilton-Jacobi equation, we obtain simultaneously a solution to the mechanical problem.

8.3. SOLUTION OF HARMONIC OSCILLATOR PROBLEM BY HAMILTON-JACOBI METHOD

Let us consider a one-dimensional harmonic oscillator. The force acting on the oscillator at a displacement q is

$$F = -kq$$

where k is force constant.

Potential energy, $\quad V = \int_0^q kq\,dq = \frac{1}{2}kq^2$

Kinetic energy, $\quad T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

Hamiltonian, $\quad H = T + V \qquad$ (conservative system)

or $\quad H = \frac{p^2}{2m} + \frac{1}{2}kq^2$

But $\quad p = \frac{\partial S}{\partial q}$, therefore

$$H = \frac{1}{2}\left[\frac{\partial S}{\partial q}\right]^2 + \frac{1}{2}kq^2 \qquad \ldots(14)$$

Hence the Hamilton-Jacobi equation corresponding to this Hamiltonian is

$$\frac{1}{2m}\left[\frac{\partial S}{\partial q}\right]^2 + \frac{1}{2}kq^2 + \frac{\partial S}{\partial t} = 0 \qquad \ldots(15)$$

As the explicit dependence of S on t is involved only in the last term of left hand side of eq.(15), a solution to this equation can be assumed in the form

$$S = S_1(q) + S_2(t) \qquad \ldots(16)$$

Thus $$\frac{1}{2m}\left[\frac{\partial S_1}{\partial q}\right]^2 + \tfrac{1}{2}kq^2 = -\frac{\partial S_2}{\partial t} \qquad ...(17)$$

Setting each side of eq. (17) equal to a constant, say α, we get

$$\frac{1}{2m}\left[\frac{\partial S_1}{\partial q}\right]^2 + \tfrac{1}{2}kq^2 = \alpha \quad \text{and} \quad -\frac{\partial S_2}{\partial t} = \alpha$$

So that $$\frac{\partial S_1}{\partial q} = \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)} \quad \text{and} \quad -\frac{\partial S_2}{\partial t} = \alpha$$

Integrating, we get

$$S_1 = \int \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)}\, dq + C_1 \quad \text{and} \quad S_2 = -\alpha t + C_2$$

Therefore, $$S = \int \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)}\, dq - \alpha t + C$$

where $C = (C_1 + C_2)$ the constant of integration. It is to be noted that C is an additive constant and will not affect the transformation, because to obtain the new position coordinate ($Q = \partial S/\partial P$ or $\beta = \partial S/\partial \alpha$) only partial derivative of S with respect to α (= P, new momentum) is required. This is why this additive constant C has no effect on transformation and is dropped. Thus

$$S = \int \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)}\, dq - \alpha t \qquad ...(18)^*$$

We designate the constant α as the new momentum P. The new constant coordinate ($Q = \beta$) is obtained by the transformation

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{\alpha - \frac{1}{2}kq^2}} - t = \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1 - \frac{kq^2}{2\alpha}}} - t$$

or $$\beta = \sqrt{\frac{m}{k}}\sin^{-1} q\sqrt{\frac{k}{2\alpha}} - t$$

Therefore, $$\sqrt{\frac{m}{k}}\sin^{-1} q\sqrt{\frac{k}{2\alpha}} = t + \beta \quad \text{or} \quad \sin^{-1} q\sqrt{\frac{k}{2\alpha}} = \sqrt{\frac{k}{m}}\,(t + \beta)$$

Writing $\omega = \sqrt{k/m}$, we obtain

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\omega\,(t + \beta) \qquad ...(19)$$

which is the *familliar solution of the harmonic oscillator*,

---

* In this expression for the Hamilton's principal function S, first part is the function of α and q and is denoted as $W(q, \alpha)$. This is called ***Hamilton's characteristic function***. Thus

$$S = W(q, \alpha) - \alpha t.$$

Now, $$p = \frac{\partial S}{\partial q} = \sqrt{2m\left(\alpha - \tfrac{1}{2}kq^2\right)} = \sqrt{2m\alpha - m^2\omega^2 q^2} \quad ...(20a)$$

Putting the value of q from (19), we get

$$p = \sqrt{2m\alpha(1-\sin^2\omega\,(t+\beta))} \quad \text{or} \quad p = \sqrt{2m\alpha}\cos\omega\,(t+\beta) \quad ...(20b)$$

The constants α and β are to be known from initial conditions. Suppose at $t = 0$, the particle is at rest i.e., $p_0 = 0$ and it is at the displacement $q = q_0$, from the equilibrium position. Then from eq. (20 a)

$$p_0 = 0 = \sqrt{2m\alpha - m^2\omega^2 q_0^2} \quad \text{or} \quad \alpha = \tfrac{1}{2}m\omega^2 q_0^2 = \tfrac{1}{2}kq_0^2 \quad ...(21)$$

Also $$H' = H + \partial S/\partial t = H - \alpha = 0 \qquad [\because \partial S/\partial t = -\alpha \text{ from } (17)]$$

This gives $H = \alpha$. But the system is conservative and hence $H = E$. Thus the ***new canonical momentum*** ($P = \alpha$) is identified as the ***total energy of the oscillator.***

Also from (21), $q_0 = \sqrt{2\alpha/m\omega^2}$ and hence the solution (19) takes the more familliar form

$$q = q_0 \sin\omega\,(t+\beta) \quad ...(22)$$

Also from (20 b) and (22) at $t = 0$, $\cos\omega\beta = 0$ and $\sin\omega\beta = 1$.

Therefore, $\omega\beta = \pi/2$ or $\beta = \pi/2\omega$

Thus the *new constant canonical coordinate*, measures the initial phase angle and in the present initial conditions the initial phase $\omega\beta = \pi/2$.

Therefore, eq. (22) is

$$q = q_0 \cos\omega t \quad ...(23)$$

In view of eq. (20 a), and then (20 b), Hamilton's principal function S from (18) is obtained to be

$$S = \int p\,dq - \alpha t = \int \sqrt{2m\alpha}\;\omega\cos\omega\,(t+\beta)\,q_0\cos\omega\,(t+\beta)\,dt - \alpha t$$

$$= 2\alpha\int \cos^2\omega\,(t+\beta)\,dt - \alpha t = 2\alpha\int[\cos^2\omega\,(t+\beta) - \tfrac{1}{2}]\,dt$$

The Lagrangian L is given by

$$L = \frac{p^2}{2m} - \tfrac{1}{2}kq^2 = \alpha\cos^2\omega\,(t+\beta) - \tfrac{1}{2}kq_0^2\sin^2\omega\,(t+\beta)$$

$$= \alpha\,[\cos^2\omega\,(t+\beta) - \sin^2\omega\,(t+\beta)] \qquad (\text{as } \alpha = \tfrac{1}{2}kq_0^2)$$

$$= 2\alpha\,\{\cos^2\omega\,(t+\beta) - \tfrac{1}{2}]$$

Therefore, $$S = \int L\,dt \quad ...(24)$$

Thus for harmonic oscillator we prove that the Hamilton's principal function is the time integral of Lagrangian. This is in agreement with the general relation, mentioned earlier.

8.4. HAMILTON-JACOBI EQUATION : *HAMILTON'S CHARACTERISTIC FUNCTION-CONSERVATIVE SYSTEMS*

In the last section, we were able to obtain the solution of Hamilton-Jacobi equation, because S could be separated into two parts : $S_1(q)$ and $S_2(t)$, where $S_1(q)$ involves the variable q only and $S_2(t)$ the variable t only. In this case, the Hamiltonian H was not involving time explicitly. However, such a separartion of variables is always possible, if the Hamiltonian H does not involve time t explicitly. This method is often called the ***method of separation of variables.***

If the Hamiltonian H is not an explicit function of time t, then the Hamilton-Jacobi equation (7) for S is obtained to be

$$H\left[q_k, \frac{\partial S}{\partial q_k}\right] + \frac{\partial S}{\partial t} = 0 \qquad ...(25)$$

Since the first term involves the dependence of S on q_k and the second term on t, we can assume the solution S in the form

$$S(q_k, \alpha_k, t) = W(q_k, \alpha_k) - \alpha_1 t \qquad ...(26)$$

Therefore, $$\frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k} \text{ and } \frac{\partial S}{\partial t} = -\alpha_1$$

and hence the Hamilton- Jacobi equation (25) assumes the form

$$H\left[q_k, \frac{\partial W}{\partial q_k}\right] = \alpha_1$$

or $$H\left(q_1, q_2, \ldots, q_n, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1 \qquad ...(27)$$

This is the ***time-independent Hamilton-Jacobi equation***. The constant of integration α_1 is thus equal to the constant value of H. For conservative system, $H = \alpha_1 = E$, where E represents the total energy of the system. Thus *for conservative system, Hamilton-Jacobi equation* is written as

$$H[q_k, \partial W/\partial q_k] = E \qquad ...(28)$$

The eq. (27) can also be obtained directly by taking W as the generating function $W(q_k, P_k)$ independent of time. The equations of transformations are

$$p_k = \partial W/\partial q_k \text{ and } Q_k = \partial W/\partial P_k \qquad ...(29)$$

Now if the new momenta P_k are all constants of motion α_k, where α_1 in particular is the constant of motion H, then $Q_k = \partial W/\partial \alpha_k$. The condition to determine W is that

$$H(q_k, p_k) = \alpha_1$$

Using $p_k = \partial W/\partial q_k$, we ontain

$$H[q_k, \partial W/\partial q_k] = \alpha_1$$

which is identical to eq. (27).

Also $$H' = H + \frac{\partial W}{\partial t}$$

But $W(q_k, P_k)$ does not involve time and hence

$$H' = H = \alpha_1 \; (= E, \text{ for conservative system}) \qquad ...(30)$$

The function W is known as ***Hamilton's characteristic function***. It generates a canonical transformation where all the new coordinates Q_k are cyclic because $H' = \alpha_1$, depending only on one of the new momenta $P_1 = \alpha_1$ and does not contain any Q_k. Now the canonical equations for new variables are

$$\dot{P}_k = -\,\partial H'/\partial Q_k = 0 \text{ or } P_k = \alpha_k, \text{ constant} \qquad ...(31)$$

and $$\dot{Q}_k = \partial H'/\partial \alpha_k = 1 \text{ for } k = 1 \text{ and } \dot{Q}_k = 0 \text{ for } k \neq 1.$$

Hence the solutions are

$$Q_1 = t + \beta_1 = \partial W/\partial \alpha_1 \quad \text{for } k = 1 \qquad ...(32a)$$

and $$Q_k = \beta_k = \partial W/\partial \alpha_k \qquad \text{for } k \neq 1 \qquad ...(32b)$$

Thus out of all the new coordinates Q_k, Q_1 is the only coordinate which is not a constant of motion. Here we observe the conjugate relationship between the time as the new coordinate and Hamiltonian (energy) as the conjugate momentum.

The Hamilton-Jacobi equation (27) determines the dependence of the Hamilton's characteristic function W on the old coordinates q_k. A complete solution of this equation will have n constants of integration and as explained earlier and in the discussion of harmonic oscillator problem, one of them is just an additive constant. Rest of the $n-1$ independent constants $\alpha_2, \alpha_3, ..., \alpha_n$ plus α_1 may then be taken as new constant momenta. First half of the equations (29), when evaluated with the initial condition $t = 0$, relates the n constants α_k to the initial values of q_k and p_k. Finally one can solve eq (31) and (32) to obtain q_k as a function of α_k, β_k and t and thus the solution to the problem is completed.*

Physical significance of the Hamilton's characteristic function W: The function W has a physical significance similar to the Hamilton's principal function S. Since $W(q_k, P_k)$ does not involve time t explicitly, its total time derivative is

$$\frac{dW}{dt} = \sum_{k=1}^{n} \frac{\partial W}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial W}{\partial P_k} \dot{P}_k$$

Since $P_k = \alpha_k$, constants, $\dot{P}_k = 0$ and therefore

$$\frac{dW}{dt} = \sum_{k=1}^{n} p_k \dot{q}_k$$

or $$W = \int \sum_k p_k \dot{q}_k \, dt = \int \sum_k p_k \, dq_k \qquad ...(33)$$

---

* Sometimes it is useful to have a set of n independent functions of α_k as the transformed momenta *i.e.*,

$$p_k = \gamma_k(\alpha_1, \alpha_2, ..., \alpha_n)$$

Now, $W = W(q_k, \gamma_k)$ and the Hamiltonian H or H' will, in general, depend on more than one of the γ_k's. The equations of motion for Q_k are

$$\dot{Q}_k = \partial H'/\partial \gamma_k = f_k$$

where f_k's are the functions of γ_k.

Therefore, $$Q_k = f_k t + \beta_k$$

Thus now all the new coordinates are linear functions of time.

which is the abbreviated action;

and $$S=\int L\,dt=\int\sum_k[p_k\dot{q}_k-H]\,dt=W-\int H\,dt$$

When H does not involve time t explicitly $\int H\,dt=\alpha_1 t$. So that

$$S=W-\alpha_1 t \text{ or } S(q_k,P_k,t)=W(q_k,P_k)-\alpha_1 t$$

or $$S(q_k,t)=W(q_k)-Et \qquad ...(34)$$

where $P_k=\alpha_k$ are constants and $\alpha_1=E$, total energy.

It is to be remarked that when the Hamiltonian does not involve time explicitly, one can solve a mechanical problem by using either Hamilton's principal function or Hamilton's characteristic function. The two functions are related by the above relation.

8.5. KEPLER'S PROBLEM : SOLUTION BY HAMILTON- JACOBI METHOD

Let a particle of mass m be moving in an inverse square central force field $[V(r)=-K/r]$. Denoting the conjugate momenta corresponding to r and θ coordinates by p_r and p_θ respectively, the Hamiltonian of the system can be written as

$$H=\frac{1}{2m}\left[p_r^2+\frac{p_\theta^2}{r^2}\right]-\frac{K}{r} \qquad ...(35)$$

As the system is consevative, the Hamiltonian will represent the total energy of the system i.e.,

$$H=\alpha_1=E \text{ (say)} \qquad ...(36)$$

Therefore, $$\frac{1}{2m}\left[p_r^2+\frac{p_\theta^2}{r^2}\right]-\frac{K}{r}=E \qquad ...(37)$$

Here, we will take $W=W(r,\theta)$ as the generating function. So that the equations of transformation relating to old momenta p_k $(k=r,\theta)$ are

$$p_r=\partial W/\partial r \text{ and } p_\theta=\partial W/\partial\theta \qquad [\because p_k=\partial W/\partial q_k] \qquad ...(38)$$

Thus , eq. (37) takes the form

$$\left[\frac{\partial W}{\partial r}\right]^2+\frac{1}{r^2}\left[\frac{\partial W}{\partial\theta}\right]^2-2m\frac{K}{r}=2mE \qquad ...(39)$$

Applying the method of separation of variables, we can write

$$W=W_1(r)+W_2(\theta) \qquad ...(40)$$

Therefore, $\frac{\partial W}{\partial r}=\frac{\partial W_1}{\partial r}$, $\frac{\partial W}{\partial\theta}=\frac{\partial W_2}{\partial\theta}$ and hence eq. (39) is

$$\left[\frac{\partial W_1}{\partial r}\right]^2+\frac{1}{r^2}\left[\frac{\partial W_2}{\partial\theta}\right]^2-\frac{2mK}{r}=2mE \qquad ...(41)$$

In a central force motion, the angular momentum is conserved. Therefore,

$$p_\theta=\text{constant, say } \alpha_2=J, \text{ angular momentum} \qquad ...(42a)$$

Thus $$p_\theta=J=\alpha_2=\partial W_2/\partial\theta \qquad ...(42b)$$

Integrating we get

$$W_2 = \alpha_2 \theta + C_1 \text{ (a constant)} \qquad \text{...(43)}$$

Substituting for $\frac{\partial W_2}{\partial \theta} = \alpha_2$ in (41), we get

$$\left[\frac{\partial W_1}{\partial r}\right]^2 + \frac{\alpha_2^2}{r^2} - \frac{2mK}{r} = 2mE \quad \text{or} \quad \frac{\partial W_1}{\partial r} = \sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}$$

or

$$W_1 = \int \sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}\, dr + C_2 \text{ (a constant)} \qquad \text{...(44)}$$

Thus the generating function W is

$$W = W_1(r) + W_2(\theta)$$

or

$$W = \int \sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}\, dr + \alpha_2 \theta + C \qquad \text{...(45)}$$

where $C = C_1 + C_2$ is an additive constant.

Now, we take the new momenta $P_k = \alpha_k (k = 1, 2)$ and hence the equations of motion in the new coordinates Q_k are

$$\dot{Q}_k = \partial H'/\partial P_k \quad \text{or} \quad \dot{Q}_k = \partial \alpha_1/\partial \alpha_k \qquad [\because H = H' = \alpha_1]$$

Hence for $k = 1$, $\quad \dot{Q}_1 = 1 \;$ or $\; Q_1 = t + \beta_1$ ...(46a)

and for $k = 2$, $\quad \dot{Q}_2 = 0 \;$ or $\; Q_2 = \beta_2$ ...(46b)

The transformation equation $Q_k = \partial W/\partial P_k$ gives

$$Q_1 = \frac{\partial W}{\partial \alpha_1} \quad \text{and} \quad Q_2 = \frac{\partial W}{\partial \alpha_2} \qquad \text{...(47)}$$

From eqs. (46) and (47), we get

$$\frac{\partial W}{\partial \alpha_1} = t + \beta_1, \quad \text{and} \quad \frac{\partial W}{\partial \alpha_2} = \beta_2 \qquad \text{...(48)}$$

Using eq. (45) ,we get (remembering $\alpha_1 = E$)

$$\frac{\partial W}{\partial \alpha_1} = \int \frac{m\, dr}{\sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}}$$

and

$$\frac{\partial W}{\partial \alpha_2} = \theta - \int \frac{\alpha_2\, dr}{r^2 \sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}} \qquad \text{...(49)}$$

Hence

$$\int \frac{m\, dr}{\sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}} = t + \beta_1 \qquad \text{...(50)}$$

and
$$\int \frac{\alpha_2\, dr}{r^2 \sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_2^2}{r^2}}} = \theta - \beta_2 \qquad ...(51)$$

Eq. (50), when integrated, gives the position as a function of time *i.e.*, $r = r(t)$. Using $r = 1/u$, $dr = -du/u^2$, we obtain eq. (51) as

$$\theta - \beta_2 = -\int \frac{du}{\sqrt{\frac{2mE}{\alpha_2^2} + \frac{2mK}{\alpha_2^2}u - u^2}} = \cos^{-1} \frac{\alpha_2^2 u - mK}{\sqrt{2mE\,\alpha_2^2 + m^2K^2}}$$

Thus
$$\frac{\alpha_2^2}{r} - mK = \sqrt{2mE\,\alpha_2^2 + m^2K^2}\,\cos(\theta - \beta_2)$$

where we have assumed that β_2 includes the constant of integration.

Therefore,
$$\frac{\alpha_2^2/mK}{r} = 1 + \sqrt{1 + \frac{2E\alpha_2^2}{mK^2}}\,\cos(\theta - \beta_2) \qquad ...(52)$$

Since $\alpha_2 = J$ from eq. (42), eq. (52) is the eq. (34) of a conic section in Chapter 4. Writing $\frac{\alpha_2^2}{mK} = l$ and $\sqrt{1 + \frac{2E\alpha_2^2}{mK}} = e$, we get the equation of the path

$$l/r = 1 + e\cos(\theta - \beta_2) \qquad ...(53)$$

which is ellipse for $e < 1$ or $E < 0$, parabola for $e = 1$ or $E = 0$ and hyperbola for $e > 1$ or $E > 0$.

Ex. 1. *Freely falling body*: *Apply the Hamilton-Jacobi method to determine the motion of a body falling vertically in a uniform gravitational field.* **(Garwal 1992)**

Solution : Let us take Z-axis along vertical direction. If the mass of the body be m, then the kinetic and potential energies at a height of z are given by

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m} \quad \text{and} \quad V = mgz$$

So that
$$H = \frac{p^2}{2m} + mgz = E \;\text{(total energy)} \qquad ...(i)$$

Here $q = z$ and $p = \partial S/\partial z$; therefore

$$H = \frac{1}{2m}\left[\frac{\partial S}{\partial z}\right]^2 + mgz \qquad ...(ii)$$

Hamilton-Jacobi equation is

$$H + \frac{\partial S}{\partial t} = 0$$

or
$$\frac{1}{2m}\left[\frac{\partial S}{\partial z}\right]^2 + mgz + \frac{\partial S}{\partial t} = 0 \qquad ...(iii)$$

The solution of this equation can be written as

$$S(z, \alpha, t) = W(z, \alpha) - \alpha t \qquad ...(iv)$$

whence $\quad \dfrac{\partial S}{\partial z} = \dfrac{\partial W}{\partial z}$ and $\dfrac{\partial S}{\partial t} = -\alpha$

Substituting these values in (iii), we obtain

$$\frac{1}{2m}\left[\frac{\partial W}{\partial z}\right]^2 + mgz - \alpha = 0$$

or $$\frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz)} \text{ or } W = \int \sqrt{2m(\alpha - mgz)}\,dz + C \qquad ...(v)$$

where C is a constant of integration.

Therefore, the Hamilton's principal function S is

$$S = \int \sqrt{2m(\alpha - mgz)}\,dz + C - \alpha t$$

This gives $$\frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{2}\int \frac{dz}{\sqrt{\alpha - mgz}} - t = \beta$$

or $$\beta + t = \frac{\sqrt{2m}}{2}\frac{2\sqrt{\alpha - mgz}}{(-mg)} \text{ or } \beta + t = -\sqrt{\frac{2}{m}}\frac{\sqrt{\alpha - mgz}}{g}$$

or $$\alpha - mgz = \frac{mg^2}{2}(\beta + t)^2 \text{ or } z = -\frac{g}{2}(\beta + t)^2 + \frac{\alpha}{mg} \qquad ...(vi)$$

If at $t = 0$, $z = z_0$ and $p = 0$, then from (v)

$$p = \partial W/\partial z = \sqrt{2m(\alpha - mgz_0)} = 0 \text{ or } \alpha = mgz_0 \qquad ...(vii)$$

Substituting for α in eq. (vi), we have

$$z = -\frac{g}{2}(\beta + t)^2 + z_0 \qquad ...(viii)$$

Since at $t = 0$, $z = z_0$, therefore, $-\dfrac{g}{2}\beta^2 = 0$ or $\beta = 0$.

Hence the equation of the freely falling body is

$$z = -\frac{1}{2}gt^2 + z_0 \qquad ...(ix)$$

8.6. ACTION AND ANGLE VARIABLES

Periodic motion is of special interest in many physical systems. Sometimes we are interested only to know the frequencies of the motion but not in the details of the orbit. We shall now develop a very interesting and powerful method to handle the problem of periodic motion by extending the Hamilton-Jacobi method. In this method, we make use of properly defined constants J_k, which form a set of n independent functions of the new momenta α_k, occurring in the solution of the Hamilton-Jacobi equation. We shall call these constants J_k as the action variables.

In order to introduce and illustrate the ideas of the action-angle variables, we consider a conservative periodic system with one degree of freedom. The Hamiltonian for a conservative system is constant and is given by

$$H = H(q, p) = \alpha_1 \quad \text{...(54a)}$$

If we solve this equation for the momentum p, we get

$$p = p\,(q, \alpha_1) \quad \text{...(54b)}$$

This equation gives the orbit, traced out by the representative point in the two dimensional (p-q) phase space, provided the Hamiltonian has the constant value α_1. A simple and important example of periodic motion is a one-dimensional harmonic oscillator, where p, given by eq. (20 a), is

$$p = \sqrt{2m\alpha - m\omega q^2}\,, \text{ where } \alpha_1 = \alpha = E \quad \text{...(55)}$$

or

$$p^2 = 2m\alpha - m\omega^2 q^2 \quad \text{or} \quad \frac{q^2}{(\omega^2/2\alpha)} + \frac{p^2}{2m\alpha} = 1 \quad \text{...(56)}$$

Hence for a harmonic oscillator the representative point in the p-q plane gives an ellipse [Fig. 8.1].

Now, we define a new variable J, given by

$$J = \oint p\, dq \quad \text{...(57)}$$

where the integration is taken over a complete period. This integral J is called ***phase integral*** or ***action variable***. This name comes from the similarity of eq. (57) to the abbreviated action $W = \int \sum_k p_k dq_k$ of eq. (33). Further one may see that J has the dimensions of *angular momentum*.

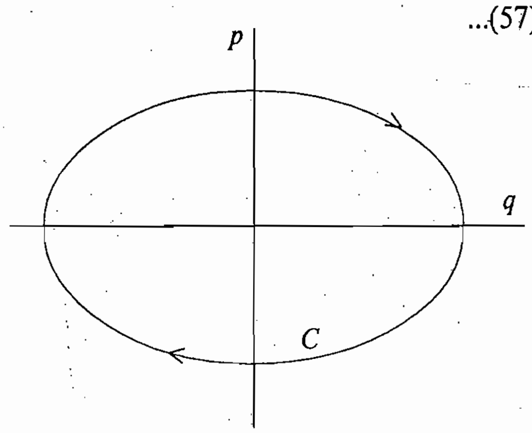


Fig. 8.1 : Phase path for one-dimensional harmonic oscillator

We observe from eqs. (54) and (57) that J is a function of α_1, because in the integration (57) q coordinate is integrated out. Therefore

$$J = J(\alpha_1) = J(H) \quad \text{...(58a)}$$

and vice-versa :

$$\alpha_1 = H = H(J) \quad \text{...(58b)}$$

Hence, the Hamilton's characteristic function W can be written as

$$W = W(q, J) \qquad [\text{as } W = W(q, P) = W(q, \alpha) = W(q, J)] \quad \text{...(59)}$$

The generalized coordinate conjugate to J is called the ***angle variable*** w and is defined by the transformation equation :

$$w = \partial W / \partial J \quad \text{...(60)}$$

Also the other transformation equation is

$$p = \partial W / \partial q \quad \text{...(61)}$$

Hence the equation of motion of w is

$$\dot{w} = \partial H(J)/\partial J = \nu\,(J) \quad \text{...(62)}$$

where ν is a constant function of the action variable J only.

The solution of eq. (62) is

$$w = \nu t + \beta \quad \text{...(63)}$$

Thus the angle variable is a linear function of time. We shall identify ν as the ***frequency*** and hence w has the dimensions of the angle and this is why it is designated as **angle** variable. Here the angle variable w is conjugate to the action variable J similar to angle θ, being conjugate to angular momentum.

Now, let us consider the change in w as q completes a cycle i.e.,

$$\Delta w = \oint \frac{\partial w}{\partial q}\, dq \qquad \text{...(64)}$$

But from (60), $w = \partial W/\partial J$, therefore

$$\Delta w = \oint \frac{\partial^2 W}{\partial q\, \partial J}\, dq = \frac{d}{dJ}\oint \frac{\partial W}{\partial q}\, dq$$

where the derivative with respect to J has been taken outside because J is constant independent of q. Now from (61) $p = \partial W/\partial q$, hence

$$\Delta w = \frac{d}{dJ}\oint p\, dq = 1 \qquad \text{...(65)}$$

by using the definition of J [eq. (57)]. We see from eq. (65) that w changes by 1 when q goes through a period. If T be the period of a complete cycle of q, then from eq. (63)

$$\Delta w = \nu \Delta t \quad \text{or} \quad \Delta w = \nu T \qquad \text{...(66)}$$

Hence from (65) and (66), we obtain

$$\nu T = 1 \quad \text{or} \quad \nu = 1/T \qquad \text{...(67)}$$

Thus the constant ν is identified as the reciprocal of the period and is, therefore, gives the ***frequency*** of motion in q. We, thus, find that *the application of action-angle variables provides an elegant procedure to determine the frequency of periodic motion without going into the details of its solution.*

8.7. PROBLEM OF HARMONIC OSCILLATOR USING ACTION-ANGLE VARIABLES (*DEDUCTION OF FREQUENCY OF MOTION*)

For example, let us apply action-angle variables to find the frequency of the harmonic oscillator. The constant action variable J for it in view of eqs. (55) and (57) is given by

$$J = \oint p\, dq = \oint \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq \qquad \text{...(68)}$$

where $\alpha = H = E$ is the total energy and $\omega = \sqrt{k/m}$.

Let $\quad q = \sqrt{\dfrac{2\alpha}{m\omega^2}}\sin\theta$. So that

$$J = \frac{2\alpha}{\omega}\int_0^{2\pi} \cos^2\theta\, d\theta = \frac{\alpha}{\omega}\int_0^{2\pi}(1 + \cos 2\theta)\, d\theta$$

or

$$J = \frac{2\pi\alpha}{\omega} = \frac{2\pi E}{\omega} \qquad \text{...(69)}$$

whence

$$\alpha = H = \omega J/2\pi \qquad \text{...(70)}$$

Use of eq. (62) gives the *frequency of harmonic oscillator* i.e.,

$$\nu = \frac{\partial H}{\partial J} = \frac{\omega}{2\pi} \quad \text{or} \quad \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{...(71)}$$

This is the familiar formula for the frequency of a simple harmonic oscillator.

Also $$w = \frac{\omega}{2\pi} t + \beta \text{ or } 2\pi w = \omega(t + \beta') \qquad \text{[from (63)]}$$

If the constant β' is defined to be β of eq. (19) then the expressions of q and p in terms of the action-angle variables are

$$q = \sqrt{\frac{J}{\pi m \omega}} \sin 2\pi w \text{ and } p = \sqrt{\frac{mJ\omega}{\pi}} \cos 2\pi w \qquad ...(72)$$

These transformation equations relate the canonical variables (q, p) to the new canonical variables (w, J).

8.8. ACTION-ANGLE VARIABLES IN GENERAL CASE

We may introduce action-angle variables to discuss the motion of a system with many degrees of freedom, provided that the Hamilton-Jacobi equation is completely separable in coordinate variables. We consider conservative system in which the Hamiltonian does not involve time explicitly. The Hamilton-Jacobi equation in such a case is given by [eq. (27)] :

$$H\left(q_1, q_2, \ldots, q_n; \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1 \qquad ...(73)$$

The variables q_k occurring in these equations are separable, if a solution of the form

$$W = W(q_1, q_2, \ldots q_n; \alpha_1, \alpha_2, \ldots, \alpha_n) = \sum_k W_k \, (q_k; \alpha_1, \alpha_2, \ldots\ldots, \alpha_n) \qquad ...(74)$$

splits the equation into n equations :

$$H_k \, (q_k; \partial W_k / \partial q_k; \alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_k \qquad ...(75)$$

Each of the equations (75) involves only one of the coordinates q_k and the corresponding partial derivative of W_k with respect to q_k.

The equation of canonical transformation has the form

$$p_k = \frac{\partial W_k(q_k; \alpha_1, \alpha_2, \ldots, \alpha_n)}{\partial q_k} \qquad ...(76)$$

Thus, it gives

$$p_k = p_k(q_k; \alpha_1, \alpha_2, \ldots, \alpha_n) \qquad ...(77)$$

For one degree of freedom, this equation assumes the form (54). In fact, eq. (77) represents the orbit equation of the projection of the representative point of the system on the (q_k, p_k) plane in the phase space. Now, action-angle variables for the system can be defined, if the orbit equations for all the (p_k, q_k) pairs describe either closed orbits or periodic functions of q_k.

Similar to eq. (57), the *action variables* J_k are defined as

$$J_k = \oint p_k dq_k \qquad ...(78)$$

where the integral is to be carried out over a complete period.

In case q_k is a cyclic coordinate, its conjugate momentum p_k is constant. Also, if q_k is angle coordinate as in rotational type of periodic motion, then integral for action variable is to be integrated from 0 to 2π i.e.,

$$J_k = p_k \int_0^{2\pi} dq_k = 2\pi p_k \qquad ...(79)$$

$$=2\int_{r_{min}}^{r_{max}} \frac{2m\alpha_1 r + 2mK}{\sqrt{2m\alpha_1 r^2 + 2mKr - \alpha_2^2 r^2}} dr - 2\alpha_2^2 \int_{r_{min}}^{r_{max}} \frac{dr}{r\sqrt{2m\alpha_1 r^2 + 2mKr - \alpha_2^2 r^2}}$$

or $$J_r = \frac{2\pi mK}{\sqrt{(-2m\alpha_1)}} - 2\pi\alpha_2 \qquad ...(viii)$$

Therefore from (*iv*) and (*ix*), we obtain

$$J_\theta + J_r = \frac{2\pi mK}{\sqrt{(-2m\alpha_1)}} \qquad ...(ix)$$

But $H = H' = \alpha_1 = E$, therefore

$$H' = E = \alpha_1 = -\frac{2\pi^2 mK^2}{(J_\theta + J_r)^2} \qquad ...(x)$$

Now, the frequencies ν_θ and ν_r are given by

$$\nu_\theta = \frac{\partial H'}{\partial J_\theta} = \frac{\partial}{\partial J_\theta}\left[-\frac{2\pi^2 mK^2}{(J_\theta + J_r)^2}\right] = \frac{4\pi^2 mK^2}{(J_\theta + J_r)^3} \qquad ...(xi)$$

and $$\nu_r = \frac{\partial H'}{\partial J_r} = \frac{\partial}{\partial J_r}\left[-\frac{2\pi^2 mK^2}{(J_\theta + J_r)^2}\right] = \frac{4\pi^2 mK^2}{(J_\theta + J_r)^3} \qquad ...(xii)$$

Thus $$\nu_\theta = \nu_r = \frac{4\pi^2 mK^2}{(J_\theta + J_r)^3} \qquad ...(xiii)$$

Thus the two frequencies are equal and the ***motion of the system is said to be degenerated.***
From (x) and (xiii), we obtain the period of orbit, given by

$$\tau = \frac{1}{\nu} = \frac{(J_\theta + J_r)^3}{4\pi^2 mK^2} = \frac{1}{4\pi^2 mK^2}\left[-\frac{2\pi^2 mK^2}{E}\right]^{3/2} = \pi K\sqrt{-\frac{m}{2E^3}} \qquad ...(ivx)$$

This formula agrees with the Kepler's third law, keeping in view that the semi-major axis a is equal to $-K/2E$.

Ex. 2. *Quantized Energy Levels of Hydrogen Atom* : *In an atom, an electron of charge – e is moving around a nucleus of charge Ze in a central force field, given by*

$$\mathbf{F} = -\frac{Ze^2}{r^2}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ *is a unit vector along* $\mathbf{r}$ *and Z is the atomic number. If according to the postulate of Bohr-Sommerfield's quantum theory, the action variables are integral multiples of Planck's constant h i.e.,*

$$J_r = \oint p_r\, dr = n_1 h \text{ and } J_\theta = \oint p_\theta\, d\theta = n_2 h$$

then prove that there will be a discrete set of energy levels, given by

$$E = -2\pi^2 mZ^2 e^4 / n^2 h^2$$

where $n = n_1 + n_2 = 1, 2, \ldots,$ *the total quantum number.*

$$H=\frac{1}{2m}\left[p_r^2+\frac{p_\theta^2}{r^2}\right]-\frac{K}{r}$$

termine the frequency by the method of action-angle variables and discuss degeneracy. Show that the of orbit is given by **(Rohilkhand 1999)**

$$\tau=\pi K\sqrt{-m/2E^3}\ .$$

lution : The Kepler's problem has been solved in Art. 8.5 by the Hamilton-Jacobi method. The action es J_k are given by

$$J_k=\oint p_k dq_k \qquad ...(i)$$

n case of Kepler's problem, the action variables are given by

$$J_\theta=\oint p_\theta\, d\theta \qquad ...(ii)$$

$$J_r=\oint p_r dr \qquad ...(iii)$$

From eq. (42 b) $p_\theta=\partial W/\partial\theta=\alpha_2$ and therefore

$$J_\theta=\oint\frac{\partial W}{\partial\theta}d\theta=\oint\alpha_2 d\theta=\int_0^{2\pi}\alpha_2 d\theta=2\pi\alpha_2 \qquad ...(iv)$$

Also from eq. (45) $\quad \dfrac{\partial W}{\partial r}=\sqrt{2m\alpha_1+\dfrac{2mK}{r}-\dfrac{\alpha_2^2}{r^2}}$

re we have replaced $E=\alpha_1$.

Therefore, $\quad J_r=\oint p_r\, dr=\oint\dfrac{\partial W}{\partial r}dr=\oint\sqrt{2m\alpha_1+\dfrac{2mK}{r}-\dfrac{\alpha_2^2}{r^2}}\,dr \qquad ...(v)$

The motion is bound and in elliptical path for negative value of the total energy E. Further the limits of on are given by r_{min} and r_{max} values of r. These values are determined by the zero of the quadratic equation in 51) *i.e.*,

$$\sqrt{2m\alpha_1+\frac{2mK}{r}-\frac{\alpha_2^2}{r^2}}=0 \quad \text{or,} \quad 2m\alpha_1 r^2+2mKr-\alpha_2^2=0 \qquad ...(vi)$$

Therefore, $\quad r=-\dfrac{K}{2\alpha_1}\pm\dfrac{1}{2\alpha_1}\sqrt{K^2+\dfrac{2\alpha_1\alpha_2^2}{m}}=\dfrac{K}{2\alpha_1}\left[-1\pm\sqrt{1+\dfrac{2\alpha_1\alpha_2^2}{mK^2}}\right]$

i.e., $\quad r_{min}=\dfrac{K}{2\alpha_1}\left[-1-\sqrt{1+\dfrac{2\alpha_1\alpha_2^2}{mK^2}}\right]$ and $r_{max}=\dfrac{K}{2\alpha_1}\left[-1+\sqrt{1+\dfrac{2\alpha_1\alpha_2^2}{mK^2}}\right] \qquad ...(vii)$

In a complete cycle of the coordinate r, it varies from r_{min} to r_{max} and back to r_{min}. Now,

$$J_r=2\int_{r_{min}}^{r_{max}}\sqrt{2m\alpha_1+\frac{2mK}{r}-\frac{\alpha_2^2}{r^2}}\,dr=2\int_{r_{min}}^{r_{max}}\frac{2m\alpha_1+\dfrac{2mK}{r}-\dfrac{\alpha_2^2}{r^2}}{\sqrt{2m\alpha_1+\dfrac{2mK}{r}-\dfrac{\alpha_2^2}{r^2}}}\,dr$$

or

Therefore fr

But $H=H'$

Now, the fr

and

Thus

Thus the t

From (x) a

This form

– *K/2E.*

Ex. 2. ***Qu***

around a nucl

where $\hat{\mathbf{r}}=\mathbf{r}$.

Sommerfield

then prove th

where $n=n_1$

Solution : Here $V(r) = -K/r = -Ze^2/r$ i.e., $K = Ze^2$

From eq. (x) of Ex. 1 (given above), total energy E is given by

$$E = -\frac{2\pi^2 mK^2}{(J_\theta + J_r)^2} = -\frac{2\pi^2 mZ^2e^4}{(J_\theta + J_r)^2} \quad ...(i)$$

But $J_r = n_1 h$ and $J_\theta = n_2 h$ and therefore

$$J_\theta + J_r = (n_1 + n_2)\, h \text{ or } J = J_\theta + J_r = nh \quad ...(ii)$$

where $n_1 + n_2 = n$ with $n = 1, 2, ...$.

Substituting the value of $J_\theta + J_r$ from (ii) in (i), we have

$$E = -2\pi^2 mZ^2e^4/n^2h^2 \quad ...(iii)$$

This relation gives the ***quantized energy levels*** for hydrogen atom. The integer n is known as ***principal quantum number***.

8.9. HAMILTON-JACOBI EQUATION – GEOMETRICAL OPTICS AND WAVE MECHANICS (*TRANSITION FROM CLASSICAL TO QUANTUM MECHANICS*)

Let us consider for simplicity a single particle system in which the forces are conservative and the Hamiltonian is a constant of motion, equal to the total energy E of the system. In general, the Hamilton-Jacobi equation [eq. (7)] is

$$H\left[q_k, \frac{\partial S}{\partial q_k}, t\right] + \frac{\partial S}{\partial t} = 0 \quad ...(91)$$

where the Hamilton's principal function S is related to the Hamilton's characteristic function W by

$$S(q_k, t) = W(q_k) - Et \quad ...(92)$$

With $\partial S/\partial q_k = \partial W/\partial q_k = p_k$, the time independent Hamilton-Jacobi equation [eq. (27)] is

$$H\left[q_k, \frac{\partial S}{\partial q_k}\right] = E \qquad \left(\because \frac{\partial S}{\partial t} = -E\right) \quad ...(93)$$

If we use Cartesian coordinate system $q_k = x, y, z$, then

$$p_x = \frac{\partial W}{\partial x}, \quad p_y = \frac{\partial W}{\partial y}, \quad p_z = \frac{\partial W}{\partial z}$$

So that
$$\mathbf{p} = \hat{\mathbf{i}}\frac{\partial W}{\partial x} + \hat{\mathbf{j}}\frac{\partial W}{\partial y} + \hat{\mathbf{k}}\frac{\partial W}{\partial z} = \nabla W \quad ...(94)$$

Suppose the particle is having mass m and moving in a potential V, then

$$H = \frac{p^2}{2m} + V = \frac{|\nabla W|^2}{2m} + V \quad ...(95)$$

and the Hamilton-Jacobi equation (93) is

$$\frac{|\nabla W|^2}{2m} + V = E \quad ...(96)$$

So that
$$|\nabla W| = \sqrt{2m(E - V)} \quad ...(97)$$

where $|\nabla W|^2 = (\nabla W)^2 = \left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 + \left(\frac{\partial W}{\partial z}\right)^2$.

In general, the motion of a system can be represented by a continuous curve in configuration space. In the present discussion, we are having a single particle in the system and hence this curve will represent the actual path of the particle in ordinary space. The function W is independent of time and hence W = constant represents a surface in the ordinary space at fixed locations. Set of surfaces are represented by various constant values of W. Eq. (94) implies that the momentum **p** has the direction perpendicular to constant W surface i.e., the path of motion of the particle is always normal to these surfaces. This is similar to the motion of rays normal to the wave surfaces in optics. Therefore, we can think that **the particle motion is associated with some form of wave motion.** Let us calculate the wave or phase velocity u of this wave motion.

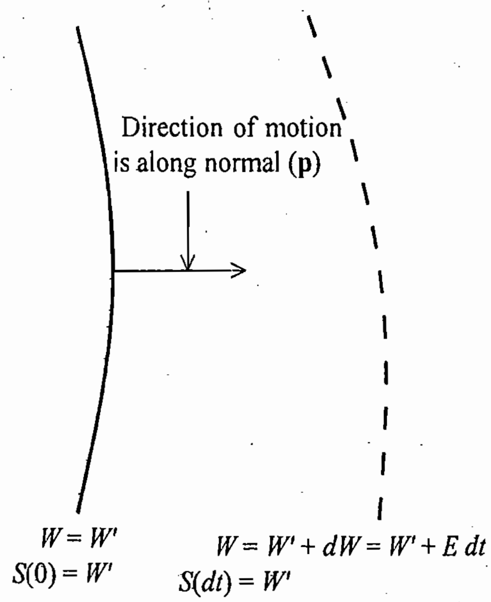


Fig. 8.2 : The Motion of the Surfaces of constant S in Configurationa space

Relation between phase velocity and particle velocity : Let us consider the relation (92)

$$S = W - Et$$

At $t = 0$, $S = S(0) = W'$ (say) which means that in the beginning the surface $S(0) = W'$ coincides with the surface $W = W'$. After time dt, the surface $S(dt) = W'$ coincides with the surface for which $W = W' + dW = W' + Edt$ (because then $S(dt) = W' + Edt - Edt = W'$) in space. In other words, in time dt the surface $S = W'$ has moved from $W = W'$ to $W = W' + Edt$ in the ordinary space so that the function W changes by

$$dW = E\,dt \qquad ...(98)$$

The motion of the surface with constant S in time is similar to the propagation of a wave-front in space.

If ds is the perpendicular distance through which the wave-front S moves in dt time, then the phase or wave velocity is given by

$$u = ds/dt \qquad ...(99)$$

Also the change dW is given by

$$dW = \frac{\partial W}{\partial s} ds = |\nabla W|\, ds \qquad ...(100)$$

because $|\nabla W| = dW/ds$ gives the maximum rate of change of W along normal (**p**). Equation (98) and (100), we get

$$|\nabla W|\, ds = E\,dt \quad \text{or} \quad \frac{ds}{dt} = \frac{E}{|\nabla W|}$$

or

$$u = \frac{E}{|\nabla W|} = \frac{E}{\sqrt{2m\,(E - V)}} \quad \text{[using (97)]} \qquad ...(101)$$

which gives the *phase velocity*. But the kinetic energy $T = E - V = \frac{1}{2} mv^2$ for the moving particle and hence the phase velocity is

$$u = \frac{E}{\sqrt{2m\,\frac{1}{2}mv^2}} = \frac{E}{mv} = \frac{E}{p} \qquad ...(102)$$

Eq.(102) is the relation connecting *the particle velocity* v to *the phase* (*wave*) *velocity* u, which is the velocity of a point on the surface of constant S. The direction of the trajectory of the particle at a point in space is determined by the direction of the momentum $\mathbf{p} = \nabla W$ and normal to the surface of constant W or S.

Optics, Classical and Wave Mechanics : We have characterized the surface of constant S as wave-fronts which propogate in space similar to wave surfaces of constant S. We have deduced an expression for wave velocity but we have nothing said regarding the wave properties e.g., period, frequency, wavelength etc. In this context, let us look closely the wave equation for light waves.

If ϕ is a scalar function (e.g., scalar electromagnetic potential), the wave equation of optics is

$$\nabla^2\phi - \frac{\mu^2}{c^2}\frac{\partial^2\phi}{\partial t} = 0 \qquad ...(103)$$

where c is the speed of light in vacuum and μ is the refractive index. In general, the refractive index μ depends upon the medium and is a function of position in space. If μ is constant in eq. (103), the solution are plane waves of the form

$$\phi = \phi_0\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \qquad ...(104)$$

where the wave number k and frequency ω are related as

$$k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c'} = \frac{\mu\omega}{c} \left[\because \mu = \frac{c}{c'} = \frac{\text{speed of light in vacuum}}{\text{speed of light in a medium}}\right]$$

Also if $k_0 = 2\pi/\lambda_0$ is the wave number in vacuum, then

$$k_0 = \frac{2\pi}{\lambda_0} = \frac{2\pi\nu}{c} = \frac{\omega}{c} \text{ and therefore } k = \mu k_0.$$

In (104) $\qquad \mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z$ and $k^2 = k_x^2 + k_y^2 + k_z^2 = \mu^2 k_0^2$

Therefore, $\qquad \mathbf{k}\cdot\mathbf{r} = k_0 f(q_k),\ q_k = x, y, z \qquad ...(105)$

and hence the plane wave solution (104) is

$$\phi = \phi_0\, e^{i[k_0 f(q_k) - \omega t]} \qquad ...(106)$$

where f is function of q_k and μ and is called **eikonal**.

We are interested in geometrical optics, where μ is not actually constant but varies slowly in the space. Therefore plane wave (104) is no longer the solution of the wave equation (103). As μ varies slowly in space, we seek the solutions close to the plane wave form i.e.,

$$\phi = \phi_0(q_k)\, e^{i[k_0 f(q_k) - \omega t]} \qquad ...(107)$$

As the time dependent part of ϕ varies as $e^{-i\omega t}$ and if it is substituted in the wave equation (103), we get,

$$\nabla^2\phi + \frac{\mu^2\omega^2\phi}{c^2} = 0$$

or $$\nabla^2\phi + \mu^2 k_0^2\phi = 0 \quad \left[\because k_0 = \frac{\omega}{c}\right] \qquad ...(108)$$

Considering only single wavelength, from (105) we have

$$k_0\nabla f = \mathbf{k} \text{ or } k^2 = [k_0\nabla f]^2$$

But $\qquad k^2 = \mu^2 k_0^2, \quad \therefore \quad (\nabla f)^2 = \mu^2$ or $\mu = |\nabla f| \qquad ...(109)$

Surfaces of constant phase are called wave-fronts and are given by

$$\Phi(q_k, t) = k_0 f(q_k) - \omega t = \text{constant} \qquad ...(110)$$

This is similar to eq. (92) [$S(q_k, t) = W(q_k) - E(t)$]. Thus W plays the role of eikonal f and the surfaces of constant S may be identified as wave surfaces of constant phase. In view of the similarity of eqs. (92) and (110), we can write

$$S \propto \Phi \quad \text{or} \quad S = a\Phi,\ W = ak_0 f \ \text{and}\ E = a\omega \qquad ...(111)$$

where a is the proportionality constant.

Thus from (109), (111) and (97), we have

$$\mu = |\nabla f| = \frac{|\nabla W|}{ak_0} = \frac{\sqrt{2m(E-V)}}{ak_0} \qquad ...(112)$$

Now we can introduce a wave function ψ to represent the wave behaviour of particle motion corresponding to the wave function ϕ in optics. Then by putting the value of μ from eq. (112) in eq. (108), the wave equation for particle motion is obtained as

$$\nabla^2\psi + \frac{2m\,(E-V)}{a^2}\psi = 0 \qquad ...(113)$$

If we put the constant $a = \hbar$ ($h/2\pi$, h is Planck's constant), then eq. (113) is

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

or
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \qquad ...(114)$$

which is the well known ***time independent Schrodinger wave equation*** for a single particle of mass m moving in a conservative force field. Thus the Schrodinger's wave (or quantum) mechanics for particle motion is related to ordinary particle (or classical) mechanics as wave (or physical) optics bears the relation to geometrical (or ray) optics.

De Broglie Relation : If ν is the frequency of the wave corresponding to the motion of a particle and λ the wavelength, then putting $u = \nu\lambda$ in relation (102), we get

$$\nu\lambda = \frac{E}{p} \quad \text{or} \quad \lambda = \frac{a\omega}{\nu p} = \frac{2\pi a}{p} \qquad ...(115)$$

where we have used $E = a\omega$ from relation (111) with $\omega = 2\pi\nu$. If we put $a = h/2\pi$ as suggested above, we have

$$\lambda = h/p = h/mv \qquad ...(116)$$

This is well known ***de Broglie relation***, giving the wavelength associated with a moving particle.

Also if we put $a = h/2\pi = \hbar$ in eq. (111), we obtain the energy of the particle

$$E = \hbar\,\omega \quad \text{or} \quad E = h\nu \qquad ...(117)$$

Fermat's Principle : The principle of least action [eq.(82), chapter 5] can be expressed as

$$\Delta\int \sqrt{2m\,(H-V)}\,ds = 0$$

But $H = E$,
$$\Delta\int \sqrt{2m\,(E-V)}\,ds = 0 \qquad ...(118)$$

Also from eq.(112), $\mu = \dfrac{1}{ak_0}\sqrt{2m\,(E-V)}$ or $\sqrt{2m\,(E-V)} = ak_0\,\mu$.

Therefore $$\Delta\int \mu\, ds = 0 \qquad ...(119)$$

which is ***the Fermat's principle of least optical path***. Since from (101) $u = E/\sqrt{2m\,(E-V)}$ or $\sqrt{2m\,(E-V)} = E/u = ak_0\,\mu$ [using eq. (112)], or $\mu = E/(aku)$, we have eq.(119) as

$$\Delta\int \frac{ds}{u} = 0 \qquad ...(120)$$

This is *another form of Fermat's principle.*

Hamilton-Jacobi Equation as the Short Wavelength Limit of Schrodinger Equation : In view of eq. (107) and (111) with $a = \hbar$, one may suggest that the wave displacement, associated with the particle motion, should have the form

$$\psi = \psi_0 e^{iS/\hbar} \qquad ...(121)$$

because $\Phi = k_0 f - \omega t = S/a = S/\hbar$.

Now if our analogy is correct, Schrodinger equation should reduce in the limit of short wavelength λ or for very small h ($\lambda = h/p$) to the Hamilton-Jacobi equation. The time dependent Schrodinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\,\frac{\partial\psi}{\partial t} \qquad ...(122)$$

Substituting (121) into (122), we have

$$-\frac{\hbar^2}{2m}\left[\frac{i}{\hbar}\psi\,\nabla^2 S - \frac{\psi}{\hbar^2}(\nabla S)^2\right] + V\psi = -\psi\,\frac{\partial S}{\partial t}$$

or $$\left[\left\{\frac{1}{2m}(\nabla S)^2 + V\right\} + \frac{\partial S}{\partial t}\right]\psi = \frac{i\hbar\psi}{2m}\nabla^2 S \qquad ...(123)$$

Since $\psi \neq 0$,

$$\left[\frac{1}{2m}(\nabla S)^2 + V\right] + \frac{\partial S}{\partial t} = \frac{i\hbar}{2m}\nabla^2 S \qquad ...(124)$$

The quantity in the bracket is the Hamiltonian H in the Hamilton-Jacobi equation. Eq. (124) may be called quantum mechanical Hamilton-Jacobi equation. In the short wavelength limit $h \to 0$, eq. (124) is exactly the Hamilton Jacobi equation, given by

$$H + \partial S/\partial t = 0 \qquad ...(125)$$

It may be remarked that Hamilton realized in 1834 the equivalence of Hamilton-Jacobi and eikonal equations and corresponding Schrodinger wave equation was deduced in 1926. One may think that if Hamilton had gone a little further in the analysis, he would have deduced the Schrodinger equation. This is not so. In the time of Hamilton, classical mechanics was considered to be rigorously correct and the experimental data were not to go beyond in the realm of quantum mechanics. In other words, Hamilton had to believe that the value of h was zero. When the experimental findings, regarding the associated wavelength with moving particle e.g., in the interference experiments of Davisson and Germer, were available, only then physical reality could be ascribed to h (known as Planck's constant). However, we note that the Hamilton-Jacobi theory can tell us how to generalize classical mechanics to quantum mechanics.

Action-Angle Variables and Sommerfield-Wilson's Rule of Quanti-sation : In Bohr's quantum theory of atom, electrons can revolve only in those orbits in which their angular momentum is an integral multiple of $h/2\pi$ i.e.,

$$J = n\, h/2\pi \ ,\ n = 1, 2, \ldots \qquad \ldots(126)$$

Thus Bohr quantized the action variable J and gave successful explanation of hydrogen spectrum in 1913. Just after this discovery, scientists realized that one can state that quantum conditions simply in terms of action variables. This led to the development of ***old quantum theory***. In classical mechanics, the action variables have continuous range of values, but this is not true in case of quantum mechanics. According to the Sommerfield-Wilson's rule of quantization, the motion of the electrons is limited to such orbits for which action variables are an integral of h, called the quantum of action i.e.,

$$J_k = \oint p_k dq_k = n_k h \qquad \ldots(127)$$

where $n_k = 1, 2, 3, \ldots$.

Thus in case of quantum mechanics, the action variables possess the discrete values. In old quantum theory, one has to solve the problem in classical mechanics using action-angle variables and then the motion is quantized by replacing the action variables J by integral multiples of h.

For example, we have already discussed the quantization of energy levels in hydrogen atom in Ex.2 (following Art. 8.8) using quantum theory. Now, let us consider the case of one-dimensional harmonic oscillator [eqs. (68), (69) and (71)], where we have evaluated the action integral :

$$J = \oint p\, dq = \frac{2\pi E}{\omega} = 2\pi E \sqrt{\frac{m}{k}} \qquad \ldots(128)$$

which is the area of the orbit of motion in phase space. According to Sommerfield-Wilson's rule, J satisfies the quantum condition

$$J = nh \quad \text{or} \quad 2\pi E \sqrt{\frac{m}{k}} = nh$$

So that

$$E = \frac{nh}{2\pi}\sqrt{\frac{k}{m}} \quad \text{or} \quad E = n\hbar\omega \quad \text{or} \quad E = nh\nu \qquad \ldots(129)$$

where $\omega = k/m = 2\pi\nu$ is the frequency of the harmonic oscillator.

Thus we see that the old quantum theory gives the solution in a very simple way except for the zero point energy. The quantization rule was applied to the cases of particle in a box, a rigid rotator, elliptic orbits of electron in hydrogen atom etc. Thus we see that the Hamiltonian dynamics in the form of action-angle variables has played a key role in the development of quantum mechanics from classical mechanics.

Poisson Brackets and Quantum Mechanics : In quantum mechanics, the dynamical variables are represented by operators which are not governed by the commutation rules of ordinary algebra. If we assume that the properties of Poisson brackets for dynamical variables in classical mechanics are also satisfied by corresponding operators in quantum mechanics, then for any two operators X and Y, it can be shown that the commutator

$$(XY - YX) = \alpha\, [X, Y]_{\text{Poisson Bracket}} \qquad \ldots(130)$$

or

$$[X, Y]_{\text{Commutator}} = \alpha\, [X, Y]_{\text{Poisson Bracket}} \qquad \ldots(131)$$

where α is constant. Now, if we assume that the operators corresponding to the classical conjugate variables q_k, p_l play an equally fundamental role in quantum mechanics, we may write

$$[q_k, p_l]_{\text{Commutator}} = (q_k p_l - p_l q_k) = \alpha\,[q_k, p_l]_{\text{Poisson Bracket}} = \alpha\delta_{kl} \qquad ...(132)$$

Postulating $\alpha = i\hbar$ ($\hbar = h/2\pi$, h = Planck's constant), we obtain the quantum mechanical relations for q_k and p_l operators *i.e.*,

$$[q_k, p_l] = i\hbar\,\delta_{kl} \qquad ...(133)$$

Also $$[q_k, q_l] = [p_k, p_l] = 0 \qquad ...(134)$$

Thus the results of classical mechanics lead us on the road to quantum mechanics and the subject of quantum mechanics has been rigorously developed by Schrodinger, Heisenberg, Dirac, Born, Pauli and others.

Some More Worked Examples

Ex. 1. *Three-Dimensional Harmonic Oscillator* : *Set up the Hamilton-Jacobi equation for a three-dimensional harmonic oscillator and solve it.*

Solution : Generalizing the Hamiltonian of one-dimensional oscillator, the Hamiltonian of a three-dimensional oscillator is given by

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + \tfrac{1}{2}k_1 q_1^2 + \tfrac{1}{2}k_2 q_2^2 + \tfrac{1}{2}k_3 q_3^2$$

where the suffixes 1,2,3 stands for the three cartesian axes respectively and the spring constants are in general different.

The system is conservative and hence the Hamiltonian H has no explicit time dependence and it is a constant of motion *i.e.*,

$$H = \alpha_1 = E \text{ (say)}$$

Hence the Hamilton's principal function S is

$$S(q_i, P_i, t) = W(q_i, P_i) - Et$$

But $p_i = \partial W/\partial q_i$,

$$H = \sum_{i=1}^{3}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial q_i}\right)^2 + \frac{1}{2}k_i q_i^2\right] = E$$

or $$\sum_{i=1}^{3}\left[\left(\frac{\partial W}{\partial q_i}\right)^2 + mk_i q_i^2\right] = 2mE \qquad ....(i)$$

which is the Hamilton-Jacobi equation for the given problem. This equation can be solved by the method of separation of variables.

Let us write $W = W_1(q_1) + W_2(q_2) + W_3(q_3)$.

Therefore, we obtain three equations for i =1, 2, 3 :

$$(\partial W/\partial q_i)^2 + mk_i q_i^2 = 2m\alpha_i \qquad ...(ii)$$

where $E = \alpha_1 + \alpha_2 + \alpha_3$.

Integrating (*ii*), we get

$$W_i = \int \left(2m\alpha_i - mk_i q_i^2\right)^{1/2} dq_i \qquad ...(iii)$$

We designate the constants α_i as the new momenta P_i. The new constant coordinates (Q_i) are obtained by the transformation

$$Q_i = \frac{\partial W}{\partial P_i} = \frac{\partial W}{\partial \alpha_i} = \frac{\sqrt{2m}}{2} \int \frac{dq_i}{\sqrt{\alpha_i - \frac{1}{2} k_i q_i^2}} \qquad \text{...}(iv)$$

But for the conservative system $H = H' = E = \alpha_1 + \alpha_2 + \alpha_3$, the equations of motion in the new coordinates are

$$\dot{Q}_i = \partial H'/\partial P_i = \partial E/\partial \alpha_i = 1$$

which gives $\quad Q_i = t + \beta_i \qquad \text{...}(v)$

From (iv) and (v)

$$t + \beta_i = \sqrt{\frac{m}{2\alpha_i}} \int \frac{dq_i}{\sqrt{1 - \frac{k_i q_i^2}{2\alpha_i}}} \quad \text{or} \quad \sqrt{\frac{m}{k_i}} \sin^{-1} q_i \sqrt{\frac{k_i}{2\alpha_i}} = t + \beta_i$$

So that $\quad q_i = \sqrt{\frac{2\alpha_i}{m\omega_i^2}} \sin \omega_i (t + \beta_i) \qquad \text{...}(vi)$

where $\quad \omega_i = \sqrt{k_i/m}$ and $i = 1, 2, 3$.

The Hamilton's principal function is

$$S(q_i, \alpha_i, t) = W(q_i, \alpha_i) - Et$$

or $\quad S(q_i, \alpha_i, t) = \sum_{i=1}^{3} W_i(q_i, \alpha_i) - Et \qquad \text{...}(vii)$

where W_i is given by (iii).

Ex. 2. ***Motion in a Plane under Central Force :*** *Obtain a complete integral of the Hamilton-Jacobi equation for the motion of a point particle in a plane under a cental force.*

Solution : The Hamiltonian of the particle in polar coordinates is

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + V(r) = E = \alpha_1 \qquad \text{...}(i)$$

because the system is conservative and $E = \alpha_1$ is a constant of motion.

Now $\quad S = W - Et$

Since $\quad p_i = \frac{\partial W}{\partial q_i}, \; p_r = \frac{\partial W}{\partial r}$ and $p_\theta = \frac{\partial W}{\partial \theta}$.

As θ is cyclic coordinate, $p_\theta = \alpha_2$ is another constant of motion.

Let us write $\; W = W_1(r) + W_2(\theta)$.

So that from (i), we obtain the time-independent Hamilton-Jacobi equation as

$$\frac{1}{2m}\left[\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_2^2}{r^2}\right] + V(r) = E \qquad \text{...}(ii)$$

So that $$W_1 = \int dr \left[2m \{E - V(r)\} - \frac{\alpha_2^2}{r^2} \right]^{1/2}$$

Also $$p_\theta = \frac{\partial W}{\partial \theta} = \frac{\partial W_2}{\partial \theta} = \alpha_2$$

Therefore, $$W_2 = \alpha_2 \theta$$

Thus $$W = \int dr \left[2m\{E - V(r)\} - \frac{\alpha_2^2}{r^2} \right]^{1/2} + \alpha_2 \theta$$

We designate α_i as the new momenta ($\alpha_1 = E = P_1$ and $\alpha_2 = p_\theta = P_2$). Now

$$Q_1 = \frac{\partial W}{\partial P_1} = \frac{\partial W}{\partial \alpha_i} = \frac{\partial W}{\partial E} = \int m\, dr \left[2m \{E - V(r)\} - \frac{\alpha_2^2}{r^2} \right]^{-1/2}$$

and $$Q_2 = \frac{\partial W}{\partial \alpha_2} = \theta - \int \alpha_2 \frac{dr}{r^2} \left[2m \{E - V(r)\} - \frac{\alpha_2^2}{r^2} \right]^{-1/2}$$

Using equation of motion

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} = \frac{\partial \alpha_1}{\partial \alpha_i}, \text{ we have}$$

$$\dot{Q}_1 = 1 \text{ and } \dot{Q}_2 = 0 \text{ or } Q_1 = \beta_1 + t \text{ and } Q_2 = \beta_2,$$

where β_1 and β_2 are constants.

Thus $$t = \int dr \left[\frac{2}{m} \{E - V(r)\} - \frac{J^2}{m^2 r^2} \right]^{-1/2} - \beta_1$$

and $$\theta = \int J \frac{dr}{r^2} \left[2m \{E - V(r)\} - \frac{J^2}{r^2} \right]^{-1/2} + \beta_2 \qquad ...(iii)$$

where we have put $\alpha_2 = p_\theta = J$.

Ex. 3. ***Fundamental Frequencies of Two-Dimensional Harmonic Oscillator*** : *Set up the action variables of a simple harmonic oscillator in two-dimensions and obtain its fundamental frequencies.*

Solution : $$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \tfrac{1}{2} k_1 q_1^2 + \tfrac{1}{2} k_2 q_2^2 = E = \alpha$$

Hamilton-Jacobi equation is

$$\sum_{i=1,2} \left[\frac{1}{2m} \left(\frac{\partial W}{\partial q_i} \right)^2 + \frac{1}{2} k_i q_i^2 - \alpha_i \right] = 0$$

where $E = \alpha = \alpha_1 + \alpha_2$.

Let $W = W_1(q_1, \alpha_1) + W_2(q_2, \alpha_2)$

Therefore, $$\frac{1}{2m}\left(\frac{\partial W_i}{\partial q_i}\right)^2 + \frac{1}{2}k_i q_i^2 = \alpha_i$$

From which we obtain

$$p_i = \frac{\partial W_i}{\partial q_i} = \pm\,[2m\,(\alpha_i - \tfrac{1}{2}k_i q_i^2)]^{1/2}, \; i = 1, 2.$$

Introducing new variables θ_i as $q_i = (2\alpha_i/k_i)^{1/2}\sin\theta_i$, and hence

$$dq_i = \left(\frac{2\alpha_i}{k_i}\right)^{1/2}\cos\theta_i\, d\theta_i$$

Now, $$J_i = \oint p_i dq_i = 2\alpha_i\sqrt{\frac{m}{k_i}}\int_0^{2\pi}\cos^2\theta_i\, d\theta_i = 2\pi\alpha_i\sqrt{\frac{m}{k_i}}$$

So that $$\alpha_i = \frac{J_i}{2\pi}\sqrt{\frac{k_i}{m}}$$

Thus $$H = \alpha = \alpha_1 + \alpha_2 = \frac{1}{2\pi}\left[J_1\sqrt{\frac{k_1}{m}} + J_2\sqrt{\frac{k_2}{m}}\right]$$

Hence the fundamental frequencies of the oscillator are

$$\nu_1 = \frac{\partial H}{\partial J_1} = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}} \quad \text{and} \quad \nu_2 = \frac{\partial H}{\partial J_2} = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}}.$$

Ex. 4 . *Projectile* : *Solve the problem of projectile of mass m in the earth's gravitational field along the Y-axis by Hamilton-Jacobi method.*

Solution : Let a mass m be projected with speed v_0 at an angle α with the hori- zontal in the earth's gravitational field. The motion is in a plane and let it be xy-plane with Y-axis as vertical. The system is conservative, hence the Hamiltonian is

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + mgy = E$$

Hamilton's principal function S is

$$S = W - Et$$

Let $$W = W_1(x) + W_2(y)$$

The Hamilton-Jacobi equation is

$$\frac{1}{2m}\left[\left(\frac{\partial W_1}{\partial x}\right)^2 + \left(\frac{\partial W_2}{\partial y}\right)^2\right] + mgy = E$$

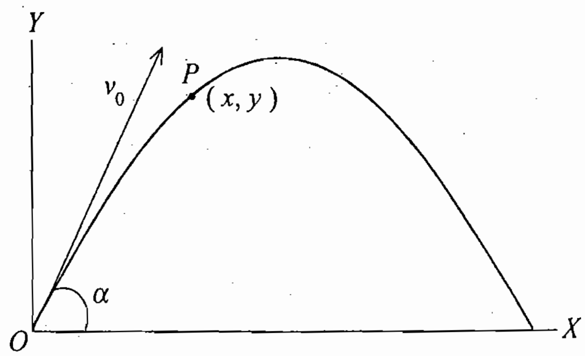


Fig. 8.3 : Projectile

because $p_x = \partial W_1/\partial x$ and $p_y = \partial W_1/\partial y$. Here $H = E = \alpha_1$ (say).

As x is cyclic coordinate,

$$p_x = \partial W_1/\partial x = \alpha_2 \text{ (a constant)} \quad \text{or} \quad W_1 = \alpha_2 x .$$

Therefore, $$\frac{\partial W_2}{\partial y} = \sqrt{2m(E - mgy) - \alpha_2^2},$$

whence $$W_2 = \int dy \sqrt{2m(E - mgy) - \alpha_2^2}$$

Thus $$W = \alpha_2 x + \int dy \sqrt{2m(E - mgy) - \alpha_2^2} \text{ and } S = \alpha_2 x + \int dy \sqrt{2m(E - mgy) - \alpha_2^2} - Et$$

The new momenta $P_1 = \alpha_1 = E$, $P_2 = \alpha_2 = p_x$,

$$Q_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial S}{\partial E} = m\int \frac{dy}{\sqrt{2m(E - mgy) - \alpha_2^2}} - t = \beta_1 \text{ (constant)}$$

Thus $$\beta_1 + t = m\int_0^y \frac{dy}{\sqrt{2m(E - mgy) - \alpha_2^2}} \text{ or } \beta_1 + t = \left.\frac{m\sqrt{2m(E - mgy) - \alpha_2^2}}{-m^2 g}\right|_0^y$$

Here, total energy $E = \frac{1}{2}mv_0^2$, $\alpha_2 = p_x = mv_x = mv_0 \cos\alpha$. Therefore,

$$\beta_1 + t = -\frac{1}{mg}\sqrt{m^2v_0^2 - 2m^2 gy - m^2 v_0^2 \cos^2\alpha} + \frac{1}{mg}\sqrt{m^2v_0^2 - m^2v_0^2\cos^2\alpha}$$

or $g(\beta_1 + t) = -\sqrt{v_0^2 \sin^2\alpha - 2gy} + v_0 \sin\alpha$ or $-\sqrt{v_0^2\sin^2\alpha - 2gy} = -v_0 \sin\alpha + (\beta_1 + t)g$

or $-2gy = -2g(\beta_1 + t)v_0 \sin\alpha + g^2(\beta_1 + t)^2$ or $y = v_0 \sin\alpha(\beta_1 + t) - \frac{1}{2}g(\beta_1 + t)^2$

If at $t = 0$, $y = 0$, then $\beta_1 = 0$. Hence

$$y = v_0 \sin\alpha t - \tfrac{1}{2}gt^2 \qquad ...(i)$$

Also, $$\beta_2 = \frac{\partial S}{\partial \alpha_2} = x - \int_0^y \frac{\alpha_2 dy}{\sqrt{2m(E - mgy) - \alpha_2^2}} = x - \frac{\alpha_2}{m^2 g}\left[\sqrt{2mE - \alpha_2^2 - 2m^2 gy)}\right]_0^y$$

or $$\beta_2 = x - \frac{v_0 \cos\alpha}{g}\sqrt{v_0 \sin^2\alpha - 2gy} + \frac{v_0 \sin\alpha \cos\alpha}{g}$$

or $$\frac{v_0 \cos\alpha}{g}\sqrt{v_0^2\sin^2\alpha - 2gy} = \frac{v_0 \sin\alpha\cos\alpha}{g} + x - \beta_2$$

whence $$y = (x - \beta_2)\tan\alpha - \frac{1}{2}\frac{g(x - \beta_2)}{v_0^2\cos^2\alpha}$$

If at $x = 0$, $y = 0$, then

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v_0^2\cos^2\alpha} \qquad ...(ii)$$

which represents a parabolic path.

Further $\dot{x} = p_x/m = v_0 \cos\alpha$ or $dx/dt = v_0 \cos\alpha$

Therefore, $\int_0^x dx = \int_0^t v_0 \cos\alpha$ or $x = v_0 \cos\alpha\, t$ ...(*iii*)

Equations (*i*), (*ii*) and (*iii*) describe completely the motion of the projectile.

Ex. 5. *The parabolic coordinates* (u, v, ϕ) *are defined by*

$$x = \sqrt{uv}\cos\phi,\ y = \sqrt{uv}\sin\phi \text{ and } z = (u-v)/2$$

Write down the Hamiltonian for planetary motion in parabolic coordinates and obtain the Hamilton-Jacobi equation. Show that the planetary motion is decomposable into completely separable Hamilton's principal function.

Solution : The Lagrangian for planetary motion is

$$L = \frac{1}{2}mv^2 + \frac{K}{r} = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{K}{r}, \text{ where } r = \sqrt{x^2 + y^2 + z^2}\,.$$

In parabolic coordinates,

$$L = \tfrac{1}{8}m\,(u+v)\left(\frac{\dot{u}^2}{u} + \frac{\dot{v}^2}{v}\right) + \frac{1}{2}muv\dot{\phi}^2 + \frac{2K}{u+v}$$

The canonical momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{m}{4}(u+v)\frac{\dot{u}}{u},\quad p_v = \frac{\partial L}{\partial \dot{v}} = \frac{m}{4}(u+v)\frac{\dot{v}}{v},\quad p_\theta = \frac{\partial L}{\partial \dot{\phi}} = muv\dot{\phi}$$

Hence the Hamiltonian $H = \sum_k p_k \dot{q}_k - L$ is obtained to be

$$H = \frac{2}{m}\frac{up_u^2 + vp_v^2}{u+v} + \frac{p_\phi^2}{2muv} - \frac{2K}{u+v}$$

where ϕ is cyclic coordinate i.e., $p_\phi = \alpha_1$, a constant.

The Hamilton-Jacobi equation is

$$H + \partial S/\partial t = 0$$

$$\frac{2}{m\,(u+v)}\left[u\left(\frac{\partial S}{\partial u}\right)^2 + v\left(\frac{\partial S}{\partial v}\right)^2\right] + \frac{1}{2muv}\left(\frac{\partial S}{\partial \phi}\right)^2 - \frac{2K}{u+v} + \frac{\partial S}{\partial t} = 0$$

or
$$u\left(\frac{\partial S}{\partial u}\right)^2 + v\left(\frac{\partial S}{\partial v}\right)^2 + \frac{1}{4}\left(\frac{1}{u} + \frac{1}{v}\right)\left(\frac{\partial S}{\partial \phi}\right)^2 - Km = \frac{1}{2}mE(u+v)$$

Writing $S = S_1(u) + S_2(v) + \alpha_1\phi - Et$ (as $p_\phi = \alpha_1 = \dfrac{\partial S}{\partial \phi}$ i.e., $S = \alpha_1\phi$), the Hamilton-Jacobi equation is separated as

$$u\left(\frac{\partial S_1}{\partial u}\right)^2 + \frac{\alpha_1^2}{4u} - \frac{1}{2}mEu - \frac{Km}{2} = \beta \qquad ...(i)$$

and
$$v\left(\frac{\partial S_2}{\partial v}\right)^2 + \frac{\alpha_1^2}{4v} - \frac{1}{2}mEv - \frac{Km}{2} = -\beta \qquad ...(ii)$$

Further $\quad \dot{x} = p_x/m = v_0 \cos\alpha \quad \text{or} \ d\cos\alpha$

Therefore, $\quad \int_0^x dx = \int_0^t v_0 \cos\alpha \quad \text{or} \quad x =$

Equations (*i*), (*ii*) and (*iii*) describe completely the m projectile. ...(*iii*)

Ex. 5. *The parabolic coordinates* (u, v, ϕ) *are defin*

$$x = \sqrt{uv}\cos\phi,\ y = \sqrt{uv}\sin\phi \ -v)/2$$

Write down the Hamiltonian for planetary motion i coordinates and obtain the Hamilton-Jacobi equation. Show that the planetary motion is de into completely separable Hamilton's principal function.

Solution : The Lagrangian for planetary motion is

$$L = \frac{1}{2}mv^2 + \frac{K}{r} = \frac{1}{2}m(\dot{x}^2 + \quad \text{where } r = \sqrt{x^2 + y^2 + z^2}.$$

In parabolic coordinates,

$$L = \frac{1}{8}m(u+v)\left(\frac{\dot{u}^2}{u} + \frac{\dot{v}^2}{v}\right) + \frac{}{v}$$

The canonical momenta are

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{m}{4}(u+v)\frac{\dot{u}}{u},\quad p_v = \cdot,\quad p_\theta = \frac{\partial L}{\partial \dot{\phi}} = muv\dot{\phi}$$

Hence the Hamiltonian $H = \sum_k p_k \dot{q}_k - L$ is obtained t

$$H = \frac{2}{m}\frac{up_u^2 + vp_v^2}{u+v} + \frac{p_\phi^2}{2muv} - \frac{}{u}$$

where ϕ is cyclic coordinate i.e., $p_\phi = \alpha_1$, a constant.

The Hamilton-Jacobi equation is

$$H + \partial S/\partial t = 0$$

$$\frac{2}{m(u+v)}\left[u\left(\frac{\partial S}{\partial u}\right)^2 + v\left(\frac{\partial S}{\partial v}\right)^2 \frac{K}{+v} + \frac{\partial S}{\partial t} = 0\right.$$

or $$u\left(\frac{\partial S}{\partial u}\right)^2 + v\left(\frac{\partial S}{\partial v}\right)^2 + \frac{1}{4}\left(\frac{1}{u} + \right.(u+v)$$

Writing $S = S_1(u) + S_2(v) + \alpha_1\phi - Et$ (as $p_\phi = \alpha_1 = \frac{\partial S}{\partial \phi}$ ilton-Jacobi equation is separated as

$$u\left(\frac{\partial S_1}{\partial u}\right)^2 + \frac{\alpha_1^2}{4u} - \frac{1}{2}mEu - \frac{Km}{2} \qquad ...(i)$$

and $$v\left(\frac{\partial S_2}{\partial v}\right)^2 + \frac{\alpha_1^2}{4v} - \frac{1}{2}mEv \cdot \frac{Km}{2} \qquad ...(ii)$$

Thus in terms of parabolic coordinates, decomposed the Hamilton-Jacobi equation. Now, we obtain from above

$$S_1 = \int du \left(\frac{mE}{2} - \frac{\alpha_1^2}{4u^2}\right)^{\frac{1}{2}} \qquad ...(iii)$$

and

$$S_2 = \int dv \left(n - \frac{\alpha_1^2}{4v^2}\right)^{\frac{1}{2}} \qquad ...(iv)$$

where $\alpha_1 = p_y$, E and β are the constan taken as new momenta.

Questions

1. Outline Hamilton-Jacobi theor solve the problem of one-dimensional Harmonic oscillator. rut 1999, 92; Garwal 92; Agra 73; Rohilkhand 95, 81)
2. When is Hamilton-Jacobi Th l? Discuss the harmonic oscillater problem using Hamilton-Jacobi method. **(Gorakhpur 1996)**
3. Discuss the Hamilton-Jacot t circumstances is the characteristic function W more useful than the special function S **(Rohilkhand 1985, 83)**
4. State and prove Hamilton for Hamilton's principal function and explain how it can be used to solve Kepler's pr le in an inverse square central force field. **(Agra 1971; Rohilkhand 78)**
5. Prove by Hamilton-Jaco rbit of a planet round the sun is an elliptic one with the sun at one of its foci. **(Rohilkhand 1987; Meerut 83, 80)**
6. Apply the Hamilton one dimensional Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ to deduce the motion of a particle. **(Rohilkhand 1985; 83)**
7. (*a*) What are essenti ton-Jacobi method. **(Meerut 2000)**
   (*b*) Use Hamilton-J n Kepler's laws of planetary motion. **(Meerut 1994)**
8. What are action-an ain how they can be used to obtain the frequencies of periodic motion. Hence dete of linear harmonic oscillator. **(Rohilkhand 1999, Meerut 95)**

Problems — [SET- I]

1. A particle of rce field whose potential difference in spherical coordinates is given by

Write down equation describing its motion. Find a complete solution of the Hamilton-J ed.

Ans : $$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right]-\frac{K\cos\theta}{r^2}+\frac{\partial S}{\partial t}=0$$

$$S=\int\sqrt{2mE+\frac{\beta}{r^2}}dr+\int\sqrt{2mk\cos\theta-p_\phi^2\mathrm{cosec}^2\theta-\beta}\;d\theta+p_\phi\phi-Et.$$

[**Hint :** $$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\phi^2}{r^2\sin^2\theta}\right)-\frac{K\cos\theta}{r^2}$$

But $p_k=\partial S/\partial q_k$ i.e., $p_r=\partial S/\partial r$, $p_\theta=\partial S/\partial\theta$ and $p_\phi=\partial S/\partial\phi$.

Substituting these, obtain the Hamilton-Jacobi equation.

Let $S=S_1(r)+S_2(\theta)+S_3(\phi)-Et$. As ϕ is a cyclic coordinate,

$\partial S/\partial\phi=\partial S_3/\partial\phi=p_\phi$, a constant and hence $S_3=p_\phi\phi$.

Now, $\partial S/\partial r=\partial S_1/\partial r$ etc. and hence Hamilton-Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S_1}{\partial r}\right)^2+\frac{1}{2mr^2}\left(\frac{\partial S_2}{\partial \theta}\right)^2+\frac{1}{2mr^2\sin^2\theta}\left(\frac{\partial S_3}{\partial \phi}\right)^2-\frac{K\cos\theta}{r^2}=E$$

or, $$r^2\left(\frac{\partial S_1}{\partial r}\right)^2-2mEr^2=\left(\frac{\partial S_2}{\partial \theta}\right)^2-\frac{1}{\sin^2\theta}\left(\frac{\partial S_3}{\partial \phi}\right)^2+2mK\cos\theta=-\beta,\text{ (say)}$$

because L.H.S. of this equation depends on r and R.H.S. on θ and ϕ. Now

$$r^2(\partial S_1/\partial r)^2-2mEr^2=-\beta\text{ i.e., }S_1=\int\sqrt{2mE+\beta/r^2}\,dr$$

Also $$\left(\frac{\partial S_2}{\partial \theta}\right)^2+\frac{1}{\sin^2\theta}\left(\frac{\partial S_3}{\partial \phi}\right)^2-2mK\cos\theta=\beta$$

or, $$\sin^2\theta\,(\partial S_2/\partial\theta)^2-2mK\cos\theta-\beta\sin^2\theta+p_\phi^2=0$$

whence $\partial S_2/\partial\theta=(2mK\cos\theta-p_\phi^2\;\mathrm{cosec}^2\theta-\beta)^{1/2}$

Therefore, $$S_2=\int\sqrt{2mK\cos\theta-p_\phi^2\mathrm{cosec}^2\theta-\beta}\;d\theta$$

Hence $S=S_1+S_2+S_3-Et$.]

2. Solve the Hamilton-Jacobi equation for the system whose Hamiltonian is given by

$$H=\frac{p^2}{2}-\frac{K}{q}$$

Ans : $S=S_1(q)+S_2(t)=\sqrt{2CK}\sin^{-1}(q/c)^{\frac{1}{2}}+[2Kq(C-q)/C]^{\frac{1}{2}}+(K/C)\,t$

[**Hint :** Hamilton-Jacobi equation is

$$\frac{1}{2}\left(\frac{\partial S_1}{\partial q}\right)^2 - \frac{K}{q} + \frac{\partial S_2}{\partial t} \quad \text{or} \quad \frac{\partial S_2}{\partial t} = \frac{K}{q} - \frac{1}{2}\left(\frac{\partial S_1}{\partial q}\right)^2 = \frac{K}{C}$$

where C is a constant.

Hence $$S_2 = \frac{K}{C}t, \frac{\partial S_1}{\partial q} = \sqrt{2\left(\frac{K}{q} - \frac{K}{C}\right)^{\frac{1}{2}}} = \frac{dq}{\sqrt{q}}\sqrt{2K}\sqrt{1-\frac{q}{C}}.$$

Put $\sqrt{q/C} = \sin\theta$ and integrate to obtain

$$S_1 = \sqrt{2KC}\,\theta - \sqrt{2KC}\sin\theta\cos\theta.]$$

3. A particle of mass m is free to move in xy-plane under the action of two simple harmonic forces $-kx$ and $-ly$. Construct the Hamilton-Jacobi partial differential equation and prove that the energy as a function of the phase integrals is

$$E = \frac{1}{2\pi}\left[J_x\sqrt{\frac{k}{m}} + J_y\sqrt{\frac{l}{m}}\right].$$

Prove also that the fundamental frequencies are

$$\nu_x = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ and } \nu_y = \frac{1}{2\pi}\sqrt{\frac{l}{m}}.$$

4. Apply the method of action-angle variables to determine the fundamental frequencies of a three-dimensional harmonic oscillator with unequal spring constants.

Ans : $\nu_1 = \frac{1}{2\pi}\sqrt{\frac{k_1}{m}}$, $\nu_2 = \frac{1}{2\pi}\sqrt{\frac{k_2}{m}}$, $\nu_3 = \frac{1}{2\pi}\sqrt{\frac{k_3}{m}}$.

5. Set up and solve the Hamilton-Jacobi equation for the motion of a particle of mass m in a uniform gravitational field along the Z-axis and XY-plane as horizontal.

Ans : $$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2\right] + mgz = E$$

$$S = W - Et = p_x x + p_y y + \int dz\sqrt{2m(E - mgz) - p_x^2 - p_y^2}$$

where $p_x = \alpha_1$, $p_y = \alpha_2$ and $E = \alpha_3$ are the constants, taken as new momenta.

$$x = p_x t, y = p_y t, z = \left[2mE - p_x^2 - p_y^2\right]^{1/2}\frac{t}{m} - \frac{1}{2}gt^2.$$

6. A body of unit mass is constrained to move on the path $y = \cosh x$ under a potential $V = \frac{1}{2}x^2$. Set up the Hamiltonian and Hamilton-Jacobi equation and finally solve it.

Ans : $$H = \frac{1}{2}(p^2\,\text{sech}^2 x + x^2),\ (\partial W/\partial x)^2 = \text{sech}^2 x + x^2 = 2E;$$

$$S = \int dx\cosh x\sqrt{2E - x^2} - Et.$$

[**Hint :** $L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-\frac{1}{2}x^2=\frac{1}{2}\dot{x}^2(1+y'^2)-\frac{1}{2}x^2=\frac{1}{2}\dot{x}^2\cosh^2x-\frac{1}{2}x^2$,

$$p=\frac{\partial L}{\partial \dot{x}}=\dot{x}\cosh^2 x\ ;\ H=p\dot{x}-L=\frac{1}{2}(p^2\operatorname{sech}^2x+x^2)]$$

7. Determine the frequencies of a harmonic oscillator of mass m, for which the Hamiltonian is given by $H=\frac{1}{2}\left[\frac{p^2}{m}+\mu q^2\right]$ by the method of action-angle variables.

Ans : $\nu=\frac{1}{2\pi}\sqrt{\mu/m}$

8. By using the method of action-angle variables show that the frequency of a simple pendulum is given by $\frac{1}{2\pi}\sqrt{\frac{g}{l}}$.

9. In case of a problem, the Hamiltonian is given by

$$H=\frac{1}{2\pi}\left[p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\phi^2}{r^2\sin^2\theta}\right]-\frac{K\mu}{r}.$$

Show that the action integrals are given by

$J_\phi=2\pi p_\phi$, $J_\theta=2\pi(\alpha_\theta-p_\phi)$ with $\alpha_\theta=\frac{\sqrt{p_\theta^2+p_\phi^2}}{\sin^2\theta}$, and $J_r=-(J_\theta+J_\phi)+\pi K\mu\sqrt{2\mu/-E}$.

Hence find an expression for the energy and show that all the three frequencies ν_r, ν_θ and ν_ϕ are identical.

10. Determine the action and angle variables for the potential energy $V(q)=V_0\tan^2(Kq)$, where V_0 and K are positive constants. Find the frequency of oscillation.

Ans : Action variable $J=\left(\sqrt{2m(E+V_0)}-\sqrt{2mV_0}\right)/K$, and angle variable $w=\nu t+\beta$; $\nu=K\sqrt{2(E+V_0)/m}$

[**Hint :** $J=\frac{1}{\pi}\int_{q_1}^{q_2}dq\sqrt{2(E-V_0\tan^2(Kq)}$, where the limits $q_1=-q_2$, and $\tan^2(Kq_2)=E/V_0$. The integration gives the desired action variable.]

[SET- II]

1. Find a complete integral of the Hamilton-Jacobi equation for the motion of a particle in the potential $V(r)=(\mathbf{a}\cdot\mathbf{r})/r^3$. Take the vector $\mathbf{a}$ along Z-axis.

Ans : $S=-Et+P_\phi\phi\pm\int\left[\beta-2ma\cos\theta-\frac{p_\phi^2}{\sin^2\theta}\right]d\theta\pm\int\left[2mE-\frac{\beta^2}{r^2}\right]dr$, where E, p_ϕ and β are the constants of motion.

2. Describe the motion in terms of parabolic coordinates for a particle moving in the potential $V(r)=-\frac{\alpha}{r}-\mathbf{F}\cdot\mathbf{r}$. Take Z-axis along $\mathbf{F}$.

Ans : $$S = -Et + p_\phi \phi + \int p_u(u)\,du + \int p_v(v)\,dv,$$

where $$p_u = \pm\sqrt{\frac{m}{2}}[(E - V_u(u)], \quad p_v = \pm\sqrt{\frac{m}{2}}[(E - V_v(v)],$$

$$V_u = \frac{p_\phi^2}{2mu^2} - \frac{m\alpha + \beta}{mu} - \frac{1}{2}Fu, \quad V_v = \frac{p_\phi^2}{2mv^2} - \frac{m\alpha - \beta}{mv} + \frac{1}{2}Fv.$$

The motion is determined by the equations

$$\frac{\partial S}{\partial E} = -t + \frac{m}{4}\int \frac{du}{p_u(u)} + \frac{m}{4}\int \frac{dv}{p_v(v)} = \beta_1,$$

$$\frac{\partial S}{\partial p_\phi} = \phi - \frac{p_\phi}{2}\int \frac{du}{u^2 p_u(u)} - \frac{p_\phi}{2}\int \frac{dv}{v^2 p_v(v)} = \beta_2,$$

$$\frac{\partial S}{\partial \beta} = \int \frac{du}{u\,p_u(u)} - \int \frac{dv}{v\,p_v(v)} = \beta_3.$$

[**Hint :** $V(r) = -\frac{\alpha}{r} - Fr\cos\theta = -\frac{\alpha}{r} - Fz = \frac{a(u) + b(v)}{u + v}$

where $a(u) = -\alpha - \frac{1}{2}Fu^2,\ b(v) = -\alpha + \frac{1}{2}Fv^2$.]

3. Prove that the motion of a particle of mass m under a non-central potential

$$V(r) = \frac{\beta^2}{2mr^2}\sec^2\theta - \frac{K}{r}$$

leads to identical integrals J_r and J_ϕ as those of Kepler's problem. Find the energy function as $E(J_r, J_\theta, J_\phi)$ and further prove that the frequencies $\nu_r = \nu_\phi$ and $\nu_\theta = 2\nu_\phi$.

[**Hint :** $J_\phi = 2\pi\alpha_\phi = 2\pi p_\phi$, and $J_r = \oint dr\sqrt{2mE + \frac{2mK}{r} - \frac{\alpha_\theta^2}{r^2}}$.

These integrals are the same as in Kepler's problem. However,

$$J_\theta = \oint d\theta\sqrt{\alpha_\theta - \alpha_\phi \operatorname{cosec}^2\theta - \beta^2\sec^2\theta}$$

and it is different. Put $u = \tan^2\theta$ and do contour integral to obtain

$$J_\theta = \pi(\alpha_\theta - \alpha_\phi - \beta) \text{ and } J_r = -2\pi\alpha_\theta + \sqrt{2\pi^2 mK^2/(-E)}.$$

Obtain that $\quad E = -2\pi^2 mK^2/(J_r + 2J_\theta + J_\phi + 2\pi\beta)^2$.

This gives $\quad \nu_r = \nu_\phi = \nu_\theta/2$.]

Objective Type Questions

1. If we make a canonical transformation from the set of variables (p_k, q_k) to new set of variables (P_k, Q_k) and the transformed Hamiltonian is identically zero, then

(*a*) the new variables are constant in time.

(*b*) the new variables are cyclic.

(*c*) the old variables (p_k, q_k) remain constant in time.

(*d*) the momentum coordinates p_k remain constant in time.

Ans : (*a*), (*b*).

2. Hamilton's principal function S and Hamilton's characteristic function W for conservative system are related as

(*a*) $S = W$, (*b*) $S = W - Et$

(*c*) $S = W + Et$ (*d*) S is not related to W.

where E is the total energy and t is the time.

Ans : (*b*).

3. For a one-dimensinol harmonic oscillator, the representative point in two-dimensional phase space traces

(a) an ellipse, (*b*) a parabola,

(*c*) a hyperbola, (*d*) always a straight line.

Ans : (*a*).

4. The action and angle variables have the dimensions of

(*a*) force and angle (*b*) angular momentum and angle

(*c*) energy and angle (*d*) are dimensionless quantities.

Ans : (*b*).

5. For a particle of mass m, moving in an inverse square force field $V(r) = -K/r$,

(*a*) the Hamiltonian of the system is $H = \dfrac{1}{2m}\left[p_r^2 + \dfrac{p_\theta^2}{r^2}\right] - \dfrac{K}{r}$

where p_r and p_θ are the conjugate momenta corresponding to r and θ coordinates,

(*b*) the action-angle variable analysis gives two equal frequencies,

(*c*) the action-angle variable analysis gives two unequal frequencies,

(*d*) the action-angle variable analysis cannot predict any frequency.

Ans : (*a*), (*b*).

Short Answer Questions

1. Establish Hamilton-Jacobi equation.
2. Give the physical significance of Hamilton's characteristic function W.
3. What are action-angle variables ?
4. What is the advantage of using action-angle variables ?
5. Show that the Hamilton-Jacobi equation is the short wavelength limit of the Shrodinger equation.
6. Deduce de Broglie relation $\lambda = h/mv$ by assuming the constant a in classical theory to be equal to $h/2\pi$.
7. Could you relate Poisson bracket in classical mechanics to commutator in quantum mechanics.
8. What is Sommerfield-Wilson's rule in relation to action variable.
9. Fill in the blank.

(*i*) Hamilton's principal function $S(q_k, t)$ and Hamilton's characteristic function $W(q_k)$ are related by the equation..........................

(*ii*) Action variable has the dimensions of..........................

(*iii*) Angle variable is a linear function of..........................

Ans. (*i*) $S(q_k, t) = W(q_k) - Et$, (*ii*) angular momentum (*iii*) time

CHAPTER 9

Small Oscillations and Normal Modes (Coupled Oscillators)

9.1. INTRODUCTION

In this chapter, we generalize the harmonic oscillator problem of one degree of freedom in the Lagrangian formulation to the case of the small amplitude oscillations of a system of several degrees of freedom near the position of equilibrium. The theory of such small oscillations is extremely important in several areas of physics, *e.g.*, molecular spectra, acoustics, vibrations of atoms in solids, coupled mechanical oscillators and electrical circuits etc. When we go from a single oscillator to the problem of two coupled oscillators, the analysis results in some interesting and surprising new features. We shall see that the motion of two coupled oscillators in general is much complicated and none of the oscillators in general executes simple harmonic motion. However, for small amplitude oscillations, we may express the general motion as a superposition of two independent simple harmonic motions, both going on simultaneously. We call these two simple harmonic motions as ***normal modes*** or simply ***modes***. Further we shall see that a system of N coupled oscillators with N degrees of freedom, has exactly N independent modes of vibrations and the general motion can be expressed as the superposition of N normal modes. Each mode has its own frequency and wavelength. We will establish a relation between the wavelength and frequency of a mode, known as the ***dispersion relation***. Now, considering exceedingly large number of particles and allowing the interparticle distance to approach zero, we obtain the system as continuous medium and its motion is dealt as waves.

9.2. POTENTIAL ENERGY AND EQUILIBRIUM – *ONE DIMENSIONAL OSCILLATOR*

In order to understand the general theory of oscillations, it is essential to know about the potential energy at the equilibrium configuration. Let us consider a conservative system in which the potential energy is a function of position only. Let the system be specified by n generalized coordinates $q_1, q_2, \ldots, q_n$, not involving time explicitly. For such a system, the potential energy is given by

$$V = V(q_1, q_2, \ldots, q_n) \qquad \ldots(1)$$

and the generalized forces are given by

$$G_k = -\frac{\partial V}{\partial q_k} \quad \text{where } k = 1, 2, \ldots, n \qquad \ldots(2)$$

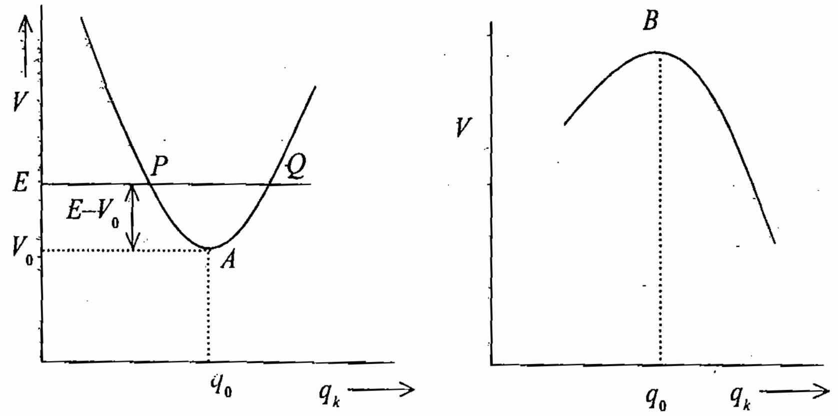
The system is said to be in equilibrium, if the generalized forces acting on the system are equal to zero, *i.e.*,

$$G_k = -\left[\frac{\partial V}{\partial q_k}\right]_0 = 0 \qquad \ldots(3)$$

Thus the potential energy has an extremum at the equilibrium configuration of the system, represented by the coordinates $q_1^0, q_2^0, \ldots, q_n^0$. Now, if the system is in equilibrium with zero initial velocities $\dot{q}_k$, the system will remain in equilibrium indefinitely. Examples of mechanical systems at equilibrium are a pendulum and a spring-mass system at rest, an egg standing on one end etc.

9.2.1. Stable, Unstable and Neutral Equilibrium

A system is said to be in ***stable equilibrium***, if a small displacement of the system from the rest position (by giving a little energy to it) results in a small bounded motion about the equilibrium position. In case, small displacement of the system from the equilibrium position results in an unbounded motion, it is in an ***unstable equilibrium***. Further, if the system on displacement has no tendency to move about or away the equilibrium position, it is said to be in ***neutral equilibrium***. An example of stable equilibrium is a pendulum in the rest position and that of an unstable equilibrium is an egg standing on one end. A coin placed flat anywhere on a table is in neutral equilibrium.



(a) Stable equilibrium (b) Unstable equilibrium

Fig. 9.1 : Potential energy curve

A graph drawn between the potential energy of the system and a particular coordinate q_k is called ***potential energy curve*** and is shown in Fig. 9.1. The positions A and B, where the generalized force $F = -\partial V/\partial q$ vanishes, are the positions of equilibrium; potential energy V is minimum (say V_0) at A [Fig 9.1(a)] and maximum at B [Fig. 9.1(b)]. Position A corresponds to the stable equilibrium, because if the system is displaced from A to Q by giving energy $(E - V_0)$ and left to itself, the system tries to come in the position of minimum potential energy. Consequently the potential energy will change to kinetic energy and at A the energy $(E - V_0)$ will be purely in the kinetic form because of the conservation law. This will change again to potential form, when the system moves towards the position P and hence a bounded motion ensues about the equilibrium position A. Obviously the position B of the maximum potential energy represents the unstable equilibrium because any energy given to the system at this position will result more and more kinetic energy when the system moves either left or right to it. In this case, the system moves away from the equilibrium position. In case of neutral equilibrium, the potential energy is independent of the coordinate and equilibrium occurs at any arbitrary value of that coordinate.

9.2.2. One-Dimensional Oscillator

The motion of the system about the position of stable equilibrium is of great interest in physics. For simplicity, let the system possess one degree of freedom with one generalized coordinate q. For small displacement from the equilibrium, we can expand the potential energy $V(q)$ in a Taylor's series about the equilibrium and consider only the lowest order terms :

$$V(q) = V(q_0) + \left[\frac{\partial V}{\partial q}\right]_0 (q - q_0) + \frac{1}{2}\left[\frac{\partial^2 V}{\partial q^2}\right]_0 (q - q_0)^2 + ... \qquad ...(4)$$

where the derivatives are to be evaluated at the equilibrium position $q = q_0$. At the position of equilibrium, we have $(\partial V/\partial q)_0 = 0$. First term, $V(q_0)$, in the series is the potential energy at the equilibrium position and we can take $V(q_0) = 0$, if we shift the origin of the potential energy to be at the minimum equilibrium value. Thus

$$V(q) = \frac{1}{2}\left[\frac{\partial^2 V}{\partial q^2}\right]_0 (q - q_0)^2 \qquad \text{...(5)}$$

If we put $(\partial^2 V/\partial q^2)_0 = K$ and take the origin of q coordinate at $q_0 = 0$, then eq. (5) becomes

$$V(q) = \tfrac{1}{2} K q^2 \qquad \text{...(6)}$$

where $K = (\partial^2 V/\partial q^2)_0$ is a positive parameter at the position of stable equilibrium.

In the present case, the generalized coordinate does not involve time explicitly and hence the kinetic energy is a homogeneous quadratic function of the generalized velocities [see eq. (39), Chapter 2] *i.e.*,

$$T = \tfrac{1}{2} m(q)\dot{q}^2 \qquad \text{...(7)*}$$

where the coefficient $m(q)$ is, in general, function of q-coordinate and may also be expanded in Taylor's series about the equilibrium position $(q_0 = 0)$:

$$m(q) = m(0) + (\partial m/\partial q)_0\, q + \ldots\ldots \qquad \text{...(8)}$$

Eq. (7) is already quadratratic in $\dot{q}$, the lowest non-vanishing approximation to T is obtained by retaining only the first term in the expansion. Thus for small oscillations, the Lagrangian of the one-dimensional oscillator is given by

$$L = T - V = \tfrac{1}{2} m(0)\, \dot{q}^2 - \tfrac{1}{2} K q^2 \qquad \text{...(9)}$$

Thus the equation of motion is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = 0$$

or $\qquad m(0)\,\ddot{q} + Kq = 0 \quad \text{or} \quad \ddot{q} + \omega^2 q = 0 \qquad \text{...(10)}$

where $\qquad \omega^2 = K/m(0)$.

Let the solution of eq. (10) be

$$q = ae^{\alpha t}$$

Now substituting the value of $\ddot{q}$ and q in eq. (10), we have

$$a\alpha^2 e^{\alpha t} + \omega^2 a e^{\alpha t} = 0 \quad \text{or} \quad ae^{\alpha t}\left(\alpha^2 + \omega^2\right) = 0$$

But in general $ae^{\alpha t}$ is not zero. Hence

$$\alpha^2 + \omega^2 = 0 \text{ or } \alpha = \sqrt{-\omega^2} = \pm i\omega$$

Therefore, $q = a_1 e^{i\omega t}$ or $q = a_2 e^{-i\omega t}$

---

* $\dot{x} = \dfrac{\partial x}{\partial q}\dot{q}$ and $T = \tfrac{1}{2} m\dot{x}^2 = \tfrac{1}{2} m \left[\dfrac{\partial x}{\partial q}\right]^2 \dot{q}^2 = \tfrac{1}{2} m(q)\,\dot{q}^2$

where $m(q) = m(\partial x/\partial q)^2$ is, in general, a function of q.

Hence the general solution of eq. (10) is

$$q = a_1 e^{i\omega t} + a_2 e^{-i\omega t} \qquad ...(11)$$

Since q is a real number, so that a_1 and a_2 must be complex conjugates. If we write

$$a_1 = \frac{1}{2}(c - id) \text{ and } a_2 = \frac{1}{2}(c + id)$$

we obtain

$$q = \frac{1}{2}(c - id)e^{i\omega t} + \frac{1}{2}(c + id)e^{-i\omega t}$$

or

$$q = c \cos \omega t + d \sin \omega t \qquad ...(12)$$

If we write $c = a \cos \phi$ and $d = -a \sin \phi$, we may express (12) in the following form:

$$q = a \cos(\omega t + \phi) \qquad ...(13)$$

In eq. (13), the constant a is called the *amplitude* of oscillation, ω the *frequency* and ϕ the *initial phase*.

9.3. TWO COUPLED OSCILLATORS

Let us consider two identical masses m, connected to two fixed walls by two massless springs of force constant k and also coupled to each other by a massless spring of force constant k' [fig. 9.2(a)]. In the figure, the system is in equilibrium and for simplicity, we assume that in this position, the springs do not exert any forces. Even if the springs exert forces in equilibrium, the vibrations would not be affected. Let the system be displaced from the equilibrium position and the motion of the two masses be restricted along the line joining the two masses (longitudinal oscillations), say along the X-axis. Thus the system has two degrees of freedom, represented by the coordinates x_1 and x_2 [fig 9.2(b)].

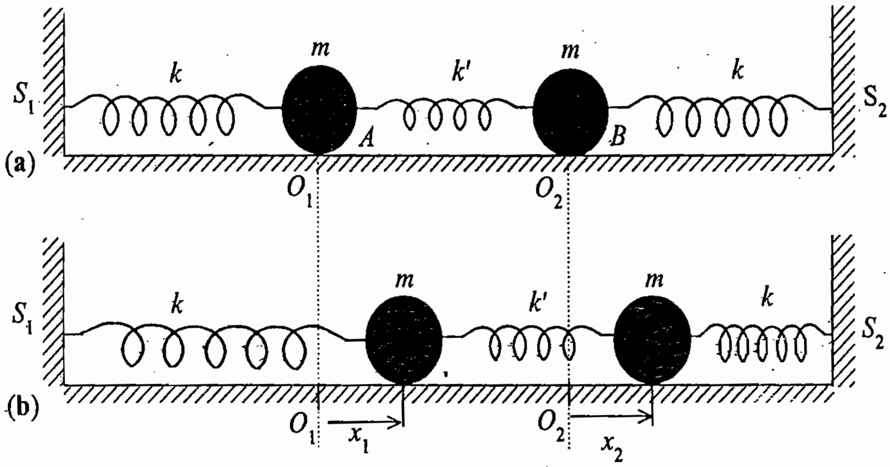


Fig. 9.2 : Two coupled oscillators – System of two equal masses connected to each other by a spring of force constant k' and to rigid supports S_1 and S_2 by springs, each of force constant k : (a) Equilibrium configuration, (b) Configuration at any instant t.

Here $q_1 = x_1$ and $q_2 = x_2$ are the two generalized coordinates. This is an example of two coupled oscillators. If the two oscillators were not connected, each would vibrate with a frequency, given by

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad ...(14)$$

If these springs are connected by a spring of force constant k', the system oscillates with different frequencies. We are interested in the calculation of these frequencies.

The kinetic energy of the oscillating system is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

and the potential energy of the system is

$$V = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k'(x_1 - x_2)^2$$

Hence the Lagrangian $L\ (= T - V)$ for the system is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k'(x_1 - x_2)^2 \qquad ...(15)$$

The Lagrange equations of motion are

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_1}\right] - \frac{\partial L}{\partial x_1} = 0 \qquad ...(16)$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}_2}\right] - \frac{\partial L}{\partial x_2} = 0 \qquad ...(17)$$

Using eq. (15), we obtain the equations of motion as

$$m\ddot{x}_1 + kx_1 + k'(x_1 - x_2) = 0 \qquad ...(18)$$

$$m\ddot{x}_2 + kx_2 - k'(x_1 - x_2) = 0 \qquad ...(19)$$

The third term in eqs. (18) and (19) is the result of coupling between the two oscillators. Eq. (18) can be solved for x_1, if we know x_2 and conversely for the equation (19). In fact these are coupled differential equations which must be solved simultaneously.

9.3.1. Solution of the Differential Equations

Eqs. (18) and (19) is a set of simultaneous linear differential equations with constant coefficients and therefore, let us try solutions of the form :

$$x_1 = a_1 e^{i\omega t} \quad \text{and} \quad x_2 = a_2 e^{i\omega t} \qquad ...(20)$$

where a_1 and a_2 are constants. Substituting for x_1 and x_2 in eqs. (18) and (19), we obtain

$$-m\omega^2 a_1 + ka_1 + k'(a_1 - a_2) = 0$$

and

$$-m\omega^2 a_2 + ka_2 - k'(a_1 - a_2) = 0$$

or

$$(\omega_0^2 + \omega_c^2 - \omega^2)\, a_1 - \omega_c^2 a_2 = 0 \qquad ...(21)$$

$$-\omega_c^2 a_1 + (\omega_0^2 + \omega_c^2 - \omega^2)\, a_2 = 0 \qquad ...(22)$$

where

$$\omega_c^2 = k'/m .$$

Eqs. (21) and (22) will have solutions, if the determinant of the coefficients a_1 and a_2 vanishes, *i.e.*,

$$\begin{vmatrix} \omega_0^2 + \omega_c^2 - \omega^2 & -\omega_c^2 \\ -\omega_c^2 & \omega_0^2 + \omega_c^2 - \omega^2 \end{vmatrix} = 0 \qquad ...(23)$$

This is called the ***secular equation***.

This may be written as

$$\left(\omega_0^2 + \omega_c^2 - \omega^2\right)^2 - \omega_c^4 = 0 \qquad ...(24)$$

which yields

$$\omega = \pm\,\omega_0 = \pm\,\omega_1 \quad \text{and} \quad \omega = \pm\sqrt{\omega_0^2 + 2\omega_c^2} = \pm\,\omega_2 \qquad ...(25)$$

Thus the general solutions of eqs. (18) and (19) are

$$x_1 = a_1 e^{i\omega_1 t} + a_1' e^{-i\omega_1 t} + b_1 e^{i\omega_2 t} + b_1' e^{-i\omega_2 t} \quad ...(26)$$

and
$$x_2 = a_2 e^{i\omega_1 t} + a_2' e^{-i\omega_1 t} + b_2 e^{i\omega_2 t} + b_2' e^{-i\omega_2 t} \quad ...(27)$$

where a's and b's are arbitrary constants, but they are not all independent. Substituting from eqs. (25) in eqs. (21) and (22), we obtain the ratio a_1/a_2 for different values of ω to be :

for $\omega = \pm\omega_1 = \pm\omega_0$, $\qquad a_1 = a_2 \quad ...(28\,a)$

and for $\omega = \pm\omega_2 = \pm\sqrt{\omega_0^2 + 2\omega_c^2}$, $\qquad a_1 = -a_2 \quad ...(28\,b)$

In view of (28), obviously eqs. (26) and (27) are

$$x_1 = a_1 e^{i\omega_1 t} + a_1' e^{-i\omega_1 t} + b_1 e^{i\omega_2 t} + b_1' e^{-i\omega_2 t} \quad ...(29\,a)$$

and
$$x_2 = a_1 e^{i\omega_1 t} + a_1' e^{-i\omega_1 t} - b_1 e^{i\omega_2 t} - b_1' e^{-i\omega_2 t} \quad ...(29\,b)$$

which can be written as

$$x_1 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \quad ...(30\,a)$$

and
$$x_2 = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \quad ...(30\,b)$$

These equations involve the same number of constants as in eqs. (29) (i.e., A_1, A_2, ϕ_1, ϕ_2 instead a_1, a_1', b_1, b_1'). We observe from eqs.(30) that each coordinate is a superposition of two simple harmonic motions of frequencies $\omega_1 (= \omega_0)$ and $\omega_2 \left(= \sqrt{\omega_0^2 + 2\omega_c^2}\right)$.

9.3.2. Normal Coordinates and Normal Modes

If $A_2 = 0$ in eq. (30),

$$x_1 = A_1 \cos(\omega_1 t + \phi_1) \text{ and } x_2 = A_1 \cos(\omega_1 t + \phi_1) \quad ...(31)$$

and if $A_1 = 0$, then

$$x_1 = A_2 \cos(\omega_2 t + \phi_2) \text{ and } x_2 = -A_2 \cos(\omega_2 t + \phi_2) \quad ...(32)$$

Thus if $A_2 = 0$, the two masses oscillate together in phase with frequency ω_1 and if $A_1 = 0$, the two masses oscillate with frequency ω_2 opposite to each other, *i.e.*, out of phase by π radians. ***The two such modes of oscillation involving a single frequency are called normal modes of vibration of the system***. Thus *for a given normal mode, all the coordinates (x_1 and x_2) oscillate with the same frequency.*

Adding eqs. (18) and (19), we obtain

$$m\frac{d^2}{dt^2}(x_1 + x_2) + k(x_1 + x_2) = 0$$

or
$$\frac{d^2}{dt^2}(x_1 + x_2) + \omega_0^2(x_1 + x_2) = 0 \quad ...(33)$$

Subtracting eq. (19) from eq. (18), we get

$$m\frac{d^2}{dt^2}(x_1 - x_2) + (k + 2k')(x_1 - x_2) = 0$$

or

$$\frac{d^2}{dt^2}(x_1 - x_2) + \left(\omega_0^2 + 2\omega_c^2\right)(x_1 - x_2) = 0 \qquad ...(34)$$

where $\omega_c^2 = k'/m$.

Let us define two new coordinates

$$X_1 = \frac{x_1 + x_2}{2} \text{ and } X_2 = \frac{x_1 - x_2}{2} \qquad ...(35)$$

Then eqs. (33) and (34) take the form

$$\frac{d^2 X_1}{dt^2} + \omega_1^2 X_1 = 0 \text{ and } \frac{d^2 X_2}{dt^2} + \omega_2^2 X_2 = 0 \qquad ...(36)$$

We see that the motion of the coupled system is now described by two uncoupled differential equations (36), each of which describes a simple harmonic motion of single frequency (ω_1 or ω_2) in terms of single coordinate (X_1 or X_2).

The solutions of the eqs. (36) are of the form

$$X_1 = A_1 \cos(\omega_1 t + \phi_1) \qquad ...(37\ a)$$

and

$$X_2 = A_2 \cos(\omega_2 t + \phi_2) \qquad ...(37\ b)$$

These two simple harmonic motions, obtained after decoupling the coupled equations, are called ***normal modes*** or simply ***modes***. Each mode of vibration has its ***normal frequency*** ($\omega_1$or ω_2) and is described by a coordinate (X_1or X_2), known as ***normal coordinate***. The amplitude and phase constant for *Mode* 1 are A_1 and ϕ_1 and for *Mode* 2, A_2 and ϕ_2 respectively.

The general motion of the oscillating system is expressed by the coordinates x_1 and x_2 [from eqs. (35)] :

$$x_1 = X_1 + X_2 \text{ and } x_2 = X_1 - X_2 \qquad ...(38)$$

So that in view of eqs. (37 *a*) and (37 *b*), we have

$$x_1 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \qquad ...(39\ a)$$

and

$$x_2 = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \qquad ...(39\ b)$$

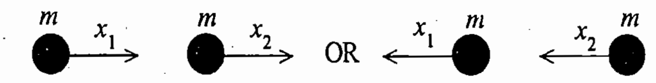
These equations are to be identified with eqs. (30).Thus ***we find that the displacement of any mass is a linear combination or superposition of two modes X_1 and X_2, oscillating simultaneously.***

It is to be noted that when the system is oscillating in one mode or the other (with A_1= 0 or A_2= 0), we observe from eqs. (31) and (32) that only in these two cases, both of the masses (or x_1 and x_2 coordinates) are executing simple harmonic motions. In all other states of vibration, the displacement of each mass (x_1 or x_2), given by eqs. (39 *a*) and (39 *b*), depends on both mode frequencies (ω_1 and ω_2) and hence the motion is no longer simple harmonic.

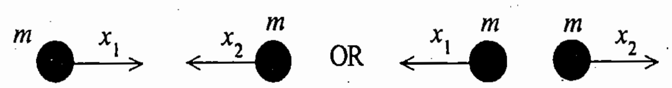
Symmetric and antisymmetric modes : If one mode is absent, then only the other mode describes the motion. For X_2= 0, X_1 coordinate is responsible for the motion. In this case from eq. (35) and (37), we have

$$X_2 = \frac{x_1 - x_2}{2} = 0 \text{ or } x_1 = x_2$$

and $$X_1 = \frac{x_1 + x_2}{2} = x_1 = x_2 = A_1 \cos(\omega_1 t + \phi_1)$$



(a) Symmetric mode (Mode 1) : $X_2 = 0, x_1 = x_2 = X_1 = A_1 \cos(\omega_1 t + \phi_1)$



(b) Antisymmetric mode (Mode 2) : $X_1 = 0, x_1 = -x_2 = X_2 = A_2 \cos(\omega_2 t + \phi_2)$

Fig. 9.3 : Normal modes of two coupled masses

Thus in **Mode 1,** both masses have equal displacements, have the same frequency ω_1 $(= \sqrt{k/m})$ and keep in phase. This is called ***symmetric mode*** and is shown in Fig. 9.3 (*a*).

For $X_1 = 0$, X_2 coordinate describes the motion. In this case

$$X_1 = \frac{x_1 + x_2}{2} = 0 \quad \text{or} \quad x_1 = -x_2$$

and $$X_2 = \frac{x_1 - x_2}{2} = x_1 = -x_2 = A_2 \cos(\omega_2 t + \phi_2)$$

Thus in **Mode 2**, both masses have equal and opposite displacements, but oscillate with the same frequency $\omega_2 (= \sqrt{(k + 2k')/m})$. This is called ***anti-symmetric mode*** and is shown in Fig. 9.3 (*b*).

We observe that in a symmetric mode, the two oscillators vibrate as if there were no coupling between them and their frequency is the same as the frequency of a single spring-mass system. In the antisymmetric mode, the coupling is working and the oscillators are vibrating out of phase with a frequency higher than the frequency of a single spring-mass system.

In order to excite a symmetric mode, the two masses are to be pulled from their equilibrium positions by equal amounts in the same direction and then released. For the excitation of an antisymmetric mode, the two masses should be pulled apart equally in opposite directions and then allowed to oscillate. In general, the motion of the two coupled oscillators will be a superposition of these two symmetric and antisymmetric modes.

9.3.3. Kinetic and Potential Energies in Normal Coordinates

Kinetic energy
$$T = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m\dot{x}_2^2$$
$$= \frac{1}{2} m\left[\dot{X}_1 + \dot{X}_2\right]^2 + \frac{1}{2} m\left[\dot{X}_1 - \dot{X}_2\right]^2$$
$$= \frac{1}{2} m\dot{X}_1^2 + \frac{1}{2} m\dot{X}_2^2 \qquad ...(40)$$

Potential energy
$$V = \frac{1}{2} kx_1^2 + \frac{1}{2} kx_2^2 + \frac{1}{2} k'(x_1 - x_2)^2$$
$$= \frac{1}{2} k\left[(X_1 + X_2)^2 + (X_1 - X_2)^2\right] + \frac{1}{2} k'\left[2X_2\right]^2$$
$$= k\left[X_1^2 + X_2^2\right] + 2k' X_2^2 \qquad ...(41)$$

Lagrangian $$L = \frac{1}{2} m \left[\dot{X}_1^2 + \dot{X}_2^2 \right] - k \left[X_1^2 + X_2^2 \right] - 2\,k'\,X_2^2 \qquad \text{...(42)}$$

Thus ***when the kinetic and potential energies are expressed in terms of normal coordinates, no cross terms of normal coordinates are present, i.e., both T and V are homogeneous quadratic functions.***

The above example of two coupled oscillators is of fundamental importance in order to understand the general theory of small oscillations and the general procedure of transferring to normal coordinates.

9.4. GENERAL THEORY OF SMALL OSCILLATIONS

The potential energy of a conservative system, specified by n generalized coordinates $q_1, q_2, ..., q_n$, is represented as [eq. (1)].

$$V = (q_1, q_2, ..., q_n) \qquad \text{...(43)}$$

We are interested in the motion of the system, when the displacements of the particles are small from the position of stable equilibrium. We denote the displacements of the generalized coordinates from equilibrium position by u_i, *i.e.*,

$$q_i = q_i^0 + u_i \qquad \text{...(44)}$$

Since q_i^0 is fixed, u_i may be taken as new generalized coordinates of the motion. Expanding the potential energy about the position of equilibrium,we obtain

$$V(q_1,..,q_n) = V(q_1^0, q_2^0, ..., q_n^0) + \sum_{i=1}^{n} \left[\frac{\partial V}{\partial q_i} \right]_0 (q_i - q_i^0) + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial^2 V}{\partial q_i\, \partial q_j} \right]_0 (q_i - q_i^0)(q_j - q_j^0) + ... \qquad \text{...(45)}$$

In consequence of equilibrium [eq. (3)], $(\partial V/\partial q_i)_0 = 0$. First term in the expansion represents the potential energy in the equilibrium position and is constant for the system. Assuming the potential energy in the equilibrium to be zero and writing $u_i = q_i - q_i^0$ and $u_j = q_j - q_j^0$, we get

$$V = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} V_{ij}\, u_i\, u_j \qquad \text{...(46)}$$

where $V_{ij} = \left[\frac{\partial^2 V}{\partial q_i\, \partial q_j} \right]_0 = \left[\frac{\partial^2 V}{\partial u_i\, \partial u_j} \right]_0$ = constant which is to be evaluated at $q_i = q_i^0$ and $q_j = q_j^0$.

The constant $V_{ij} = V_{ji}$ form a symmetric matrix V. In eq. (46), we retain the terms quadratic in the coordinates.

The kinetic energy of the system is given by

$$T = \sum_i \sum_j \frac{1}{2} m_{ij}\, \dot{q}_i\, \dot{q}_j = \sum_i \sum_j \frac{1}{2} m_{ij}\, \dot{u}_i\, \dot{u}_j \qquad \text{...(47)}$$

because the generalized coordinates do not involve time explicitly and therefore the kinetic energy is a homogeneous quadratic function of generalized velocities.

The coefficients are, in general, functions of generalized coordinates and therefore expanding m_{ij} in Taylor's series, we get

$$m_{ij}(q_1, ..., q_n) = m_{ij}(q_0^1, ..., q_0^n) + \sum_{k=1}^{n} \left[\frac{\partial m_{ij}}{\partial q_k} \right]_0 u_k + ... \qquad \text{...(48)}$$

In eq. (47), the term is already quadratic in the u_i's, we obtain the lowest non-vanishing approximation to

T in quadratic form only by retaining the first term in the expansion. If the constant values of the function m_{ij} are denoted by T_{ij}, then the kinetic energy is

$$T = \frac{1}{2}\sum_i \sum_j T_{ij}\dot{u}_i\dot{u}_j \qquad ...(49)$$

Obviously the constants T_{ij} are elements of symmetric matrix $\boldsymbol{T}$. Now, the Lagrangian L (= $T - V$) can be written as

$$T = \frac{1}{2}\sum_i \sum_j \left[T_{ij}\dot{u}_i\dot{u}_j - V_{ij}u_iu_j\right] \qquad ...(50)$$

Using u_i's as generalized coordinates, the Lagrange's equations

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_i} = 0$$

take the form

$$\sum_{j=1}^{n}\left[T_{ij}\ddot{u}_j + V_{ij}u_j\right] = 0 \qquad ...(51)$$

For $i = 1, 2, ..., n$, eqs. (51) represent n equations which are to be solved to obtain the motion near the position of equilibrium.

9.4.1. Secular Equation and Eigen value Equation

We try an oscillatory solution of eq. (51) in the form

$$u_i = Ca_ie^{i\omega t} \qquad ...(52)$$

where Ca_i is the complex amplitude of the oscillation for each coordinate u_i, the factor C being used for convenience as a scale factor, the same for all the coordinates.

Substituting for u_j from eq. (52) into eq. (51), we obtain

$$\sum_{j=1}^{n}\left[V_{ij}a_je^{i\omega t} - \omega^2T_{ij}a_je^{i\omega t}\right] = 0 \text{ or } e^{i\omega t}\sum_{j=1}^{n}\left[V_{ij}a_j - \omega^2\, T_{ij}a_j\right] = 0$$

In general, $e^{i\omega t}$ is not zero, hence

$$\sum_{j=1}^{n}\left[V_{ij}a_j - \omega^2\, T_{ij}a_j\right] = 0 \qquad ...(53\,a)$$

or in matrix form

$$\boldsymbol{Va} - \omega^2\boldsymbol{Ta} = 0 \qquad ...(53\,b)$$

where the matrix V, T and $\boldsymbol{a}$ are

$$V = \begin{pmatrix} V_{11} & V_{12} \ldots V_{1n} \\ V_{21} & V_{22} \ldots V_{2n} \\ \ldots & \ldots \\ V_{n1} & V_{n2} \ldots V_{nn} \end{pmatrix}, \; \boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} \ldots T_{1n} \\ T_{21} & T_{22} \ldots T_{2n} \\ \ldots & \ldots \\ T_{n1} & T_{n2} \ldots T_{nn} \end{pmatrix} \text{ and } \boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \ldots \\ a_n \end{pmatrix}$$

Eqs. (53) represent n linear, homogeneous, algebraic equations in a_j and ω, *i.e.*,

$$\begin{aligned}
&\left[V_{11}-\omega^2 T_{11}\right]a_1+\left[V_{12}-\omega^2 T_{12}\right]a_2+\ldots+\left[V_{1n}-\omega^2 T_{1n}\right]a_n=0\\
&\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\\
&\left[V_{n1}-\omega^2 T_{n1}\right]a_1+\left[V_{n2}-\omega^2 T_{n2}\right]a_2+\ldots+\left[V_{nn}-\omega^2 T_{nn}\right]a_n=0
\end{aligned} \qquad \ldots(53\text{ c})$$

Let us assume that inverse of $\boldsymbol{T}$ matrix exists. Multiplying eq. (53 b) by $\boldsymbol{T}^{-1}$, we get

$$\boldsymbol{T}^{-1}\boldsymbol{V}\boldsymbol{a}-\omega^2\boldsymbol{T}^{-1}\boldsymbol{T}\boldsymbol{a}=0$$

Since $\boldsymbol{T}^{-1}\boldsymbol{T}=\boldsymbol{I}$, unit matrix and $\boldsymbol{T}^{-1}\boldsymbol{V}=\boldsymbol{P}$ (say), then

$$\boldsymbol{P}\boldsymbol{a}-\omega^2\boldsymbol{I}\boldsymbol{a}=0 \text{ or } (\boldsymbol{P}-\omega^2\boldsymbol{I})\boldsymbol{a}=0 \qquad \ldots(54)$$

Eq. (54) is the ***eigen value equation***. Here ω^2 are the eigenvalues of $\boldsymbol{P}$ and $\boldsymbol{a}$ is the eigenvector with n components.

9.4.2. Solution of the Eigenvalue Equation

The eigenvalues are obtained by solving the determinant

$$|\boldsymbol{P}-\omega^2\boldsymbol{I}|=0 \qquad \ldots(55\text{ a})$$

or

$$|\boldsymbol{V}-\omega^2\boldsymbol{T}|=0$$

or

$$\begin{vmatrix}
V_{11}-\omega^2 T_{11} & V_{12}-\omega^2 T_{12}\ldots\ldots V_{1n}-\omega^2 T_{1n}\\
V_{21}-\omega^2 T_{21} & V_{22}-\omega^2 T_{22}\ldots\ldots\ldots\ldots\ldots\ldots\\
\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
V_{n1}-\omega^2 T_{n1}\ldots\ldots & \ldots\ldots\ldots\ldots V_{nn}-\omega^2 T_{nn}
\end{vmatrix}=0 \qquad \ldots(55\text{ b})$$

Eq. (55) is called ***secular equation***. This determinantal condition is in effect an algebraic equation of nth degree for ω^2 and the roots of the determinant provide n frequencies $(\omega_1^2, \omega_2^2, \ldots, \omega_n^2)$. These values are the ***normal mode frequencies***, mentioned in the last article.

For each of the value of ω^2, say ω_k^2 ($k = 1, 2, \ldots, n$) in the kth mode of vibration, eqs. (53 c) may be solved for amplitudes a_i. Corresponding to this ω_k^2, we denote the amplitudes by a_{ik} ($i = 1, 2, \ldots, n$). Thus a_{ik} is the amplitude in the kth mode of the ith coordinate. Only for $\omega_k^2 > 0$, the motion is oscillatory about the position of stable equilibrium.

In order to find the amplitudes a_{ik}, we use eqs.(54) for a particular value of ω, say ω_1 and then we know $a_{11}, a_{21}, \ldots, a_{n1}$. Similarly for ω_2, $a_{12}, a_{22}, \ldots, a_{n2}$ and for ω_n, $a_{1n}, a_{2n}, \ldots, a_{nn}$ are known. More correctly speaking, we may find $n-1$ amplitudes for a particular frequency. For example, for frequency ω_k, we can determine all the amplitudes except one, say $a_{2k}, a_{3k}, \ldots, a_{nk}$ except a_{1k}. In other words, we may determine the coefficients a_{ik} in terms of a_{1k} in the form of ratios :

$$\frac{a_{2k}}{a_{1k}}, \frac{a_{3k}}{a_{1k}}, \ldots, \frac{a_{nk}}{a_{1k}} \qquad \ldots(56)$$

A general solution of equation of motion (53) involves a superposition of oscillations with all the permitted frequencies. Thus if the system is displaced slightly from the equilibrium position and then released, it performs small amplitude oscillations about the equilibrium position with frequencies $\omega_1, \omega_2, ..., \omega_n$. The solutions of the secular equation (55) are therefore often called as ***the frequencies of free vibrations*** or as the ***resonant frequencies*** of the system.

The general solution may now be written as

$$u_i = \sum_{k=1}^{n} C_k\, a_{ik}\, e^{i\omega_k t} \qquad ...(57)$$

where we have used index k for summation for displacements due to all the allowed frequencies. Corresponding to the normal frequency ω_k (kth mode of vibration), the eigenvector is $\boldsymbol{a}_k$ with n components given by the matrix

$$\boldsymbol{a}_k = \begin{bmatrix} a_{1k} \\ a_{2k} \\ \cdots \\ a_{nk} \end{bmatrix} \qquad ...(58)$$

For the oscillating system, there are n eigen vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_k, ..., \boldsymbol{a}_n$, where $\boldsymbol{a}_k$ is given by (58). Thus in all there are $n \times n$ eigenvector components for the system, which may be represented by the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots. a_{1n} \\ a_{21} & a_{22} \ldots. a_{2n} \\ \multicolumn{2}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ a_{n1} & a_{n2} \ldots. a_{nn} \end{bmatrix} \qquad ...(59)$$

Obviously one may say that for each solution ω_k^2 of the secular equation (55), there are two resonant frequencies $+\omega_k$ and $-\omega_k$. The eigenvector $\boldsymbol{a}_k$ is the same for the two frequencies, but the scaling factors C_k^+ and C_k^- may be much different. Thus the general solution should be

$$u_i = \sum_{k=1}^{n} a_{ik}\left[C_k^+ e^{+i\omega_k t} + C_k^- e^{-i\omega_k t}\right] \qquad ...(60)$$

The actual motion is the real part of the complex solution (60) which can be expressed as

$$u_i = \sum_{k=1}^{n} f_k a_{ik} \cos(\omega_k t + \phi_k) \qquad ...(61)$$

where f_k and ϕ_k are determined from initial conditions.

9.4.3. Small Oscillations in Normal Coordinates

Let us define

$$u_i = \sum_{k=1}^{n} a_{ik} Q_k \qquad ...(62)$$

In terms of single column matrices

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{Q} = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{bmatrix}$$

we have, $u = AQ$ ...(63)

The potential energy V can be written as

$$V = \frac{1}{2}\sum_i \sum_j V_{ij}\, u_i u_j = \frac{1}{2}\sum_i \sum_j u_i V_{ij}\, u_j \text{ or } V = \frac{1}{2}\bar{u}\, V u \qquad ...(64)$$

where $\bar{u}$ is the transpose of u or single row matrix.

From eq. (63) $\bar{u} = \overline{A\,Q} = \bar{Q}\,\bar{A}$

Therefore $V = \frac{1}{2}\bar{Q}\,\bar{A}\, V A\, Q$ ...(65)

The kinetic energy K similarly is

$$T = \sum_i \sum_j \dot{u}_i T_{ij} \dot{u}_j = \frac{1}{2}\bar{\dot{Q}}\,\bar{A}\, T A\, \dot{Q} \qquad ...(66)$$

From eq. (53), writing $\omega_k^2 = \lambda_k$,

$$\sum_{j=1}^{n}\left[V_{ij} a_{jk} - \lambda_k T_{ij} a_{jk}\right] = 0 \qquad ...(67)$$

The complex conjugate of this equation is

$$\sum_{i=1}^{n}\left[V_{ij}\, a_{il}^* - \lambda_l^*\, T_{ij}\, a_{il}^*\right] = 0 \qquad ...(68)$$

As a_{ij} are real, we eliminate V_{ij} from (67) and (68) by multiplying the former by a_{il} and summing over i and the latter by a_{jk} and summing over j. Thus

$$(\lambda_k - \lambda_l^*)\ \sum_i \sum_j a_{jk}\, T_{ij}\, a_{il} = 0 \qquad ...(69)$$

If all λ_k are distinct, *i.e.*, $(\lambda_k - \lambda_l^*)$ is not zero, then

$$\sum_i \sum_j a_{jk}\, T_{ij}\, a_{il} = 0 \qquad ...(70)$$

The coefficients a_{jk} in eq. (67) cannot be completely determined, because this is a set of linear equations. This indeterminacy can be removed by requiring that

$$\sum_i \sum_j a_{jk}\, T_{ij}\, a_{ik} = 1 \qquad ...(71)$$

The two equations (70) and (71) can be combined into one by means of Kronecker delta symbol δ_{kl}, *i.e.*,

$$\sum_i \sum_j a_{jk}\, T_{ij}\, a_{il} = \delta_{kl} \qquad ...(72)$$

Eqs. (70) and (71) can be written as

$$\bar{A}\ T A = I \qquad ...(73)$$

Writing $\lambda_l = \lambda_k \delta_{lk}$, we obtain from eq. (67)

$$\sum_{j=1}^{n} V_{ij}\, a_{jk} = \sum_{j=1}^{n} T_{ij} a_{jk} \lambda_k \delta_{lk} \qquad ...(74\ a)$$

which is in matrix notation

$$V A = T A\, \lambda \qquad ...(74\ b)$$

Multiplying by $\overline{A}$ from left, we get

$$\overline{A}\,V A = \overline{A}\,T A\,\lambda \qquad \text{...(75)}$$

But $\overline{A}\,T A = I$ [eq. (73)],

$$\overline{A}\,V A = \lambda \qquad \text{...(76)}$$

In view of eq. (76), eq. (65) is obtained to be

$$V = \frac{1}{2}\overline{Q}\,\lambda\,Q = \frac{1}{2}(Q_1, Q_2 \ldots, Q_n)\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix} \bullet \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix}$$

$$= \frac{1}{2}(\lambda_1 Q_1^2 + \lambda_2 Q_2^2 + \ldots + \lambda_n Q_n^2)$$

$$= \frac{1}{2}\sum_{k=1}^{n}\lambda_k Q_k^2 = \frac{1}{2}\sum_{k=1}^{n}\omega_k^2 Q_k^2 \qquad \text{...(77)}$$

Similarly from eqs. (66) and (73), we have

$$T = \frac{1}{2}\overline{\dot{Q}}\,I\,\dot{Q} = \frac{1}{2}\sum_{k=1}^{n}\dot{Q}_k^2 \qquad \text{...(78)}$$

We see from eqs. (77) and (78) that in the new coordinates, both the potential and kinetic energies are the sums of squares only without any cross terms.

Now, the Lagrangian $L = T - V$ is

$$L = \frac{1}{2}\sum_{k=1}^{n}\dot{Q}_k^2 - \frac{1}{2}\sum_{k=1}^{n}\omega_k^2 Q_k^2 \qquad \text{...(79)}$$

Hence the Lagrangian equations

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{Q}_k}\right] - \frac{\partial L}{\partial Q_k} = 0$$

for the new coordinates are

$$\ddot{Q}_k + \omega_k^2 Q_k = 0 \qquad \text{...(80)}$$

which are n equations for $k = 1, 2, \ldots, n$.

Thus each new coordinate executes simple harmonic motion with a single frequency and therefore, Q_1, $Q_2, \ldots, Q_n$ are called ***normal coordinates***. The frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are referred as ***normal frequencies***.

The solution of eq. (80) is

$$Q_k = f_k \cos(\omega_k t + \phi_k) \qquad \text{...(81)}$$

From eqs. (81) and (62), we see

$$u_i = \sum_{k=1}^{n} a_{lk} Q_k = \sum_{k=1}^{n} f_k\, a_{lk} \cos(\omega_k t + \phi_k) \qquad \text{...(82)}$$

Thus (81) could have been obtained directly from (61) and (62).

It may be reminded again that each normal coordinate corresponds to a vibration of the system with only one frequency and these component oscillations are called as the ***normal modes of vibration***. *In each mode all the particles vibrate with the same frequency and with the same phase**, the relative amplitudes being determined by *the matrix elements* a_{ik}. *The complete motion is then composed of sum of the normal modes* weighted with proper amplitude and phase factors contained in the scaling factors C_k's.

9.5. EXAMPLES OF TWO COUPLED OSCILLATORS

We have already discussed longitudinal oscillations of two coupled masses. We discuss below some other important examples of two coupled oscillators.

(1) Two coupled pendulums : Consider two identical pendulums as shown in Fig 9.4. Each pendulum has a bob of mass m with an effective length l. The two bobs of the pendulums are connected by a light spring of force constant k. The relaxed length of the spring is equal to the distance between the two bobs at equilibrium. We shall consider small amplitude oscillations, restricted to the plane in equilibrium configuration. Thus the system of two coupled pendulums, under consideration, has two degrees of freedom.

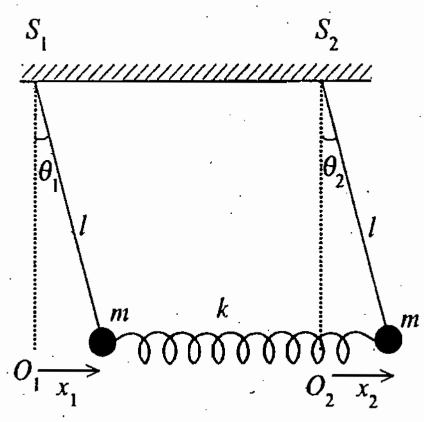


Fig. 9.4 : Two coupled pendulums

Let the system of two coupled pendulums be allowed to oscillate so that x_1 and x_2 represent displacements from the equilibrium positions O_1 and O_2 respectively. If θ_1 and θ_2 be the angular displacements at any instant t, then the potential energy of the system is given by

$$V = mgl\,(1-\cos\theta_1) + mgl\,(1-\cos\theta_2) + \frac{1}{2}k\,(x_1 - x_2)^2$$

where the potential energy in the equilibrium configuration is assumed to be zero. For small amplitude oscillations.

$$1-\cos\theta_1 = 1-\left(1-\theta_1^2/2\right) = \theta_1^2/2 = x_1^2/2l^2$$

and similarily $1-\cos\theta_2 = x_2^2/2l^2$, where $\theta_1 = x_1/l$ and $\theta_2 = x_2/l$.

Thus
$$V = \frac{1}{2}\frac{mg}{l}x_1^2 + \frac{1}{2}\frac{mg}{l}x_2^2 + \frac{1}{2}k(x_1 - x_2)^2 \qquad ...(83)$$

The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \qquad ...(84)$$

The V and T matrices for the system are

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \text{ and } T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

Here
$$V_{11} = \left[\frac{\partial^2 V}{\partial x_1^2}\right]_{\substack{x_1=0\\x_2=0}} = k + \frac{mg}{l}, \quad V_{12} = \left[\frac{\partial^2 V}{\partial x_1\,\partial x_2}\right]_0 = -k,$$

$$V_{21} = \left[\frac{\partial^2 V}{\partial x_2 \partial x_1}\right]_0 = -k, \quad V_{22} = \left[\frac{\partial^2 V}{\partial x_2^2}\right]_0 = k + \frac{mg}{l}$$

* The particles may be out of phase, if the a's have opposite sign.

Since $$T = \frac{1}{2}\left[T_{11}\dot{x}_1^2 + T_{12}\,\dot{x}_1\,\dot{x}_2 + T_{21}\,\dot{x}_1\,\dot{x}_2 + T_{22}\dot{x}_2^2\right],$$

$$T_{11} = m = T_{22} \text{ and } T_{12} = T_{21} = 0$$

Thus $$V = \begin{pmatrix} k + \dfrac{mg}{l} & -k \\ -k & k + \dfrac{mg}{l} \end{pmatrix} \text{ and } T = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad ...(85)$$

The normal frequencies are determined from the equation

$$|V - \omega^2 T| = 0$$

Thus $$\begin{vmatrix} k + \dfrac{mg}{l} - m\omega^2 & -k \\ -k & k + \dfrac{mg}{l} - m\omega^2 \end{vmatrix} = 0$$

or $$\left[k + \frac{mg}{l} - m\omega^2\right]^2 - k^2 = 0 \text{ or } \left[\frac{mg}{l} - m\omega^2\right]\left[2k + \frac{mg}{l} - m\omega^2\right] = 0 \qquad ...(86)$$

which gives

$$\omega^2 = \omega_1^2 = \frac{g}{l} \text{ and } \omega^2 = \omega_2^2 = \frac{g}{l} + \frac{2k}{m}$$

or $$\omega_1 = \pm\sqrt{\frac{g}{l}} \text{ and } \omega_2 = \pm\sqrt{\frac{g}{l} + \frac{2k}{m}}$$

Thus the normal frequencies of the system are

$$\omega_1 = \sqrt{\frac{g}{l}} \text{ and } \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}} \qquad ...(87)$$

To determine the eigenvectors, we use the equation

$$[V - \omega_k^2\, T]\, a_k = 0$$

or $$\begin{pmatrix} k + \dfrac{mg}{l} - m\omega_k^2 & -k \\ -k & k + \dfrac{mg}{l} - m\omega_k^2 \end{pmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \end{pmatrix} = 0$$

For $\omega^2 = \omega_1^2 = g/l$, we have

$$\begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = 0 \quad \text{or} \quad \frac{a_{21}}{a_{11}} = 1$$

If $a_{11} = \alpha$, then $a_{21} = \alpha$.

For $\omega^2 = \omega_2^2 = \dfrac{g}{l} + \dfrac{2k}{m}$, we have

$$\begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = 0 \quad \text{or} \quad \frac{a_{22}}{a_{12}} = -1$$

If $a_{12} = \beta$, then $a_{22} = -\beta$.

Thus the eigenvectors are

$$a_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} \beta \\ -\beta \end{pmatrix} \qquad ...(88)$$

Now, the matrix A is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix}$$

Therefore, the transpose of A matrix i.e., $\overline{A}$ is

$$\overline{A} = \begin{bmatrix} \alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

We impose the condition $\overline{A}\ T\ A = I$ i.e.,

$$\begin{pmatrix} \alpha & \alpha \\ \beta & -\beta \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 2m\alpha^2 & 0 \\ 0 & 2m\beta^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

whence $\alpha = \beta = 1/\sqrt{2m}$.

Thus the eigenvecfors are

$$a_1 = \frac{1}{\sqrt{2m}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad a_2 = \frac{1}{\sqrt{2m}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad ...(89)$$

The generalized coordinates x_1 and x_2 are related to normal coordinates Q_1 and Q_2 by using the relation :

$$u_i = \sum_{k=1}^{2} a_{ik} Q_k$$

where for $i = 1, 2$, $u_1 = x_1$ and $u_2 = x_2$.

Therefore,

$$x_1 = a_{11} Q_1 + a_{12} Q_2 \quad \text{and} \quad x_2 = a_{21} Q_1 + a_{22} Q_2$$

or

$$x_1 = \frac{1}{\sqrt{2m}} Q_1 + \frac{1}{\sqrt{2m}} Q_2 \quad \text{and} \quad x_2 = \frac{1}{\sqrt{2m}} Q_1 - \frac{1}{\sqrt{2m}} Q_2 \qquad ...(90)$$

Hence the normal coordinates Q_1 and Q_2 are

$$Q_1 = \sqrt{2m}\,(x_1 + x_2) \quad \text{and} \quad Q_2 = \sqrt{2m}\,(x_1 - x_2) \qquad ...(91)$$

Further the normal coordinates Q_1 oscillates with frequency ω_1 and Q_2 with ω_2. So that

$$Q_1 = f_1 \cos(\omega_1 t + \phi_1) \text{ and } Q_2 = f_2 \cos(\omega_2 t + \phi_2) \qquad ...(92)$$

Thus
$$x_1 = \frac{f_1}{\sqrt{2m}} \cos(\omega_1 t + \phi_1) + \frac{f_2}{\sqrt{2m}} \cos(\omega_2 t + \phi_2) \qquad ...(93\,a)$$

and
$$x_2 = \frac{f_1}{\sqrt{2m}} \cos(\omega_1 t + \phi_1) - \frac{f_2}{\sqrt{2m}} \cos(\omega_2 t + \phi_2) \qquad ...(93\,b)$$

Putting $f_1/\sqrt{2m} = A_1$ and $f_2/\sqrt{2m} = A_2$, we get

$$x_1 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \qquad ...(94\,a)$$

$$x_2 = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \qquad ...(94\,b)$$

Thus the displacement of a pendulum is obtained by the superposition of harmonic oscillations of ω_1 and ω_2 frequencies.

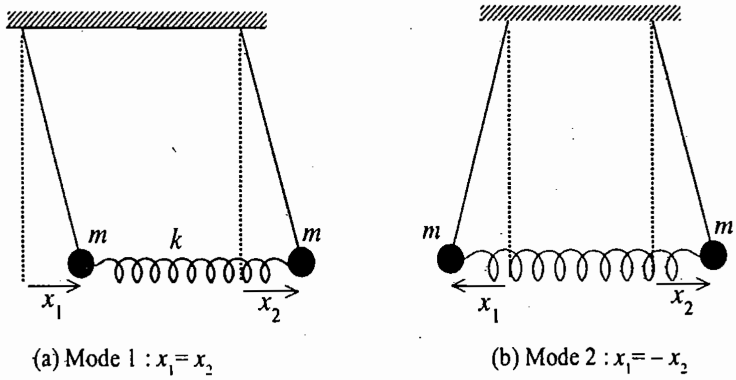


Fig. 9.5 : Normal modes of two coupled pendulums : (a) in same phase, (b) out of phase.

Eqs. (87), (89), (91) and (92) completely describe the motion. Eqs. (93) [or (94)] are the result of eqs. (91) and (92).

We may discuss the normal modes of vibration similar to Art. 9.3.2. If we put $Q_2 = 0$ (or f_2 or A_2), then from eq. (91)

$$x_1 = x_2$$

which means that in Mode 1 (Q_1), the two pendula oscillate with the same frequency $\omega_1 = \sqrt{g/l}$ in the same phase.

If we put $Q_1 = 0$ (or f_1 or A_1), then from eq. (91), we have

$$x_1 = -x_2$$

This means that in Mode 2 (Q_2), the two pendula oscillate exactly out of phase (with a phase difference of π) with the same frequency $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$. It is to be noted that in Mode 1 ($x_1 = x_2$), there is no stretching or compression of the spring so that ω_1 does not depend on spring constant k, while in Mode 2, due to compression or stretching of the spring the force constant contributes in ω_2.

(2) Double Pendulum : A double pendulum consists of a pendulum of mass m_1 and length l_1 to which a second pendulum of mass m_2 and length l_2 is suspended [Fig 9.6]. The motion is considered in a plane so that the system has two degrees of freedom. If (x_1, y_1) and (x_2, y_2) be the coordinates of the masses m_1 and m_2 respectively, then from the figure, we have

$$x_1 = l_1 \sin\theta_1,\ x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

and
$$y_1 = l_1 \cos\theta_1,\ y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

where θ_1 and θ_2 are the angles, made by the lengths of the pendulums with the vertical. These are taken as generalized coordinates.

Thus the potential energy of the system is

$$V = -m_1 g y_1 - m_2 g y_2$$
$$= -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

where the potential energy is considered to be zero at O.

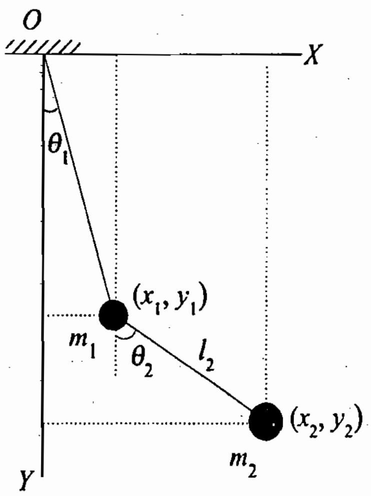


Fig. 9.6 : Double pendulum

For small θ_1 and
$$\cos\theta_1 = 1 - \theta_1^2/2\,\theta_2,$$

and
$$\cos\theta_2 = 1 - \theta_2^2/2$$

Therefore, $V = -m_1 g l_1 - m_2 g (l_1 + l_2) + \frac{1}{2} m_1 g l_1 \theta_1^2 + \frac{1}{2} m_2 g l_2 \theta_2^2$

The V matrix is
$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

Here
$$V_{11} = \left[\partial^2 V/\partial\theta_1^2\right]_{\substack{\theta_1=0\\ \theta_2=0}} = (m_1 + m_2) g l_1,$$

$$V_{12} = V_{21} = 0 \text{ and } V_{22} = \left[\partial^2 V/\partial\theta_2^2\right]_{\substack{\theta_1=0\\ \theta_2=0}} = m_2 g l_2$$

Therefore,
$$V = \begin{pmatrix} (m_1 + m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix} \qquad ...(95)$$

The kinetic energy of the system is

$$T = \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]$$

As θ_1 and θ_2 are small $\cos(\theta_1 - \theta_2) \simeq 1$,

$$T = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2$$

Since for two degrees of freedom

$$T = \frac{1}{2} T_{11} \dot{\theta}_1^2 + \frac{1}{2} T_{12} \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} T_{21} \dot{\theta}_2 \dot{\theta}_1 + \frac{1}{2} T_{22} \dot{\theta}_2^2,$$

therefore, $T_{11} = [m_1 + m_2]\, l_1^2,\ T_{12} = T_{21} = m_2 l_1 l_2$ and $T_2 = m_2 l_2^2$

Thus
$$T = \begin{pmatrix} (m_1 + m_2)\, l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix} \qquad ...(96)$$

The normal mode frequencies are determined from the equation

$$|V-\omega^2 T|=0 \text{ or } \begin{bmatrix} (m_1+m_2)gl_1-\omega^2(m_1+m_2)l_1^2 & -\omega_2^2 m_2 l_1 l_2 \\ -\omega^2 m_2 l_1 l_2 & m_2 g l_2-\omega^2 m_2 l_2^2 \end{bmatrix}=0$$

Dividing by l_1 in the first row and $m_2 l_2$ in the second row, we get

$$[(m_1+m_2)g-\omega^2(m_1+m_2)l_1](g-\omega^2 l_2)-\omega^4 m_2 l_1 l_2=0$$

or
$$\omega^4 m_1 l_1 l_2-\omega^2 g(m_1+m_2)(l_1+l_2)+(m_1+m_2)g^2=0$$

or
$$\omega^2=\frac{g(m_1+m_2)(l_1+l_2)\pm\sqrt{[g(m_1+m_2)(l_1+l_2)]^2-4m_1(m_1+m_2)l_1 l_2 g^2}}{2m_1 l_1 l_2} \quad ...(97)$$

which gives two normal mode frequencies.

Now, we consider following three special cases for the determination of frequencies.

Case I : When $m_1 >> m_2$, then $m_1+m_2 \simeq m_1$

Now from eq. (97), we have

$$\omega^2=\frac{gm_1(l_1+l_2)\pm gm_1\sqrt{(l_1+l_2)^2-4\,l_1 l_2}}{2\,m_1\,l_1\,l_2}$$

whence the two normal frequencies are

$$\omega_1^2=g/l_2 \text{ and } \omega_2^2=g/l_1 \quad ...(98)$$

Case II : When $m_1 << m_2$, then $m_1+m_2 \simeq m_1$

Now from eq. (97), we obtain

$$\omega^2=\frac{gm_2(l_1+l_2)\pm g\sqrt{[m_2(l_1+l_2)]^2-4m_1m_2l_1l_2}}{2\,m_1l_1l_2}=\frac{gm_2(l_1+l_2)\left[1\pm\left(1-\frac{4m_1l_1l_2}{m_2(l_1+l_2)^2}\right)\right]^{\frac{1}{2}}}{2\,m_1\,l_1\,l_2}$$

$$=\frac{gm_2(l_1+l_2)\left[1\pm\left(1-\frac{2m_1l_1l_2}{m_2(l_1+l_2)^2}\right)\right]}{2\,m_1\,l_1\,l_2}$$

Hence $\quad \omega_1^2=\frac{gm_2}{m_1}\left[\frac{1}{l_1}+\frac{1}{l_2}\right]$ and $\omega_2^2=\frac{g}{l_1+l_2}$ ...(99)

Case III : When $m_1=m_2=m$ and $l_1=l_2=l$,

$$\omega^2=\frac{4gml\pm\sqrt{(4gml)^2-8m^2l^2g^2}}{2ml^2}=\frac{2g\pm g\sqrt{2}}{l}=\frac{g}{l}(2\pm\sqrt{2})$$

Thus the two normal frequencies are

$$\omega_1^2 = \frac{g}{l}\left[2+\sqrt{2}\right] \text{ and } \omega_2^2 = \frac{g}{l}\left[2-\sqrt{2}\right] \quad ...(100)$$

One can calculate the eigenvectors in any case following the example of two coupled pendulums.

9.6. VIBRATIONS OF A LINEAR TRIATOMIC MOLECULE

Let us consider a linear triatomic molecule of the type AB_2 (*e.g.*, CO_2) in which A atom is in the middle and B atoms are at the ends [Fig. 9.7]. The mass of A atom is M and that of each of the B atom is m. The interatomic force between A and B atom is approximated by elastic force of spring force constant k. The motion of the three atoms is constrained along the line joining them. There are three coordinates marking the positions of three atoms on the line. If x_1, x_2 and x_3 are the positions of the three atoms at any instant from some arbitrary origin, then

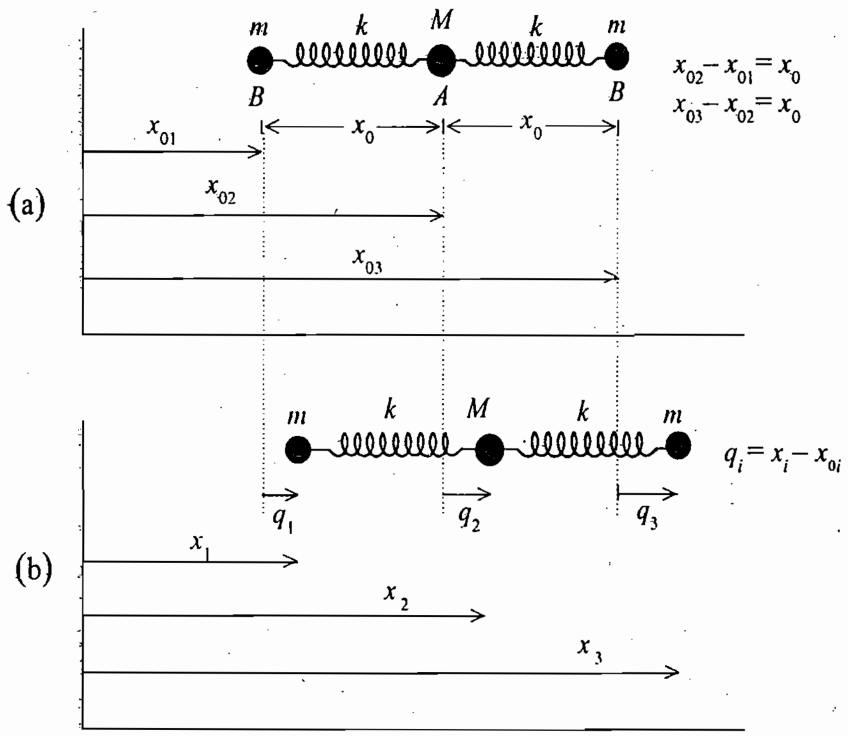


Fig. 9.7 : Longitudinal oscillations of a linear symmetric triatomic molecule : (a) Equilibrium configuration, (b) Configuration at any instatnt t

$$T = \frac{1}{2}m\left[\dot{x}_1^2+\dot{x}_3^2\right]+\frac{1}{2}M\dot{x}_2^2$$

and
$$V = \frac{1}{2}k(x_2-x_1-x_0)^2+\frac{1}{2}k(x_3-x_2-x_0)^2$$

where x_0 is the distance between any A and B atoms in the equilibrium configuration.

Let us define the generalized coordinates as

$$q_1 = x_1 - x_{01}, q_2 = x_2 - x_{02}, q_3 = x_3 - x_{03},$$

where
$$x_{02} - x_{01} = x_{03} - x_{01} = x_0.$$

Then
$$T = \frac{1}{2}m\,[\dot{q}_1^2+\dot{q}_3^2]+\frac{1}{2}M\dot{q}_2^2$$

and
$$V = \frac{1}{2}k(q_2-q_1)^2+\frac{1}{2}k(q_3-q_2)^2$$

Thus the T and V matrices are

$$T = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \text{ and } V = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \quad ...(101)$$

The secular equation is

$$|V - \omega^2 I| = \begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0 \quad ...(102)$$

whence $$\omega^2 (k - m\omega^2)[k(M + 2m) - \omega^2 Mm] = 0$$

The solutions of this equation are

$$\omega_1 = 0,\ \omega_2 = \sqrt{\frac{k}{m}} \text{ and } \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)} \quad ...(103)$$

The first eigen value $\omega_1 = 0$ corresponds to non-oscillatory motion and refers to translatory motion of the molecule as a whole rigidly.

To determine the eigenvectors, we use the equation

$$(V - \omega_k^2\, T)\, \boldsymbol{a}_k = 0 \text{ or } \begin{pmatrix} k - m\omega_k^2 & -k & 0 \\ -k & 2k - M\omega_k^2 & -k \\ 0 & -k & k - m\omega_k^2 \end{pmatrix} \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \end{pmatrix} = 0$$

Let us now discuss the eigen vectors for the three modes of vibrations.

(1) For $\omega_1 = 0$,

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = 0$$

or $$a_{11} - a_{21} = 0, -a_{11} + 2a_{21} - a_{31} = 0,\ -a_{21} + a_{31} = 0$$

or $$a_{11} = a_{21} = a_{31} = \alpha \quad \text{(say)}.$$

Thus for $\omega_1 = 0$, the eigen vector is given by

$$\boldsymbol{a}_1 = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \quad ...(104)$$

(2) For $\omega_2 = \sqrt{k/m}$,

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 2k - \dfrac{Mk}{m} & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = 0$$

or $$a_{22} = 0, \; -a_{12} - a_{32} = 0$$

Therefore, $a_{22} = 0$, $a_{12} = -a_{32} = \beta$ (say)

$$\text{Thus, for } \omega_2 = \sqrt{k/m},\; \boldsymbol{a}_2 = \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix} \qquad \text{...(105)}$$

(3) For $$\omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)},$$

$$\begin{pmatrix} -\frac{2mk}{M} & -k & 0 \\ -k & -\frac{kM}{m} & -k \\ 0 & -k & -\frac{2mk}{M} \end{pmatrix} \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

which yields

$$\frac{2m}{M} a_{13} + a_{23} = 0,\; a_{13} + \frac{M}{m} a_{23} + a_{33} = 0,\; a_{23} + \frac{2m}{M} a_{33} = 0$$

Therefore, $a_{13} = a_{33} = \gamma$ (say) and $a_{23} = -(2m/M)\gamma$

$$\text{Thus for } \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)},\; \boldsymbol{a}_3 = \begin{pmatrix} \gamma \\ -\frac{2m}{M}\gamma \\ \gamma \end{pmatrix} \qquad \text{...(106)}$$

Now, the A matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -\frac{2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \qquad \text{...(107)}$$

We impose the condition

$$\overline{A}\, T A = I$$

i.e.,

$$\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -\frac{2m}{M}\gamma & \gamma \end{pmatrix} \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & -\frac{2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or $$\begin{pmatrix} \alpha^2(2m+M) & 0 & 0 \\ 0 & 2\beta^2 m & 0 \\ 0 & 0 & 2\gamma^2 m\left(1+\dfrac{2m}{M}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $$\alpha = \frac{1}{\sqrt{2m+M}},\ \beta = \frac{1}{\sqrt{2m}}, \gamma = \frac{1}{\sqrt{2m(1+2m/M)}}.$$

Hence the eigen vectors are

$$a_1 = \frac{1}{\sqrt{2m+M}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, a_2 = \frac{1}{\sqrt{2m}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, a_3 = \frac{1}{\sqrt{2m\left(1+\dfrac{2m}{M}\right)}}\begin{bmatrix} 1 \\ -\dfrac{2m}{M} \\ 1 \end{bmatrix} \quad ...(108)$$

Thus in case (1), $a_{11} = a_{21} = a_{31}$ means that the displacements of three atoms are the same in the same direction [Fig. 9.8 (a)]. This is what expected from translatory motion.

In case (2) $a_{22} = 0$, and $a_{12} = -a_{32}$ implies that in this mode, the middle atom does not vibrate and the end atoms (B) oscillate with equal amplitudes but in opposite direction. In case (3),

$$a_{13} = a_{33} = \gamma \text{ and } a_{23} = -\left(\frac{2m}{M}\right)\gamma$$

show that end atoms oscillate in phase with equal amplitudes, while the central atom vibrates in opposite phase with different amplitude.

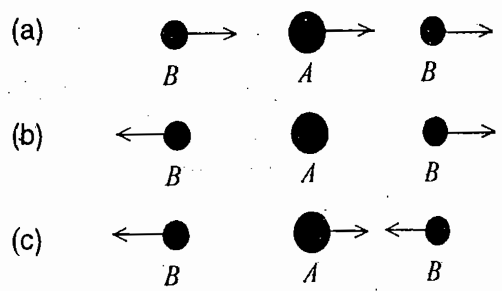


Fig. 9.8 : Longitudinal normal modes of the triatomic molecule :

(a) Mode 1, all the three atoms are displaced equally in the same direction,
(b) Mode 2, A atom does not vibrate and B atoms oscillate with equal amplitudes but in opposite directions,
(c) B atoms vibrate in phase with equal amplitudes and the middle atom A vibrates in opposite phase with different amplitude.

The generalized co-ordinates q_1, q_2 and q_3 are related to the normal coordinates Q_1, Q_2 and Q_3 by using the relation

$$q_i = \sum_{k=1}^{3} a_{ik}\, Q_k \text{ where } i = 1, 2, 3.$$

Therefore $$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \beta & 0 & -\dfrac{2m}{M}\gamma \\ \gamma & -\beta & \gamma \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \quad ...(109)$$

Further the normal coordinate Q_1 oscillates with frequency $\omega_1 = 0$, Q_2 with $\omega_2 = \sqrt{\dfrac{k}{m}}$ and Q_3 with $\omega_3 = \sqrt{\dfrac{k}{m}\left(1+\dfrac{2m}{M}\right)}$. So that

$$Q_1 = f_1 \cos(\omega_1 t + \phi_1),\ Q_2 = f_2 \cos(\omega_2 t + \phi_2) \text{ and}$$

$$Q_3 = f_3 \cos(\omega_3 t + \phi_3) \qquad ...(110)$$

Thus $\quad q_1 = \alpha\, f_1 \cos(\omega_1 t + \phi_1) + \beta\, f_2 \cos(\omega_2 t + \phi_2) + \gamma f_3 \cos(\omega_3 t + \phi_3)$

or $\quad x_1 = A \cos(\omega_1 t + \phi_1) + B \cos(\omega_2 t + \phi_2) + C \cos(\omega_3 t + \phi_3) + x_{01}$

But $\omega_1 = 0$, therefore

$$x_1 = A' + B \cos(\omega_2 t + \phi_2) + C \cos(\omega_3 t + \phi_3) + x_{01} \qquad ...(111a)$$

Similarly, $$x_2 = A' - \frac{2m}{M} C \cos(\omega_3 t + \phi_3) + x_{02} \qquad ...(111b)$$

and $$x_3 = A' - B \cos(\omega_2 t + \phi_2) + C \cos(\omega_3 t + \phi_3) + x_{03} \qquad ...(111c)$$

where A' represents a constant corresponding to rigid translation and x_{0i} the equilibrium position of an atom.

Thus we observe from eqs. (111) that any general longitudinal vibration of a molecule, if it does not involve a rigid translation, is some linear combination of the normal modes ω_2 and ω_3. The amplitudes of the normal modes and their phases relative to each other may be determined by the initial conditions.

Ex. 1. *Two identical simple pendulums, each of length* 0.5 *m, are connected by a light spring* [Fig. 9.9]. *The force constant of the spring is* $2Nm^{-1}$ *and the mass of each bob is* 0.1 *kg. If one pendulum is clamped, calculate the period of other pendulum. When the clamp is removed, determine the periods of two normal modes of the system.* ($g = 9.8$ m sec^{-2})

Solution : When A pendulum is clamped, B pendulum will oscillate under two restoring forces, (i) spring force ($-kx$) and (ii) gravitational force, $-mg \sin\theta = -mgx/l$ (for small oscillations). Thus the equation of motion of pendulum B is

$$m\frac{d^2x}{dt^2} = -kx - \frac{mgx}{l} \quad \text{or} \quad \frac{d^2x}{dt^2} + \left[\frac{k}{m} + \frac{g}{l}\right] x = 0$$

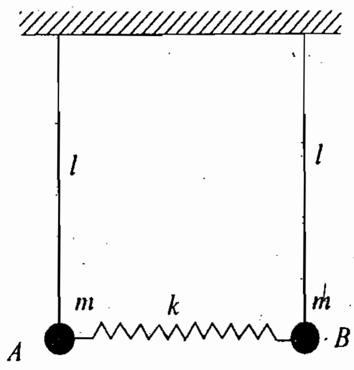


Fig. 9.9

which represents a simple harmonic motion of period, given by

$$T = 2\pi \Big/ \sqrt{\frac{k}{m} + \frac{g}{l}} = 2\pi \sqrt{\frac{2}{0.1} + \frac{9.8}{0.5}} = \mathbf{0.99\ sec.}$$

When the clamp is removed, then the system will act as two coupled oscillators, discussed in Sec. 9.5.

For normal Mode 1, $$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{g/l}} = \frac{2\pi}{\sqrt{9.8/0.5}} = \mathbf{1.42\ sec}$$

For normal Mode 2,

$$T_2 = \frac{2\pi}{\omega_2} = \frac{2\pi}{\sqrt{(2k/m) + (g/l)}} = \frac{2\pi}{\sqrt{(2 \times 2/0.1) + (9.8/0.5)}} = \mathbf{0.75\ sec.}$$

9.7. TRANSVERSE OSCILLATIONS OF N-COUPLED MASSES ON AN ELASTIC STRING : *MANY COUPLED OSCILLATORS*

Let us consider a flexible light elastic string to which N identical particles, each of mass m, are attached at equal distances. The string is fixed at $x = 0$ and $x = l$ and the particles are situated at $x = a, 2a, \ldots, na$ so that $l = (N+1)\,a$ [Fig. 9.10]. This is an example of many coupled oscillators. The string is stretched with tension force F, present at all points of the string at all times. We consider only small transverse oscillations, confined to the plane of the paper. Let the transverse displacement of the nth particle be ψ_n. Thus the system of particles, having displacements $\psi_1, \psi_2, \ldots, \psi_n, \ldots, \psi_N$, has the kinetic energy T, given by

$$T = \frac{1}{2} m \sum_{n=1}^{N} \dot{\psi}_n^2 \qquad \ldots(112)$$

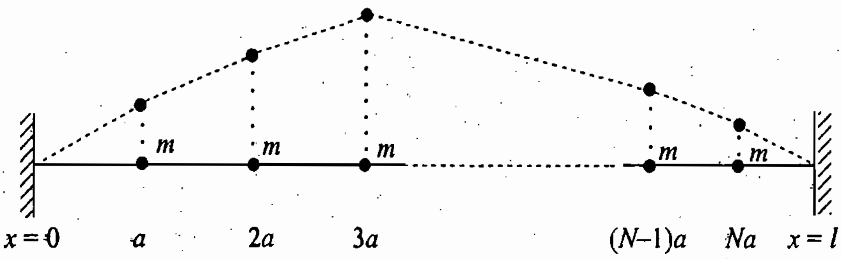


Fig. 9.10. System of N particles each of mass m on a light elastic string

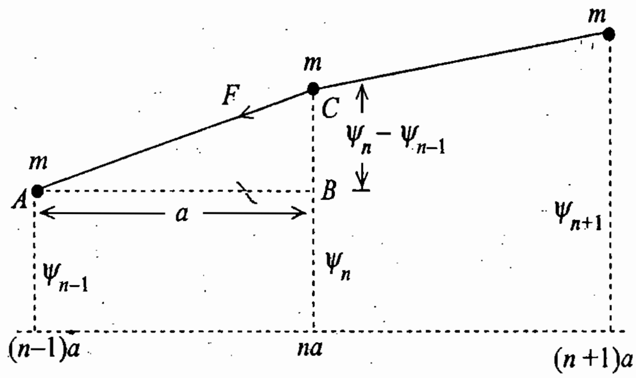


Fig. 9.11. Transverse oscillations of particles on a string

In order to find the potential energy we calculate the change in length of the string between $(n-1)$th and nth particles during oscillations of the system. The extension of the length of the string between the two particles [Fig. 9.11] is

$$\delta s = AC - AB = \sqrt{(\psi_n - \psi_{n-1})^2 + a^2} - a$$

$$= a\left[1 + \left(\frac{\psi_n - \psi_{n-1}}{a}\right)^2\right]^{\frac{1}{2}} - a = \frac{1}{2a}(\psi_n - \psi_{n-1})^2$$

where we have neglected the higher order terms in Bionomial expansion for small displacements. In this extension, the work done is $F\,\delta s$. Hence the total work done in displacing all the particles from their equilibrium positions becomes the potential energy of the system, *i.e.*,

$$V = \frac{F}{2a} \sum_{n=1}^{N+1} (\psi_n - \psi_{n-1})^2 \qquad \ldots(113)$$

where at the end points $\psi_0 = \psi_{N+1} = 0$.

The Lagrangian for the system is

$$L = \sum_{n=1}^{N+1}\left[\frac{1}{2}m\dot{\psi}_n^2 - \frac{F}{2a}(\psi_n - \psi_{n-1})^2\right] \quad ...(114)$$

The Lagrangian equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}_n}\right) - \frac{\partial L}{\partial \psi_n} = 0, \quad n = 1, 2, ..., N+1 \quad ...(115)$$

i.e.,

$$m\ddot{\psi}_n + \frac{F}{a}(\psi_n - \psi_{n-1}) + \frac{F}{a}(\psi_{n+1} - \psi_n) = 0$$

where ψ_n is occurring in two terms of the potential energy V.

or

$$\ddot{\psi}_n + \frac{F}{ma}(2\psi_n - \psi_{n-1} - \psi_{n+1}) = 0 \quad ...(116)$$

or

$$\ddot{\psi}_n + \omega_0^2(2\psi_n - \psi_{n-1} - \psi_{n+1}) = 0 \quad ...(117)$$

where $\omega_0^2 = F/ma$.

The equation of motion for the nth particle shows that it depends on the displacements ψ_{n-1} and ψ_{n+1} of its neighbouring particles. Thus this is an example of interaction with the *nearest neighbour* where the spring or elastic forces are acting between neighbouring particles only.

If we consider a harmonic solution of the form

$$\psi_n = C\,a_n e^{i\omega t}, \quad ...(118)$$

we obtain

$$-\omega^2 a_n + \omega_0^2\,(2a_n - a_{n-1} - a_{n+1}) = 0$$

or

$$\omega_0^2\,a_{n+1} + (\omega^2 - 2\omega_0^2)\,a_n + \omega_0^2\,a_{n-1} = 0 \quad ...(119)$$

Let us assume that the amplitude of the nth particle, situated at a distance $x = na$ from the fixed end, is represented for a particular ω (or mode of vibration) :

$$a_n = A'\sin kna \quad ...(120)^*$$

where A' and k are constants .

Substitute in eq. (119)

$$\omega_0^2\sin k\,(n+1)\,a + (\omega^2 - 2\omega_0^2)\sin kna + \omega_0^2\sin k(n-1)\,a = 0$$

or

$$\omega^2\sin kna + \omega_0^2\,[\sin k\,(n+1)\,a + \sin k\,(n-1)\,a - 2\sin kna] = 0$$

---

* If we assume a solution of eq. (119) in the exponential form, then

$$a_n = Be^{ikna} = Be^{ikx}$$

where $n = 1, 2, ..., N$.

This will not satisfy the condition $a_0 = a_{N+1} = 0$.

However the equations of motion are linear, hence a superposition of solutions is also a solution. We may take

$$a_n = \frac{B'}{2i}\left(e^{ikna} - e^{-ikna}\right) = A'\sin kna$$

or $$\omega^2 \sin kna + \omega_0^2 (2 \sin kna \cos ka - 2 \sin kna) = 0$$

or $$\omega^2 = 2\omega_0^2 (1 - \cos ka)$$

or $$\omega = 2\omega_0 \sin \frac{ka}{2} \qquad ...(121)$$

where we have taken positive square root only because negative frequencies have no physical meaning.

Now the problem is to find the value of k for a particular mode of vibration. This we determine from the boundary conditions, *i.e.*, at $x = 0$ and $x = (N + 1)\,a$

$$\psi_0 = \psi_{N+1} = 0 \text{ i.e., } a_0 = a_{N+1} = 0 \qquad ...(122)$$

Substituting these boundary conditions in (120), we get

$$a_0 = A' \sin 0 = 0 \text{ (for } n = 0)$$

and $$a_{N+1} = A' \sin k\,(N + 1)\,a = 0 \text{ or } \sin k\,(N + 1)\,a = 0.$$

Therefore, $$k\,(N + 1)\,a = r\pi \text{ or } k = \frac{r\pi}{(N+1)\,a} \qquad ...(123)$$

where $r = 1, 2, 3, ..., N$.

Hence eq. (121) is

$$\omega_r = 2\omega_0 \sin \frac{r\pi}{2(N+1)} \qquad ...(124)$$

This relation gives normal mode frequency for a particular value of r, *i.e.*, frequency for rth mode of vibration, and therefore we have designated the frequency with the suffix r.

Now we can write the amplitude in the rth mode of vibration of the nth particle as

$$a_{nr} = A_r' \sin \frac{r\pi n}{N+1},\; r = 1, 2, ..., N. \qquad ...(125)$$

when all the particles are oscillating in the rth mode, the actual displacement of nth particle is given by

$$\psi_n = A_r \sin \frac{rn\pi}{N+1} \cos (\omega_r t + \phi_r) \qquad ...(126)$$

where we have taken the real part of $\psi_n = C_r\, a_{nr}\, e^{i\omega_r t}$.

The general solution for the displacement of the nth particle is obtained by the super position of various modes for different values of r, *i.e.*,

$$\psi_n = \sum_{r=1}^{N} A_r \sin \frac{rn\pi}{N+1} \cos (\omega_r t + \phi_r) \qquad ...(127)$$

Number of normal modes : The string with N particles can vibrate transversely in N normal modes with frequencies ω_r, where $r = 1, 2, ..., N$. We show in Fig. 9.12 the normal or independent modes of vibration for a string equipped with $N = 1, 2, 3, ...$ particles. For $N = 1$, there is one normal mode of vibration. For $N = 2, 3, 4,....$ there are two, three,....modes of vibration respectively. For N particles, there are N independent modes of vibration.

Representation of various modes : We discuss various modes for *sufficient number of particles*. The particle displacements in the rth mode are given by

$$\psi_n = \left(A_r \sin \frac{rn\pi}{N+1}\right) \cos (\omega_r t + \phi_r) \qquad ...(126)$$

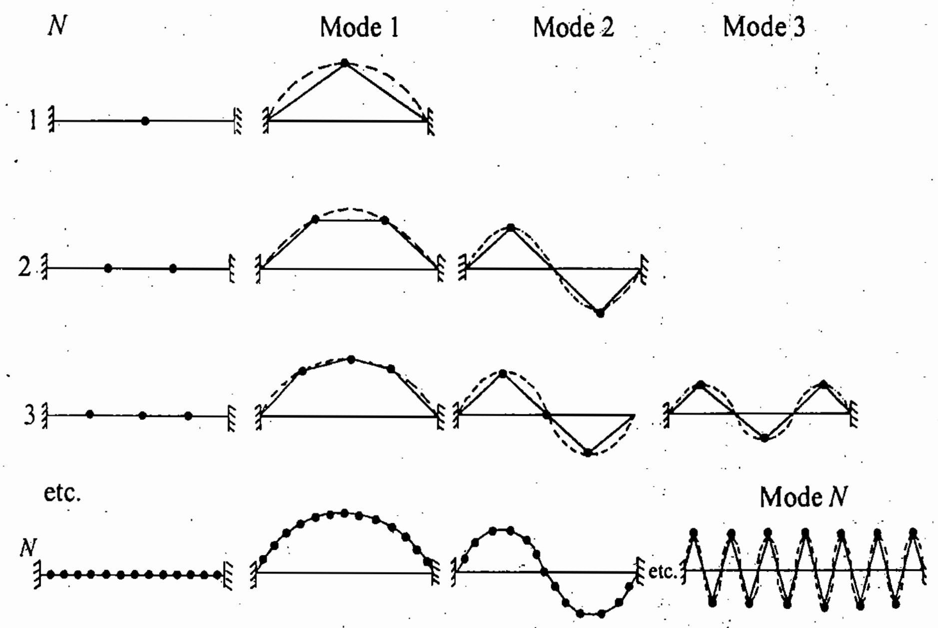


Fig. 9.12. Normal modes of a string with N particles :string with 1, 2, 3,..., N particles have 1, 2, 3,..., N modes of vibration respectively. In mode r, string has r half wavelengths i.e., $l = rl/2$ or $l = 2l/r$

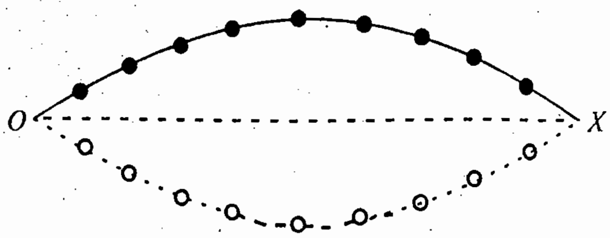
where $n = 1, 2, 3, \ldots, N$ represents various particles.

When $r = 1$, all the particles are vibrating with the same frequency ω_1, in Mode 1 and the amplitude, $A_1 \sin\frac{n\pi}{N+1}$, is varying with the particle number n. The value of amplitude increases as n changes from $n = 1$ to onwards and becomes nearly maximum for $n = N/2$. Further it decreases and becomes zero at $n = N+1$ or $x = l$. Thus we see that in Mode 1, all the masses execute simple harmonic motions in phase with frequency ω_1, with different amplitudes in a loop [Fig. 9.13(a)]. For $r = 2$, *i.e.*, Mode 2, the frequency of vibration of all the particles is ω_2.

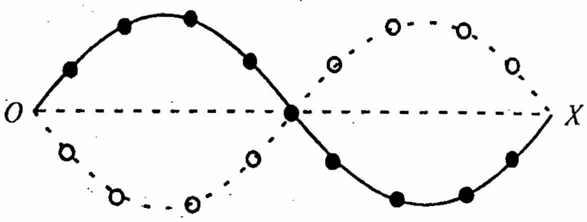
The amplitude $\left(A_2 \sin\frac{2n\pi}{N+1}\right)$ first increases and then decreases with particle number n. It is zero in the middle (when N is odd there is a particle at the centre at rest) and again a similar viriation in amplitude occurs. Now the string with particle vibrates in two loops [Fig. 13(b)]. The frequency of vibration $\omega_2\left(=2\omega_0 \sin\frac{2\pi}{2(N+1)}\right)$ in Mode 2 is approximately twice that of ω_1 $\left(=2\omega_0 \sin\frac{\pi}{2(N+1)}\right)$ in Mode 1. In Mode 3, ($r = 3$). The frequency of vibration is approximately thrice to that in Mode 1 and the string with particles vibrates in 3 loops and so on. It is to be observed that the particles of the first loop are out of phase to those in the second loop and in phase with those in the third loop. The distance over which one complete cycle occurs gives wavelength λ. Thus for Mode 1, $\lambda_1/2 = l$ or $\lambda_1 = 2l$, for Mode 2, $\lambda_2 = l$ or $\lambda_2 = 2l/2$, for Mode 3, $\lambda_3 = 2l/3$ and for Mode r, $\lambda_r = 2l/r$. Now let us relate the constant k for Mode r to the wavelength. From eq. (123),

$$k = \frac{r\pi}{(N+1)a} = \frac{r\pi}{l} = \frac{2\pi}{2l/r} = \frac{2\pi}{\lambda_r} \quad [\because (N+1)\,a = l] \qquad \ldots(128)$$

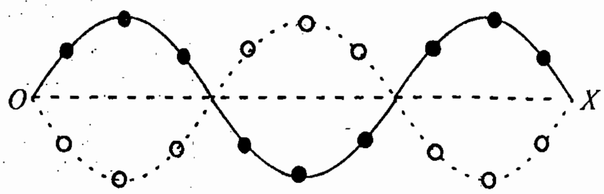
This k is called *wave number*.



(a) Mode 1



(b) Mode 2



(c) Mode 3

Fig. 9.13. Instantaneous positions of particles in Mode 1, Mode 2 and Mode 3 are represented by solid sine curves. Dotted sine curves represent the position of particles at any other instant. Note that in any mode, all the particles are passing their equilibrium positions simultaneously.

Thus the frequency in terms of wave number (k) or wavelength (λ) is expressed as follows :

$$\omega = 2\omega_0 \sin\frac{ka}{2} \text{ or } \omega = 2\omega_0 \sin\frac{\pi a}{\lambda} \qquad ...(129)$$

If we write $\qquad 2\omega_0 = 2\sqrt{\dfrac{F}{ma}} = \omega_m$, then

$$\omega = \omega_m \sin\frac{ka}{2} \qquad ...(130)$$

Dispersion relation : A relation between the frequency (ω) and the wave number (k) is called *dispersion relation*. Relation (130) represents the dispersion relation for the string with massive particles. A graph, drawn between ω and k, is called *dispersion curve*. For a string with 9 particles, the dispersion relation is shown in Fig. 9.14. The wave number (k) can have value, given by

$$k = \frac{r\pi}{(N+1)a} = \frac{r\pi}{10a} \qquad ...(131)$$

where k can have 9 values corresponding to $r = 1, 2, ..., 9$. We have drawn the curve between ω and k, *i.e.*, the dispersion curve for the system of 9 particles on the string [Fig. 9.14].

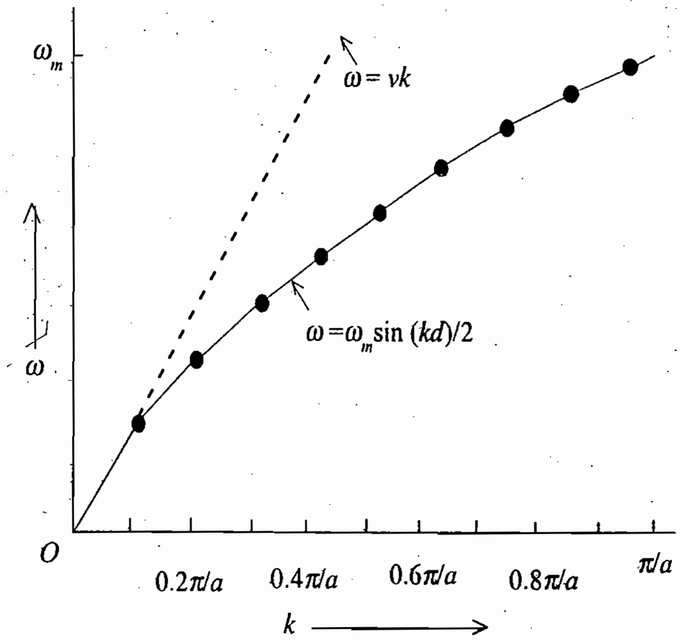


Fig. 9.14. Dispersion relation for 9 particles on the string. Nine points on the curve represent nine frequencies corres-ponding to the nine modes of the string with 9 particles

Alternative Treatment :

Kinetic energy $\quad T = \frac{1}{2} m\dot{\psi}_1^2 + \frac{1}{2} m\dot{\psi}_2^2 + \ldots + \frac{1}{2} m\dot{\psi}_N^2 \qquad$ ...(*i*)

Potential energy $\quad V = \frac{F}{a}\left(\psi_1^2 + \psi_2^2 + \ldots + \psi_N^2 - \psi_1\psi_2 - \psi_2\psi_3 - \ldots - \psi_{N-1}\psi_N\right) \qquad$ ...(*ii*)

Hence T and V matrices are

$$T = \begin{pmatrix} m & 0 & 0 \ldots & 0 \\ 0 & m & 0 \ldots & 0 \\ 0 & 0 & m \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & m \end{pmatrix}, \quad V = \begin{pmatrix} 2F/a & -F/a & 0 & \ldots & 0 \\ -F/a & 2F/a & F/a & \ldots & 0 \\ 0 & -F/a & 2F/a & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2F/a \end{pmatrix}$$

The normal mode frequencies are determined from the equation

$$|V - \omega^2 T| = 0 \text{ or } \begin{vmatrix} \frac{2F}{a} - m\omega^2 & -\frac{F}{a} & 0 & \ldots & 0 \\ -\frac{F}{a} & \frac{2F}{a} - m\omega^2 & -\frac{F}{a} & \ldots & 0 \\ 0 & -\frac{F}{a} & \frac{2F}{a} - m\omega^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \frac{2F}{a} - m\omega^2 \end{vmatrix} = 0 \qquad \text{...(iii)}$$

whose solution gives

$$\omega_r^2 = 4\omega_0^2 \sin^2 \frac{ka}{2} \text{ or } \omega_r = 2\omega_0 \sin\frac{ka}{2} \quad \text{...(iv)}$$

where $$\omega_0^2 = F/ma \text{ and } k = \frac{r\pi}{(N+1)a} \quad \text{...(v)}$$

Normal modes in terms of eigen vectors are found from the equation

$$(V - \omega_r^2 T)\, \boldsymbol{a}_r = 0$$

or
$$\begin{pmatrix} \frac{2F}{a} - m\omega_r^2 & -\frac{F}{a} & 0 & \ldots & 0 \\ -\frac{F}{a} & \frac{2F}{a} - m\omega_r^2 & -\frac{F}{a} & \ldots & 0 \\ 0 & -\frac{F}{a} & \frac{2F}{a} - m\omega_r^2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \frac{2F}{a} - m\omega_r^2 \end{pmatrix} \begin{pmatrix} a_{1r} \\ a_{2r} \\ a_{3r} \\ \ldots \\ a_{Nn} \end{pmatrix} = 0 \quad \text{...(vi)}$$

which gives
$$(\omega_r^2 - 2\omega_0^2)\, a_{1r} + \omega_0^2\, a_{2r} = 0$$
$$\omega_0^2 a_{1r} + (\omega_r^2 - 2\omega_0^2)\, a_{2r} + \omega_0^2\, a_{3r} = 0 \quad \text{...(vii)}$$
$$\omega_0^2 a_{2r} + (\omega_r^2 - 2\omega_0^2)\, a_{3r} + \omega_0^2\, a_{4r} = 0$$
$$\text{- -}$$

where $\omega_0^2 = F/ma$.

The general equation is

$$\omega_0^2\, a_{(n+1)r} + (\omega_r^2 - 2\omega_0^2)\, a_{nr} + \omega_0^2\, a_{(n-1)r} = 0 \quad \text{...(viii)}$$

Now, $$\omega_r^2 - 2\omega_0^2 = 4\omega_0^2 \sin^2 \frac{ka}{2} - 2\omega_0^2 = -2\omega_0^2 \cos ka \quad \text{[using (iv)]}$$

Hence eq. (*viii*) is

$$a_{(n+1)r} - 2\cos ka\, a_{nr} + a_{(n-1)r} = 0 \quad \text{...(ix)}$$

This is to be satisfied by the condition $a_{0r} = a_{(N+1)r} = 0$ ...(*x*)

Eq. (*ix*) is satisfied by the solution of the form

$$a_{nr} = A_r' \sin kna \quad \text{...(xi)}$$

The boundary condition (*x*) gives

$$k = r\pi/[(N+1)\,a] \quad \text{[as eq. (123)].}$$

The constants A_r' can be determined from the orthogonality condition (72):

$$\sum_i \sum_n a_{nr}\, T_{in}\, a_{is} = \delta_{rs} \quad \text{...(xii)}$$

Here $T_{in} = m\,\delta_{in}$, hence eq. (*xii*) is

$$\sum_n m\, a_{nr}\, a_{rs} = \delta_{rs} \qquad ...(xiii)$$

Substitute from (*xi*),

$$\sum_{n=1}^{N} m\, A_r' A_s' \sin\left(\frac{nr\pi}{N+1}\right) \sin\left(\frac{ns\pi}{N+1}\right) = \delta_{rs}$$

or
$$m\, A_r'\, A_s' \sum_{n=1}^{N} \sin\left(\frac{nr\pi}{N+1}\right) \sin\left(\frac{ns\pi}{N+1}\right) = \delta_{rs}$$

or
$$m\, A_r'\, A_s'\, \tfrac{1}{2}\delta_{rs}(N+1) = \delta_{rs}$$

or
$$m\, A_r'^2 = \frac{2}{N+1} \text{ or } A_r' = \sqrt{\frac{2}{m\,(N+1)}} \qquad ...(xiv)$$

Hence
$$a_{nr} = \sqrt{\frac{2}{m\,(N+1)}} \sin\frac{nr\pi}{N+1} \qquad ...(xv)$$

where $r = 1, 2, ..., N$.

The general solution is the same as eq. (127), *i.e.*,

$$\psi_n = \sum_{r=1}^{N} A_r \sin\frac{nr\pi}{N+1} \cos(\omega_r t + \phi_r) \qquad ...(xvi)$$

9.8. TRANSITION FROM DISCRETE TO A CONTINUOUS SYSTEM : WAVES ON A STRING

If the number of particles, attached on the string, become very large and the distance between two consecutive particles (i.e., a) tends to zero, then the system becomes a massive continuous string. It is shown below that the coupled oscillations of the particles on a string in the continuous limit become ***waves***.

The equation of motion of the nth particle on the string [eq. (116)] can be written in the following form :

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left(\frac{\psi_{n+1} - \psi_n}{a} - \frac{\psi_n - \psi_{n-1}}{a}\right) \qquad ...(132)$$

If we make the distance between two consecutive particles to be small and denote a by δx, then

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left(\frac{\psi_{n+1} - \psi_n}{\delta x} - \frac{\psi_n - \psi_{n-1}}{\delta x}\right)$$

or
$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left[\left(\frac{\delta\psi}{\delta x}\right)_{n+1} - \left(\frac{\delta\psi}{\delta x}\right)_n\right]$$

If we represent the position of the nth particle by x (*i.e.*, $na = x$) and take the limit $\delta x \to 0$, then

$$\frac{d^2\psi_n}{dt^2} = \frac{F}{m}\left[\left(\frac{d\psi}{dx}\right)_{x+dx} - \left(\frac{d\psi}{dx}\right)_x\right] \qquad ...(133)$$

Since ψ is also changing with x, the value of $\frac{d\psi}{dx}$ at $x + dx$ will be given by

$$\left(\frac{d\psi}{dx}\right)_{x+dx} = \frac{d}{dx}(\psi + d\psi) = \frac{d\psi}{dx} + \frac{d^2\psi}{dx^2}dx$$

Therefore, $$\frac{d^2\psi}{dt^2} = \frac{F}{m}\frac{d^2\psi}{dx^2}dx = \frac{F}{\rho}\frac{d^2\psi}{dx^2} \qquad ...(134)$$

where $m/dx = \rho$, the mass per unit length of the string. Eq. (134) represents the *wave equation*. The quantity F/ρ has the dimensions of the square of the wave velocity (v) and therefore we denote $v^2 = F/\rho$ *i.e.*, the **wave velocity** in the string is given by

$$v = \sqrt{\frac{F}{\rho}} \qquad ...(135)$$

Thus the wave equation from (134) is obtained to be

$$\frac{d^2\psi}{dt^2} = v^2\frac{d^2\psi}{dx^2} \qquad ...(136)$$

The particle displacement in the rth mode is given by

$$\psi_n = A_r \sin\frac{nr\pi}{N+1}\cos(\omega_r t + \phi_r) \qquad ...(137)$$

But $(N+1)\, a = l$ and $na = x$,

$$\psi = A_r \sin\frac{r\pi x}{l}\cos(\omega_r t + \phi_r)$$

or $$\psi = A_r \sin kx \cos(\omega_r t + \phi_r) \qquad ...(138)$$

In fact, this is the equation of standing waves in a string, fixed at both ends. At $t = 0$, if we take the particle displacement to be maximum, then we have $\phi_r = 0$. In such a case,

$$\psi_n = a \sin kx \cos \omega_r t \qquad ...(139)$$

When N becomes large, the form of the dispersion relations for first modes ($r << N$) is

$$\omega = 2\omega_0 \sin\frac{ka}{2} = 2\sqrt{\frac{F}{ma}}\frac{ka}{2} = \sqrt{\frac{F}{m/a}}\,k = \sqrt{\frac{F}{\rho}}\,k \text{ or } \omega = vk \qquad ...(140)$$

where we have used $\sin\theta = \theta$ for small θ because $\theta = \frac{ka}{2} = \frac{r\pi}{2(N+1)}$ is small for $r << N$.

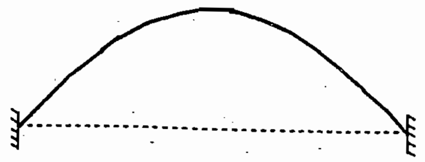
Such a relation holds good for waves between ω and k in a continuous medium and corresponds to the usual relation

$$v = \nu\lambda \qquad ...(141)$$

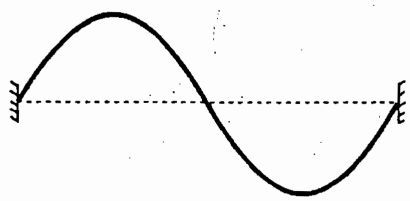
because $$\omega = vk \text{ or } 2\pi\nu = v\frac{2\pi}{\lambda}$$

or $$\nu = v/\lambda \text{ or } v = \nu\lambda.$$

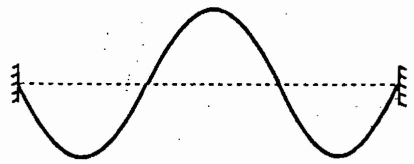
When N is very large, different modes are represented by eq. (139) and at any instant first few modes are shown in fig. 9.15. Actually these are the various modes of vibration of a stretched string, fixed at its ends (with $\lambda_r = 2l/r$). The frequency in Mode 1 ($r = 1$) is given by



(a) Mode 1 : $\lambda_1 = 2l,\ \omega_1 = 2\pi v/\lambda_1 = \pi v/l$



(b) Mode 2 : $\lambda_2 = l,\ \omega_2 = 2\pi v/\lambda_2 = 2\pi v/l = 2\omega_1$



(c) Mode 3 : $\lambda_3 = 2l/3,\ \omega_3 = 2\pi v/\lambda_3 = 3\pi v/l = 3\omega_1$

Fig. 9.15. First three modes of vibration of a continuous uniform string fixed at its two ends

$$\omega_1 = v k_1 = \frac{\pi v}{l} \left[\because\ k = \frac{r\pi}{l} = \frac{\pi}{l} \text{ for } r = 1\right] \qquad ...(142)$$

This is called *fundamental mode of vibration* of the string.

For Mode 2, $\omega_2 = \dfrac{2\pi v}{l}$; for Mode 3, $\omega_3 = \dfrac{3\pi v}{l}$ and in general ($r \ll N$)

$$\omega_r = r\omega_1 \qquad ...(143)$$

In fact such a relation holds good for a stretched string.

Questions

1. What do you mean by stable and unstable equilibriums ? Establish the Lagrangian and deduce the Lagrange's equations of motion for small oscillations of a system in the neighbourhood of stable equilibrium. **(Meerut 2001)**
2. Deduce the eigen value equation for small oscillations. How will you obtain the eigen values (ω^2) and eigenvectors from this equation ?
3. What do you understand by normal modes of vibration ? Explain the meaning of normal coordinates and normal frequencies. Show that when the kinetic and potential energies are expressed in terms of normal coordinates, both kinetic and potential energies are homogeneous quadratic functions.
4. Consider the case of two coupled pendulums as shown in Fig. 9.4. Determine (*a*) ***T*** and ***V*** matrices, (*b*) the normal frequencies, (*c*) the normal coordinates, (*d*) the equation of motion, (*e*) the eigenvectors and (*f*) the general solution.
5. Two identical harmonic oscillators are coupled together. Set up the equations of motion and obtain the general solutions. Describe the two normal modes. **(Mumbai April 2003, 2000)**
6. Write short note on normal modes and eigen frequencies for small oscillations. **(Meerut 2001)**
7. Discuss the vibrations of a linear triatomic molecule. **(Meerut 1999)**
8. Outline in brief the Lagrangian for continuous systems. **(Meerut 1999)**

Problems

[SET- I]

1. If an object of mass m is suspended to a rigid support with the help of a spring of force constant k, it vibrates with a frequency 2 Hz. Now two identical objects A and B, each of mass m, are joined together by a spring of force constant k' and then they are connected to rigid supports S_1 and S_2 by two identical springs, each of force constant k [Fig 9.2]. Next, if A is clamped, B vibrates with a frequency 2.5 Hz. Calculate the frequencies of the two modes of vibration. Find also the ratio k'/k of the two force constants.
[**Ans :** 2 Hz, 2.92 Hz; 9/16 .]

2. Determine the normal mode frequencies of a pair of coupled pendulums as shown in Fig. 9.9 if the two pendulums are of different masses M and m with the same length l. Given that the pendulum of mass M is started oscillating with amplitude a, find the maximum amplitude of the other pendulum in the subsequent oscillations.

$$\left[\textbf{Ans.}: \omega_1^2 = \frac{g}{l}, \omega_2^2 = \frac{g}{l} + \frac{k}{M} + \frac{k}{m}; \frac{2Ma}{M+m}.\right]$$

3. A system of two coupled oscillators is shown in Fig 9.16. Find expressions for normal mode frequencies. If the force constant of the middle spring is the geometric mean of the side springs, what are the modified expressions for the normal mode frequencies ?

$\left[\textbf{Ans.}: \sqrt{(k + k_1 + k_2)/m}, \sqrt{k/m}.\right]$.

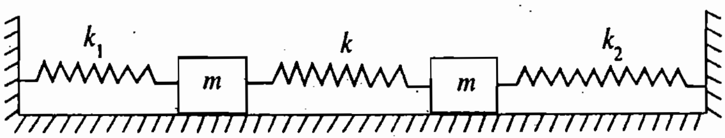


Fig. 9.16

4. Two equal masses (m) are connected to each other with the help of a spring of force constant k and then the upper mass is connected to a rigid support by an identical spring as shown in Fig. 9.17. The system is allowed to oscillate in the vertical direction. Show that the frequencies of two normal modes are $\omega^2 = (3 \pm \sqrt{5})\, k/2m$ and the ratios of the amplitudes of two masses in the two modes are $\frac{1}{2}(\sqrt{5} \pm 1)$.

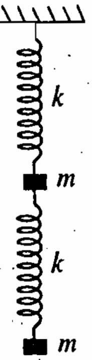


Fig. 9.17

5. Determine the normal mode frequency of the Lagrangian, given by

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}(\omega_1^2 x^2 + \omega_2^2 y^2) + \alpha\, xy$$

$$\left[\textbf{Ans.}: \tfrac{1}{2}(\omega_2^1 + \omega_2^2) \mp \sqrt{(\omega_2^2 - \omega_1^2)^2 + 4\alpha^2}.\right]$$

6. Three masses, which are connected by springs, move along a circle. The point O is fixed [Fig. 9.18]. Set up the Lagran-gian. Determine the normal frequencies and normal coordinates. Find the eigen-vibrations (displacement time relations) of the system. Write down the Lagrangian in normal coordinates.

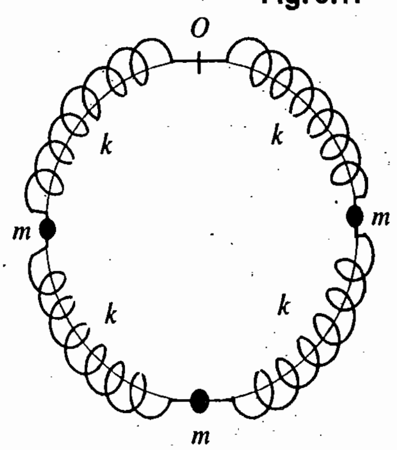


Fig. 9.18

[**Ans. :** $L = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$

$$-\frac{1}{2} K\left[x_1^2 + (x_2 - x_1)^2 + (x_3 - x_2)^2 + x_3^2\right];$$

$$\omega_1 = \sqrt{\left(2-\sqrt{2}\right)k/m},\ \omega_2 = \sqrt{2K/m},\ \omega_3 = \sqrt{\left(2+\sqrt{2}\right)k/m};$$

$$Q_1 = A\cos(\omega_1 t + \phi_1),\ Q_2 = B\cos(\omega_2 t + \phi_2),\ Q_3 = C\cos(\omega_3 t + \phi_3);$$

$$x_1 = \frac{1}{2}(q_1 + \sqrt{2}q_2 + q_3),\ x_2 = \frac{1}{\sqrt{2}}(q_1 - q_3),\ x_3 = \frac{1}{2}(q_1 - \sqrt{2}q_2 + q_3);$$

$$L = \frac{1}{2} m \sum_{i=1,2,3} (\dot{Q}_i^2 - \omega_i^2 Q_i^2).]$$

7. A particle of mass m is attached to a light rod, pivoted at the point O as shown in Fig. 9.19. The rod has motion in the verticle plane. Unstretched lengths of the springs are l_1 and l_2. Set up the Lagrangian and find the period of small oscillations.

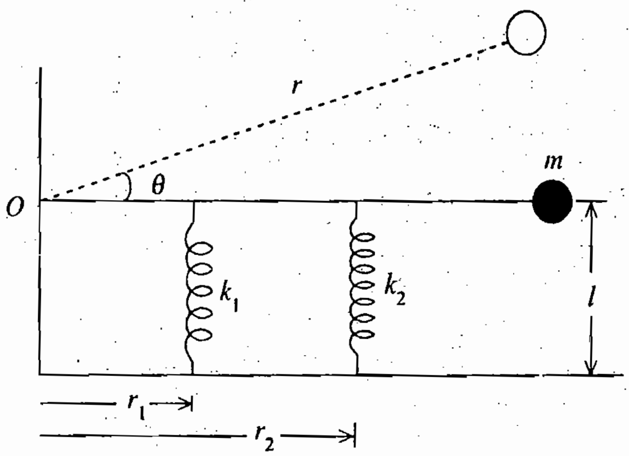


Fig. 9.19

[**Ans.:** $L = \frac{1}{2} mr^2\dot{\theta}^2 - \frac{1}{2}k_1(l - l_1 + r_1\theta)^2 + \frac{1}{2}k_2(l - l_2 + r_2\theta)^2 + mgr\theta;$

$$\text{Period} = 2\pi\left[mr^2/(k_1r_1^2 + k_2r_2^2)\right]^{\frac{1}{2}}]$$

[SET- II]

1. Find the Lagrangian for the circuit shown in Fig. 9.20. Find the normal frequencies of the system.

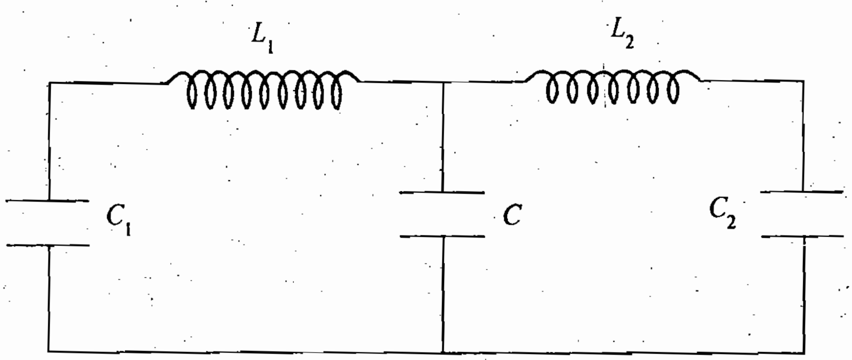


Fig. 9.20

[**Ans :** $L = \frac{1}{2}\left(L_1\dot{q}_1^2 + L_2\dot{q}_2^2\right) - \frac{1}{2}\left[\frac{q_1^2}{C_1} + \frac{q_2^2}{C_2} + \frac{(q_1+q_2)^2}{C}\right]$;

$$\omega_1^2 = \frac{1}{2}\left(p_1^2 + p_2^2 - \sqrt{(p_2^2 - p_1^2)^2 + 4f^2}\right),\quad \omega_2^2 = \frac{1}{2}\left(p_1^2 + p_2^2 + \sqrt{(p_2^2 - p_1^2)^2 + 4f^2}\right)$$

where $r_1^2 = \frac{1}{L_1}\left(\frac{1}{C} + \frac{1}{C}\right).\ p_2^2 = \frac{1}{L_2}\left(\frac{1}{C} + \frac{1}{C_2}\right), f = \frac{1}{C\sqrt{L_1L_2}}.$]

2. **Triple pendulum :** Show that the normal mode frequencies of a triple pendulum, shown in Fig 9.21, are given by

$$\omega_1 = \omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{g}{l} - \frac{k}{m}}$$

Further show that the generalized coordinates θ_i are related to the normal coordinates Q_i as

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}\gamma & \beta & \alpha\\ -2\gamma & 0 & \alpha\\ \gamma & -\beta & \alpha\end{pmatrix}\begin{pmatrix}Q_1\\ Q_2\\ Q_3\end{pmatrix}$$

where $\alpha = 1/\sqrt{3ml^2}$, $\beta = 1/\sqrt{2ml^2}$ and $\gamma = 1/\sqrt{6ml^2}$.

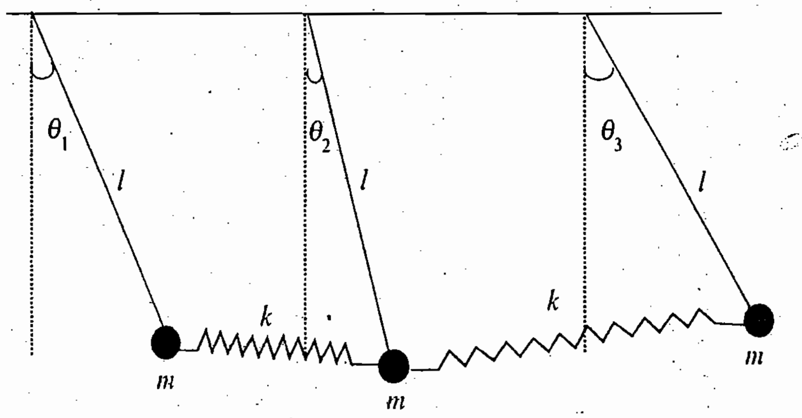


Fig. 9.21. Triple pendulum

[**Hint :** $T = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right)$, $V = \frac{1}{2}\left[(mgl + kl^2)(\theta_1^2 + \theta_2^2 + \theta_3^2) - 2kl^2(\theta_1\theta_2 + \theta_2\theta_3 + \theta_1\theta_3)\right]$.

3. A hemispherical bowl of radius $2b$ and mass $4m$ rests on a smooth table such that the plane of its rim is horizontal. In this bowl, there is a perfectly rough solid sphere of radius b and mass m. The system is set in motion in such a way that the two centres of sphere and bowl remain in a vertical plane in which when the system was in equilibrium. Show that the normal frequencies for small oscillations will be obtained from the equation :

$$156\, b^2\omega^2 - 260\, bg\omega + 75g^2 = 0.$$

4. A system of N identical masses is coupled by $N+1$ springs each of force constant K [Fig. 9.22]. The free ends of the system are rigidly fixed at S_1 and S_2. The spring-mass system is allowed to oscillate longitudinally. If x_{n-1}, x_n and x_{n+1} are the displacements of $(n-1)$th, nth and $(n+1)$th masses respectively

[Fig 9.23], show that the equation of motion of nth mass is

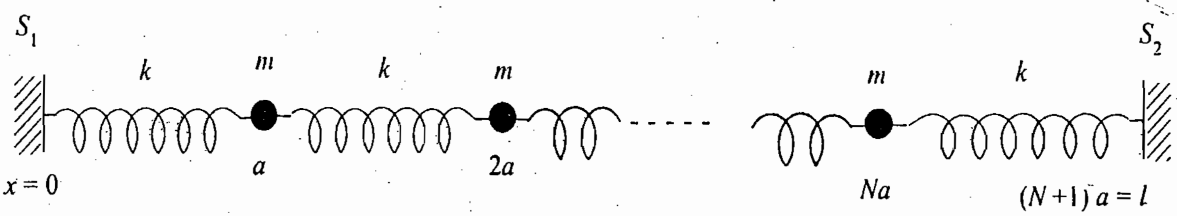


Fig. 9.22. A system of N masses coupled by N +1 springs between two rigid supports

$$\ddot{x} + 2\omega_0^2 x_n - \omega_0^2 (x_{n-1} + x_{n+1}) = 0,$$

where $\omega_0^2 = k/m$.

Further show that the displacement in the rth or kth mode of vibration can be given by

$$x_n = a \sin ka \cos \omega t$$

where $k = r\pi/l$.

Find the dispersion relation for the system.

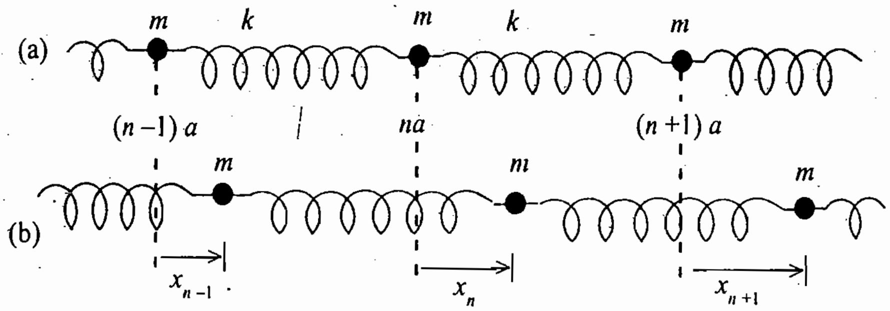


Fig. 9.23 : (a) Spring mass system in equilibrium, (b) Longitudinal oscillations of spring-mass system

[Ans : $\omega = 2\omega_0 \sin ka/2$]

Objective Type Questions

1. An example of stable equilibrium is
   (*a*) a hanging spring-mass system in the stationary position,
   (*b*) a pendulum in the rest position,
   (*c*) an egg standing on one end,
   (*d*) a book placed flat any where on a table.
   Ans. : (*a*), (*b*).

2. Identify the points of unstable equilibrium for the potential shown in Fig. 9.24 :
   (*a*) p and s
   (*b*) q and t
   (*c*) r and t
   (*d*) r and s **(Gate 2004)**
   Ans : (*c*)

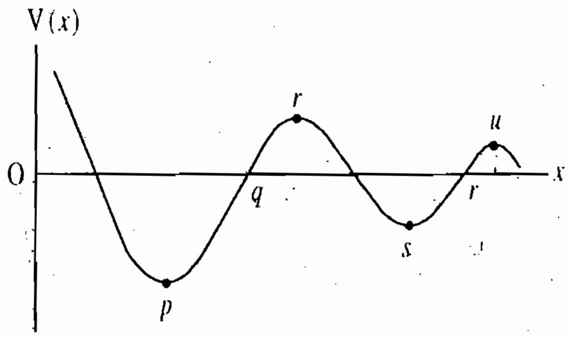


Fig. 9.24

3. Consider the motion of a particle in the potential $V(x)$ shown in Fig. 9.25 :
   (*i*) Suppose the particle has a total energy $E = V_1$ in the figure. Then the speed of the particle is zero when it is at

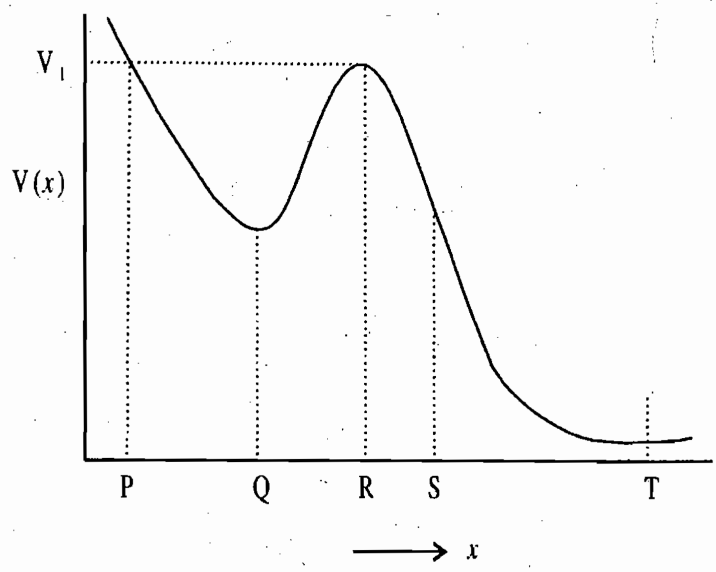


Fig. 9.25

(*a*) point *P* (*b*) point *Q*
(*c*) point *S* (*d*) point *T* **(Gate 2003)**

Ans : (*b*)

(*ii*) Which one of the following statement is NOT correct about the particle ?

(*a*) It experiences no force when its position corresponds to the point *Q* on the curve.

(*b*) It experiences no force when its position corresponds to the point *R* on the curve.

(*c*) Its speed is the largest when it is at *S*.

(*d*) It will be in a closed orbit between *P* and *R*, if $E < V_1$. **(Gate 2003)**

Ans : (*c*)

4. In case of two coupled identical pendulums, oscillating in a plane,

(*a*) each pendulum always execute simple harmonic motion,

(*b*) two pendulums may execute simple harmonic motions,

(*c*) the general motion can be expressed as a superposition of two simple harmonic motions of the same frequency,

(*d*) the general motion can be expressed as a superposition of two simple harmonic motions of different frequencies.

Ans. : (*b*), (*d*).

5. In case of a symmetric mode for two coupled identical spring-mass systems,

(*a*) both masses have equal displacements in phase,

(*b*) both masses have equal and opposite displacements,

(*c*) both masses vibrate with the frequency of single spring-mass system,

(*d*) both masses vibrate with a frequency different to the frequency of single spring-mass system.

Ans. : (*a*), (*c*).

6. In case of two coupled identical pendulums, in general,

(*a*) the potential energy is a homogeneous quadratic function, when expressed in terms of actual displacements.

(*b*) the potential energy is not a homogeneous quadratic function, when expressed in terms of actual displacements,

(*c*) the potential energy is a homogeneous quadratic function, when expressed in terms of normal coordinates.

(*d*) the potential energy is not a homogeneous quadratic function, when expressed in terms of normal coordinates.

Ans. : (*b*), (*c*).

7. In case of double pendulum, if the masses as well as thread lengths are equal $(m_1 = m_2 = m, l_1 = l_2 = l)$,
   (*a*) two normal mode frequencies are equal to that of a pendulum of length $2l$ and mass m.
   (*b*) two normal mode frequencies are equal to that of single pendulum of length l and mass m.
   (*c*) two normal mode frequencies are different.
   (d) the sum of squared frequencies of normal modes is equal to four times the squared frequency of a single pendulum of length l and mass m.

   Ans. : (*c*), (*d*).

8. In case of a linear triatomic of molecule XY_2 type, the eigen frequencies ω_1, ω_2 and ω_3 can be represented as
   (*a*) $\omega_1 = \omega_2 = \omega_3$ (*b*) $\omega_1 = 0,\ \omega_2 = \omega_3$
   (*c*) $\omega_1 = 0,\ \omega_2 \neq \omega_3$ (*d*) $\omega_1 \neq \omega_2 \neq \omega_3$.

   Ans. : (*c*), (*d*).

Short Answer Questions

1. What are coupled oscillators ?
2. What do you understand by normal modes in relation to coupled oscillators ?
3. What are normal coordinates and normal frequencies ?
4. Could a system of two coupled pendulums oscillate in a single mode ? Could a pendulum of the system oscillate under the combined effect of both of the modes ?
5. In case of two coupled pendulums when only one mode is present, show that (*i*) each pendulum executes simple harmonic motion, (*ii*) both pendulums oscillate with the same frequency, and (*iii*) both pendulums pass through their equilibrium positions simultaneously.
6. Any of the two coupled pendulums oscillate always in smiple harmonic motion. Whether is it true or false ? Explain.
7. What do you understand by modulation and phenomenon of beats in case of a two-body coupled oscillator ?
8. What is the advantage of discussing the motion of coupled oscillators in the normal modes ?
9. 4 identical beads are attached at equal distances on a string under tension. Draw diagrams of various modes of transverse vibrations of the system.
10. Suppose that in Fig. 9.2 the two masses are weakly coupled ($K >> K'$). Show that the ratio of the difference of mode frequencies to the lower mode frequency is nearly K'/K.
11. What is dispersion relation ? Draw the dispersion relation for 9 beads attached at equal distances on a string under tension.
12. Fill in the blanks :
    (*i*) A system of N coupled oscillators with N degrees of freedom has ....independent modes of vibration.
    (*ii*) When the kinetic and polential energies are expressed in terms of normal coordinates, no...........of normal coordinates are present.
    (*iii*) In a mode, all the particles vibrate with the ....frequency.

    Ans. (*i*) N, (*ii*) cross terms, (*iii*) same.

CHAPTER 10 Dynamics of a Rigid Body

10.1. GENERALIZED COORDINATES OF A RIGID BODY

A rigid body is defined as a system of particles in which the distance between any two particles remains fixed throughout the motion. Thus a system of N particles is said to be a rigid body if it is subjected to holonomic constraints of the form

$$r_{ij} = C_{ij} \quad \text{...(i)}$$

where r_{ij} is the distance between ith and jth particles and C_{ij} is the constant. In a rigid body motion, the deformations, occurring in actual bodies, are neglected and a rigid body maintains its shape during its motion.

We cannot obtain the actual number of degrees of freedom just by subtracting the number of constraint equations from $3N$ because there are $\frac{1}{2} N(N-1)$ possible constraint equations of the form (1). Obviously for large value of N, these constraint equations are more in number than $3N$. In fact, the equations represented by (1) are not all independent.

We can show in the following two ways that *the number of degrees of freedom for the general motion of a rigid body is six i.e.*, six independent coordinates are needed to specify the motion.

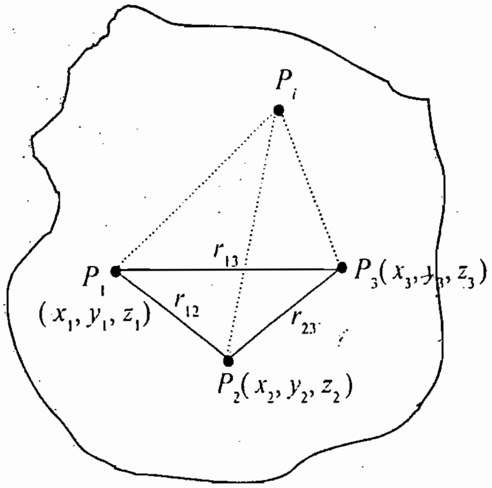


Fig. 10.1. Degrees of freedom for the motion of a rigid body – 3 reference points P_1, P_2, P_3 of the rigid body with three equations of rigid constraints allow the body to have six degrees of freedom

(1) Let us consider three non-collinear particles P_1, P_2 and P_3 in a rigid body (Fig. 10.1). As each particle has three degrees of freedom, nine degrees of freedom in total is required. From (1), the three equations of constraints, expressed in terms of coordinates of the points relative to an arbitrary origin fixed in the body, are

$$r_{12} = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{1/2} = C_{12} \text{ (constant length)}$$
$$r_{23} = [(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2]^{1/2} = C_{23}$$
$$r_{13} = [(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2]^{1/2} = C_{13} \quad \text{...(2)}$$

Hence the number of degrees of freedom of this three particle system is reduced to $9 - 3 = 6$. The position of any other particle in the body, say P_i, needs three coordinates and obviously there are three equations of constraints, because the distances of P_i from P_1, P_2 and P_3 are fixed. Hence any other particle will not add any new degree of freedom to six degrees of freedom of the three-particle system. Thus the motion of a rigid body be specified by six degrees of freedom. In other words, we need six independent or generalized coordinates to specify the motion of a rigid body.

(2) To look the situation in other way, the position of the particle P_1 needs three coordinates. Relative to P_1, the position of P_2 can be specified by only two coordinates because of one constraint equation $r_{12} = C_{12}$. The third particle P_3 relative to P_1 and P_2 has only one degree of freedom because of two constraints $r_{13} = C_{13}$ and $r_{23} = C_{23}$. Thus the three particles (non-collinear) of the rigid body have $3 + 2 + 1 = 6$ degrees of freedom. It is to be noted that (1) particle P_2 relative to P_1 is constrained to move on the surface of a sphere and its position can be specified by two angles, and (2) particle P_3 relative to P_1 and P_2 can only rotate about the axis jouning P_1 and P_2 which can be specified by third angle. Intutively, one can think that rigid body should possess three translational and three rotational degrees of freedom. Therefore, in order to describe the motion of a rigid body, we usually choose three of these coordinates to be the coordinates of a point in the body (generally the centre of mass) and the remaining three to be the three angles (usually three Eulerian angles*) which describe the rotation of the body about the point.

In addition to the constraints of rigidity, if the body has additional constraints, this will further reduce the number of degrees of freedom and hence the number of independent generalized coordinates.

10.2. BODY AND SPACE REFERENCE SYSTEMS

We may describle the motion of a rigid body by using two coordinate systems –

(1) Body coordinate system : A coordinate system, fixed in the rigid body, is called a body coordinate system and its axes are called body set of axes.

(2) Space coordinate system : The axes of such a coordinate system are fixed in the space are called space set of axes.

In Fig. 10.2, XYZ represents the space reference system with origin O and $X'Y'Z'$ the body coordinate system, fixed in the rigid body with origin O'. We choose the origin O' of the body set of axes to coincide with the centre of mass of the rigid body. Clearly three coordinates are required to specify the origin O' of this body set of axes relative to the origin O of the space reference system.

Let $\mathbf{R}$ (X, Y, Z) be the position vector of O' relative to O. Further, for the general motion of the rigid body, the orientation of the body set of axes $X'Y'Z'$ is described by three angles relative to a coordinate system with

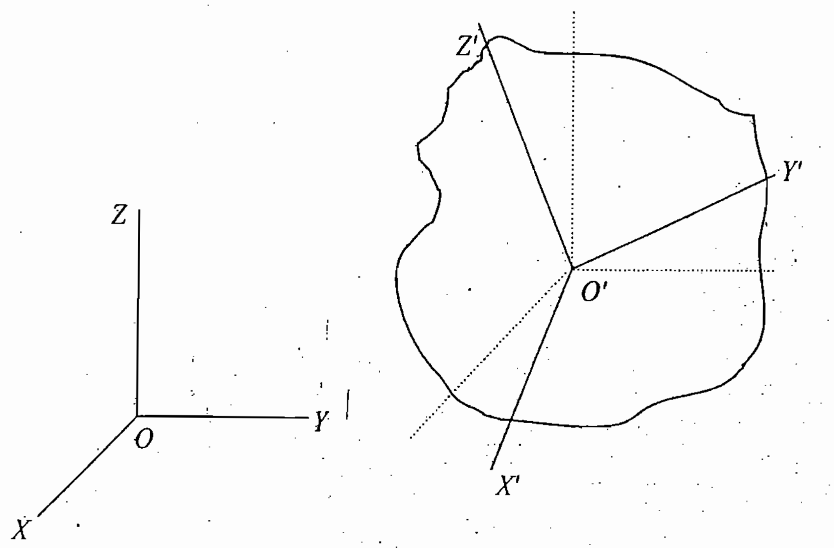


Fig. 10.2. XYZ – Space set of axes ; $X'Y'Z'$ – Body set of axes

* We describe Euler's angles in Art. 10.3.

common origin O' and axes parallel to the space set of axes (XYZ). Thus three coordinates of the origin O' and these three angles constitute six independent coordinates which provide complete configuration of the rigid body in motion at any instant of time.

For convenience, first consider the origins of space set of axes and body set of axes to be the same (O). In order to specify the orientation of the body set of axes, we may use the direction cosines of body set of axes ($X'Y'Z'$) relative to the space set of axes (XYZ).

Let $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ be the unit vectors along X, Y, Z axes and $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$, $\hat{\mathbf{k}}'$ along X', Y', Z' axes respectively. If C_{11}, C_{12}, C_{13} be the direction cosines of the X' axis (or $\hat{\mathbf{i}}'$ unit vector) with respect to X, Y, Z axes respectively, then

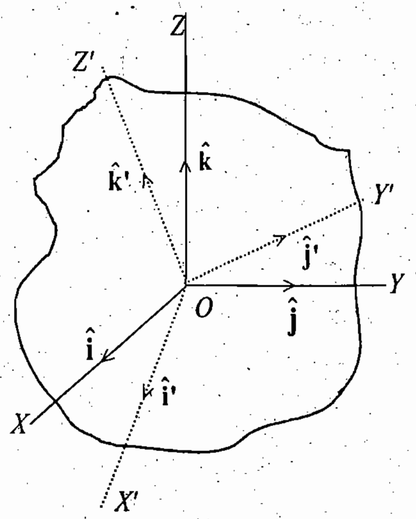


Fig. 10.3. X Y Z - Space set of axes ; X'Y'Z' - Body set of axes

$$\begin{aligned} C_{11} &= \cos(X', X) = \cos(\hat{\mathbf{i}}', \hat{\mathbf{i}}) = \hat{\mathbf{i}}' \cdot \hat{\mathbf{i}} \\ C_{12} &= \cos(X', Y) = \cos(\hat{\mathbf{i}}', \hat{\mathbf{j}}) = \hat{\mathbf{i}}' \cdot \hat{\mathbf{j}} \qquad \text{...(3)} \\ C_{13} &= \cos(X', Z) = \cos(\hat{\mathbf{i}}', \hat{\mathbf{k}}) = \hat{\mathbf{i}}' \cdot \hat{\mathbf{k}} \end{aligned}$$

Thus $$\hat{\mathbf{i}}' = C_{11}\hat{\mathbf{i}} + C_{12}\hat{\mathbf{j}} + C_{13}\hat{\mathbf{k}} \qquad \text{...(4 a)}$$

or $$\hat{\mathbf{i}}' = (\hat{\mathbf{i}}'\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{i}}'\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{i}}'\cdot\hat{\mathbf{k}})\hat{\mathbf{k}} \qquad \text{...(4 a)'}$$

Similarly, $$\hat{\mathbf{j}}' = C_{21}\hat{\mathbf{i}} + C_{22}\hat{\mathbf{j}} + C_{23}\hat{\mathbf{k}} \qquad \text{...(4 b)}$$

and $$\hat{\mathbf{k}}' = C_{31}\hat{\mathbf{i}} + C_{32}\hat{\mathbf{j}} + C_{33}\hat{\mathbf{k}} \qquad \text{...(4 c)}$$

where C_{21}, C_{22}, C_{23} are the direction cosines of Y' -axis and C_{31}, C_{32}, C_{33} those of Z'-axis with respect to X, Y, Z axes respectively.

These sets of nine direction cosines then completely specify the orientation of the X', Y', Z' axes with respect to X, Y, Z axes. With the help of these direction cosines, we can also relate the coordinates of a given point from one system to another. If $\mathbf{r}$ be the position vector of a point with coordinates (x, y, z) and (x', y', z') in the two systems, then

$$\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad \text{...(5 a)}$$

and $$\mathbf{r} = x'\hat{\mathbf{i}}' + y'\hat{\mathbf{j}}' + z'\hat{\mathbf{k}}' \qquad \text{...(5 b)}$$

Now, $$x' = (\mathbf{r} \cdot \hat{\mathbf{i}}') = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (C_{11}\hat{\mathbf{i}} + C_{12}\hat{\mathbf{j}} + C_{13}\hat{\mathbf{k}})$$

or $$x' = C_{11}x + C_{12}y + C_{13}z \qquad \text{...(6 a)}$$

Similarly, $$y' = C_{21}x + C_{22}y + C_{23}z \qquad \text{...(6 b)}$$

and $$z' = C_{31}x + C_{32}y + C_{33}z \qquad \text{...(6 c)}$$

If $\mathbf{A}$ is any vector, then its components along X'-axis are related to its components A_x, A_y, A_z in the unprimed frame as

$$A_{x'} = (\mathbf{A} \cdot \hat{\mathbf{i}}') = (C_{11}A_x + C_{12}A_y + C_{13}A_z) \qquad \text{...(7)}$$

and so on.

We may invert the process and express $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ or any vector in terms of their components along X', Y', Z' axes. Thus

$$\hat{\mathbf{i}} = (\hat{\mathbf{i}} \cdot \hat{\mathbf{i}}')\hat{\mathbf{i}}' + (\hat{\mathbf{i}} \cdot \hat{\mathbf{j}}')\hat{\mathbf{j}}' + (\hat{\mathbf{i}} \cdot \hat{\mathbf{k}}')\hat{\mathbf{k}}'$$

or

$$\hat{\mathbf{i}} = C_{11}\hat{\mathbf{i}}' + C_{21}\hat{\mathbf{j}}' + C_{31}\hat{\mathbf{k}}'$$

$$\hat{\mathbf{j}} = C_{12}\hat{\mathbf{i}}' + C_{22}\hat{\mathbf{j}}' + C_{32}\hat{\mathbf{k}}' \qquad ...(8)$$

$$\hat{\mathbf{k}} = C_{13}\hat{\mathbf{i}}' + C_{23}\hat{\mathbf{j}}' + C_{33}\hat{\mathbf{k}}'$$

Similarly, x, y, z components of $\mathbf{r}$ vector are

$$x = C_{11}x' + C_{21}y' + C_{31}z'$$

$$y = C_{12}x' + C_{22}y' + C_{32}z' \qquad ...(9)$$

$$z = C_{13}x' + C_{23}y' + C_{33}z'$$

These nine direction cosines are not all independent and the relations between them arise because the unit vectors in a coordinate system are orthogonal to each other with unit magnitude. If we consider $\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'$ unit vectors in X' Y' Z' coordinate system, then

$$\hat{\mathbf{i}}' \cdot \hat{\mathbf{i}}' = 1 = C_{11}^2 + C_{12}^2 + C_{13}^2 \qquad ...(10\ a)$$

$$\hat{\mathbf{j}}' \cdot \hat{\mathbf{j}}' = 1 = C_{21}^2 + C_{22}^2 + C_{23}^2 \qquad ...(10\ b)$$

$$\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}' = 1 = C_{31}^2 + C_{32}^2 + C_{33}^2 \qquad ...(10\ c)$$

$$\hat{\mathbf{i}}' \cdot \hat{\mathbf{j}}' = 0 = C_{11}C_{21} + C_{12}C_{22} + C_{13}C_{23} \qquad ...(10\ d)$$

$$\hat{\mathbf{j}}' \cdot \hat{\mathbf{k}}' = 0 = C_{21}C_{31} + C_{22}C_{32} + C_{23}C_{33} \qquad ...(10\ e)$$

$$\hat{\mathbf{k}}' \cdot \hat{\mathbf{i}}' = 0 = C_{31}C_{11} + C_{32}C_{12} + C_{33}C_{13} \qquad ...(10\ f)$$

These relations in direction cosines are six in number and reduce the number nine to three independent direction cosines which in fact correspond to three degrees of freedom, needed to specify the rotation about the common origin. Thus we cannot take these nine direction cosines as generalized coordinates to set up the Lagrangian and equations of motion for the rigid body because they are not all independent. In order to have three generalized coordinates corresponding to three degrees of freedom, we need a set of three independent functions of the direction cosines. A number of such sets of three generalized coordinates have been described in the literature, but the most common and important ones are the Euler's angles.

10.3. EULER'S ANGLES

We are interested in knowing three independent parameters to specify the orientation of body set of axes relative to the space set of axes. For this purpose, we use three angles. These angles may be chosen in various ways, but the most commonly used set of three angles are the Euler's angles, represented by ϕ, θ and ψ.

We can reach an arbitrary orientation of the body set of axes X' Y' Z' from space set of axes (X Y Z) by making three successive rotations performed in a specific order.

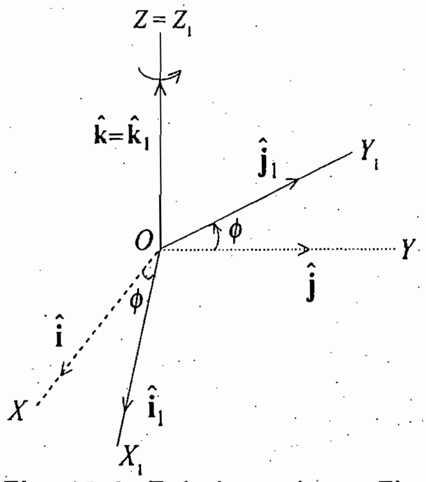


Fig. 10.4. Euler's angles – First rotation ϕ, defining precession angle.

(1) First rotation (ϕ) : First the space set of axes is rotated through an angle ϕ counter-clockwise about the Z-axis so that Y-Z plane takes the new position Y_1- Z_1 and this new plane Y_1- Z_1 contains the Z'-axis of the body coordinate system. Now the new position of the coordinate system is $X_1\ Y_1\ Z_1$ (with $Z = Z_1$) [Fig. 10.4]. If $\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'$ are the unit vectors along X, Y, Z axes and $\hat{\mathbf{i}}_1, \hat{\mathbf{j}}_1, \hat{\mathbf{k}}_1$ along X_1, Y_1, Z_1 axes respectively, then the transformation to this new set of axes from space set of axes is represented by the equations

$$\begin{aligned} \hat{\mathbf{i}}_1 &= \cos\phi\ \hat{\mathbf{i}} + \sin\phi\ \hat{\mathbf{j}} \\ \hat{\mathbf{j}}_1 &= -\sin\phi\ \hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}}_1 &= \hat{\mathbf{k}} \end{aligned} \quad \text{...(11)*}$$

or

$$\begin{pmatrix} \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_1 \\ \hat{\mathbf{k}}_1 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{pmatrix} \quad \text{...(12)}$$

Thus XYZ axes are transformed to $X_1\ Y_1\ Z_1$ by the matrix of transformation

$$\boldsymbol{D} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{...(13)}$$

The angle ϕ is called the ***precession angle.***

(2) Second rotation (θ) : Next intermediate axes $X_1\ Y_1\ Z_1$ are rotated about X_1 axis counter-clockwise through an angle θ to the position $X_2\ Y_2 Z_2$ so that Y_1, Z_1 axes acquire the positions Y_2, Z_2 with $Z_2 = Z'$ [Fig. 10.5]. This also results the plane $X_2.Y_2$ in plane $X'\ Y'$. If $\hat{\mathbf{i}}_2, \hat{\mathbf{j}}_2, \hat{\mathbf{k}}_2$ are unit vectors along X_2, Y_2, Z_2 axes respectively, then

$$\begin{aligned} \hat{\mathbf{i}}_2 &= \hat{\mathbf{i}}_1 \\ \hat{\mathbf{j}}_2 &= \cos\theta\ \hat{\mathbf{j}}_1 + \sin\theta\ \hat{\mathbf{k}}_1 \\ \hat{\mathbf{k}}_2 &= -\sin\theta\ \hat{\mathbf{j}}_1 + \cos\theta\ \hat{\mathbf{k}}_1 \end{aligned}$$

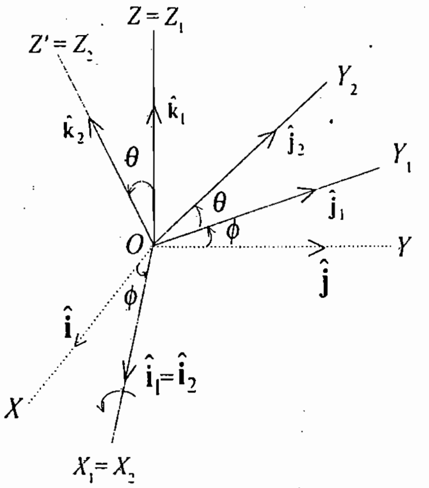


Fig. 10.5. Euler's angles - Second rotation θ, defining nutation angle

---

$$* \quad \hat{\mathbf{i}}_1 = (\hat{\mathbf{i}}_1 \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{i}}_1 \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{i}}_1 \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$$

$$= \cos\phi\ \hat{\mathbf{i}} + \cos\left(\frac{\pi}{2} - \phi\right)\hat{\mathbf{j}} + \cos\frac{\pi}{2}\hat{\mathbf{k}}$$

$$= \cos\phi\ \hat{\mathbf{i}} + \sin\phi\ \hat{\mathbf{j}}$$

$$\hat{\mathbf{j}}_1 = (\hat{\mathbf{j}}_1 \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{j}}_1 \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{j}}_1 \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$$

$$= \cos\left(\frac{\pi}{2} + \phi\right)\hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}} = -\sin\phi\ \hat{\mathbf{i}} + \cos\phi\ \hat{\mathbf{j}}.$$

or
$$\begin{pmatrix}\hat{\mathbf{i}}_2\\ \hat{\mathbf{j}}_2\\ \hat{\mathbf{k}}_2\end{pmatrix}=\begin{pmatrix}1 & 1 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\hat{\mathbf{i}}_1\\ \hat{\mathbf{j}}_1\\ \hat{\mathbf{k}}_1\end{pmatrix} \quad ...(14)$$

In this case the matrix of transformation is

$$C=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix} \quad ...(15)$$

The angle θ is called the ***nutation angle***. The $X_2 = X_1$ axis is at the intersection of the X-Y and X_2-Y_2 planes and is called the ***line of nodes***.

(3) Third rotation (ψ) : Finally the third rotation is performed about $Z_2 = Z'$ axis through an angle ψ counter-clockwise so that X_2, Y_2 axes coincide $X_3 = X'$, $Y_3 = Y'$ [Fig 10.6 (a)].

Thus these three rotations ϕ, θ and ψ bring the space set of axes to coincide with body set of axes. The ϕ, θ and ψ are the Euler's angles and completely specify the orientation of the $X'\,Y'\,Z'$ system relative to the $X\,Y\,Z$ system. These ϕ, θ and ψ angles can be taken as three generalized coordinates. Now

$$\hat{\mathbf{i}}_3=\hat{\mathbf{i}}'=\hat{\mathbf{i}}_2\cos\psi+\hat{\mathbf{j}}_2\sin\psi$$
$$\hat{\mathbf{j}}_3=\hat{\mathbf{j}}'=-\hat{\mathbf{i}}_2\sin\psi+\hat{\mathbf{j}}_2\cos\psi$$
$$\hat{\mathbf{k}}_3=\hat{\mathbf{k}}'=\hat{\mathbf{k}}_2$$

or
$$\begin{pmatrix}\hat{\mathbf{i}}'\\ \hat{\mathbf{j}}'\\ \hat{\mathbf{k}}'\end{pmatrix}=\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\mathbf{i}}_2\\ \hat{\mathbf{j}}_2\\ \hat{\mathbf{k}}_2\end{pmatrix} \quad ...(16)$$

So that the transformation matrix is

$$B=\begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix} \quad ...(17)$$

The angle ψ is called the ***body angle***.

In this way we have reached at the body set of axes after three successive rotations of space set of axes. We may write the complete matrix of transformations A as

$$\begin{pmatrix}\hat{\mathbf{i}}'\\ \hat{\mathbf{j}}'\\ \hat{\mathbf{k}}'\end{pmatrix}=A\begin{pmatrix}\hat{\mathbf{i}}\\ \hat{\mathbf{j}}\\ \hat{\mathbf{k}}\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}=A\begin{pmatrix}x\\ y\\ z\end{pmatrix} \quad ...(18)$$

But using (12), (13), (14), (15), (16) and (17), we get

$$\begin{pmatrix}\hat{\mathbf{i}}'\\ \hat{\mathbf{j}}'\\ \hat{\mathbf{k}}'\end{pmatrix}=\begin{pmatrix}\hat{\mathbf{i}}_3\\ \hat{\mathbf{j}}_3\\ \hat{\mathbf{k}}_3\end{pmatrix}=B\begin{pmatrix}\hat{\mathbf{i}}_2\\ \hat{\mathbf{j}}_2\\ \hat{\mathbf{k}}_2\end{pmatrix}=B\,C\begin{pmatrix}\hat{\mathbf{i}}_1\\ \hat{\mathbf{j}}_1\\ \hat{\mathbf{k}}_1\end{pmatrix}=B\,C\,D\begin{pmatrix}\hat{\mathbf{i}}\\ \hat{\mathbf{j}}\\ \hat{\mathbf{k}}\end{pmatrix} \quad ...(19)$$

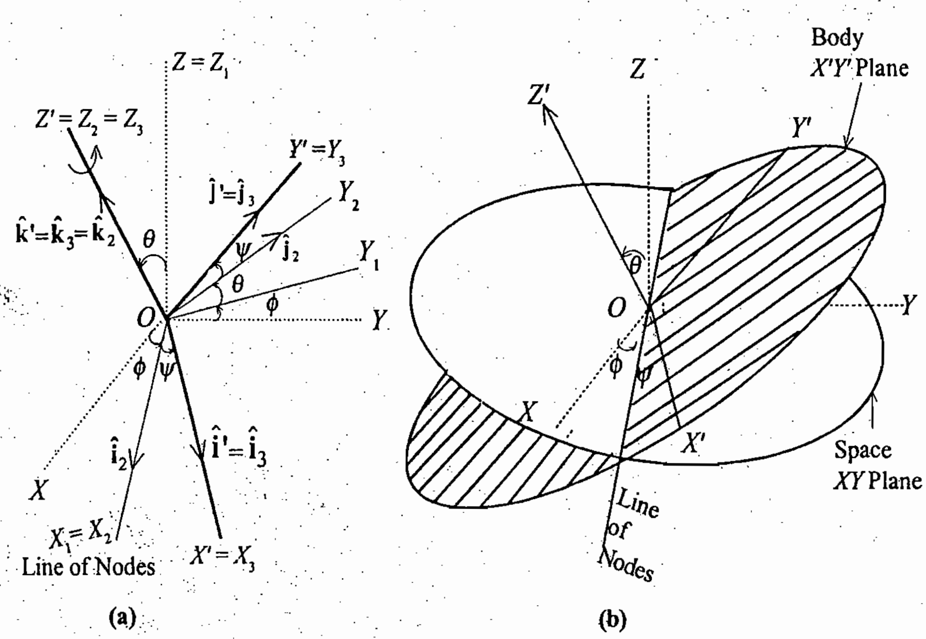


Fig. 10.6. (a) Euler's angles – Third rotation ψ, defining body angle, (b) The three Eulerian angle ϕ, θ and ψ in different planes

From (18) and (19) we see that the complete matrix of transformation from space set of axes to body set of axes is

$$A = BCD \qquad \text{....(20)}$$

The inverse transformation from body set of axes to space set of axes will be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Now

$$A = BCD = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \qquad \text{...(21)}$$

The inverse transformation matrix from body set of axes to space set of axes is given by $A^{-1} = A_T$ because A represents a proper orthogonal matrix. Thus

$$A^{-1} = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & -\sin\theta\cos\phi \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix} \qquad \text{...(22)}$$

10.4. INFINITESIMAL ROTATIONS AS VECTORS – ANGULAR VELOCITY

A finite rotation or angular displacement cannot be represented by a vector.

This is because the sum of two such rotations, one after the other, is not given by the parallelogram law of addition of vectors. For illustration, suppose the point P of the sphere in fig 10.7 moves to position Q by a 90°

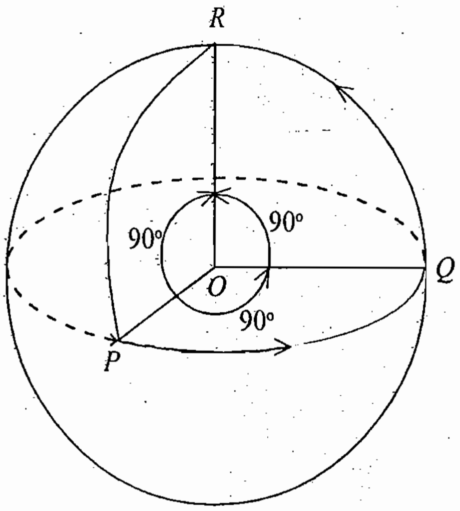


Fig. 10.7. Finite rotation cannot be represented as vector

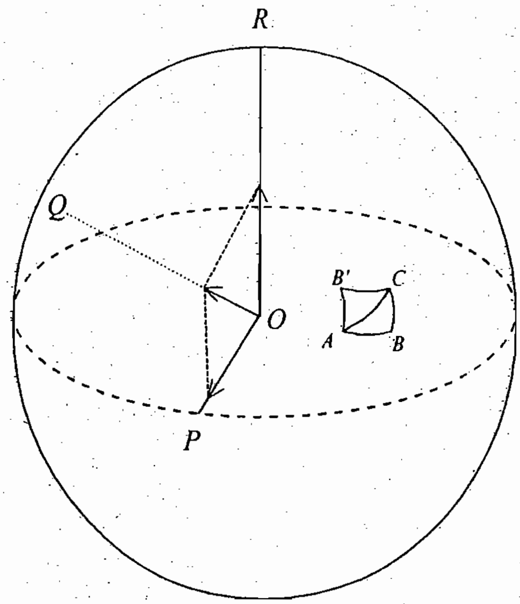


Fig. 10.8. Representation of infinitesimal rotation as vectors

rotation about OR axis. Let this rotation be represented by $\overrightarrow{OR}$. Further rotation from Q to R by 90° may be represented by $\overrightarrow{OP}$. We observe that to bring the point P to the point R, we can perform a single rotation of 90° about QO and hence the total rotation (P to Q and Q to R) is to be represented by the vector $\overrightarrow{QO}$. As $\overrightarrow{QO}$ is not the diagonal of the parallelogram formed by $\overrightarrow{OR}$ and $\overrightarrow{OP}$, we conclude that ***finite rotations or angular displacements cannot be represented as vector quantities.***

If the rotations are infinitesimal, we can represent them by vectors. In Fig 10.8, we consider very small rotations about axes OR and OP. For this purpose, we move any point A on the sphere first to B about OR axis and then to C about OP axis. The total movement AC could be done by a single rotation about an axis OQ, which is in the direction of the resultant of the original vectors along OR and OP. This addition is also commutative. Now, if $d\alpha$ is the infinitesimal rotation occurring between t and $t + dt$ times, then the instantaneous angular velocity ω of the rigid body is defined by the rotation fig. 10.9.

$$\omega\, dt = d\alpha \text{ or } \omega = \frac{d\alpha}{dt} \qquad \text{...(23)}$$

where the vector ω is along the axis of the infinitesimal rotation.

The instantaneous linear velocity of a point at $\mathbf{r}$ relative to the origin O is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad \text{...(24)}$$

where the magnitude of the instantaneous linear velocity is $\omega r \sin \beta$.

10.5. COMPONENTS OF ANGULAR VELOCITY

An infinitesimal rotation can be represented by a vector. If ϕ, θ, and ψ represent the Euler's angles, then the three time derivatives, $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ represent the angular speeds about the space Z-axis, line of nodes and

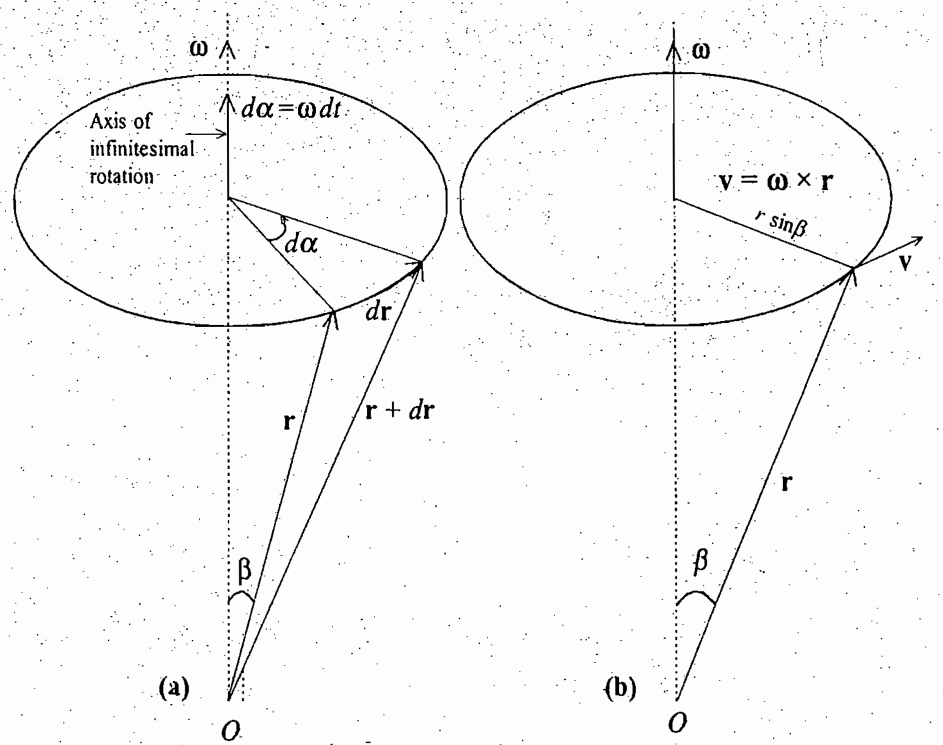


Fig. 10.9 : (a) Infinitesimal rotation $d\alpha$ and angular velocity $\omega = d\alpha/dt$.
(b) Representation of linear velocity $v = \omega \times r$

body Z' axis respectively. We denote these three angular speeds by ω_ϕ, ω_θ and ω_ψ and may consider as the three components of the angular velocity $\boldsymbol{\omega}$. However, these three components of $\boldsymbol{\omega}$ are not all either along the space set of axes or the body set of axes. Therefore, it is not convenient to use these components to describe the motion of a rigid body. Rigid body equations will be described in terms of the body set of axes. Thus we must calculate the angular velocity vector $\boldsymbol{\omega}$ ($\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$) in the body coordinate system. In order to obtain the angular velocity components $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ along the body set of axes, we should resolve $\dot\phi$, $\dot\theta$ and $\dot\psi$ along these axes X', Y', Z' i.e.,

$$\dot\phi_{x'} = \dot\phi \sin\theta \sin\psi, \dot\theta_{x'} = \dot\theta \cos\psi, \dot\psi_{x'} = 0 \text{ along } X'\text{-axis},$$

$$\dot\phi_{y'} = \dot\phi \sin\theta \cos\psi, \dot\theta_{y'} = -\dot\theta \sin\psi, \dot\psi_{y'} = 0 \text{ along } Y'\text{-axis},$$

$$\dot\phi_{z'} = \dot\phi \cos\theta ; \dot\theta_{z'} = 0, \dot\psi_{z'} = \dot\psi \text{ along } Z'\text{-axis}.$$

Thus the component of $\boldsymbol{\omega}$ ($\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$) along the body set of axes are

$$\omega_{x'} = \dot\phi_{x'} + \dot\theta_{x'} + \dot\psi_{x'} \text{ or } \omega_{x'} = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$

$$\omega_{y'} = \dot\phi_{y'} + \dot\theta_{y'} + \dot\psi_{y'} \text{ or } \omega_{y'} = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi \qquad \text{...(25)}$$

$$\omega_{z'} = \dot\phi_{z'} + \dot\theta_{z'} + \dot\psi_{z'} \text{ or } \omega_{z'} = \dot\phi \cos\theta + \dot\psi$$

These equations are known as ***Euler's geometrical equations***. We may use these equations to describe the motion of a rigid body in the body coordinate system.

In further discussion, we shall be using body set of axes, fixed in the rigid body. Therefore, it will be convenient to omit the prime sign to denote the body set of axes. In this system, x_i, y_i, z_i, the distances of any ith particle of the rigid body from the fixed origin, are constant in time and the calculations regarding rigid body motion become easier.

10.6. ANGULAR MOMENTUM AND INERTIA TENSOR

Let us consider the motion of a rigid body rotating about a fixed point O in the body [Fig. 10.10]. At any instant of time, the body will be rotating with angular velocity $\boldsymbol{\omega}$ about the instantaneous axis through O. A particle P of the body, having the position vector $\mathbf{r}_i$ with respect to O, has an instantaneous velocity $\mathbf{v}_i$ relative to O, given by

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i \qquad ...(26)$$

Let $OXYZ$ be the body set of axes fixed in the body. The angular velocity $\boldsymbol{\omega}$ has components ω_x, ω_y, ω_z along X, Y, Z axes respectively and is given by

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}}$$

The angular momentum of the particle P about the point O is given by

$$\mathbf{J}_p = \mathbf{r}_i \times m_i\mathbf{v}_i$$

where m_i is the mass of the particle P.

Hence the angular momentum $\mathbf{J}$ of the entire body about the point O can be obtained by summing $\mathbf{J}_p$ for all the particles of the body, *i.e.*,

$$\mathbf{J} = \sum_{i=1}^{N} \mathbf{r}_i \times m_i\mathbf{v}_i = \sum_i m_i\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \qquad ...(27)$$

where we have used eq. (26). Using the definition of vector triple product $[\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}]$, we have

$$\mathbf{J} = \sum_i m_i[(\mathbf{r}_i \cdot \mathbf{r}_i)\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i]$$

or
$$\mathbf{J} = \sum_i m_i[\mathbf{r}_i^2\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i] \qquad ...(28)$$

whose direction, in general, is not along $\boldsymbol{\omega}$.

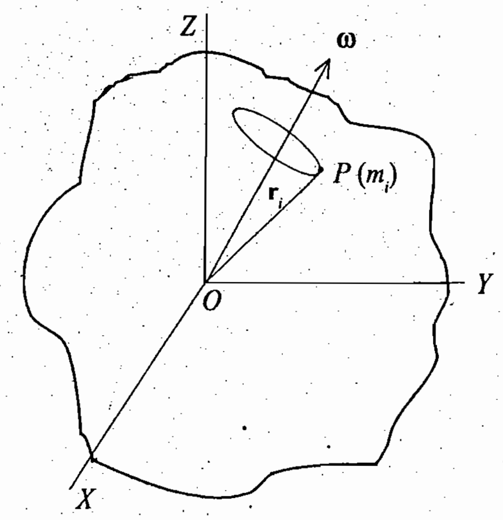


Fig. 10.10 : Rotating rigid body

If J_x, J_y, J_z are the components of angular momentum along X, Y, Z axes respectively, then

$$J_x = \sum_i m_i[r_i^2\omega_x - (x_i\omega_x + y_i\omega_y + z_i\omega_z)x_i]$$

or
$$J_x = \omega_x\sum_i m_i(r_i^2 - x_i^2) - \omega_y\sum_i m_ix_iy_i - \omega_z\sum_i m_ix_iz_i \qquad ...(29\ a)$$

Similarly,
$$J_y = -\omega_x\sum_i m_ix_iy_i + \omega_y\sum_i m_i(r_i^2 - y_i^2) - \omega_z\sum_i m_iy_iz_i \qquad ...(29\ b)$$

and
$$J_z = -\omega_x\sum_i m_ix_iz_i - \omega_y\sum_i m_iy_iz_i + \omega_z\sum_i m_i(r_i^2 - z_i^2) \qquad ...(29\ c)$$

Eq. (29) can be written as

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \qquad ...(30\ a)$$
$$J_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \qquad ...(30\ b)$$
$$J_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \qquad ...(30\ c)$$

where
$$I_{xx} = \sum_i m_i(r_i^2 - x_i^2) = \sum_i m_i(y_i^2 + z_i^2) \qquad ...(31\ a)$$

$$I_{yy} = \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + z_i^2) \qquad ...(31\ b)$$

$$I_{zz} = \sum_i m_i(r_i^2 - z_i^2) = \sum_i m_i(x_i^2 + y_i^2) \quad ...(31\ c)$$

$$I_{xy} = -\sum_i m_i x_i y_i = I_{yx} \quad ...(32\ a)$$

$$I_{xz} = -\sum_i m_i x_i z_i = I_{zx} \quad ...(32\ b)$$

$$I_{yz} = -\sum_i m_i y_i z_i = I_{zy} \quad ...(32\ c)$$

The quantity $I_{xx}[= \sum_i m_i(y_i^2 + z_i^2)]$ is called the ***moment of inertia*** of the body about X-axis. Similarly, I_{yy} and I_{zz} define the moments of inertia of the body about the Y and Z axes respectively. The quantities I_{xy}, I_{xz},....etc., which involve the sums of products of coordinates, are called ***products of inertia***.

Any component of the angular momentum vector may be written as

$$J_\alpha = \sum_\beta I_{\alpha\beta}\,\omega_\beta \quad ...(33)$$

where α, $\beta = x, y, z$.

Eq. (30) may be written in the matrix notation as

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{xz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad ...(34\ a)$$

or $$J = I\omega \quad ...(34\ b)$$

The nine elements I_{xx}, I_{xy},..., I_{xz} of the (3 × 3) matrix may be regarded as components of a single entity I. This entity I is called ***inertia tensor***. Since $I_{xy} = I_{yx}$ etc., I is a ***symmetric tensor***. If we denote x, y, z by x_1, x_2, x_3 respectively, then in general any element of the inertia tensor is given by

$$I_{\alpha\beta} = I_{\beta\alpha} = \sum_{i=1}^{N} m_i[\delta_{\alpha\beta} r_i^2 - x_{i\alpha} x_{i\beta}] \quad ...(35)$$

where α, $\beta = 1,2,3$.

In eqs. (31) and (32), the matrix elements are in the suitable form, if the rigid body is composed of discrete particles. In case of a continuous body, the summation sign is replaced by mass or volume integration. Thus the diagonal element I_{xx} is

$$I_{xx} = \int (r^2 - x^2)\, dm = \int \rho(\mathbf{r})(r^2 - x^2)\, dV = \int \rho(\mathbf{r})\,(y^2 + z^2)\, dV \quad ...(36\ a)$$

where dm is the mass of an infinitesimal volume element dV and ρ ($\mathbf{r}$) is the mass density at $\mathbf{r}$ within dV.

The off-diagonal element I_{xy} is

$$I_{xy} = -\int xy\, dm = -\int \rho(\mathbf{r})\, xy\, dV \quad ...(36\ b)$$

In general, the matrix element $I_{\alpha\beta}$ can be written as

$$I_{\alpha\beta} = \int \rho(\mathbf{r})(r^2\delta_{\alpha\beta} - x_\alpha x_\beta) dV \quad ...(37)$$

10.7. PRINCIPAL AXES – PRINCIPAL MOMENTS OF INERTIA

If we choose the axes of the coordinate system fixed in the body with respect to which off-diagonal elements disappear and only the diagonal elements remain in the inertia tensor, then such axes are called the ***principal axes*** of the body and the corresponding moments of inertia as the ***principal moments of inertia***. In general the directions of the principal axes are different to those of arbitrary axes fixed in the body.

If we denote the principal axes by X', Y', Z', then the inertial tensor is

$$I' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad ...(38)$$

where we have denoted $I_{x'x'} = I_1, I_{y'y'} = I_2$ and $I_{z'z'} = I_3$. If ω_1, ω_2, ω_3 be the components of angular velocity and J_1, J_2, J_3 those of angular momentum about the principal axes, then from eq. (34) we obtain for the principal axes

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad ...(39\text{ a})$$

or $$J_1 = I_1\omega_1,\ J_2 = I_2\omega_2 \text{ and } J_3 = I_3\omega_3 \qquad ...(39\text{ b})$$

Thus, in general, if a rigid body is rotating about a principal axis, the angular momentum **J** and angular velocity **ω** are directed along any of the principal axes and we may write

$$\mathbf{J} = I\boldsymbol{\omega} \qquad ...(40)$$

where I is the scalar, being the moment of inertia about this axis. The angular momentum **J** and angular velocity **ω** are along a principal axis and hence each will have three components along the axes of any arbitrary coordinate system (X, Y, Z) fixed in the body (body coordinate system) *i.e.*,

$$\mathbf{J} = J_x\hat{\mathbf{i}} + J_y\hat{\mathbf{j}} + J_z\hat{\mathbf{k}} = I(\omega_x\hat{\mathbf{i}} + \omega_y\hat{\mathbf{j}} + \omega_z\hat{\mathbf{k}})$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are unit vectors along X, Y, Z axes respectively.

Thus $$J_x = I\omega_x,\ J_y = I\omega_y,\ J_z = I\omega_z \qquad ...(41)$$

Eqs. (30) and (41) give (using the symmetry property of inertia tensor)

$$J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = I\omega_x$$

$$J_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z = I\omega_y$$

$$J_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z = I\omega_z$$

or $$(I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0$$

$$I_{xy}\omega_x + (I_{yy} - I)\omega_y + I_{yz}\omega_z = 0$$

$$I_{xz}\omega_x + I_{yz}\omega_y + (I_{zz} - I)\omega_z = 0$$

or $$\begin{pmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} - I & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} - I \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \qquad ...(42)$$

For these equations to have non-trivial solutions, the determinant of the coefficients must vanish, *i.e.*,

$$\begin{vmatrix} I_{xx}-I & I_{xy} & I_{xz} \\ I_{xy} & I_{yy}-I & I_{zy} \\ I_{xz} & I_{zy} & I_{zz}-I \end{vmatrix} = 0 \qquad ...(43)$$

This is called ***secular*** or ***characteristic equation*** and its solutions are called ***eigen values***. Eq. (43) is cubic in I, the three solutions $I = I_1, I_2, I_3$ are the ***three principal moments of inertia***.

The direction of any one principal axis is determined by substituting for I equal to one of the three roots, say $I = I_1$, in eqs. (42) and determine the ratios for $\omega_x : \omega_y : \omega_z$, *i.e.*,

$$\frac{\omega_y}{\omega_x} = \lambda_1, \frac{\omega_z}{\omega_x} = \lambda_2 \text{ (say)}. \text{ Thus } \boldsymbol{\omega} = \omega_x\hat{\mathbf{i}} + \lambda_1\omega_x\hat{\mathbf{j}} + \omega_x\lambda_2\hat{\mathbf{k}}$$

$$\therefore \qquad \hat{\boldsymbol{\omega}} = (\hat{\mathbf{i}} + \lambda_1\hat{\mathbf{j}} + \lambda_2\hat{\mathbf{k}})\Big/\sqrt{1+\lambda_1^2+\lambda_2^2}\,.$$

Hence we can determine the direction of ω or the direction of the principal axis corresponding to I_1. Similarly, if we substitute I_2 or I_3, we may find the direction of the corresponding principal axis by using eqs. (42). The magnitude of angular velocity is arbitrary and we are free to take its any value.

10.8 . ROTATIONAL KINETIC ENERGY OF A RIGID BODY

Let a rigid body be rotating about an axis passing through a fixed point in the body with an angular velocity ω. The velocity $\mathbf{v}_i$ of *i*th particle of mass m_i of the body is given by

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

The kinetic energy of this particle is given by

$$T_i = \frac{1}{2}m_i v_i^2 = \frac{1}{2}m_i\mathbf{v}_i \bullet \mathbf{v}_i$$

Total kinetic energy of the entire body is given by

$$T = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i\mathbf{v}_i \bullet \mathbf{v}_i = \frac{1}{2}\sum_i \mathbf{v}_i \bullet m_i\mathbf{v}_i = \frac{1}{2}\sum_i (\boldsymbol{\omega} \times \mathbf{r}_i) \bullet m_i\mathbf{v}_i$$

$$= \frac{1}{2}\sum_i \boldsymbol{\omega} \bullet (\mathbf{r}_i \times m_i\mathbf{v}_i) \;[\because (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})]$$

$$= \frac{1}{2}\boldsymbol{\omega} \bullet \sum_i (\mathbf{r}_i \times m_i\mathbf{v}_i) \text{ (because } \boldsymbol{\omega} \text{ is the same for all particles.)}$$

But $\qquad \mathbf{J} = \sum_i (\mathbf{r}_i \times m_i\mathbf{v}_i)$ about the fixed point, hence

$$T = \frac{1}{2}\boldsymbol{\omega} \bullet \mathbf{J} \qquad ...(44)$$

This is a scalar and can be written in the form

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J} = \frac{1}{2}\omega_x J_x + \frac{1}{2}\omega_y J_y + \frac{1}{2}\omega_z J_z$$

$$= \frac{1}{2}\omega_x(I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z) + \frac{1}{2}\omega_y(I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z) + \frac{1}{2}\omega_z(I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z)$$

$$= \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2 + I_{xy}\omega_x\omega_y + I_{xz}\omega_x\omega_z + I_{yz}\omega_y\omega_z \qquad ...(45)$$

The kinetic energy may be written in the compact form as

$$T = \frac{1}{2} \sum_{\alpha,\beta=1}^{3} I_{\alpha\beta}\, \omega_\alpha \omega_\beta \qquad ...(46)$$

In case, I_1, I_2, I_3 are the principal moments of inertia, the angular momentum **J** and angular velocity **ω** are related by eq. (39) *i.e.,*

$$J_1 = I_1\omega_1,\ J_2 = I_2\omega_2,\ J_3 = I_3\omega_3$$

where J_1, J_2, J_3 are the components of **J** along three principal axes.

Hence the kinetic energy in a coordinate system of principal axes is given by

$$T = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{J} = \frac{1}{2}\,\omega_1 J_1 + \frac{1}{2}\,\omega_2 J_2 + \frac{1}{2}\,\omega_3 J_3$$

$$= \frac{1}{2}\, I_1\omega_1^2 + \frac{1}{2}\, I_2\omega_2^2 + \frac{1}{2}\, I_3\omega_3^2$$

$$= \frac{1}{2} \sum_{\alpha,\beta=1}^{3} I_\alpha\, \omega_\alpha^2 \qquad ...(47)$$

We may express the kinetic energy of rotation in the usual form, used in elementary mechanics, as

$$T = \frac{1}{2} I\omega^2 \qquad ...(48)$$

where I is the moment of inertia about the axis of rotation. The proof is as follows :

Let $\hat{\mathbf{n}}$ be a unit vector along **ω**. So that

$$\boldsymbol{\omega} = \omega\,\hat{\mathbf{n}} \text{ and then } T = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{J} = \frac{1}{2}\,\omega\,(\hat{\mathbf{n}}\cdot\mathbf{J}) \qquad ...(49)$$

But from eq. (28), we have

$$\mathbf{J} = \sum_i m_i[r_i^2\omega\,\hat{\mathbf{n}} - (\mathbf{r}_i\cdot\omega\,\hat{\mathbf{n}})\mathbf{r}_i]$$

So that $\hat{\mathbf{n}}\cdot\mathbf{J} = \hat{\mathbf{n}}\cdot\left(\sum_i m_i\omega\,[r_i^2\hat{\mathbf{n}} - (r_i\cdot\hat{\mathbf{n}})\,\mathbf{r}_i]\right) = \omega\sum_i m_i[r_i^2(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) - (\mathbf{r}_i\cdot\hat{\mathbf{n}})(\mathbf{r}_i\cdot\hat{\mathbf{n}})]$

But $\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} = 1$ and $r_i^2 = \mathbf{r}_i\cdot\mathbf{r}_i$. Hence

$$\hat{\mathbf{n}}\cdot\mathbf{J} = \omega\sum_i m_i[(\mathbf{r}_i\cdot\mathbf{r}_i) - (\mathbf{r}_i\cdot\hat{\mathbf{n}})\,(\mathbf{r}_i\cdot\hat{\mathbf{n}})] = \omega\sum_i m_i\mathbf{r}_i\cdot[(\mathbf{r}_i - \hat{\mathbf{n}}\,(\mathbf{r}_i\cdot\hat{\mathbf{n}})] = \omega\sum_i m_i\mathbf{r}_i\cdot[\hat{\mathbf{n}}\times(\mathbf{r}_i\times\hat{\mathbf{n}})]$$

because $\hat{\mathbf{n}}\times(\mathbf{r}_i\times\hat{\mathbf{n}}) = \mathbf{r}_i(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}) - \hat{\mathbf{n}}\,(\mathbf{r}_i\cdot\hat{\mathbf{n}}) = \mathbf{r}_i - \hat{\mathbf{n}}\,((\mathbf{r}_i\cdot\hat{\mathbf{n}})$.

As $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, $\hat{\mathbf{n}}\cdot\mathbf{J} = \omega\sum_i m_i(\vec{\mathbf{r}}_i\times\hat{\mathbf{n}})\cdot(\mathbf{r}_i\times\hat{\mathbf{n}}) = \omega\sum_i m_i\,|\mathbf{r}_i\times\hat{\mathbf{n}}|^2 = \omega\sum_i m_i r_{i0}^2$ ...(50)

where $r_{i0} = r_i\sin\theta_i = |\,\mathbf{r}_i\times\hat{\mathbf{n}}\,|$ is the perpendicular distance of the ith particle from the axis of rotation. Defining $I = \sum_i m_i r_{i0}^2$, the moment of inertia about the axis of rotation, the component of **J** along the axis of rotation, say J_0, is

$$J_0 = \hat{\mathbf{n}}\cdot\mathbf{J} = I\omega \qquad ...(51)$$

Substituting for $\hat{\mathbf{n}}\cdot\mathbf{J} = I\omega$ in eq. (49), the expression for kinetic energy is obtained as

$$T = \frac{1}{2} I\omega^2$$

which proves the relation (48).

10.9. SYMMETRIC BODIES

In most cases in rigid body dynamics, the body has some regular shape and the principal axes may be found by determining the symmetry of the body. The axis of symmetry is one of the principal axis. For example, in case of a circular cylinder, its axis of cylindrical symmetry is one of the principal axes. Suppose the axis of cylindrical symmetry is the Z-axis, then the contribution from (x, y, z) is cancelled from $(-x, -y, z)$ because $I_{xz} = -\Sigma m\, xz = 0$, and $I_{yz} = -\Sigma m\, yz = 0$ and then $J_z = I_{zz}\omega_z = I_3\omega_z$. Thus Z-axis or axis of cylindrical symmetry is the principal axis. Obviously the other two principal axes are in the X-Y plane with $I_1 = I_2$. In general, a rigid body is said to be *symmetric*, if two of its principal moments of inertia are equal. It may happen that all the three principal moments of inertia are equal i.e., $I_1 = I_2 = I_3$. This is the case for a sphere and a cube with origin at the centre. These bodies may be called *totally symmetric*.

10.10. MOMENTS OF INERTIA FOR DIFFERENT BODY SYSTEMS

Let I be the inertia tensor, defined in a body coordinate system with origin O and I^c the inertia tensor in a centre of mass coordinate system with origin O_c [fig. 10.11]. Also let the axes of the two coordinate systems be parallel to the corresponding axes. The components of the inertia tensor I are given by

$$I_{xx} = \sum m_i(r_i^2 - x_i^2) = \sum m_i(y_i^2 + z_i^2) \text{ and } I_{xy} = -\sum m_i x_i y_i \qquad ...(52)$$

If the centre of mass O_c of the system is at a distance $\mathbf{R}$ (x, y, z) from the origin O, the relation between $\mathbf{r}_i$ and $\mathbf{r}_i^c$ is

$$\mathbf{r}_i = \mathbf{r}_{ic} + \mathbf{R} \qquad ...(53\text{ a})$$

or in component form $\quad x_i = x_{ic} + X \qquad ...(53\text{ b})$

Substituting in (52), we have

$$I_{xx} = \sum m_i[(y_{ic} + Y)^2 + (z_{ic} + Z)^2] \text{ and } I_{xy} = -\sum m_i(x_{ic} + X)(y_{ic} + Y)$$

or

$$I_{xx} = \sum m_i(y_{ic}^2 + z_{ic}^2) + (Y^2 + Z^2)\sum m_i + 2Y\sum m_i y_{ic} + 2Z\sum m_i z_{ic}$$

and

$$I_{xy} = -\sum m_i x_{ic} y_{ic} - XY\sum m_i - X\sum m_i x_{ic} - Y\sum m_i y_{ic}$$

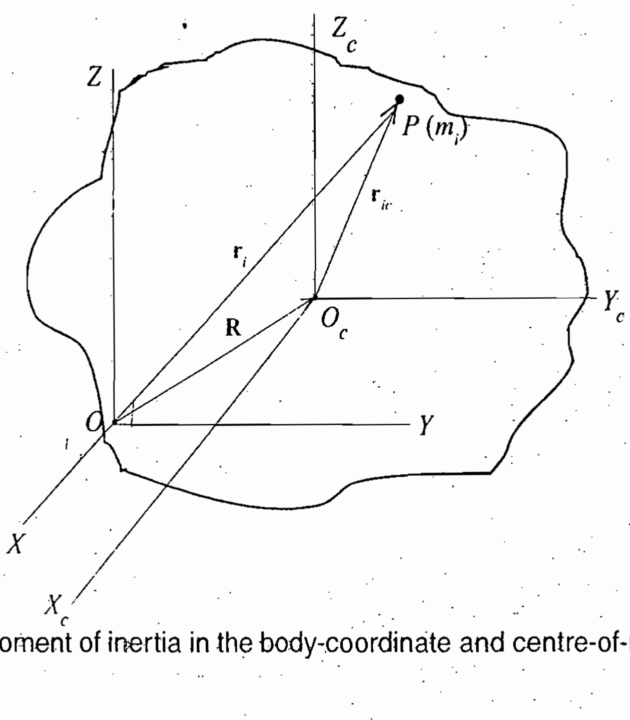


Fig. 10.11 : Moment of inertia in the body-coordinate and centre-of-mass systems

But because of the property of centre of mass [eq. (49), Chapter 1];

$$\sum m_i \mathbf{r}_{ic} = 0 \text{ or } \sum m_i x_{ic} = \sum m_i y_{ic} = \sum m_i z_{ic} = 0$$

Therefore, $$I_{xx} = M(Y^2 + Z^2) + I^c_{xx} \quad ...(54\,a)$$

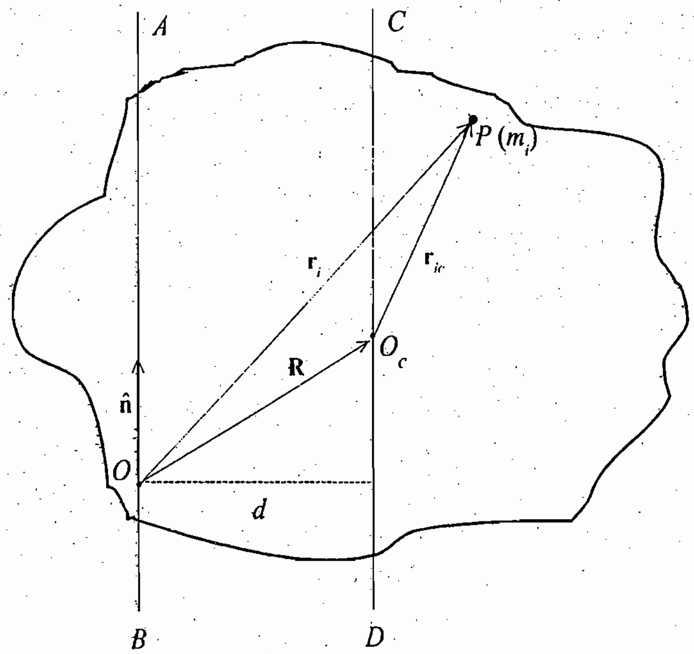


Fig. 10.12 : Presentation of vectors involved in relating the moments of inertia about parallel axes

and $$I_{xy} = -MXY + I^c_{xy} \quad ...(54\,b)$$

where I^c_{xx} and I^c_{xy} are the components of the inertia tensor I^c.

Thus the components of the inertia tensor with respect to an arbitrary origin are obtained from those with respect to the centre of mass by adding the contribution of a particle of mass M at $\mathbf{R}$. Thus for any given body we need only to compute the moments and products of inertia with respect to centre of mass and those with respect to any other origin may be obtained by using eqs. (54).

A similar treatment can be done to relate the moment of inertia (I) about an arbitrary axis (AB through O) to the moment of inertia (I_c) about a parallel axis (CD) passing through the centre of mass (O_c).

If $\hat{\mathbf{n}}$ is the unit vector along OA, then

$$I = \sum m_i(\mathbf{r}_i \times \hat{\mathbf{n}})^2 = \sum m_i[(\mathbf{r}_{ic} + \mathbf{R}) \times \hat{\mathbf{n}}]^2$$

$$= \sum m_i(\mathbf{r}_{ic} \times \hat{\mathbf{n}})^2 + (\mathbf{R} \times \hat{\mathbf{n}})^2 \sum m_i + 2\sum m_i(\mathbf{R} \times \hat{\mathbf{n}}) \cdot (\mathbf{r}_{ic} \times \hat{\mathbf{n}})$$

But the last term is $-2(\mathbf{R} \times \hat{\mathbf{n}}) \cdot (\hat{\mathbf{n}} \times \sum m_i \mathbf{r}_{ic}) = 0$

Therefore, $$I = I_c + M(\mathbf{R} \times \hat{\mathbf{n}})^2 \quad ...(55)$$

If d be the minimum distance between the parallel axes, then obviously $|\mathbf{R} \times \hat{\mathbf{n}}| = d$ and hence

$$I = I_c + Md^2 \quad ...(56)$$

This is known as the ***theorem of parallel axes*** in literature.

10.11. EULER'S EQUATIONS OF MOTION FOR A RIGID BODY

(1) Newtonian method : If a rigid body is rotating under the action of a torque $\boldsymbol{\tau}$ with one point fixed, then the torque is expressed as

$$\boldsymbol{\tau} = \left[\frac{d\mathbf{J}}{dt}\right]_s \qquad ...(57)$$

where **J** is the angular momentum and its time derivative refers to the space set of axes, represented by the subscript s, because the equation holds in an inertial frame.

The body coordinate system is rotating with an instantaneous angular velocity $\boldsymbol{\omega}$. The time derivatives of angular momentum **J** in the body coordinate and space coordinate systems are related as

$$\left[\frac{d\mathbf{J}}{dt}\right]_s = \left[\frac{d\mathbf{J}}{dt}\right]_b + \boldsymbol{\omega} \times \mathbf{J} \qquad ...(58)$$

Thus
$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt} + \boldsymbol{\omega} \times \mathbf{J} \qquad ...(59)$$

where we have dropped the body subscript because we shall represent the physical quantities of right hand side in the body coordinate system.

We choose principal axes for body set of axes. If I_1, I_2 and I_3 are the principal moments of inertia, then

$$\mathbf{J} = I_1\omega_1\hat{\mathbf{i}} + I_2\omega_2\hat{\mathbf{j}} + I_3\omega_3\hat{\mathbf{k}} \qquad ...(60)$$

where $\boldsymbol{\omega} = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} + \omega_3\hat{\mathbf{k}}$ is the angular velocity with components ω_1, ω_2 and ω_3 along the principal axes.

As the principal moments of inertia and body base vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are constants in time with respect to the body coordinate system, we find that in the body coordinate system, using (60) the time derivative of **J** is

$$\frac{d\mathbf{J}}{dt} = I_1\dot{\omega}_1\hat{\mathbf{i}} + I_2\dot{\omega}_2\hat{\mathbf{j}} + I_3\dot{\omega}_3\hat{\mathbf{k}} \qquad ...(61)$$

Substituting in (59), we obtain

$$\boldsymbol{\tau} = I_1\dot{\omega}_1\hat{\mathbf{i}} + I_2\dot{\omega}_2\hat{\mathbf{j}} + I_3\dot{\omega}_3\hat{\mathbf{k}} + (\omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} + \omega_3\hat{\mathbf{k}}) \times (I_1\omega_1\hat{\mathbf{i}} + I_2\omega_2\hat{\mathbf{j}} + I_3\omega_3\hat{\mathbf{k}}) \qquad ...(62)$$

Writing $\boldsymbol{\tau} = \tau_1\hat{\mathbf{i}} + \tau_2\hat{\mathbf{j}} + \tau_3\hat{\mathbf{k}}$, we can obtain the x, y, z components of the torque $\boldsymbol{\tau}$ as

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 \qquad ...(63\ a)$$

$$\tau_2 = I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 \qquad ...(63\ b)$$

$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 \qquad ...(63\ c)$$

Eqs. (63) are known as ***Euler's equations*** for the motion of a rigid body with one point fixed under the action of a torque. These equations can also be derived from Lagrange's equations,when the generalized forces G_k are the torques and Euler's angles (ϕ, θ, ψ) are the generalized coordinates.

(2) Lagrange's method : When a rigid body is rotating with one point fixed, Euler's angles completely describe the orientation of the rigid body. In case of the rotating rigid body, we take the Euler's angles ϕ, θ, ψ as the generalized coordinates and components of the applied torque as the generalized forces corresponding to these angles. For conservative system, Lagrangian for the system is

$$L = T(\dot{\phi}, \dot{\theta}, \dot{\psi}, \phi, \theta, \psi) - V(\phi, \theta, \psi) \qquad ...(64)$$

where T is the rotational kinetic energy and is given by

$$T = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \qquad ...(65)$$

where the body axes are taken as principal axes.

In view of eq. (25), the angular velocity components ω_1, ω_2 and ω_3 along the principal axes can be written as

$$\begin{aligned} \omega_1 &= \dot{\phi}\ \sin\theta \sin\psi + \dot{\theta}\ \cos\psi \\ \omega_2 &= \dot{\phi}\ \sin\theta \cos\psi - \dot{\theta}\ \sin\psi \\ \omega_3 &= \dot{\phi}\ \cos\theta + \dot{\psi} \end{aligned} \qquad ...(66)$$

The Lagrange's equation for ψ coordinate is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\psi}}\right] - \frac{\partial L}{\partial \psi} = 0$$

But for $L = T - V$, given by (64),

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\psi}}\right] - \frac{\partial T}{\partial \psi} = -\frac{\partial V}{\partial \psi} \qquad ...(67)$$

because $\partial V/\partial \dot{\psi} = 0$

However, the angle ψ is the angle of rotation about the principal Z-axis and is one of the generalized coordinates in the present problem. The generalized force $\left[G_\psi = -\partial V/\partial \psi\right]$ cooresponding to the generalized coordinate ψ is obviously the Z-component of the impressed torque *i.e.*,

$$\tau_3 = G_\psi = -\frac{\partial V}{\partial \psi} \qquad ...(68)$$

Thus eq. (67) assumes the form

$$\tau_3 = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\psi}}\right] - \frac{\partial T}{\partial \psi}$$

or

$$\tau_3 = \frac{d}{dt}\left[\sum_i \frac{\partial T}{\partial \omega_1}\frac{\partial \omega_1}{\partial \dot{\psi}}\right] - \frac{\partial}{\partial \psi}\left[\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi}\right] \qquad ...(69)$$

But from (65), we get

$$T = \tfrac{1}{2}\sum_i I_i\omega_i^2$$

Therefore, $\dfrac{\partial T}{\partial \omega_i} = I_i\omega_i$

From (66), we obtain

$$\frac{\partial \omega_1}{\partial \dot{\psi}} = \frac{\partial \omega_2}{\partial \dot{\psi}} = 0 \text{ and } \frac{\partial \omega_3}{\partial \dot{\psi}} = 1$$

So that
$$\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} = I_3\omega_3 \qquad ...(70)$$

Also from (66), we get

$$\frac{\partial \omega_1}{\partial \psi} = -\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi = \omega_2,$$

$$\frac{\partial \omega_2}{\partial \psi} = -\dot{\phi}\sin\theta\sin\psi - \dot{\theta}\cos\psi = -\omega_1, \qquad \frac{\partial \omega_3}{\partial \psi} = 0$$

Hence
$$\sum_i \frac{\partial T}{\partial \omega_1}\frac{\partial \omega_i}{\partial \psi} = \frac{\partial T}{\partial \omega_1}\frac{\partial \omega_1}{\partial \psi} + \frac{\partial T}{\partial \omega_2}\frac{\partial \omega_2}{\partial \psi} + \frac{\partial T}{\partial \omega_3}\frac{\partial \omega_3}{\partial \psi}$$

$$= I_1\omega_1\omega_2 + I_2\omega_2(-\omega_1) = -(I_2 - I_1)\,\omega_1\omega_2 \qquad ...(71)$$

Substituting the values from (63) or (72) and (71) in (69), we get

$$\tau_3 = \frac{d}{dt}(I_3\omega_3) + (I_2 - I_1)\,\omega_1\omega_2$$

or
$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 \qquad ...(72)$$

which is the third Euler's equation obtained earlier. One may obtain the other two Euler's equations by simply cyclic permutation. Note that these two equations do not correspond to θ and ϕ coordinates.

In case a rigid body is rotating about a fixed axis, say principal Z-axis, then

$$\omega_1 = \omega_2 = 0 \text{ and } \omega_3 = \omega$$

Therefore, from eqs. (63) or (72) we have the equations of motion as

$$\tau_1 = \tau_2 = 0$$

and
$$\tau_3 = I_3\dot{\omega} \text{ or } \tau = I\dot{\omega} \qquad ...(73)$$

where we have put $\tau_3 = \tau$ and $I_3 = I$ corresponding to Z-axis.

Instantaneous angular momentum about Z-axis is

$$J_3 = I_3\omega_3 \text{ or } J = I\omega \qquad ...(74)$$

and instantaneous rotational kinetic energy is

$$T = \frac{1}{2}\,\boldsymbol{\omega}\cdot\mathbf{J} = \frac{1}{2}I\omega^2 \qquad ...(75)$$

10.12. TORQUE-FREE MOTION OF A RIGID BODY

(1) Equations of motion : When a rigid body is not subjected to any net torque, the Euler's equations of motion of the body with one point fixed reduce to

$$I_1\dot{\omega}_1 = (I_2 - I_3)\,\omega_2\omega_3 \qquad ...(76a)$$

$$I_2\dot{\omega}_2 = (I_3 - I_1)\,\omega_3\omega_1 \qquad ...(76b)$$

$$I_3\dot{\omega}_3 = (I_1 - I_2)\,\omega_1\omega_2 \qquad ...(76c)$$

In case the body is not subjected to any net forces or torques, its centre of mass is either at rest or moves with uniform velocity. Obviously we may discuss the rotational motion of the rigid body in a reference system in which the centre of mass is stationary and choose the centre of mass as fixed point and origin for the principal axes in the body. In such a case, we obtain from (76) two integrals of motion, describing the kinetic energy and angular momentum as constant in time.

If we multiply eqs. (76) by ω_1, ω_2, ω_3 respectively and then add, we obtain

$$I_1\,\omega_1\dot{\omega}_1 + I_2\,\omega_2\dot{\omega}_2 + I_3\,\omega_3\dot{\omega}_3 = (I_2 - I_3 + I_3 - I_1 + I_1 - I_2)\,\omega_1\omega_2\omega_3 = 0$$

or
$$\frac{d}{dt}\left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2\right) = 0$$

or
$$\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J} = \text{constant} \qquad ...(77)$$

which is the ***principle of conservation of total rotational kinetic energy*** in absence of external torque.

As
$$\tau = \frac{d\mathbf{J}}{dt} = 0$$

$$\mathbf{J} = I_1\omega_1\hat{\mathbf{i}} + I_2\omega_2\hat{\mathbf{j}} + I_3\omega_3\hat{\mathbf{k}} = \text{constant} \qquad ...(78)$$

describes another constant of motion, representing ***the principle of conservation of angular momentum.***

(2) Geometrical description of the rigid body motion : In case of torque-free motion of a rigid body, we have written above eqs. (76) and consequently two integrals of motion (77) and (78). Now we describe an interesting geometrical description of the motion, called as ***Pointsot's construction***. In this context, first we shall describe inertia ellipsoid.

(A) Inertia ellipsoid : The kinetic energy of a rotating rigid body in a coordinate system of principal axes is given by [from eqs. (44), and (47)]

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{J} = \frac{1}{2}\sum_{\alpha=1}^{3} I_\alpha\omega_\alpha^2$$

where the angular velocity $\boldsymbol{\omega}$ is expressed as $\boldsymbol{\omega} = \omega\,\hat{\mathbf{n}} = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} + \omega_3\hat{\mathbf{k}}$ and I_1, I_2, I_3 are the principal moments of inertia and I is the moment of inertia about the axis (instantaneous) of rotation. Thus we have

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = I\omega^2 = 2T \qquad ...(79)$$

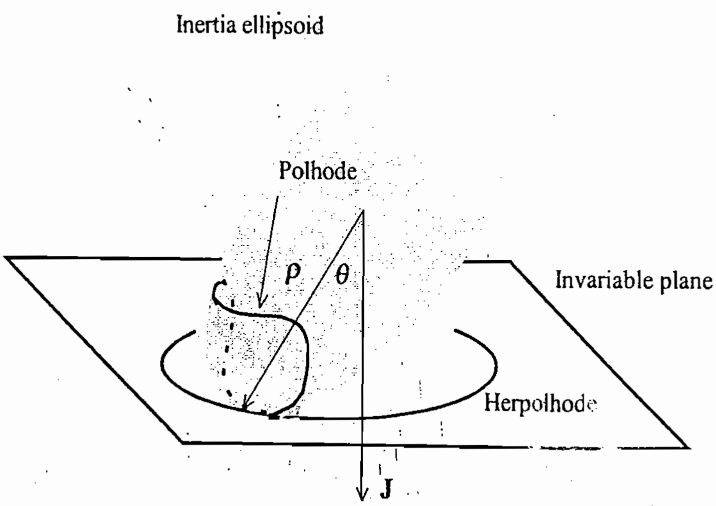


Fig. 10.13 : The motion of the inertia ellipsoid relative to the invariable plane

Let us define a vector

$$\rho = \frac{\hat{\mathbf{n}}}{\sqrt{I}} = \frac{\boldsymbol{\omega}}{\omega\sqrt{I}} = \frac{\boldsymbol{\omega}}{\sqrt{I\omega^2}} = \frac{\boldsymbol{\omega}}{\sqrt{2T}} \quad ...(80)$$

and

$$\rho = \rho_1\hat{\mathbf{i}} + \rho_2\hat{\mathbf{j}} + \rho_3\hat{\mathbf{k}} \quad ...(81)$$

So that $\rho_1 = \frac{\omega_1}{\sqrt{2T}}$ etc. are the components of ρ vector along principal axes.

Hence eq. (79) is

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1 \quad ...(82)$$

This equation represents an ellipsoid in ρ-space which is called as ***inertia ellipsoid*** (Fig. 10.13). As the direction of the axis of rotation changes in time, the ρ vector along the same direction moves accordingly and its tip moves on the surface of the inertial ellipsoid.

(B) Invariable plane : Let a rigid body be rotating about a fixed point O. The body is not subjected to any external force or torque. Therefore, in absence of external torque, the angular momentum vector **J** is constant and has a fixed direction in space [Fig. 10.14]. The line along the direction of the angular momentum vector is known as ***invariable line.***

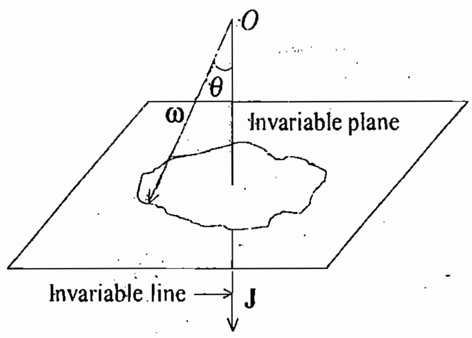


Fig. 10.14 : Invariable line and plane

For force free motion, the kinetic energy is also constant [eq. (77)] and hence

$$\boldsymbol{\omega} \cdot \mathbf{J} = 2T = \text{constant} \quad ...(83)$$

Thus the projection of ω on **J** ($\omega \cos\theta$) is constant and therefore the tip of $\boldsymbol{\omega}$ describes a plane, called as the ***invariable plane***. Now, as the body rotates, an observer, fixed in the body coordinate system, would see a rotation or ***precession*** of the angular velocity vector $\boldsymbol{\omega}$ with time about the angular momentum vector **J**.

(C) Motion of the inertia ellipsoid on invariable plane : Since from eq. (80), $\rho = \omega / \sqrt{2T}$, this gives for force-free motion

$$\rho \cdot \mathbf{J} = \frac{\boldsymbol{\omega} \cdot \mathbf{J}}{\sqrt{2T}} = \sqrt{2T} = \text{constant} \quad ...(84)$$

because the kinetic energy $\left(\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{J}\right)$ is the constant of motion. Therefore the tip of ρ also describes an *invariable plane* in ρ-space. In fact this plane is the tangent plane at the point ρ. Let us prove this statement. From eq. (82)

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1 = F(\rho) \text{ (say)}$$

Now,

$$\frac{\partial F}{\partial \rho_1} = 2I_1\rho_1, \quad \frac{\partial F}{\partial \rho_2} = 2I_2\rho_2, \quad \frac{\partial F}{\partial \rho_3} = 2I_3\rho_3$$

Therefore,

$$\nabla_\rho F = 2(I_1\rho_1\hat{\mathbf{i}} + I_2\rho_2\hat{\mathbf{j}} + I_3\rho_3\hat{\mathbf{k}})$$

$$= \frac{2}{\sqrt{2T}}(I_1\omega_1\hat{\mathbf{i}} + I_2\omega_2\hat{\mathbf{j}} + I_3\omega_3\hat{\mathbf{k}}) = \sqrt{\frac{2}{T}}\,\mathbf{J}.$$

Thus the normal at the point ρ on the ellipsoid (in case force-free motion) is along the constant angular momentum vector **J** and the tangent plane at the point ρ is perpendicular to **J**. But the invariable plane is normal to the vector **J** and hence the tangent plane at ρ is the invariable plane.

The distance between the origin of the ellipsoid and the tangent plane at the point ρ is given by [Fig. 10.13].

$$\rho \cos\theta = \frac{\boldsymbol{\rho} \cdot \mathbf{J}}{J} = \frac{\boldsymbol{\omega} \cdot \mathbf{J}}{J\sqrt{2T}} = \frac{\sqrt{2T}}{J} = \text{constant}$$

In the present problem, we find that the *distance between the origin of the ellipsoid and the invariable plane remains constant in time*. Thus as the angular velocity vector $\boldsymbol{\omega}$ and hence ρ changes with time, the inertia ellipsoid rolls (without slipping) on the invariable plane* with the centre of the ellipsoid at a constant height above the plane. The curve traced on the invariable plane by the point of contact with the ellipsoid is called the ***herpolhode*** and the corresponding curve described on the ellipsoid is called the ***polhode***. In other words, we can say that ***the polhode rolls without slipping on the herpolhode in the invariable plane***. The polhode is a closed curve on the inertia ellipsoid because the inertia ellipsoid would move in order to maintain the height of its origin above the invariable plane. However, the herpolhode, in general, is not a closed curve on the invariable plane.

We have discussed the Poinsot's geometrical construction to describe the force-free motion of a rigid body. The values of kinetic energy T and angular momentum **J** determine the direction of the invariable plane and the height of the centre of the ellipsoid above it. Hence one may trace out the polhode and the herpolhode. The direction of the angular velocity $\boldsymbol{\omega}$ is the same as that of the vector ρ and the instantaneous orientation of the body is given by the orientation of the ellipsoid, which is fixed in the body.

Let us discuss the Poinsot's geometrical discussion for a symmetrical rigid body for which $I_1 = I_2$. In this case, the inertia ellipsoid is an ellipsoid of revolution. The ρ vector and hence the angular velocity vector $\boldsymbol{\omega}$

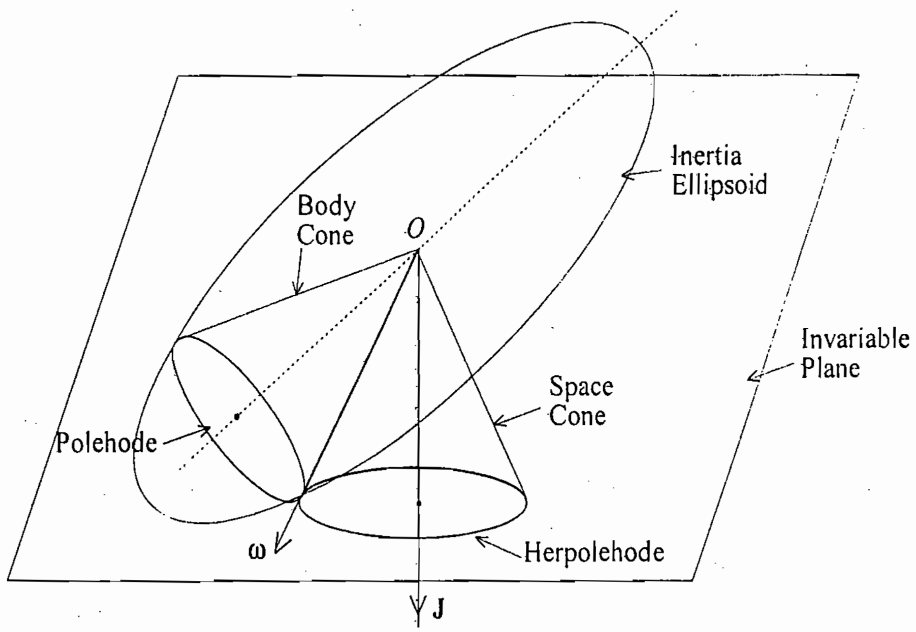


Fig. 10.15 : Motion of the inertia ellipsoid for a symmetrical body ($I_1 = I_2$)

* The point of contact of the inertia ellipsoid and the invariable plane is the tip of the vector $\rho\left[= \omega/\sqrt{2T} = \left(\omega/\sqrt{2T}\right)\hat{\mathbf{n}}\right]$, which corresponds to the direction of instantaneous axis of rotation. This direction of the rotating body is momentarily at rest and hence the point of contact ρ is stationary momentarily. This explains why ***the rolling occurs without slipping***.

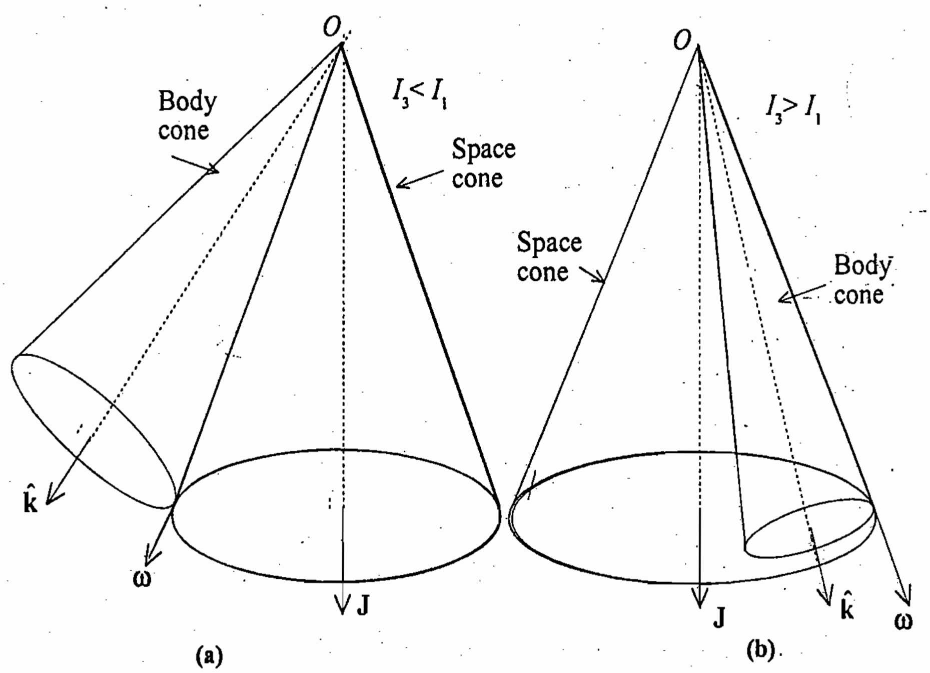


Fig. 10.16 : Body cone rolling around a space cone without slipping ; (a) outside ($I_3 < I_1$) (b) inside ($I_3 > I_1$).

will remain constant in magnitude. Consequently the polhode is a circle about the symmetry axis of the ellipsoid and herpolhode is a circle on the invariable plane. An observer sees that the angular velocity vector ω moves on the surface of a cone. This is called **body cone** and its intersection with the inertia ellipsoid is the polhode. An observer, fixed in the space, sees also the angular velocity vector ω to move on the surface of a cone, called as **space cone.** The intersection of this space cone with the invariable plane gives the herpolhode. In this way, ***the free motion of a symmetrical rigid body is described as the rolling of body cone on the space one.*** If $I_3 < I_1$, the body cone is outside the space cone and if $I_3 > I_1$, the body cone rolls around the inside of the space cone [Fig. 10.16 (*a*), (*b*)]. In both cases the two cones are tangent to each other along the instantaneous axis of rotation. In any case, the direction of the angular velocity vector ω ***precesses*** in time about the axis of symmetry of the body.

Poinsot's geometrical discussion, described above, is in accordance with that obtained by using Euler's equations for a rotating rigid body. Below we discuss the force-free motion of a symmetrical rigid body by using these equations of motion.

10.13. FORCE-FREE MOTION OF A SYMMETRICAL TOP

Now, we use Euler's equations to discuss the force-free motion of a symmetrical rigid body, that is a symmetrical top, for which $I_1 = I_2$ and the third principal axis (Z-axis) is the axis of symmetry of the body. In such a case, eqs. (76) take the form :

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_1 - I_3)\, \omega_2 \omega_3 \\ I_1 \dot{\omega}_2 &= (I_3 - I_1)\, \omega_3 \omega_1 \qquad \text{...(85)} \\ \dot{\omega}_3 &= 0 \end{aligned}$$

On integrating the last equation, we obtain

$$\omega_3 = \text{constant}$$

Thus the component of the angular velocity (ω) along the symmetry axis is constant with time for force-free motion of the rigid body.

If we now put

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3 = \text{constant}, \quad ...(86)$$

then the first two equations of (85) can be written as

$$\dot{\omega}_1 = -\Omega\omega_2 \quad ...(87a)$$

$$\dot{\omega}_2 = \Omega\omega_1 \quad ...(87b)$$

Differentiating eq. (87a), we get

$$\ddot{\omega}_1 = -\Omega\dot{\omega}_2 \quad ...(88)$$

Substituting for $\dot{\omega}_2$ from (87 b), we obtain

$$\ddot{\omega}_1 = -\Omega^2\omega_1 \text{ or } \ddot{\omega}_1 + \Omega^2\omega_1 = 0 \quad ...(89)$$

This is well known differential equation, used to deal simple harmonic motion, and its solution is

$$\omega_1 = \omega_c \sin \Omega t \quad ...(90)$$

where ω_c is some constant and we have chosen initial conditions $\omega_1 = 0$ at $t = 0$ so that the phase constant is zero.

Substituting for $\dot{\omega}_1$ from (90) in (87 a), we get

$$\omega_c \Omega \cos \Omega t = \Omega\omega_2 \text{ or } \omega_2 = \omega_c \cos \Omega t \quad ...(91)$$

Squaring (90) and (91), we get

$$\omega_1^2 + \omega_2^2 = \omega_c^2 \quad ...(92)$$

This is the equation of a circle with radius ω_c. If $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors along the two principal axes other than the symmetric axis, then the vector

$$\omega' = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}}$$

or

$$\omega' = (\omega_c \sin \Omega t)\hat{\mathbf{i}} + (\omega_c \cos \Omega t)\hat{\mathbf{j}} \quad ...(93)$$

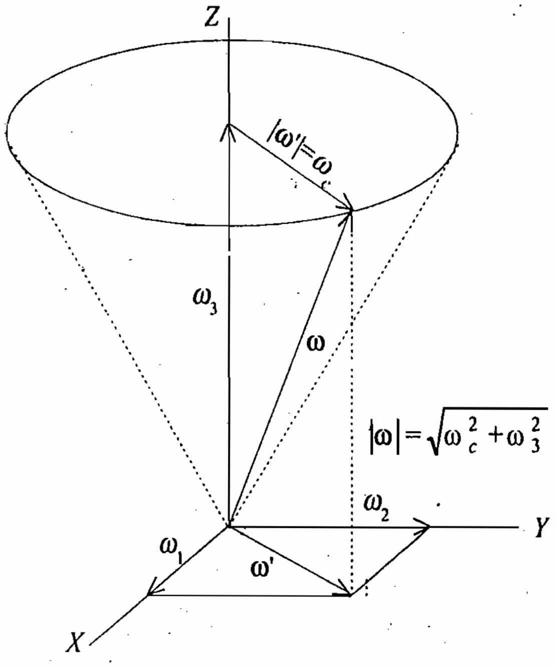


Fig. 10.17 : Force-free motion of a symmetrical top ($I_1 = I_2$)

has a constant magnitude $|\omega'| = \omega_c$ and rotates about the third principal axis (symmetry axis) with an angular frequency

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3$$

as shown in Fig. 10.17. Hence the total angular velocity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \omega_1\hat{\mathbf{i}} + \omega_2\hat{\mathbf{j}} + \omega_3\hat{\mathbf{k}} = \omega' + \omega_3\hat{\mathbf{k}} \quad ...(94)$$

with $$|\boldsymbol{\omega}| = \sqrt{\omega_c^2 + \omega_3^2} = \text{constant.}$$

Thus the angular velocity vector $\boldsymbol{\omega}$ has constant magnitude and precesses about the Z-axis (axis of symmetry) with constant angular frequency Ω. The precessional motion is shown in fig. 10.17 as obtained in Poinsot's construction. We observe that $\boldsymbol{\omega}$ ***vector moves on the surface of a cone about the axis of symmetry*** with the constant angular frequency Ω. This precessional motion is with respect to the principal (body) axes, fixed in the body, which are themselves rotating in space with the large frequency ω (relative to Ω). It is to be seen from eq. (86) $[\Omega = (I_3 - I_1)\,\omega_3/I_1]$ that as I_1 is closer to I_3, the precessional frequency Ω will be slower compared to the rotation frequency ω. The constants ω_c and ω_3 can be determined, if the constants of the motion, T the kinetic energy and the magnitude of the angular momentum J, are known. Both T and J can be expressed in terms of ω_c and ω_3 as

$$T = \frac{1}{2}I_1\omega_c^2 + \frac{1}{2}I_c\omega_3^2 \quad ...(95)$$

and $$J^2 = I_1^2\omega_c^2 + I_3^2\omega_3^2 \quad ...(96)$$

which can be solved to obtain the values of ω_c and ω_3.

Frequency and period of precession of angular velocity vector $\boldsymbol{\omega}$ (or the axis of rotation) about the axis of symmetry (Z-axis) are given by

$$f = \frac{\Omega}{2\pi} = \frac{\omega_3}{2\pi}\frac{I_3 - I_1}{I_1} \quad ...(97)$$

and $$T = \frac{2\pi}{\omega_3}\frac{I_1}{I_3 - I_1} \quad ...(98)$$

Rotation of the earth : One very important example of the application of the theory discussed above, is the case of rotating earth. The rotational motion of the earth may be considered as that of a free body because the external torques of the earth are very weak and therefore one would expect the precession of axis of rotation around the axis of symmetry. The earth is symmetrical about the polar axis and flattened slightly at the poles (shape of oblate spheroid). Consequently I_3 is little greater than I_1 and hence the precessional frequency is

$$\Omega = \frac{I_3 - I_1}{I_1}\omega_3 = \frac{\omega_3}{306} \quad ...(99)$$

Since ω_3 is practically equal to the magnitude of $\boldsymbol{\omega}$, *i.e.*, $\omega_3 \simeq \omega$, the angular frequency of the earth,

$$\frac{2\pi}{\omega_3} \simeq \frac{2\pi}{\omega} = 1 \text{ day} \quad ...(100)$$

Hence the ***precessional period of the earth*** is given by

$$T = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega_3}\cdot\frac{I_1}{I_3 - I_1} = \frac{2\pi}{\omega_3} \times 306 = 306 \text{ days} \quad ...(101)$$

If some circumstance disturbed the axis of rotation from the figure axis of the earth, one would therefore expect the axis of rotation to precess around the figure axis, i.e., around the North Pole once every 306 days. However, the measured value is 440 days. The disagreement is not due to lack of knowledge of I_1 or I_3 but due to the fact that the earth is not a perfect rigid body and does not possess the shape of oblate spheroid. In fact, the shape of earth resembles a flattened pear. Careful measurements tell us that the earth's rotation axis precesses about the North Pole in a circle of radius 10 m with a period of 440 days. As the latitude depends on the position of the axis of rotation, a measurable change in latitude results.

10.14. MOTION OF A HEAVY SYMMETRICAL TOP

Let us consider a spinning symmetrical top in a uniform gravitational field with one point O on the symmetry axis fixed in space. Such a top is called a heavy symmetrical top and its examples are child's top, gyroscope etc. Let G be the centre of gravity of the top and l be the distance from the fixed point O to $C.G.$ We take the symmetry axis as one of the principal axes and choose it Z-axis fixed in the body so that X, Y are the other two principal axes [Fig. 10.18] and $I_1 = I_2$. The force acting on the top is Mg, the force due to gravity. Let $X_0\, Y_0\, Z_0$ be the fixed set of axes; X_0 and Y_0 are in the horizontal plane and Z_0 is vertical. As O is the fixed point of the top, the motion can be described in terms of the three Euler's angles ϕ, θ and ψ:

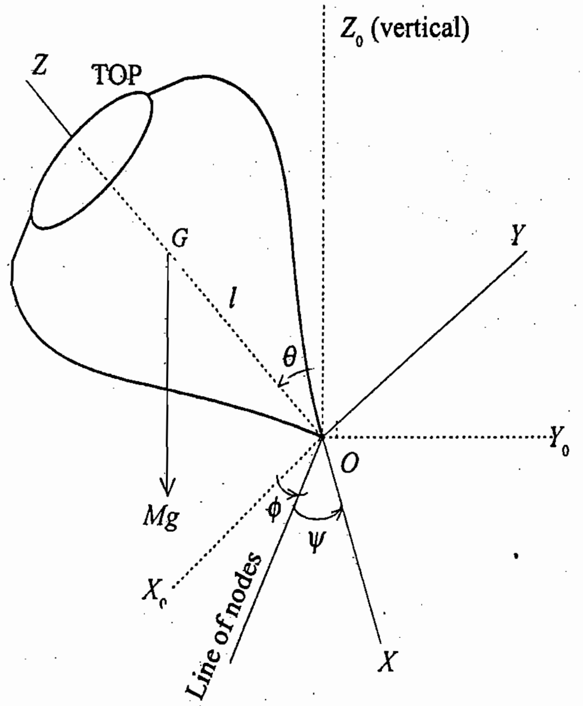


Fig. 10.18 : Euler's angles specifying the orientation of a heavy symmetrical top ($I_1 = I_2$)

(*i*) θ is the angle of inclination of Z-axis from the vertical (Z_0-axis).

(*ii*) ϕ is the azimuth of the top about vertical (Z_0-axis), *i.e.*, the angle in the horizontal plane between X_0 and line of nodes, and

(iii) ψ is the rotation angle of the top about its own Z-axis i.e., the angle between line of nodes and X-axis (body axis).

The Lagrangian for the top is

$$L = T - V = \frac{1}{2} I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 - Mgl \cos\theta$$

where I_3 is the principal moment of inertia about the symmetry axis. Here from eq. (25)

$$\omega_1^2 + \omega_2^2 = \dot\theta^2 + \dot\phi^2 \sin^2\theta \text{ and } \omega_3^2 = (\dot\psi + \dot\phi \cos\theta)^2$$

So that
$$L = \tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3(\dot\psi + \dot\phi \cos\theta)^2 - Mgl \cos\theta \quad ...(102)$$

First integrals of motion : We see that in the expression for the Lagrangian, ϕ and ψ coordinates do not appear. Hence ϕ and ψ are the cyclic coordinates and therefore the generalized momenta are the constants in time i.e., the two first integrals of motion are

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi \cos\theta) = I_3\omega_3 = I_1 a \text{ (say)} \quad ...(103)$$

and
$$p_\phi = \frac{\partial L}{\partial \dot\phi} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\, \dot\phi + I_3 \dot\psi \cos\theta = I_1 b \quad ...(104)$$

where we have expressed the two constants of motion in terms of new coordinates a and b.

Since the system is conservative, another first integral is the total energy E, remaining constant in time, *i.e.*,

$$E = T + V = \tfrac{1}{2} I_1(\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \tfrac{1}{2} I_3\omega_3^2 + Mgl \cos\theta \quad ...(105)$$

From (103), we obtain

$$I_3\dot\psi = I_1 a - I_3 \dot\phi \cos\theta \quad ...(106)$$

Substituting for $I_3\dot\psi$ in (104), we get

$$(I_1 \sin^2\theta + I_3 \cos^2\theta)\, \dot\phi + (I_1 a - I_3\dot\phi \cos\theta) \cos\theta = I_1 b$$

or
$$I_1\dot\phi \sin^2\theta + I_1 a \cos\theta = I_1 b \text{ or } \dot\phi = \frac{b - a\cos\theta}{\sin^2\theta} \quad ...(107)$$

Substituting from (107) for $\dot\phi$ in (106), we get

$$\dot\psi = \frac{I_1 a}{I_3} - \left[\frac{b - a\cos\theta}{\sin^2\theta}\right] \cos\theta \quad ...(108)$$

Substituting for $\dot\phi$ and $\dot\psi$ in eq. (105), we get

$$E = \tfrac{1}{2} I_1\dot\theta^2 + \tfrac{1}{2} I_1 \sin^2\theta \left[\frac{b - a\cos\theta}{\sin^2\theta}\right]^2 + \tfrac{1}{2} I_1 a^2 + Mgl\cos\theta \quad [\because I_3\omega_3 = I_1 a]$$

or
$$E - \tfrac{1}{2} I_1^2 a^2 = \tfrac{1}{2} I_1\dot\theta^2 + \tfrac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

If we denote $E - \frac{1}{2} I_1^2 a^2 = E'$, then

$$E' = \tfrac{1}{2} I_1\dot\theta^2 + \tfrac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \quad ...(109)$$

which is the sum of kinetic energy ($\frac{1}{2} I_1\dot\theta^2$) and an ***effective potential***, $V(\theta)$, given by

$$V(\theta) = \tfrac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \quad ...(110)$$

Thus $$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta) \quad ...(111)$$

whence $$\frac{d\theta}{dt} = \dot{\theta} = \left[\frac{2}{I_1}[E' - V(\theta)]\right]^{1/2} \quad ...(112)$$

or $$\int_0^t dt = \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{(2/I_1)[E' - V(\theta)]}}$$

or $$t = t(\theta) \quad ...(113)$$

This equation, in principle, can be solved to obtain θ as a function of time, *i.e.*, $\theta = \theta(t)$. We may substitute $\theta = \theta(t)$ in eqs. (107) and (108) and obtain after integration $\phi = \phi(t)$ and $\psi = \psi(t)$. Thus we have obtained all the three Eulerian angles ϕ, ψ, θ which specify the orientation of the rigid body at any time. We find that the problem looks to be completely solved. However, the solution for θ as a function of time involves the use of elliptic integrals and procedure becomes complex. Hence we discuss alternative way which deals the phenomenon qualitatively.

(1) Steady precession : We write from (110) the expression for effective potential $V(\theta)$, *i.e.*,

$$V(\theta) = \frac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

We plot V against θ in Fig. 10.19, where $V = \infty$ at $\theta = 0$ and π. Thus the physically acceptable range of the graph is $0 < \theta < \pi$ and we should have at least a minimum in the energy diagram for some value (θ_0) of θ in between 0 and π . For any energy value $E' = E'_1$, the motion is limited between $\theta = \theta_1$ and $\theta = \theta_2$, as shown in figure. This means that the symmetry axis OZ of the rotating top can vary its inclination θ to the vertical (Z_0) such that $\theta_1 \le \theta \le \theta_2$. This is called **nutation.** In the special case, if the energy of the top is such that $E' = E'_0 = V_{min}$, then θ has only one value $\theta = \theta_0$ [Fig. 10.19]. Thus corresponding to this energy, the precession angle θ remains constant. In other words, this is the case of **steady precession** in which the symmetry axis of the top or gyroscope describes a right circular cone about the vertical axis (Z_0). This is also called the case of **precession** without nutation. We discuss first this special case in a bit detail and later the general case.

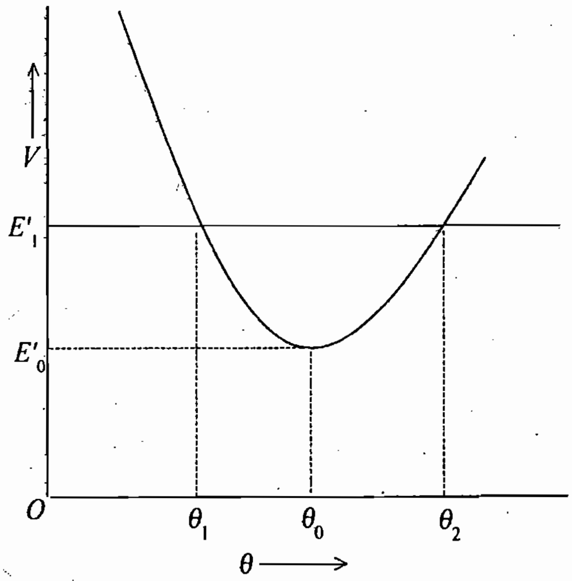


Fig. 10.19 : Plot of effective potential $V(\theta)$ versus θ (0 m θ m π) for a heavy symmetrical top

The value of θ_0 can be obtained by setting the derivative of the effective potential $V(\theta)$ equal to zero at $\theta = \theta_0$ *i.e.*,

$$\left[\frac{dV}{d\theta}\right]_{\theta_0} = I_1 a \frac{b - a\cos\theta_0}{\sin\theta_0} - I_1 \frac{(b - a\cos\theta_0)^2}{\sin^3\theta_0} - Mgl\sin\theta_0 = 0 \quad ...(114)$$

Let us define $$c = b - a\cos\theta_0 \quad ...(115)$$

Hence eq. (114) is

$$(\cos\theta_0)\, c^2 - (a\sin^2\theta_0)\, c + (Mgl/I_1)\sin^4\theta_0 = 0 \quad ...(116)$$

This is a quadratic equation in c and the solution is

$$c = \frac{a \sin^2\theta_0}{2 \cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4\,Mgl \cos\theta_0}{I_1 a^2}}\right] \qquad ...(117)$$

From (115), c is a real quantity. Thus for $\theta_0 < \pi/2$,

$$\sqrt{1 - \frac{4\,Mgl\cos\theta_0}{I_1 a^2}} \geq 0 \text{ or } I_1^2 a^2 \geq 4\,Mgl\, I_1 \cos\theta_0$$

But from (103) $I_1 a = I_3 \omega_3$, hence

$$I_3^2 \omega_3^2 \geq 4\,Mgl\, I_1 \cos\theta_0 \text{ or } \omega_3 \geq \frac{2}{I_3}\sqrt{Mgl\, I_1 \cos\theta_0}$$

or $$\omega_3 \geq (\omega_3)_{min} \qquad ...(118)$$

where $$(\omega_3)_{min} = \frac{2}{I_3}\sqrt{Mgl\, I_1\cos\theta_0} \qquad ...(119)$$

Thus a steady precession at the angle of inclination θ_0 of the symmetry axis (Z_3) is possible, only if the angular velocity ω_3 of the spinning top about this axis is greater than or equal to its minimum value calculated from (119). Obviously, when $\omega_3 > (\omega_3)_{min}$, c will have two values.

For $\theta = \theta_0$, from eq. (107)

$$\dot{\phi}_0 = \frac{b - a\cos\theta_0}{\sin^2\theta_0} = \frac{c}{\sin^2\theta_0} \qquad ...(120)$$

Since from (117) for $\omega_3 > (\omega_3)_{min}$, c has two values and hence the precessional angular velocity $\dot{\phi}_0$ has two values. Greater value of c results in fast precession and smaller value in slow precession i.e., $\dot{\phi}_{0f}$ and $\dot{\phi}_{0s}$. If ω_3 is very large, a is also very large (because $I_1 a = I_3\omega_3$) and the term $(4\,Mgl \cos\theta_0 / I_1 a^2)$ in (117) is very small. Using Bionomial expansion and neglecting higher order terms, we have

$$1 \pm \left[1 - \frac{4\,Mgl\cos\theta_0}{I_1 a^2}\right]^{1/2} = 1 \pm \left[1 - \frac{2\,Mgl\cos\theta_0}{I_1 a^2}\right] \cong 2 \text{ or } 2\,Mgl\cos\theta_0/I_1 a^2 \qquad ...(121)$$

Therefore, from (117)

$$c = \frac{a\sin^2\theta_0}{\cos\theta_0}\text{ (greater) and } c = \frac{Mgl\sin^2\theta_0}{I_1 a}\text{ (smaller)}$$

Hence from (120)

$$\dot{\phi}_{0f} = \frac{a}{\cos\theta_0} = \frac{I_3\omega_3}{I_1\cos\theta_0} \qquad ...(122)$$

and $$\dot{\phi}_{0s} = \frac{Mgl}{I_1 a} = \frac{Mgl}{I_3\omega_3} \qquad ...(123)$$

Eq. (123) is the well known result in the elementary gyroscopic theory and this is the slow precession rate $\dot{\phi}_{0s}$ which is usually observed.

Thus when the symmetry axis of the top is at $\theta = \theta_0$ with $\theta_0 < \pi/2$, and the top is rotating at large frequency ω_3 (which must be much greater than the minimum allowed value, given by (118)), the symmetry axis (Z) can precess about the fixed axis (vertical Z_0) with two possible frequencies given by (122) and (123). When ω_3 is large, we have a ***fast*** spinning top only then Bionomial expansion could have been done.

When $\theta_0 > \pi/2$, the fixed tip of the top is at a position above the centre of mass and the top is hanging with its symmetry axis with some angle from the vertical. Also from eq. (117), if $\theta_0 > \pi/2$, the radical is always positive and there is no restriction on the minimum value of ω_3. Further as the radical is greater than 1, the values of $\dot{\phi}_{0f}$ and $\dot{\phi}_{0s}$ have opposite signs, *i.e.*, for $\theta_0 > \pi/2$, the fast precession $\dot{\phi}_{0f}$ is in the same direction as that for $\theta_0 < \pi/2$ and slow precession $\dot{\phi}_{0s}$ takes place in the opposite direction as that for $\theta_0 < \pi/2$.

(2) Nulation : θ motion : In relation to the effective potential $V(\theta)$ versus θ graph [fig. 10.19], we discussed that the motion of the symmetry axis of the top is limited between $\theta_1 < \theta < \theta_2$ for any given E' of the top. As θ varies between θ_1 and θ_2, the value of $\dot{\phi}$, given by eq. (107),

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \quad ...(124)$$

may or may not change sign. If there is no change in the sign of $\dot{\phi}$, the top precesses monotonically around the fixed (Z_0) axis while the symmetry (Z) axis of the top oscillates between $\theta = \theta_1$ and $\theta = \theta_2$. **The variation in the angle θ is referred to as the nulation of the symmetry axis of the top and is an up and down motion of the top.** The curve traced by the symmetry axis of the top on a unit sphere in the space set of axes is shown in Fig. 10.20 (*a*). The polar coordinates of a point on this curve are identical with the Euler's angle θ, ϕ. In the graph as shown, for a particular value of the energy $E' = E_1'$, θ motion is bound between the angles θ_1 and θ_2, which are the roots of the equation

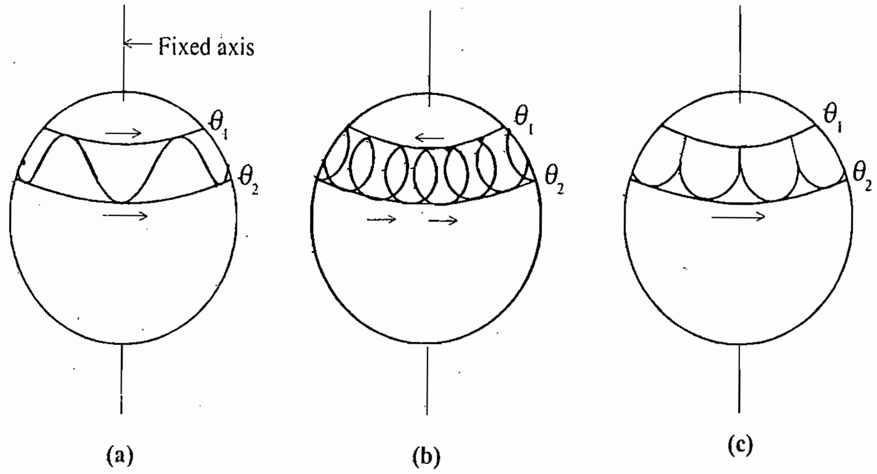


Fig. 10.20 : The possible shapes for the locus of the figure axis on the unit sphere

$$E_1' = \frac{1}{2} I_1 \frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \quad ...(125)$$

because at θ_1 and θ_2, $E_1' = V(\theta)$.

At the angle θ_0, the effective potential $V(\theta)$ has a minimum value which correponds to the point of stable equilibrium. θ_1 and θ_2 are the turning angles corresponding to the energy E_1'. From (112). $\dot{\theta} = 0$ (when $E_1' =$

$V(\theta)$) at $\theta = \theta_1$ and $\theta = \theta_2$ and the symmetry axis of the top is moving up and down between the bounding circles θ_1 and θ_2.

Looping motion : If the value of b and a are such that $\dot{\phi}$ changes its sign between the limiting values of θ, the precessional velocity must have opposite signs at $\theta = \theta_1$ and $\theta = \theta_2$. Thus the nutational precessional motion is a looping motion of the symmetry axis on the unit sphere [Fig. 10.20 (*b*)].

Cusplike motion : Finally, if the values of a and b are such that at $\theta = \theta_1$,

$$(b - a\cos\theta)_{\theta=\theta_1} = 0 \qquad \text{....(126)}$$

then
$$(\dot{\phi})_{\theta=\theta_1} = 0 \qquad \text{[from eq. (107)]} \qquad \text{....(127 a)}$$

and
$$(\dot{\theta})_{\theta=\theta_1} = 0 \qquad \text{....(127 b)}$$

(because $\dot{\theta} = 0$ at $\theta = \theta_1$ as discussed above). Thus in this case at the boundary circle $\theta = \theta_1$, both $\dot{\theta}$ and $\dot{\phi}$ vanish and the resulting motion of the symmetry axis on the unit sphere is ***cusplike***, as shown in Fig 10.20 (*c*).

10.15. FAST TOP

We consider a fast top, which is spinning very rapidly about its symmetry axis with the initial conditions as follows :

At $t = 0$, $\theta = \theta_1$, $\qquad \dot{\theta} = \dot{\phi} = 0$ and $\dot{\psi} = \omega_3$ ...(128)

In fact this is the last case [Fig. 10.20 (*c*)], discussed above, and corresponds to the usual method of starting the top. First the top is set to spin about its symmetry axis, then it is given an initial tilt ($\theta = \theta_1$) and released so that at $t = 0$, $\theta = \theta_1$, $\dot{\theta} = \dot{\phi} = 0$. In this case from (126)

$$\cos\theta_1 = \frac{b}{a} \qquad \text{...(129)}$$

When the top is released in this manner, the top always starts to fall under gravity, and continues to fall until the other bounding angle ($\theta = \theta_2$) is reached, precessing the meanwhile. In this duration, the loss of potential energy appears in the kinetic form in view of the conservation principle [eq. (105)] and consequently $\dot{\theta}$ and $\dot{\phi}$ differ from the initial zero values. The symmetry axis of the top then begins to rise again to θ_1 [Fig.10.20 (*c*)].

In order to discuss some quantitative predictions, we assume that the initial kinetic energy is large in comparison to the maximum change in the potential energy, *i.e.*,

$$\frac{1}{2} I_3 \omega_3^2 >> 2\,Mgl \qquad \text{...(130)}$$

This is called the case of ***fast top***. In this situation, we obtain below the expressions for the frequency of nutation, the amplitude of nutation and the average frequency of precession.

Frequency of nutational motion : The potential energy in the neighbourhood of the position of the stable equilibrium (θ_0) can be expanded in a Taylor's series :

$$V(\theta) = V(\theta_0) + \left[\frac{dV}{d\theta}\right]_{\theta_0} (\theta - \theta_0) + \frac{1}{2}\left[\frac{d^2V}{d\theta^2}\right]_{\theta_0} (\theta - \theta_0)^2 + \ldots \qquad \text{...(131)}$$

At the position of stable equilibrium

$$\left[\frac{dV}{d\theta}\right]_{\theta=\theta_0} = 0 \qquad ...(132)$$

$V(\theta_0)$ is the value of potential energy at the position of equilibrium (θ_0) and is therefore constant. It may be taken as zero. We write using eq. (110)

$$K = \left[\frac{d^2V}{d\theta^2}\right]_{\theta_0} = I_1 a^2 \qquad ...(133)$$

where we have dropped for fast top (ω_3 or a large) the terms due to gravitational potential energy and also due to small $\dot{\phi}$ at $\theta = \theta_0$.

Hence eq. (131) can be written as

$$V(\theta) = \frac{1}{2}K(\theta - \theta_0)^2 \qquad ...(134)$$

Thus eq. (111) is

$$E' = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}K(\theta - \theta_0)^2 \qquad ...(135)$$

Since E' is constant with time, we obtain after differentiation

$$I_1\ddot{\theta} + K(\theta - \theta_0) = 0 \qquad ...(136)$$

This represents a simple harmonic motion with solution

$$\theta - \theta_0 = \theta_m \cos(\omega t + \alpha) \qquad ...(137)$$

where θ_m is the amplitude and ω is the frequency of nutation, given by

$$\omega = \sqrt{\frac{K}{I_1}} = \sqrt{\frac{I_1 a^2}{I_1}} = a$$

But $I_1 a = I_3\omega_3$,

$$\therefore \qquad \omega = \frac{I_3\omega_3}{I_1} = a \qquad ...(138)$$

Thus the frequency of nutation for fast top is more for faster spinning top.

As θ_m is the amplitude of the sinusoidal nutational motion about the mean position θ_0, obviously

$$\theta_0 = \theta_1 + \theta_m \text{ or } \theta_m = \theta_0 - \theta_1$$

Since at $t = 0$, $\theta = \theta_1$, from eq.(137), we obtain

$$\theta_1 - \theta_0 = \theta_m \cos\alpha \text{ or } \theta_1 - \theta_0 = (\theta_0 - \theta_1)\cos\alpha \text{ or } \cos\alpha = -1 \text{ or } \alpha = \pi$$

Thus the nutational motion is represented by

$$\theta - \theta_0 = \theta_m \cos(\omega t + \pi) \text{ or } \theta = \theta_0 - \theta_m \cos\omega t$$

or

$$\theta = \theta_1 + \theta_m(1 - \cos\omega t) \qquad ...(139)$$

Amplitude of nutation : Since θ_m is the amplitude, the maximum nutational angle θ_2 is given by

$$\theta_2 = \theta_1 + 2\theta_m$$

As θ_1 and θ_2 are the roots of eq. (125) and the total energy E' must be the same at θ_1 and θ_2, we have

$$(E')_{\theta=\theta_1} = (E')_{\theta=\theta_2} \qquad \text{...(140)}$$

At $\theta=\theta_1$, $\qquad \dot{\phi} = \dfrac{b - a\cos\theta_1}{\sin^2\theta_1} = 0,$

therefore from (125) and (110) we obtain

$$Mgl\cos\theta_1 = Mgl\cos\theta_2 + \frac{1}{2}I_1\left[\frac{b - a\cos\theta_2}{\sin\theta_2}\right]^2$$

or

$$Mgl\cos\theta_1 = Mgl\cos(\theta_1 + 2\theta_m) + \frac{1}{2}I_1 a^2\left[\frac{(b/a) - \cos(\theta_1 + 2\theta_m)}{\sin(\theta_1 + 2\theta_m)}\right]^2 \qquad \text{...(141)}$$

(because $\theta_2=\theta_1+2\theta_m$)

As θ_m is small, $\cos 2\theta_m \simeq 1$ and $\sin 2\theta_m \simeq 2\theta_m$.

Now, $\cos(\theta_1+2\theta_m) = \cos\theta_1\cos 2\theta_m - \sin\theta_1\sin 2\theta_m \simeq \cos\theta_1 - 2\theta_m\sin\theta_1$ and hence $\cos\theta_1 - \cos(\theta_1 + 2\theta_m) \simeq 2\theta_m\sin\theta_1$

Also as $\cos\theta_1 = b/a$, eq. (141) for small θ_m is obtained to be

$$Mgl\cos\theta_1 = Mgl\cos\theta_1 - 2\theta_m\, Mgl\sin\theta_1 + \frac{1}{2}I_1 a^2\left[\frac{2\theta_m\sin\theta_1}{\sin\theta_1}\right]^2$$

or

$$\frac{1}{2}I_1 a^2 (2\theta_m)^2 = (2\theta_m)\, Mgl\sin\theta_1 \text{ or } \theta_m = \frac{Mgl\sin\theta_1}{I_1 a^2}$$

Thus

$$\theta_m = \frac{Mgl\sin\theta_1}{I_3^2\omega_3^2} \qquad (\text{because } I_3\omega_3 = I_1 a) \qquad \text{...(142)}$$

Thus the amplitude of rotation goes down as $1/\omega_3^2$, *i.e.*, **faster the top is spun, the less is the nutation.**

Average frequency of precession : The precessional angular velocity is given by [eq. (107)]

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \frac{a[(b/a) - a\cos\theta]}{\sin^2\theta}$$

But $b/a = \cos\theta_1$ and $\theta = \theta_1 + \theta_m(1 - \cos\omega t)$,

$$\dot{\phi} = \frac{a\{\cos\theta_1 - \cos\{\theta_1 + \theta_m(1 - \cos\omega t)\}\}}{\sin^2[\theta_1 + \theta_m(1 - \cos\omega t)]}$$

As $\theta_m(1-\cos\omega t)$ is small, the numerator is the difference term, given by

$$\cos\theta_1 - \cos\{\theta_1 + \theta_m(1-\cos\omega t)\} = \cos\theta_1 - \cos\theta_1\cos[\theta_m(1-\cos\omega t)] + \sin\theta_1\sin[\theta_m(1-\cos\omega t)]$$

$$= \cos\theta_1 - \cos\theta_1 + \sin\theta_1[\theta_m(1-\cos\omega t)] = \sin\theta_1[\theta_m(1-\cos\omega t)]$$

because for small $\theta_m(1-\cos\omega t)$, we have taken $\cos[\theta_m(1-\cos\omega t)] = 1$ and $\sin[\theta_m(1-\cos\omega t)] = \theta_m(1-\cos\omega t)$.

Thus

$$\dot{\phi} = \frac{a\sin\theta_1[\theta_m(1-\cos\omega t)]}{\sin^2\theta_1} \text{ or } \dot{\phi} = \frac{a\theta_m}{\sin\theta_1}(1-\cos\omega t) \qquad \text{...(143)}$$

Thus the rate of precession is not uniform but varies harmonically with time with the frequency of nutation ($\omega = a = I_3\omega_3/I_1$).

Further, the **average precessional angular frequency** (or velocity) is obtained as

$$\langle \dot{\phi} \rangle = \frac{\int_0^T \dot{\phi}\, dt}{\int_0^T dt} = \frac{\int_0^T a\theta_m (1-\cos\omega t)dt}{\sin\theta_1\, T}$$

$$= \frac{a\theta_m T}{T\sin\theta_1} = \frac{a\theta_m}{\sin\theta_1} = \frac{I_3\omega_3}{I_1\sin\theta_1}\theta_m$$

But $$\theta_m = \frac{Mgl\, I_1}{I_3\,\omega_3}\sin\theta_1, \text{ hence } \langle\dot{\phi}\rangle = \frac{Mgl}{I_3\omega_3} \quad ...(144)$$

Thus ***the average precession frequency decreases as the top is spun fast initially.***

Now we are in a position to describe a complete picture of the motion of the top, when initially the axis of the symmetry of the top has zero velocity. As the spinning top is released, in the beginning the top falls under the action of gravity. This develops a precessional velocity in the top which is directly proportional to the extent of its fall. This makes the symmetry axis of the top to move sideways about the vertical axis. In addition to the precession, the fall also results in a periodic nutational motion of the figure (symmetry) axis. ***As the top is spun faster and faster initially,* (i) *the amplitude of nutation decreases rapidly,* (ii) *the frequency of nutation increases* and (iii) *the precession about the vertical becomes slower*.** In practice, if the top is spinning sufficiently fast, the nutation is damped out by friction at the pivot and hence becomes unobservable. Therefore ***the top appears to process uniformly about the vertical axis.*** As the precession is regular only in appearance, it is called ***pseudo-regular precession.***

10.16. SLEEPING TOP

If the top is spinning sufficiently fast and is in the vertical position, the axis of the top will remain fixed in the vertical position. The top in this condition is called **sleeping top**.

For a sleeping top initially $\theta = 0$ and $\dot{\theta} = 0$. From (103) and (104), we have

$$p_\psi = p_\phi = I_3(\dot{\psi} + \dot{\phi}) = I_3\omega_3 = I_1 a = I_1 b$$

Therefore, $$a = b \quad ...(145)$$

Hence the effective potential $V(\theta)$ is [from eq. (110)] :

$$V(\theta) = \frac{1}{2}I_1\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta = \frac{1}{2}I_1 a^2\left[\frac{(1-\cos\theta)^2}{\sin^2\theta} + \frac{2Mgl\cos\theta}{I_1 a^2}\right]$$

But $$I_1 a^2 = I_1^2 a^2/I_1 = I_3^2\omega_3^2/I_1,$$

$\therefore$ $$V(\theta) = \frac{I_3^2\omega_3^2}{2I_1}\left[\frac{(1-\cos\theta)^2}{\sin^2\theta} + \lambda\cos\theta\right] \quad ...(146)$$

where $\lambda = 2Mgl/I_1 a^2 = 2I_1 Mgl/I_3^2\omega_3^2$.

If we draw the potential energy curve (V versus θ), we find that the shape of the curve depends on the value of λ.

Since $\sin^2\theta = 1 - \cos^2\theta = (1 - \cos\theta)(1 + \cos\theta)$,

$$V(\theta) = \frac{I_3^2\omega_3^2}{2I_1}\left[\frac{1-\cos\theta}{1+\cos\theta} + \lambda\cos\theta\right]$$

For minima in $V - \theta$ curve,

$$\frac{dV}{d\theta} = \frac{I_3^2\omega_3^2}{2I_1}\left[\frac{\sin\theta}{1+\cos\theta} + \frac{(1-\cos\theta)\sin\theta}{(1+\cos\theta)^2} - \lambda\sin\theta\right] = 0$$

or $$\sin\theta\left[\frac{1}{1+\cos\theta} + \frac{(1-\cos\theta)}{(1+\cos\theta)^2} - \lambda\sin\theta\right] = 0 \qquad ...(147)$$

which gives $\sin\theta = 0$ or $\theta = 0$ i.e., there can be minima at $\theta = 0$. Further for minima at $\theta = 0$,

$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=0} > 0 \text{ or } \left[\cos\theta\left\{\frac{1}{1+\cos\theta} + \frac{(1-\cos\theta)}{(1+\cos\theta)^2} - \lambda\right\}\right]_{\theta=0} > 0$$

or $$\tfrac{1}{2} - \lambda > 0 \text{ or } \tfrac{1}{2} > \lambda \qquad ...(148)$$

Thus at $\theta = 0$, (or top with its symmetry axis vertical), the spinning motion is stable for

$$\tfrac{1}{2} > \lambda \text{ or } \tfrac{1}{2} > \frac{2I_1Mgl}{I_3\omega_3^2}$$

which is valid for large ω_3 or rapidly spinning top. In other words, for vertical spinning top,

$$\omega_3^2 > 4I_1Mgl/I_3 \qquad ...(149)$$

The minimum spin angular frequency of the top below which the top cannot spin stably about vertical axis is when $\omega_3 = \omega_{min}$, given by

$$\omega^2_{\min} = \frac{4I_1Mgl}{I_3} \text{ or } \omega_{\min} = \left[\frac{4I_1Mgl}{I_3}\right]^{1/2} \qquad ...(150)$$

Thus if initially $\omega_3 > \omega_{\min}$, a top with its axis vertical will spin continuously and this is why it is called a **sleeping top**. This $\omega_{\min}$ is the critical angular velocity, above which vertical motion of the top is possible. The expression (150) is identical with eq. (119) for the minimum frequency for uniform precession with $\theta_0 = 0$. In practice, friction gradually reduces the spin frequency (ω_3) and when it is below the critical angular velocity ($\omega_{\min}$), then precession combined with nutation will be introduced. Further reduction of energy will finally cause the top fall down.

10.17. GYROSCOPE

A gyroscope is a heavy symmetrical body (top) in the form of a heavy disc or flywheel, rotating at a very high speed about its axle (figure axis). It is mounted in gimbals so that the disc and axle are both free to turn as a whole about one or more special axes, keeping the centre of gravity of the moving system to be stationary.

A gyroscope is illustrated in fig. 10.21. A disc D, having its axle AB mounted in gimbals, is given a spin angular velocity ω. If the outer gimbal is turned through an angle, the spin axis of the disc will tend to point in the same direction as previously [fig. 10.21 (b)]. This is in accordance with the ***principle of conservation of angular momentum*** ($J = I\omega$ = constant). If we move the gyroscope around the room, we will see that AB

always points in the same direction. We have assumed of course that the friction in gimbal bearings is negligible. Let the gyro-axis AB be horizontal and parallel to east west direction. We shall observe that the axis AB gradually inclines itself relative to the horizontal and after six hours it becomes vertical. This apparent rotation of the gyro-axis AB is due to the rotation of the earth. As the apparatus moves, the gyro-axis AB remains fixed in space.

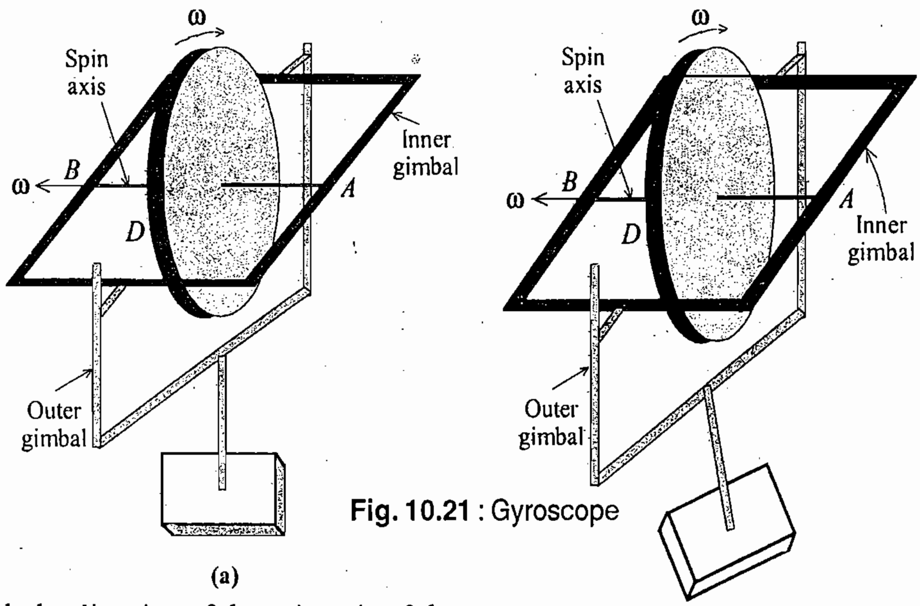


Fig. 10.21 : Gyroscope

In general, the direction of the spin axis of the gyroscope remains fixed even when the outer gimbal, which is attached to some object, moves freely in space. Due to this property, a gyroscope finds many applications in devices where we want to maintain the direction. Gyro-compass, working on this principle, is used in ships, aeroplanes and submarines to find direction. Other practical applications are found in missiles, satellites or other moving vehicles.

Effect of externsl torque on gyroscope : If external torque is suitably applied on the gyroscope, it will undergo the precession and nutation motions, as described earlier. For rapid motion of the disc, the condition of the fast top is satisfied so that the amplitude of nutation is always very small and is damped out by the method of mounting (friction). Then only precession is observed in the gyroscope and in such a case the mathematical treatment becomes relatively much simpler. This generalization can be done by considering the case of heavy symmetrical top.

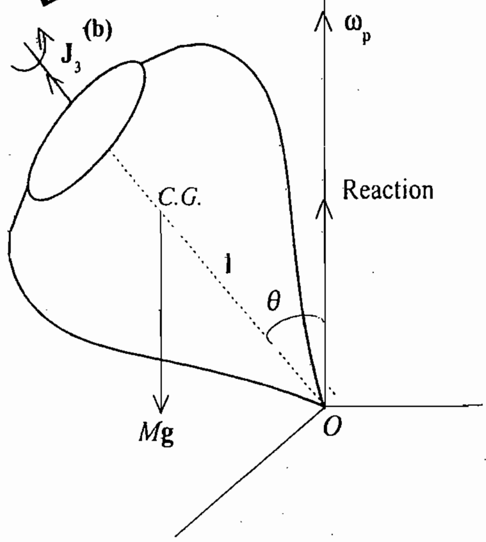


Fig. 10.22 : Precession under gravitational torque.

If l is the position vector along the figure axis from the fixed point O to the centre of gravity [Fig. 10.22], then the gravitational torque τ acting on the top is

$$\tau = \mathbf{l} \times M\mathbf{g} \qquad ...(151)$$

where g is the acceleration due to gravity in the downward direction and the magnitude of the torque is

$Mgl \sin\theta$. If J_3 is the angular momentum vector along the figure axis ($\mathbf{J}_3 = I_3\omega_3$) and ω_p, the precession vector, along the vertical, then the sense and magnitude of precession is given by

$$\omega_p \times \mathbf{J}_3 = \tau \qquad \text{...(152 } a\text{)}$$

or
$$\omega_p \times \mathbf{J}_3 = \mathbf{l} \times M\mathbf{g}$$

or
$$\omega_p J_3 \sin\theta = l\,Mg \sin\theta$$

or
$$\omega_p = \frac{Mgl}{J_3} = \frac{Mgl}{I_3\omega_3} \qquad \text{...(152 } b\text{)}$$

which is the same expression, as deduced earlier for the mean precession frequency. In the present discussion, τ is perpendicular to $\mathbf{J}_3$, hence $\mathbf{J}_3$ will remain constant in magnitude similar to speed remains constant in circular motion under the action of centripetal force. From (152) we see that the precession rate is proportional to the torque. However, in case of a non-spinning body, it is the angular acceleration which is proportional to the torque.

Now any torque about the fixed point or centre of mass can be put in the form $\mathbf{r} \times \mathbf{F}$ similar to eq. (151), the resulting precession rate for a fast spinning gyroscope can always be derived from eq. (152). In almost all the technological applications of the gyroscope, its equilibrium behaviour can be deduced from eq. (152), *i.e.*,

$$\omega_p \times \mathbf{J}_3 = \mathbf{r} \times \mathbf{F} \qquad \text{...(153)}$$

If a gyroscope is free from external torques, the spin axis of the gyroscope will always maintain its original direction relative to an inertial frame and we have described this behaviour in the beginning of this article. As shown by eq. (152), through the precession phenomena, the gyroscope can sense the applied torques, because in absence of the external torque, a gyroscope stops precessing.

Ex. 1. *Two point masses, each of mass m, are connected by a massless rigid rod of length 2a, forming a dumbbell. This dumbbell is rotating with constant angular velocity ω about an axis which makes an angle θ with the rod. Find the magnitudes and the directions of the angular momentum and the torque, applied to the system. How are the results affected, if an identical dumbbell is symmetrically fixed with the first dumbbell and the system is rotated with an angular velocity ω about the same axis.*

Solution : Let the dumbbell [Fig. 10.23] rotate with an angular velocity ω about the axis AOB in an inertial coordinate system. The angular momentum **J** of the two masses is

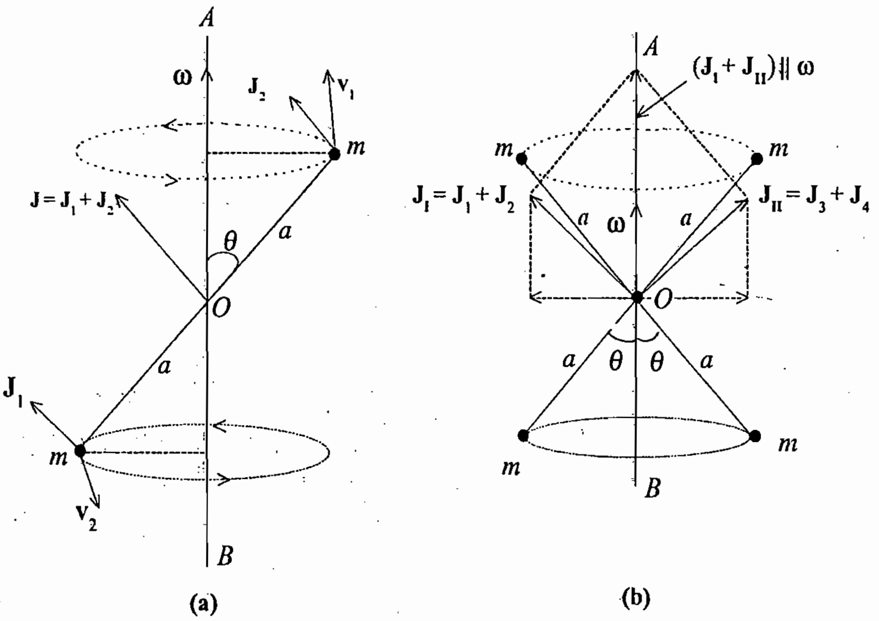


Fig. 10.23

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = m\mathbf{r}_1 \times (\boldsymbol{\omega} \times \mathbf{r}_1) + m\mathbf{r}_2 \times (\boldsymbol{\omega} \times \mathbf{r}_2)$$

As shown in fig. 10.23, $\mathbf{J}_1$ and $\mathbf{J}_2$ point in the same direction, but the total angular momentum $\mathbf{J}$ of the system is not along $\boldsymbol{\omega}$. The magnitude of $\mathbf{J}$ is

$$J = ma^2\omega \sin\theta + ma^2\omega\sin\theta = 2ma^2\omega\sin\theta = I\omega\sin\theta$$

where $I = 2ma^2$ is the moment of inertia of the dumbbell about an axis perpendicular to the length of the connecting rod.

Since $\mathbf{J}$, which is continuously changing direction, is not constant and hence to maintain the motion, a torque $\mathbf{T}$ is acting on the system. The torque $\mathbf{T}$ is given by

$$\mathbf{T} = \frac{d\mathbf{J}}{dt} = \dot{\mathbf{J}}$$

where $\mathbf{T} = \dot{\mathbf{J}}$ is a vector in the direction in which the tip of the vector $\mathbf{J}$ is moving. Obviously

$$\dot{\mathbf{J}} = \boldsymbol{\omega} \times \mathbf{J}$$

in analogy to the relation $\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$. Hence the magnitude of the applied torque is

$$\mathbf{T} = |\dot{\mathbf{J}}| = \omega J \sin(90-\theta) = \omega J\cos\theta = 2ma^2\omega^2 \sin\theta\cos\theta$$

and its instantaneous direction is perpendicular to the plane of $\boldsymbol{\omega}$ and $\mathbf{J}$. Fig. 10.23(b) shows the case when the two identical dumbbells, given in the problem, are moving symmetrically. In this case the perpendicular components of the angular momenta of the two dumbbells will cancel and the components parallel to $\boldsymbol{\omega}$ add up so that $\mathbf{J}$ and $\boldsymbol{\omega}$ are in the same direction. If there are no resisting forces to the rotating system, the system, once rotated, will rotate indefinitely and in this case there is no need of the external torque.

Ex. 2. *Calculate the inertia tensor for the system of four point masses* 1 *gm*, 2 *gm*, 3 *gm and* 4 *gm*, *located at the points* (1,0,0), (1,1,0), (1,1,1) *and* (1, 1, –1) *cm.*

Solution : $$I_{xx} = \sum_{i=1}^{4} m_i(y_i^2 + z_i^2) = 1\times 0 + 2\times 1 + 3\times 2 + 4\times 2 = 16 \text{ gm-cm}^2$$

Similarly, $I_{yy} = \sum_i^4 m_i(x_i^2 + z_i^2) = 17 \text{ gm-cm}^2$ and $I_{zz} = \sum_i m_i(x_i^2 + y_i^2) = 19 \text{ gm-cm}^2$

Also $I_{xy} = I_{yx} = -\sum_{i=1}^{4} m_i x_i y_i = -[0 + 2\times1\times1 + 3\times1\times1 + 4\times1\times1] = -9 \text{ gm-cm}^2$

Similarly, $I_{xz} = I_{zx} = -\sum_i m_i x_i z_i = 1 \text{ gm-cm}^2$ and $I_{yz} = I_{zy} = -\sum_i m_i y_i z_i = 1 \text{ gm-cm}^2$

Thus the inertia tensor I is

$$I = \begin{pmatrix} 16 & -9 & 1 \\ -9 & 17 & 1 \\ 1 & 1 & 19 \end{pmatrix}.$$

Ex. 3. *Consider a rectangular parallelopiped of uniform density* ρ, *mass* M *with sides* a, b *and* c. *For origin* O *at one corner, find the moments and products of inertia of the parallelopiped by taking the coordinate axes along the edges. If* $a = b = c$ *(case of a cube), determine the inertia tensor.*

Solution : $$I_{xx} = \int \rho(y^2+z^2)\,dV = \int_0^c\int_0^b\int_0^a \rho(y^2+z^2)\,dx\,dy\,dz = \rho a\int_0^c\int_0^b (y^2+z^2)\,dy\,dz$$

$$= \rho\, a \int_0^c \left[\frac{b^3}{3} + z^2 b\right] dz$$

$$= \rho\, a \left[\frac{b^3 c}{3} + \frac{c^3 b}{3}\right]$$

$$= \frac{\rho\, abc}{3}(b^2 + c^2)$$

$$= \frac{M}{3}(b^2 + c^2),$$

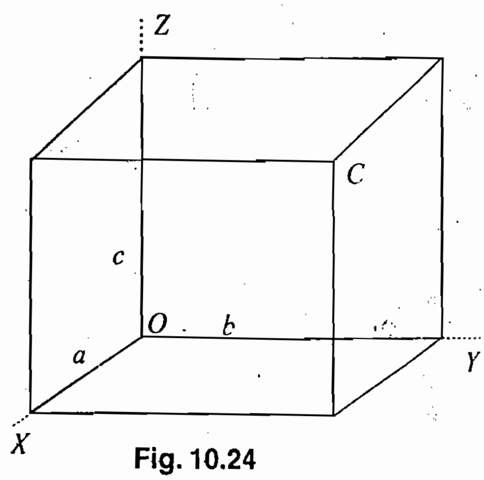


Fig. 10.24

where $M = \rho\, abc$.

Similarly, $I_{yy} = \dfrac{M}{3}(c^2 + a^2)$, $I_{zz} = \dfrac{M}{3}(a^2 + c^2)$

Also $I_{xy} = -\int_0^c \int_0^b \int_0^a \rho\, x\, y\, dx\, dy\, dz = -\rho\, c \int_0^b \int_0^a xy\, dx\, dy = -\rho\, c \dfrac{a^2}{2}\dfrac{b^2}{2} = -\dfrac{M}{4} ab = I_{yx}$

Similarly, $\quad I_{xz} = I_{zx} = -\dfrac{1}{4} Mac$ and $I_{yz} = I_{zy} = -\dfrac{1}{4} Mbc$

For a cube $a = b = c$, then $I_{xx} = \dfrac{2}{3} Ma^2 = I_{yy} = I_{zz}$

and $\quad I_{xy} = -\dfrac{M}{4} a^2 = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy}$.

Therefore, $$\boldsymbol{I} = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}.$$

Ex. 4 . *Consider a homogeneous cube of density ρ, mass M and side a. Taking origin O at one corner and axes along the edges of the cube, determine the inertia tensor, the principal axes and their associated moments of inertia.*

Solution : The inertia tensor $\boldsymbol{I}$ for the cube under consideration [Fig. 10.25] can be evaluated as above *i.e.*,

$$I = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} = \begin{pmatrix} 8\lambda & -3\lambda & -3\lambda \\ -3\lambda & 8\lambda & -3\lambda \\ -3\lambda & -3\lambda & 8\lambda \end{pmatrix}$$

with $$\lambda = \frac{Ma^2}{12}.$$

To find the the principal moments of inertia, we solve the secular equation

$$\begin{vmatrix} 8\lambda - I & -3\lambda & -3\lambda \\ -3\lambda & 8\lambda - I & -3\lambda \\ -3\lambda & -3\lambda & 8\lambda - I \end{vmatrix} = 0$$

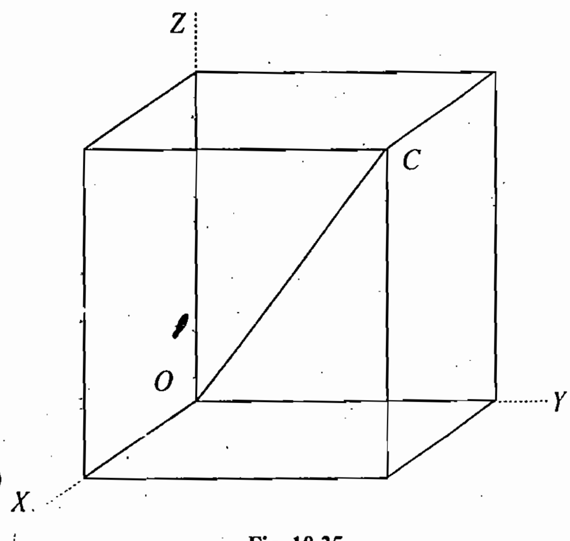


Fig. 10.25

Subtract the second row from first, we obtain

$$\begin{vmatrix} 11\lambda - I & -(11\lambda - I) & 0 \\ -3\lambda & 8\lambda - I & -3\lambda \\ -3\lambda & -3\lambda & 8\lambda - I \end{vmatrix} = 0 \text{ or } (11\lambda - I)\begin{vmatrix} -1 & -1 & 0 \\ -3\lambda & 8\lambda - I & -3\lambda \\ -3\lambda & -3\lambda & 8\lambda - I \end{vmatrix} = 0$$

or $(11\lambda - I)\,[(8\lambda - I)^2 - (3\lambda)^2 - 1\{(3\lambda)^2 + 3\lambda(8\lambda - I)\}] = 0$

or $(11\lambda - I)(22\lambda^2 - 13\lambda I + I^2) = 0$ or $(11\lambda - I)(11\lambda - I)(2\lambda - I) = 0$

Therefore, $I_1 = I_2 = 11\lambda$ and $I_3 = 2\lambda$

or $$I_1 = I_2 = \frac{11}{12} Ma^2 \text{ and } I_3 = \frac{1}{6} Ma^2$$

which are the expressions for the principal moments of inertia.

Since the two roots are identical, $I_1 = I_2$, the principal axis associated with the root I_3 is the axis of symmetry. The moment of inertia tensor, when consi-dered principal axes, is

$$I = \frac{Ma^2}{12}\begin{pmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Now we evaluate the directions of the principal axes associated with the three roots. First we evaluate the direction of the principal axis associated with the root I_3. For this, we substitute $I = I_3 = Ma^2/6 = 2\lambda$ in eq.(42). Then

$$(8\lambda - 2\lambda)\,\omega_x - 3\lambda\,\omega_y - 3\lambda\,\omega_z = 0$$
$$-3\lambda\,\omega_x + (8\lambda - 2\lambda)\,\omega_y - 3\lambda\,\omega_z = 0$$
$$-3\lambda\,\omega_x - 3\lambda\,\omega_y + (8\lambda - 2\lambda)\,\omega_z = 0$$

or $$2\omega_x - \omega_y - \omega_z = 0,\ -\omega_x + 2\omega_y - \omega_z = 0,\ -\omega_x - \omega_y + 2\omega_z = 0$$

From which, we obtain

$$\frac{\omega_x}{\omega_y} = 1,\ \frac{\omega_y}{\omega_z} = 1,\ \frac{\omega_z}{\omega_x} = 1 \text{ i.e., } \omega_x : \omega_y : \omega_z = 1:1:1$$

If we take unit vector $\hat{\omega}$ along ω corresponding to $I_3 = Ma^2/6$, then

$$\hat{\omega} = \frac{1}{\sqrt{3}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

which is obviously along the diagonal OC of the cube $[r = a(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})]$. As $I_1 = I_2$, axes associated with I_1 and I_2 have any mutually perpendicular directions in a plane perpendicular to this diagonal.

Ex. 5 . *Ellipsoid of inertia* : *If the moments and products of inertia of a rigid body with respect to a coordinate system XYZ with origin O are* $I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}$, *prove that the moment of inertia of the body about an axis, making angles* α, β, γ *with the X, Y, Z axes respectively is given by*

$$I = I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\gamma + 2I_{xy}\cos\alpha\cos\beta + 2I_{xz}\cos\alpha\cos\gamma + 2I_{yz}\cos\beta\cos\gamma.$$

Further show that the above equation can be represented in the form of ellipsoid, given by

$$I_{xx}\rho_x^2 + I_{yy}\rho_y^2 + I_{zz}\rho_z^2 + 2I_{xy}\rho_x\rho_y + 2I_{yz}\rho_y\rho_z + 2I_{zx}\rho_z\rho_x = 1.$$

Solution : Let OA be the axis of rotation and I be the moment of inertia of the body about this axis [Fig. 10.26]. If $\hat{\mathbf{n}}$ be a unit vector along OA, then

$$\hat{\mathbf{n}} = \cos\alpha\,\hat{\mathbf{i}} + \cos\beta\,\hat{\mathbf{j}} + \cos\gamma\,\hat{\mathbf{k}}$$

The moment of inertia of a particle of mass m about the axis OA is given by

$$= m\,PN^2 = m\,|\,\mathbf{r}\times\hat{\mathbf{n}}\,|^2$$

But $\mathbf{r}\times\hat{\mathbf{n}} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x & y & z \\ \cos\alpha & \cos\beta & \cos\gamma \end{pmatrix}$

$$= (y\cos\gamma - z\cos\beta)\,\hat{\mathbf{i}} + (z\cos\alpha - x\cos\gamma)\,\hat{\mathbf{j}} + (x\cos\beta - y\cos\alpha)\,\hat{\mathbf{k}}$$

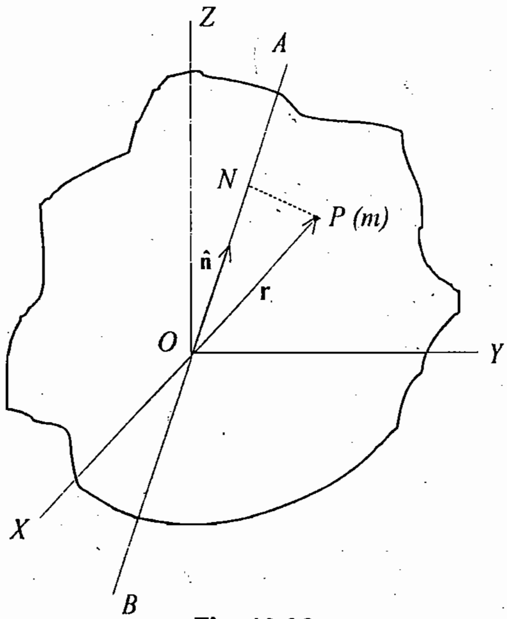


Fig. 10.26

Hence the total moment of inertia of the body about OA is given by

$$I = \Sigma\, m\,|\,\mathbf{r}\times\hat{\mathbf{n}}\,|^2 = \Sigma m\,(y^2+z^2)\cos^2\alpha + \Sigma m(x^2+z^2)\cos^2\beta + \Sigma m\,(x^2+y^2)\cos^2\gamma$$
$$- 2\,\Sigma mxy\cos\alpha\cos\beta - 2\,\Sigma mxz\cos\alpha\cos\gamma - 2\,\Sigma myz\cos\beta\cos\gamma$$

or $I = I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\gamma + 2I_{xy}\cos\alpha\cos\beta + 2I_{xz}\cos\alpha\cos\gamma + 2I_{yz}\cos\beta\cos\gamma$ ...(i)

This is the desired relation.

If we define a vector $\rho = \hat{\mathbf{n}}/\sqrt{I}$, then

$$\rho_x = \frac{\cos\alpha}{\sqrt{I}},\ \rho_y = \frac{\cos\beta}{\sqrt{I}},\ \rho_z = \frac{\cos\gamma}{\sqrt{I}} \qquad ...(ii)$$

Dividing eq. (*i*) by I and substituting from (*ii*), we get

$$I_{xx}\,\rho_x^2 + I_{yy}\,\rho_y^2 + I_{zz}\,\rho_z^2 + 2I_{xy}\,\rho_x\rho_y + 2I_{yz}\,\rho_y\rho_z + 2I_{zx}\,\rho_z\rho_x = 1$$

which represents an ellipsoid in the coordinates ρ_x, ρ_y, ρ_z. This is called ***ellipsoid of inertia*** or ***momental ellipsoid.***

Note : If the coordinate axes are rotated to coincide with the principal axes of the ellipsoid, the equation becomes

$$I_1\,\rho_1^2 + I_2\,\rho_2^2 + I_3\,\rho_3^2 = 1$$

where ρ_1, ρ_2, ρ_3 represent the coordinates of the new axes.

Ex. 6. *Find the kinetic energy of rotation of a rigid body with respect to principal axes in terms of Euler's angles and interpret the results when* $I_1 = I_2$.

Solution : The kinetic energy is given by

$$T = \tfrac{1}{2}[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2]$$

where I_1, I_2, I_3 are the principal moments of inertia and ω_1, ω_2, ω_3 are the components of angular velocity along these axes.

Substituting for ω_1, ω_2, ω_3 in terms of Euler's angles ϕ, θ, ψ from eq. (25), we obtain

$$T=\frac{1}{2}I_1[\dot{\phi}\sin\theta\sin\psi+\dot{\theta}\cos\psi]^2+\frac{1}{2}I_2[\dot{\phi}\sin\theta\cos\psi+\dot{\theta}\sin\psi]^2+\frac{1}{2}I_3[\dot{\phi}\cos\theta+\dot{\psi}]^2$$

If $I_1 = I_2$, then

$$T=\frac{1}{2}I_1[\dot{\phi}^2\sin^2\theta+\dot{\theta}^2]+\frac{1}{2}I_3[\dot{\phi}\cos\theta+\dot{\psi}]^2.$$

Questions

1. (*a*) How will you assign the generalized coordinates for the motion of a rigid body. **(Meerut 1999)**
   (*b*) For a rigid body consisting of N particles, how many generalized coordinates will have to be specified? **(Meerut 1999)**
   (*c*) Define Euler's angles for the orientation of a rigid body. **(Meerut 1999)**
2. A rigid body which is symmetrical about an axis has one point fixed on the axis. Discuss the rotational motion of the body assuming that there are no forces, other than the reaction forces, acting at the fixed point. **(Gorakhpur 1996)**
3. Define Euler's angles and obtain an expression for the complete transfor-mation matrix. **(Meerut 1983)**
4. Discuss Euler's angles as the generalized coordinates for a rigid body motion. Obtain an expression for the angular velocity of a rigid body in terms of Euler's angles **(Meerut 2001)**
5. A rigid body is rotating about an axis through the origin. Deduce relations connecting the components of total angular momentum with the components of the angular velocity. **(Gorakpur 1995)**
5. Find the relation between the angular momentum vector, the inertia tensor and the angular velocity vector. **(Kanpur 2003)**
6. What do you mean by inertia tensor. Explain what do you understand by principal axes and the principal moments of inertia. How will you determine the principal moments of inertia of a rigid body and directions of principal axes.
7. Define inertia tensor. Give its physical significance. **(Kanpur 1998)**
8. Derive an expression for the rotational kinetic energy of a rigid body. **(Gorakhpur 1995)**
9. Obtain Euler's equations of motion for a rotating rigid body with a fixed point. **(Kanpur 2003)**
10. (*a*) Show that the angular momentum **J** of a rotating rigid body is given by

    $$\mathbf{J}=\boldsymbol{I}\boldsymbol{\omega},$$

    where $\boldsymbol{\omega}$ is the angular velocity. Show that I is tensor of second rank. **(Meerut 1982)**

    (*b*) Show that the kinetic energy of a rotating rigid body can be expressed as $T=\frac{1}{2}\mathbf{J}\cdot\boldsymbol{\omega}$
11. (*a*) Show that the kinetic energy of a rotating rigid body in a coordinate system of principal axes is given by

    $$T=\frac{1}{2}(I_1\omega_1^2+I_2\omega_2^2+I_3\omega_3^2)$$

    (*b*) If I is the moment of inertia about the axis of rotation, prove that the kinetic energy can be expressed as

    $$T=\frac{1}{2}I\omega^2.$$
12. Discuss the theory of a spinning symmetrical top under gravity. **(Gorakpur 1995)**

13. Obtain the condition that heavy symmetrical top under the action of gravity which starts spinning initially with its symmetry axis vertical, may continue to spin in the same way for an indefinite period. **(Meerut 1980)**

14. Calculate the angular velocity of precession and spin when nutation is absent. **(Kanpur 2003)**

15. Write short notes on the following :
(*i*) Nutation (*ii*) Gyoscope **(Gorakpur 1996)**
(*iii*) Motion of a symmetrical top **(Kanpur 1999)**

Problems

1. Find the number of degrees of freedom for a rigid body which (a) can move freely in space, (b) has one point fixed, (c) has two points fixed.
Ans. : (*a*) 6, (*b*) 3, (*c*) 1.

2. (*a*) Consider a homogeneous cube of density ρ, mass M and side a. For origin O at one corner and axes along the edges of the cube, determine the inertia tensor.

$$\textbf{Ans. : } I = \frac{Ma^2}{12}\begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}.$$

(b) Determine the inertia tensor, if the origin O is the centre of mass in the above problem and the axes are parallel to the edges. Could you correlate the inertia tensors in the two cases ? Explain.

$$\textbf{Ans. : } I^c = \frac{Ma^2}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{; yes.}$$

$$\left[\textbf{Hint: } I_{xx} = I_{xx}^c + M(y^2+z^2) = \frac{Ma^2}{6} + M\left[\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right] = \frac{Ma^2}{6} + \frac{Ma^2}{2} = \frac{2}{3}Ma^2\right.$$

$$\left. \textit{and } I_{xy} = I_{xy}^c - MXY = -\frac{Ma^2}{4}\right]$$

3. Prove that the principal moments of inertia for a system consisting of two particles of masses m_1 and m_2 connected by a massless rigid rod of length a are $I_1 = I_2 = m_1 m_2 a^2/(m_1 + m_2)$ and $I_3 = 0$.

4. A rigid body consists of three particles of masses 2, 1, 4 gram located at (1,–1,1), (2, 0, 2), (–1, 1, 0) cm respectively. Find the angular momentum of the body if it is rotated about the origin with angular velocity $\omega = -3\hat{i} - 2\hat{j} + 4\hat{k}$. Find also the principal moments of inertia and directions of the principal axes of the system.

Ans : $\mathbf{J} = -6\hat{j} + 42\hat{k}$ gm-cm^2/sec; $I_1 = 18$, $I_2 = 13 - \sqrt{73}$, $I_3 = 13 + \sqrt{73}$ gm-cm^2; Along $\hat{j} + \hat{k}$, $\frac{1}{6}(1+\sqrt{73})\,\hat{i} - \hat{j} + \hat{k}$, $\frac{1}{6}(1-\sqrt{73})\,\hat{i} - \hat{j} + \hat{k}$ vectors.

5. (*a*) Find the moments of inertia and products of inertia of a uniform square plate of length a and mass M about the X, Y, Z axes, shown as in Fig. 10.27. **(GATE 2001)**

(*b*) In the above problem, find also the principal moments of inertia and the directions of the principal axes for the plate. **(GATE 2001)**

(*c*) Deduce the equation for the ellipsoid of inertia in the above problem.

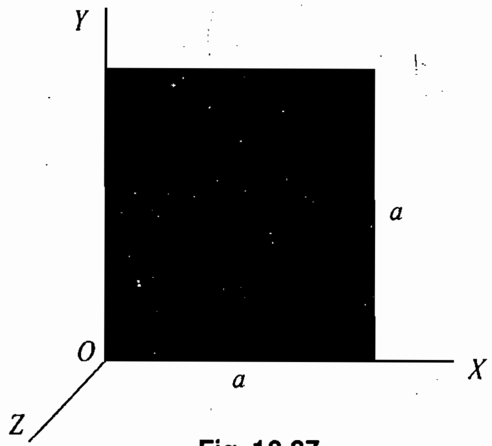


Fig. 10.27

Ans.:

(*a*) $I_{xx} = I_{yy} = \frac{1}{3} Ma^2,\ I_{zz} = \frac{2}{3} Ma^2$

$I_{xy} = I_{yx} = -\frac{1}{4} Ma^2,\ I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0.$

(*b*) $I_1 = \dfrac{Ma^2}{12},\ I_2 = \dfrac{7}{12} Ma^2, I_3 = \dfrac{2}{3} Ma^2, \hat{\mathbf{i}} + \hat{\mathbf{j}},\ \hat{\mathbf{i}} - \hat{\mathbf{j}},\ \hat{\mathbf{k}}.$

6. Find the principal moments of inertia at the centre of a uniform rectangular plate of sides a and b.

Ans.: $I_1 = \frac{1}{12} Ma^2,\ I_2 = \frac{1}{12} Mb^2, I_3 = \frac{1}{12} M(a^2 + b^2).$

7. Find the moments of inertia and products of inertia of a uniform solid sphere $x^2 + y^2 + z^2 = a^2$ in the first octant.

Ans.: $I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} Ma^2, I_{xy} = I_{yz} = I_{zx} = -2Ma^2/5\pi.$

8. Determine the principal moments of inertia of a uniform cylinder of radius a and hight h at the centre.

Ans.: $I_1 = I_2 = \frac{1}{12} M(3a^2 + h^2),\ I_3 = \frac{1}{2} Ma^2.$

9. Find the principal moments of inertia at the centre of the ellipsoid, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Ans.: $I_1 = \dfrac{M}{5}(b^2 + c^2),\ I_2 = \dfrac{M}{5}(a^2 + c^2),\ I_3 = \dfrac{M}{5}(a^2 + b^2).$

10. If T be the kinetic energy, **G** be the external torque about the instantaneous axis of rotation and $\boldsymbol{\omega}$ the angular velocity, then prove that $\dfrac{dT}{dt} = \mathbf{G} \cdot \boldsymbol{\omega}$.

[**Hint:** According to Euler's equations, $G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2$ or $I_1\dot{\omega}_1 = G_1 + (I_3 - I_2)\omega_2\omega_3$ etc.

$$T = \frac{1}{2} I_1\omega_1^2 + \frac{1}{2} I_2\omega_2^2 + \frac{1}{2} I_3\omega_3^2$$

$$\therefore \frac{dT}{dt} = \omega_1(I_1\dot{\omega}_1) + \omega_2(I_2\dot{\omega}_2) + \omega_3(I_3\dot{\omega}_3)$$

$$= \omega_1[G_1 + (I_2 - I_3)\omega_2\omega_3 + \omega_2[G_2 + (I_3 - I_1)\omega_2\omega_3] + \omega_3[G_3 + (I_1 - I_2)\ \omega_2\ \omega_3]$$
$$= \omega_1 G_1 + \omega_2 G_2 + \omega_3 G_3 = \mathbf{G}\cdot\boldsymbol{\omega}\ ]$$

11. From Euler's equations of motion for a rigid dody, having no external torque about a fixed point, prove that

$T = \frac{1}{2} I_1\omega_1^2 + \frac{1}{2} I_2\omega_2^2 + \frac{1}{2} I_3\omega_3^2 =$ constant, and $\mathbf{J} = I_1\omega_1\hat{\mathbf{i}} + I_2\omega_2\hat{\mathbf{j}} + I_3\omega_3\hat{\mathbf{k}} =$ constant,

where the terms have standard meaning.

12. A compound pendulum of mass M is oscillating about a horizontal axis. This axis makes angles α, β, γ with respect to the principal axes of inertia. If the principal moments of inertia are I_1, I_2, I_3 respectively and the distance from the centre of mass to the axis of rotation is l, show that for small oscillations the period (T) is given by

$$T = 2\pi\sqrt{\frac{Mgl}{I}}$$

where $I = Ml^2 + I_1\cos^2\alpha + I_2\cos^2\beta + I_3\cos^2\gamma$.

13. A rigid body which is symmetric about an axis has one point fixed on this axis. Describe the rotational motion of the body assuming that there are no forces acting other than the reaction force at the fixed point. Calculate the precessional frequency and period in the case of earth rotating about its axis. (Polar diameter =12,710 km; equatorial diameter =12,754 km.)

Ans. : $f = 0.00345$ rad/day, $T = 290$ days .

[**Hint :** $f = \frac{1}{T} = \frac{\omega_3}{2\pi}\frac{I_3 - I_1}{I_1}$

The earth is an oblate spheroid for which $a = b$ and c differs slightly from a or b. Thus for $a = b$ from problem 9

$$\frac{I_3 - I_1}{I_1} = \frac{a^2 - c^2}{a^2 + c^2} = \frac{(a-b)(a+c)}{a^2+c^2}$$

But c slightly differs from a, hence

$$\frac{a+c}{a^2+c^2} Q \frac{2a}{2a^2} = \frac{1}{a}, \quad \therefore \frac{I_3 - I_1}{I_1} = \frac{a-c}{a} = 1 - \frac{c}{a}$$

Therefore, $$f = \frac{\omega_3}{2\pi}\left(1 - \frac{c}{a}\right)$$

Here $a = 6377$ km, $c = 6355$ km, and therefore

$$f = \frac{\omega}{2\pi}\left(1 - \frac{6355}{6377}\right) = \frac{\omega}{2\pi} \times 0.00345.$$

But $\omega = 2\pi$ rad/day for earth. $f = 0.00345$ rad/day; $T = 1/f = 290$ days.]

14. Show that the energy equation for a spinning top with one point fixed can be expressed as

$$E' = I_1\frac{\dot{Q}^2}{2} + V(\theta) \qquad ...(i)$$

where $V(\theta) = Mgl\cos\theta + \frac{I_1}{2}\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2$

Writing $\cos\theta = u$, show that eq. (i) can be represented as

$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 = f(u)$ with $\alpha = 2E'/I_1$ and $\beta = 2Mgl/I_1$.

Hence prove that $\dot{\theta} = 0$ at those values of u for which

$$f(u) = (1-u^2)(\alpha - \beta u) - (b - au)^2 = 0.$$

Show that this equation has three real roots u_1, u_2, u_3 but in general not all the angles corresponding to these are real. Hence or otherwise find the condition for a sleeping top.

15. A top consists of a uniform solid cone of height 50 mm and base radius 20 mm. It is spinning with its vertex fixed at 120 revolutions per second. Find the period of precession of the axis about the vertical.

Ans. : 1.55 sec.

Objective Type Questions

1. In case of a rigid body, having N particles, the number of degrees of freedom is

   (*a*) N (*b*) $3N$ (*c*) 3 (*d*) ∞

   Ans. : (*c*).

2. If a rigid body is rotating with an angular velocity $\boldsymbol{\omega}$ about an instantaneous axis through a fixed point in the body, the angular momentum vector **J** about the same point

   (*a*) will be always in the direction of $\boldsymbol{\omega}$.
   (*b*) may be in the direction of $\boldsymbol{\omega}$.
   (*c*) may have different direction to that of $\boldsymbol{\omega}$.
   (*d*) will be always perpendicular to $\boldsymbol{\omega}$.

   Ans : (*b*), (*c*).

3. A sphere of mass M and radius r slips on a rough horizontal surface. At some instant, it has horizontal velocity v and rotational velocity $v/2r$. The translational velocity after the sphere starts pure rolling is

   (*a*) v (*b*) $6v/7$ (*c*) zero (*d*) $v/2$.

   Ans : (*b*).

4. If I_1, I_2 and I_3 represent the principal moments of inertia of a rigid body and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity with components along the three principal axes,

   (*a*) the z-component of the torque acting on the body in general is

   $$T_3 = I_3 \dot{\omega}_3$$

   (*b*) the z-component of the torque acting on the body in general is

   $$T_3 = I_3 \dot{\omega}_3 + (I_2 - I_1)\, \omega_1 \omega_2$$

   (*c*) for torque free motion of the rigid body, always we have

   $$I_3\, \omega_3 = \text{constant}$$

   (d) for torque free motion of the rigid body, in general we have

   $$I_3 \dot{\omega}_3 = (I_1 - I_2)\, \omega_1 \omega_2 .$$

   Ans. : (*b*), (*d*).

5. A heavy symmetrical top is rotating under the action of a gravitational angular momentum along the figure axis.

   (*a*) The torque is perpendicular to $\mathbf{J}_3$.

   (*b*) The angular momentum $\mathbf{J}_3$ will change in magnitude.

   (*c*) The angular momentum $\mathbf{J}_3$ will remain constant in magnitude.

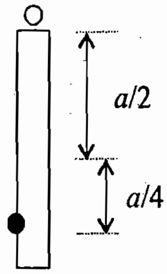
   (*d*) The angular momentum $\mathbf{J}_3$ will remain constant in magnitude as well as in direction.

   Ans. : (*a*), (*c*).

6. A particle of mass m is attached to a thin uniform rod of length a and mass 4 m. The distance of the particle from the centre of mass of the rod is $a/4$. The moment of inertia of the combination about an axis passing through O normal to the rod is

(*a*) $\frac{64}{48} ma^2$ (*b*) $\frac{91}{48} ma^2$

(*c*) $\frac{27}{48} ma^2$ (*d*) $\frac{51}{48} ma^2$.



(GATE 2004)

Ans. (*b*)

Short Answer Type Questions

1. How many generalized coordinates are needed to specify the motion of a rigid body ?
2. What are body and space coordinate systems in relation to the motion of a rigid body ?
3. What are Euler's angles ?
4. Write the matrix of transformation from space set of axes to body set of axes.
5. What do you understand by inertia tensor ? **(Kanpur 2002)**
6. Find the relation between the angular momentum vector, the inertia tensor and the angular velocity vector ? **(Kanpur 2003)**
7. What are principal axes and principal moments of inertia of a rigid body ?
8. Show that the kinetic energy of a rigid body can be represented as

$$T = \frac{1}{2}\, \omega \,.\, \mathbf{J}$$

9. What are Euler's equations of motion for a rigid body with a fixed point ? **(Kanpur 2003)**
10. Discuss the to torque-free motion of a rigid body.
11. What is inertia ellipsoid ? Explain invariable plane.
12. What do you understand by nutation ? **(Kanpur 2001)**
13. What do you understand by precession ?
14. What is sleeping top ?
15. What is gyroscope ?
16. Fill in the blanks :

(i) The general motion of a rigid body has .... degrees of freedom.

(ii) In case of inertia tensor I_{xy} = - - -.

(iii) For a rigid body angular momentum vector (**J**) and angular velocity vector (ω) are ....

Ans. : (*i*) 6, (*ii*) I_{yx} (*iii*) not always in the same direction.

CHAPTER 11

Noninertial and Rotating Coordinate Systems

11.1. NONINERTIAL FRAMES OF REFERENCE

In Chapter 1, we described that Newton's laws of motion are valid in inertial frame of reference and these inertial frames are unaccelerated. The accelerated frames are called as noninertial frames because in such a frame, a force-free particle will seem to have an acceleration. If we do not consider the acceleration of the frame but apply Newton's laws to the motion of the force free-particle, then it will appear that a force is acting on it. This means that in the accelerated frames, Newton's law of inertia is not valid. Thus a noninertial frame of reference is defined as a frame of reference in which Newton's first law does not hold true. An observer of a rotating frame will also see a force on a force-free particle and hence all rotating frames are also noninertial.

11.2. FICTITIOUS OR PSEUDO FORCE

Suppose that S is an inertial frame and another frame S' is moving with an acceleration $\mathbf{a}_0$ relative to S. The acceleration of a particle P, on which no external force is acting, will be zero in the frame S; but in frame S' the observer will find that an acce leration $-\mathbf{a}_0$ is acting on it. Thus, in frame S' the observed force on the particle is $-m\mathbf{a}_0$, where m is the mass of the particle. *Such a force, which does not really act on the particle but appears due to the acceleration of the frame, is called a **fictitious** or **pseudo** force*. Hence fictitious force on the particle P is

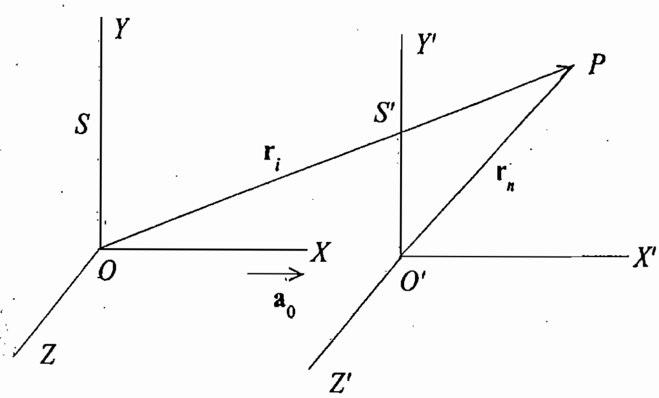


Fig. 11.1 : Non-inertial (accelerated) frames

$$\mathbf{F}_0 = -m\mathbf{a}_0 \qquad \text{...(1)}$$

Here, the accelerated frame S' is noninertial.

Now, if a force $\mathbf{F}_i$ is applied on the particle and $\mathbf{a}_i$ is the observed acceleration in the inertial frame (S), then according to Newton's law

$$\mathbf{F}_i = m\,\mathbf{a}_i \qquad \text{...(2)}$$

Suppose frame S' coincides at $t = 0$ with the initial frame S. Then at any time t, the position vectors of a particle $\mathbf{r}_i$ and $\mathbf{r}_n$ in the inertial and noninertial frames respectively are connected as

$$\mathbf{r}_i = \mathbf{r}_n + \frac{1}{2}\,\mathbf{a}_0 t^2$$

where $\mathbf{a}_0$ is the acceleration of the frame S' with respect to S. Double differentiation with respect to time t gives

$$\frac{d^2\mathbf{r}_i}{dt^2} = \frac{d^2\mathbf{r}_n}{dt^2} + \mathbf{a}_0 \qquad ...(3)$$

As $\frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{a}_i$ is the acceleration in the inertial frame, and $\frac{d^2\mathbf{r}_n}{dt^2} = \mathbf{a}_n$,

the acceleration observed in the noninertial frame, we can write eq. (3) as

$$\mathbf{a}_i = \mathbf{a}_n + \mathbf{a}_0 \qquad ...(4)$$

or $\qquad \mathbf{a}_i - \mathbf{a}_0 = \mathbf{a}_n$

or $\qquad m\mathbf{a}_i - m\mathbf{a}_0 = m\mathbf{a}_n \qquad ...(5)$

If we define the force on the particle in the accelerated system according to Newton's second law *i.e.*, $m\mathbf{a}_n = \mathbf{F}_n$, then using eqs. (1) and (2), we get

$$\mathbf{F}_n = \mathbf{F}_i + \mathbf{F}_0 \qquad ...(6)$$

where $\mathbf{F}_i$ $(= m\mathbf{a}_i)$ is the real force acting on the particle and $\mathbf{F}_0(= - m\mathbf{a}_0)$ is the fictitious force. Thus, the observer in the accelerated frame will measure the resultant (total) force which is the sum of real and fictitious forces on the particle *i.e.*,

Total force = True force + Fictitious force

For example, suppose that a box is falling in the gravitational field of the earth with an acceleration $\mathbf{a}_0 = -g\,\hat{\mathbf{n}}$, where g is the acceleration due to gravity and $\hat{\mathbf{n}}$ is a unit vector in the upward direction. Now, if we consider a particle, falling freely inside the box, the fictitious force on the particle is $\mathbf{F}_0 = - m\mathbf{a}_0 = mg\,\hat{\mathbf{n}}$. As the real force on the particle due to the attraction of the earth is $-mg\,\hat{\mathbf{n}}$, the force observed by the observer inside the box is

$$\mathbf{F}_n = \mathbf{F}_i + \mathbf{F}_0 = - mg\,\hat{\mathbf{n}} + mg\,\hat{\mathbf{n}} = 0$$

If the particle has no initial velocity relative to the box, it will seem to remain suspended in mid-air at the same place inside the box.

Now, suppose that the box is moved with an acceleration $\mathbf{a}_0 = g\,\hat{\mathbf{n}}$ in the upward direction relative to the ground. In such a case, the real force $(\mathbf{F}_i)$ and fictitious force $(\mathbf{F}_0)$ on the particle are given by

$$\mathbf{F}_i = - mg\,\hat{\mathbf{n}}, \ \mathbf{F}_0 = - m\,\hat{\mathbf{a}}_0 = - mg\,\hat{\mathbf{n}}$$

Hence the total force in the accelerated frame (box) is

$$\mathbf{F}_n = \mathbf{F}_i + \mathbf{F}_0 = - mg\,\hat{\mathbf{n}} - mg\,\hat{\mathbf{n}} = - 2\,mg\,\hat{\mathbf{n}}$$

This means that the observer, stationed in the box having an acceleration g upward, will measure a force $2mg$ downward on the particle.

We also consider the example of a lift (Fig. 11.2) which is moving relative to the ground

(a) with constant velocity upward or downward,

(b) with an acceleration $\mathbf{a}_0 = (= a_0\,\hat{\mathbf{n}})$ upward,

and (c) with an acceleration $- \mathbf{a}_0(= - a_0\,\hat{\mathbf{n}})$ downward.

Let us determine the weight of a man of mass m, standing on the lift. We assume here the earth to be stationary and a frame fixed with it to be inertial one. In all the cases, the force in the inertial frame (real force) is

$$\mathbf{F}_i = - mg\,\hat{\mathbf{n}}$$

The observer in case (*a*) is moving up or down with constant velocity so that the acceleration of the frame (attached with the lift) relative to the ground is zero, *i.e.*, $\mathbf{a}_0 = 0$. Thus the frame of the lift is inertial and in both frames the force on the man is same *i.e.*,

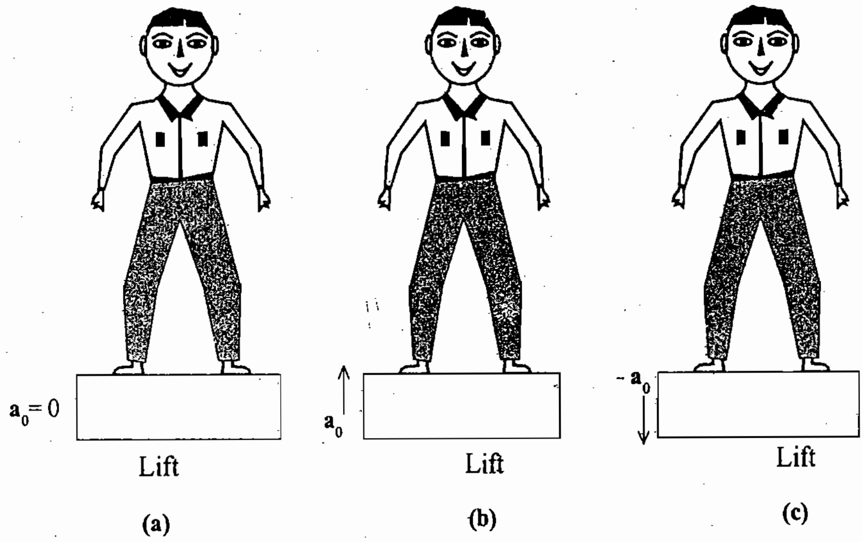


Fig. 11.2 : Lift moving—(a) up or down with uniform velocity ($a_0 = 0$), (b) with an acceleration $\mathbf{a}_0$ ($= a_0\,\hat{\mathbf{n}}$) upward, (c) with an acceleration $\mathbf{a}_0$ ($= -a_0\,\hat{\mathbf{n}}$) downward

$$\mathrm{F(Lift)} = \mathbf{F}_i = -mg\,\hat{\mathbf{n}}$$

Thus in the first case, the weight of the man is mg acting downward.

In case (b), $\mathbf{F}_0 = -ma_0 = -ma_0\,\hat{\mathbf{n}}$ and the observed force on the mass in the frame of the lift (accelerated upward) is

$$\mathbf{F}_n = \mathbf{F}_i + \mathbf{F}_0 = -mg\,\hat{\mathbf{n}} - ma_0\,\hat{\mathbf{n}} = -m\,(g + a_0)\,\hat{\mathbf{n}}$$

In case (c), $\mathbf{F}_0 = -m\mathbf{a}_0 = ma_0\,\hat{\mathbf{n}}$ and the force in the accelerated (lift) frame downward is

$$\mathbf{F}_n = \mathbf{F}_i + \mathbf{F}_0 = -mg\,\hat{\mathbf{n}} + ma_0\,\hat{\mathbf{n}} = -m\,(g - a_0)\,\hat{\mathbf{n}}$$

Thus when the lift is accelerated upward, the weight of the man is $m\,(g + a_0)$ and hence he becomes heavier, while for the lift accelerating downward, the man loses his weight $[m\,(g - a_0)]$. If the lift falls freely ($a_0 = g$), then the weight of the man becomes zero *i.e.,* he feels weightlessness.

11.3. CENTRIFUGAL FORCE

Let us consider a mass m, moving on the circumference of a circle of radius r with an angular velocity ω. For example, consider a stone attached at the end of a string. In an inertial frame, the centripetal force acting on the mass m is given by

$$\mathbf{F}_i = -m\omega^2\mathbf{r} \qquad \ldots(7)$$

where $\mathbf{r}$ is directed outward from the centre O. In case of rotating string with stone, this centripetal force is provided by the tension $\mathbf{T}$ of the string. So that

$$\mathbf{F}_i = \mathbf{T} = -m\omega^2\mathbf{r} \qquad \ldots(8)$$

Now suppose that a frame is rotating with an angular velocity ω relative to the inertial frame so that in the rotating frame the mass m is at rest. In this noninertial (rotating) frame, the observed acceleration ($\mathbf{a}_n$) of the mass m is zero

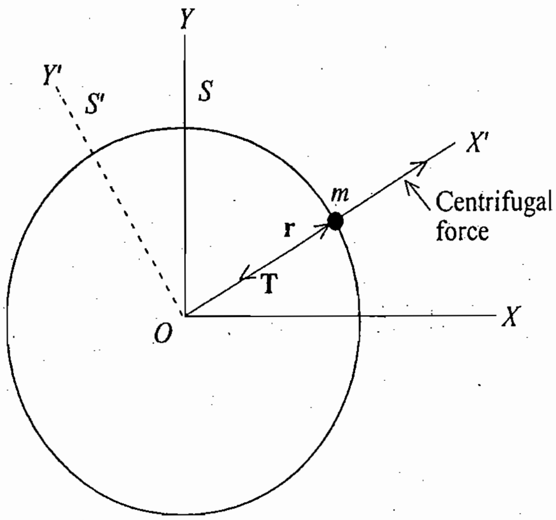


Fig. 11.3 : Centrifugal force

i.e., $\mathbf{a}_n = 0$ and consequently the total force ($\mathbf{F}_n$) is given by

$$\mathbf{F}_n = m\mathbf{a}_n = 0 \qquad ...(9)$$

Now $$\mathbf{F}_i + \mathbf{F}_0 = \mathbf{F}_n \qquad ...(10)$$

i.e., $$-m\omega^2\mathbf{r} + \mathbf{F}_0 = 0$$

Thus $$\mathbf{F}_0 = m\omega^2\,\mathbf{r} \qquad ...(11)$$

This fictitious force ($\mathbf{F}_0$) is directed away from the centre (along **r**) and is called the **centrifugal force.**

In the rotating noninertial frame, the acceleration ($\mathbf{a}_n$) or total force ($\mathbf{F}_n$) on the mass (m) is zero and $\mathbf{F}_i$ = T for stone-string arrangement, we have from eq. (10)

$$\mathbf{T} + \mathbf{F}_0 = 0$$

This means that in the noninertial frame the centrifugal force is balanced by the inward tension in the string. In general, in the rotating frame, the centrifugal force is equal and opposite to the actual force and both are acting on the same particle. Remember that the centrifugal force is a pseudo force and appears in the rotating frame due to its rotation and it is not to be confused with the reactionary force acting on the centre away from it (on the hand in the stone-string arrangement).

Ex. 1. *Calculate the effective weight of an astronaut ordinarily weighing 60 kg, when his rocket moves vertically (a) upward (b) downward with 6g acceleration.* **(Kanpur 1990)**

Solution : Real force $\mathbf{F}_i = -mg\,\hat{\mathbf{n}} = -60\,g\,\hat{\mathbf{n}}$

The effective weight of the astronaut in the rocket is the force $\mathbf{F}_n$ experienced in the accelerated frame (attached with the rocket). Now,

$$\mathbf{F}_n = \mathbf{F}_i + \mathbf{F}_0$$

(a) Here, $\mathbf{F}_0 = -m\mathbf{a}_0 = -60 \times 6\,g\,\hat{\mathbf{n}}$ $\quad [\because \mathbf{a}_0 = 6\,g\,\hat{\mathbf{n}}]$

Therefore, $\mathbf{F}_n = -60\,g\,\hat{\mathbf{n}} - 60 \times 6\,g\,\hat{\mathbf{n}} = -420\,g\,\hat{\mathbf{n}}$ = **420 kg** downward.

(b) Here, $\mathbf{F}_0 = -m\mathbf{a}_0 = -60 \times (-6g\,\hat{\mathbf{n}})$ $\quad [\because \mathbf{a}_0 = -6\,g\,\hat{\mathbf{n}}]$

Hence, $\mathbf{F}_n = -60\,g\,\hat{\mathbf{n}} + 60 \times 6\,g\,\hat{\mathbf{n}} = 300\,g\,\hat{\mathbf{n}}$ = **300 kg upward.**

Ex. 2. *A 1 kg. stone at the end of a 2 m long string makes 5 revolutions per second. Calculate the force on the stone as measured in an inertial frame and in a frame which is rotating with the string.*

Solution : In an inertial frame, the desired force to rotate the stone in the circular path is the centripetal force $-m\omega^2 r$, *i.e.*, force on the stone in inertial frame $= -m\omega^2\mathbf{r} = -1(2 \times 3.14 \times 5)^2 \times 2 = -1974$ newtons = 1974 newtons towards the centre (centripetal). This force is supplied by the tension of the string.

In case of the noninertial frame, rotating with the stone, the acceleration of the stone is zero. Thus the total force in this frame is zero. To have equilibrium of the stone in the rotating frame, the inward force of tension on the stone is balanced by the centrifugal (fictitious) force, amounting $m\omega^2 r = 1974$ newtons.

11.4. UNIFORMLY ROTATING FRAMES

We know that the earth itself rotates about its axis in 24 hours. Therefore, any frame fixed with the earth will also rotate with it and so it will be a noninertial frame.

Suppose that a frame $S'(X_r, Y_r, Z_r)$ is rotating with an angular velocity ω relative to an inertial frame $S(X_i, Y_i, Z_i)$. For simplicity, we assume that both of the frames have common origin O and common Z-axis. In case of the earth, the common origin O may be considered as the centre of the earth, Z-axes as coinciding with its rotational axis and the frame S' as rotating with earth relative to the non-rotating frame S.*

The position vector of a particle P in both frames will be the same^{+}, i.e., $\mathbf{R}_i = \mathbf{R}_r = \mathbf{R}$, because the origins are coincident. Now, if the particle P is stationary in the frame S, the observer in the rotating frame S' will see that the particle is moving oppositely with linear velocity $-\,\omega \times \mathbf{R}$. Thus, if the velocity of the particle in the frame S is $\left(\dfrac{d\mathbf{R}}{dt}\right)_i$, then its velocity $\left(\dfrac{d\mathbf{R}}{dt}\right)_r$ in the rotating frame will be given by

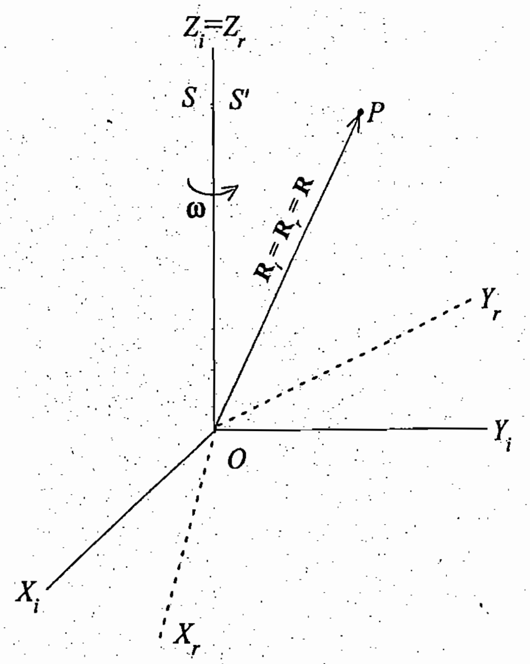


Fig. 11.4 : Uniformly rotating frame

$$\left(\frac{d\mathbf{R}}{dt}\right)_r = \left(\frac{d\mathbf{R}}{dt}\right)_i - \omega \times \mathbf{R}$$

or
$$\left(\frac{d\mathbf{R}}{dt}\right)_i = \left(\frac{d\mathbf{R}}{dt}\right)_r + \omega \times \mathbf{R} \quad ...(12)$$

In fact this equation holds for all vectors and relates the time derivatives of a vector in the frames S and S'. Therefore, relation (12) may be written in the form of operator equation :

$$\left(\frac{d}{dt}\right)_i = \left(\frac{d}{dt}\right)_r + \omega \times \mathbf{R} \quad ...(13)$$

Writing $d\mathbf{R}/dt = \mathbf{v}$ for the velocity of the particle, we have

$$\mathbf{v}_i = \mathbf{v}_r + \omega \times \mathbf{R} \quad ...(14)$$

Now, if we operate eq. (13) on velocity vector $\mathbf{v}_i$, we have

$$\left(\frac{d\mathbf{v}_i}{dt}\right)_i = \left(\frac{d\mathbf{v}_i}{dt}\right)_r + \omega \times \mathbf{v}_i$$

Substituting the value of $\mathbf{v}_i$ in the right hand side of this relation from eq. (14), we obtain

$$\left(\frac{d\mathbf{v}_i}{dt}\right)_i = \left[\frac{d}{dt}(\mathbf{v}_r + \omega \times \mathbf{R})\right]_r + \omega \times (\mathbf{v}_r + \omega \times \mathbf{R})$$

$$= \left(\frac{d\mathbf{v}_r}{dt}\right)_r + \frac{d\omega}{dt} \times \mathbf{R} + \omega \times \left(\frac{d\mathbf{R}}{dt}\right)_r + \omega \times \mathbf{v}_r + \omega \times (\omega \times \mathbf{R})$$

If we write the acceleration $\dfrac{d\mathbf{v}}{dt} = \mathbf{a}$ and $\left(\dfrac{d\mathbf{R}}{dt}\right)_r = \mathbf{v}_r$, then

$$\mathbf{a}_i = \mathbf{a}_r + 2\omega \times \mathbf{v}_r + \omega \times (\omega \times \mathbf{R}) + \frac{d\omega}{dt} \times \mathbf{R}$$

* Actually, this frame is rotating about the sun. But the assumption of S to be an inertial frame will not add any perceptible error in our measurements on earth.

+ The components may have different values, but the direction and magnitude of the position vector will be the same in the two frames.

For earth, ω is constant, so $\frac{d\omega}{dt} = 0$. Then

$$\mathbf{a}_i = \mathbf{a}_r + 2\omega \times \mathbf{v}_r + \omega \times (\omega \times \mathbf{R}) \qquad ...(15)$$

If m is the mass of the particle, then force in the rotating frame is

$$m\mathbf{a}_r = m\mathbf{a}_i - 2m\omega \times \mathbf{v}_r - m\omega \times (\omega \times \mathbf{R})$$

But $m\mathbf{a}_r = \mathbf{F}_i + \mathbf{F}_0$, therefore fictitious force $\mathbf{F}_0$ is given by

$$\mathbf{F}_0 = -2m\omega \times \mathbf{v}_r - \omega \times (\omega \times \mathbf{R}) \qquad ...(16)$$

where $-2m\omega \times \mathbf{v}_r$ is the Coriolis force and $-m\omega\times(\omega \times \mathbf{R})$, the centrifugal force.

The centrifugal force is the only fictitious force, acting on a particle which is at rest ($\mathbf{v}_r = 0$) in the rotating frame. The centrifugal force may be written as

$$-m\,\omega \times (\omega \times \mathbf{R}) = m\omega^2\mathbf{r} \qquad ...(17)$$

where $\mathbf{r}$ is the vector from the axis of the earth to the particle and normal to it, because

$$\omega \times (\omega \times \mathbf{R}) = (\omega \cdot \mathbf{R})\,\omega - (\omega \cdot \omega)\,\mathbf{R} = \omega^2 R \sin\phi\,\hat{\mathbf{k}} - \omega^2 R\,(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{k}}\sin\phi)$$

$$= -\omega^2 R\cos\phi\,\hat{\mathbf{i}} = -\omega^2\mathbf{r} \qquad [\because\ \omega = \omega\hat{\mathbf{k}} \text{ and } \mathbf{R} = R(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{k}}\sin\phi)]$$

Coriolis force ($-2m\omega \times \mathbf{v}_r$) is a fictitious force which acts on a particle only if it is moving with respect to the rotating frame. Thus, in the rotating frame if a particle moves with velocity $\mathbf{v}_r$, then it always experiences a force ($-2m\omega \times \mathbf{v}_r$) perpendicular to its path opposite to the direction of vector product $\omega \times \mathbf{v}_r$. Due to the Coriolis force a moving particle in the northern hemisphere is deflected towards the right of its path. In the southern hemisphere, the deflection is towards the left of the path. The effect of the Coriolis force is appreciable when it acts horizontally or has a horizontal component because in the vertical direction its effect is masked by the large gravitational force. Since this force is a comparatively small force, its effect is observable only when a particle travels large horizontal distances. When the velocity $\mathbf{v}_r$ of the moving particle in the rotating frame is horizontal, the horizontal component of Coriolis force will be only due to the vertical component $\omega \sin\phi$ of angular velocity ω where ϕ is the latitude of the place. Hence the magnitude of horizontal Coriolis force is $2m\omega v \sin\phi$ and is zero for $\phi = 0$, *i.e.*, at the equator.

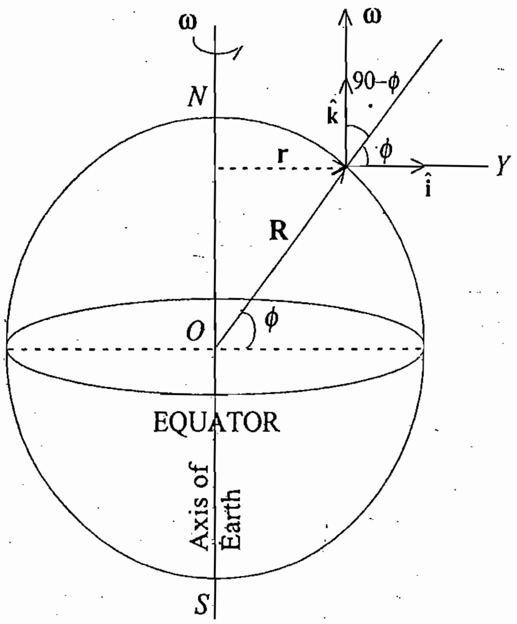


Fig. 11.5

An important effect of the Coriolis force is that the freely falling bodies deviate from their true vertical path. This deviation is always towards east in either of the hemisphere of the earth. In such a case the vertical fall means a velocity along vertical direction. Due to the horizontal component of Earth's angular velocity, Coriolis force starts to act on it in the horizontal direction and deviates the freely falling particle from the true vertical direction. The derivation of an expression for this deviation is given in next section.

11.5. FREE FALL OF A BODY ON EARTH'S SURFACE

Let us consider the free fall of a body from a height h on the surface of the earth at a latitude ϕ. The earth is rotating about its axis with an angular velocity ω. At the point P of the earth, take X-axis vertically, Y-axis along east and Z-axis along north. [Fig. 11.6]. If $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$, are unit vectors along these axes, then the angular velocity $\boldsymbol{\omega}$ can be represented as

$$\boldsymbol{\omega} = \omega \cos\left[\frac{\pi}{2} - \phi\right] \hat{\mathbf{i}} + \omega \cos \phi \hat{\mathbf{k}}$$

or $$\boldsymbol{\omega} = \omega (\sin \phi \hat{\mathbf{i}} + \cos \phi \hat{\mathbf{k}}) \quad ...(18)$$

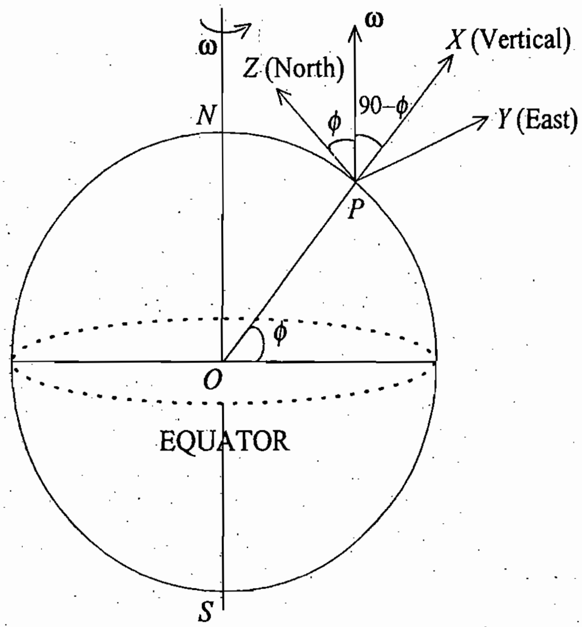


Fig. 11.6

As the effective value of the acceleration due to gravity g is the combined effect of the centripetal acceleration and acceleration in the inertial frame, then substituting $\mathbf{a}_i - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = -\hat{\mathbf{i}}\, g$ in the equation

$$\mathbf{a}_i = \mathbf{a}_r + 2\, \boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}), \text{ we get}$$

$$-\hat{\mathbf{i}}\, g = \mathbf{a}_r + 2\, \boldsymbol{\omega} \times \mathbf{v}_r \quad ...(19)$$

Here the velocity of the body $\mathbf{v}_r$ is almost along X-axis with negligible y and z components and hence we can have

$$\mathbf{v}_r = \hat{\mathbf{i}} \frac{dx}{dt} \quad ...(20)$$

Writting $\mathbf{a}_r$ in component form, we get from eq. (19)

$$-\hat{\mathbf{i}}\, g = \hat{\mathbf{i}} \frac{d^2x}{dt^2} + \hat{\mathbf{j}} \frac{d^2y}{dt^2} + \hat{\mathbf{k}} \frac{d^2z}{dt^2} + 2\omega\, (\sin \phi\, \hat{\mathbf{i}} + \cos \phi\, \hat{\mathbf{k}}) \times \hat{\mathbf{i}} \frac{dx}{dt}$$

or $$-\hat{\mathbf{i}}\, g = \hat{\mathbf{i}} \frac{d^2x}{dt^2} + \hat{\mathbf{j}} \left[\frac{d^2y}{dt^2} + 2\omega \frac{dx}{dt} \cos \phi\right] + \hat{\mathbf{k}} \frac{d^2z}{dt^2}$$

Now, the three component equations are

$$\frac{d^2x}{dt^2} = -g \quad ...(21a)$$

$$\frac{d^2y}{dt^2} = -2\omega \frac{dx}{dt} \cos \phi \quad ...(21b)$$

$$\frac{d^2z}{dt^2} = 0 \quad ...(21c)$$

Integrating eq. (21a), we get

$$\frac{dx}{dt} = -gt + C$$

But initially at $t = 0$; $\frac{dx}{dt} = 0$, therefore $C = 0$.

Thus $$\frac{dx}{dt} = -gt. \quad ...(22)$$

Integrating it further, we get

$$x = -\tfrac{1}{2} g t^2 + C'$$

Initially at $t = 0$, the distance of the stone from the earth $x = h$. This gives $h = C'$, and hence

$$x = -\tfrac{1}{2} g t^2 + h$$

Finally, when at $t = T$, the stone touches the ground $x = 0$. Therefore

$$h - \tfrac{1}{2} g T^2 = 0 \quad \text{or} \quad T = \sqrt{2h/g} \qquad \text{...(23)}$$

Now,
$$\frac{d^2y}{dt^2} = -2\omega \frac{dx}{dt} \cos\phi \quad \text{or} \quad \frac{d^2y}{dt^2} = 2\omega g t \cos\phi$$

Integrating it, we get

$$\frac{dy}{dt} = \omega g t^2 \cos\phi \quad \text{and hence} \quad y = \frac{\omega g t^3}{3} \cos\phi \qquad \text{...(24)}$$

The constants of integration in (24) have been taken zero, because initially the stone has no displacement and velocity in the Y-direction is zero.

Finally, at $t = T$, the maximum horizontal displacement will be given by

$$Y = \frac{\omega g T^3}{3} \cos\phi \qquad \text{...(25)}$$

Substituting the value of T from eq. (23), we get

$$Y = \frac{\omega g}{3}\left[\frac{2h}{g}\right]^{3/2} \cos\phi \quad \text{or} \quad Y = \left[\frac{8}{9g}\right]^{1/2} h^{3/2}\, \omega \cos\phi \qquad \text{...(26)}$$

Thus a freely falling body at latitude ϕ is displaced horizontally due east by Coriolis force by an amount given by eq. (26). At the equator the easterly deflection is obtained to be

$$Y = \left[\frac{8}{9g}\right]^{1/2} h^{3/2}\, \omega \qquad \text{...(27)}$$

11.6. FOUCAULT'S PENDULUM

Experimentally the fact, that the earth rotates about its axis and the frame attached to it is a noninertial frame, is demonstrated by the Foucault's pendulum. The related experiment was first performed by Foucault in 1851.

In Foucault's pendulum, a very heavy bob is suspended by means of a long strong wire similar to a simple pendulum. Thus if it is once vibrated, it vibrates for considerably large time with a large period (≈17 sec). Foucault took in his experiment a bob of 28 kg mass and a wire of 70 metres long. The upper end of the wire is attached to a rigid support in such a way that the pendulum may vibrate with equal freedom in any direction.

When Foucault's pendulum is oscillated in the northern hemisphere of the earth, its plane of oscillation rotates from east to west

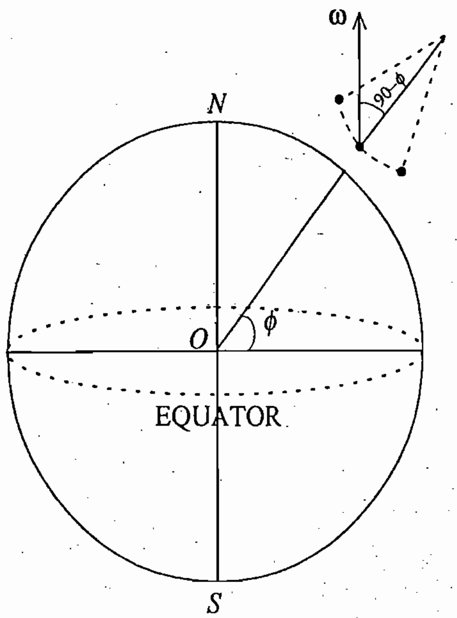


Fig. 11.7 : Foucault's pendulum

(or clockwise as seen from above). It is not possible that the point of support might have turned the plane of vibration. The only possibility is this that the floor, *i.e.*, the earth under the pendulum is rotating. If such a pendulum is imagined to vibrate at the north pole of the earth, its plane of oscillation will remain fixed in an inertial frame or solar frame of reference. However, the earth under the pendulum is rotating once every 24 hours, therefore an observer on the earth will see that the plane of oscillation is turning from east to west, opposite to the earth's rotation. At any other latitude ϕ, the angular velocity ω can be resolved into two components : vertical component $\omega \sin \phi$ and horizontal component $\omega \cos \phi$ in the north-south direction. Obviously the later component will not have any perceptible effect on the pendulum. The vertical component will make the plane of oscillation to rotate with an angular velocity $\omega \sin \phi$. Thus the time of one rotation of the plane of oscillation will be given by

$$T = \frac{2\pi}{\omega \sin \phi} \qquad ...(28)$$

The earth rotates once in 24 hours with an angular velocity $\boldsymbol{\omega}$, hence

$$24 \text{ hours} = \frac{2\pi}{\omega} \quad \text{or,} \quad T = \frac{24 \text{ hours}}{\sin \phi} \qquad ...(29)$$

The rotation of the plane of oscillation will be clockwise in the northern hemisphere and opposite to it in the southern hemisphere. Actually, this rotation, as seen by the observer on the earth, is due to Coriolis force. When the pendulum moves along a horizontal line, the Coriolis force acts in the perpendicular direction to its velocity and hence the plane of oscillation turns. Thus, ***the rotation of the plane of oscillation of Foucault's pendulum demonstrates the fact that the earth rotates about its axis.***

Ex. 1. *A bullet is fired horizontally in the north direction with a velocity of 500 m/sec. at 30° N latitude. Calculate the horizontal component of Coriolis acceleration and the consequent deflection of the bullet as it hits a target 250 metres away. Also determine the vertical displacement of the bullet due to gravity. If the mass of the bullet is 10 gm. find the Coriolis force.* **(Ajmer 1988)**

Solution : If X-axis is taken vertically, Z-axis towards north and Y-axis along east, then the velocity of the bullet is $v = 500$ km/sec. and angular velocity $\boldsymbol{\omega} = \omega(\hat{\mathbf{k}} \cos 30° + \hat{\mathbf{i}} \sin 30°)$, because the angular velocity vector $\boldsymbol{\omega}$ of the earth is directed parallel to its axis and is inclined at 30° to the horizontal.

Here, $\omega = \dfrac{2\pi}{24 \times 60 \times 60} = 7.2 \times 10^{-5}$ rad./sec.

Hence Coriolis acceleration

$= 2\,\boldsymbol{\omega} \times \mathbf{v} = 2\omega(\hat{\mathbf{k}} \cos 30° + \hat{\mathbf{i}} \sin 30°) \times 500\,\hat{\mathbf{k}}$

$= -2 \times 7.2 \times 10^{-5} \times 500 \times \frac{1}{2}\,\hat{\mathbf{j}} =$ **0.036 m/sec² towards west.**

Time of journey, $t = \dfrac{250}{500} = \dfrac{1}{2}$ sec.

Deflection of the bullet due to the Coriolis acceleration

$$= \frac{1}{2}at^2 = \frac{1}{2} \times 0.036 \times \left(\frac{1}{2}\right)^2 = 4.5 \times 10^{-3} \text{ m.}$$

Vertical displacement of the bullet due to the gravity

$$= \frac{1}{2}gt^2 = \frac{1}{2} \times 9.8 \times \left(\frac{1}{2}\right)^2 = \mathbf{1.23 \text{ m.}}$$

Coriolis force $= -2m\,\boldsymbol{\omega} \times \mathbf{v}$

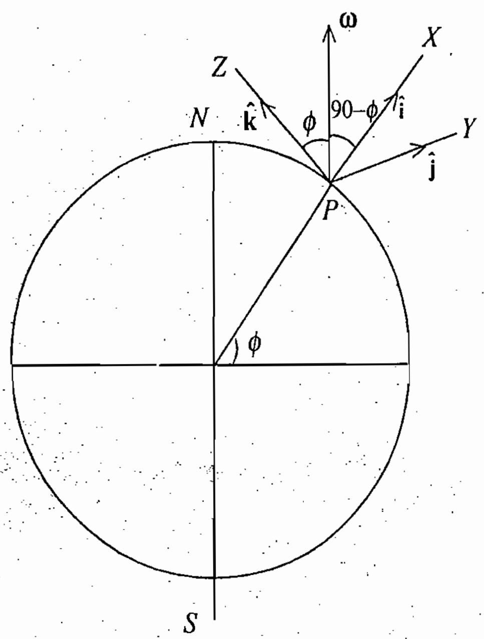


Fig. 11.8

$$= 2 \times 0.01 \times 7.2 \times 10^{-5} \times 5 \times 10^{2} \times \tfrac{1}{2}\,\hat{\mathbf{i}} = 3.6 \times 10^{-4}\,\hat{\mathbf{i}}$$

$= 3.6 \times 10^{-4}$ **newton towards east.**

Ex. 2. *Prove that the observed acceleration due to gravity g_ϕ at the latitude ϕ is related to its value g by the relation*

$$g_\phi^2 = (g\cos\phi - \omega^2 R\cos\phi)^2 + (g\sin\phi)^2 \qquad \textbf{(Ajmer 1988; Rajasthan 83)}$$

Solution : If a particle is at rest at ϕ latitude, then it is not acted by Coriolis force. Therefore,

$$\mathbf{a}_i = \mathbf{a}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$$

If Z-axis is taken along the axis of the earth and X-axis perpendicular to it, then

$$\mathbf{a}_r = \mathbf{a}_i - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$$

or
$$\mathbf{g}_\phi = -g\,(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{k}}\sin\phi) - \omega\hat{\mathbf{k}} \times [\omega\hat{\mathbf{k}} \times \mathbf{R}\,(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{k}}\sin\phi)]$$

$$= -g\,(\hat{\mathbf{i}}\cos\phi + \hat{\mathbf{k}}\sin\phi) + \omega^2\mathbf{R}\cos\phi\,\hat{\mathbf{i}} = -\hat{\mathbf{i}}\,(g\cos\phi - \omega^2\mathbf{R}\cos\phi) - \hat{\mathbf{k}}\,g\sin\phi$$

Therefore
$$g_\phi^2 = \mathbf{g}_\phi \cdot \mathbf{g}_\phi = (g\cos\phi - \omega^2 R\cos\phi)^2 + (g\sin\phi)^2.$$

Ex. 3. *Prove that the plane of oscillation of Foucault's pendulum rotates* 150 *sin ϕ per hour, where ϕ is the latitude of the place.*

Solution : Period of one rotation of plane of oscillation of Foucault's pendulum is given by

$$T = \frac{24 \text{ hours}}{\sin\phi}$$

If the plane of oscillation be rotating with an angular velocity θ per hour, then

$$T = \frac{2\pi}{\theta} = \frac{360^\circ}{\theta} \text{ hour}$$

Therefore, $\dfrac{360^\circ}{\theta} = \dfrac{24}{\sin\phi}$ or $\boldsymbol{\theta = 15^\circ \sin\phi}$ **per hour.**

Questions

1. What are noninertial frames and fictitious forces ? Is the centrifugal force fictitious one ? **(Kanpur 1984, 79)**
2. What are fictitious forces ? Illustrate with examples. Find out the fictitious acceleration of the sun in a frame fixed with the earth and rotating about its axis. **(Agra 1977)**
3. What are fictitious forces ? How are these related to noninertial frames ? What are effects of Coriolis force due to the earth's rotation ? Explain. **(Ajmer 1989)**
4. What are Coriolis force ? Show that the total Coriolis force acting on a body of mass m in a rotating frame is $-2m\,\boldsymbol{\omega} \times \mathbf{v}$, where $\boldsymbol{\omega}$ is the angular velocity of rotating frame and $\mathbf{v}$ is the velocity of the body in rotating frame. **(Ajmer 1988)**
5. A reference frame A rotates with respect to another reference frame B withuniform angular velocity $\boldsymbol{\omega}$. If the position, velocity and acceleration of a particle in frame A are represented by $\mathbf{R}$, $\mathbf{v}_a$ and $\mathbf{f}_a$ respectively, show that the acceleration of that particle in frame B is given by

$$\mathbf{f}_b = \mathbf{f}_a + 2\boldsymbol{\omega} \times \mathbf{v}_a + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}).$$

Interpret this equation with reference to the motion of bodies on earth's surface.

6. Explain inertial frames of reference. Frame of reference R rotates about its origin fixed in an inertial frame of reference I. Find how velocities and accelerations in the two reference frames are related to each other. What are pseudo forces ? Explain. **(Agra 1992)**

7. A stone is allowed to fall under gravity from the top of a h metre high towerat the equator. Show that the horizontal displacement of the stone due to the earth's rotation is given by

$$y = \left(\frac{8}{9g}\right)^{1/2} h^{3/2}\,\omega\,.$$

8. An elevator is descending at a constant speed. A passenger drops a coin to the floor. What acceleration would (*i*) the passenger and (*ii*) a person at rest with respect to elevator shaft that observe for the falling coin ?

9. Under what circumstances would your weight be zero ? Does your answer depend upon the choice of the frame of reference ?

10. What is Foucault's pendulum ? From this how it is proved that the earth rotates. **(Ajmer 1989)**

11. A simple pendulum is oscillating at a place with latitude $\theta°$. Explain (using the expression of Coriolis force) why the plane of oscillation does not remain fix. Also obtain the expression for angular speed of precession of plane of oscillation. **(Mumbai, Nov. 2000)**

Problems

[SET- I]

1. Calculate the fictitious and total force on a body of mass 2.5 kg relative to a frame moving vertically upwards on earth with an acceleration of 10 m./sec^2. **(Mysore 1981)**

   Ans. : 25 N downwards, 49.5 N downwards.

2. What is the fictitious force and total force acting on a freely falling body of 3 kg mass with reference to a frame of reference moving with a downward acceleration of 4 m/sec^2.

   Ans. : 12 newtons upward ; 17.4 newtons downward.

3. A rocket is moving upward with acceleration 3g. Calculate the effective weight of a man sitting in it, if his actual weight is 75 kg. **(Agra 1982)**

   Ans. : 300 kg. wt.

4. A 2 kg. stone at the end of 2 metres long string makes 3 revolutions in 4 seconds. Calculate the forces on the stone as measured in an inertial frame and in a frame which is rotating with the string.

   Ans. : 88 newtons (centripetal), 88 newtons (centrifugal).

5. A body is attached to the lower end of a spring suspended from the roof of a lift. When the lift is at rest, the body produces an elongation of 2 cm in the spring. When the lift moves down with an acceleration, the elongation is changed to 1.95 cm. Find the acceleration of the lift. **(Mysore 1981)**

   Ans. : 0.245 m s^{-2}.

6. A massless string pulls a mass of 50 kg upwards against gravity. The string would break if subjected to a tension greater than 600 newtons. What is the maximum acceleration with which mass can be moved upwards ? **(Lucknow 1980)**

   Ans. : 2.2 ms^{-2}.

7. Find the horizontal component of Coriolis force acting on a body of mass 0.1 kg moving northward with a horizontal velocity of 100 m/sec at 30° N latitude on the earth.

   Ans. : 7.2×10^{-4} N towards east.

8. A person in a jet plane is flying along the equator due east with a speed of 450 m/sec. What is his Coriolis acceleration ?
Ans. : 6.56×10^{-3} m/sec^2.

9. A stone is dropped with zero initial velocity from the top of a 100 m high tower at the equator. Calculate horizontal displacement of the stone due to earth's rotation. ($g = 10$ m/s^2 and $\omega = 7.2 \times 10^{-5}$ rad/s).
Ans. : 2.15 cm.

10. Calculate the eastward deflection of a stone falling freely from a height 50 m above the ground at the latitudes (*i*) 60° N, (*ii*) 45° N and (*iii*) at equator. ($\omega = 7.3 \times 10^{-5}$ rad/s and $g = 9.8$ m/s^2)
Ans. : (*i*) 0.388 cm (*ii*) 0.549 cm (*iii*) 0.776 cm.

11. *A* suggests that if a stone is thrown vertically upwards in the northern hemi-sphere so as to attain a maximum height h. On its return to the earth, it will have no horizontal displacement. He argues that during the return journey the stone experiences exactly equal and opposite Coriolis force as it does during the onwards journey and, therefore, the net displacement on the ground will be zero. Person *B* disagrees to this view. What is your answer to this riddle? Support your answer with proper derivation.
Ans. : The net displacement will be double of the displacement at the maximum height towards west.

12. A body is thrown vertically upward with a velocity u. Prove that it will fall back on a point displaced to the west by a distance equal to $\frac{4}{3}\left(\frac{8h^3}{g}\right)^{1/2} \omega \cos \phi$, where ϕ is the latitude and $h = u^2/2g$.
(Ajmer 1989; Rajasthan 84)

13. A body of mass 10 gm is at rest in an inertial frame. Determine its motion in a frame rotating with an angular speed of 10 rad/sec. The particle is situated at a distance of 5 cm from the axis of rotation. Find the Coriolis and centrifugal forces.
Ans. : Centrifugal force = 0.05 N directed outwardly ; Coriolis force = 0.1 N directed inwardly. The addition of the two fictitious forces is 0.05 N inwardly, *i.e.*, this explains the observed circular motion and it seems a force (centripetal) 0.05 N is acting towards the axis of rotation.

14. What is the fictitious acceleration of the sun relative to a frame rotating with the earth. (Distance between the sun and the earth = 1.5×10^{11} m) **(Agra 1981)**
Ans. : 7.8×10^2 m/sec^2 towards earth .
[**Hint :** In the rotating frame with the earth the total fictitious acceleration on the sun $= -2\boldsymbol{\omega} \times \mathbf{v}_r - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = 2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R}) = -\omega^2 \mathbf{R}$.]

15. In how much time will the plane of oscillation of a Foucault's pendulum turn through 90° at 30° latitude.
Ans. : 12 hours.

16. A Foucault's pendulum is oscillating at any time in the east-west direction. Calculate the time in which its plane of oscillation will start pointing 18° north of east, if the latitude of the place be 45°.
Ans. : 32.2 hours.

17. A 100 kg mass is moving horizontally along the latitude of 60° in the northern hemisphere at speed 50 m/s. Find the magnitude of Coriolis force acting on it and show its direction by drawing a diagram. Also find the horizonal component of the force. **(Mumbai, Nov. 2000)**
Ans. 7.2 N; 6.23 N.

[SET- II]

1. Apply the principle of conservation of angular momentum about the centre of the earth to show that a stone dropped from rest and from a height h above the ground will have the same deviation in the east direction as would be given by a consideration of the Coriolis force.

[**Hint :** The principle of conservation of angular momentum gives

$$m\omega_0(R+h)^2 = m\omega\,(R+h-\tfrac{1}{2}gt^2)^2$$

Therefore, $$\frac{d\theta}{dt} = \omega = \frac{\omega_0(R+h)^2}{(R+h-\frac{1}{2}gt^2)^2} = \frac{\omega_0\left(1+\frac{h}{R}\right)^2}{\left(1+\frac{h-\frac{1}{2}gt^2}{R}\right)^2}$$

$$= \omega_0(1+gt^2/R) \qquad \text{(Applying Bionomial theorem)}$$

Hence, $$\theta = \int_0^T \omega\, dt = \omega_0 T + \frac{\omega_0 g T^3}{3R}\text{; Deflection } = R(\theta - \omega_0 T) = \tfrac{1}{3}\,\omega_0 g T^3$$

where we have considered the free fall at the equator.]

2. A locomotive is travelling towards north at a latitude ϕ along a straight level track with velocity v. Show that the ratio of the forces on the two rails is approximately

$$1 + 4\,\omega v h \sin\phi/(ga).$$

where h is the height of the centre of gravity above the rails and $2a$ is the width between the rails.

3. If a projectile is fired due east from a point on the surface of the earth at a latitude ϕ with a velocity v_0 and at an angle of elevation above the horizontal of θ, show that the lateral deflection of the projectile when it strikes the earth is

$$d = \frac{4v_0^3}{g^2}\,\omega \sin\phi \sin^2\theta \cos\theta$$

where ω is the angular velocity of the earth and g is the acceleration due to gravity.

If the range of the projectile is R_0 for $\omega = 0$, show also that the change of range due to the rotation of the earth is

$$R - R_0 = \sqrt{\frac{2R^3}{g}}\,\omega \cos\phi\,[\cot^{1/2}\phi - \frac{1}{3}\tan^{3/2}\theta].$$

Objective Type Questions

1. Non-inertial frames
   (*a*) are accelerated frames.
   (*b*) are unaccelerated frames.
   (*c*) are those frames in which a force-free particle moves with constant velocity.
   (*d*) cannot be rotating frames.
   Ans. : (*a*).

2. A particle is observed from two frames S_1 and S_2. S_2 moves with an acceleration relative to S_1. If F_1 and F_2 be the pseudo-forces on the particle when seen from S_1 and S_2 frames, the possibility is
   (*a*) $F_1 = 0, F_2 \neq 0$ (*b*) $F_1 \neq 0,\ F_2 = 0$
   (*c*) $F_1 \neq 0,\ F_2 \neq 0$ (*d*) $F_1 = 0,\ F_2 \neq 0$
   Ans. (*a*), (*b*), (*c*).

3. A body is kept on the floor of an elevator at rest. The elevator starts to descend with an acceleration of 11 m/s^2. The displacement of the body during the first two seconds is (g =10 m/sec^2)
   (*a*) 0 m (*b*) 2 m
   (*c*) 20 m (*d*) 42 m
   Ans. (c).
4. A particle is at rest in a rotating frame. The pseudo force acting on the particle in the rotating frame is
   (*a*) zero (*b*) only the centrifugal force
   (*c*) only the Coriolis forces (*d*) the combination of both (*b*) and (*c*)
   Ans. (*b*).
5. The Corolis force causes a moving particle
   (*a*) in the northern hemisphere to deflect towards the right of its path.
   (*b*) in the southern hemisphere to deflect towards the right of its path.
   (*c*) in the northern hemisphere to deflect towards the left of its path.
   (*d*) in the southern hemisphere to deflect towards the left to its path.
   Ans. (*a*), (*d*).
6. A rigid frictionless rod rotates anticlockwise in a vertical plane with angular velocity $\vec{\omega}$. A bead of mass m moves outward along the rod with constant velocity $\vec{u_0}$. The bead will experience a Coriolis force.
   (*a*) $2mu_0\omega\hat{\theta}$ (*b*) $-2mu_0\omega\hat{\theta}$
   (*c*) $4mu_0\omega\hat{\theta}$ (*b*) $-mu_0\omega\hat{\theta}$ **(GATE 2004)**
   Ans. (*b*).
7. The plane of oscillation of a Foucault's pendulum rotates
   (*a*) 15° per hour at the equator. (*b*) 15° per hour at the pole.
   (*c*) 7.5° per hour at the latitude 60°. (*d*) 30° per hour at the latitude 60°.
   Ans. (*b*).

Short Type Questions

1. What do you mean by non-incrtial frame of reference ?
2. What are fictious forces ? Discuss centifugal force.
3. Differentiate between real and fictitious forces.
4. What is Coriolis force ?
5. Prove that a frame of reference fixed with the earth is non-inertial frame.
6. A bullet in fired horizontally in the north direction at a latitude ϕ. In what direction the bullet will be deflected due to coriolis force ? Explain.
7. Calculate the fictitious force and total force acting on a freely falling body of mass 20 kg with reference to a frame moving with a downward acceleration of 6 m/sec^2 .
   Ans. 120 *n* upwards; 76 *n* downwards.
8. Fill in the blanks :
   (*i*) A stone falls under gravity from the top of a tower at the equator. The horizontal displacement of the stone due to earth's rotation will be..................... direction.
   (*ii*) The time of one rotation of the plane of oscillation of Foucault's pendulum at latitude ϕ is T=........
   Ans. (*i*) east (*ii*) $2\pi/\omega \sin\phi$.

CHAPTER 12

Special Theory of Relativity-Lorentz Transformations

12.1. INTRODUCTION

In Chapter 1, we discussed the concept of space and time and gave the idea of inertial frames. An inertial frame is an unaccelerated coordinate system. In such a frame, if force is not acting on a particle, it will remain either at rest or will move with the constant velocity, *i.e.*, the law of inertia (Newton's first law) holds true. We have also shown that a frame, moving with constant velocity relative to an inertial frame, is also inertial one.

In Newtonian mechanics, space and time are completely separable and the transformations connecting the space-time coordinates of a particle are the Galilean transformations. These transformations are valid as far as Newton's laws are concerned, but fail in the field of electrodynamics. Principle of relativity, when applied to the electromagnetic phenomena, asserts that the speed of light in vacuum is a constant of nature. This starement has been confirmed by several experiments and led Einstein to formulate the special theory of relativity. In view of this theory, space and time are not independent of each other and the correct transformation equations are Lorentz transformations. In this chapter, first we discuss the Galilean transformations and their failures and then we deduce the Lorentz transformations. Finally, we discuss the consequences of Lorentz transformations, namely length contraction, time dilation, simultaneity, velocity addition etc.

12.2. GALILEAN TRANSFORMATIONS

At any instant, the coordinates of a point or particle in space will be different in different coordinate systems. The equations which provide the relationship between the coordinates of two reference systems are called ***transformation equations***.

In Chapter 1, we have shown that a frame S' which is moving with constant velocity v relative to an inertial frame S, is itself inertial.

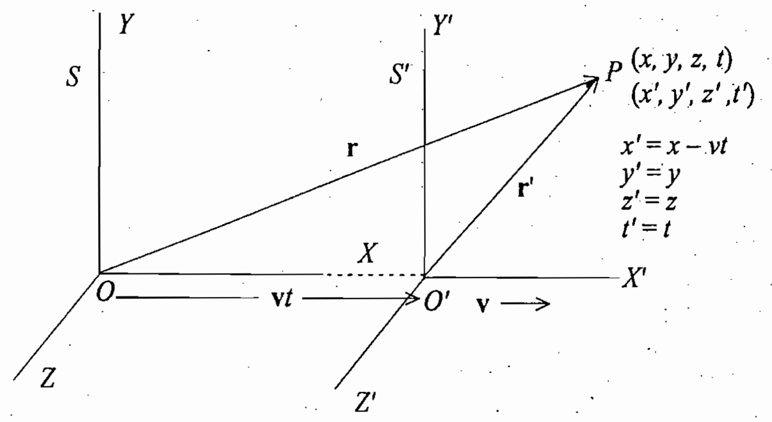


Fig. 12.1 : Representation of Galilean transformations

For convenience, if we assume (*i*) that the origins of the two frames coincide at $t = 0$, (*ii*) that the coordinate axes of the second frame are parallel to that of the first and (*iii*) that the velocity of the second frame relative to the first is v along X-axis, then the position vectors of a particle at any instant t in the two frames are related by the equation

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t \qquad ...(1)$$

In the component form, the coordinates are related by the equations

$$x' = x - vt \;;\; y' = y \;;\; z' = z \qquad ...(2)$$

Eq. (1) or (2) expresses the transformation of coordinates from one inertial frame to another and they are referred as ***Galilean transformations***.

The form of eq. (1) or (2) depends, of course, on the relative motion of two frames of reference, but it also depends upon certain assumptions regarding the nature of time and space. It is assumed that the time t is independent of any particular frame of reference i.e., if t and t' be the times recorded by the observers O and O' of an event occurring at P, then $t' = t$. If we add the equation $t' = t$, then the ***Galilean transformation equations*** are expressed as

$$x' = x - vt \;;\; y' = y \;;\; z' = z;\; t' = t \qquad ...(3)$$

We can also consider that frame S is moving with velocity $- v$ along the negative X-axis with respect to S' frame. Then the transformation equations from frame S' to S are

$$x = x' + vt',\; y = y' \quad z = z' \quad t = t' \qquad ...(3')$$

These are known as ***inverse Galilean transformations***.

The other assumption, regarding the nature of the space, is that the distance between two points (or two particles) is independent of any particular frame of reference. Evidently, this assumption is expressed by the form of the transformation eq. (1) or (3). If a rod has length L in the frame S with the end coordinates x_1 and x_2, then $L = x_2 - x_1$.

If at the same time the end coordinates of the rod in S' are x_1' and x_2', then $L' = x_2' - x_1'$. But for any time t, from eq. (3), we have

$$x_2' - x_1' = x_2 - x_1$$

Therefore, $$L' = L \qquad ...(4)$$

Thus the length or distance between two points is invariant under Galilean transformations.

Differentiating eq.(1) with respect to time, we get

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} + \frac{d\mathbf{r}'}{dt} = \mathbf{v} + \frac{d\mathbf{r}'}{dt'} \qquad [\because t = t']$$

or $$\mathbf{u} = \mathbf{v} + \mathbf{u}' \qquad ...(5)$$

where $\mathbf{u}$ and $\mathbf{u}'$ are the observed velocities in S and S' frames respectively.

Eq. (5) transforms the velocity of a particle from one frame to another and is known as ***Galilean*** (or ***classical***) ***law of addition of velocities***.

Again differentiating eq. (5) with respect to time t, we have

$$\frac{d\mathbf{u}}{dt} = 0 + \frac{d\mathbf{u}'}{dt} = \frac{d\mathbf{u}'}{dt'} \qquad [\because t = t']$$

or $$\mathbf{a} = \mathbf{a}' \qquad ...(6)$$

Hence *according to Galilean transformations, the accelerations of a particle relative to S and S' frames are equal.*

It is to be mentioned that the Galilean transformations are based basically on two assumptions :

(1) ***There exists a universal time t which is the same in all reference systems.***

(2) ***The distance between two points in various inertial systems is the same.***

Thus if any two events occur simultaneously for any observer, then they must occur simultaneously for all observers. In other words, *the time interval between two given events must be identical for all systems of reference.*

12.3. PRINCIPLE OF RELATIVITY

Absolute velocity of a body has no meaning. The velocity has a meaning only when it is measured relative to some other body or frame of reference. If two bodies are moving with uniform relative velocity, it is impossible to decide which of them is at rest or which of them is moving. This is known as ***principle of relativity***. However, acceleration has an absolute meaning. For example, if we are sitting in a windowless accelerated aircraft, we can perform an experiment and measure its acceleration. But if the aircraft is moving with uniform velocity, we cannot measure its velocity. Of course, we measure its velocity relative to a body outside. Thus the principle of relativity can be alternatively stated as follows :

It is impossible to perform an experiment which will measure the state of uniform velocity of a system by observations, confined to that system.

The motion of a body itself has no meaning unless, we do not know with respect to which this motion has been measured. This led Newton to think about the absolute space and it represents an absolute frame with respect to which every motion should be measured. However, in view of this principle of relativity, we cannot perform an experiment which will measure the uniform velocity of a reference system relative to the absolute frame by observations confined to that system.

In the unaccelerated windowless ship all experiments performed inside it will appear the same whether this ship is stationary or in uniform motion. Newton stated the principle of relativity as follows :

The motions of bodies included in a given space are the same among themselves whether that space is at rest or moving uniformly forward in a straight line.

Study of the physical laws involves the measurements of accelerations, forces etc among bodies. The principle of relativity can be stated in an elegant form as follows :

The basic laws of physics are identical in all inertial systems which move with uniform velocity with respect to one another.

This principle is called ***Galilean*** or ***Newtonian principle of relativity*** and sometimes it is named as ***hypothesis of Galilean invariance.*** In fact, the principle of relativity is a fundamental postulate and is entirely consistent with the theory of special relativity. If any two inertial systems, moving with constant relative velocity, are connected by Galilean transformations, the principle of relativity is modified as :

The basic laws of physics are invariant in form in two reference systems connected by Galilean transformations.

This statement is somewhat special than the principle of relativity in the sense that it means the assumptions that the time and the space intervals are independent of the frame of reference. We shall see later in the theory of special relativity that the Galilean transformations are not correct, but the appropriate exact transformation equations are the Lorentz transformation equations for connecting any two frames in uniform relative motion. Thus, the principle of relativity may be stated as :

The basic laws of physics are invariant in form in two inertial frames connected by Lorentz transformations.

12.4. TRANSFORMATION OF FORCE FROM ONE INERTIAL SYSTEM TO ANOTHER

Suppose that the force F on a particle of mass m in the frame S is represented by Newton's second law

$$\mathbf{F} = m\mathbf{a}$$

But according to the postulate that the laws of physics are the same in the frame S and in another frame S', which is in uniform motion relative to S, we have

$$\mathbf{F}' = m'\mathbf{a}'$$

in the frame S'.

We have shown in the last article that the acceleration of the particle is the same in two inertial frames, connected by Galilean transformations, *i.e.*,

$$\mathbf{a}' = \mathbf{a}$$

where in the deduction basically we have assumed the invariance of space and time separately.

In Newtonian mechanics, the mass is independent of velocity and hence

$$m = m'$$

Thus,

$$\mathbf{F} = m\mathbf{a} = m'\mathbf{a}' = \mathbf{F}'$$

This means that if the relation $\mathbf{F} = m\mathbf{a}$ (Newton's second law) is used to define the force, then in all inertial systems, the force $\mathbf{F}$ will have the same magnitude and direction, independent of the relative velocities of the reference frames. Further, the Newton's equation has the same form in the inertial frame S as well as in the frame S'. We mean this statement that Newton's second law is invariant under Galilean transformations. As Newton's first law ($\mathbf{F} = 0$) can be deduced from second law and third law involves forces. We may also say that *Newton's laws of motion* (so called laws of mechanics) *are invariant under Galilean transformations.*

12.5. COVARIANCE OF THE PHYSICAL LAWS

If the form of a law is not changed by certain coordinate transformation (i.e., *if it is the same law in terms of either set of coordinates*), *we call that the law is* **invariant** or **covariant** *with respect to the coordinate transformation* under consideration. Newton's laws of motion are covariant with respect to Galilean transformations. Mathematically, suppose a phenomenon is described in system S by an equation

$$f(x, y, z, t) = 0 \qquad \text{...(7)}$$

Then the covariance of the equations means that in the system S', it will have the form

$$f(x', y', z', t') = 0 \qquad \text{...(8)}$$

The principle of relativity asserts that the laws of physics are covariant in all inertial systems, moving with constant relative velocity. It is to be mentioned that the Galilean transformations satisfy the principle of relativity as far as Newton's laws of motion are concerned, but as we shall see later, these transformations do not satisfy this principle for propagation of electromagnetic waves.

12.6. PRINCIPLE OF RELATIVITY AND SPEED OF LIGHT

According to the principle of relativity, basic laws of physics remain the same in all inertial systems. If the principle of relativity is extended to electrodynamics, Maxwell's fundamental equations should remain the same in any two inertial systems with uniform constant relative motion. It follows from Maxwell's equations that the electromagnetic waves are propagated in vacuum with a constant velocity $c = 3 \times 10^8$ m/sec in all directions irrespective of the motion of the source. Light waves are basically electromagnetic waves and hence *according to the principle of relativity, the velocity of light must be the same with value c in all inertial systems, independent of the motion of the light source.*

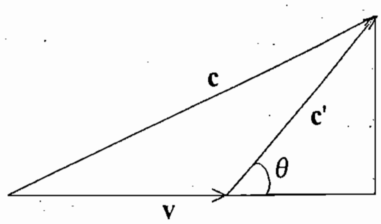


Fig. 12.2

It can be shown that the idea of constancy of speed of light contradicts the Galilean transformations. Let S be a frame of reference with a source of light at the origin O. In this system, the velocity of light is c in all the directions. Now, let a frame S' be moving with constant velocity $\mathbf{v} = v\,\hat{\imath}$ along X-axis. In the frame

S', the velocity of light, using Galilean transformations, will be given by

$$\mathbf{c}' = \mathbf{c} - \mathbf{v} \quad ...(9)$$

The speed of the light signal along X-axis ($\theta = 0$) will be noted in S' as

$$c' = c - v \quad ....(9\text{ a})$$

and along Y'-axis ($\theta = \pi/2$) as

$$c' = \sqrt{c^2 - v^2} \quad (\text{as } c^2 = c'^2 + v^2) \quad ...(9\text{ b})$$

Hence, if we use Galilean transformations, we find that the speed of light is not constant in all inertial systems and this contradicts the principle of relati-vity. Further S must be a ***preferred*** or ***absolute*** frame in which the speed of light is c and hence any other inertial frame (S') should be less suitable. This leads the possibility of defining ***absolute motion***. If we accept the principle of relativity in the fields of electromagnetism and optics, we should revise the concepts of space and time. However, it seems necessary that before adopting a radical departure from the classical ideas of space and time, one should be sure for the truth of the new step by experiments. Michelson Morley experiments were performed to detect the influence of the motion of the earth with respect to so called absolute frame. Negative results were obtained from these experiments and this led finally the acceptance of the principle of relativity.

Ex. *A man can row a boat with a velocity V in still water. How much time does he take to cross a stream of width d and return to the same point if velocity of the stream is v ? How much time does he take a distance d upstream and return ? Calculate in first order (v^2/V^2) the difference between the two times.*

Solution : According to the problem, the man should row the boat perpendicular to the direction of flow so as to return at the same point. If the velocity V of the boat makes an angle θ with the direction of the flow, then it can be resolved into two components :

(*i*) $V\cos\theta$, and (*ii*) $V\sin\theta$.

Now, the component $V\cos\theta$ should balance the velocity of the stream, *i.e.*,

$$V\cos\theta = v \quad \text{or} \quad \cos\theta = v/V.$$

If t be the time required in crossing and returning, then

$$t = \frac{d}{V\sin\theta} + \frac{d}{V\sin\theta} = \frac{2d}{V\sin\theta} = \frac{2d}{V\sqrt{1-\cos^2\theta}} = \frac{2d}{V\sqrt{1-v^2/V^2}} = \frac{2d}{\sqrt{V^2-v^2}}$$

In the second case, time required is

$$t' = \frac{d}{V-v} + \frac{d}{V+v} = \frac{2dV}{V^2-v^2}$$

Now, $$t = \frac{2d}{V(1-v^2/V^2)^{1/2}} = \frac{2d}{V}\left(1-\frac{v^2}{V^2}\right)^{-1/2} = \frac{2d}{V}\left(1+\frac{v^2}{2V^2}\right)$$

and $$t' = \frac{2dV}{V^2-v^2} = \frac{2d}{V(1-v^2/V^2)} = \frac{2d}{V}\left(1-\frac{v^2}{V^2}\right)^{-1} = \frac{2l}{V}\left(1+\frac{v^2}{V^2}\right)$$

where we have used Binomial expansion and retained the terms in first order in v^2/V^2.

Hence, $$t' - t = \frac{2d}{V}\frac{v^2}{2V^2} = \frac{dv^2}{V^3}.$$

12.7. THE MICHELSON-MORLEY EXPERIMENTS

Michelson-Morley experiment was designed to determine the motion of the earth relative to a frame of reference in which the speed of light is c in all the directions. This frame is the previledge or absolute frame of reference. The earth itself cannot be the absolute frame of reference because it is moving around the sun in its orbit with a speed $v = 3 \times 10^4$ m/sec and therefore at any time if the earth is identified with the motion of the preferred frame, it will have a speed of 6×10^4 m/sec after 6 months relative to the absolute (inertial) frame of reference. In any case, the earth has a speed at least $v = 3 \times 10^4$ m/sec relative to an inertial frame at some instant or the other during 6 months of the year. Assuming that the absolute frame is situated at the centre of the sun and the earth is moving with a speed 3×10^4 m/sec relative to it. Therefore, an observer on earth will measure a speed $(c - v)$ for light moving along the direction of its motion and $(c + v)$ for light moving in opposite direction. We shall see how this assumption (*the change of velocity of light due to the motion of the observer*) was contradicted by the famous experiments of Michelson and Morley, conducted in the year 1880.

In principle, in the experiment of Michelson and Morley, a ray of light starts from a monochromatic source of light S and falls on a half-silvered glass plate P, where a part of light is reflected towards the mirror M_1 and part of it is transmitted towards the mirror M_2 [Fig 12.3(a)]. PM_1 and PM_2 are mutually perpendicular directions and they are nearly equal. Let $PM_1 = PM_2 = l$. The two rays, reflected from the mirrors M_1 and M_2 unite again at P and interference fringes are obtained. These fringes may be seen through the telescope T. The entire apparatus is floated on mercury, contained in a large vessel so that the interferometer may be rotated in any desired direction.

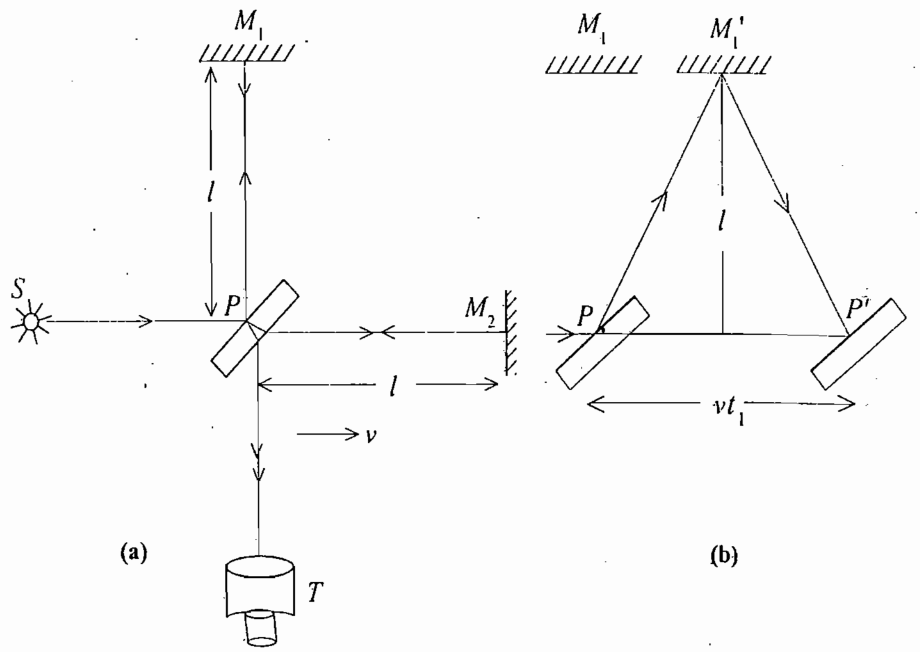


Fig. 12.3 : Michelson-Morley Experiment

In the experiment the instrument is first set up in such a way that arm PM_2 is parallel to the motion of the earth in space. Therefore, the apparatus moves in the direction PM_2 with velocity v relative to the absolute frame. After reflection at P, the ray moves towards the mirror M_1. In the time, the ray reaches the mirror, it moves to the position M_1' so that the reflection occurs at this position M_1'. If t_1 is the time taken by the ray in traversing the path $PM'_1 P'$, in this duration plate P will move a distance $PP' = vt_1$. Hence from fig 12.3 (b), we have

$$PM_1' = \sqrt{PN^2 + M_1'N^2} = \sqrt{(vt_1/2)^2 + l^2}$$

Hence, distance $PM_1'P' = 2PM_1' = 2\sqrt{l^2 + v^2t_1^2/4}$.

This distance $PM_1'P'$ has been traversed by the ray in the absolute frame, in which the velocity of light is c in all the directions. Hence

$$ct_1 = 2\sqrt{l^2 + v^2t_1^2/4} \quad \text{or} \quad c^2t_1^2 = 4l^2 + v^2t_1^2 \quad \text{or} \quad t_1^2 = \frac{4l^2}{c^2 - v^2}$$

Thus
$$t_1 = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c(1 - v^2/c^2)^{1/2}} = \frac{2l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$= \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right) \qquad \left[\text{by Binomial theorem for } \frac{v^2}{c^2} \ll 1\right]$$

The transmitted light ray at P is travelling with velocity $(c - v)$ relative to the mirror M_2, because the mirror is moving with the velocity of the earth (v). This ray after reflection at M_2 will travel with a velocity $(c + v)$ relative to the plate P, because now the plate is moving opposite to the ray. If t_2 be the time taken by the transmitted ray to travel the distance l to the mirror M_2 and back, then

$$t_2 = \frac{l}{c-v} + \frac{l}{c+v} \quad (\text{where } PM_2 = l)$$

$$= \frac{2lc}{c^2 - v^2} = \frac{2l}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1} = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) \quad [\text{using Binomial series for } v^2/c^2 \ll 1]$$

Hence the difference in times taken to traverse by the two paths is

$$\Delta t = t_2 - t_1 = \frac{2l}{c}\left(1 + \frac{v^2}{c^2}\right) - \frac{2l}{c}\left(1 + \frac{v^2}{2c^2}\right) = \frac{2l}{c}\cdot\frac{v^2}{2c^2} = \frac{lv^2}{c^3}$$

Therefore, the difference in the distance travelled by the two rays of light, *i.e.*, the path difference = $c\Delta t = lv^2/c^2$.

Finally, the whole apparatus is turned through 90° so that the other arm PM_1 becomes coincident with the earth's velocity (v) in space [Fig. 12.4]. This causes the difference of path in the opposite direction and hence the displacement of fringes should correspond to the path difference $2lv^2/c^2$. In experiment, distance l was taken nearly 11 metres. Hence the displacement of the fringes should correspond to the path difference

$$\frac{2 \times 11 \times (3 \times 10^4)^2}{(3 \times 10^8)^2} = 2200 \times 10^{-1} \text{ m.}$$

The shift for yellow colour (λ = 5000 Å) is expected to be 0.4 (= $2200 \times 10^{-10}/5800 \times 10^{-10}$) of a fringe-width. The expected shift of 0.4 of a fringe could be measured easily in the experiment. However, no shift of fringes was observed. The experiment was repeated several times, but such displacement was never seen.

One may argue that accidently the absolute frame has the same velocity 3×10^4 m /sec with respect to sun as does the earth. Therefore, at this instant, the earth is at rest in the absolute reference system. However, Michelson and Mor-ley repeated their experiment six months later and a four times magnified effect could be expected ; but again nothing was observed [Fig. 12.5]. The experiments since then have been repeated several times under different circumstances and always the same negative result was obtained.

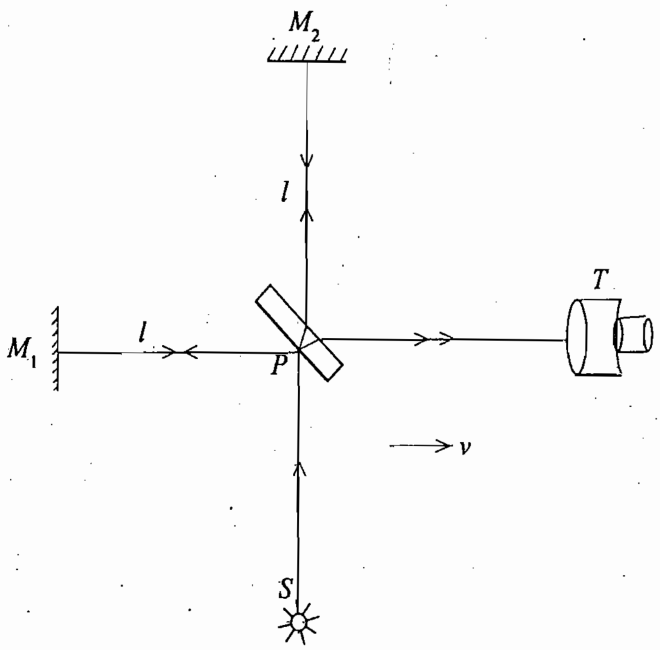


Fig. 12.4 : Michelson-Morley apparatus rotated by $\pi/2$.

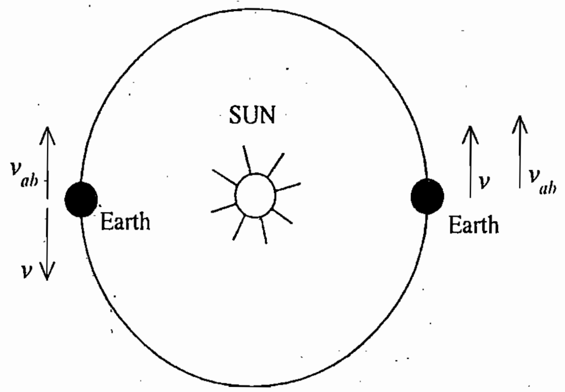


Fig. 12.5

The negative results of Michelson-Morley suggest that the value of v relative to the absolute frame should be zero. In other words, ***the speed of light in vacuum must be the same*** (c) ***in all inertial frames.*** It does not depend upon the motion of the observer or source expected on the basis of Galilean transformations. The negative results of Mechelson-Morley experiments show the validity of the principle of relativity in the fields of elctrodynamics and optics. In fact, *the principle of relativity is a fundamental truth applicable to all areas of physics.*

12.8. ETHER HYPOTHESIS

In the 19th century, physicists made a false analogy between light waves and sound waves or other purely mechanical disturbances. In order to propagate the sound waves a material medium (*e.g.*, air) is necessary. If we say that the speed of sound in air is 332 m/sec, it means that this is the speed which is measured with respect to reference frame fixed in the air. Therefore, these physicists postulate the existence of a hypothetical medium for transmission of light and called it *ether*. It was supposed to fill all the space. To explain the very high speed of light, the density of the ether was supposed to be vanishingly small while its elastic moduli were assumed to be quite large. These workers considered that there is a fixed frame of reference of ether in which light travels with velocity c $(= 3 \times 10^8$ m/sec) in all directions. Since the earth is moving at a speed of $v = 3 \times 10^4$ m/sec. around the sun in its orbit, the supporters of ether theory reasoned that there must be times of the year when the earth has a velocity of at least 3×10^4 m/sec with respect to the ether. The negative results of Michelson-Morley experiments suggest that the effects of ether are undetectable and therefore, ether theory must be discarded. In fact, what Einstein said that there is no necessity of any material medium for the propagation of light waves and analogy between the electromagnatic waves and mechanical waves is not correct.

In conclusion, *the Michelson-Morley experiments discard the idea of a priviledged (absolute) frame of reference or ether and suggest that* ***the velo-city of light c is constant in vaccum in all inertial frames.*** The later fact is the root of the relativistic discussion of physical laws.

12.9. POSTULATES OF SPECIAL THEORY OF RELATIVITY

We have seen that an extension of the principle of relativity to the fields of electrodynamics and optics demands the contancy of speed of light c in vacuum in all inertial systems. This fact is verified by the negative results of Michelson-Morley experiments. In 1905, Einstein propounded the special theory relativity which makes a conceptual change in the ideas of space and time. According to Newtonian mechanics, time and space are invariant for all observers, while both are variable at high speeds for various observers in special theory of relativity.

The two fundamental postulates of the special theory of relativity are the following :

(1) *All the laws of physics have the same form in all inertial systems, moving with constant velocity relative to one another*. This postulate is just the principle of relativity.

(2) *The speed of light is constant in vacuum in every inertial system*. This postulate is an experimental fact and asserts that the speed of light does not depend on the direction of propagation in vacuum and the relative velocity of the source and the observer. In fact, *the second postulate is contained in the first because it predicts the speed of light c to be constant of nature.*

The name special theory of relativity comes from the fact that this theory permits the independence of the physical laws of those coordinate systems which are moving with constant velocity relative to one another. Later, Einstein propounded his *general theory of relativity* which allows for the independence of the physical laws of all coordinate systems, having any general relative motion.

These two postulates of special theory of relativity look to be very simple, but they have revolutionised the physics with far reaching consequences. First we deduce transformation equations, connecting any two inertial systems moving with constant relative velocity. The transformation should be such that they are applicable to both Newtonian mechanics and electromagnetism. Such transformations were deduced by Einstein in 1905 and are known as Lorentz transformations because Lorentz deduced them first in his theory of electromagnetism.

12.10. LORENTZ TRANSFORMATIONS

Suppose that S and S' be the two inertial frames of reference. S' is moving along positive direction of X-axis with velocity v relative to the frame S. Let t and t' be the times recorded in two frames. For our convenience, we will assume that the origins O and O' of the two co-ordinate systems coincide at $t=t'=0$.

Now suppose that a source of light is situated at the origin O in the frame S, from which a wavefront of light is emitted at $t = 0$. When the light reaches at the point P, let the positions and times, measured by the observers O and O', be (x, y, z, t) and (x', y', z', t') respectively. If the velocity of light is c, then the time measured by the light signal in traversing the distance OP in frame S is

$$t = \frac{OP}{c} = \frac{(x^2 + y^2 + z^2)^{1/2}}{c} \quad \text{or} \quad x^2 + y^2 + z^2 = c^2t^2. \qquad ...(10)$$

This equation represents the equation of wavefront in frame S. According to the special theory of relativity, the velocity of light will be c in the second frame S'. Hence, in frame S', the time required by the light signal in travelling the distance $O'P$ is given by

$$t' = \frac{O'P}{c} = \frac{(x'^2 + y'^2 + z'^2)^{1/2}}{c} \quad \text{or} \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad ...(11)$$

which is the equation of the wavefront in frame S'.

Now transformation equations relating x, y, z, t and x', y', z', t' should be such that eq. (11) transforms to eq. (10). The Galilean transformations connect the measurements in the two frames according to the following equations :

$$x' = x - vt \; ; \quad y' = y; \quad z' = z \; ; \quad t' = t$$

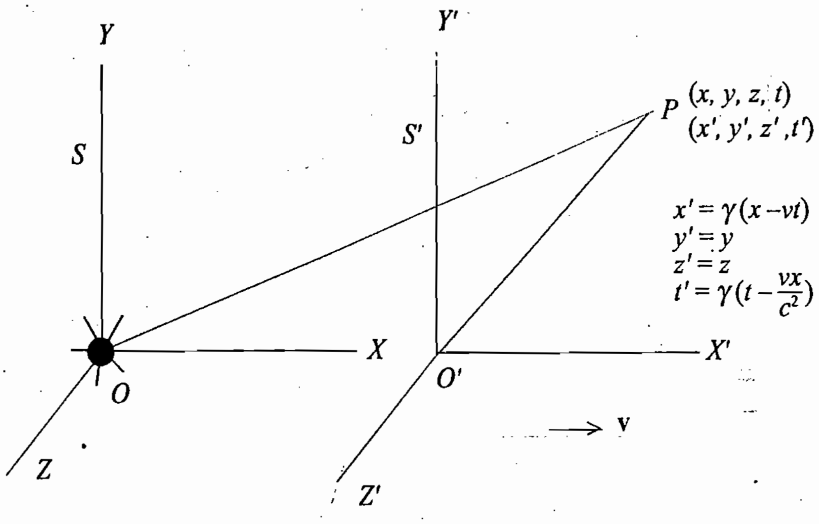


Fig. 12.6 : Representation of Lorentz transformations

Substituting these values in eq. (11), we get

$$(x - vt)^2 + y^2 + z^2 - c^2t^2 = 0$$

or
$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 - c^2t^2 = 0 \qquad ...(12)$$

This equation is certainly not in agreement with eq. (10) because it contains an extra term $(-2xvt + v^2t^2)$. Thus the Galilean transformation fails. Further $t \neq t'$, (because $t = OP/c$ and $t' = O'P/c$) which does not agree with the Galilean transformation equations. If the principle of the constancy of the speed of light is valid in all frames, there should exist some transformation which reduces to the Galilean one for $v/c \to 0$ and which transform $x'^2 + y'^2 + z'^2 - c^2t^2 = 0$ into $x^2 + y^2 + z^2 - c^2t^2 = 0$:

When we look at the eq. (10) and eq. (12), we find that the terms of y and z are in agreement. Hence we can say $y' = y$ and $z' = z$. The extra term $(-2xvt + v^2t^2)$ indicates that transformations in x and t should be modified so that this extra term is cancelled.

We note that for the observer O, the distance $OO' = vt$ and therefore when $x' = 0$ (point O'), $x = vt$. This suggests the transformation $x' = \alpha(x - vt)$ because only then for $x' = 0$, $x = vt$. Since t' is different from t and may be depending on x, so that in general we may also assume that $t' = \alpha'(t + fx)$. Here α, α' and f are constants, to be determined (for Galilean transformations $\alpha = \alpha' = 1$ and $f = 0$). Now substitution for x', y', z' and t' in (11), we have

$$\alpha^2(x - vt)^2 + y^2 + z^2 = c^2\alpha'^2(t + fx)^2$$

or
$$\alpha^2(x^2 - 2vxt + v^2t^2) + y^2 + z^2 = c^2\alpha'^2(t^2 + 2fxt + f^2x^2)$$

or
$$x^2(\alpha^2 - f^2\alpha'^2c^2) - 2xt\,(\alpha^2v + fc^2\alpha'^2) + y^2 + z^2 = \left(\alpha'^2 - \frac{\alpha^2v^2}{c^2}\right)c^2t^2 \qquad ...(13)$$

This result obtained from applying transformations from S' to S, must be identical to eq. (10). Therefore,

$$\alpha^2 - f^2\alpha'^2c^2 = 1 \ ...(i)\ ; \qquad \alpha^2v + fc^2\alpha'^2 = 0 \ ...(ii)\ ; \qquad \alpha'^2 - \frac{\alpha^2v^2}{c^2} = 1 \ ...(iii)$$

Substituting the value of $f = -\dfrac{\alpha^2v}{\alpha'^2c^2}$ from eq. (ii) in eq. (i), we get

$$\alpha^2 - \frac{\alpha^4v^2}{\alpha'^2c^2} = 1 \quad \text{or} \quad 1 - \frac{\alpha^2v^2}{\alpha'^2c^2} = \frac{1}{\alpha^2}$$

But from (iii) $\alpha'^2 = 1 + \alpha^2v^2/c^2$, hence

$$1 - \frac{\alpha^2v^2/c^2}{1+\alpha^2v^2/c^2} = \frac{1}{\alpha^2} \text{ or } \frac{1}{1+\alpha^2v^2/c^2} = \frac{1}{\alpha^2} \text{ or } \alpha^2 = 1 + \frac{\alpha^2v^2}{c^2}$$

or
$$\alpha^2 = \frac{1}{1-v^2/c^2} \text{ or } \alpha = \frac{1}{\sqrt{1-v^2/c^2}}$$

Therefore,
$$\alpha'^2 = 1 + \frac{v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2} \text{ or } \alpha' = \frac{1}{\sqrt{1-v^2/c^2}}$$

Thus from (*ii*) $f = -v/c^2$

Therefore, $x' = \alpha(x - vt) = (x - vt)\big/\sqrt{1-v^2/c^2}$ and $t' = \alpha'(t + fx) = \left(t - vx/c^2\right)\big/\sqrt{1-v^2/c^2}$

Thus, the new transformation equations, which are in agreement with the invariance of velocity of light c, are

$$x' = \frac{x - vt}{\sqrt{1-v^2/c^2}}; \quad y' = y, \quad z' = z; \quad t' = \frac{t - vx/c^2}{\sqrt{1-v^2/c^2}} \qquad ...(14)$$

These equations are called ***Lorentz transformations***, because they were first obtained by Dutch physicist H. Lorentz.

We note that *when* $v \ll c$ i.e. $v/c \to 0$, *we get the Galilean transformations from the Lorentz transformations*. In most of the cases, which we encounter on earth, c is a velocity very large compared with the great majority of velocities *i.e.*, $v \ll c$ so that the results of Lorentz transformations do not differ to any great extent from those of the Galilean transformations; but *from a theoretical point of view the Lorentz transformations represent a most profound conceptual change specially in relation to space and time.*

For convenience, sometimes we put $\beta = v/c$ and

$$1\big/\sqrt{1-v^2/c^2} = 1\big/\sqrt{1-\beta^2} = \gamma.$$

Hence the transformations are written as

$$x' = \gamma(x - vt); \; y' = y; \; z' = z, \; t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad ...(15)$$

In the derivation of these equations, we assumed that frame S' is moving in positive X-direction with velocity v relative to the frame S. But if we say that frame S is moving with $-v$ velocity relative to S' along negative X-direction, then the transformations are

$$x = \gamma(x' + vt'); \; y = y'; \; z = z'; \; t = \gamma\left(t' + \frac{vx'}{c^2}\right) \qquad ...(16)$$

These are known as ***inverse Lorentz transformations.***

From practical point of view at low speeds, there is no difference between the Lorentzian and Galilean transformations and we use the later in most of the problems which we encounter. However, when we have to deal with very fast particles having velocities comparable to c, such as electrons in the atoms, cosmic ray particles, we must use the Lorentz transformations.

12.11. CONSEQUENCES OF LORENTZ TRANSFORMATIONS

Now, let us discuss the consequences of Lorentz transformations regarding the lengths of the bodies and the time intervals between given events.

(1) Length contraction : In order to measure the length of an object in motion, relative to an observer, the positions of the two end points must be recorded simultaneously. Consider a frame S relative to which a rod is moving with velocity v along the X-axis. Let us associate a frame S' with the rod so that the rod is at rest in S'. If in this frame, the x-coordinates of the ends of the rod are x_1' and x_2', then

$$l_0 = x_2' - x_1'$$

This length l_0 has been measured by a stationary observer relative to the rod and is called the ***proper length*** of the rod. It is not necessary that the observer O' should measure the positions of the end points of the rod simultaneously, because the rod is at rest relative to him.

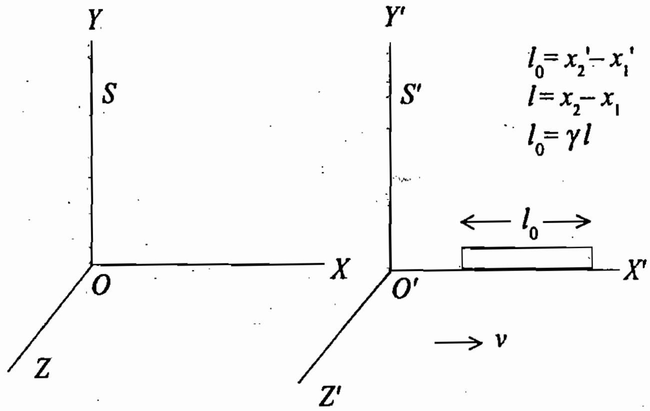


Fig. 12.7 : Contraction of moving rod

If the X-coordinates of the end points of that rod in frame S are measured to be x_1 and x_2 at the same time t, then in this frame the observed length of the rod is

$$l = x_2 - x_1$$

According to Lorentz transformations

$$x_1' = \gamma(x_1 - vt) \text{ and } x_2' = \gamma(x_2 - vt)$$

$$\therefore \quad x_2' - x_1' = \gamma(x_2 - x_1) \text{ or } l_0 = \gamma l$$

Thus $$l = l_0\sqrt{1 - v^2/c^2} \qquad ...(17)$$

As the factor $\sqrt{1 - v^2/c^2}$ is smaller than unity, we have $l < l_0$. This means that the length of the rod (l), as measured by an observer relative to which the rod is in motion, is smaller than its proper length. Such a contradiction of length in the direction of motion relative to an observer is called ***Lorentz-Fitzgerald contradiction***. However, there will be no change in length in the perpendicular direction of motion.

If the rod is at rest in the frame S, then its proper length is

$$l_0 = x_2 - x_1$$

Now the observer of S' at time t' measures the end coordinates of the rod as x_2' and x_1', then according to Lorentz transformations, we have

$$x_1 = \gamma(x_1' + vt') \text{ and } x_2 = \gamma(x_2' + vt')$$

Hence, $$x_2 - x_1 = \gamma(x_2' - x_1') \text{ or } l_0 = \gamma l \text{ or } l = l_0\sqrt{1 - v^2/c^2}.$$

This means that a rod at rest in S appears to be contracted to the observer O'. Thus, *a length is contracted, if there is relative motion between it and the observer.*

(2) Simultaneity : If two events occur at the same time in a frame, they are said to be *simultaneous*. Suppose that S' frame is moving relative to S along positive direction of X-axis with velocity v. Let two events occur simultaneously in frame S at the points P_1 and P_2 with the coordinates (x_1, y_1, z_1, t_1) and $(x_2,$

y_2, z_2, t_2) respectively as measured by the observer O of the S-frame. As the two events are simultaneous in frame S, we have $t_2 = t_1$. If t_1' and t_2' are the corresponding times of the same two events are measured by O' of frame S', then the use of Lorentz transformation equations gives

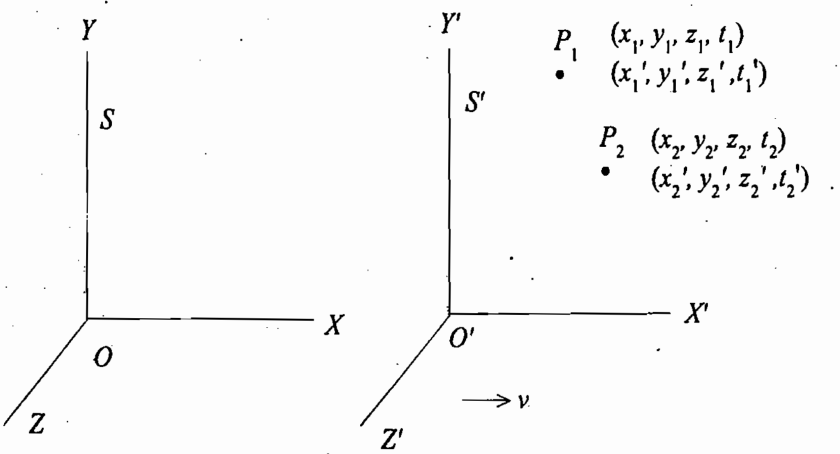


Fig. 12.8 : Representation of two events in two inertial frames

$$t_1' = \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad t_2' = \frac{t_2 - vx_2/c^2}{\sqrt{1 - v^2/c^2}}$$

Therefore, $$t_2' - t_1' = \frac{t_2 - t_1}{\sqrt{1 - v^2/c^2}} - \frac{(v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

As $t_2 = t_1$ or $t_2 - t_1 = 0$, we have

$$t_2' - t_1' = -\frac{(v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}} \qquad ...(18)$$

We observe that $t_2' - t_1' \neq 0$. This means that the two events at two different points P_1 and P_2 which are simultaneous for O in frame S are not simultaneous for the observer O' of the frame S', moving with speed v along X-axis relative to S. Thus the simultaneity is not absolute, but relative.

(3) Time Dilation : Let a frame S' be moving along X-axis with velocity v relative to S. Now, if a clock being at rest in the frame S', mea- sures the time t_1' and t_2' of two events occurring at a fixed position x' in this frame, then the interval of time between these events is

$$\Delta t' = t_2' - t_1' = \Delta t_0 \text{ (say)}$$

Now, according to Lorentz transformations, we have

$$t_1 = \gamma(t_1' + vx'/c^2) \quad \text{and} \quad t_2 = \gamma(t_2' + vx'/c^2)$$

Therefore $\quad t_2 - t_1 = \gamma(t_2' - t_1')$ or $\Delta t = \gamma \Delta t_0$

Thus $$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} \qquad ...(19)$$

As $\quad 1/\sqrt{1 - v^2/c^2} > 1, \ \Delta t > \Delta t_0$.

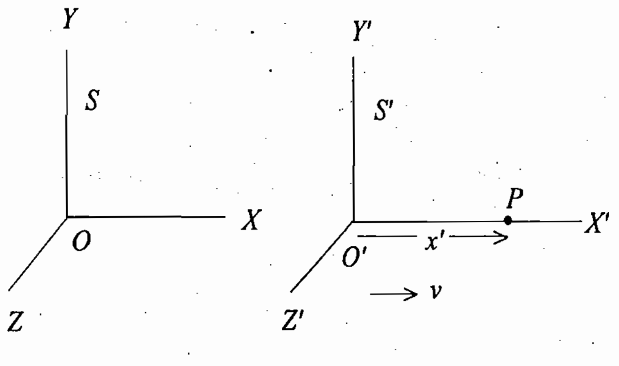


Fig. 12.9 : Two events occur in frames S' at a fixed position x'

Thus, the time interval, measured in the frame S is larger than the time interval in the frame S', in which the two events are occurring at a certain point x'. This effect is called ***time dilation*** (lengthening of time interval). This means *to stationary observer the moving clock will appear to go slow.* In consequencc we may say that the two events appear to take a longer time when they occur in a body in motion relative to the observer than when the body is at rest relative to the observer. *The time interval between any two events which has been measured relative to a frame in which the same two events take place at a certain point, is minimum and is called* ***proper time interval.*** It is represented by Δt_0 or $\Delta\tau$, given by

$$\Delta\tau = \Delta t_0 = \Delta t\sqrt{1 - v^2/c^2}$$

For example, an observer sitting in a moving rocket, takes the readings of the needle of his clock at 9 a.m. and 10 a.m., then this Δt =1 hour is the proper interval of time.

Now, if $\Delta\tau$ is the decay half life of mesons of radioactive matter as measured in the frame S' in which the particle is at rest, then

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} \qquad ...(20)$$

is the decay half life observed in a frame S in which the particles are moving with velocity v.

Verification of Time Dilation : (1) A muon or μ-meson is an elementary particle, whose mean life is 2.2×10^{-6} sec in a frame in which it is at rest. When cosmic rays fall on the upper part of the atmosphere, π-mesons are produced and they decay with a very short life to μ-mesons. The velocity of the μ-mesons should be less than the speed of light c, so the expected distance, travelled by them, is less than $2.2 \times 10^{-6}c$ $= (2.2 \times 10^{-6} \times 3 \times 10^{8})$ or 0.66 km. Thus, in absence of any dilation effect, the μ-mesons should not travel a distance greater than 0.66 km and hence there should be no possibility for these μ-mesons, created at the top of the atmosphere, to reach theearth's surface (a distance near about 10 km). But experimentally on the earth's surface an appreciable number of μ-mesons can be detected after travelling about 10 km of the atmosphere. This becomes possible only when we consider the life-time of a fast μ-meson to be much longer than the measured life-time of a μ-meson at rest. The μ-mesons of cosmic rays, detected on the earth's surface, have an energy of the order of corresponding to v = $0.999c$ or γ = 30 so that their average life according to the special theory of relativity is

$$\Delta t = \Delta\tau\Big/\sqrt{1-v^2/c^2} = \gamma\,\Delta\tau = 30 \times 2.2 \times 10^{-6} \approx 6.6 \times 10^{-5}\text{ sec.}$$

Hence the distance travelled by a μ-meson is nearly $6.6 \times 10^{-5} \times 0.999\,c$ or 20 km.

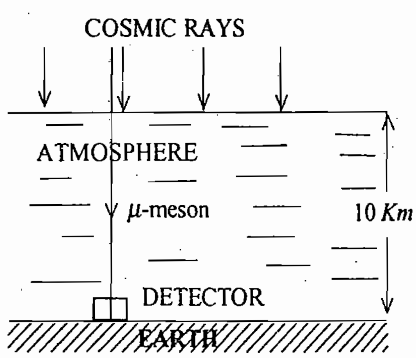


Fig. 12.10

Thus there is sufficient possibility that some μ-mesons reach on the earth's surface.

(2) It is known that π^+ meson is positively charged particle, having mass 273 times that of an electron. π^+ meson decays into a μ^+ meson (mass = 215 m_e) and a neutrino with a mean life about 2.5×10^{-8} sec in a frame in which it is at rest. If the velocity of the π^+ mesons in the laboratory frame be 0.9 c, then the expected life time in this frame is

$$\Delta t = \frac{2.5 \times 10^{-8}}{[1-(0.9c/c)^2]^{1/2}} = \frac{2.5 \times 10^{-8}}{(1-0.81)^{1/2}} = 5.7 \times 10^{-8} \text{ sec.}$$

Therefore, on an average, before decaying π^+ meson will travel a distance twice (distance = $5.7 \times 10^{-8} \times 0.9\ c$) as we would expect non-relativistically (distance = $2.5\times10^{-8} \times 0.9c$). If the velocity of pions is close to the velocity of light (c), expected life time in the laboratory frame may be many times than the proper value. Experiments have been done for the life time of π^+ mesons and the results are in good agreement with the predicted time dilation for the appropriate velocity.

The π^+ mesons, produced at a target, are passed through a slit and the muons, generated in the decay, are observed by a detector. Therefore number of muons, detected per second, is proportional to the flux (number of particles arriving per second) of pions and hence the detector shows the reading proportional to the flux of pion beam. If Δt is the mean life time, then the number of particles, left unchanged, after time t is given by

$$N = N_0\, e^{-t/\Delta t} = N_0\, e^{-x/v\Delta t} \qquad ...(21)$$

where N_0 is the initial number of particles and $t = x/v$; v is the velocity of the particles and x the distance, travelled by the pions in time t. The original flux of pions is obtained by reducing their velocity to zero by allowing them to enter a solid block and measuring the muons emitted. Hence from eq. (21) the value of mean life time (Δt) of moving pions can be calculated. To determine the proper mean life time $\Delta\tau$, pions are brought to rest by introducing a solid at the beginning of the path. The time required between the entrance into the stopping solid and the emission of muons is determined by electronic device. This gives the proper mean life $\Delta\tau$ of π^+ mesons. The two values of mean life time of pions (Δt and $\Delta\tau$) satisfy the time dilation relation

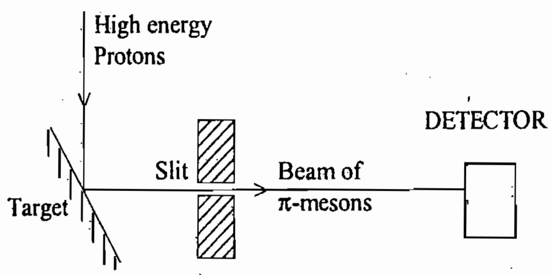


Fig. 12.11

$$\Delta t = \Delta\tau \Big/ \sqrt{1 - v^2/c^2}$$

Thus it will be seen that for the pions of larger velocity, life-time (Δt) will be dilated accordingly.

Twin Paradox in Special Relativity : Consider two twins A and B, each 20 years of age. Twin A remains at rest at the origin O and twin B takes a round trip space voyage to a star with velocity $v = 0.99c$ relative to A. The star is 10 light years away from O. We want to determine the age of A and B as B finishes his journey.

According to A, the time taken by B in the round trip is

$$t_1 = \frac{20 \text{ light years}}{0.99c} = \frac{20 \text{ years} \times c}{0.99c} = 20.2 \text{ years.}$$

Thus according to A his own age, as B completes the journey, is 20 + 20.2 = 40.2 years.

According to B, the time of journey (proper time interval) is given by

$$t_2 = 20 \text{ years} \times \sqrt{1 - v^2/c^2} = 20 \times \sqrt{(1-0.99)^2} = 2.8 \text{ years.}$$

Hence according to B, his own age after the journey is 20 + 2.8 = 22.8 years.

These two statements are different. After the space journey, will one of the two twins look younger than the other. This is twin paradox in the theory of special relativity.

(4) Addition of Velocities : Let the coordinates of a particle in frame S be (x, y, z, t) and in frame S' (x', y', z', t'), then the components of its velocity in two frames can be written as

$$u_x = dx/dt, \quad u_y = dy/dt, \quad u_z = dz/dt \qquad \text{in } S$$

and $$u'_x = dx'/dt', \quad u_y' = dy'/dt', \quad u_z' = dz'/dt' \qquad \text{in } S'.$$

According to the inverse Lorentz transformations

$$x = \gamma(x' + vt'), y = y', z = z', t = \gamma(t' + vx'/c^2)$$

Therefore, $$dx = \gamma(dx' + vdt'), dy = dy', dz = dz' \text{ and } dt = \gamma(dt' + vdx'/c^2)$$

$\therefore$ $$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma\left(dt' + \dfrac{vdx'}{c^2}\right)} = \frac{\dfrac{dx'}{dt'} + v}{1 + \dfrac{v}{c^2}\dfrac{dx'}{dt'}}$$

or $$u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \qquad \text{...(22 a)}$$

Also, $$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \dfrac{vdx'}{c^2}\right)} = \frac{dy'/dt'}{\gamma\left(1 + \dfrac{v}{c^2}\dfrac{dx'}{dt'}\right)}$$

or $$u_y = \frac{u'_y}{\gamma(1 + vu'_x/c^2)} \qquad \text{...(22 b)}$$

Similarly, $$u_z = \frac{u_z}{\gamma(1 + vu'_x/c^2)} \qquad \text{...(22 c)}$$

This is the ***relativistic law of addition of velocities*** while in classical mechanics $u_x = u_x' + v$, $u_y' = u_y$ and $u_z' = u_z$. We get the later (Galilean) equations, when v is much less than the speed of light c.

If we take the Lorentz transformations, we can prove that

$$u_x' = \frac{u_x - v}{1 - vu_x/c^2}; \quad u_y' = \frac{u_y}{\gamma(1 - vu_x/c^2)}; \quad u_z' = \frac{u_z}{\gamma(1 - vu_x/c^2)} \qquad \text{...(23)}$$

In case a particle (as photon) is moving with a velocity c in the frame S' and S' is moving with velocity c relative to S along positive X-axis direction, then from eq. (22 a), we have

$$u_x = \frac{c + c}{1 + cc/c^2} = c \qquad [\text{because } u_x' = c, v = c]$$

Thus the speed of photon in the second frame is also c, *i.e.*, the velocity of light is the same for all inertial frames whatever their relative speeds may be. This result is in accordance with the Michelson-Morley experiment. In fact, the Lorentz transformations and hence the law of addition of velocities have been deduced by assuming the velocity of light constant for all inertial observers.

Ex. 1. *Determine the length and the orientation of a rod of length* 10 *metres in a frame of reference which is moving with* 0.6*c velocity in a direction making* 30° *angle with the rod.*

Solution : The proper length of the rod along the direction of the moving frame l_{x_0} = 10 cos 30°. Therefore, the length measured in the moving frame is

$$l_x = l_{x_0}\sqrt{1 - v^2/c^2} = 10 \cos 30^\circ \sqrt{(1 - (0.6c)^2)/c^2}$$

$$= 10 \cos 30^\circ \sqrt{1 - 0.36} = 8 \cos 30^\circ = 4\sqrt{3}\,\text{m}.$$

Since the length does not contract perpendicular to the direction of mation of the moving frame, $l_y = l_{y_0}$ = 10 sin 30° = 5 m.

Hence the length of the rod observed in the moving frame, is given by

$$l = \sqrt{l_x^2 + l_y^2} = \sqrt{48 + 25} = \sqrt{73} = \mathbf{8.54\ m}.$$

If the rod makes an angle θ with X-axis of the moving frame, then

$$\tan\theta = \frac{l_y}{l_x} = \frac{5}{4\sqrt{3}} = 0.72 \text{ or } \theta = \tan^{-1}(0.72) = \mathbf{35.8^\circ}.$$

Ex. 2. *Show that if* V_0 *is the rest volume of a cube of side* l_0, *then*

$$V_0(1 - \beta^2)^{1/2}$$

is the volume viewed from a reference frame moving with uniform velocity v in a direction parallel to an edge of the cube.

Solution : One side of the cube = l_0; Rest volume of the cube $V_0 = l_0^3$.

Let the edges of the cube lie parallel to X, Y, Z axes. Consider that one of its edges lie along the X-axis of the moving frame.

Now, the edge along X-axis as observed by an observer in moving frame is $l = l_0\sqrt{1-\beta^2}$, where $\beta^2 = v^2/c^2$.

But there is no contraction along Y and Z directions, hence the volume of the cube as observed by the observer in the moving frame is

$$l_0\sqrt{1-\beta^2}\cdot l_0 \cdot l_0 = l_0^3\sqrt{1-\beta^2} = V_0\sqrt{1-\beta^2}.$$

Ex. 3. *A meson has a speed* 0.8 *c relative to the ground. Find how far the meson travels relative to the ground. if its speed remains constant and the time of its flight, relative to the system, in which it is at rest, is* 2×10^{-8} *sec.*

Solution : Time of flight of meson relative to the earth is

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}}.$$

Proper time of flight of meson $\Delta\tau = 2 \times 10^{-8}$ sec. Hence

$$\Delta t = \frac{2 \times 10^{-8}}{\sqrt{1 - (0.8c/c)^2}} = \frac{2 \times 10^{-8}}{0.6} = 3.33 \times 10^{-8} \text{ sec.}$$

Distance travelled by the meson relative to the ground

$$= v\,\Delta t = 0.8 \times 3 \times 10^8 \times 3.33 \times 10^{-8} = \mathbf{8\ m}.$$

Ex. 4. (*a*) *What is the mean life of a burst of* π^+ *mesons travelling with* $v = 0.73c$, *if the proper mean life time is* 2.3×10^{-8} *sec ?*

(*b*) *What is the distance travelled at* $v = 0.73c$ *during one mean life ?*

(*c*) *What distance will be travelled without relativistic effect ?*

Solution : (a) $$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}} = \frac{2.5 \times 10^{-8}}{\sqrt{1-(0.73)^2}} = \frac{2.5 \times 10^{-8}}{\sqrt{0.4671}}$$

$$= \mathbf{3.6\times10^{-8}\ sec.}$$

(*b*) Distance travelled by the burst of π^+ meson

$$= v\,\Delta t = 0.73 \times 3 \times 10^8 \times 3.6 \times 10^{-8} \simeq \mathbf{7.9\ m.}$$

(*c*) Distance travelled without relativistic effect

$$= v\Delta\tau = 0.73 \times 3 \times 10^8 \times 2.5 \times 10^{-6} \simeq \mathbf{5.5\ m.}$$

Ex. 5. *A beam of particles of half life* 2×10^{-6} *sec travels in the laboratory with speed 0.96 time the speed of light. How much distance does the beam travel before the flux falls to* $\frac{1}{2}$ *times the initial flux ?* **(Kanpur 1990)**

Solution : If Δt is the observed half life, then

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}} = \frac{2\times10^{-6}}{\sqrt{1-(0.73)^2}} = \frac{2\times10^{-6}}{0.28}\ \text{sec.}$$

Evidently, in this observed half life the flux will fall to $\frac{1}{2}$ times the initial flux. In this time the distance travelled by the beam is

$$v\,\Delta t = 0.96 \times 3 \times 10^8 \times \frac{2\times10^{-6}}{0.28} = \mathbf{2{,}000\ m.}$$

Ex. 6. *A clock keeps correct time. With what speed should it be moved relative to an observer so that it may seem to lose 2 minutes in 24 hours ?* **(Agra 2000, 1991)**

Solution : Let the clock be at rest in a frame S. In this frame, the clock should show $\Delta\tau$ = 23 hours 58 minutes. The observer, relative to which frame S is moving with speed v, will measure the time Δt =24 hours = 24 × 60 minutes in his clock. Now

$$\Delta t = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}}$$

or $$1 - \frac{v^2}{c^2} = \left[\frac{\Delta\tau}{\Delta t}\right]^2 = \left[\frac{23\text{ hrs }58\text{ mts}}{24\text{ hrs}}\right]^2 = \left[1 - \frac{2}{24\times60}\right]^2 = 1 - \frac{2\times2}{24\times60} = 1 - \frac{1}{360}\ \text{approx.}$$

Hence $$v^2 = \frac{c^2}{360} = \frac{(3\times10^8)^2}{360} \text{ or } v = \frac{3\times10^8}{\sqrt{360}} = \mathbf{1.58 \times 10^7\ m/s}.$$

Ex. 7. *In the laboratory one particle A has velocity* $v_x = +2 \times 10^8$ *m/sec and another particle B has velocity* $v_x = -2\times10^8$ *m/sec. Calculate the velo- city of A relative to B.* **(Kanpur 1978)**

Solution : If we assume that the particle B, which is moving with velocity -2×10^8 m/sec relative to the laboratory is at rest in frame S. Clearly relative to this frame, the frame S' of the laboratory is moving with 2 × 10^8 m/sec. Hence the velocity of the particle A relative to the frame S (*i.e.*, relative to the particle B) is

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} = \frac{2\times10^8 + 2\times10^8}{1 + \frac{(2\times10^8)^2}{(3\times10^8)^2}} = \frac{4\times10^8}{1+4/9} = \mathbf{2.77 \times 10^8\ m/sec.}$$

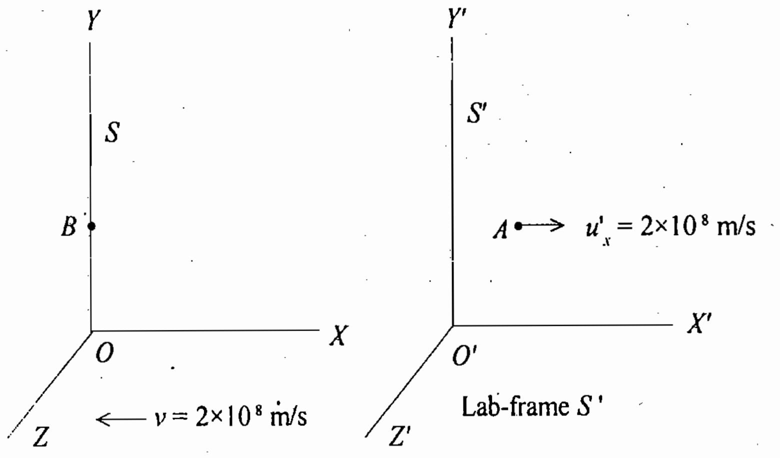


Fig. 12.12 : Particle A has velocity $u'_x = 2\times10^8$ m/s in S-frame (Lab-frame). Particle B is at rest in S-frame which is moving with v = 2 × 10^8 m/s along – ve X-axis relative to S' i.e., B is moving opposite to A with 2 × 10^8 m/s in the Lab-frame.

Ex. 8. *A spaceship moving away from the earth with velocity* 0.5*c fires a rocket whose velocity relative to the space is* 0.5*c* (*a*) *away from the earth,* (*b*) *towards the earth. Calculate thr velocity of the rocket as observed from the earth in two cases.* **(Kanpur 1988; Delhi 94)**

Solution : Let the velocity of the rocket as observed from thr earth be u.

(*a*) In the 1st case, $u = \dfrac{u' + v}{1 + \dfrac{u'v}{c^2}}$

Here, $u' = 0.5c$, $v = 0.5c$, Therefore

$$u = \frac{0.5c + 0.5c}{1 + \frac{0.5c \times 0.5c}{c^2}} = \frac{c}{1.25} = \mathbf{0.8\ c}.$$

(*b*) In the 2nd case, $u' = -0.5c$, Hence

$$u = \frac{-0.5c + 0.5c}{1 - \frac{0.5c \times 0.5c}{c^2}} = 0.$$

Ex. 9. *A particle has a velocity* 6×10^7 *m/sec in the X-Y plane at an angle of* 60° *with X-axis in the system S. Determine the magnitude and direction of its velocity in system S', when S' has a velocity* 3×10^7 *m/sec along the positive X-axis.*

Solution : The x and y components of the velocity in S' ; frame are given by

$$u_{x'} = \frac{u_x - v}{1 - u_x v / c^2}, \quad u_{y'} = \frac{u_y\sqrt{1 - v^2 / c^2}}{(1 - u_x v / c^2)}.$$

Here, $u_x = 6\times10^7 \cos 60^o = 3\times10^7$ m/sec. $u_y = 6 \times 10^7 \sin 60^o = 3\sqrt{3} \times10^7$ m/sec and $v = 3\times10^7$ m/sec.

Therefore, $$u_{x'} = \frac{3 \times 10^7 - 3 \times 10^7}{1-\frac{3 \times 10^7 \times 3 \times 10^7}{(3 \times 10^8)^2}} = 0$$

Also, $$u_{y'} = \sqrt{1 - \left[\frac{3 \times 10^7}{3 \times 10^8}\right]^2} \times \frac{3\sqrt{3} \times 10^7}{\left[1-\frac{3 \times 10^7 \times 3 \times 10^7}{(3 \times 10^8)^2}\right]}$$

$$= \sqrt{\frac{99}{100}} \times \frac{3\sqrt{3} \times 10^7 \times 100}{99} = \sqrt{\frac{3}{11}} \times 10^8 = 5.2 \times 10^7 \text{ m/sec.}$$

Hence the velocity in S' frame is

$$\mathbf{u}' = 0 \times \hat{\mathbf{i}} + 5.2\times10^7\hat{\mathbf{j}} \text{ or } \mathbf{u}' = 5.2\times10^7\hat{\mathbf{j}}$$

This means that the particle will appear to an observer in S' to be moving along the Y-axis with velocity 5.2×10^7 m/sec.

12.12. ABERRATION OF LIGHT FROM STARS

In 1727, James Bradley observed that a star at the zenith* (directly overhead) appeared to move in a nearly circular orbit with a period of 1 year with an angular diameter of 41" of arc. He also observed that stars in other positions have a similar motion in general elliptical. This phenomenon is called ***aberration***.

Obviously this kind of apparent variation in the positions of stars is a consequence of earth's motion around the sun. It arises from the finite speed of light and from the speed of earth in its orbit about the sun. In order to explain this phenomenon, let us consider a frame of reference S in which the sun and distant star are at rest. (The distant star may be moving with constant velocity relative to the sun and hence this has nothing to do with the observed circular or elliptical motion). The earth in its orbit around the sun has a speed v (= 3×10^4 m/sec). Let the instantaneous direction of motion of the earth with respect to the frame S be taken as X-axis and the distant star P be situated in the XY-plane [Fig. 12.14]. The light from the star P is coming towards the earth-sun system (the entire orbit of the earth around the sun may be treated as point; in comparison to the distance of the star); making an angle θ with the X-axis. Thus the velocity components of the incoming light ray from the star in S are

$$u_x = - c \cos \theta, \; u_y = - c \sin \theta$$

Let the frame of the earth be S'. In this frame (for the observer on the earth), the velocity components of the incoming light from P are obtained by Lorentz transformations :

$$u_{x'} = \frac{u_x - v}{1-cu_x/c^2} = -\frac{c \cos \theta + v}{1 + v \cos \theta/c} \text{ and } u_{y'} = \frac{u_y\sqrt{1-v^2/c^2}}{1-vu_x/c^2} = -\frac{c \sin \theta\sqrt{1-v^2/c^2}}{1+v \cos \theta/c}.$$

* Zenith is the direction perpendicular to the plane of the orbit (called the elliptic) around the sun.

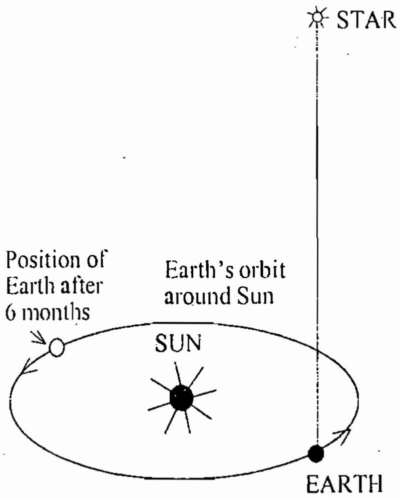


Fig. 12.13 : Aberration of light from star

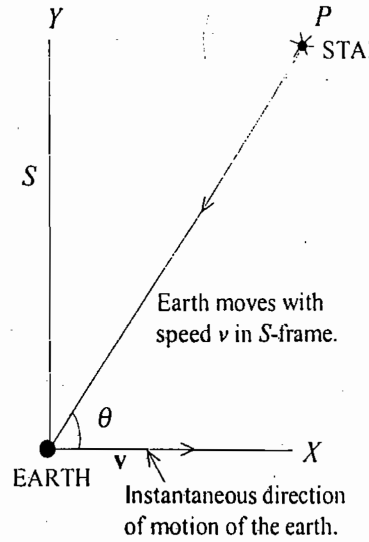


Fig. 12.14

The apparent angle θ' which the incoming light from the star P makes with the common instantaneous X-axis for an earth observer is given by

$$\tan\theta' = \frac{u_{y'}}{u_{x'}} = \frac{c\sin\theta\sqrt{1-v^2/c^2}}{c\cos\theta + v}$$

$$\text{or}\quad \tan\theta' = \frac{\tan\theta\sqrt{1-v^2/c^2}}{1+(v/c)\sec\theta} \qquad ...(24)$$

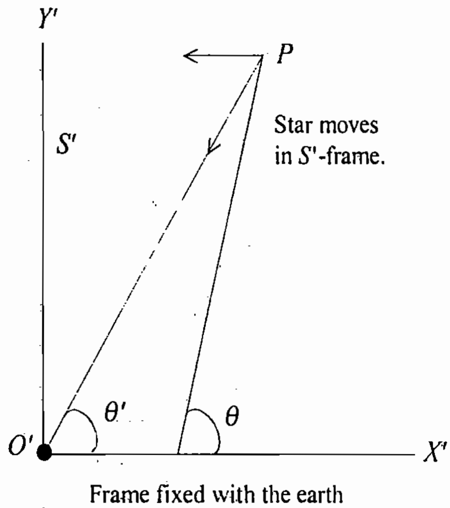


Fig. 12.15

According to the special theory of relativity eq. (24) gives an exact expression for the aberration. In order to have an idea for the magnitude of the expected aberration, we note that $\frac{v}{c} = \frac{3\times10^4}{3\times10^8} \simeq 10^{-4}$ and hence we may neglect the terms in higher order in v/c. Putting $\theta' = \theta + \Delta\theta$ ($\Delta\theta$ is small),

$$\tan(\theta+\Delta\theta) = \frac{\tan\theta}{1+(v/c)\sec\theta} = \tan\theta\,(1+\frac{v}{c}\sec\theta)^{-1} = \tan\theta\,(1-\frac{v}{c}\sec\theta)$$

$$\text{or}\quad \tan(\theta+\Delta\theta) - \tan\theta = -\frac{v\sin\theta}{c\cos^2\theta} \quad\text{or}\quad \frac{\sin\Delta\theta}{\cos(\theta+\Delta\theta)\cos\theta} = -\frac{v\sin\theta}{c\cos^2\theta} \quad\text{or}\quad \Delta\theta = -\frac{v}{c}\sin\theta.$$

Thus we find that there is an aberration [$-(v/c)\sin\theta$] in the observed direction of star due to the motion of the earth around the sun. θ' is smaller than θ by $\Delta\theta$, which is obtained to be negative.

After six months when the direction of motion of the earth relative to the sun is in the opposite direction, $\Delta\theta$ will be positive with the same magnitude. Hence the total angular shift in the position of the star in the period of six months is obtained to be

$$2\Delta\theta = 2\frac{v}{c}\sin\theta$$

This shift is maximum for $\theta = \pi/2$ (overhead star) and is equal to

$$2\frac{v}{c} = \frac{2\times 3\times 10^4}{3\times 10^8} = 2\times 10^{-4} \text{ radian} = 41''$$

Thus the special theory of relativity explains satisfactorily the phenomenon of aberration of light from stars. Classifically for a star directly overhead, the maximum aberration occurs when the earth's velocity is perpendicular to the line of aberration [Fig. 12.16]. The aberration $\Delta\theta$ is given by

$$\Delta\theta = \frac{v}{c} = 20.5''$$

Twice of this *i.e.*, 41" is the observed angular diameter of the apparent orbit of the star. It is to be mentioned that the classical formula for aberration differs from the relativistic formula in terms of second and higher order of v/c. At an inclination θ, the exact classical formula is

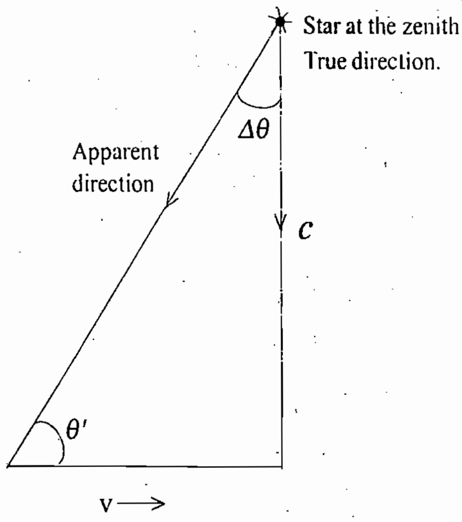


Fig. 12.16

$$\tan\theta' = \frac{\tan\theta}{1+(v/c)\sec\theta}$$

which differs by the multiplier $\sqrt{1-v^2/c^2}$ to the relativistic formula.

12.13. RELATIVISTIC DOPPLER'S EFFECT

Consider two frames S and S', where S' is moving with velocity $v\hat{\mathbf{i}}$ relative to S. At the origin of the frame S, a source of light is situated, having frequency ν. An observer is at rest in the frame S' (*i.e.*, the observer is moving with velocity $v\hat{\mathbf{i}}$ relative to S ; the frame S' is not shown). The observer of S' receives signals of light sent to him from the origin O of frame S. Let the receipt of the signals be the two events A and B, as shown in Fig. 12.17 at an interval of time $\Delta t'$, then according to Lorentz transformations, the time interval of the two events Δt in the frame S will be given by

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}} \qquad \text{...(25 a)}$$

Since the two events are occurring at a certain point (or observer), $\Delta t'$ is the proper time interval for the two events.

As the receiver is stationary in his frame, $\Delta x' = 0$ and then from the transformation $x = \dfrac{1}{\sqrt{1-v^2/c^2}}(x'+vt')$,

we have

$$\Delta x = \frac{v.\Delta t'}{\sqrt{1-v^2/c^2}} \qquad \text{...(25 b)}$$

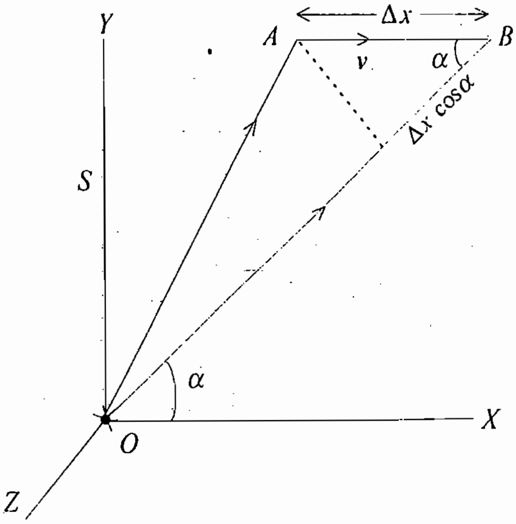


Fig. 12.17 : Relativistic Doppler's effect

If T be the period of the signals sent from O, then ν = $1/T$. This means that the two successive signals are sent at t = 0 and t = T. As the second signal moves a distance $\Delta x \cos\alpha$ farther than the first signal, the actual time interval between the events A and B as measured in S is

$$\Delta t = T + \frac{\Delta x \cos \alpha}{c}$$

$$\frac{\Delta t'}{\sqrt{1 - v^2/c^2}} = T + \frac{v \, \Delta t' \cos \alpha}{c \sqrt{1 - v^2/c^2}} \quad \text{[By using eq. (25)]} \qquad ...(26)$$

The observer of S' measures the time $T' = \Delta t'$ (of the same events A and B) for the period of the signals, then from relations (25) and (26), we have

$$\frac{T'}{\sqrt{1 - v^2/c^2}} = T + \frac{v \, T' \cos \alpha}{c \sqrt{1 - v^2/c^2}} \quad \text{or} \quad T = \frac{T'}{\sqrt{1 - v^2/c^2}} \left[1 - \frac{v \cos \alpha}{c} \right] \qquad ...(27)$$

Using $\nu = 1/T$ and $\nu' = 1/T'$ (observed frequency in S'), we have

$$\nu' = \frac{\nu}{\sqrt{1 - v^2/c^2}} \left[1 - \frac{v \cos \alpha}{c} \right] \qquad ...(28)$$

where α is the direction of the signals as seen in S. Eq. (28) is known as *Doppler's formula*. This relates the observed frequency ν' in S' frame to the transmitted frequency ν in S frame.

Now, if the observer is moving relative to frame S along X-axis, then $\cos \alpha = \cos 0 = 1$. So that

$$\nu' = \frac{\nu}{\sqrt{1 - v^2/c^2}} \left(1 - \frac{v}{c} \right) \qquad ...(29)$$

This is the expression for the ***longitudinal Doppler's effect***.

Since $c = \nu\lambda$, corresponding expression for the observed wavelength is

$$\lambda' = \frac{\lambda \sqrt{1 - v^2/c^2}}{1 - v/c} \qquad ...(30)$$

If the receiver is receding from the source, then v is positive and ν' is less than ν. If the receiver is approaching the source, v is negative and ν' is greater than ν.

If we consider only the terms of first order in v/c, we obtain

$$\nu' = \nu \left(1 - \frac{v}{c} \right) \qquad ...(31)$$

which is the ***non-relativistic Doppler's formula***.

Verification of Relativistic Doppler's Formula : In order to verify the relativistic Doppler's formula, H. E. Ives and G. R. Stilwell performed spectroscopic experiments on the beams of hydrogen atoms in excited electronic states. The atoms were accelerated as molecular hydrogen ions H_2^+ by an intense electric field. Atomic hydrogen was produced as a break up product of the ions. The atoms were having the velocity of the order of $v = 0.005c$. (Value of v was obtained from the accelerating potential applied to the original ions.) Ives and Stilwell looked for a shift in the mean wavelength of a certain spectral line emitted by hydrogen atoms. The spectrum of the light emitted in the direction of motion of atoms was directly seen in a spectrograph. The light emitted opposite to the direction of motion of atoms was reflected on the slit of the spectrograph by means of a mirror. Fig. 12.18 shows qualitatively the experimental photograph. The central line (λ_0) is obtained due to the atoms at rest which may always be present in substantial amount. The shifted lines λ_1 and λ_2 on the higher and lower wavelength sides are corresponding to the light

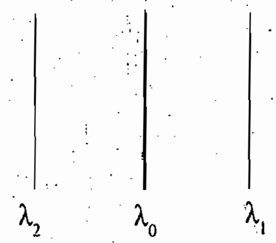


Fig. 12.18

emitted in the direction opposite to, and along, the motion of the atoms respectively. Thus the mean wavelength of the two displaced lines is

$$\lambda_m = (\lambda_1 + \lambda_2)/2$$

Therefore, the shift in the mean position of the displaced lines is given by

$$\lambda_m - \lambda_0 = \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_0 \qquad ...(32)$$

Now, from the relation (30), derived on the basis of theory of relativity, we have

$$\lambda_1 = \frac{\lambda_0\sqrt{1 - v^2/c^2}}{1 + v/c} \quad \text{and} \quad \lambda_2 = \frac{\lambda_0\sqrt{1 - v^2/c^2}}{1 - v/c}$$

Hence
$$\lambda_m = \frac{\lambda_1 + \lambda_2}{2} = \frac{\lambda_0}{2}\frac{2\sqrt{1-v^2/c^2}}{1-v^2/c^2} = \frac{\lambda_0}{\sqrt{1-v^2/c^2}}$$

$$= \lambda_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx \lambda_0\left(1 + \frac{v^2}{2c^2}\right)$$

whence
$$\lambda_m - \lambda_0 = \lambda_0 v^2/2c^2. \qquad ...(33)$$

Thus according the theory of special relativity there is a shift of the order of v^2/c^2 in the mean position of the displaced lines. Experimentally the observed shift according to eq. (32) was 0.074 Å, while the calculated value for this shift from eq. (33) was 0.072 Å. This shows an excellent confirmation of the theory of the relativistic Doppler's effect.

Recessional Red Shift : From relation (30), we have

$$\lambda' = \frac{\lambda\sqrt{1 - v^2/c^2}}{1 - v/c} = \lambda\sqrt{\frac{1 + v/c}{1 - v/c}}.$$

Thus, if the relative velocity v is positive then the value of observed wavelength will increase and therefore the spectral lines will shift towards the red end of the spectrum. If v is to be negative, the spectral line will shift towards the violet end of the spectrum.

We know that in the spectrum of every element, lines of difinite wavelength are obtained. In the spectrum of distant galaxies and stars, the spectral lines of the known elements are not found in their normal positions but are found shifted to the red end of the spectrum. This means that the value of the wavelength has been increased from real value and the galaxy is receding from the observer. This displacement of the spectral lines towards the red end of the spectrum is called ***recessional red shift***. Doppler's effect for the distant stars tells us that mostly the galaxies are receding from earth and their velocities are approximately proportional to their distances from the earth. Thus, if v be the velocity of a galaxy, situated at a distant r, then

$$v \propto r \quad \text{or} \quad v = Hr \qquad ...(34)$$

where H is Huble's constant, whose value is 3×10^{-18} per sec approximately.

Ex. 1. *An astronaut wishes to determine his velocity of approach as he nears the moon. He sends a radio signal of frequency* 5×10^3 *MHz and compares this frequency with its echo, observing a difference of* 43 *KHz. What is the velocity of the space vehicle relative to the moon ? Use the terms of first order in v/c.*

Solution : Let the velocity of the space vehicle be v.

First, the frequency received on the moon is

$$\nu' = \nu\,(1+ v/c) \qquad \text{[from eq. (31)]}$$

because relative to the space vehicle the moon is coming towards it, *i.e.*, v is negative.

The moving observer receives the reflected signal of frequency ν'' from the moon so that v is still negative. Hence the received frequency is

$$\nu'' = \nu'\,(1+ v/c) = \nu\,(1+ v/c)(1+ v/c) \quad \text{or} \quad \nu'' = \nu\,(1+ 2v/c)$$

[neglecting the terms of second order of v^2/c^2]

or $$\nu''- \nu = \frac{2v}{c}\nu\,.$$

Here $\nu''- \nu = 43\times10^3$ Hz, $\nu = 5 \times 10^9$ Hz and $c = 3 \times 10^8$ m/sec.

Therefore, $$43 \times 10^3 = \frac{2v}{3\times10^8}\times 5\times10^9 \quad \text{or} \quad v = \frac{43\times10^3\times3\times10^8}{2\times5\times10^9} = \mathbf{1.3 \times 10^3\ m/sec.}$$

Ex. 2. *The spectral line of* $\lambda = 5000$ Å *in the light coming from a distant star is observed at* 5200 Å. *Find the recessional velocity of the star. Find also the distance of the galaxy.*

Solution : If we consider the first order in v/c, then the observed wavelength is

$$\lambda' = \frac{\lambda}{1- v/c} = \lambda\left(1-\frac{v}{c}\right)^{-1} = \lambda\left(1+\frac{v}{c}\right) \quad \text{or} \quad \lambda' - \lambda = \frac{v}{c}\lambda$$

Therefore, $$v = \frac{\lambda' - \lambda}{\lambda}c = \frac{5200-5000}{5000}\times3\times10^8 = \mathbf{1.2 \times 10^7\ m/sec.}$$

Thus the recessional velocity of the galaxy is 1.2×10^7 m/sec.

Now, $$v = rH \text{ or distance } r = \frac{v}{H} = \frac{1.2\times10^7}{3\times10^{-18}} = \mathbf{4 \times 10^{24}\ m.}$$

Ex. 3. *In an experiment, protons were accelerated through a potential difference of* 20 *KV, after which they moved with constant velocity through a region where neutralization to* H*-atoms and associated light emission was taking place. The* H_β *line, having* $\lambda_0 = 4861.33$ Å *for an atom at rest is observed in a spectrometer. The optical axis of the spectrometer was parallel to the motion of the protons. Along with* H_β *line two Doppler's shifted spectral lines corresponding to emission of light parallel and antiparallel to the direction of motion of protons were observed. Calculate the second order shift, i.e., difference of the mean of the two displaced lines over that of the* H_β *line due to the atoms at rest.*

Solution : Required shift $\lambda_m - \lambda_0 = \lambda_0\, v^2/2c^2$. [from eq. (33)]

Protons were accelerated through a potential difference of 20,000 volts, so that the energy obtained by each proton is 20,000 electron volts. Equating it to the kinetic energy of a proton, we get

$$\tfrac{1}{2}mv^2 = 20{,}000 \text{ eV} \quad \text{(using non-relativistic kinetic energy formula)}$$

or $\tfrac{1}{2} \times 1.67 \times 10^{-27}\times v^2 = 20{,}000 \times 1.6 \times 10^{-19}$, whence $v^2 = \dfrac{2\times20000\times1.6\times10^{-19}}{1.67\times10^{-27}}$

Thus the desired shift $= \dfrac{\lambda_0 v^2}{2c^2} = \dfrac{4861.33\times10^{-10}\times20000\times1.6\times10^{-19}}{2\times(3\times10^8)^2\times1.67\times10^{-27}} = \mathbf{0.1\times 10^{-10}\ m = 0.1Å.}$

Questions

1. What do you understand by frame of reference ? What is an inertial frame ? Show that a frame of reference having a uniform rectilinear motion relative to an inertial frame is also inertial.
2. What are Galilean transformations ?

   A frame of reference S' is moving with constant velocity relative to another frame. Write down the transformation of x, y, z, t to x', y', z', t' in the Galilean form. At time $t = 0$, both frames are coincident.

   Obtain also the transformations of velocity and acceleration. **(Agra 1984)**
3. Discuss the basic assumptions implied in the Galilean transformations. Use these transformations to show that the distance between two points is in-variant in two inertial frames.
4. Discuss the principle of relativity and the invariance of speed of light. Use this principle to deduce Lorentz transformations. Discuss the relativity of simultaneity. **(Agra 2000, 1994)**
5. Describe the Michelson-Morley's experiment. What was the purpose of this experiment and what was the conclusion ? What significant change this experiment could introduce in the Galilean theory of relativity ? **(Agra 1992)**
6. Enunciate the principle of the special theory of relativity and derive Lorentz transformations. **(Meerut 2001, 1999; Agra 2001, 1999, 95)**
7. Show by direct application of Lorentz transformations that $x^2 + y^2 + z^2 - c^2t^2$ is invariant. **(Agra 1999, 92)**
8. (*a*) Show how Lorentz transformation equations are superior to Galilean transformations. **(Rohilkhand 1980)**

   (*b*) Prove that when $v << c$, Lorentz transformations reduces to Galilean one. **(Agra 1991)**

   (*c*) Prove the following statement :

   To the stationary observer, the moving clock appears to go show. **(Agra 1991)**
9. (*a*) Derive the expressions for the Lorentz space-time transformations. **(Delhi 1989)**

   (*b*) What is meant by relativistic length contraction and time dilation ? What are proper length and proper time interval ? **(Meerut 1999; Agra 2001, 1990; Kanpur 80)**
10. What do you understand by time dilation ? What is proper interval of time ? Briefly discuss one experiment in support of time dilation in special relativity. **(Agra 1986, 85; Allahabad 84; Kanpur 78)**
11. Obtain Einstein's formula for addition of velocities. **(Agra 1990; Delhi 90; Rohilkhand 80)**
12. Derive the relativistic law of addition of velocities. (*i*) Hence show that c is the ultimate speed (*ii*) Prove that the law is in conformity with the principle of constancy of speed of light. **(Meerut 1994)**
13. An elementary particle called the neutrino is moving with the speed of light ($u = c$). An observer is travelling with velocity v towards the neutrino. According to the moving observer what is the velocity of the neutrino ?
14. Obtain an expression for relativistic Doppler's effect. **(Delhi 1990)**
15. Show that when velocity of light is added to the velocity of light we get velocity of light. **(Meerut 2001)**

Problems

[SET- I]

(Galilean Transformations)

1. A body is projected at an angle to the horizontal. What is the path in a frame of reference, which is moving with

(*i*) a velocity equal to the horizontal component of the velocity of the body, (*ii*) double of this velocity, and (*iii*) which is stationary ?

Ans : (*i*) A vertical straight line, (*ii*) A parabola in the reversed direction, (*iii*) Same parabola in the direction of the body.

2. The position of a particle P in a coordinate system S is measured as

$$\mathbf{r} = (6t^2 - 4t)\,\hat{\mathbf{i}} + (-3t^3)\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}\ \text{m}$$

(*a*) Determine the constant relative velocity of system S' with respect to S if the position of P is measured as

$$\mathbf{r}' = (6t^2 + 3t)\,\hat{\mathbf{i}} + (-3t^3)\,\hat{\mathbf{j}} + 3\,\hat{\mathbf{k}}\ \text{m}$$

(*b*) Show that the acceleration of the particle is the same in both systems.

Ans. : (*a*) $-7\,\hat{\mathbf{i}}$.

3. Prove that the Galilean transformation of a position vector is expressed by $\mathbf{r} = \mathbf{r}' + \mathbf{v}t + \mathbf{R}$, where $\mathbf{v}$ is the velocity of the frame S' relative to S and $\mathbf{R}$ the position vector of the origin O' as measured in S at $t' = 0$. **(Agra 1980)**

4. A pilot is supposed to fly due east from P to Q and then back again to P due west. The velocity of the plane in air is $\mathbf{v}_0$ and the velocity of the air relative to the ground is $\mathbf{u}$. The distance between P and Q is l and the plane's air speed v_0 is constant.

(*i*) If $u = 0$ (still air), show that the time for the round trip is $T_0 = 2l/v_0$.

(*ii*) If the air velocity is due east (or west), show that the time for a round trip is

$$T_E = \frac{T_0}{\sqrt{1 - u^2/v_0^2}}.$$

(*iii*) If the air velocity is due north (or south), then show that the time for a round trip is

$$T_N = \frac{T_0}{\sqrt{1 - u^2/v_0^2}}.$$

5. An aeroplane has a speed of 135 km/hour in still air. It is moving straight north so that it is at all time directly above a north-south highway. An observer on the ground tells the pilot by wireless that a 70 km/hour wind is blowing, but neglects to tell him the direction of the wind. The pilot observes that inspite of the wind he can travel 135 km in one hour along the high way. In other words, his ground speed is the same as if there were no wind. (*i*) What is the direction of the wind ? (*ii*) What is the heading of the plane, *i.e.*, the angle between its axis and the highway ?

Ans. : (*i*) Wind is blowing from a direction 75° E of S, (*ii*) 30° E of N. Substituting W for E in the above gives another solution.

6. The pilot of an aeroplane wishes to reach a point 200 miles east of his present position. A wind blows 30 miles per hour from the north-west. Calculate his vector velocity with respect to the moving air mass if his schedule requires him to arrive at his destination in 40 minutes. **(Agra 1972)**

Ans. : $300\,\hat{\mathbf{i}} - \dfrac{30}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$ miles/h, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are the unit vectors along east and north directions respectively.

7. A pilot of an aeroplane wishes to reach a place 100 km east to its present position. The velocity of the wind is 15 km/hour from the north-east. Find his velocity relative to the wind, if he reaches in 20 minutes at the required place.

Ans. : $\left(300 + \dfrac{15}{\sqrt{2}}\right)\hat{\mathbf{i}} + \dfrac{15}{\sqrt{2}}\hat{\mathbf{j}}$.

(Michelson-Morley Experiment)

8. In a frame moving with a velocity v show that the times taken by a wave to travel equal distances d starting from a point and back along and perpendicular to the direction of travel differ by an amount dv^2/V^3, where V is the velocity of the wave. (Assume $v \ll V$).

9. In the Michelson-Morley experiment, what is the expected fringe-shift, if the effective length of each path is 6 metres and light has 6000 Å wave-length ? (Speed of earth $v = 3 \times 10^4$ m/sec.) **(Agra 1985)**

Ans. : 0.2 fringe-width.

(Lorentz Transformations)

10. With the help of the Lorentz transformation equations, find an expression $(x^2 - c^2t^2)$ in terms of x' and t'.

Ans. : $(x'^2 - c^2t'^2)$.

11. Show by the direct application of Lorentz transformations that

$$x^2 + y^2 + z^2 + w^2$$

is invariant, where $w = ict$, $i = \sqrt{-1}$. **(Kanpur 1980, 75)**

(Length Contraction)

12. A Hypothetical train moving with a speed of $0.6c$ passes by the platform of a small station without being slowed down, the observer on the platform note that the length of the train is just equal to the length of the platform which is 200 m.

(*i*) Find the rest length of the train.

(*ii*) Find the length of the platform as measured by the observer in the train. **(Agra 1999)**

Ans. : (*i*) 250 m, (*ii*) 160 m.

13. What will be the apparent length of a meter stick measured by an observer at rest, when the stick is moving along its length with a velocity equal to

(*i*) c, (*ii*) $c/\sqrt{2}$ (*iii*) $\sqrt{3}\,c/2$ (*iv*) $c/2$. **(Agra 1978, 77)**

Ans : (*i*) 0. (*ii*) $1/\sqrt{2}$ m, (*iii*) 1/2 m, (*iv*) $\sqrt{3}/2$ m.

14. A rod has a length of 1 m. It is moving in a space-ship moving with a velocity $0.4\,c$ relative to the earth. Calculate its length as measured by an observer (*i*) on space-ship; (*ii*) on earth. **(Agra 1990)**
Ans : (*i*) 1 m; (*ii*) 0.92 m.

15. Calculate the percentage contraction of a rod moving with a velocity $0.8\,c$ in a direction inclined at $60°$ to its own length.
Ans : 8.3%.

16. A rod is seen to move with velocity $0.6c$, in a direction inclined at $60°$ to the rod by an observer, situated in the laboratory and the length of the rod is measured as 0.8 m. What is the true length of the rod and what is the true direction of its motion ?
Ans : 0.85 m ; $\tan\theta = 1.38$.

17. A vector in system S' is represented by $8\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$. How can the vector be represented in system S while S' is moving with velocity $0.8\,c\,\hat{\mathbf{i}}$ relative to S.
Ans : $4.8\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$.

18. Show that the circle $x^2 + y^2 = r_0^2$ in frame S, appears to be ellipse in a frame S' which is moving with velocity $v\,\hat{\mathbf{i}}$ relative to S.

(Time Dilation)

19. The half life of a particular particle as measured in the laboratory comes out to be 4×10^{-8} sec, when its speed is $0.8\,c$ and 3×10^{-8} sec, when its speed is $0.6\,c$. Explain this. **(Agra 1971)**
Ans. : In both cases, the proper half life comes to be 2.4×10^{8} sec. This means that a particle has its own life time independent of observers in motion.

20. A stationary μ meson decays in 2.2×10^{-6} sec. What will be its length of path, if it is moving towards the earth with velocity $0.99\,c$? In a frame of reference fixed on the meson, what distance will the earth travel before the meson decays ? **(Delhi 1990)**
Ans. : 4.6×10^{3} m; 6.5×10^{2} m .

21. A beam of pions has velocity $v = 0.6\,c$. The pion has a half life of 1.8×10^{-8} sec. How long will it take for the pions to decay ? How far will they travel in this time ?
Ans. : 2.25×10^{-8} sec; 4.05 m.

22. The proper mean life of the μ meson is approximately 2×10^{-6} sec. Suppose that a burst of μ mesons is produced at some height in the atmosphere travels downward at $v = 0.99\,c$. If 1% of those in the original burst survive to reach the earth, estimate the initial height.
Ans. : 2×10^{4} m.

23. Calculate the velocity of a watch when it seems to be slowed down by 1 minute in 1 hour. **(Kanpur 1989)**
Ans. : 5.5×10^{7} m/sec.

24. A clock gives correct time. With what speed should it be moved relative to an observer so that it may seem to lose 5 minutes in 24 hours ?
Ans. : 2.5×10^{7} m/sec.

25. If a rocket travels with a constant velocity of $0.8\,c$ from the earth to a star 4 light years distance, what will be (*i*) the time taken for the trip according to estimates made on the earth, (*ii*) the time according to a passenger, (*iii*) the distance from the earth to the star according to a passenger, and (*iv*) the velocity of the earth and the star as measured by a passenger during the trip ?
Ans. : (*i*) 5 years, (*ii*) 3 years, (*iii*) 2.4 light years, (*iv*) 0.8 *c*.

(Addition of Velocities)

26. A nuclear particle was observed to break into two fragments which moved in opposite directions. The velocity of each fragment was found to be 0.8 c relative to the laboratory. What was the velocity of one fragment relative to the other ?
Ans. : 0.98 c.

27. Two electrons, each of velocity 0.8 c, move towards each other. Find the relative velocity of one electron with respect to the other. **(Agra 1993; Delhi 92; Rohilkhand 80)**
Ans. : 0.97 c.

28. Rockets A and B are observed from the earth to be travelling with velocities 0.8 c and 0.7 c in the same direction. What is the velocity of B as seen by an observer in A. **(Lucknow 1980)**
Ans. : 0.227c.

29. A particle has a velocity $u' = 3\,\hat{\mathbf{i}} + 4\,\hat{\mathbf{j}} + 2\,\hat{\mathbf{k}}$ m/sec in a coordinate system moving with velocity 0.8 c relative to laboratory frame along +ve direction of X-axis. Find **u** in the laboratory frame.

Ans. : $(2.4 \times 10^8\,\hat{\mathbf{i}} + 2.4\,\hat{\mathbf{j}} + 7.2\,\hat{\mathbf{k}})$ m/sec.

(Doppler's Effect)

30. A spectral line of wavelength 4×10^{-7}m, in the spectrum of light from a star is found to be displaced from its normal position towards the red end of the spectrum by an amount 10^{-10} m. What velocity of the star would account for this ? (Use terms in v/c of first order).

Ans. : 750 m/sec.

31. Excited Fe^{57} nuclei sometimes decay to produce γ-ray photons of frequency 3.46×10^{18} Hz. Determine the frequency of the photon emitted at an angle of 60° in the laboratory frame relative to the direction of the Fe^{57} nucleus, when it is moving with a velocity 6×10^7 m/sec.

Ans. : 3.7×10^{18} Hz (nearly)

32. Light from a distant star is observed to have a wavelength shift of 0.1%. What is the speed of the star? **(Delhi 1990)**

Ans. : 3×10^5 m/sec.

[SET- II]

1. If ψ represents a scalar function of position and time, show that the wave equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial\psi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t}=0$$

is not invariant under Galilean transformation, but invariant under Lorentz transformation.

2. A light beam is emitted at an angle θ' relative to the X-axis in system S'. Show that the angle θ as measured in system S is

$$\cos\theta = \frac{\cos\theta' + (v/c)}{1 + (v/c)\cos\theta'}$$

where v is the speed of system S' along the X- X' axes.

3. (*i*) The frame S' moves relative to S with a velocity v along the common X-axis and the two frames coincide when the clocks at their origins record $t = t' = 0$. If the velocities of a given particle are **u** (u_x, u_y, u_z) and $\mathbf{u}' = (u_x', u_y', u_z')$ as seen in the two frames respectively, show that

$$\sqrt{1-\frac{u'^2}{c^2}} = \frac{\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}}{(1-vu_x/c^2)}.$$

(*ii*) An observer moving along the X-axis of a reference system S with a velocity sees a body of proper volume V_0, moving along the X-axis of S with a velocity u. Show that the apparent volume of the body measured by the observer is

$$V = V_0\sqrt{c^2-u^2}\sqrt{c^2-v^2}\Big/\left(c^2-uv\right).$$

[**Hint :** The volume is $V_0\sqrt{1-\frac{u'^2}{c^2}}$ and use the relation of part (i).]

4. Consider three inertial frames S, S' and S''. Let S' move with velocity v with respect to S, and let S'' move with velocity v' with respect to S'. All velocities are in the same direction .

(*i*) Write the transformation equations relating x, y, z, t with x', y', z', t' and also those relating x', y', z', t' with x'', y'', z'', t''. Combine these equations to get the relations between x, y, z, t and x'', y'', z'', t''.

(*ii*) Show that these relations are equivalent to a direct transformation from S to S'' in which the relative velocity v'' of S'' with respect to S is given by the relativistic addition theorem :

$$v'' = \frac{v+v'}{1+vv'/c^2}.$$

(*iii*) Discuss how the above analysis proves that two successive Lorentz transformations are equivalent to one direct transformation ?

5. Find the wavelength shift, if any, in the Doppler effect for the sodium D_2 line (5890Å) emitted from a source moving in a circle with constant speed $0.1c$ measured by an observer fixed at the centre of the circle.

Ans. : 29 Å.

6. A particle moves at speed v at an angle θ with the X-axis in the S-frame. Find its speed and direction relative to the X'-axis of the S'-frame.

$$\textbf{Ans :}\ v' = \frac{\left\{v^2 - 2uv\cos\theta + V^2 - \left(\frac{Vv\sin\theta}{c}\right)^2\right\}^{1/2}}{1-(uV\cos\theta)/c^2}$$

$$\tan\theta' = \frac{v\sin\theta\sqrt{1-v^2/c^2}}{(v\cos\theta - V)}.$$

7. Two rods having the same length l_0 move lengthwise towards each other parallel to a common axis with the same velocity v relative to the laboratory frame. What is the length of each rod in the frame fixed to the other rod ?

Ans. : $l = l_0(1-v^2/c^2)/(1+v^2/c^2)$.

8. Two relativistic particles move at right angles to each other in the laboratory frame of reference, one with velocity v_1, and the other with velocity v_2. Determine their relative velocity.

Ans. : $v = \sqrt{v_1^2 + v_2^2 - (v_1 v_2/c^2)}$.

9. At two points of the reference frame S two events occurred separated by a time interval Δt. Demonstrate that if these events obey the cause and effect relationship in the frame S, they obey that relationship in any inertial frame S'.

Objective Type Questions

1. Choose the correct statement :
   (*a*) Galilean transformations are consistent with the constancy of speed of light in all initial frames.
   (*b*) Galilean transformations are inconsistent with the constancy of speed of light in all inertial frames.
   (*c*) Lorentz transformations are consistent with the constancy of speed of light in all inertial frames.
   (*d*) Lorentz transformations are inconsistent with the constancy of speed of light in all inertial frames.

   Ans. : (*b*) and (*c*).

2. Frame S' is moving with speed v along X-axis relative to S. A rod is stationary in frame S with length l along X-axis. The length as observed in frame S' is

   (*a*) $l/\sqrt{1 - v^2/c^2}$ (*b*) $l\sqrt{1 - v^2/c^2}$ (*c*) l (*d*) l/v.

   Ans. : (*b*).

3. In the laboratory one particle A has the velocity v and another particle B has velocity $-v$ opposite to each other. The velocity of A relative to B is

   (*a*) $\dfrac{2v}{1 + v^2/c^2}$ (*b*) $\dfrac{2v}{1 - v^2/c^2}$ (*c*) $2v$ (*d*) $\dfrac{2v}{\sqrt{1 - v^2/c^2}}$.

   Ans. : (*a*).

4. Aberration of light from stars is caused due to
   (*a*) the travelling of light in the atmosphere.
   (*b*) the elliptical orbit of the earth around the sun.
   (*c*) the finite speed of light and the speed of earth in its orbit around the sun.
   (*d*) the scattering of light by the air particles.

   Ans. : (*c*).

5. The recessional red shift
   (*a*) is a relativistic phenomenon.
   (*b*) is a non-relativistic phenomenon.
   (*c*) is caused by the motion of a galaxy away from an observer.
   (*d*) is caused by the motion of a galaxy towards an observer.

   Ans. : (*a*), (*c*).

Short Type Questions

1. What is principle of relativity ? Explain.
2. What do understand by the covariance of physical laws ?
3. How does the principle of relativity lead the constancy of speed of light in all inertial frames ?

4. Why were Michelson Morley experiments performed ?
5. Discuss the importance of negative results of Michelson-morley experiments.
6. Why is interferometer related by $\pi/2$ angle in Michelson-Morley experiment ?
7. State the fundamental postulates of special theory or relativity.
8. Show that Lorentz transformation equations are superior to Galilean transformations.
9. Prove that at low velocity ($v << c$), Lorentz transformation reduces to Galilcan one.
10. What do you understand by Lorentz-Fitzgerald contraction ?
11. What is time dilation ? Explain the time dilation effect for μ-mesons falling towards earth from sky.
12. What do you understand by proper length and proper time interval ?
13. Write down velocity transformation equations at relativistic velocities. What can be the maximum velocity of a particle ?
14. Moving clock appears to go slow. Explain.
15. What is aberration of light ? Explain in brief.
16. The spectral line of $\lambda = 5000$ Å in the light coming from a distant star is observed at 5100 Å. What is the recessional velocity of the star ?

    Ans. : 6×10^6 m/s.
17. Fill in the blanks :
    (*i*) In view of the Galilean transformations, if any two events occur simult-aneously in an inertial frame, then they must........................relative to all inertial frames.
    (*ii*) In view of Lorentz transformation, if two events at two different points are simultaneous in one inertial frame, they are................in another frame in constant relative motion with respect to first frame.

    Ans. : (*i*) occur simultaneously, (*ii*) not simultaneous.

CHAPTER 13

Relativistic Mechanics

13.1. INTRODUCTION

When the speed of a material particle is comparable with the speed of light, the formulations of classical mechanics are not consistent with the experimental facts. In Newtonian mechanics, the linear momentum of a particle of mass m_0 is defined as $m_0\mathbf{v}$, where m_0 is independent of the particle velocity ($\mathbf{v}$). In the next article, we shall show that the non-relativistic momentum is not conserved in collisions, where the particles have relativistic speeds. The law of conservation of momentum is a fundamental law of nature and it is assumed that this law holds correct even at relativistic speeds. Hence to deal the problems at relativistic speeds, it is necessary to modify the classical definition of momentum so that the law of conservation of momentum is still valid. This will emerge out that the mass of a body varies with velocity relative to an observer. A most important consequence of this result is obtained in the form of Einstein's mass-energy relation, $E = mc^2$. In the present chapter first we deal with relativistic momentum, relativistic energy and the relation between them. We shall also discuss force in the special relativity. Next the transformation equations of momentum-energy, force etc. will be given. Finally we shall deal with the relativistic formulation of the Lagrangian and Hamiltonian.

13.2. CONSERVATION OF MOMENTUM AT RELATIVISTIC SPEEDS–VARIATION OF MASS WITH VELOCITY

We assume that the law of conservation of momentum is valid even at relativistic speeds. Below, first we shall show that with the Newtonian definition of momentum ($\mathbf{p} = m_0\mathbf{v}$), the momentum is not conserved in collision problem at relativistic speeds and then we shall give the necessary modification so that the fundamental law of conservation of momentum retains its validity at any speed.

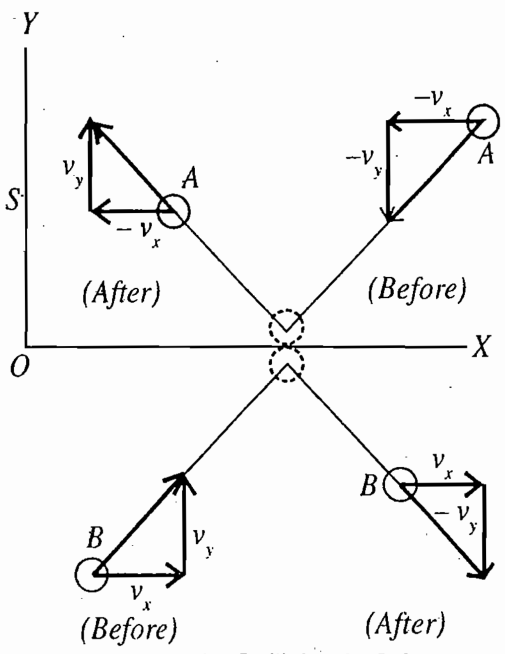


Fig. 13.1 : Collision in S-frame

Consider the collision of two particles A and B of equal masses m_0 in X-Y plane [Fig. 13.1].Let the two particles initially have equal and opposite velocities in a frame S. Therefore, before the collision, x and y components of the velocity for the particle A are $-v_x$ and $-v_y$ and for B are v_x and v_y. Now, if they collide elastically, their x-components of velocity will not change due to collision but the y-components will have opposite sign.

The changes of the momenta of the particles A and B along Y-direction due to collision will be given by

$$\Delta p_a = m_0 v_y - (-m_0 v_y) = 2m_0 v_y \quad \text{and}$$

$$\Delta p_b' = -m_0 v_y - (m_0 v_y) = -2m_0 v_y$$

Hence the total change in the momentum along Y-direction after the collision is

$$\Delta p_a + \Delta p_b = 2m_0 v_y - 2m_0 v_y = 0$$

Thus the Newtonian momentum is conserved along Y-axis. This is also evident from Fig.13.1 that the velocities of the particles along X-axis are not changed by the collision and therefore in X-direction the change in momentum is also zero.

Next, we see the collision of the same particles from a frame S' which is moving with velocity v_x along X-axis relative to S (Fig. 13.2). Now, according to Lorentz transformations, x- and y-components of the velocities of the particles in frame S' will be seen as :

(1) For particle A, having $-v_x$ and $-v_y$ velocity components in S frame

$$v'_x(A) = \frac{-v_x - v_x}{1 + v_x^2/c^2} = -\frac{2v_x}{1 + v_x^2/c^2} \qquad \text{...(1a)}$$

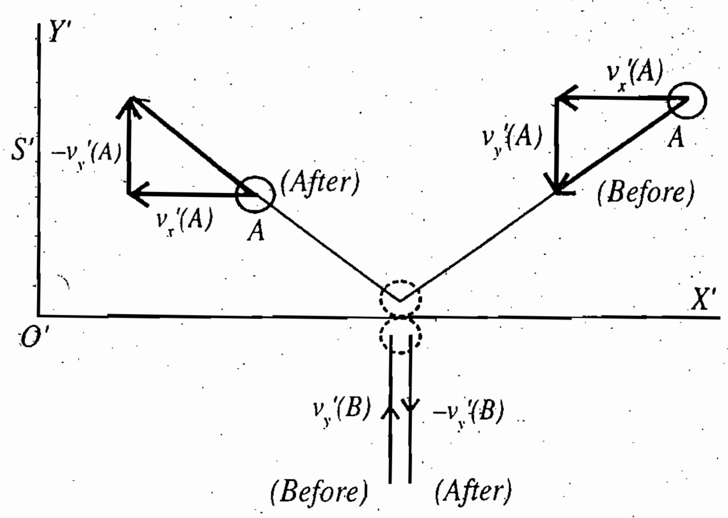


Fig. 13.2 : Collision in S'-frame

[because $u'_x = \dfrac{u_x - v}{1 - vu_x/c^2}$ and here $u_x = -v_x$ in S and $v = v_x$]

and $$v'_y(A) = \frac{-v_y}{\gamma(1 + v_x^2/c^2)} \qquad \text{...(1b)}$$

(2) For particle B, having $+v_x$ and $+v_y$ velocity components in frame S,

$$v'_x(B) = \frac{v_x - v_x}{1 - v_x^2/c^2} = 0 \quad \text{and} \quad v'_y(B) = \frac{v_y}{\gamma(1 - v_x^2/c^2)} \qquad \text{...(2)}$$

By the collision the x-components of the velocities of the particles in frame S' are not changed [Fig. 13.2.] so that the change in momentum of the system along X-axis is zero. But after the collision the signs of $v'_y(A)$ and $v'_y(B)$ are changed and hence according to Newtonian mechanics, the change in y-components of the momenta of the particles will be given by

$$\Delta p'_a = \frac{2m_0 v_y}{\gamma(1 + v_x^2/c^2)} \quad \text{and} \quad \Delta p'_b = -\frac{2m_0 v_y}{\gamma(1 - v_x^2/c^2)} \quad \text{i.e.,} \quad \Delta p'_a + \Delta p'_b \neq 0 \qquad \text{...(3)}$$

Thus in frame S', there occurs a change in total momentum of the system. Hence, *if the masses of the particles are assumed to be constant at relativistic speeds then the law of conservation of momentum is not valid in all inertial frames.*

For the validity of the conservation of momentum, one has to assume that the mass of a particle depends upon its velocity relative to the frame of reference, chosen for description. Now, if in frame S' the masses of the particles A and B have become m_1 and m_2 respectively, then for the conservation of the momentum, we have

$$\Delta p'_a + \Delta p'_b = \frac{2m_1 v_y}{\gamma(1+v_x^2/c^2)} - \frac{2m_2 v_y}{\gamma(1-v_x^2/c^2)} = 0 \quad i.e., \quad \frac{m_1}{m_2} = \frac{1+v_x^2/c^2}{1-v_x^2/c^2} \qquad ...(4)$$

From eq. (1*a*), $(v'_x)^2 = \dfrac{4v_x^2}{(1+v_x^2/c^2)^2}$. This can be written as

$$1 - \frac{(v'_x)^2}{c^2} = 1 - \frac{4v_x^2/c^2}{(1+v_x^2/c^2)^2} = \frac{(1-v_x^2/c^2)^2}{(1+v_x^2/c^2)^2} \qquad ...(5)$$

From eqs. (4) and (5): $\dfrac{m_1}{m_2} = \dfrac{1}{\sqrt{1-v'^2_x/c^2}}$ or $m_1 = \dfrac{m_2}{\sqrt{1-v'^2_x/c^2}}$

If we consider that the particle B is at rest in frame S' so that $m_2 = m_0$ and A is moving with velocity $v'_x = v$ in this frame, then

$$m_1 = \frac{m_0}{\sqrt{1-v^2/c^2}} \qquad ...(6)$$

When $v=0$, $m_1 = m_0$, this is called the ***rest mass*** or ***proper mass*** of the particle. We have considered in our description the particles of equal masses, *i.e.*, correctly of equal rest masses. Thus, the mass of a particale in a frame, in which it has the speed v, will be given by

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}} \qquad ...(7)$$

This relation represents the ***variation of mass with velocity***. This result is an important consequence of special theory of relativity and has been verified by several experiments in connection with high energy particles. In 1908, Bucherer showed from his experiment that e/m, ratio of charge and mass, for fast moving electrons was smaller than that for slow moving electrons, because the mass of electrons with higher speed is more and the charge e remains constant.

Thus the relativistic momentum is given by

$$\mathbf{p} = m\,\mathbf{v} = \frac{m_0 \mathbf{v}}{\sqrt{1-v^2/c^2}} \qquad ...(8)$$

With this definition of momentum, our law of conservation of momentum will hold even at relativistic velocities for the two-body collision of identical particles. This may be shown that even if particle B has a different mass from particle A, the above definition will hold in order to conserve the momentum. For $v/c \ll 1$, the expression of momentum reduces to the classical definition $\mathbf{p} = m_0 \mathbf{v}$.

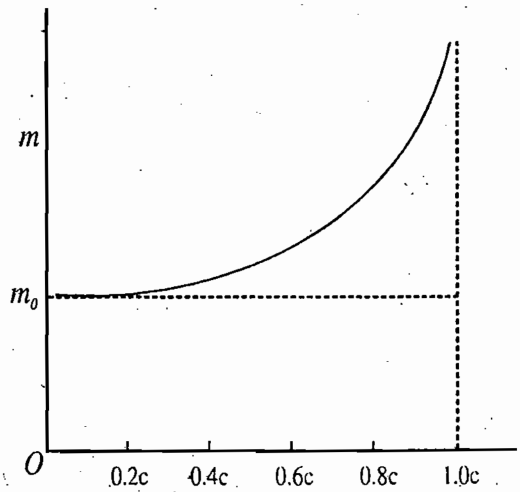


Fig. 13.3 : Variation of mass with velocity

Ultimate speed of particles : Now, it can be proved theoretically that no material particle can attain the speed of light c, *i.e.*, c is the ultimate speed of particles. From eq. (7), if a graph is plotted between the mass m of a particle and its speed v (Fig. 13.3), then in the condition $v \ll c$, $m = m_0$ which is equal to the classical value, but when the value of v

becomes greater than $c/2$, the mass of the particle m increases and this departure from the classical value cannot be neglected. When the particle speed v is approaching the speed of light c, its mass tends to be infinite. Hence the force, applied to accelerate the particle, increases its mass while speed remains nearly constant. This shows that by accelerating a material particle, it cannot attain the speed equal to or greater than the speed of light. Thus the ultimate speed of any material particle is always less than the speed of light c.

13.3. RELATIVISTIC ENERGY : MASS-ENERGY RELATION ($E = mc^2$)

Suppose a force $F = \frac{d}{dt}(mv)$ be acting on a particle of mass m so that its kinetic energy increases. The gain in kinetic energy will be equal to the work done on the particle. If the force displaces the particle through a distance dr along its line of action, then the infinitesimal gain in the kinetic energy is

$$dE_k = Fdr = \frac{d(mv)}{dt}dr = vd(mv) \qquad \left[\text{because } v = \frac{dr}{dt}\right]$$

If the particle starts from rest ($v = 0$) and acquires velocity v under the action of the force, then the gain in the kinetic energy by the particle will be given by

$$E_k = \int dE_k = \int_0^v vd(mv)$$

Integrating this equation by parts, we obtain

$$E_k = vmv\Big|_0^v - \int_0^v mvdv = mv^2 - \int_0^v \frac{m_0 vdv}{\sqrt{1-v^2/c^2}} \; *$$

$$= \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} + m_0c^2\sqrt{1-\frac{v^2}{c^2}} - m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 = mc^2 - m_0c^2$$

Thus $$E_k = (m-m_0)c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 \qquad ...(9)$$

where $$m = m_0\Big/\sqrt{1-v^2/c^2}$$

Eq. (9) is the expression for **relativistic kinetic energy**. It shows that *the gain in kinetic energy corresponds to an increase in mass.*

The quantity m_0c^2, occurring in relation (9), is due to the rest mass of the particle and is called the ***rest energy*** or ***proper energy*** E_0 of the particle, *i.e.*, $E_0 = m_0c^2$. Thus, the ***total energy*** of the particle, when it is moving with velocity v, is

$$E = \text{kinetic energy}(E_k) + \text{Rest energy}(E_0)$$

$$= (m-m_0)c^2 + m_0c^2 = mc^2$$

Thus $$E = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} = mc^2 \qquad ...(10)$$

* For integrating put $1 - v^2/c^2 = \alpha$ and hence $vdv = -c^2 d\alpha/2$.

This energy E is called the ***relativistic energy*** (total energy) of a particle, having relativistic mass m. Thus, there exists a very close relation between mass and energy, unknown in classical physics. This is well known ***Einstein's mass-energy relation***.

The relativistic kinetic energy can be expressed as

$$E_k = E - E_0 = (m - m_0)c^2 = m_0c^2(1 - v^2/c^2)^{-1/2} - m_0c^2$$

$$E_k = m_0c^2\left[1 + \frac{1}{2}.\frac{v^2}{c^2} + \frac{3}{8}.\frac{v^4}{c^4} + \ldots\right] - m_0c^2 \qquad \text{(Using Bionomial theorem)}$$

In the limit $v^2/c^2 << 1$, we have

$$E_k = m_0c^2\left(1 + \frac{v^2}{2c^2}\right) - m_0c^2 = \frac{1}{2}m_0v^2 \qquad \ldots(11)$$

This relation is the classical result for the kinetic energy.

We see from eq. (9) and (10) that the increase in kinetic energy or total energy ΔE of a particle is associated with a corresponding increase in mass Δm according to the relation,

$$\Delta E = \Delta mc^2 \qquad \ldots(12)$$

It is known that one kind of energy, *e.g.*, kinetic energy can be converted in other forms and hence all forms of energy must be associated with them some mass. According to Einstein, eq. (12) is the most important consequence of the special theory of relativity. He considers that an amount of energy ΔE in any form is equivalent to a mass $\Delta m = \Delta E / c^2$ and conversely, any mass Δm is equivalent to an energy $\Delta E = \Delta mc^2$. This is called the ***principle of equivalence of mass and energy***. Thus there is the possibility that mass can be changed into energy and vice-versa. The truth of this fact has been verified by a number of experiments.

In the language of Einstein, the mass of a body is the measure of the quantity of its energy. This means that a system of inertial mass m is equivalent to an energy $E = mc^2$. Further the rest mass of a body cannot be distinguished from the mass due to the energy possessed by it. Thus, there is the possibility that the rest mass of a body is due to some form of energy and an interchange between rest mass and energy may occur.

13.4. EXAMPLES OF MASS-ENERGY CONVERSION

(1) Electron-positron annilhilation : A positron is an elementary particle, having rest mass equal to that of an electron and charge (+ e) equal and opposite to that of an electron. When an electron and a positron come together, they can annihilate with the production of γ-rays. It is found that the energy associated with the γ-rays is equal to the total energy $(2mc^2)$ associated with the masses of both particles. Similarly, it has been observed that from γ-rays a pair of electron and positron is produced. These observed phenomena of eletron-position annihilation and pair production confirm most directly the principle of mass-energy equivalence.

(2) Nuclear energy : (*i*) An enormous amount of destructive energy, obtained from an atom bomb, is a consequence of change of mass into energy . In a U^{235} atom bomb, the nucleus of uranium atom breaks into two parts, having total rest mass slightly less than the original mass of the atom so that the difference of the two masses Δm appears in the form of energy $\Delta E = \Delta mc^2 = \Delta m \times (3 \times 10^8)^2$ joules per atom. A huge amount of energy is released by the disintegration of a large number of U^{235} atoms in an atom bomb.

(*ii*) Similarly, in some nuclear reactions two light nuclei combine to produce other nuclei, whose total mass is slightly less than the mass of the original nuclei. The difference of these two masses is released in the form of energy. For example, consider the nuclear reaction :

$$\text{Li} + \text{H}^2 \rightarrow \text{He}^4 + \text{He}^4$$
$$\text{Lithuim} + \text{deutron} \rightarrow 2 \text{ Helium atoms}$$

In fact the mass of left hand side atoms is in excess tc the mass of right hand side atoms by an amount $\Delta m = 0.02381$ amu*. The calculated energy $\Delta E = \Delta mc^2$ is 22.17 MeV and the observed value 22.2 MeV are in very close agreement.

(*iii*) The most important source of solar energy is considersd the fusion of protons to form helium. The energy release per helium atom, formed from four protons and two electrons, can be calculated from the net change in mass, *i.e.*,

$\Delta m =$ (mass of the 4 protons + mass of 2 electrons) – mass of helium atom

$$= 4 \times 1.6725 \times 10^{-27} + 2 \times 0.911 \times 10^{-30} - 6.647 \times 10^{-27} = 0.045 \times 10^{-27} \text{ kg}.$$

Thus the release of energy per atom

$$\Delta E = \Delta mc^2 = 0.045 \times 10^{-27} \times (3 \times 10^8)^2 = 25 \text{ MeV} \quad [1 \text{ eV} = 1.6 \times 10^{-19} \text{J}]$$

The sun radiates energy at the rate of 3.8×10^{26} J/sec, so its mass is decreasing $3.8 \times 10^{26} / (3 \times 10^8)^2 = 4 \times 10^9$ kg/sec.

(3) A number of examples may be quoted, showing the principle of mass-energy equivalence. An internal energy of a body in any form, increases its mass. Thus, a compressed spring is heavier than in the uncompressed state; a hot body loses mass in becoming cold. In such cases, the change of energy is small, hence the change in mass $[\Delta m = \Delta E / (3 \times 10^8)^2]$ is extremely small and is not measurable.

(4) **Binding energy :** The amount of energy needed to separate the particles (neutrons and protons) of a nucleus to infinite distances is called the *binding energy* of the nucleus. Thus, it is expected that mass of a nucleus to be smaller than the mass of the constituents of the nucleus by an amount $\Delta m = \Delta E / c^2$, where ΔE is the binding energy of the nucleus.

13.5. RELATION BETWEEN MOMENTUM AND ENERGY AND CONSERVATION LAWS

If a particle of rest mass m_0 moves with a velocity v, its momentum p is given by

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = m_0 v\gamma \quad \text{or} \quad p^2 = m_0^2 v^2 \gamma^2 \qquad \text{...(13)}$$

where
$$\gamma^2 = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{c^2}{c^2 - v^2} \quad \text{or} \quad \gamma^2 v^2 = \gamma^2 c^2 - c^2 \qquad \text{...(14)}$$

---

* **Atomic mass unit** (*amu*) is the unit of mass, used in atomic physics and is equal to 1/12th of the rest mass of C^{12} atom.

$$1 \text{ amu} = 1.6604 \times 10^{-27} \text{ kg}$$

Equivalent energy (1 amu) $\Delta E = 1.6604 \times 10^{-27} \times (3 \times 10^8)^2 = 1.49 \times 10^{10}$ J $= \dfrac{1.49 \times 10^{-10}}{1.6 \times 10^{-19}}$ eV $=$ **931 MeV.**

Substituting for $\gamma^2 v^2$ in (13) from eq. (14), we have

$$p^2 = m_0^2\gamma^2 c^2 - m_0 c^2 \text{ or } p^2c^2 = m_0^2\gamma^2 c^4 - m_0^2 c^4 \text{ or } m_0^2\gamma^2 c^4 = p^2c^2 + m_0^2 c^4$$

But $E = mc^2 = m_0\gamma c^2$, therefore,

$$E^2 = p^2c^2 + m_0^2c^4 \qquad ...(15)$$

The relation connects the total energy (E) of a particle to its momentum (p). The quantity $m_0^2c^4$ is constant and hence in relativity a particle has definite energy corresponding to a specific value of its momentum. Thus, the law of conservation of energy will be valid in a collision (even inelastic) where the law of conservation of momentum is assumed to be correct. Thus, according to the law of conservation of momentum in a two particle collision

$$P_1 + P_2 = P_1' + P_2' \qquad ...(16)$$

and according to the law of conservation of energy

$$E_1 + E_2 = E_1' + E_2' \qquad ...(17)$$

where E_1, E_2 are the relativistic energies of the particles before the collision and E_1', E_2' after the collision.

From relation (15), we get

$$E^2 - p^2c^2 = m_0^2c^4 \qquad ...(18)$$

The quantity $E^2 - p^2c^2$ is Lorentz invariant. If we transform from one reference frame S to frame S' with $p \to p'$ and $E \to E'$, then the invariance of (18) means that

$$E'^2 - p'^2c^2 = E^2 - p^2c^2 = m_0^2c^4$$

This can be proved easily by applying the transformation equations (see. Art. 13.6).

We want to emphasize here that the rest mass m_0 of a particle is invariant under Lorentz transformations.

13.6. TRANSFORMATION OF MOMENTUM AND ENERGY

The momentum p of a particle, having rest mass m_0 and velocity v, is given by

$$\mathbf{p} = m\mathbf{v} \text{ or } \mathbf{p} = m_0\mathbf{v}\gamma$$

where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{dt}{d\tau}$; $d\tau$ is the proper time interval for the particle and dt is the time interval observed relative to a frame S.

Now, $p_x = m_0 v_x \gamma = m_0 \dfrac{dx}{dt}\dfrac{dt}{d\tau}$ or $p_x = m_0 \dfrac{dx}{d\tau}$

Similarly, $p_y = m_0 \dfrac{dy}{d\tau}$ and $p_z = m_0 \dfrac{dz}{d\tau}$ ...(19)

and $E = \dfrac{m_0c^2}{\sqrt{1 - v^2/c^2}} = m_0c^2 \dfrac{dt}{d\tau}$ or $\dfrac{E}{c^2} = m_0 \dfrac{dt}{d\tau}$ ...(20)

The rest mass m_0 and proper interval time $d\tau$ are Lorentz invariants, it follows from eq. (19) and (20) that p_x, p_y, p_z and E/c^2 must transform from frame S to frame S' under Lorentz transformations exactly as $x, y, z, t,$ transform *i.e.*, similar to the transformations

$$x' = \gamma(x + vt); y' = y'; z' = z \text{ and } t' = \gamma(t - vx/c^2)$$

Thus the transformations of energy and momentum are

$$p_x' = \gamma (p_x - vE/c^2)\ ;\ p_y' = p_y\ ;\ p_z' = p_z\ ;\ E' = \gamma (E - p_x v) \quad ...(21)$$

where t has been replaced by E/c^2.

The inverse transformations are

$$p_x = \gamma (p_x' + vE'/c^2)\ ;\ p_y = p_y'\ \ p_z = p_z'\ \ E = \gamma (E' + p_x' v) \quad ...(22)$$

Now, if the momentum and energy of a particle are known, we can determine its velocity as

$$v_x = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{p_x}{m_0}\frac{m_0 c^2}{E} = \frac{c^2 p_x}{E} \qquad \left(p_x = m_0 \frac{dx}{d\tau}\right)$$

Thus, $\quad \mathbf{v} = \dfrac{c^2 \mathbf{p}}{E} \quad$ or $\quad \mathbf{p} = \mathbf{v}\dfrac{E}{c^2}$ ...(23)

Alternatively, $\mathbf{p} = m\mathbf{v} = \dfrac{E}{c^2}\mathbf{v}$ (because $E = mc^2$)

13.7. PARTICLES WITH ZERO REST MASS

The relativistic energy E of a particle of rest mass m_0 and momentum p is given by

$$E = (p^2c^2 + m_o^2 c^4)^{1/2} \quad ...(24)$$

When $\quad m_0 = 0,\ E = pc$ or $p = E/c$ ...(25)

But $\quad p = v\dfrac{E}{c^2}$, hence $\dfrac{Ev}{c^2} = \dfrac{E}{c}$

Therefore, $\ v = c$ ...(26)

Thus, ***a particle of zero rest mass travels with the speed of light***. If m is the equivalent mass of such a particle (as photon) of energy E, then

$$E = mc^2 \text{ or } m = E/c^2 \text{ and } p = mc = E/c \quad ...(27)$$

In case of a photon of frequency ν, $E = h\nu$, where h is Planck's constant. Therefore, for a **photon** the *relativistic mass* (m) and *momentum* (p) will be given by

$$m = \frac{E}{c^2} = \frac{h\nu}{c^2} \text{ and } p = \frac{h\nu}{c} = \frac{h}{\lambda} \qquad (\because c = \nu\lambda) \quad ...(28)$$

Conversely, one can prove that the *particles, moving with speed of light, possess zero rest mass.*

13.8. FORCE IN RELATIVISTIC MECHANICS

The definition of force $\mathbf{F} = d\mathbf{p}/dt$ is also maintained in relativistic mechanics. Thus

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = \frac{dm}{dt}\mathbf{v} + m\frac{d\mathbf{v}}{dt} \text{ or } \mathbf{F} = \frac{dm}{dt}\mathbf{v} + m\mathbf{a} \quad ...(29)$$

Thus *the force in general is not along the direction of the acceleration vector.*

The rate of doing work is given by

$$\frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v} = \frac{d\mathbf{p}}{dt}\cdot\mathbf{v} = \mathbf{v}\cdot\frac{d}{dt}\left(\frac{m_0\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right)$$

$$= m_0 \mathbf{v} \cdot \left[\frac{\dot{\mathbf{v}}}{\sqrt{1-v^2/c^2}} + \frac{\mathbf{v}(v\dot{v}/c^2)}{(1-v^2/c^2)^{3/2}} \right]$$

$$= m_0 \frac{v\dot{v}}{\sqrt{1-v^2/c^2}} + \frac{m_0 v^3 \dot{v}/c^2}{(1-v^2/c^2)^{3/2}} = \frac{m_0}{(1-v^2/c^2)^{3/2}}$$

$$= \frac{d}{dt} \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} = \frac{d}{dt}(mc^2) \qquad ...(30)$$

because $v^2 = v_x^2 + v_y^2 + v_z^2$ or $v\dot{v} = v_x \dot{v}_x + v_y \dot{v}_y + v_z \dot{v}_z = \mathbf{v} \cdot \dot{\mathbf{v}}$

Thus *the rate of work done is equal to the rate of change of relativistic energy.*

13.9. LORENTZ TRANSFORMATION FOR FORCE

Suppose a particle be instantaneous at rest at x' in frame S'. The force vector in S and S' frames can be written as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \text{ and } \mathbf{F}' = \frac{d\mathbf{p}'}{dt'}$$

or in components form

$$\mathbf{F} = F_x \hat{\mathbf{i}} + F_y \hat{\mathbf{j}} + F_z \hat{\mathbf{k}} = \frac{dp_x}{dt}\hat{\mathbf{i}} + \frac{dp_y}{dt}\hat{\mathbf{j}} + \frac{dp_z}{dt}\hat{\mathbf{k}}$$

and $$\mathbf{F}' = F'_x \hat{\mathbf{i}} + F'_y \hat{\mathbf{j}} + F'_z \hat{\mathbf{k}} = \frac{dp'_x}{dt}\hat{\mathbf{i}} + \frac{dp'_y}{dt}\hat{\mathbf{j}} + \frac{dp'_z}{dt}\hat{\mathbf{k}}$$

The transformation equations are

$$p_x = \gamma (p'_x + vE'/c^2), \; p_y = p'_y, \; p_z = p'_z \text{ and } t = \gamma\left(t' + vx'/c^2\right)$$

Hence $$dp_x = \gamma\left(dp_x' + \frac{vdE'}{c^2}\right), dp_y = dp'_y, dp_z = dp'_z \text{ and } dt = \gamma dt'$$

But in S'-frame

$$E' = \sqrt{p'^2 c^2 + m_0^2 c^4} \quad \text{or} \quad dE' = p' dp' c^2 / (m_0^2 c^4 + p'^2 c^2)^{1/2}$$

In S'-frame $p' = 0$, therefore, $dE' = 0$. Hence

$$\frac{dp_x}{dt} = \frac{dp'_x}{dt'}, \; \frac{dp_y}{dt} = \frac{1}{\gamma}\frac{dp'_y}{dt'}, \; \frac{dp_z}{dt} = \frac{1}{\gamma}\frac{dp'_z}{dt'}$$

Thus the transformation equations are

$$F_x = F'_x, \; F_y = \frac{F'_y}{\gamma}, \; F_z = \frac{F'_z}{\gamma} \qquad ...(31)$$

13.10. EQUILIBRIUM OF A RIGHT-ANGLED LEVER

The equilibrium of a right-angled lever provides an interesting example of the transformation of force from one system to another. In S-frame, the lever is in static equilibrium under the action of F_x and F_y forces as shown in Fig 13.4. Thus

$$F_x l_y = F_y l_x \qquad ...(32)$$

S' frame is moving with velocity v along X-axis. It is expected that to an observer at rest in S'-frame the lever should remain in equilibrium. The transformation equations for length and force are

$$l'_x = l_x\sqrt{1-v^2/c^2} \text{ and } l'_y = l_y \ ; \ F'_x = F_x \text{ and}$$

$$F'_y = F_y\sqrt{1-v^2/c^2} \qquad ...(33)$$

In S'-frame, the observer finds a net torque T' on the lever, given by

$$T' = F'_x l'_y - F'_y l'_x$$

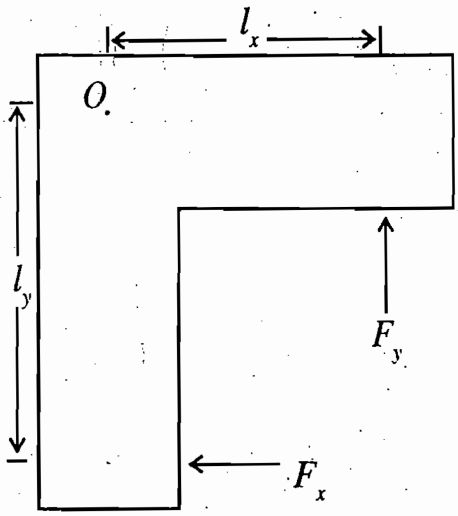


Fig. 13.4 : Equilibrium of a right-angled lever

or
$$T' = F_x l_y - (F_y\sqrt{1-v^2/c^2})(l_x\sqrt{l-v^2/c^2}) = \frac{F_x l_y v^2}{c^2} \qquad ...(34)$$

However, this torque results in no rotation because F'_x is doing the work at the rate $F'_x v = F_x v$. This is equivalent to $F_x v / c^2$ mass developed per unit time and hence a momentum per unit time $(F_x v / c^2)\, v$. This is the force developed in the frame S' whose moment or torque is

$$= \frac{F_x v^2}{c^2} l_y \qquad ...(35)$$

The two torques balance each other and hence the static equilibrium of the right angled lever is explained.

Ex. 1. *An electron and a positron practically at rest come together and annihilate each other. Calculate the energy released.*

Solution : Rest mass of the electron and positron

$$M_0 = 2m_e = 2 \times 0.9 \times 10^{-30} \text{ kg.}$$

Energy released $M_0c^2 = 2 \times 0.9 \times 10^{-30} \times 9 \times 10^{16} = \mathbf{1.62 \times 10^{-13}\ J.}$

Ex. 2. *The rest masses of a proton and a neutron are* 1.6725×10^{-27} *kg* and 1.6748×10^{-27} *kg. Calculate the binding energy.*

Solution : A nucleus of deutron consists of one proton and a neutron

$$\Delta m = \text{(Mass of one proton + one neutron)} - \text{Mass of the deutron}$$

$$= (1.6725 + 1.6748) \times 10^{-27} - 3.3433 \times 10^{-27} = 4 \times 10^{-30} \text{ kg.}$$

Therefore, binding energy $\Delta E = \Delta mc^2$

$$= 4 \times 10^{-30} \times (3 \times 10^8)^2 = 3.6 \times 10^{-13} \text{ joule}$$

$$= \frac{3.6 \times 10^{-13}}{1.6 \times 10^{-19}} \text{ eV} = \mathbf{2.25\ MeV.}$$

Ex. 3. *Find the energy in electron volts released when a neutron decays into a proton and an electron.*

$$(m_n = 1.6747 \times 10^{-27} kg;\ m_p = 1.6725 \times 10^{-27} kg;\ m_e = 0.9 \times 10^{-30} kg.)$$

Solution : Change of mass in the decaying process can be represented as

$$\Delta m = \text{Mass of the neutron} - \text{(Mass of proton + Mass of electron)}$$

$$= 1.6747 \times 10^{-27} - [1.6725 \times 10^{-27} + 0.9 \times 10^{-30}]$$

$$= 1.6747 \times 10^{-27} - 1.6734 \times 10^{-27} = 0.0013 \times 10^{-27} \times (3 \times 10^8)^2 \text{ kg.}$$

Hence, energy liberated

$$\Delta E = \Delta mc^2 = 0.0013 \times 10^{-27} \times (3 \times 10^8)^2 \text{ J} = 13 \times 9 \times 10^{-15} \text{ J}$$

$$= \frac{13 \times 9 \times 10^{-15}}{1.6 \times 10^{-19}} \text{eV} = \mathbf{7.3 \times 10^5 \text{ eV.}}$$

Ex. 4. *Calculate the speed of an electron which has kinetic energy 2 MeV.* **(Agra 1991)**

Solution : Kinetic energy $= (m - m_0)c^2 = 2$ MeV

Therfore $\quad m = m_0 + \frac{2\text{MeV}}{c^2} = 0.91 \times 10^{-30} + \frac{2 \times 10^6 \times 1.6 \times 10^{-19}}{(3 \times 10^8)^2}$

$$= 0.91 \times 10^{-30} + 3.45 \times 10^{-30} = 4.36 \times 10^{-30} \text{kg.}$$

But $\quad m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$ or $\frac{v^2}{c^2} = 1 - \left(\frac{m_0}{m}\right)^2$, therefore

$$v = c\sqrt{1 - \left(\frac{m_0}{m}\right)^2} = 3 \times 10^8 \sqrt{1 - \left(\frac{0.91 \times 10^{-30}}{4.36 \times 10^{-30}}\right)^2} = \mathbf{2.93 \times 10^8 \text{ m/sec.}}$$

Ex. 5. *The rest mass of a proton is* 1.6725×10^{-27} *kg. Find its mass and momentum, when it is moving with* 2.7×10^8 *m/sec velocity. If it collides with a stationary nucleus of mass* 2.7×10^{-26} *kg and coalesces, find the velocity of the combined particle.*

Solution : $m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.6725 \times 10^{-27}}{\sqrt{1 - \left(\frac{2.7 \times 10^8}{3 \times 10^8}\right)^2}} = \frac{1.6725 \times 10^{-27}}{\sqrt{1 - 0.81}} = 3.84 \times 10^{-27}$ kg.

Momentum $p = mv = 3.84 \times 10^{-27} \times 2.7 \times 10^8 = \mathbf{1.04 \times 10^{-18} \text{ kg-m/sec.}}$

If the velocity of the final particle is V, then using the law of conservation of momentum, we have

$$p = (m_0 + M)V$$

[As V will be much lower than c, hence we will consider only rest masses of the particles.]

Therfore, $\quad V = \frac{1.04 \times 10^{-18}}{(1.6725 \times 10^{-27} + 2.5 \times 10^{-26})} = \frac{1.04 \times 10^{-18}}{2.66725 \times 10^{-26}} = \mathbf{3.9 \times 10^7 \text{ m/sec.}}$

Ex. 6. *Express for the momentum of a photon in terms of wavelength* λ. *How much is the rest mass of the photon. Calculate the relativistic mass of the photon of wavelength* 5000 Å.

Solution : Momentum of the photon

$$p = \frac{E}{c} = \frac{h\nu}{c} = \frac{h}{\lambda} \qquad [\text{as } c = \nu\lambda]$$

Also $\quad p = \dfrac{m_0 v}{\sqrt{1 - v^2/c^2}}$ or $m_0 = \dfrac{p\sqrt{1 - v^2/c^2}}{v}$

For photon, $v = c$, therefore, $1 - v^2/c^2 = 1 - 1 = 0$; hence, $m_0 = 0$. Thus the rest mass of photon is **zero**.

Relativistic mass $m = \dfrac{p}{c} = \dfrac{h}{c\lambda} = \dfrac{6.62 \times 10^{-34}}{3 \times 10^8 \times 5000 \times 10^{-10}} = \mathbf{4 \times 10^{-36}\,kg}$.

Ex. 7. *Show that the rest mass of a particle of momentum p and kinetic energy T given by*

$$m_0 = \frac{p^2c^2 - T^2}{2Tc^2}. \qquad \textbf{(Agra 1998, 90)}$$

Solution : We know that the relativistic energy E is given by

$$E = E_k + E_0 \quad \text{or} \quad E = T + m_0c^2 \qquad ...(i)$$

Also $\quad E^2 = p^2c^2 + m_0^2c^4 \qquad ...(ii)$

Squaring (i) and equating with (ii), we get

$$T^2 + m_0^2c^4 + 2Tm_0c^2 = p^2c^2 + m_0^2c^4 \quad \text{or} \quad m_0 = \frac{p^2c^2 - T^2}{2Tc^2}.$$

Ex. 8. *Show that if E and p are relativistic energy and momentum in S-frame, than*

$$E'^2 - p'^2c^2 = E^2 - p^2c^2 = m_0^2c^4$$

where E' and p' are the corresponding quantities in S'-frame. Prove that $E^2 = p^2c^2 + m_0^2c^4$ *is invariant under Lorentz transformations.*

Solution : The transformation of energy and momentum from S to S' frame are

$$p'_x = \gamma(p_x - vE/c^2),\, p'_y = p_y,\, p'_z = p_z \text{ and } E' = \gamma(E - vp_x)$$

The relation $E^2 = p^2c^2 + m_0^2c^4$ is Lorentz invariant means that for a particle in S-frame if

$$E^2 = p^2c^2 + m_0^2c^4,$$

then for the same particle in S'-frame, moving with constant velocity relative to S, we have

$$E'^2 = p'^2c^2 + m_0^2c^4$$

In other words

$$E'^2 - p'^2c^2 = E^2 - p^2c^2 = m_0^2c^4$$

Applying transformation relations

$$E'^2 - p'^2c^2 = E'^2 - c^2\left(p'^2_x + p'^2_y + p'^2_z\right)$$

$$= \gamma^2(E - vp_x)^2 - c^2\left[\gamma^2\left(p_x - \frac{vE}{c^2}\right)^2 + c^2p_y^2 + c^2p_z^2\right]$$

$$= -p_x^2\gamma^2c^2\left(1 - \frac{v^2}{c^2}\right) + \gamma^2E^2\left(1 - \frac{v^2}{c^2}\right) + c^2p_y^2 + c^2p_z^2$$

$$= E^2 - c^2(p_x^2 + p_y^2 + p_z^2) = E^2 - p^2c^2$$

Thus $\quad E'^2 - p'^2c^2 = E^2 - p^2c^2 = m_0^2c^4$

Ex. 9. *A π -meson of rest mass m_π decays into a μ -meson of mass m_μ and a neutrino of mass m_ν. Show that total energy of the μ -meson is*

$$\frac{1}{2m_\pi}\left[m_\pi^2 + m_\mu^2 + m_\nu^2\right]c^2$$

Solution : π -meson m -meson + neutrino (ν)

Thus $\quad E_\pi = m_\pi c^2 = E_\mu + E_\nu$

The π -meson is at rest initially and after decay, μ -meson and neutrino will have equal and opposite momentum (p and $-p$) so that final net momentum is zero. In view of momentum energy relations

$$E_\mu^2 = p^2c^2 + m_\mu^2c^4, E_\nu^2 = p^2c^2 + m_\nu^2c^4$$

where m_μ and m_ν are the rest masses.

Therefore, $E_\mu^2 - E_\nu^2 = (m_\mu^2 - m_\nu^2)c^4$

Also $\quad E_\mu + E_\nu = m_\pi c^2 \quad$ ...(*i*)

Hence, $\quad \dfrac{E_\mu^2 - E_\nu^2}{E_\mu + E_\nu} = \dfrac{(m_\mu^2 - m_\nu^2)c^2}{m_\pi}$ or $E_\mu - E_\nu = \dfrac{(m_\mu^2 - m_\nu^2)c^2}{m_\pi} \quad$ ...(*ii*)

From eqs. (*i*) and (*ii*), we get

$$E_\mu = \frac{1}{2m_\pi}\left[m_\pi^2 + m_\mu^2 - m_\nu^2\right]c^2.$$

Ex. 10. *An excited atom of total mass M, at rest with respect to an inertial frame, goes over into a lower state with an energy smaller by . It emits a photon and thereby undergoes a recoil. The frequency of the photon will not be exactly $\nu = \Delta W/h$ but smaller. Compute this frequency.* **(Agra 1995)**

Solution : Momentum of the emitted photon $p = h\nu/c$.

As the atom is at rest initially in the inertial frame, it will get a backward momentum $-p$ on the emission of the photon. Thus the loss of energy used in recoiling the nucleus

$$\Delta W_1 = \frac{1}{2}m_0v^2 = \frac{m_0^2v^2}{2m_0} = \frac{p^2}{2m_0} = \frac{1}{2m_0}\left(\frac{h\nu}{c}\right)^2$$

where we have assumed that the mass of the atom is not changed by the emission of photon.

Emitted energy of the photon $= h\nu$

Therefore, $\Delta W = h\nu + \dfrac{1}{2m_0}\left(\dfrac{h\nu}{c}\right)^2 = h\nu\left[1 + \dfrac{h\nu}{2m_0c^2}\right]$ or $h\nu = \Delta W\left[1 + \dfrac{h\nu}{2m_0c^2}\right]^{-1} = \Delta W\left[1 - \dfrac{h\nu}{2m_0c^2}\right]$,

or $\quad h\nu\left[1 + \dfrac{\Delta W}{2m_0c^2}\right] = \Delta W$ or $h\nu = \Delta W\left[1 + \dfrac{\Delta W}{2m_0c^2}\right]^{-1} = \Delta W\left[1 - \dfrac{\Delta W}{2m_0c^2}\right]$

Therefore $$\nu = \frac{\Delta W}{h} - \frac{(\Delta W)^2}{2m_0hc^2}$$

Thus, the frequency of the photon is smaller than $\Delta W/h$ by an amount $[(\Delta W)^2/2m_0hc^2]$.

Ex. 11. *Show that a photon cannot give rise to an electron-positron pair in free space in the absence of an external field.* **(Agra 1995, Rohilkhand 80)**

Solution : Suppose that a photon of energy $E = h\nu$ produces an electron-positron pair in free space. Let the velocity of the each particle be v and mass m_0 (treated as non-relativistic). Principle of conservation of energy gives

$$E = \tfrac{1}{2}m_0v^2 + \tfrac{1}{2}m_0v^2 + 2m_0c^2 = m_0v^2 + 2m_0c^2 \qquad \text{...(i)}$$

Principle of conservation of momentum gives

$$\frac{h\nu}{c} = m_0v + m_0v \text{ or } \frac{E}{c} = 2m_0v \text{ or } E = 2m_0vc \qquad \text{...(ii)}$$

where for convenience, we assume that the two particles move along the direction of the photon.

From (*i*) and (*ii*), we obtain

$$m_0v^2 + 2m_0c^2 = 2m_0vc \quad \text{or} \quad v^2 - 2vc + 2c^2 = 0$$

This gives $v = c \pm ic$

Thus the velocity of any of the created particles is not real, indicating the process to be imaginary. Thus a photon cannot produce an electron-positron pair in free space in the absence of an external agency.

13.11. THE LAGRANGIAN AND HAMILTONIAN OF A PARTICLE IN RELATIVISTIC MECHANICS

In the non-relativistic mechanics, the canonical momentum components (in cartesian coordinate system) of a particle are given by

$$p_x \frac{\partial L}{\partial \dot{x}} = \frac{m_0\dot{x}}{\sqrt{1-u^2/c^2}} + qAx \qquad \text{...(36)}$$

where L is the Lagrangian.

In relativistic mechanics, we assume a similar definition for the momentum components, given by

$$p_x = \frac{m_0\dot{x}}{\sqrt{1-\beta^2}} = \frac{\partial L}{\partial \dot{x}} \qquad \text{...(37)}$$

$$p_y = \frac{m_0\dot{y}}{\sqrt{1-\beta^2}} = \frac{\partial L}{\partial \dot{y}} \qquad \text{...(38)}$$

$$p_z = \frac{m_0\dot{z}}{\sqrt{1-\beta^2}} = \frac{\partial L}{\partial \dot{z}} \qquad \text{...(39)}$$

where $\beta = \frac{u}{c}$ and $u^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$; u is the speed of the particle in the Lorentz (inertial) frame under consideration.

Integrating (37), we get

$$L = -m_0c^2\sqrt{1-\beta^2} - V \qquad ...(40)^*$$

where V is the constant of integration and may be taken as the potential energy of the particle as a function coordinates [$V = V(x, y, z)$] only. The justification of this assumption and correctness of the form of the Lagrangian L in (40) can be shown, when we use the definition (40) of the Lagrangian in the Lagrange's equations and obtain the correct relativistic equation of motion of the particle $(F_x = dp_x/dt)$.

The Lagrange's equation for x coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial x} = 0. \qquad ...(41)$$

From (40) $\dfrac{\partial L}{\partial x} = -\dfrac{\partial V}{\partial x}$

and $$\frac{\partial L}{\partial \dot{x}} - \frac{\partial}{\partial \dot{x}}(-m_0c^2\sqrt{1-\beta^2 - V}) - m_0c^2\frac{\partial}{\partial \dot{x}}\left(\sqrt{1-\frac{\dot{x}+\dot{y}^2+\dot{z}^2}{c^2}}\right) = \frac{m_0\dot{x}}{\sqrt{1-\beta^2}} = p_x$$

Thus from (41)

$$\frac{dp_x}{dt} + -\frac{\partial V}{\partial x} = 0 \text{ or } \frac{dp_x}{dt} = -\frac{\partial V}{\partial x} = F_x$$

It is to be noted that in relativistic mechanics the Lagrangian is no longer equal to $(T - V)$.

We may extend the definition (40) of the Lagrangian to a system of many particles and change from cartesian coordinates to any generalized set of coordinates q_i. The canonical momenta p_i will be defined similar to nonrelativistic case as

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad ...(42)$$

so that the relation between the cyclic coordinates and conservation of the corresponding momenta remains the same.

The definition of the **Hamiltonian** is given similar to the non-relativistic case as

$$H = \sum p_i\dot{q}_i - L \qquad ...(43)$$

If L does not contain the time explicitly, the Hamiltonian H represents the constant of motion. For a single particle moving under conservative force, the Hamiltonian is given by

$$H = \sum_{\alpha=x,y,z} p_\alpha\dot{\alpha} - L$$

$$= p_x\dot{x} + p_y\dot{y} + p_z\dot{z} + m_0c^2\sqrt{1-\beta^2} + V$$

$$= \frac{m_0\dot{x}^2}{\sqrt{1-\beta^2}} + \frac{m_0\dot{y}^2}{\sqrt{1-\beta^2}} + \frac{m_0\dot{z}^2}{\sqrt{1-\beta^2}} + m_0c^2\sqrt{1-\beta^2} + V$$

$$= \frac{m_0v^2}{\sqrt{1-\beta^2}} + m_0c^2\sqrt{1-\beta^2} + V = \frac{m_0c^2}{\sqrt{1-\beta^2}} + V$$

$$= mc^2 + V = E + V = T + E_0 + V = \text{Total energy} \qquad ...(44)$$

$$^*L = \int \frac{m_0\dot{x}d\dot{x}}{\sqrt{1-u^2/c^2}} \text{ with } u^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2.$$

Thus in case of a single particle moving in a conservative force field, the Hamiltonian H still represents the total energy. However, the identification of H with energy for a Lagrangian of the form (40) cannot proceed along the same lines of nonrelativistic case because neither L is equal to $T-V$, nor $\sum p_i\dot{q}_i$ is equal to $2T$.

One may express the single particle Hamiltonian in the form given below :

$$H = mc^2 + V = E + V$$

But $\quad E^2 = p^2c^2 + m_0^2c^4$, hence

$$H = \sqrt{p^2c^2 + m_0^2c^4} + V \qquad ...(45)$$

13.12. RELATIVISTIC LAGRANGIAN AND HAMILTONIAN OF A CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD–VELOCITY DEPENDENT POTENTIALS

Let us consider a charged particle moving with velocity **u** in an electromagnetic field. The force acting on a particle with charge q in electric field **E** and magnetic induction **B** is given by

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{u} \times \mathbf{B}) \qquad ...(46)$$

The x- component of the force can be expressed [Chapter 2, eq. (65)] as

$$F_x = \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}}\right) - \frac{\partial U}{\partial x} \qquad ...(47)$$

where $U = q(\phi - \mathbf{u} \cdot \mathbf{A})$ is the velocity dependent potential; **A** and ϕ are vector and scalar potentials respectively ($\mathbf{B}$ = curl $\mathbf{A}$, $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} + \text{grad}\,\phi$).

Therefore, the Lagrangian for a charged particle in an electromagnetic field in the relativistic case is given by

$$L = -m_0c^2\sqrt{1-\beta^2} - U \quad \text{or} \quad L = -m_0c^2\sqrt{1-u^2/c^2} - q\phi + q(\mathbf{u} \cdot \mathbf{A}) \qquad ...(48)$$

In such a case of the velocity dependent potential, the canonical momenta are

$$p_x \frac{\partial L}{\partial x} = \frac{\mathrm{m}_0\dot{\mathrm{x}}}{\sqrt{1-u^2/c^2}} + qAx \qquad ...(49)$$

Hence the Hamiltonian H is obtained as

$$H = \sum_{\alpha = x,y,z} p_\alpha\dot{\alpha} - L = P_x\dot{x} + P_y\dot{y} + P_z\dot{z} + m_0c^2\sqrt{1-\beta^2} + q\phi - q(\mathbf{u}.\mathbf{A})$$

$$= \frac{m_0u^2}{\sqrt{1-\beta^2}} + m_0c^2\sqrt{1-\beta^2} + q\phi = \frac{m_0c^2}{\sqrt{1-\beta^2}} + q\phi = mc^2 + q\phi$$

$$= E + q\phi = E' \text{ (total energy)} \qquad ...(50)$$

But $\quad E = \sqrt{p^2c^2 + m_0^2c^4}$

Also $\quad P_x = \frac{m_0\dot{x}}{\sqrt{1-u^2/c^2}} + qA_x = m\dot{x} + qA_x = p_x + qA_x$

or $\quad \mathbf{P} = \mathbf{p} + \mathbf{q}\,\mathbf{A}$, *i.e.*, $\mathbf{p} = \mathbf{P} - q\,\mathbf{A} \qquad ...(51)$

Thus the relativistic Hamiltonian for a charged particle in an electromagnetic field has the form

$$H = \sqrt{(\mathbf{P} - q\mathbf{A})^2 c^2 + m_0^2 c^4} + q\phi \qquad ...(52)$$

where **P** is the vector of the canonical momenta conjugate to the cartesian position coordinates of the particle. In fact, H represents the total energy and this form of the Hamiltonian serves as the starting point of the relativistic quantum theory of the electron.

Questions

1. Establish the relation giving variation of mass with velocity of a particle. **(Agra 2001, 1999, 95, 92)**
2. Using the special theory of relativity derive the expression for the momentum and kinetic energy of a particle in motion. Hence obtain the relation between energy, momentum and rest mass. **(Agra 1994)**
3. Prove the relativistic formula $m = m_0 / \sqrt{1 - v^2 / c^2}$ and then $E = mc^2$. **(Garwal 1993)**
4. What is mass-energy equivalence ? Obtain the relation $E = mc^2$. **(Delhi 1990)**
5. Establish the mass-energy relation $E = mc^2$. Show that in special theory of relativity, a particle has a definite energy corresponding to a specific value of its momentum. **(Agra 1995)**
6. Derive relativistic expression for the kinetic energy of a particle. Show that it reduces to the classical expression $\frac{1}{2}mv^2$ when $v << c$. **(Agra 1971; Lucknow 80)**
7. What is relativistic energy ? Prove the relation $E^2 - p^2c^2 = m_0^2 c^4$.

   Derive an expression for the velocity of a particle in terms of its relativistic momentum and energy. **(Agra 1981, 78; Allahabad 84)**
8. Find the relativistic expression for the kinetic energy of a particle of rest mass m_0, moving with velocity **u**. Obtain the expression

   $E^2 = p^2c^2 + m_0^2 c^2$ **(Delhi 1989; Agra 92)**
9. Prove that

   (*i*) Momentum of a particle of velocity **v** and relativistic energy E is given by $\mathbf{p} = E\mathbf{v} / c^2$.

   (*ii*) $E^2 - p^2c^2$ is invariant under Lorentz transformation. **(Mumbai 2002; Kanpur 1981)**
10. Obtain the relativistic energy and momentum transformation equations. **(Delhi 1990)**
11. Comment on mass-energy equivalence. Discuss the observations which support this concept. Explain why an electron-positron pair cannot be created without the presence of nucleus. **(Agra 1995)**
12. Discuss the equilibrium of a right angled lever in a moving inertial frame. **(Agra 1998, 93)**
13. Assuming the law of conservation of momentum to be correct in every inertial frame, show that by the use of transformation of energy and momentum, the relativistic energy is conserved in a two particle system.
14. Discuss the relativistic Lagrangian formulation of relativistic mechanics. **(Meerut 1999)**
15. Prove that the relativistic Hamiltonian is equal to the total energy of the system.

Problems

[SET- I]

1. By what factor is the density of an object increased when it is moving with velocity v ?

   Ans. : By the factor $[1/(1 - v^2/c^2)]$.

2. The velocity of an object is such that its mass increases by 10%.
(*i*) By what fraction does its length in the direction of motion decrease?
(*ii*) If its rest energy is W_0, what is its kinetic energy ?
Ans. : (*i*) 9.1%, (*ii*) $W_0/10$.
3. A body of specific heat 0.2 is heated to the temperature 100°C. Calculate the percentage by which the mass increases. **(Kanpur 1979)**
Ans. : 9×10^{-11} %.
4. Find the velocity of an electron accelerated through a voltage of 10^5 volts. (Assume rest energy of the electron = 0.5 MeV).
Ans. : 1.66×10^8 m/sec.
5. Calculate the speed of an electron accelerated to potential of one million electron volts. **(Agra 1979)**
Ans. : 0.94 c.
6. At what velocity the mass of a particle will be double of its rest mass ?
Ans. : $2.6\times10^8\, m/\sec$. **(Kanpur 1988; Lucknow 79)**
7. Calculate the velocity of a particle when its rest mass energy is double to its kinetic energy. **(Kanpur 1989)**
Ans. : 0.79 c.
8. How many kilowatt-hours of energy would be liberated by the complete conversion of 4 mgm. mass ?
Ans. : 10^5 KWh.
9. Calculate the increase in total rest mass, if two particles each of mass 1 gm are moving with 1.5 km/sec velocity in opposite directions collide and come to rest.
Ans. : 2.5×10^{-14} kg.
10. What is the kinetic energy of a proton (rest mass = 1.67×10^{-27} kg) which is moving with velocity 2.7×10^8 m/sec. (Give your answer in eV) **(Lucknow 1980)**
Ans. : 1746 MeV.
11. Calculate the rest mass of a particle of kinetic energy 50 MeV and momentum 130 MeV/c.
Ans. : 2.56×10^{-28} kg.
12. Calculate the amount of work done in accelerating an electron from rest to a velocity 2.4×10^8 m/sec. (Assume the rest energy of the electron = 0.5 MeV).
Ans. : 0.335 MeV.
13. A particle at rest breaks into two particles of rest masses in the ratio 1 : 2. If the heavier particle moves with velocity 1.8×10^8 m/sec, find the velocity of the lighter particle ? Find also the velocity of the lighter particle relative to the heavier one.
Ans. : 2.5×10^8 m/sec. 2.8×10^8 m/sec.
14. If a particle of rest mass m_0, moving with velocity v, collides and sticks with a stationary particle of rest mass M_0, show that the speed of the composite particle will be given by $\gamma m_0 v/(M_0+\gamma m_0)$.
15. Calculate the momentum of a photon of energy 10^{-19} J.
Ans. : 3.3×10^{-28} kg-m/sec.
16. An electron travelling with speed 2.70×10^8 m/sec experiences a force 2.64×10^{-15} newton. Deduce the acceleration. **(Agra 1970 S)**
Ans. : 1.27×10^{15} m/sec^2 if we consider the particle in circular motion; 2.12×10^{14} m/sec^2 for rectilinear motion.

17. Calculate the recoil momentum in the laboratory of an Fe nucleus recoiling due to the emission of a 14 KeV photon? Is the momentum of the nucleus is relativistic ?

    Ans. : 7.5×10^{-27} kg-m/sec.

18. Deduce the minimum energy of gamma-ray photons (in MeV) which can cause electron-positron pair production. $(m_0 = 9.1 \times 10^{-31}$ kg). **(Agra 1980, 71; Delhi 90)**

    Ans. : 1.02 MeV.

19. If E and p are the relativistic energy and relativistic momentum of a particle respectively, show that the velocity v of the particle is given by $v = dE/dp$.

20. Calculate the rest mass of a particle whose momentum is 130 MeV/c when its kinetic energy is 50 MeV.

    Ans. : 2.56×10^{-28} kg.

21. Show that a particle with rest mass zero travels with speed of light.

22. An excited nucleus of rest mass m_0 is at rest with respect to a chosen inertial frame. It goes over to the lower state whose energy is smaller by ΔE. As a result it emits a γ-ray photon and undergoes a recoil. Show that the frequency of γ-ray photon is given by

$$\nu = \frac{\Delta E}{h}\left[1 - \frac{\Delta E}{2m_0c^2}\right]$$ **(Meerut 1993; Agra 1972)**

23. Show that the de Broglie wavelength for a material particle of rest mass m_0 and charge q accelerated from rest through a potential difference V relativistically is given by

$$\lambda = \frac{h}{\sqrt{2m_0qV(1 + qV/2m_0c^2)}}$$

    [**Hint :** $E = T + m_0c^2 = qV + m_0c^2$ and $E^2 = p^2c^2 + m_0^2c^4$.

    Hence $p = \sqrt{2m_0qV(1 + qV/2m_0c^2)}$ and $\lambda = h/p$].

24. Calculate the de Broglie wevelength of an electron having a kinetic energy 1 MeV. **(Kanpur 1976)**

    Ans. : 86×10^{-3} Å.

25. Find the velocity that an electron must be given so that its momentum is 10 times its rest mass times the speed of light. What is the energy at this speed ? **(Agra 1996)**

    Ans. : $v = 0.995c$; 8.14×10^{-13} J

[SET- II]

1. Show that the transformation equation for force components from S'- frame to S-frame are

$$F_x = \frac{F_x' + (v/c^2)\mathbf{u}'.\mathbf{F}'}{1 + u_x'v/c^2}, \quad F_y = \frac{F_y'}{\gamma(1 + u_x'v/c^2)}, \quad F_z = \frac{F_z'}{\gamma(1 + u_x'v/c^2)}$$

   where $\mathbf{u}'$ is the velocity in S'-frame.

2. (*a*) Show that the force on a particle, having instantaneous velocity $\mathbf{u}$, can be expressed as

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} + \frac{\mathbf{u}(\mathbf{F} \cdot \mathbf{u})}{c^2}$$

Further show that when $\mathbf{F}$ is parallel to $\mathbf{u}$,

$$F = \frac{m_0 a}{(1-u^2/c^2)^{3/2}} \quad \left(a = \frac{du}{dt}\right)$$

and when **F** is perpendicular to **u**,

$$\mathbf{F} = m\mathbf{a}\cdot$$

(*b*) Show that although **a** (= $d\mathbf{u}/dt$) and **F** are not parallel, the angle between them is always less than 90°.

3. Show that in relativistic mechanics the speed v and distance x travelled by a constant force F will be given by

$$v = \frac{c(F/m_0c)t}{\sqrt{1+(F/m_0c)^2 t^2}} \quad \text{and} \quad x = \frac{m_0c^2}{F}\left[\sqrt{1+\left(\frac{F}{m_0c}\right)^2 t^2} - 1\right]$$

where the particle starts from rest at $x = 0$, $t = 0$.

4. Show that in a region in which there is uniform magnetic field **B**, a particle of charge q entering at right angles to the field moves in a circle of radius r, given by

$$r = \frac{m_0 u}{qB\sqrt{1-u^2/c^2}} = \frac{p}{qB}$$

where $p = mu$ is the momentum. Hence compute the radius of 10 MeV electron moving at right angles to a uniform magnetic field of strength 2 weber/m^2.

Ans. : 1.8 cm.

5. The nucleus of a carbon atom initially at rest in the laboratory goes from one state to another by emitting a photon of energy 4.43 MeV. The atom in its final state has rest mass of 12.00 amu. (*a*) Determine the momentum of the carbon atom after the decay, as measured in the laboratory frame. (*b*) What is the kinetic energy (in MeV) of the carbon atom after the decay as measured in the laboratory system ? (1 *amu* = 931.478 MeV)

Ans. : (*a*) 2.36×10^{-21} kg-m/sec (*b*) 8.78×10^{-4} MeV.

6. A gamma ray, passing near a nucleus, creates an electron-positron pair, which enter a magnetic field of intensity 0.1 weber/m^2. The magnetic field is perpendicular to the flight paths of both particles, which are observedto be circles of radius 4 cm and 10 cm respectively. Find the energy of the incident gamma ray.

Ans. : 4.38 MeV.

7. The photon energy in the frame S is equal to E. Find its energy E' in frame S', moving with a velocity v relative to the frame S in the photon's motion direction. At what value of v is the energy of the photon equal to $E' = E/2$.

Ans. : $E' = E\sqrt{(1-\beta)/(1+\beta)}$, where $\beta = v/c$, $v = \frac{3}{5}c$.

8. A beam of relativistic particles with kinetic energy E_k strike an absorbing target. The beam current equals I and the charge and rest mass of each particle are equal to e and m_0 respectively. Find the pressure developed by the beam on the target surface and the power liberated there.

Ans : $(l/ec)\sqrt{E_k(E_k+2m_0c^2)}$; $E_k l/e$.

9. A neutron with kinetic energy $E_k = 2m_0c^2$, where m_0 is its rest mass, strikes another stationary neutron. Find the combined kinetic energy of both neutrons in the frame of their centre of mass and the momentum of each neutron in that frame. Find also the velocity of centre of mass of this system.

Ans. : $E_k' = 2m_0c^2\ (1+\sqrt{E_k/2m_0c^2}-1) = 777$ MeV;

$$p' = \sqrt{\frac{1}{2}m_0E_k} = 940 \text{ MeV/c} \quad v = c\sqrt{E_k/(E_k+2m_0c^2)} = 2.12\times10^8 \text{ m/s.}$$

Objective Type Questions

1. The expression for the relativistic energy of a particle is

(*a*) mc^2 (*b*) $\sqrt{p^2c^2+m_0^2c^4}$

(*c*) $(m - m_0)c^2$ (*d*) $p^2c^2+m_0^2c^4$

Ans. : (*a*), (*b*).

2. An electron gains energy so that its mass becomes $2m_0$. Its speed is

(*a*) $\frac{\sqrt{3}}{2}c$ (*b*) $\frac{3}{4}c$

(*c*) $\frac{3}{2}c$ (*d*) $\sqrt{\frac{3}{2}}c$ **(GATE 2004)**

Ans. : (*a*).

3. Choose the correct statement/s :

(*a*) The rest mass of a photon of frequency ν is zero.

(*b*) The rest mass of a photon of frequency ν is $h\nu/c^2$.

(*c*) The relativistic mass of a photon of frequency is zero.

(*d*) The relativistic mass of a photon of frequency is $h\nu/c^2$.

Ans. : (*a*), (*d*).

4. The relativistic energy of a particle

(*a*) is independent of the frame of reference.

(*b*) is different in different inertial frames.

(*c*) depends on the momentum of the particle.

(*d*) is independent of the momentum of the particle.

Ans. : (*b*), (*c*).

5. Suppose a particle be instantaneons at rest in frame S'; frame S' is moving with speed v relative to S along X-axis. The components of force in the two frames are related as

(*a*) $F_x = F'_x/\sqrt{1-v^2/c^2}$, $F_y = F'_y$, $F_z = F'_z$

(*b*) $F_x = F'_x$, $F_y = F'_y/\sqrt{1-v^2/c^2}$, $F_z = F'_z/\sqrt{1-v^2/c^2}$

(*c*) $F_x = F'_x$, $F_y = F'_y\sqrt{1-v^2/c^2}$, $F_z = F'_z\sqrt{1-v^2/c^2}$

(*d*) $F_x = F'_x$, $F_y = F'_y$, $F_z = F'_z$

Ans. : (*c*).

6. The law of conservation of momentum;

(*a*) is valid at relativistic speeds.

(*b*) is not valid at relativistic speeds.

(*c*) is valid at nonrelativistic speeds.

(*d*) is not valid at nonrelativistic speeds.

Ans. : (*a*), (*c*).

Short Type Questions

1. Show that the ultimate speed of any material particle is always less than the speed of light.
2. Show that the gain in kinetic energy of a particle corresponds to increase in its mass.
3. What is relativistic kinetic energy ? Show that at low speeds, the relativistic expression of kinetic energy leads to the classical result.
4. Show that the relativistic energy E and relativistic momentum p are related as $E^2 = p^2c^2 + m_0^2c^4$
5. Deduce the transformations of energy and momentum ?
6. Show that the particles, moving with speed of light, possess zero rest mass.
7. Show that in relativistic mechanics, in general the force is not along the direction of acceleration vector.
8. Discuss the equilibrium of right-angled lever in inertial frames.
9. Determine the velocity of a particle when its rest mass energy is double to its kinetic energy.
   Ans. : 2.37×110^8 *m/s*.
10. Fill in the blanks :
    (*i*) The annihilation of electron and positron results in the production................
    (*ii*) The transformation of energy from one inertial frame to another is.........................

Ans. : (*i*) γ-rays, (*ii*) $E' = \dfrac{E - p_x v}{\sqrt{1-v^2/c^2}}$.

CHAPTER 14

Four Dimensional Formulation—Minkowski Space

14.1. INTRODUCTION

In accordance with the two postulates of the special theory of relativity, namely the constancy of the speed of light in vacuum and the invariance of the basic laws of physics in inertial frames, we deduced earlier the Lorentz transformations. These transformations connect the space-time coordinates of an event in two inertial frames S and S' and are given by

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z \text{ and } t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, the frame S' is moving with constant velocity v along X-axis relative to the frame S.

We find that in relativistic mechanics, the space and time coordinates depend on each other. The time coordinate of one inertial system depends on both the space and time coordinates of another system $[t' = g\,(t - vx/c^2)]$. Therefore, instead of treating the space and time coordinates separately, it is natural to seek the way so that both the coordinates are dealt together similarly. In fact, H. Minkowski was the first to develop a procedure in which the time coordinate is treated similar to the three space coordinates.

14.2. MINKOWSKI SPACE AND LORENTZ TRANSFORMATIONS

Minkowski considered a four dimensional cartesian space in which the position is specified by three coordinates x, y, z and the time is referred by a fourth coordinate ict. If we write $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_4 = ict$, then an event is represented by the position vector (x_1, x_2, x_3, x_4) in this four dimensional space. Of course the fourth dimension, referring to time, is imaginary. This four dimensional space is called ***Minkowski*** or ***world space***. It is also referred as ***space-time continuum*** and sometimes briefly as ***four-space***. The square of the magnitude of the position vector in such a four-space has the form

$$s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = x^2 + y^2 + z^2 - c^2t^2 \qquad ...(1)$$

Lorentz transformations are designed so that the speed of light remains constant in S and S' inertial frames (S' is moving with constant velocity v relative to S) and this condition is equivalent to require that the position vector in the four-space is held invariant under the transformations, *i.e.*,

$$s'^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2$$

or

$$s^2 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

or

$$s^2 = \sum_{\mu=1}^{4} x_\mu'^2 = \sum_{\mu=1}^{4} x_\mu^2 \qquad ...(2)$$

This equation is analogous to the distance-preserving orthogonal transformation for rotation from one frame of reference to another in three dimensional space. Thus the coordinates, x_1, x_2, x_3, x_4, chosen above, form an orthogonal coordinate system in four dimensions and eq. (2) implies that the transformations which we are seeking, correspond to a rotation in a fourdimensional space. In fact, these orthogonal transformations in the four-dimensional Minkowski space are the Lorentz transformations.

Deduction of Lorentz Tansformations : In order to prove the statement that the Lorentz transformations can be regarded as orthogonal transformations due to rotation of axes in the Minkowski space, we deduce these transformations in the four-space.

The frame S' is moving with constant velocity v along X-axis relative to the inertial frame S and hence we may have

$$y' = y \quad \text{and} \quad z' = z \quad \text{or} \quad x'_2 = x_2 \quad \text{and} \quad x'_3 = x_3 \qquad ...(3)$$

Thus from (2), the transformations should be such that

$$x'^2_1 + x'^2_4 = x^2_1 + x^2_4 \qquad ...(4)$$

In order to keep this requirement, we consider two orthogonal coordinate systems X_1X_4 and $X'_1\,X'_4$ in the same plane (plane of the paper) with the same origin O. The axes of $X'_1\,X'_4$ system correspond to rotation θ with respect to those of X_1X_4 system, *i.e.*, the axes of the former coordinate system are inclined with the later through an angle θ. We observe that

$$OP^2 = x_1^2 + x_4^2 = x'^2_1 + x'^2_4$$

where the coordinates in two coordinate systems are related as

$$\left.\begin{aligned} x'_1 &= x_1\cos\theta + x_4\sin\theta \\ x'_4 &= -x_1\sin\theta + x_4\cos\theta \end{aligned}\right\} \qquad ...(5)$$

In matrix notation

$$\begin{pmatrix} x'_1 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} \qquad ...(6)$$

Also,
$$\left.\begin{aligned} x_1 &= x'_1\cos\theta - x'_4\sin\theta \\ x_4 &= x'_1\sin\theta + x'_4\cos\theta \end{aligned}\right\} \qquad ...(7)$$

When $\quad x'_1 = 0,\ x_1 = -x'_4\sin\theta$

and $\quad x_4 = x'_4\cos\theta$

So that
$$\tan\theta = -\frac{x_1}{x_4} = -\frac{x}{ict} = \frac{iv}{c} \qquad ...(8)$$

where $x'_1 = x' = 0$ corresponds to the coordinate of the point O' (S'-frame) relative to O (S-frame); *i.e.*, $x = vt$ or $\dfrac{x}{t} = v$.

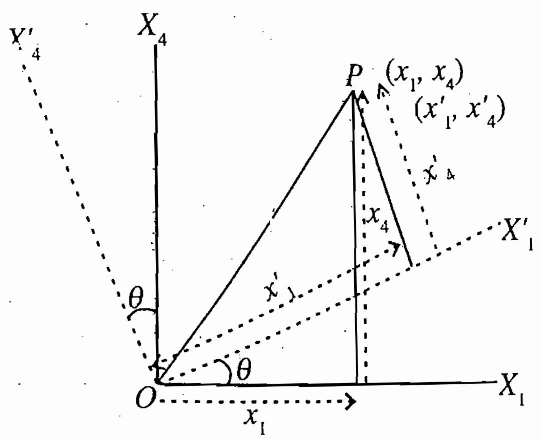


Fig. 14.1 : Rotation of orthogonal coordinates axes and invariance of $OP_2 = x_1^2 + x_4^2 - x'^2_1 + x'^2_4$

Therefore from (8),

$$\sin\theta = \frac{iv/c}{\sqrt{1-v^2/c^2}} = \frac{i\gamma v}{c} \text{ and } \cos\theta = \frac{1}{\sqrt{1-v^2/c^2}} = \gamma \quad \text{(say)}$$

Hence eqs. (5) can be expressed as

$$x'_1 = \gamma x_1 + i\gamma \frac{v}{c} x_4 = \gamma\left(-x_1 + i\frac{v}{c}x_4\right) \text{ and } x'_4 = -i\gamma\frac{v}{c}x_1 + \gamma x_4 = \gamma\left(-\frac{iv}{c}x_1 + x_4\right).$$

If we add $x'_2 = x_2$, and $x'_3 = x_3$, the transformation equations are

$$x'_1 = \gamma\left(x_1 + i\frac{v}{c}x_4\right), x'_2 = x_2, x'_3 = x_3, \text{ and } x'_4 = \gamma\left(-i\frac{v}{c}x_1 + x_4\right) \quad ...(9)$$

In fact, these are the Lorentz transformations. This may be seen by putting $x_1 = x, x_2 = y, x_3 = z$ and $x_4 = ict$ in eq. (9), *i.e.*,

$$x' = \gamma(x - vt),\ y' = y,\ z' = z \text{ and } t' = \gamma\left(t - vx/c^2\right) \quad ...(10\,a)$$

In matrix notation, the Lorentz transformations from S-frame to S'-frame can be represented as

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad ...(10b)$$

or $$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \quad ...(10c)$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. and $a_{\mu\nu}$ are the elements of the above square matrix .

The inverse Lorentz transformations are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} \quad ...(11a)$$

or $$x_\mu = \sum_{\nu=1}^{4} a_{\nu\mu} x'_\nu \quad ...(11b)$$

because $\sum_\nu a_{\nu\mu} x'_\nu = \sum_\nu a_{\nu\mu} \sum_\lambda a_{\nu\lambda} x_\lambda = \sum_\lambda \sum_\nu a_{\nu\mu} a_{\nu\lambda} x_\lambda = \sum_\lambda \delta_{\mu\lambda} x_\lambda = x_\mu$.

Remember that for orthogonal transformations

$$\sum_{\nu} a_{\mu\nu} a_{\lambda\nu} = \sum_{\nu} a_{\nu\mu} a_{\nu\lambda} = \delta_{\mu\lambda}$$

Here x_μ and x'_μ satisfy the condition (2), *i.e.*,

$$\sum_{\mu=1}^{4} x'^{2}_{\mu} = \sum_{\mu=1}^{4} x^2_\mu$$

The four coordinates x_1, x_2, x_3 and x_4 or x, y, z and ict, define the position vector in the four-space and may be termed as ***four-position vector***. We shall discuss more about four-vectors later.

14.3. WORLD POINT AND WORLD LINE

A physical event in Minkowski space is described by a point with four coordinates (x_1, x_2, x_3, x_4) $(x_4 = ict)$. This point in the four-space is called ***world point.*** In this space, the motion of a particle (*i.e.*, a particle at various instants) corresponds to a line, known as ***world line.*** A particle in uniform rectilinear motion corresponds to a straight world line. The relative position (in space-time) of one event with respect to another would be represented by ***line element***, joining the two events.

In order to show the interdependence of space and time more clearly and to represent them geometrically, we consider only one space axis, X-axis and ignore Y and Z axes. The time axis is represented perpendicular to X-axis by $T = ct$, so that the dimensions of the coordinates are the same.

The Lorentz transformations for x and t are

$$x' = \gamma(x - vt) \quad \text{or} \quad x' = \gamma(x - \beta T) \qquad ...(12)$$

and

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{or} \quad T' = \gamma(T - \beta x), \qquad ...(13)$$

where $\beta = v/c$.

We observe that *the Lorentz transformations for space and time in this form possess symmetry*. In this X-T coordinate system in the Minkowski space, we have represented the motion of a particle by a world line [Fig.14.2]. The inclination α of a tangent at any point E of a world line is given by

$$\tan\alpha = \frac{dx}{dT} = \frac{dx}{cdt} = \frac{u}{c} \qquad ...(14)$$

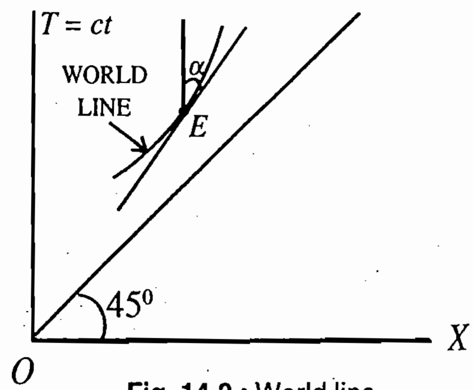


Fig. 14.2 : World line

where we must have $u < c$ for a material particle. This means that $\alpha < 45^\circ$ for a material particle. If the particle velocity (u) is constant, $\tan\alpha$ is also constant. Hence the world line for a particle moving with constant velocity is a straight line. For light signal, $u = c$ and therefore $\alpha = 45^\circ$. Thus the world line for light signal is a straight line making an angle 45° with the X-axis.

14.4. SPACE-TIME INTERVALS

Let us consider two inertial frames S and S'. Frame S' is moving with constant velocity v along X-axis relative to S. In system S, the square of the interval s_{12} between two events $E_1(x_1, y_1, z_1, t_1)$ and $E_2(x_2, y_2, z_2, t_2)$ is defined by

$$s_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 \qquad ...(15)$$

In S'-frame if the same two events have the coordinates (x_1, y_1, z_1, t_1) and (x'_2, y'_2, z'_2, t'_2), the square of the interval s'^2_{12} is

$$s'^2_{12} = (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c^2(t'_2 - t'_1)^2 \qquad ...(16)$$

The interval s_{12} may be looked upon as a measure of separation between the two events in the Minkowski four dimensinal space. This interval is invariant under Lorentz transformations from one inertial system to another. This means that

$$s'^2_{12} = s^2_{12} \qquad ...(17)$$

The proof is given below :

According to Lorentz transformations

$$x'_1 = \gamma\ (x_1 - vt_1),\ y'_1 = y_1,\ z'_1 = z_1 \quad \text{and } t'_1 = \gamma\,(t_1 - vx_1/c^2)$$

and
$$x'_2 = \gamma\ (x_2 - vt_2),\ y'_2 = y_2,\ z'_2 = z_2 \text{ and } t'_2 = \gamma\ (t_2 - vx_2/c^2)$$

Therefore,

$$x'_2 - x'_1 = \gamma[(x_2 - x_1) - v(t_2 - t_1)],\ y'_2 - y'_1 = y_2 - y_1,\ z'_2 - z'_1 = z_2 - z_1$$

and
$$t'_2 - t'_1 = \gamma\ [(t_2 - t_1) - v\,(x_2 - x_1)/c^2]$$

Now,
$$\begin{aligned} s'^2_{12} &= (x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 - c^2(t'_2 - t'_1)^2 \\ &= \gamma^2[(x_2 - x_1) - v(t_2 - t_1)]^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &\qquad - c^2\gamma^2[(t_2 - t_1) - v(x_2 - x_1)/c^2]^2 \\ &= (x_2 - x_1)^2\gamma^2(1 - v^2/c^2) + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2\gamma^2(1 - v^2/c^2) \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2 = s^2_{12} \end{aligned}$$

We note from the above analysis that the time interval between two events is not the same in the two inertial systems (*i.e.*, $t'_2 - t'_1 = t_2 - t_1$) and the space interval is also different [because $(x'_2 - x'_1)^2 + (y'_2 - y'_1)^2 + (z'_2 - z'_1)^2 \neq (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$]. Thus in the special theory of relativity, the time and space intervals are not invariant separately, while in Newtonian mechanics (Galilean transformations), the time interval and space interval between two events are separately invariant and do not depend on the frame of reference . However, we see that the space-time interval s_{12} in the theory of relativity is independent of the frame of reference and absolutely specifies the separation between two events in the Minkowski world. It may be noted that regarding the interval s_{12} between two events, nothing is to be observable and obviously it is a mathematical entity. In the theory of relativity, the time and space lose their mutually independent significance and in fact they occur as constituents of the more fundamental entity, space-time.

Three dimensions of space and one-dimension of time merge together to form what is known as ***four dimensional space-time continuum.***

Space like Intervals : The square of interval is represented as

$$s_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2$$

or

$$s_{12}^2 = r_{12}^2 - c^2(t_2 - t_1)^2$$

If $r_{12}^2 > c^2(t_2 - t_1)^2$ the square of the ***space-time*** interval is positive, *i.e.*, $s_{12}^2 > 0$ and the interval is said to be ***space-like***. In such a case, we cannot find a frame in which the two events may take place at the same place. If it happens $r_{12}^2 = 0$ and then s_{12}^2. will be negative and the interval is no more space-like.

For events having space-like interval, we may find an inertial frame in which the two events occur at the same time or simultaneously $(t_2 - t_1 = 0)$. In such a case r_{12}^2 would have an appropriate value to give the same value of s_{12}^2, *i.e.*,

$$s_{12}^2 = r_{12}^2 - c^2(t_2 - t_1)^2 = r'^2_{12} = s'^2_{12} \qquad \text{...(18)}$$

In case of space-like interval, the order of occurrence of two events in time is not definite. Obviously in some frame, one event may occur earlier $(t_1 < t_2)$, in another the other may appear earlier $(t_1 > t_2)$ and in yet another both events may occur simultaneously $(t_1 = t_2)$. We find that past, present and future cannot be defined relative to these events. Thus for space-like interval, the two events have no causal relationship. [For a causal relation (cause and effect) between two events, one must always occur after the other. For example, a bullet fired from a gun can never hit the target before it is actully fired.]

It is to be noted that for space-like interval

$$r_{12}^2 > c^2(t_2 - t_1)^2 \quad \text{or} \quad \frac{r_{12}}{c} > (t_2 - t_1)$$

Thus *for space-like interval, the time separation between the two events is less than the time taken by light in covering the distance between them. Consequently, the two events cannot be connected by any real physical process.*

Time-like Interval : The square of space-time interval is given by

$$s_{12}^2 = r_{12}^2 - c^2(t_2 - t_1)^2$$

If $c^2(t_2 - t_1)^2 > r_{12}^2$, the square of the interval is negative *i.e.*, $s_{12}^2 < 0$.

In such a case the interval is imaginary and said to be ***time-like***. For events having a time-like interval, we can not find a frame of reference in which the two events are simultaneous, because in such a case $t_2 - t_1 = 0$ and the interval is no more time-like. However, one may find a frame in which the two events occur at a point, *i.e.*,

$$s_{12}^2 = r_{12}^2 - c^2(t_2 - t_1)^2 = -c^2(t'_2 - t'_1)^2 = s'^2_{12} \qquad \text{...(19)}$$

Obviously $t'_2 - t'_1 = \Delta t'$ is the minimum time between two events (which are taking place at a point in S'-frame) and is the ***proper time interval*** $\Delta\tau$, *i.e.*,

$$-c^2\Delta\tau^2 = r_{12}^2 - c^2\Delta t^2 \qquad (\Delta t = t_2 - t_1)$$

or $$\Delta\tau^2 = \Delta t^2 - \frac{r_{12}^2}{c^2} = \Delta t^2\left(1-\frac{u^2}{c^2}\right) \quad \text{or} \quad \Delta t = \frac{\Delta\tau}{\sqrt{1-u^2/c^2}} \qquad \text{...(20)}$$

where $u = r_{12}/\Delta t$ is the particle velocity, in case a particle moves the distance r_{12} uniformly in Δt time in S-frame. The time of motion Δt in S-frame corresponds to $\Delta\tau$ time (proper) in a frame (S') attached with the particle. (If a particle is moving with uniform velocity v along X-axis relative to S-frame, we may fix a frame S' with the particle itself. The time interval for distance Δx travelled in S-frame is Δt and $\Delta x/\Delta t = v$ represents the particle velocity in S-frame.)

For two events with time-like interval, order of time of occurrence is definite, because they cannot occur simultaneously in any reference frame. Consequently, there is causal relation between such events. Further for time-like interval,

$$c^2(t_2-t_1)^2 > r_{12}^2 \quad \text{or} \quad t_2 - t_1 > \frac{r_{12}}{c}$$

Thus for time-like interval, the time separation between two events is more than the time taken by light in covering the distance between them and therefore, the two events can be connected by a real physical process, such as the motion of a particle.

Light-like Interval : If the interval $s_{12} = 0$, it is said to be ***light-like*** or ***singular***. For such an interval

$$(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 = c^2(t_2-t_1)^2$$

World Regions and Light Cone : The square of the interval between two events $E_1(x_1, y_1, z_1, t_1)$ and $E_2(x_2, y_2, z_2, t_2)$ is given by

$$s_{12}^2 = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - c^2(t_2-t_1)^2$$

which is Lorentz invariant.

The interval is said to be space-like, time-like and light-like corresponing to s_{12}^2 to be positive, negative and zero. This represents the separation between two events in the Minkowski four dimensional world. The characteristics of the interval can clearly be demarcated in certain distinct regions in the Minkowski space. These regions are known as ***world regions***.

In order to study the world regions, we consider one of the two events at the origin O (0, 0, 0, 0) and the other E (x, y, z, ct). Then the square of the interval between the event E and event O will be given by

$$s^2 = x^2 + y^2 + z^2 - c^2t^2 = r^2 - c^2t^2$$

We now define the various world regions as (see Fig . 14.3) :

(1) Region A, $s^2 > 0$

(2) Region B, $s^2 < 0$

(3) Sheet C, $s^2 = 0$

The sheet C which divides the Minkowski world into the regions A and B is given by the equation

$$s^2 = 0 \text{ or } x^2 + y^2 + z^2 = (ct)^2 \text{ or } r = ct \qquad \text{...(21)}$$

In case of three dimensions, the equation

$$x^2 + y^2 = z^2 \qquad \text{...(22)}$$

represents the surface of a cone with the apex at the origin and its axis along Z-axis (with semi-vertical angle $\pi/4$). In analogy to eq. (22) in three-dimensional case, the sheet C, represnted by eq. (21) is referred as cone, with its apex at the origin O of the world of events and its axis along time $T = ct$ with semi-vertical angle equal to $\pi/4$ (Fig. 14.3).

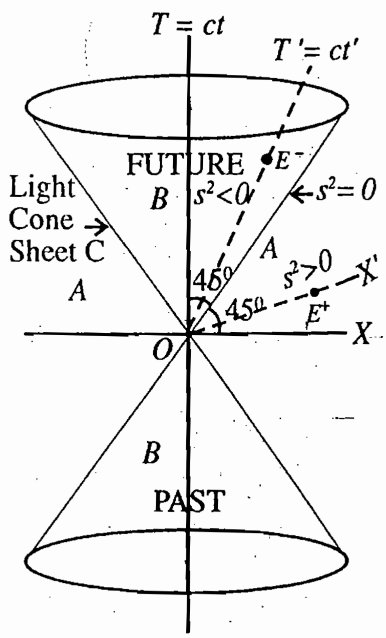


Fig. 14.3 : Light Cones

The value of s^2 is zero on the surface of this cone and therefore it is called as ***null cone***. Eq. (21) represents the totality of all those world points which are, sooner or later, reached by a light signal which is once at O. Thus, the null cone constitutes the space-time representation of the propagation of light and hence it is also called ***light cone***.

For convenience let us consider only X and T axes. World region A corresponds to space-like interval $(s^2 > 0)$. In this region, corresponding to any point E^+, we can find a frame of reference S' (referring X' axis) in which the events O and E^+ are simultaneous $(t' = 0)$. World region B (shaded region) corresponds to time-like interval $(s^2 < 0)$. In this region, corresponding to any point E^- we can find a frame of reference S' (referring T' axis) in which the events O and E^- occur at the same place $(x' = 0)$. However, event E^- occurs at a later instant than O in stystem S'. Hence the events in the upper cone are absolutely in future relative to O and this region is called ***absolute future***. The events in the lower cone are absolutely in past relative to O and this region is referred as ***absolute past***.

14.5. FOUR-VECTORS

A vector in four dimensional Minkowski space is called a ***four-vector***. Its components transform from one frame to another similar to Lorentz transformations.

An event in four domensional space is represented by a world point (x_1, x_2, x_3, x_4). The Lorentz transformations from S-frame to S'-frame correspond to orthogonal transformations in the four-space and are represented as

$$x'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} x_\nu \quad \text{or} \quad \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{...(23)}$$

with the condition

$$\sum_{\mu=1}^{4} x'^2_\mu = \sum_{\mu=1}^{4} x^2_\mu \qquad \text{...(24)}$$

We may represent the position vector of a world point by

$$x_\mu = (x_1, x_2, x_3, x_4) = (\mathbf{r}, ict) \qquad \text{...(25)}$$

where (x_1, x_2, x_3) or (x, y, z) represent the position vector $\mathbf{r}$ of a point in three dimensional space and $x_4 = ict$ or $x_4' = iT$. $\mathbf{r}(= x, y, z)$ is the space part and ict is the time part of the four dimensional position vector x_μ.

A four-vector A_μ is a vector in four dimensional space with components A_1, A_2, A_3 and A_4 and is represented as

$$A_\mu = (A_1, A_2, A_3, A_4) = (\mathbf{A}, iA_t) \qquad ...(26)$$

where $\mathbf{A}(= A_1, A_2, A_3)$ is the space component and $A_4(= iA_t)$ is the time component. These components transform from S-frame to S'-frame similar to Lorentz transformations, *i.e.*,

$$\begin{pmatrix} A'_1 \\ A'_2 \\ A'_3 \\ A'_4 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \qquad ...(27a)$$

or
$$A'_\mu = \sum_{\nu=1}^{4} a_{\mu\nu} A'_\nu \qquad ...(27b)$$

i.e., $A'_1 = \gamma(A_1 + i\beta A_4), A'_2 = A_2, A'_3 = A_3, A'_4 = \gamma(-i\beta A_1 + A_4)$ ...(27c)

These transformations are governed by the condition

$$\sum_{\mu=1}^{4} A'^2_\mu = \sum_{\mu=1}^{4} A^2_\mu \quad \text{or} \quad A'_\mu A'_\mu = A_\mu A_\mu \qquad ...(28)$$

The square of the magnitude of the four vector is given by

$$A_\mu A_\mu = A_1^2 + A_2^2 + A_3^2 + A_4^2 \qquad ...(29a)$$

or
$$\sum_{\mu=1}^{4} A_\mu^2 = A_1^2 + A_2^2 + A_3^2 - A_t^2 \qquad ...(29b)$$

Let two vectors A_μ and B_μ be $A_\mu = (A_1, A_2, A_3, A_4)$ with $A_4 = iA_t$ and $B_\mu = (B_1, B_2, B_3, B_4)$ with $B_4 = iB_t$.

The scalar product of the four-vectors A_μ and B_μ is defined as

$$A_\mu B_\mu = A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4 \qquad ...(30a)$$

or
$$A_\mu B_\mu = A_1 B_1 + A_2B_2 + A_3B_3 - A_t B_t \qquad ...(30b)$$

This scalar product is invariant under Lorentz transformations *i.e.*,

$$A'_\mu B'_\mu = A_\mu B_\mu$$

because
$$A'_1 B'_1 + A'_2 B'_2 + A'_3 B'_3 + A'_4 B'_4$$

$$= \gamma^2(A_1 + i\beta A_4)(B_1 + i\beta B_4) + A_2B_2 + A_3B_3 + \gamma^2(-i\beta A_1 + A_4)(-i\beta B_1 + B_4)$$

$$= A_1B_1\gamma^2(1-\beta^2) + A_2B_2 + A_3B_3 + A_4B_4\gamma^2(-\beta^2 + 1)$$

$$= A_1B_1 + A_2B_2 + A_3B_3 + A_4B_4$$

Note : Some authors define the square of magnitude of the four-vector A_μ as

$$A_\mu A_\mu = A_t^2 - A_1^2 - A_2^2 - A_3^2 \qquad ...(31)$$

14.6. EXAMPLES OF FOUR-VECTORS

(1) Position four-vector x_μ : It is expressed as

$$x_\mu = (x_1, x_2, x_3, x_4) = (\mathbf{r}, ict) \qquad ...(32)$$

(2) Four-velocity or velocity four-vector u_μ **:** The components of the velocity four-vector u_μ are defined as

$$u_1 = \frac{dx_1}{d\tau} = \frac{dx_1}{dt}\frac{dt}{d\tau} = \frac{dx}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{u_x}{\sqrt{1-u^2/c^2}}$$

$$u_2 = \frac{dx_2}{d\tau} = \frac{dx_2}{dt}\frac{dt}{d\tau} = \frac{dy}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{u_y}{\sqrt{1-u^2/c^2}}$$

$$u_3 = \frac{dx_3}{d\tau} = \frac{dx_3}{dt}\frac{dt}{d\tau} = \frac{dz}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{u_z}{\sqrt{1-u^2/c^2}}$$

$$u_4 = \frac{dx_4}{d\tau} = \frac{d(ict)}{dt}\frac{dt}{d\tau} = \frac{ic}{\sqrt{1-u^2/c^2}}$$

where $$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-u^2/c^2}}.$$

Hence $$u_\mu = (u_1, u_2, u_3, u_4)$$

or $$u_\mu = \left(\frac{u_x}{\sqrt{1-u^2/c^2}}, \frac{u_y}{\sqrt{1-u^2/c^2}}, \frac{u_z}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}}\right) \qquad ...(33a)$$

i.e., $$u_\mu = \left(\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}, \frac{ic}{\sqrt{1-u^2/c^2}}\right) \qquad ...(33b)$$

where $\mathbf{u} = d\mathbf{r}/dt$ is the three dimensional velocity vector.

The square of the magnitude of the velocity four vector is given by

$$u_\mu u_\mu = \frac{u^2}{1-u^2/c^2} - \frac{c^2}{1-u^2/c^2} = -c^2 \qquad ...(34a)$$

which is Lorentz invariant.

Note : In the literature, sometimes the time in which light moves in vacuum 1 meter is taken as the unit of time. In such a case, c is to be replaced by 1 in the expressions. For example, for this unit of time

$$u_\mu u_\mu = -1 \qquad ...(34b)$$

and if one defines $u_\mu u_\mu = u_t^2 - u_1^2 - u_2^2 - u_3^2$, then $u_\mu u_\mu = 1$. ...(34c)

(3) Momentum four vector p_μ : The components of four-momentum p_μ are defined by

$$p_1 = m_0 u_1 = \frac{m_0 u_x}{\sqrt{1-u^2/c^2}} = m u_x = p_x$$

$$p_2 = m_0 u_2 = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} = m u_y = p_y$$

$$p_3 = m_0 u_3 = \frac{m_0 u_z}{\sqrt{1-u^2/c^2}} = m u_z = p_z$$

$$p_4 = m_0 u_4 = \frac{m_0 ic}{\sqrt{1-u^2/c^2}} = imc = i\frac{E}{c}$$

Hence,

$$p_\mu = (p_1, p_2, p_3, p_4) = (p_x, p_y, p_z, imc) = (\mathbf{p}, iE/c) \text{ with } \mathbf{p} = m\mathbf{u} \qquad \text{...(35a)}$$

The square of the magnitude of the four-momentum is given by

$$p_\mu p_\mu = p^2 - \frac{E^2}{c^2} = -(E^2 - p^2c^2)/c^2 \quad \text{or} \quad p_\mu p_\mu = -m_0^2 c^2 \qquad \text{...(35b)}$$

This p_μ is also called ***energy- momentum four-vector.***

(4) Acceleration four-vector a_μ : Its components are defined by

$$a_1 = \frac{du_1}{d\tau} = \frac{du_1}{dt}\frac{dt}{d\tau} = \frac{d}{dt}\left(\frac{u_x}{\sqrt{1-u^2/c^2}}\right)\frac{1}{\sqrt{1-u^2/c^2}}$$

$$= \frac{1}{\sqrt{1-u^2/c^2}}\left(\frac{u_x}{\sqrt{1-u^2/c^2}} + \frac{u_x u\dot{u}}{c^2\left(1-u^2/c^2\right)^{3/2}}\right)$$

But $\qquad u^2 = u_x^2 + u_y^2 + u_z^2$ and hence, $u\dot{u} = u_x\dot{u}_x + u_y\dot{u}_y + u_z\dot{u}_z = \mathbf{u}\cdot\dot{\mathbf{u}}$

Therefore, $\qquad a_1 = \dfrac{u_x}{1-u^2/c^2} + \dfrac{u_x(\mathbf{u}\cdot\dot{\mathbf{u}})}{c^2\left(1-u^2/c^2\right)^2}$

Similarly, $\qquad a_2 = \dfrac{\dot{u}_y}{1-u^2/c^2} + \dfrac{u_y(\mathbf{u}\cdot\dot{\mathbf{u}})}{c^2\left(1-u^2/c^2\right)^2}, \quad a_3 = \dfrac{\dot{u}_z}{1-u^2/c^2} + \dfrac{u_z(\mathbf{u}\cdot\dot{\mathbf{u}})}{c^2\left(1-u^2/c^2\right)^2}$

Also, $\qquad a_4 = \dfrac{du_4}{d\tau} = \dfrac{du_4}{dt}\dfrac{dt}{d\tau} = \dfrac{d}{dt}\left(\dfrac{ic}{\sqrt{1-u^2/c^2}}\right)\dfrac{1}{\sqrt{1-u^2/c^2}} = \dfrac{i(\mathbf{u}\cdot\dot{\mathbf{u}})}{c\left(1-u^2/c^2\right)^2}$

Thus, $$a_\mu = \left(\frac{\mathbf{a}}{1-u^2/c^2} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{c^2\left(1-u^2/c^2\right)^2}, \frac{i(\mathbf{u}\cdot\mathbf{a})}{c\left(1-u^2/c^2\right)^2} \right) \qquad ...(36)$$

where $\mathbf{a} = \dot{\mathbf{u}} = \dot{u}_x\hat{\mathbf{i}} + \dot{u}_y\hat{\mathbf{j}} + \dot{u}_z\hat{\mathbf{k}}$.

(5) **Four-force or Minkowski force** F_μ **:** The four-force F_μ is represented as

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{d}{d\tau}(m_0 u_\mu) = m_0\frac{du_\mu}{d\tau} = m_0\frac{d^2x_\mu}{d\tau^2} \qquad ...(37)$$

This equation is called the ***Minkowski force equation*** and is presented in a form similar to Newton's equation.

In the limit $u<<c$, the three-dimensional components are obtained as

$$F_k = \frac{dp_k}{dt} = m_0\frac{d^2x_k}{dt^2} \qquad (dt \sim d\tau)$$

which is the classical Newton's equation.

The components of four-force F_μ are

$$F_1 = \frac{dp_1}{d\tau} = \frac{dp_1}{dt}\frac{dt}{d\tau} = \frac{dp_x}{dt}\frac{1}{\sqrt{1-u^2/c^2}} = \frac{F_x}{\sqrt{1-u^2/c^2}}$$

Similarly, $F_2 = \dfrac{F_y}{\sqrt{1-u^2/c^2}}$, $F_3 = \dfrac{F_z}{\sqrt{1-u^2/c^2}}$ and $F_4 = \dfrac{dp_4}{d\tau} = \dfrac{dp_4}{dt}\dfrac{dt}{d\tau} = \gamma\dfrac{d}{dt}\left(\dfrac{iE}{c}\right) = \dfrac{i\gamma}{c}\dfrac{dE}{dt}$

Thus, $$F_\mu = \left(\frac{\mathbf{F}}{\sqrt{1-u^2/c^2}}, \frac{i\gamma}{c}\frac{dE}{dt} \right) \qquad ...(38)$$

The four-force may be expressed in terms of four-acceleration vector as

$$F_\mu = \frac{dp_\mu}{d\tau} = \frac{d}{d\tau}\left(m_0 u_\mu\right) = m_0\frac{du_\mu}{d\tau} = m_0 a_\mu$$

Hence, $$F_\mu = \left(\frac{m_0\mathbf{a}}{1-u^2/c^2} + \frac{m_0\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{c^2\left(1-u^2/c^2\right)^2}, \frac{im_0(\mathbf{u}\cdot\mathbf{a})}{c\left(1-u^2/c^2\right)^2} \right) \qquad ...(39)$$

Since (38) and (39) are the same, equating the space part of F_μ, we obtain

$$\mathbf{F} = \frac{m_0\mathbf{a}}{\left(1-u^2/c^2\right)^{1/2}} + \frac{m_0\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{c^2\left(1-u^2/c^2\right)^{3/2}} \qquad ...(40)$$

or $$\mathbf{F} = m\mathbf{a} + \frac{m\mathbf{u}(\mathbf{u}\cdot\mathbf{a})}{c^2-u^2} \qquad ...(41)$$

where $\mathbf{F} = d\mathbf{p}/dt$ is the three-dimensional force vector and in general is not equal to $m\mathbf{a}$.

The fourth component of F_μ in (39) can be written as

$$\frac{im_0(\mathbf{u}\cdot\mathbf{a})}{c\left(1-u^2/c^2\right)^2}=\frac{i\gamma}{c}(\mathbf{F}\cdot\mathbf{u}) \qquad ...(42)$$

because using (40)

$$\mathbf{F}\cdot\mathbf{u}=\frac{m_0(\mathbf{u}\cdot\mathbf{a})}{\sqrt{1-u^2/c^2}}+\frac{m_0(\mathbf{u}\cdot\mathbf{u})(\mathbf{u}\cdot\mathbf{a})}{c^2\left(1-u^2/c^2\right)^{3/2}}=\frac{m_0(\mathbf{u}\cdot\mathbf{a})}{\sqrt{1-u^2/c^2}}\left[1+\frac{u^2}{c^2-u^2}\right]$$

$$=\frac{m_0(\mathbf{u}\cdot\mathbf{a})}{\sqrt{1-u^2/c^2}}\frac{1}{1-u^2/c^2}=\frac{m_0(\mathbf{u}\cdot\mathbf{a})}{\left(1-u^2/c^2\right)^{3/2}}$$

Thus the fourth component of F_μ from (38) and (42) is

$$\frac{i\gamma}{c}\frac{dE}{dt}=\frac{i\gamma}{c}(\mathbf{F}\cdot\mathbf{u}) \text{ or } \frac{dE}{dt}=\mathbf{F}\cdot\mathbf{u} \qquad ...(43)$$

The right hand side of eq. (43) represents the power and the left hand side for a single particle $dE/dt = d(mc^2)/dt$ represents the rate of change of energy. This is in accordance with the conservation of energy.

Thus the four-force F_μ is represented as

$$F_\mu=\left(\frac{\mathbf{F}}{\sqrt{1-u^2/c^2}},\frac{i(\mathbf{F}\cdot\mathbf{u})}{c\sqrt{1-u^2/c^2}}\right) \qquad ...(44)$$

The Minkowski force equation is

$$F_\mu=\frac{dp_\mu}{d\tau}=m_0\frac{du_\mu}{d\tau} \qquad ...(45)$$

represents the fundamental equations of mechanics in the covariant four-vector form with the components given by (44). The first three equations are the three equations of motion and the fourth equation expresses the theorem of conservation of energy.

In relativistic mechanics, the concept of force has no longer any absolute meaning as it has in the Newtonian mechanics because in two inertial frames S and S' in constant relative motion, F and F' will have different values and directions.

Ex. 1. *In frame S, two events have the space-time coordinates* (0, 0, 0, 0) *and* (5c, 0, 0, 3), *where time co-ordinate is in seconds. Find the space-time interval between them. Calculate the velocity of a frame in which*

(*a*) *the two events are simultaneous,*

(*b*) *the first event occurs* 1 *sec earlier than the second,*

(*c*) *the second event occurs* 1 *sec earlier than the first.*

What is the limit for the maximum time interval between these events ?

Solution : $s_{12}^2=\Delta x^2+\Delta y^2+\Delta z^2-c^2\Delta t^2=25c^2-9c^2=16c^2$

Therefore, space-time interval $s_{12}=4c$, which is space-like.

(*a*) Let the velocity of a frame S' relative to S be v so that the two events are simultaneous (i.e., $\Delta t' = 0$) in the frame S'. Therefore,

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 0 \text{ or } \Delta t = \frac{v\Delta x}{c^2} \text{ or } 3 = \frac{v \times 5c}{c^2}, \text{ whence, } v = \mathbf{0.6c}$$

(*b*) In S'-frame, the first event occurs 1 sec earlier than the second *i.e.,* $\Delta t' = 1$ sec. Therefore

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 0 \text{ or } 1 = \frac{1}{\sqrt{1 - v^2/c^2}}\left(3 - \frac{v \times 5c}{c^2}\right)$$

Squaring and arranging the terms, we get the equation

$$13\frac{v^2}{c^2} - 15\frac{v}{c} + 4 = 0, \text{ hence } \frac{v}{c} = 0.7c \text{ or } 0.4c.$$

For having $\Delta t' = +1$ sec, $v = \mathbf{0.4c}$. because $v = 0.7c$ will make $\Delta t'$ negative.

(c) If the second event occurs 1 sec earlier than the first in frame S'; then $\Delta t' = -1$sec. So that

$$-1 = \frac{1}{\sqrt{1 - v^2/c^2}}\left(3 - \frac{v \times 5c}{c^2}\right).$$

On squaring and arranging the terms we will get the same equation, as obtained above, and for making $\Delta t' = -1$ sec, the first value *i.e.,* $v = \mathbf{0.7c}$ will be admissible.

There is no limit for the time interval of the two events whose space-time interval is spece-like; the time interval may have any value between $\Delta t' = -\infty$ to $\Delta t' = +\infty$.

Ex. 2. *Prove that the three-dimensional volume element dx dy dz is not invariant under Lorentz transformations while the four dimensional volume element dx dy dz dt is invariant.* **(Rohilkhand 1991)**

Solution : In S-frame let the three-dimensional volume element be

$$dV = dx\, dy\, dz$$

and four dimensional volume element be

$$dV_\mu = dx\, dy\, dz\, dt$$

In S'-frame, let the corresponding volume elements be dV' and dV'_μ. Let the volume element be at rest in frame S', *i.e.,* $dV' = dV_0$. Now, Lorentz transformations are

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z.$$

If the observer of S-frame observes dx element at the same time, then

$$dx' = \gamma dx; \text{ also } dy' = dy \text{ and } dz' = dz$$

Hence $dx'\, dy'\, dz' = \gamma\, dx\, dy\, dz$ (t same)

or
$$dx\, dy\, dz = \frac{dx'\, dy'\, dz'}{\gamma} \quad \text{or} \quad dV = dV_0\sqrt{1 - \beta^2}$$

Therefore, $dV \neq dV_0$ or $dV'\, dV'$

Now, suppose in S'-frame O' observes any two events at x' at dt' interval of time, then $dt' = d\tau$ is the proper interval of time and from $t = \gamma(t' + vx'/c^2)$, we have

$$dt = \gamma\ dt' = \gamma\ d\tau \quad (x' \text{ same})$$

Therefore, $dx'\,dy'\,dz'\,dt' = \gamma\ dx\,dy\,dz\ \dfrac{dt}{\gamma} = dx\,dy\,dz\,dt$ or $dV'_\mu = dV_\mu$

This proves the given statement.

Ex. 3. ***Wave number Vector*** k_μ : *Find the components of wave number four vector. What is its norm ?*

Solution : The phase of a plane monochromatic wave is invariant. Let f be the phase in frame S. Then

$$\mathbf{k}\cdot\mathbf{r} - \omega t = f \text{ (invariant)}$$

where $\quad \mathbf{k} = k_1\,\hat{\mathbf{i}} + k_2\,\hat{\mathbf{j}} + k_3\,\hat{\mathbf{k}}$ and $\mathbf{r} = x_1\,\hat{\mathbf{i}} + x_2\,\hat{\mathbf{j}} + x_3\,\hat{\mathbf{k}}$

Here $\mathbf{k}\cdot\mathbf{r} = k_1x_1 + k_2x_2 + k_3x_3$ and $-\omega t = \dfrac{i\omega}{c} ict = k_4x_4$, where $k_4 = i\omega/c$ and $x_4 = ict$.

Thus $\mathbf{k}\cdot\mathbf{r} - \omega t = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4$ or $k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = f$ *i.e.*, $k_\mu x_\mu = f$ (scalar)

Since x_μ is a four vector and its product with $k_\mu = (k_1, k_2, k_3, k_4)$ is a scalar, k_μ must be a four-vector i.e., $k_\mu = (\mathbf{k}, i\omega/c)$

The norm of the four-vector k_μ is given by

$$k_\mu k_\mu = k^2 - \omega^2/c^2$$

But for plane monochromatic wave $k = \omega/c$; hence $k_\mu k_\mu = 0$.

14.7. CONSERVATION OF FOUR-MOMENTUM - APPLICATION OF FOUR VECTORS

The four-momentum of a particle is defined by

$$p_\mu = (\mathbf{p},\ iE/c) \text{ or } (\mathbf{p},\ imc)$$

The space part of the four-momentum is the linear momentum $\mathbf{p}$ and its time part contains the energy E. $\mathbf{p}$ and E are relativistic momentum and relativistic energy respectively.

In a two particle collision, according to the conservation of four momentum

$$p_\mu^1(B) + p_\mu^2(B) = p_\mu^1(A) + p_\mu^2(A) \quad ...(46)$$

where A and B refer for after and before the collision respectively.

The above equation is equivalent to the conservation of linear momentum and conservation of energy of two particle system, *i.e.*,

$$\mathbf{p}_1(B) + \mathbf{p}_2(B) = \mathbf{p}_1(A) + \mathbf{p}_2(A) \quad ...(47)$$

and $$i\frac{E_1(B)}{c} + i\frac{E_2(B)}{c} = i\frac{E_1(A)}{c} + i\frac{E_2(A)}{c}$$

or $$E_1(B) + E_2(B) = E_1(A) + E_2(A) \quad ...(48)$$

where $\mathbf{p}_1$, E_1 and $\mathbf{p}_2$, E_2 refer to two particles.

(1) Decay of unstable particles : Many atomic nuclei and several unstable particles like π-mesons, K-mesons etc. decay spontaneously into other particles. Let us consider a nucleus of rest mass m_0, decaying spontaneously into two components of rest masses m_1 and m_2. Here, we apply the law of conservation of four momentum.

Let the four-momentum of the nucleus be $p_\mu = (\mathbf{p},\ iE/c)$ and those of the fragments after the decay p_μ^1 $(\mathbf{p}_1,\ iE_1/c)$ and p_μ^2 $(\mathbf{p}_2,\ iE_2/c)$.

Conservation of four-momentum demands

$$p_\mu = p_\mu^1 + p_\mu^2 \qquad ...(49)$$

which gives $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$ ...(50)

and $$E = E_1 + E_2 \qquad ...(51)$$

It we consider that the nucleus is at rest before the decay, then

$$p_\mu = (0, im_0 c) \qquad ...(52)$$

Therefore, eqs. (50) and (51) assume the form

$$\mathbf{p}_1 + \mathbf{p}_2 = 0 \qquad ...(53)$$

and $$E_1 + E_2 = m_0 c^2 \qquad ...(54)$$

As $$E_1 = \frac{m_1 c^2}{\sqrt{1-\frac{v_1^2}{c^2}}} \text{ and } E_2 = \frac{m_2 c^2}{\sqrt{1-\frac{v_2^2}{c^2}}},$$

we observe $E_1 > m_1 c^2$ and $E_2 > m_2 c^2$ or $E_1 + E_2 > (m_1 + m_2)c^2$.

Hence from eq. (54)

$$m_0 > m_1 + m_2$$

Thus the rest of the decaying nucleus is greater than the sum of the rest masses of the resulting components.

From (53) $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{q}$ (say)

Then $$E_1 = \sqrt{q^2c^2 + m_1^2c^4} \text{ and } E_2 = \sqrt{q^2c^2 + m_2^2c^4}$$

Therefore, eq. (54) is

$$\sqrt{q^2c^2 + m_1^2c^4} + \sqrt{q^2c^2 + m_2^2c^4} = m_0c^2$$

whence $$m_0 = \sqrt{m_1^2 + \frac{q^2}{c^2}} + \sqrt{m_2^2 + \frac{q^2}{c^2}} \qquad ...(55)$$

If the rest masses m_0, m_1 and m_2 are known, one can determine the value of the momentum q and hence the energies E_1 and E_2 of the resulting particles in the decay process. The individual energies may be determined directly by the idea of four-vectors. From eq. (49)

$$p_\mu^1 = p_\mu - p_\mu^2 \text{ or } p_\mu^1 p_\mu^1 = (p_\mu - p_\mu^2)(p_\mu - p_\mu^2)$$

or $$p_\mu^1 p_\mu^1 = p_\mu p_\mu + p_\mu^2 p_\mu^2 - 2p_\mu p_\mu^2$$

or $$-m_1c^2 = -m_0^2c^2 - m_2^2c^2 + 2m_0E_2 \qquad ...(56)$$

because $$p_\mu p_\mu^2 = (0, im_0c)(\mathbf{p}_2; iE_2/c) = -m_0E_2$$

Thus $$E_2 = \frac{m_0^2 + m_2^2 - m_1^2}{2m_0}c^2 \quad ...(57)$$

Similarly, $$E_1 = \frac{m_0^2 + m_1^2 - m_2^2}{2m_0}c^2 \quad ...(58)$$

If in the decay process, mass of one of the resulting particle is known, say m_1, then m_2 can be determined for known momentum q. From (58)

$$m_2^2 = m_0^2 + m_1^2 - \frac{2m_0E_1}{c^2} \quad ...(59)$$

where $$E_1 = \sqrt{q^2c^2 + m_1^2c^4}.$$

The **decay of $\pi^{\pm}$ mesons** (at rest) to $\mu^{\pm}$ mesons is represented as

$$\pi^{\pm} \to \mu^{\pm} + \nu \text{ (neutrino)} \quad ...(60)$$

If **q** the momentum of one of the resulting particle is known, then from (55)

$$m_{\pi}^{\pm} = \sqrt{(m_{\mu}^{\pm})^2 + \frac{q^2}{c^2}} + \frac{q}{c} \qquad \text{for } m_{\nu} = 0 \quad ...(61)$$

The momentum $|\mathbf{q}|$ of m_{μ}^{+} was found to be 29.80 ± 0.04 MeV/c by deflecting in a magnetic field in experiment.

For $m_{\mu}^{+} = 105.65 \pm 0.05$ MeV, the mass of m_{π}^{+} is found to be

$$m_{\pi}^{+} = 139.58 \pm 0.05 \text{ MeV}$$

Also, from (59) $m_{\pi}^{\pm}$ and $m_{\mu}^{\pm}$ can be related as

$$m_{\mu}^2 = m_{\pi}^2 - \frac{2m_{\pi}q}{c}$$

which gives the eq. (61).

(2) Threshold energy for production of particles : If a high energetic particle hits a target, it is possible that additional particles may be produced such that the sum of the rest masses of the particles after the collision may be greater than the sum of the rest masses before the collision. For such a reaction, the striking particle must have a certain minimum kinetic energy known as the *reaction threshold energy* (T_m). This is the minimum energy of the striking particle to produce the particles of zero velocity in the final state. We are giving below two examples for the production of new particles.

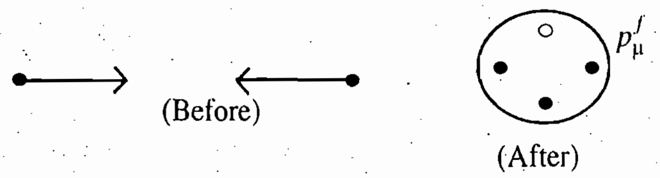
(i) Production of antiprotons : Antiprotons ($\overline{P}$) are produced in large accelerators by the following reaction :

$$P + P \to P + P + P + \overline{P}$$

Thus when an energetic proton collides with a proton at rest (for example, in a hydrogen target placed in the beam of protons) and if the incident proton has enough energy, a proton–antiproton ($P + \overline{P}$) pair may be produced, in addition to the two original protons.

We discuss the collision problem in the centre of mass (CM) frame of reference. We represent the four-momentum of the incident protons (a) as

(i) Centre of mass system :



(ii) Laboratory System :

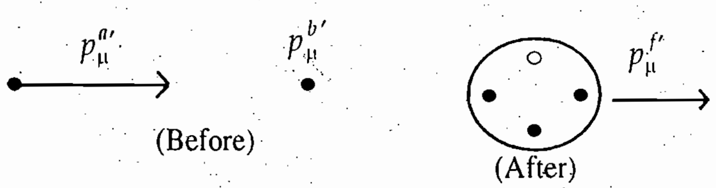


Fig. 14.4 : Protons are represented by solid small circles and antiprotons by open circles

p_μ^a and that of the target proton (b) as p_μ^b. If the incident proton has just enough energy to make the reaction go, the final state after the collision will consist of three protons and an antiproton at rest in the CM frame [Fig.14.4].

If we call p_μ^f the total four-momentum in the final state, then according to the law of conservation of four-momentum in the CM-frame

$$p_\mu^a + p_\mu^b = p_\mu^f \qquad p_\mu^{f\,\prime} \qquad \ldots(62)$$

where $\quad p_\mu^f = (0, 4iM_0c)$ and M_0 represents the rest mass of proton or antiproton.

Now, $\quad (p_\mu^a + p_\mu^b)\,(p_\mu^a + p_\mu^b) = p_\mu^f p_\mu^f$

or
$$p_\mu^a p_\mu^a + p_\mu^b p_\mu^b + 2p_\mu^a p_\mu^b = p_\mu^f p_\mu^f \qquad \ldots(63)$$

Here $\quad p_\mu p_\mu = p^2 - \dfrac{E^2}{c^2} = -\dfrac{1}{c^2}(E^2 - p^2c^2) = -\dfrac{M_0^2c^4}{c^2} = -M_0^2c^2$

Also from $\quad p_\mu^f = (0, 4iM_0c)$,

$$p_\mu^f p_\mu^f = -16\,M_0^2c^2$$

Similarly, $\quad p_\mu^b p_\mu^b = -\,M_0^2c^2$

Thus from (63), we obtain

$$-2M_0^2c^2 + 2p_\mu^a p_\mu^b = -16M_0^2c^2 \text{ or } p_\mu^a p_\mu^b = -7M_0^2c^2,$$

If $p_\mu^{a'}$ and $p_\mu^{b'}$ refer for the four-momenta in the laboratory frame, then the invariance of the scalar product in the Lab and CM systems demands

$$p_\mu^a p_\mu^b = p_\mu^{a'} p_\mu^{b'}$$

But $\quad p_\mu^{a'} = \left(\mathbf{p},\ iE/c\right)$ and $p_\mu^{b'} = (0, iM_0c)$

Therefore, $\quad p_\mu^{a'} p_\mu^{b'} = -M_0E = p_\mu^a p_\mu^b$

Since $p_\mu^a p_\mu^b = -7M_0^2c^2$

$$-M_0E = -7M_0^2c^2 \text{ or } E = 7M_0c^2 \quad ...(64)$$

This is the total minimum energy of the incident proton in the laboratory frame, which will be able to produce a pair of proton and antiproton. However, the rest energy of the incident proton is M_0c^2. Therefore, the least kinetic energy of the incident proton is

$$T = 7M_0c^2 - M_0c^2 \text{ or } T = 6M_0c^2 \quad ...(65)$$

Substituting $M_0 = 1.67 \times 10^{-27}$ kg for the mass of a proton, one obtains the value of minimum kinetic energy to produce the pair of proton and antiproton to be 5.6 BeV. Experimentally antiprotons have been produced with the incident proton kinetic energy of the order of 6 BeV.

(ii) Production of π^0 mesons : π^0- mesons are produced when high energy proton beam collides with protons at rest of the target.

$$P + P \rightarrow P + P + \pi^0$$

In the case, the conservation of momentum in the CM frame demands

$$P_\mu^a + P_\mu^b = P_\mu^f$$

where $P_\mu^f = [0, i(2M_0 + m_0)c]$ with M_0 proton mass and m_0, π^0-meson mass.

Now $(P_\mu^a + P_\mu^b)(P_\mu^a + P_\mu^b) = P_\mu^f P_\mu^f$ or $P_\mu^a P_\mu^a + P_\mu^b P_\mu^b + 2P_\mu^a P_\mu^b = P_\mu^f P_\mu^f$

Here $P_\mu^a P_\mu^a = P_\mu^b P_\mu^b = -M_0^2c^2$ and $P_\mu^f P_\mu^f = -(2M_0 + m_0)^2 c^2$

Hence, $2P_\mu^a P_\mu^b = -(2M_0 + m_0)^2 c^2 + 2M_0^2c^2$

Also, $2P_\mu^{a'} P_\mu^{b'}$ (Lab frame) $= 2P_\mu^a P_\mu^b$ (CM frame)

$$= (\mathbf{p}, iE/c)(0, iM_0c) = -M_0E$$

Hence,

$$2M_0E = (2M_0 + m_0)^2c^2 - 2M_0^2c^2 = 2M_0^2c^2 + 4M_0m_0c^2 + m_0^2c^2$$

or $$E = M_0c^2 + 2m_0c^2 + \frac{m_0^2c^2}{2M_0}$$

Hence the minimum kinetic energy of the incident proton to produce π^0-meson is

$$T = E - M_0c^2 = 2m_0c^2 + \frac{m_0^2}{2M_0}c^2$$

The mass of a π^0-meson is equivalent to 134 MeV. This gives

$$T = 2 \times 134 + \frac{1}{2} \times \frac{m_0}{M_0} \times 134 = 278 \text{ MeV}$$

14.8. COVARIANT FORMULATION OF LAGRANGIAN AND HAMILTONIAN

We have disccussed in the previous chapter the relativistic formulation of the Lagrangian and Hamiltonian of a particle where the time t is treated as a parameter distinct from the space coordinates. A covariant formulation demands that the space and time coordinates be considered as entirely similar coordinates in four dimensional Minkowaski space.

A covariant form of ***Hamilton's principle*** is,

$$\delta S = \int L(x_\mu, u_\mu, \tau)\, d\tau$$

where L and S are world scalars. Here, L is a function of four-position vector x_μ, four-velocity vector $u_\mu\ (= dx_\mu/d\tau)$ and the proper time τ, instead $x_i(x, y, z), \dot{x}_i$ and t. This representation of Hamilton's principle is identically the same for all inertial frames.

We discuss the covariant Lagrangian formulation for two cases :

(1) A freely moving particle

(2) A particle moving in an external electromagnetic field.

(1) For freely moving particle : From the covariant form of the Hamilton's principle (66), we can deduce easily the Lagrange's equation for a single particle in the covariant form as follows :

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial u_\mu}\right) - \frac{\partial L}{\partial x_\mu} = 0 \qquad ...(67)$$

Let a free particle of rest mass m_0 move with four-velocity u_μ. The non- relativistic Lagrangian $\frac{1}{2}mu^2$ suggests the covariant form of the Lagrangian as

$$L = \sum_\mu \frac{1}{2} m_0 u_\mu u_\mu \qquad ...(68)$$

For such a definition of L, $\dfrac{\partial L}{\partial x_\mu} = 0, \dfrac{\partial L}{\partial u_\mu} = m_0 u_\mu = p_\mu$

These momentum components are in the covariant form similar to non-relativistic canonical momenta. Hence from (67), we obtain

$$\frac{d}{d\tau}(m_0 u_\mu) = \frac{dp_\mu}{d\tau} = 0 \qquad ...(69)$$

This is the *covariant equation of motion of a free particle.*

The space part of eq. (69) is

$$\frac{d}{dt}\left(\frac{m_0 \mathbf{u}}{\sqrt{1-\beta^2}}\right) = \frac{d}{dt}(m\mathbf{u}) = \frac{d\mathbf{p}}{dt} = 0$$

and the time part is

$$\frac{d}{dt}(m_0 u_4) = \frac{dp_4}{dt} = \frac{d}{dt}\left(\frac{im_0 c}{\sqrt{1-\beta^2}}\right) = 0 \text{ or } \frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right) = \frac{d}{dt}(mc^2) = \frac{dE}{dt} = 0$$

Thus the space part of (69) expresses the constancy of the momentum and time part the constancy of the total energy.

For consistent formulation, the covariant form of the Hamiltonian is to be defined as

$$H = \sum_\mu p_\mu u_\mu - L \qquad ...(70)$$

In case of a free particle,

$$H = \sum_\mu m_0 u_\mu u_\mu - \frac{1}{2}\sum_\mu m_0 u_\mu^2 = \sum_\mu \frac{1}{2} m_0 u_\mu^2 = -\frac{1}{2} m_0 c^2 \left(\text{as } \sum_{\mu=1}^{4} u_\mu^2 = -c^2\right) \qquad ...(71)$$

This expression $\left(-\frac{1}{2}m_0c^2\right)$ for the Hamiltonion is different from the total energy $E = m_0c^2 \Big/ \sqrt{1-\beta^2}$.

This non-equality of the Hamiltonian and the total energy should be a serious objection to the covariant formulation. However, this is not the case because the total energy E may be determined from the fourth component of the four momentum p_μ :

$$p_4 = \frac{iE}{c} = \frac{\partial L}{du_4} = m_0 u_4 = m_0 \frac{ic}{\sqrt{1-\beta^2}}$$

Therefore, $$E = \frac{m_0c^2}{\sqrt{1-\beta^2}} = mc^2$$

Thus the covariant formulation determines the energy as a derivative of the Lagrangian.

Corresponding to the Hamilton's equations, the canonical equations are

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x_\mu}, \quad \frac{dx_\mu}{d\tau} = \frac{\partial H}{\partial p} \qquad ...(72)$$

which retain their validity.

(2) For charged particle moving in an electromagnetic field : In case of a charged particle, moving in an electric field **E** and magnetic induction **B**, suitable form of the covariant Lagrangian is

$$L = \sum_\mu \left(\tfrac{1}{2}m_0 u_\mu u_\mu + qA_\mu u_\mu\right) \qquad ...(73)$$

which is Lorentz scalar (invariant).

The canonical four-momentum vector is

$$P_\mu = \frac{\partial L}{\partial u_\mu} = m_0 u_\mu + qA_\mu \qquad ...(74)$$

Substituting in the covariant form of the Lagrange's eq. (67), we get

$$\frac{d}{d\tau}(m_0 u_\mu + qA_\mu) = \frac{\partial}{\partial x_\mu}\left(\sum_\nu qA_\nu u_\nu\right)$$

or $$\frac{d}{d\tau}(m_0 u_\mu + qA_\mu) = \sum_\nu q u_\nu \frac{\partial A_\nu}{\partial x_\mu} \qquad ...(75)$$

The spatial part of eq. (75) is

$$\frac{d}{d\tau}(m_0 u_i + qA_i) = \frac{q\mathbf{u}}{\sqrt{1-\beta^2}} \cdot \frac{\partial \mathbf{A}}{\partial x_i} + \frac{qic}{\sqrt{1-\beta^2}} \frac{\partial (i\phi/c)}{\partial x_i}$$

But $$\frac{d}{d\tau}(m_0 u_i + qA_i) = \frac{dt}{d\tau}\frac{d}{d\tau}(m_0 u_i + qA_i) = \frac{1}{\sqrt{1-\beta^2}}\frac{d}{d\tau}(m_0 u_i + qA_i)$$

Hence $$\frac{d}{dt}(m_0 u_i) = -q\frac{dA_i}{dt} + q\mathbf{u} \cdot \frac{\partial \mathbf{A}}{\partial x_i} - \frac{q\partial\phi}{\partial x_i}$$

For X-component

$$\frac{d}{dt}\left(\frac{m_0 u_x}{\sqrt{1-\beta^2}}\right) = q\left(\frac{d}{dt}\frac{\partial}{\partial \dot{x}} - \frac{\partial}{\partial x}\right)(\phi - \mathbf{u} \cdot \mathbf{A})$$

or $$\frac{d(mu_x)}{dt} = \frac{dp_x}{dt} = F_x = q[E_x + (\mathbf{u} \times \mathbf{B})] \quad ...(76)$$

The time part is

$$\frac{1}{\sqrt{1-\beta^2}} \frac{d}{dt}\left(\frac{m_0 ic}{\sqrt{1-\beta^2}} + q\frac{i\phi}{c}\right) = q\frac{\mathbf{u}}{\sqrt{1-\beta^2}} \cdot \frac{\partial \mathbf{A}}{\partial (ict)} + \frac{qic}{\sqrt{1-\beta^2}} \frac{\partial (i\phi/c)}{\partial (ict)}$$

or $$\frac{d}{dt}\left(\frac{m_0 c^2}{\sqrt{1-\beta^2}}\right) = -\frac{qd\phi}{dt} - q\mathbf{u} \cdot \frac{\partial \mathbf{A}}{\partial t} + \frac{q\partial\phi}{\partial t}$$

But $$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} = \mathbf{u} \cdot \nabla\phi$$

where $\frac{\partial\phi}{\partial t} = 0$ as ϕ has no explicit time dependence.

Therefore, $$\frac{d}{dt}(mc^2) = -q\mathbf{u} \cdot \nabla\phi - q\mathbf{u} \cdot \frac{\partial \mathbf{A}}{\partial t}$$

or $$\frac{d}{dt}(mc^2) = -q\mathbf{u} \cdot \mathbf{E} \qquad \left[\text{because } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi\right]. \quad ...(77)$$

Thus the spatial part is the Lorentz force and the time part is the rate of change of energy.

It is to be noted that the canonical four-momentum vector

$$P_\mu = \frac{\partial L}{\partial u_\mu} = m_0 u_\mu + qA_\mu \quad ...(78)$$

has its first three components as

$$P_i = m_0 u_i + qA_i$$

So that $$P_x = \frac{m_0 u_x}{\sqrt{1-\beta^2}} + qA_x$$

which agrees with P_x of Sec 13.12. The fourth component of P_μ is

$$P_4 = m_0 u_4 + qA_4 = \frac{m_0 ic}{\sqrt{1-\beta^2}} + q\frac{i\phi}{c}$$

$$= \frac{i}{c}(mc^2 + q\phi) = iE'/c \quad ...(79)$$

where $E' = mc^2 + q\phi$ is the total energy (Sec.13.12.) of the particle.

The Hamiltonian is

$$H = \sum_\mu P_\mu u_\mu - L = \sum_\mu (m_0 u_\mu + qA_\mu)\, u_\mu - \sum_\mu \left(\tfrac{1}{2} m_0 u_\mu u_\mu + qA_\mu u_\mu\right)$$

$$= \sum_\mu \tfrac{1}{2} m_0 u_\mu u_\mu = -\tfrac{1}{2} m_0 c^2 \quad ...(80)$$

Thus H is a world scalar and is not equal to the total energy. However, the fourth component of the canonical momentum is proportional to the total energy.

14.9. GEOMETRICAL INTERPRETATION OF LORENTZ TRANSFORMATIONS : *MINKOWSKI DIAGRAMS*

Lorentz transformations can be interpreted geometrically and expressed by drawings, known as *Minkowski diagrams*.

Let us consider two frames of reference S and S'. S' frame is moving with velocity v relative to S along the common direction of their X-axes. For geometrical interpretation of Lorentz transformations, we consider only X-axis and perpendicular to it T-axis with $T = ct$ corresponding to S-frame. The coordinates of an event are then represented by (x, T). The Lorentz transformation for x and t are

$$x' = \gamma(x - \beta T) \qquad \text{...(81)}$$

$$T' = \gamma(T - \beta x) \qquad \text{...(82)}$$

where $\beta = v/c$ and $\gamma = 1\big/\sqrt{1 - v^2/c^2}$

Initially the origin $(O_{S'})$ of S'-frame coincider with the origin (O_S) of S-frame. The equation of motion of the origin $O_{S'}$ relative to O_S can be written as

$$x = vt \quad \text{or} \quad x = (v/c)T \qquad (T = ct) \qquad \text{...(83)}$$

We can obtain this equation from eq. (81) by setting $x' = 0$. Eq. (83) rep-epresents a straight line of the slope

$$\tan\theta = v/c = \beta$$

$(\angle TOT' = \theta)$ .This is the world line of the observer $O_{s'}$, as seen by the observer O_S. This will also be the world line of a particle moving with velocity v (at rest in S'-frame) along the X-axis of S-frame.

If a light signal is transmitted initially (at $t = t' = 0$), then for light $v = c$ and the world line for it will be a straight line $x = T$ with slope $\beta = 1$, making an angle of 45° with the X-axis [Fig 14.5]. Speeds less than c are indicated by world lines between that of light and the T-axis. World lines of material particles are between the T-axis and the world line of light.

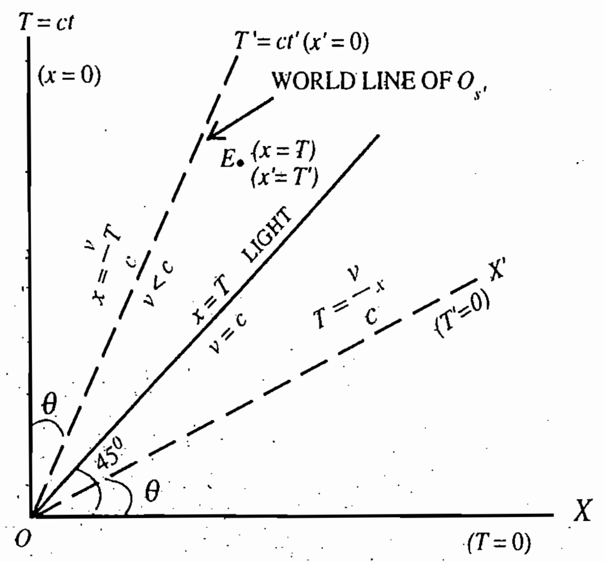


Fig 14.5

For S-frame, T-axis corresponds to $x = 0$ and X-axis to $T = 0$ or $t = 0$. Similarly, the world line $x' = 0$ is the T' (or ct') axis corresponding to S'-frame. Now for $T' = 0$,

$$T = \beta x \text{ or } T = \frac{v}{c}x \qquad \text{...(84)}$$

If we draw the world line $T = (v/c)x$ (for $T' = 0$), this will represent the X'-axis of S'-frame. Fig. 14.5 shows T'- and X'-axes on the T-X Minkowski diagram.

Note that the X'- axis is inclined with X-axis by the same angle θ as the T'- axis with T-axis. [Fig 14.5] where $\tan\theta = v/c$.

If (x, T) be the coordinates of an event in frame S and (x', T') of the same event in frame S', then let us see that (81) and (82) are the correct transformation equations relating geometically the coordinates of T-X (orthogonal) and T'-X' systems.

We know that the Lorentz transformations are characterized by

$$c^2t'^2 - x'^2 = c^2t^2 - x^2 = \text{ constant}$$

or $$T'^2 - x'^2 = T^2 - x^2 = 1 \qquad \text{...(85)}$$

where we have chosen the common constant equal to unity by a suitable choice of units for measurements. Thus

$$T^2 - x^2 = 1 \qquad \text{...(86)}$$

$$T'^2 - x'^2 = 1 \qquad \text{...(87)}$$

For the O_S observer of S-frame, T- and X-axes are rectangular and the world line corresponding to eq. (86) is hyperbola [Fig 14.6] with apex at the point (0, 1). The length OA is equal to 1, $x = T$ (OC) and $x = -T(OC')$ are the asymptotes of the hyperbola. Coresponding to T'-and X'-axes, the hyperbola of Fig. 14.6 satisfies the eq. (87). In the T'–X' system (corresponding to S'-frame), A' is the apex of the hyperbola with $OA' = 1$. It is seen directly from the figure that the unit lengths of X and X' axes are not equal.

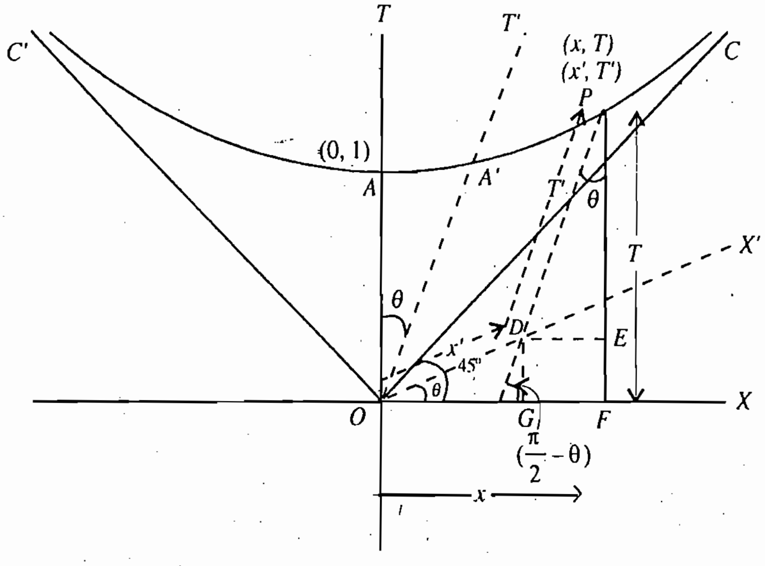


Fig. 14.6

The coordinates of A' can be obtained in T-X system by solving the equations : $T^2 - x^2 = 1$ (hyperbola) and $x = \beta T$ (T'-axis) *i.e.*, $\left(\dfrac{\beta}{\sqrt{1-\beta^2}}, \dfrac{1}{\sqrt{1-\beta^2}}\right)$ with the length $OA' = \sqrt{\dfrac{1+\beta^2}{1-\beta^2}}$. This length is unity (1) is T'-X' system. Hence the scale factor involved in the transformation from T'-X' (S'- frame) to T-X (S-frame) system :

$$f = \sqrt{\frac{1+\beta^2}{1-\beta^2}} \text{ or } f = \sqrt{\frac{1+\tan^2\theta}{1-\tan^2\theta}} = \frac{1}{\sqrt{\cos 2\theta}} \qquad \text{...(88)}$$

Now, if we refer any world point $P(x,T)$ in T-X system and (x', T') in T'-X' system, the equations connecting the coordinates of two systems are :

$$T = PF = (PE + EF) = \frac{PD\cos\theta + OD\sin\theta}{\sqrt{\cos 2\theta}} \text{ or } T = \frac{T'\cos\theta + x'\sin\theta}{\sqrt{\cos 2\theta}}$$

or
$$ct = \frac{ct'\cos\theta + x'\sin\theta}{\sqrt{\cos 2\theta}}$$

and
$$x = (OG + GF) = \frac{x'\cos\theta + T'\sin\theta}{\sqrt{\cos 2\theta}} \text{ or } x = \frac{x'\cos\theta + ct'\sin\theta}{\sqrt{\cos 2\theta}}$$

But $\tan\theta = v/c$, hence

$$\sin\theta = \frac{v/c}{\sqrt{1+v^2/c^2}} = \frac{\beta}{\sqrt{1+\beta^2}} \text{ and } \cos\theta = \frac{1}{\sqrt{1+v^2/c^2}} = \frac{1}{\sqrt{1+\beta^2}}$$

Thus
$$ct = \left(\frac{ct'}{\sqrt{1+\beta^2}} + \frac{x'\beta}{\sqrt{1+\beta^2}}\right)\sqrt{\frac{1+\beta^2}{1-\beta^2}} \text{ and } x = \left(\frac{x'}{\sqrt{1+\beta^2}} + \frac{ct'\beta}{\sqrt{1+\beta^2}}\right)\sqrt{\frac{1+\beta^2}{1-\beta^2}}$$

or
$$x = \gamma(x' + vt') \text{ and } t = \gamma\left(t' + \frac{vx}{c^2}\right)$$

which are just the desired Lorentz transformations. Thus *we have obtained a geometrical representation in the Minkowski world.*

We note that as the physical parameter v/c of the Lorentz transformations can have values in the range -1 to $+1$, the corresponding parameter θ ($\tan\theta = v/c$) will take up values from $-\pi/4$ to $+\pi/4$. This means that T'-axis sweeps out the range from one asymptote to the other asymptote of the hyperbola. The axes of T-X system are mutually perpendicular, while those of T'-X' system are oblique. From this one may have false idea that frame S is in the privileged position than S'-frame. This idea is completely incorrect because we may start with the S'-frame with T' and X' axes as orthogonal axes. Now the axes of T-X system will be oblique axes as shown in Fig14.7 because now we consider S-frame is moving with $-v$ velocity relative to S along common X-axis. Obviously the scale factor f is the same and the equations connecting the coordinates of two systems are [with $\tan(-\theta) = -v/c$ or $\tan\theta = v/c$]:

$$T' = PD - DE = f(T\cos\theta - x\sin\theta)$$

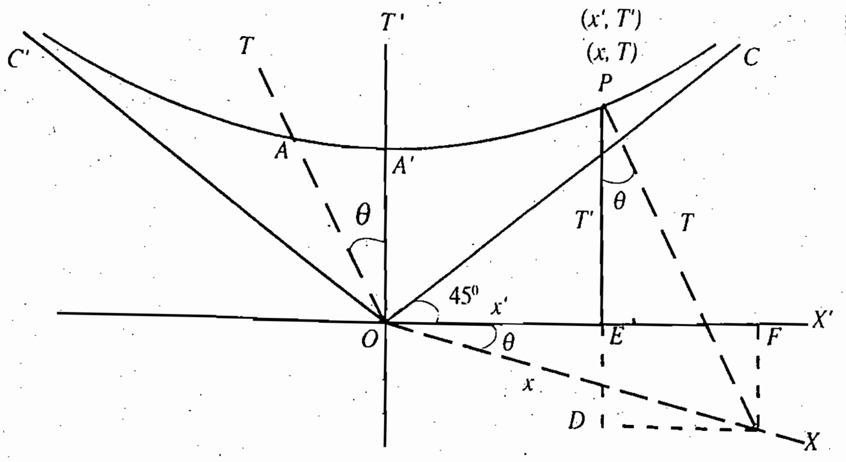


Fig. 14.7

and $$x' = OF - EF = f(x\cos\theta - T\sin\theta)$$

or $$x' = \gamma(x - vt) \text{ and } t' = \gamma(t - vx/c^2).$$

Thus again we obtain the Lorentz transformations.

14.10. GEOMETRICAL REPRESENTATION OF SIMULTANEITY, LENGTH CONTRACTION AND TIME DILATION

Let us consider two frames S and S'. Frame S' moves with velocity v relative to S along its positive X-axis.

(1) Simultaneity : The two events are said to be simultaneous in system S' if they have the same time coordinate T'. Thus any two events lie parallel to X'-axis, they are simultaneous

in system S'. In Fig.14.8, E_1 and E_2 events are simultaneous in system S' but they are not simultaneous in system S because they have different time coordinates T_1 and T_2 in system S. In the same way, two events E_3 and E_4 are simultaneous in frame S but not in S' because they have the same time coordinate T in S and different time coordinates T_1' and T_2' in S'.

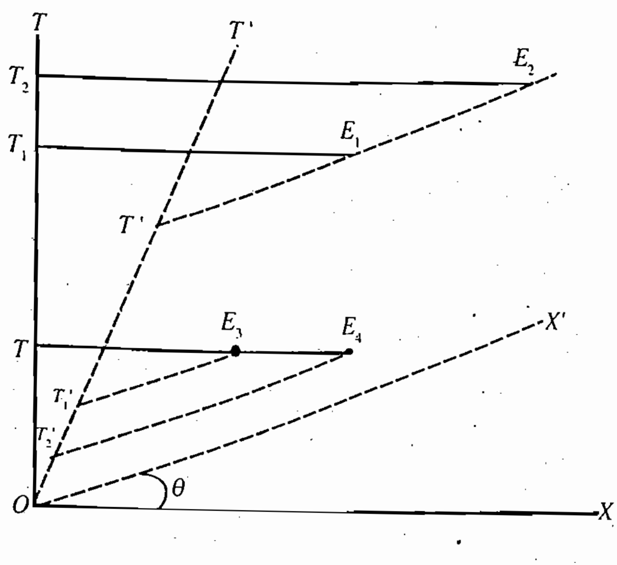


Fig 14. 8 : Geometrical representation of simultaneity

(2) Length contraction : Let a rod of length l_0 be at rest in the frame S, placed parallel to the X-axis [fig.14.9]. As time increases, the world line of each end point is a vertical line parallel to T-axis (AC and BD lines). The length of the rod is the distance between its end points measured simultaneously ($l_0=AB$). In the system S, the rod is at rest and its length is the distance between the intersecting points (A and B) of the world lines with the X-axis (or any line parallel to X-axis, because these points represent simultaneous events in S or (T-X system). From the point of view of the system S' the rod is moving with a velocity $-v$, *i.e.*,in the negative direction of X'-axis. The length of the rod l in S' would then be the difference between the simultaneous measurements of the coordinates of its ends (corrected for scale factor f). Thus

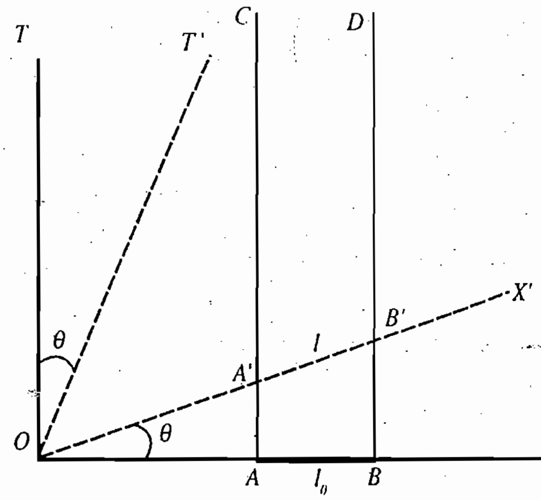


Fig 14.9 : Geometrical representation of length contraction

$$l_0 = fl\cos\theta = \sqrt{\frac{1+\beta^2}{1-\beta^2}}.l\frac{1}{\sqrt{1+\beta^2}} \text{ or } l = l_0\sqrt{1-\beta^2} = l_0\sqrt{1-v^2/c^2}$$

which is the same result as we obtained in Sec 12.11.

If the rod of length l_0 (AB) is at rest in S' placed along X'-axis, the world lines of its end points are parallel to T'-axis [Fig.14.10]. In system S, the rod is in motion with velocity v, the measured length is the distance $A'B' = l$ between intersecting points of these world lines with X-axis or any line parallal to X-axis because this will give the length in S measuring its end points simultaneously. Hence

$$\frac{l}{\sin\left(\frac{\pi}{2}-2\theta\right)} = \frac{fl_0}{\sin\left(\frac{\pi}{2}+\theta\right)} \text{ or } l = fl_0\frac{\cos 2\theta}{\cos\theta}$$

Thus $$l = \frac{l_0}{\sqrt{\cos 2\theta}}\cdot\frac{\cos 2\theta}{\cos\theta} = l_0\frac{\sqrt{\cos 2\theta}}{\cos\theta}$$

$$= l_0\sqrt{\frac{1-\beta^2}{1+\beta^2}}\cdot\frac{\sqrt{1+\beta^2}}{1} = l_0\sqrt{1-v^2/c^2}$$

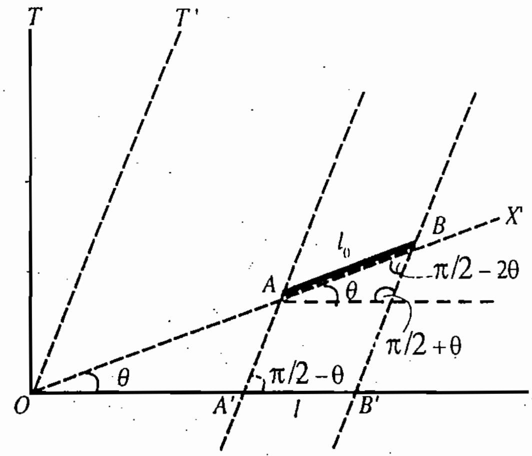


Fig. 14.10 : Length Contraction

which is the expected result.

(3) Time Dilation : Let us now discuss a geometric illustration of the relativity of the time intervals [Fig.14.11]. For this purpose, let us consider a clock at rest in frame S. The clock ticks the events E_1 and E_2 of its needle at the interval $\Delta T = c\Delta\tau$, where $\Delta\tau$ is the proper interval of time in system S. E_1E_2 line perpendicular to X-axis is the world line for the clock of system S. In frame S', relative to which the clock is in motion, the clock ticks each time at a different place. The time interval of these events E_1 and E_2 recorded in system S', is $T'_1T'_2 = \Delta T' = c\Delta t'$.

From the Fig.14.11,we see that

$$\frac{\Delta T}{\sin\left(\frac{\pi}{2}-2\theta\right)}=\frac{f\Delta T'}{\sin\left(\frac{\pi}{2}+\theta\right)} \quad \text{or} \quad \Delta t'=\frac{\Delta\tau\cos\theta}{f\cos 2\theta}$$

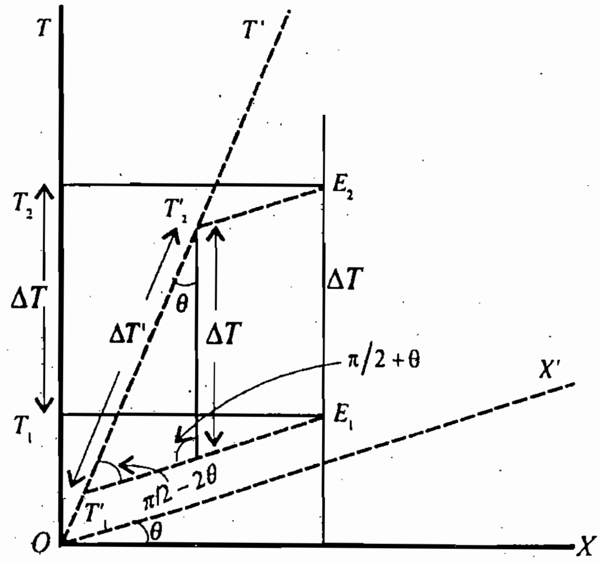


Fig 14.11 : Geometrical representation of time dilation

or $$\Delta t'=\frac{\Delta\tau\cos\theta}{\sqrt{\cos 2\theta}}=\Delta\tau\frac{1}{\sqrt{1+\beta^2}}\sqrt{\frac{1+\beta^2}{1-\beta^2}}. \qquad \left[\because f=\frac{1}{\sqrt{\cos 2\theta}}\right]$$

or $$\Delta t'=\frac{\Delta\tau}{\sqrt{1-v^2/c^2}}.$$

which is the time-dilation result. Thus the time of any two ticks of the needle of the clock in system S will be recorded more in the clock of system S' and hence to the observer of S', relative to which the clock of S is moving, will appear to go slow. The reciprocal nature of the time dilation result for a clock at rest in S' moving relative to S, can be shown similarly.

The mutual reciprocality of the results of length contraction and time dilation in two inertial frames shows that in the Minkowski space, though one frame has rectangular axes and the other oblique, the conclusions are totally consistent with the fact that all the inertial frames are on an equal footing.

Questions

1. What is Minkowski space ? Show that the Lorentz transformations can be regarded as transformations due to a rotation of axes in the four-dimensional Minkowski space. Hence deduce the Lorentz transformations. **(Agra 2003, 2000)**

2. Discuss the principle of relativity and the invariance of speed of light.Usethis principle to deduce the Lorentz transformations in four dimensional space. Discuss the relativity of simultaneity. **(Agra 2001, 1994 S)**

3. Discuss Minkowski four-dimensional space. Write velocity acceleration and momentum as four-vectors and hence write Lorentz transformations of these four-vectors. Discuss the conservation of four-momentum. **(Agra 1995)**

4. Considering the conservation of four-momentum determine the relation-ship between mass, momentum and energy. **(Agra 2002, 01, 1993)**
5. (*a*) Explain Minkowski's four dimensional space. **(Rohilkhand 1993, 92)**
   (*b*) Write notes on 'Four-dimensional space-like and time-like intervals and their significance'. **(Agra 1999)**
   (*c*) Show that four-dimensional volume element $dx\,dy\,dz\,dt$ is invariant under Lorentz transformations. **(Rohilkhand 1990)**
6. Discuss space-like and time-like intervals. Discuss the time order of two events in the two cases of intervals. **(Agra 1994S, 75; Rohilkhand 1979)**
7. Define a four-vector. What are velocity, momentum and force four vectors.
8. Define a four-vector. How are the components of the four-momentum vector related to the three-momentum of a particle ? **(Meerut 1998)**
9. What is a four-vector ? Show that the scalar product of two four vectors is invariant under Lorentz transformations. **(Agra 2000)**
10. (*a*) Discuss the principle of conservation of four momentum. Discuss its use in collision problem. **(Agra 2003, 2000)**
    (*b*) Show that when an energetic proton collides with a proton at rest, a proton-antiproton will be produced only when the least kinetic energy of the incident proton is $6M_p c^2$, where M_p is the rest mass of proton. **(Agra 2000)**
11. Explain Minkowski's four-dimensional formalism bringing out the significance of the fourth component of momentum and the equation of motion. **(Banaras 1970)**
12. Considering the conservation of four momentum, determine the relationship between energy, momentum and mass. **(Agra 1999)**
13. Explain the meaning of Minkowski force and discuss the equlibrium of a right handed lever in a moving intertial frame. **(Agra 1999)**
14. A particle with charge q and mass m has the covariant Lagrangian given by

$$L = -\frac{1}{2} m u_\nu u_\nu - q A_\nu u_\nu$$

where u_ν and A_ν are the velocity and potential four-vectors respectively. Find out the Hamiltonian of the particle and show that it is Lorentz invariant.

Problems

1. A proton has velocity $0.999c$ in the laboratory frame. Find the energy and momentum as observed in a frame travelling in the same direction with a velocity $0.99c$ relative to the laboratory frame. (Rest mass of proton = 1.67×10^{-27} kg)
   Ans. : 7.97×10^{-19} kg-m/sec ; 1.5 BeV.
2. Show by the direct application of Lorentz transformations that $x^2 + y^2 + z^2 + \omega^2$ is invariant, where $\omega = ict$, $i = \sqrt{-1}$. **(Kanpur 1980, 75)**
3. In a frame S, two events have the space-time coordinates (0, 0, 0, 0) and ($10c$, 0, 0, 6). Find the spce-time interval between them. Calculate the velocity of a frame in which
   (*i*) the two events are simultaneous,
   (*ii*) the first event occurs 8 sec. earlier than the second, and
   (*iii*) the second event occurs 8 sec earlier than the first.
   Ans. : (*i*) 0.6 c, (*ii*) $-$ 0.18 c, (*iii*) 0.91 c.
4. Show that the Lorentz transformation can be represented in the vector form as

$$\mathbf{r}' = \mathbf{r} + (\gamma - 1)\frac{(\mathbf{r} \cdot \mathbf{v})\mathbf{v}}{v^2} - \gamma\, \mathbf{v} t.$$

Also the velocity transformation is

$$\mathbf{V}' = \frac{\mathbf{V} + (\gamma - 1)(\mathbf{V}\cdot\mathbf{v})(\mathbf{v}/v^2) - \gamma\mathbf{v}}{\gamma(1 - \mathbf{v}\cdot\mathbf{V}/c^2)}$$

5. Two particles with rest masses m_1 and m_2 move along the X-axis of an inertial system with velocities v_1 and v_2 respectively. They collide and coalesce to form a single particle. Show that the rest mass m_3 and velocity v_3 of the resulting single particle is given by

$$m_3^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma_1\gamma_2\left[1 - \frac{v_1v_2}{c^2}\right] \text{ and } v_3 = \frac{\gamma_1 m_1 v_1 + \gamma_2 m_2 v_2}{\gamma_1 m_1 + \gamma_2 m_2}$$

where $\gamma_1 = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2}$ and $\gamma_2 = \left(1 - \frac{v_2^2}{c^2}\right)^{-1/2}$

6. A particle of initial kinetic energy T_0 and rest energy E_0 strikes like particle at rest. Show that the kinetic energy T of the particle deflected at an angle θ is

$$T = \frac{T_0\cos^2\theta}{1 + \frac{T_0}{2E_0}\sin^2\theta}.$$

7. Derive an expresion for the threshold energy of a bombarding particle which strikes the target at rest in the following type reaction :

$$a + b \to c + d + e + \ldots\ldots\ldots\ldots\ldots$$

with $m_a + m_b < m_c + m_d + m_e + \ldots\ldots\ldots\ldots$

and m_α ($\alpha = a, b, \ldots\ldots\ldots$) being the rest masses of $a, b, c, \ldots\ldots\ldots$particles.

Ans. : $T = \Delta m\left(1 + \frac{m_1}{m_2} + \frac{\Delta m}{2m_2}\right)c^2,$

where $\Delta m = m_c + m_d + m_e + \ldots\ldots\ldots\ldots - (m_1 + m_2)$.

8. Determine the threshold energy for the reaction

$$P + \overline{P} \to \Lambda_0 + \overline{\Lambda}_0$$

where P and $\overline{P}$ denote the proton and antiproton respectively, while Λ_0 and $\overline{\Lambda}_0$ represent the neutral lembda particle and antilembda particle. Given, rest mass of the proton or antiproton = 1836 m_e, rest mass of the lembda or antilembda particle = 2182 $m_e^\bullet$, rest mass of electron $m_e = 0.911 \times 10^{-30}$ kg.
Ans. : 763 MeV.

9. X-rays of wavelength 0.5 Å are scattered by free electrons in a block of carbon through 90°. Find the momentum of the (*i*) incident photons, (*ii*) scattered photons, (*iii*) recoil electrons and the energy of recoil electrons.

Ans. : (*i*) 1.32×10^{-23} kg-m/s, (*ii*) 1.26×10^{-23} kg-m/s, (*iii*) 1.82×10^{-23} kg-m/s, (*iv*) 1.92×10^{-16} joules.

Objective Type Questions

1. In Minkowski space
(*a*) the space interval between two points is invariant.
(*b*) the time interval between two points is invariant.
(*c*) the space-time interval between two points is invariant.
(*d*) the space-time intervel between two points is different for different observers.
Ans. : (*c*).

2. In case of space-like interval there exists a frame of reference in which for two events,
   (*a*) one event may occur earlier than the second ($t_1 < t_2$)
   (*b*) the other event may occur earlier than the first ($t_1 > t_2$)
   (*c*) both events may occur simultaneously ($t_1 = t_2$)
   (*d*) both events can not occur simultaneously ($t_1 \neq t_2$).
   Ans. : (*a*), (*b*), (*c*).
3. Choose the correct statement/s :
   (*a*) For space-like interval, two events cannot be connected by any real physical process.
   (*b*) For time-like interval, two events cannot be connected by any real physical process.
   (*c*) For space-like interval, two events can be connected by any real physical process.
   (*d*) For time-like interval, two events can be connected by any real physical process.
   Ans. : (a), (d).
4 The scalar product
   (*a*) $p_\mu u_\mu$ is invariant. (*d*) $p_\mu p_\mu$ is invariant.
   (*c*) $u_\mu u_\mu$ is invariant. (*d*) $p_\mu u_\mu$ is not invariant.
   Ans. : (*a*), (*b*), (*c*).
5. Choose the correct statements :
   (*a*) For the surface of the light cone, the square of the space-time interval (s^2) is zero.
   (*b*) For the surface of the light cone, s^2 cannot be zero.
   (*c*) Inside the light cone, $s^2 < 0$.
   (*d*) Inside the light cone, $s^2 > 0$.
   Ans : (*a*), (*c*).

Short Type Questions

1. What is Minkowski Space ?
2. What are world point and world line ?
3. What is space-time interval ? Show that this interval is invariant under Lorentz transformation.
4. What are space-like and time-like intervals ?
5. Show that for space-like interval, any two events cannot be connected by any real physical process.
6. What are world regions ?
7. What do you understand by light cone ? What are absolute future and absolute past ?
8. What are four-vectors ?
9. Show that $A'\mu\, B'\mu = A\mu\, B\mu$
10. Write the transformation equations for momentum four-vector.
11. What is Minkowski force equation
12. What is four-force ? Show that it can be expressed as

$$F\mu = \left(\frac{\mathbf{F}}{\sqrt{1-\mathrm{u}^2/c^2}}, \frac{i\,\mathbf{F}\,.\,\mathbf{u}}{c\sqrt{1-\mathrm{u}^2/c^2}}\right)$$

13. Discuss the conservation of four-momentum in two-particle collision.
14. Using the idea of conservation of four-momentum, show that the value of minimum kinetic energy to produce a pair of proton and antiprotons is 5.6 BeV.
15. Fill in the blanks :
    (*i*) For space-like interval, the time separation between two events is....................than the time taken by light in covering the distance between them.
    (*ii*) The value of square of the space-time interval is....................on the surface of the sight cone.
    Ans. (*i*) less (*ii*) zero.

CHAPTER 15 Covariant Formulation of Electrodynamics

15.1. D'ALEMBERTIAN OPERATOR $\square^2$

Three dimensional gradient operator is defined as

$$\nabla = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \text{ or } \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

We extend this definition to define the four dimensional gradient operator, given by

$$\nabla_\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{i}{c}\frac{\partial}{\partial t}\right) \qquad ...(1)$$

This operator behaves as a **four-vector**. If ϕ is a scalar in four dimensions, *i.e.*, Lorentz invariant, then $\nabla_\mu \phi$ is a four-vector field.

D'Alembertian operator ($\square^2$) is the scalar product of the four dimensional gradient operator with itself, *i.e.*,

$$\nabla_\mu \nabla_\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{i}{c}\frac{\partial}{\partial t}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{i}{c}\frac{\partial}{\partial t}\right)$$

or

$$\square^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \text{ or } \square^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \qquad ...(2)$$

Invariance of D'Alembertian Operator : *Let us consider two inertial coordinate systems S and S'. The system S' is moving with constant velocity v relative to S along X-axis. The D'Alembertian operator is invariant under Lorentz transformations, i.e.,*

$$\square^2 (\text{in } S) = \square'^2 (\text{in } S').$$

Proof : We may write

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}$$

Inverse Lorentz transformations are

$$x = \gamma(x' + vt'),\ y = y',\ z = z',\ t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Therefore, $\dfrac{\partial x}{\partial x'} = \gamma$, $\dfrac{\partial x}{\partial t'} = \gamma v$, $\dfrac{\partial t}{\partial t'} = \gamma$, $\dfrac{\partial t}{\partial x'} = \dfrac{\gamma v}{c^2}$

Hence $$\frac{\partial}{\partial x'} = \gamma\frac{\partial}{\partial x} + \frac{\gamma v}{c^2}\frac{\partial}{\partial t}$$

and $$\frac{\partial^2}{\partial x'^2} = \frac{\partial}{\partial x'}\left(\gamma\frac{\partial}{\partial x} + \frac{\gamma v}{c^2}\frac{\partial}{\partial t}\right)$$

$$= \frac{\partial x}{\partial x'}\frac{\partial}{\partial x}\left(\gamma\frac{\partial}{\partial x}\right) + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}\left(\gamma\frac{\partial}{\partial x}\right) + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x}\left(\frac{\gamma v}{c^2}\frac{\partial}{\partial t}\right) + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}\left(\frac{\gamma v}{c^2}\frac{\partial}{\partial t}\right)$$

or $$\frac{\partial^2}{\partial x'^2} = \gamma^2\frac{\partial^2}{\partial x^2} + \frac{2\gamma^2 v}{c^2}\frac{\partial^2}{\partial t\partial x} + \left(\frac{\gamma v}{c^2}\right)^2\frac{\partial^2}{\partial t^2}.$$

Also, $$\frac{\partial^2}{\partial y'^2} = \frac{\partial^2}{\partial y^2}, \quad \frac{\partial^2}{\partial z'^2} = \frac{\partial^2}{\partial z^2}$$

Similarly, $$\frac{\partial^2}{\partial t'^2} = \gamma^2 v^2\frac{\partial^2}{\partial x^2} + 2\gamma^2 v\frac{\partial^2}{\partial x\partial t} + \gamma^2\frac{\partial^2}{\partial t^2}$$

or $$-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = -\gamma^2\frac{v^2}{c^2}\frac{\partial^2}{\partial x_1^2} - \frac{2\gamma^2 v}{c^2}\frac{\partial^2}{\partial x\partial t} - \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t^2}$$

Thus $$\Box'^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}$$

$$= \gamma^2\frac{\partial^2}{\partial x^2} + \frac{2\gamma^2 v}{c^2}\frac{\partial^2}{\partial t\partial x} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\gamma^2 v^2}{c^2}\frac{\partial^2}{\partial x^2} - \frac{2\gamma^2 v}{c^2}\frac{\partial^2}{\partial t\partial x} - \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t^2}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box^2 \qquad ...(3)$$

15.2. MAXWELL'S FIELD EQUATIONS

In S. I. system, Maxwell's Field equations for electormagnetic field in vacuum or empty space are given by

(*i*) $\text{div}\,\mathbf{E} = \dfrac{\rho}{\epsilon_0}$, (*ii*) $\text{div}\,\mathbf{B} = 0$,

(*iii*) $\text{curl}\,\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$, (*iv*) $\text{curl}\,\mathbf{B} = \mu_0\left(\epsilon_0\dfrac{\partial \mathbf{E}}{\partial t} + \mathbf{j}\right)$ ...(4)

where **E** is electric field strentgth, **B** magnetic induction, ρ charge density and **j** current density. The quantities ϵ_0 and μ_0 are the permittivity and permeability of the free space.

If the charges are moving with velocity $\boldsymbol{u}$, then the current density is given by

$$\mathbf{j} = \rho\boldsymbol{u} \qquad ...(5)$$

From Maxwell's field equation (ii)

$$\nabla\cdot\mathbf{B} = 0$$

This means that **B** can be expressed as

$$\mathbf{B} = \nabla\times\mathbf{A} \quad \text{(because div curl } \mathbf{A} = 0) \qquad ...(6)$$

where **A** is called ***vector potential.***

From eq, (*iii*), we have

$$\nabla\times\mathbf{E} = -\frac{\partial(\nabla\times\mathbf{A})}{\partial t} = \nabla\times\left(-\frac{\partial\mathbf{A}}{\partial t}\right)$$

Therefore, $$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \text{grad}\ \phi \quad ...(7)$$

where ϕ is called the ***scalar potential*** function.

Thus from eqs. (6) and (7) we may determine **B** and **E** in terms of vector potential **A** and scalar function ϕ.

Equation of Continuity–Conservation of Charge : Form Maxwell's field equation (*iv*)

$$\text{div curl}\ \mathbf{B} = \mu_0\left(\epsilon_0\ \text{div}\frac{\partial\mathbf{E}}{\partial t} + \text{div}\,\mathbf{j}\right) = \mu_0\left[\epsilon_0\frac{\partial}{\partial t}(\text{div}\,\mathbf{E}) + \text{div}\,\mathbf{j}\right]$$

But div curl **B** = 0, hence

$$\epsilon_0\frac{\partial}{\partial t}(\text{div}\,\mathbf{E}) + \text{div}\,\mathbf{j} = 0$$

or $$\frac{\partial\rho}{\partial t} + \text{div}\,\mathbf{j} = 0 \qquad \left(\text{because div}\,\mathbf{E} = \frac{\rho}{\epsilon_0}\right) \quad ...(8)$$

Eq. (8) is known as the ***equation of continuity*** and represents the law of conservation of charge.

Gauge Transformation : The solution of Maxwell's field equations gives the expressions for the electromagnetic field vectors **E** and **B** in terms of vector potential **A** and scalar potential ϕ as

$$\mathbf{B} = \text{curl}\,\mathbf{A} \text{ and } \mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \text{grad}\ \phi \quad ...(9)$$

However, these solutions are not unique, because in the former case, an addition of the gradient of scalar function s to **A** does not change the magnetic field vector. Thus if we define

$$\mathbf{A}' = \mathbf{A} + \text{grad}\ s, \quad ...(10)$$

then $\mathbf{B}' = \text{curl}\,\mathbf{A}' = \text{curl}\,(\mathbf{A} + \text{grad}\ s) = \text{curl}\,\mathbf{A} + \text{curl grad}\ s = \text{curl}\,\mathbf{A} = \mathbf{B}$ (because curl grad s = 0)

Further if we require that the new electric field vector **E'** does not change when **A** is replaced by **A'**, then ϕ is to be changed to ϕ'. In such a case

$$\mathbf{E}' = -\frac{\partial\mathbf{A}'}{\partial t} - \text{grad}\ \phi' = -\frac{\partial}{\partial t}(\mathbf{A} + \text{grad}\ s) - \text{grad}\ \phi' = -\frac{\partial\mathbf{A}}{\partial t} - \text{grad}\left(\phi' + \frac{\partial s}{\partial t}\right)$$

Now if we define

$$\phi' + \frac{\partial s}{\partial t} = \phi \quad \text{or} \quad \phi' = \phi - \frac{\partial s}{\partial t} \quad ...(11)$$

Then $$\mathbf{E}' = -\frac{\partial\mathbf{A}}{\partial t} - \text{grad}\ \phi = \mathbf{E}$$

The equations (10) and (11) which leave the electromagnetic field vectors **E** and **B** to be the same, are called ***gauge transformation***.

Therefore, the Maxwell's field equations (4) will remain unchanged (or invariant) under gauge transformation.

15.3. MAXWELL'S EQUATIONS IN TERMS OF ELECTROMAGNETIC POTENTIALS A AND ϕ

Maxwell's field equations are :

(*i*) $\text{div } \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ (*ii*) $\text{div } \mathbf{B} = 0$

(*iii*) $\text{curl } \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (*iv*) $\text{curl } \mathbf{B} = \mu_0\left(\epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} + \mathbf{j}\right)$

From (*ii*) and (*iii*), the electromagnetic field vectors **E** and **B** are expressed as

$$\mathbf{B} = \text{curl } \mathbf{A} \text{ and } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi \qquad ...(12)$$

Now, $\text{div } \mathbf{E} = \text{div}\left(-\dfrac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi\right) = -\dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) - \nabla^2\phi$

and $\text{curl } \mathbf{B} = \text{curl curl } \mathbf{A} = \text{grad div } \mathbf{A} - \nabla^2 \mathbf{A}$

Substituting for div **E** and curl **B** in Maxwell's equations (*i*) and (*iv*), we obtain

$$-\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \frac{\rho}{\epsilon_0}$$

and $$\text{grad div } \mathbf{A} - \nabla^2 \mathbf{A} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \mathbf{A}}{\partial t} - \text{grad } \phi\right) + \mu_0 \mathbf{j} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \frac{1}{c^2}\text{grad}\left(\frac{\partial \phi}{\partial t}\right) + \mu_0 \mathbf{j}$$

or $$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}(\text{div } \mathbf{A}) = -\frac{\rho}{\epsilon_0}$$

and $$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\text{div}\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \mathbf{j}$$

Thus $$\Box^2 \mathbf{A} - \text{grad}\left(\text{div } \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\mu_0 \mathbf{j} \qquad ...(13)$$

and $$\Box^2 \phi + \frac{\partial}{\partial t}\left(\text{div } \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\epsilon_0} \qquad ...(14)$$

Eqs. (13) and (14) are the ***equations of motion for the electromagnetic potentials*** **A** and ϕ. These equations represent the Maxwell's equations in terms of the potentials **A** and ϕ and in fact the four equations are reduced to two, eqs. (13) and (14).

Eqs. (13) and (14) are the coupled equations and these equations become in much simplified form, if we define div **A** such that

$$\text{div } \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \qquad ...(15)$$

It may be noted that eq. (12) specifies the curl of **A** and div **A** is still unspecified. Hence one may choose div **A** according to eq. (15). This requirement (15) is called the ***Lorentz condition*** and expresses a relation between **A** and ϕ.

When the Lorentz condition is imposed on eqs. (13) and (14), the Maxwell's field equations assume the form

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{j} \qquad ...(16)$$

$$\Box^2 \phi = -\frac{\rho}{\epsilon_0} \qquad ...(17)$$

Eqs. (16) and (17) are uncoupled second order differential equations and are known as ***D'Alembertian equations for electromagnetic potentials***.

Lorentz condition puts restriction on the gauge function s, appearing in the gauge transformation, given by

$$\mathbf{A}' = \mathbf{A} + \text{grad}\, s \text{ and } \phi' = \phi - \frac{\partial s}{\partial t}$$

If the Lorentz condition for $\mathbf{A}'$ and ϕ' is satisfied, then

$$\text{div}\, \mathbf{A}' + \frac{1}{c^2}\frac{\partial \phi'}{\partial t} = 0 \text{ or } \text{div}\,(\mathbf{A} + \text{grad}\, s) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\phi - \frac{\partial s}{\partial t}\right) = 0$$

or
$$\text{div}\, \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla^2 s - \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} = 0 \text{ or } \text{div}\, \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} + \Box^2 s = 0.$$

Thus the Lorentz condition under gauge transformation is satisfied, if

$$\Box^2 s = 0 \qquad ...(18)$$

We find that the Lorentz condition is invariant under those gauge transformations for which the condition (18) is satisfied and the relevant gauge transformation is known as ***Lorentz gauge***.

15.4. CURRENT FOUR VECTOR

Charges occur in units of elementary charge e (= 1.6×10^{-19} coulomb). The total number of charges on a body cannot depend of the state of the motion of the observer. Hence we may state that total electric charge in an isolated system is relativistically invariant.

Let the charge in dV volume be dq. If $dq/dV = \rho$ be the charge density, then

$$dq = \rho\, dV \quad \text{or} \quad dq = \rho\, dx_1\, dx_2\, dx_3$$

Multiply both sides by four vector dx_μ, we get

$$dq\, dx_\mu = \rho\, dx_\mu\, dx_1\, dx_2\, dx_3 \text{ or } dq\, dx_\mu = \rho \frac{dx_\mu}{dt} dx_1\, dx_2\, dx_3\, dt \qquad ...(19)$$

Now, $\quad dx_1\, dx_2\, dx_3\, dt = dx_1\, dx_2\, dx_3 \dfrac{dx_4}{ic} \qquad (x_4 = ict)$

which is invariant or scalar.

In eq. (19) the charge is invariant (or scalar) on left hand side and hence left hand side represents a four-vector. The quantity $dx_1\, dx_2\, dx_3\, dt$ is scalar on right hand side and therefore $\rho\, dx_\mu/dt$ will be a four-vector. We represent this four vector by j_m *i.e.*,

$$j_\mu = \rho \frac{dx_\mu}{dt}$$

Therefore, $\quad j_1 = \rho \dfrac{dx_1}{dt} = \rho u_1, j_2 = \rho \dfrac{dx_2}{dt} = \rho u_2, j_3 = \rho \dfrac{dx_3}{dt} = \rho u_3$, and $j_4 = \rho \dfrac{dx_4}{dt} = \rho \dfrac{d(ict)}{dt} = ic\rho$.

The four-vector $j_\mu = (j_1, j_2, j_3, ic\rho) = (\mathbf{j}, ic\rho)$ is called current four vector. This will transform from S-frame to S'-frame, which is moving with constant velocity v along X-axis relative to S, similar to a four-vector as

$$\begin{pmatrix} j_1' \\ j_2' \\ j_3' \\ j_4' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \\ j_3 \\ j_4 \end{pmatrix} \qquad \ldots(20)$$

or $$j_1' = \gamma(j_1 + i\beta j_4) = \gamma(j_1 + i\frac{v}{c} ic\rho) = \gamma(j_1 - v\rho),\ j_2' = j_2, j_3' = j_3,$$

$$j_4' = \gamma(-i\frac{v}{c} j_1 + \gamma\, j_4) \text{ or } ic\rho' = \gamma(-i\frac{v}{c} j_1 + \gamma ic\rho) \text{ or } \rho' = \gamma\,(\rho - v\, j_1/c^2)$$

Thus the transformation equations for current and charge density are

$$j_1' = \gamma(j_1 - v\rho),\ j_2' = j_2,\ j_3' = j_3,\ \rho' = \gamma(\rho - vj_1/c^2) \qquad \ldots(21)$$

15.5. TRANSFORMATIONS OF ELECTROMAGNETIC POTENTIALS A AND ϕ (FOUR VECTOR POTENTIAL)

Maxwell's field equations can be expressed as

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{j} \qquad \ldots(22)$$

$$\Box^2 \phi = -\frac{\rho}{\epsilon_0} \qquad \ldots(23)$$

Eq. (23) can be written as

$$\Box^2\left(\frac{i\phi}{c}\right) = -\frac{i}{c}\frac{\mu_0\rho}{\mu_0\,\epsilon_0} = -\mu_0(ic\rho) \qquad \left(\text{because } c^2 = \frac{1}{\mu_0\,\epsilon_0}\right)$$

Thus we may write

$$\Box^2 \mathbf{A} = -\mu_0 j \qquad \ldots(24)$$

$$\Box^2\left(\frac{i\phi}{c}\right) = -\mu_0 j_4 \qquad \ldots(25)$$

Since $\Box^2$ is Lorentz invariant, **A** and $i\phi/c$ will transform as **j** and $j_4 = ic\rho$. However, the later quantities represent the components of current four-vector j_μ. Hence $(\mathbf{A}, i\phi/c)$ is a four vector and called as ***four vector potential*** A_μ, *i.e.*,

$$A_\mu = (\mathbf{A}, i\phi/c) = (A_1, A_2, A_3, A_4) \qquad \ldots(26)$$

where $A_4 = i\phi/c$.

Hence the Maxwell's field equations can be written in one single equation

$$\Box^2 A_\mu = -\mu_0 j_\mu \qquad \ldots(27)$$

The transformation equations from S-frame to S'-frame for the components of the four vector potential are

$$\begin{pmatrix} A_1' \\ A_2' \\ A_0' \\ \frac{i\phi'}{c} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A \\ \frac{i\phi}{c} \end{pmatrix} \qquad \ldots(28a)$$

i.e., $$A_1' = \gamma\left(A_1 - \frac{v\phi}{c^2}\right), A_2' = A_2\ , A_3' = A_3\ ,$$

$$\frac{i\phi'}{c} = \gamma\,(-i\frac{v}{c}A_1 + \frac{i\phi}{c}) \quad \text{or} \quad \phi' = \gamma\,(\phi - vA_1) \qquad ...(28b)$$

These are the ***transformation equations for electromagnetic potentials*** **A** and ϕ.

The inverse transformation equations are

$$A_1 = \gamma\,(A_1' + v\phi/c^2), A_2 = A_2', A_3 = A_3', \phi = \gamma\,(\phi' + vA_1') \qquad ...(29)$$

The Lorentz condition can be expressed as

$$\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0 \text{ or } \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial(i\phi/c)}{\partial(ict)} = 0$$

or $$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 \quad \text{or} \quad \sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial x_\mu} = 0 \qquad ...(30)$$

This equation expresses the ***Lorentz condition in covariant form.***

15.6. COVARIANCE OF MAXWELL'S FIELD EQUATIONS INTERMS OF FOUR VECTORS

The covariance (or invariance) of Maxwell's field equations means that these equations have the same form in any two inertial systems.

The Maxwell's field equations in terms of electromagnetic potentials **A** and ϕ are obtained as

$$\Box^2\mathbf{A} = -\mu_0\mathbf{j} \qquad ...(31)$$

and $$\Box^2\phi = -\rho/\epsilon_0 \qquad ...(32)$$

with the Lorentz condition $\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = 0$ ...(33)

The electromagnetic four vector potential A_μ and current four vector $\mathbf{j}_\mu$ are represented as

$$A_\mu = (\mathbf{A}, i\phi/c) \text{ or } A_\mu = (A_1, A_2, A_3, A_4,) \text{ with } A_4 = i\phi/c \qquad ...(34)$$

and $$j_\mu = (\mathbf{j}, ic\rho) \text{ or } j_\mu = (j_1, j_2, j_3, j_4) \text{ with } j_4 = ic\rho \qquad ...(35)$$

These four vectors transform like Lorentz transformations and the transformation equations are as follows :

$$A'_1 = \gamma\,(A_1 - v\phi/c^2) \qquad j_1' = \gamma\,(j_1 - v\rho)$$
$$A'_2 = A_2 \qquad j_2' = j_2$$
$$A'_3 = A_3 \qquad j_3' = j_3$$
$$\phi' = \gamma\,(\phi - vA_1) \qquad \rho' = \gamma\,(\rho - vj_1/c^2)$$

In terms of A_μ and j_μ, the Maxwell's field equations are represented in the form

$$\Box^2 A_\mu = -\mu_0 j_\mu \qquad ...(36a)$$

with the Lorentz condition

$$\sum_{\mu=1}^{4}\frac{\partial A_\mu}{\partial x_\mu} = 0 \qquad ...(36b)$$

Let the eq. (36) be in S-frame. The covariance of Maxwell's field equations requires that in any inertial frame S', eq. (36) must have the same form *i.e.*,

$$\Box^2 A_\mu' = -\mu_0 j'_\mu \qquad ...(37a)$$

with Lorentz conditon $\sum_{\mu=1}^{4} \frac{\partial A'_\mu}{\partial x'_\mu} = 0$ ...(37b)

In order that the statement (37) is true, let us consider

$$\Box'^2 A_1' = \Box^2 \gamma\left(A_1 - \frac{v\phi}{c^2}\right) \qquad \text{(because } \Box'^2 = \Box^2\text{)}$$

$$= \gamma\left[\Box^2 A_1 - \frac{v}{c^2}\Box^2\phi\right] = \gamma\left[-\mu_0 j_1 - \frac{v}{c^2}\left(-\frac{\rho}{\epsilon_0}\right)\right]$$

$$= -\gamma\left[\mu_0 j_1 - \mu_0 v\rho\right] = -\mu_0\gamma\left[j_1 - v\rho\right] = -\mu_0 j_1'$$

$$\Box'^2 A_2' = \Box^2 A_2 = -\mu_0 j_2 = -\mu_0 j_2'$$

$$\Box'^2 A_3' = \Box^2 A_3 = -\mu_0 j_3 = -\mu_0 j_3'$$

$$\Box'^2 A_4' = \Box^2 \frac{i\phi'}{c} = \Box^2 \frac{i}{c}\gamma(\phi - vA_1) = \frac{i\gamma}{\epsilon_0 c}\left[-\frac{\rho}{\epsilon_0} + v\mu_0 j_1\right]$$

$$= \frac{i\gamma}{\epsilon_0 c}\left[\rho - \frac{vj_1}{c^2}\right] = -\frac{i\rho'}{c} = -\mu_0 j_4'$$

Also $\quad A_\mu' = \sum_\nu a_{\mu\nu} A_\nu, \quad x_\mu' = \sum_\nu a_{\mu\nu} x_\nu$, hence $\frac{\partial x'_\mu}{\partial x_\nu} = a_{\mu\nu}$

Therefore, $\quad \frac{\partial A'_\mu}{\partial x'_\mu} = \sum_{\mu\nu} a_{\mu\nu} \frac{\partial A_\nu}{\partial x'_\mu} = \sum_{\mu\nu} \frac{\partial x'_\mu}{\partial x_\nu}\frac{\partial A_\nu}{\partial x'_\mu}$

$$= \sum_\nu \frac{\partial A_\nu}{\partial x_\nu} = \sum_\mu \frac{\partial A_\mu}{\partial x_\mu}$$

But $\quad \sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = 0$, hence $\sum_\mu \frac{\partial A'_\mu}{\partial x'_\mu} = 0$

Thus eqs. (37) are same as eqs. (36). This means that the *Maxwell's equations are covariant under Lorentz transformations.*

15.7. THE ELECTROMAGNETIC FIELD TENSOR

The electrromagnetic field vectors **E** and **B** can be expressed in terms of electromagnetic potential **A** and ϕ as

$$\mathbf{B} = \nabla \times \mathbf{A} \text{ and } \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi \qquad ...(38)$$

The vector **B** has three components B_x, B_y, B_z (or B_1, B_2, B_3) and **E** has E_x, E_y, E_z (or E_1, E_2, E_3) components. **A** also has three components A_x, A_y, A_z (or A_1, A_2, A_3).

Now $$\mathbf{B} = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ A_1 & A_2 & A_3 \end{bmatrix}$$

Therefore, $$B_x = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3} = F_{23} \quad ...(39a)$$

$$B_y = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1} = F_{31} \quad ...(39b)$$

$$B_z = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = F_{12} \quad ...(39c)$$

Also from (38) $E_x = \dfrac{\partial A_1}{\partial t} - \dfrac{\partial \phi}{\partial x}$

or $$\frac{iE_x}{c} = -\frac{i}{c}\frac{\partial A_1}{\partial t} - \frac{i}{c}\frac{\partial \phi}{\partial x} = \frac{\partial A_1}{\partial (ict)} - \frac{\partial (i\phi/c)}{\partial x_1}$$

Thus $$\frac{iE_x}{c} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1} = F_{41} \text{ (say)} \quad ...(40a)$$

Similarly, $$\frac{iE_y}{c} = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2} = F_{42} \quad ...(40b)$$

and $$\frac{iE_z}{c} = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3} = F_{43} \quad ...(40c)$$

where in general $$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \quad ...(41)$$

We observe that $F_{\mu\nu} = -F_{\nu\mu}$ and $F_{\mu\mu} = 0$ ...(42)

Thus we can form an antisymmetric tensor whose components are given by

$$F_{\mu\nu} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} = \begin{bmatrix} 0 & B_z & -B_y & -\dfrac{iE_x}{c} \\ -B_z & 0 & B_x & -\dfrac{iE_y}{c} \\ B_y & -B_x & 0 & -\dfrac{iE_z}{c} \\ \dfrac{iE_x}{c} & \dfrac{iE_y}{c} & \dfrac{iE_z}{c} & 0 \end{bmatrix} \quad ...(43)$$

This is called the ***electromagnetic field tensor of rank two.***

15.8. LORENTZ TRANSFORMATIONS OF ELECTRIC AND MAGNETIC FIELDS

If the Maxwell's field equations are to be covariant under Lorentz transformations, then the electromagnetic field tensor must have the same form in all inertial systems. Thus if in S-frame,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \quad \text{...(44)}$$

then in S'-frame, we must have

$$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} \quad \text{...(45)}$$

The Lorentz transformation for x_μ and A_ν are

$$x'_\mu = \sum_\lambda a_{\mu\lambda} x_\lambda \quad \text{...(46)}$$

and $$A'_\nu = \sum_\sigma a_{\nu\sigma} A_\sigma \quad \text{...(47)}$$

The inverse transformations of x_μ are

$$x_\lambda = \sum_\mu a_{\mu\lambda} x'_\mu \quad \text{...(48)}$$

Therefore, $$\frac{\partial x_\lambda}{\partial x'_\mu} = a_{\mu\lambda} \quad \text{...(49)}$$

Now, $$F'_{\mu\nu} = \frac{\partial A'_\nu}{\partial x'_\mu} - \frac{\partial A'_\mu}{\partial x'_\nu} = \frac{\partial}{\partial x'_\mu}\left(\sum_\sigma a_{\nu\sigma} A_\sigma\right) - \frac{\partial}{\partial x'_\nu}\left(\sum_\lambda a_{\mu\lambda} A_\lambda\right)$$

$$= \sum_\sigma a_{\nu\sigma} \frac{\partial A_\sigma}{\partial x'_\mu} - \sum_\lambda a_{\mu\lambda} \frac{\partial A_\lambda}{\partial x'_\nu} = \sum_{\sigma,\lambda} a_{\nu\sigma} \frac{\partial A_\sigma}{\partial x_\lambda} \cdot \frac{\partial x_\lambda}{\partial x'_\mu} - \sum_{\lambda,\sigma} a_{\mu\lambda} \frac{\partial A_\lambda}{\partial x_\sigma} \cdot \frac{\partial x_\sigma}{\partial x'_\nu}$$

$$= \sum_{\sigma,\lambda} a_{\nu\sigma} a_{\mu\lambda} \frac{\partial A_\sigma}{\partial x_\lambda} - \sum_{\lambda,\sigma} a_{\mu\lambda} a_{\nu\sigma} \frac{\partial A_\lambda}{\partial x_\sigma} = \sum_{\sigma,\lambda} a_{\nu\sigma} a_{\mu\lambda} \left(\frac{\partial A_\sigma}{\partial x_\lambda} - \frac{\partial A_\lambda}{\partial x_\sigma}\right)$$

$$= \sum_{\sigma,\lambda} a_{\mu\lambda} a_{\nu\sigma} F_{\lambda\sigma} \quad \text{...(50)}$$

where $$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

and $$\begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} = \begin{pmatrix} 0 & B_z & -B_y & \frac{-iE_x}{c} \\ -B_z & 0 & B_x & \frac{-iE_y}{c} \\ B_y & -B_x & 0 & \frac{-iE_z}{c} \\ \frac{-iE_x}{c} & \frac{iE_y}{c} & \frac{-iE_z}{c} & 0 \end{pmatrix}$$

Therefore, for $\mu = 1$ and $\nu = 2$, we have

$$F'_{12} = \sum_{\sigma,\lambda} a_{1\lambda} a_{2\sigma} F_{\lambda\sigma}$$

$$= \sum_{\lambda} a_{1\lambda}(a_{21}F_{\lambda 1} + a_{22}F_{\lambda 2} + a_{23}F_{\lambda 3} + a_{24}F_{\lambda 4}) = \sum_{\lambda} a_{1\lambda} F_{\lambda 2}$$

$$= a_{11}F_{12} + a_{12}F_{22} + a_{13}F_{32} + a_{14}F_{42}$$

Thus $\qquad B'_z = \gamma\, B_z + i\beta\gamma\, i\, E_y/c = \gamma\,(B_z - vE_y/c^2)$

Similarly, it is easy to obtain all the tensor components in S'-frame which provide the transformation equations for the components of electric field **E** and magnetic induction **B** as follows :

$$B_x' = B_x, B_y' = \gamma\,(B_y + vE_z/c^2), B_z' = \gamma\,(B_z - vE_y/c^2)$$

and $$E_x' = E_x, E_y' = \gamma\,(E_y - vB_z), E_z' = \gamma\,(E_z + vB_y) \qquad ...(51)$$

15.9. COVARIANT FORM OF MAXWELL'S FIELD EQUATION IN TERMS OF ELECTROMAGNETIC FIELD TENSOR

Maxwell's equations in free space are given by

$$(i)\ \text{div } \mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad (ii)\ \text{div } \mathbf{B} = 0$$

$$(iii)\ \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (iv)\ \text{curl } \mathbf{B} = \mu_0\left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{j}\right) \qquad ...(52)$$

Let us consider eqs. (i) and (iv)

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j} \text{ and } \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

or $$\nabla \times \mathbf{B} = \frac{\partial(i\mathbf{E}/c)}{\partial(ict)} + \mu_0 \mathbf{j} \text{ and } \nabla \cdot \frac{i\mathbf{E}}{c} = \frac{i\rho}{\epsilon_0 c}$$

or $$\nabla \times \mathbf{B} - \frac{\partial(i\mathbf{E}/c)}{\partial x_4} = \mu_0 \mathbf{j} \text{ and } \nabla \cdot \frac{i\mathbf{E}}{c} = \mu_0 j_4$$

Writing in component form, the two equations are

$$\left.\begin{array}{llllll}
0 & +\dfrac{\partial B_z}{\partial x_2} & -\dfrac{\partial B_y}{\partial x_3} & +\dfrac{\partial(iE_x/c)}{\partial x_4} & = \mu_0 j_1 \\[2ex]
-\dfrac{\partial B_z}{\partial x_1} & +\ 0 & +\dfrac{\partial B_x}{\partial x_3} & +\dfrac{\partial(-iE_y/c)}{\partial x_4} & = \mu_0 j_2 \\[2ex]
\dfrac{\partial B_y}{\partial x_1} & -\dfrac{\partial B_x}{\partial x_2} & +\ 0 & +\dfrac{\partial(-iE_z/c)}{\partial x_4} & = \mu_0 j_3 \\[2ex]
\dfrac{\partial(iE_x/c)}{\partial x_1} & +\dfrac{\partial(iE_y/c)}{\partial x_2} & +\dfrac{\partial(iE_z/c)}{\partial x_3} & +\ 0 & = \mu_0 j_4
\end{array}\right\} \qquad ...(53)$$

Introducing the electromagnetic field tensor given by

$$F_{\mu\nu} = \begin{pmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{pmatrix} = \begin{pmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{pmatrix} \qquad ...(54)$$

Hence eqs. (53) are

$$\frac{\partial F_{11}}{\partial x_1}+\frac{\partial F_{12}}{\partial x_2}+\frac{\partial F_{13}}{\partial x_3}+\frac{\partial F_{14}}{\partial x_4}=\mu_0 j_1$$

$$\frac{\partial F_{21}}{\partial x_1}+\frac{\partial F_{22}}{\partial x_2}+\frac{\partial F_{23}}{\partial x_3}+\frac{\partial F_{24}}{\partial x_4}=\mu_0 j_2$$

$$\frac{\partial F_{31}}{\partial x_1}+\frac{\partial F_{32}}{\partial x_2}+\frac{\partial F_{33}}{\partial x_3}+\frac{\partial F_{34}}{\partial x_4}=\mu_0 j_3$$

$$\frac{\partial F_{41}}{\partial x_1}+\frac{\partial F_{42}}{\partial x_2}+\frac{\partial F_{43}}{\partial x_3}+\frac{\partial F_{44}}{\partial x_4}=\mu_0 j_4$$

These equations and hence the Maxwell's field equations (*i*) and (*iv*) are obtained in the compact form

$$\sum_{\nu=1}^{4}\frac{\partial F_{\mu\nu}}{\partial x_\nu}=\mu_0 j_\mu \qquad ...(55)$$

Maxwell's equations (ii) and (iii) are written as

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=0 \text{ and } \nabla\cdot\mathbf{B}=0$$

First of these equations can be written as

$$\nabla\times\mathbf{E}+ic\frac{\partial\mathbf{B}}{\partial x_4}=0 \text{ or } \nabla\times\left(\frac{-i\mathbf{E}}{c}\right)+\frac{\partial\mathbf{B}}{\partial x_4}=0$$

Thus the two equations can be written as

$$\frac{\partial(-iE_z/c)}{\partial x_2}+\frac{\partial(iE_y/c)}{\partial x_3}+\frac{\partial B_x}{\partial x_4}=0$$

$$\frac{\partial(iE_z/c)}{\partial x_1}+\frac{\partial(iE_x/c)}{\partial x_3}+\frac{\partial B_y}{\partial x_4}=0$$

$$\frac{\partial(-iE_y/c)}{\partial x_1}+\frac{\partial(iE_x/c)}{\partial x_2}+\frac{\partial B_z}{\partial x_4}=0$$

and $$\frac{\partial B_x}{\partial x_1}+\frac{\partial B_y}{\partial x_2}+\frac{\partial B_z}{\partial x_3}=0$$

which can be written as

$$0+\frac{\partial F_{34}}{\partial x_2}+\frac{\partial F_{42}}{\partial x_3}+\frac{\partial F_{23}}{\partial x_4}=0$$

$$\frac{\partial F_{43}}{\partial x_1}+0+\frac{\partial F_{14}}{\partial x_3}+\frac{\partial F_{31}}{\partial x_4}=0$$

$$\frac{\partial F_{24}}{\partial x_1}+\frac{\partial F_{41}}{\partial x_2}+0+\frac{\partial F_{12}}{\partial x_4}=0$$

and
$$\frac{\partial F_{23}}{\partial x_1}+\frac{\partial F_{31}}{\partial x_2}+\frac{\partial F_{12}}{\partial x_3}+0=0$$

These equations and hence the Maxwell's equations (*ii*) and (*iii*) can be written in compact form as

$$\frac{\partial F_{\lambda\mu}}{\partial x_\nu}+\frac{\partial F_{\mu\nu}}{\partial x_\lambda}+\frac{\partial F_{\nu\lambda}}{\partial x_\mu}=0 \qquad ...(56)$$

Equations (55) and (56) represent the Maxwell's field equations in terms of electromagnetic field tensor $F_{\mu\nu}$ defined by (54). As tensor equations are invariant under coordinate transformations, eqs. (55) and (56) represent the ***covariant form of the Maxwell's field equations.***

15.10. LORENTZ FORCE ON A CHARGED PARTICLE

Consider two inertial frames of references S and S'. S' is moving along X-axis with velocity $\mathbf{v}=v\,\hat{i}$ relative to S. Let a charge q be instantaneously at rest in frame S'. Let **B** and **E** be the magnetic field and electric field repectively in frame S. Let us find the force on the charge q in frame S. Since the charge is moving relative to frame S, electrostatic and magnetic both forces are acting on the particle. However, the particle is at rest in frame S', only electrostatic force $\mathbf{F}'=q\mathbf{E}'$ is acting on the particle, where $\mathbf{E}'$ is the electric field seen in the S' frame. Now let us find the force on the charge q in frame S.

The $\mathbf{F}'$ in S'-frame can be written in the component form as

$$F_x'=qE_x',\quad F_y'=qE_y',\quad F_z'=qE_z'$$

Transformations of the force and electric field components are

$$F_x=F_x',\quad F_y=F_y'/\gamma,\quad F_z=F_z'/\gamma$$

or
$$F_x=qE_x',\quad F_y=qE_y'/\gamma,\quad F_z=qE_z'/\gamma$$

and
$$E_x'=E_x,\quad E_y'=\gamma(E_y-vB_z),\quad E_z'=\gamma(E_z+vB_y)$$

Therefore, $F_x=qE_x$, $F_y=q(E_y-vB_z)$ and $F_z=q(E_z+vB_y)$

Remember that the charge is invariant. Thus

$$\mathbf{F}=F_x\hat{\mathbf{i}}+F_y\hat{\mathbf{j}}+F_z\hat{\mathbf{k}}$$

$$=q\mathbf{E}+q(E_y-vB_z)\hat{\mathbf{j}}+q(E_z+vB_y)\hat{\mathbf{k}}$$

$$=q\mathbf{E}+qv\,\hat{\mathbf{i}}\times(B_x\hat{\mathbf{i}}+B_y\hat{\mathbf{j}}+B_z\hat{\mathbf{k}})$$

$$=q\mathbf{E}+q(\mathbf{v}\times\mathbf{B}) \qquad ...(57)$$

This is the expression for Lorentz force on a particle of charge q, moving with velocity **v** in both electric and magnetic fields **E** and **B**.

15.11. LORENTZ FORCE IN COVARIANT FORM

Let a charge q be moving with velocity $\mathbf{u}$ under the joint action of a magnetic inductions $\mathbf{B}$ and electric field $\mathbf{E}$. The force on the charged particle is

$$\mathbf{F} = q\mathbf{E} + q(\mathbf{u} \times \mathbf{B})$$

The force f per unit volume is given by

$$\mathbf{f} = \rho\mathbf{E} + \rho(\mathbf{u} \times \mathbf{B}) \qquad \left(\frac{q}{V} = \rho\right)$$

or $$\mathbf{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} \qquad (\rho\mathbf{u} = \mathbf{j})$$

In component form

$$f_1 = \rho E_x + B_z j_2 - B_y j_3$$
$$f_2 = \rho E_y + B_x j_3 - B_z j_1$$
$$f_3 = \rho E_z + B_y j_1 - B_x j_2$$

Using electromagnetic field tensor components $F_{\mu\nu}$, the above equations are obtained as

$$f_1 = F_{11} j_1 + F_{12} j_2 + F_{13} j_3 + F_{14} j_4$$
$$f_2 = F_{21} j_1 + F_{22} j_2 + F_{23} j_3 + F_{24} j_4 \qquad ...(58)$$
$$f_3 = F_{31} j_1 + F_{32} j_2 + F_{33} j_3 + F_{34} j_4$$

where $j_\mu = (\mathbf{j}, ic\rho)$ is the corrent four vector.

Eqs. (58) can be written as

$$f_\lambda = \sum_{\nu=1}^{4} F_{\lambda\nu} j_\nu \text{ with } \lambda = 1,2,3,$$

Obviously the right hand side of this equation is the space compnent of a four vector f_μ, expressed as

$$f_\mu = \sum_{\nu=1}^{4} F_{\mu\nu} j_\nu \qquad ...(59)$$

This f_μ is called ***force-density four vector.***

But $\displaystyle\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu$, therefore,

$$f_\mu = \frac{1}{\mu_0} \sum_{\nu=1}^{4} F_{\mu\nu} \sum_{\lambda=1}^{4} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda} \text{ or } f_\mu = \frac{1}{\mu_0} \sum_{\nu,\lambda=1}^{4} F_{\mu\nu} \frac{\partial F_{\nu\lambda}}{\partial x_\lambda} \qquad ...(60)$$

which is the tensor form of force density four vector. This represents the ***covariant form of the Lorentz force.***

Physical interpretation of the fourth component of the force-density four vector. The fourth component of the force-dinsity four vector, given by eqs. (59) is

$$f_4 = F_{41} j_1 + F_{42} j_2 + F_{43} j_2 + F_{44} j_4$$
$$= \frac{i}{c} E_x j_1 + \frac{i}{c} E_y j_2 + \frac{i}{c} E_z j_3$$
$$= \frac{i}{c}(\mathbf{E} \cdot \mathbf{j}) = \frac{i}{c}\rho(\mathbf{E} \cdot \mathbf{u}) \quad (\text{as } \mathbf{j} = \rho\,\mathbf{u}) \qquad ...(61)$$

Apart from the factor (i/c), the quantity $\rho(\mathbf{E} \cdot \mathbf{u})$ represents the rate of work done by the electric field on the charge per unit volume. Thus *the Lorentz force equation in the covariant form gives the rate of change of* *linear momentun per unit volume as the space part and the rate of change of mechanical energy per unit volume as its time part.*

Questions

1. Show that the D'Alembertian operator, defined by

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

is invariant under Lorentz transformations. **(Rohilkhand 1984, 79)**

2. Establish the covariant form of Maxwell's electromagnetic field equations by four vectors. Does it represent the covariant formulation of electro-dynamics. **(Agra 1992)**

3. Define four-current and four-vector potentials. How electric and magnetic fields are combined to form the various components of electromagnetic fields tensor ? Hence derive the transformations of **E** and **B** field.

4. What is four vector potential ? Show that the Maxwell's field equation can be written in one single equation, given by $\Box^2 A_\mu = -\mu_0 J_\mu$, where A_μ is the four vector potential and J_μ is the current four vector. Discuss the covariance of Maxwell's field equations. **(Agra 2000)**

5. (a) Express Maxwell's field equations in tensor form and thereby define electromagnetic field tensor. How does this information lead to the covariance of the theory. **(Agra 2002, 01)**

   (b) Show that $c^2B^2 - E^2$ is invariant under lorentz transformation. **(Agra 2001)**

6. Show that the four vector potential of electrodynamics can be expressed as the Lienard-Wiechert potentials. Define the electromagnetic field tensor F^{ij} and prove that the space part of the four-vector $qF^{ij}u_j$ derived from this tensor represents the Lorentz force acting on a particle of charge q. **(Agra 1995, 94)**

   [**Hint :** Four vector potential

   $$A_\mu = (A_1, A_2, A_3, A_4) = (\mathbf{A}, i\phi/c)$$

   Lienard-Wiechert potentials depend on the velocity **v** of the particle and have the form :

   $$\mathbf{A} = \frac{\mu_0 ec}{4\pi}\left(\frac{\mathbf{v}/c}{R - (\mathbf{v}\cdot\mathbf{R})/c}\right)_{\text{Ret.}}$$

   $$\phi = \frac{e}{4\pi\,\epsilon_0}\left(\frac{1}{R - (\mathbf{v}\cdot\mathbf{R})/c}\right)_{\text{Ret.}}$$

   Space part of $qF^{ij}u_j \left(i.e., q\sum_j F^{ij}u_j\right)$ is

   $$= q\sum_{j=1}^{3} F^{ij}u_j \text{ or } V\frac{q}{V}\sum F^{ij}u_j = V\sum F^{ij}\rho u_j = V\sum F^{ij} j_j = Vf^i = F^i \quad \text{(Sec. 15.11)}$$

   i.e., $\mathbf{F} = q[\mathbf{E} + \mathbf{v}\times\mathbf{B}]$

7. Discuss the Lorentz invariance of Maxwell's field equations. **(Agra 1962)**

8. Write the Maxwell's equations in terms of scalar and vector potentials. Show that these equations are invariant under gauge transformations. Discuss the significance of this transformation. **(Rohilkhand 1983)**

9. Obtain the four vector potential of electrodynamics and derive the electro-magnetic field tensor from it. Express the Maxwell's equations in four dimensional formalism and thus prove their Lorentz invariance. **(Agra 1999)**

10. Use the transformation properties of the field strength tensor to find the Lorentz transformations for the electric and magnetic fields. **(Meerut 1971)**

11. Derive an expression for the Lorentz force on a charged particle in an electromagnetic field with the help of Lorentz transformation method. **(Meerut 1981, 80)**

Problems

1. If **E** and **B** are respectively electric and magnetic induction vectors, then show that $\mathbf{E}^2 - \mathbf{B}^2c^2$ and $\mathbf{E}\cdot\mathbf{B}$ are invariant under Lorentz transformation. **(Agra 2002, 1981; Meerut 86)**

2. (*a*) Show that $c^2\rho^2 - (j_x{}^2 + j_y{}^2 + j_z{}^2)$ is an invariant quantity equal to $c^2\rho_0{}^2$.
   (*b*) Show that if $E = cB$ in one inertial frame, then $E' = cB'$ in any other inertial frame.
   (*c*) If The electric and magnetic fields are perpendicular to one another in one inertial frame, they are perpendicular in other inertial frames.

3. A particle of charge q moves with uniform velocity **u** in inertial frame S. Consider a frame S', moving with uniform velocity **u'** with respect to S in which the charge is at rest and the force on the charge is $\mathbf{F}' = q\mathbf{E}'$. Show that the force on the particle in frame S is the Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{u}\times\mathbf{B})$, by using the transformations for the components of force and the transformations for the components of **E** and **B**.

4. Show that if the source charge (*i.e.*, the source of field) moves with speed v relative to a test charge (*i.e.*, the charge acted by the field), either toward or away from it along the line connecting the two charges, the force on the test charge is $(1 - v^2/c^2)$ times the ordinary coulomb force.

5. Show that if the source charge moves with a speed v relative to a test charge at right angles to their transverse separation, the force on the test charge, at the instant the line connecting them is at right angles to **v**, is $1/\sqrt{1 - v^2/c^2}$ times the ordinary coulomb force.

6. Find the force in the laboratory frame as well as in the proper frame for two electrons, moving along the axis of a linear accelerater in parallel paths separated by 5×10^{-9} m at speed $v = 0.999$c. The line connecting the two charges is perpendicular to the direction of their motion.

   Ans. : $F_y' = 9.22\times10^{-12}$ N, 4.12×10^{-13} N.

Objective Type Questions

1. Which one of the following remains invariant under Lorentz transformations:

   (*a*) $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} - \frac{1}{c^2}\frac{\partial}{\partial t}$ (*b*) $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

(c) $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ (d) $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ **(GATE 2004)**

Ans : (c).

2. For gauge transformation
(a) the electric and magnetic field vectors do not change.
(b) the electric field vector changes but not the magnetic field vector.
(c) the magnetic field vector changes but not the electric field vector.
(d) both the electric and magnetic field vectors change.
Ans : (a).

3. Choose the correct statement/s :

(a) $\sum_\mu \frac{\partial A_\mu'}{\partial x_\mu'} = \sum_\mu \frac{\partial A_\mu}{\partial x_\mu}$ (b) $\sum_\mu \frac{\partial A_\mu'}{\partial x_\mu'} \neq \sum_\mu \frac{\partial A_\mu}{\partial x_\mu}$

(c) Lorentz condition is $\sum_\mu \frac{\partial A_\mu}{\partial x_\mu} = 0$. (d) Lorentz condition is $\Box^2 A_\mu = -\mu_0 j_\mu$.

where the letters with prime superscript refer to S'-frame.

Ans. : (a), (c).

4. Lorentz transformations

(a) for magnetic field vector are

$$B_x' = B_x, B_y' = \gamma\left(B_y + \frac{vE_y}{c^2}\right) B_z' = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$$

(b) for electric field vector are

$$E_x' = E_x,\ E_y' = \gamma(E_y - v\beta_z),\ E_z' = \gamma(E_z + v\beta_y)$$

(c) $B_x' = B_x,\ B_y' = \gamma(B_y - vE_z),\ B_z' = \gamma(B_z + vE_y)$

(d) $E_x' = E_x,\ E_y' = \gamma,\ \left(E_y - \frac{vB_z}{c^2}\right),\ E_z' = \gamma\left(E_z + \frac{vB_z}{c^2}\right)$.

Ans : (a), (b).

5. Covariant form of the Maxwell's field equations $\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ in terms of electromagnetic field tensor is

(a) $\Box^2 A_\mu = -\mu_0 j_\mu$ (b) $\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 j_\mu$

(c) $\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{\rho}{\epsilon_0}$ (d) $\frac{\partial F_{\lambda\mu}}{\partial x_\gamma} + \frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} = 0$

Ans. : (b).

6. Lorentz force equation in the covariant form gives
(a) the rate of change of linear momentum per unit volume as the space part.
(b) the rate of change of linear momentum per unit volume as the time part.
(c) the rate of change of mechanical energy per unit volume as its time part.
(d) the rate of change of mechanical energy per unit volume as its space part.
Ans : (a), (c).

Short Type Questions

1. What is D'Alembertian operator ?
2. Show that the D'Alembertian operator is invariant under Lorentz transformations.
3. What are gauge transformations ?
4. Express the Maxwell field equations in a single equation.
5. What are the transformation equations for electromagnetic potentials ?
6. What is Lorentz condition ? Express it in the covariant form.
7. Show that the Maxwells equations are covariant under Lorentz transformations.
8. What is electromagnetic field tensor ?
9. Express the covariant form of Maxwell's field equations.
10. Write the Lorentz force in covariant form.
11. Fill in the blanks :

    (*i*) The current four vector is j_μ = (..............).

    (*ii*) The fourth component of the force-density four vector is.........

    Ans. : (*i*) $(j_1, j_2, j_3, ic\rho)$, (*ii*) $\frac{i}{c}\rho(\boldsymbol{E}\cdot\mathbf{u})$

CHAPTER 16 Nonlinear Dynamics and Chaos

16.1. INTRODUCTION

Most of the practical mechanical and electrical systems are nonlinear in nature. A theoretical analysis of such a system leads to the solution of nonlinear differential equations. In case of nonlinear differential equatons, principle of superposition is not applicable. Nonlinear equations are generally difficult to solve. General methods for obtaining solutions of nonlinear equations are not available. This has resulted in the development of large number of different techniques for nonlinear systems. In fact, for most of the nonlinear differential equations, closed form solutions can not be written down. Even then, a lot of useful information about the solutions can be obtained from the equations themselves.

In this chapter, we plan to introduce the nonlinear oscillations and chaos. First we shall give some examples of nonlinear systems and then we present a qualitative analysis in terms of phase trajectories and limit cycles. Poincare was the first to understand the possibility of completely irregular **chaotic** behaviour of solutions of nonlinear differential equations which are characterized by an extreme sensitivity to initial conditions. For given shightly different initial conditions, solutions can grow exponentially apart with time so that the system soon becomes effectively unpredictable or **chaotic.** It is important that this chaos still involves deterministic solutions to deterministic equations. They are referred as chaotic because, although deterministic, they are not predictable because they are highly sensitive to initial conditions. **Chaos** exhibits extensive randomness tempered by some regularity. Chaotic trajectories develop from the motion of nonlnear system, which is nonperiodic, but still some what predictable. In this chapter, we also discuss some of the properties of chaotic motion and some examples.

16.2. NONLINEAR DIFFERENTIAL EQUATIONS

In case of simple harmonic motion, the restoring force is protional to $-x$ (the diplacement) and it does not depend on x^2, x^3..... etc. A differential equation which contains the terms only in first power of $x, \frac{dx}{dt}, \frac{d^2x}{dt^2}$..... etc. is said to be **linear** in x and its time derivatives. If any higher powers of x or its derivatives occur, the differential equation is said to be **nonlinear.** The examples of linear differential equations are

(i) $\frac{d^2x}{dt^2} + \omega_0^2 x = 0$ (harmonic oscillator) ...(1)

(ii) $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$ (damped harmonic oscillator) ...(2)

etc. (where $\omega_0 = \sqrt{k/m}$ and b are constants) and those of nonlinear differential equations are

(i) $\frac{d^2x}{dt^2} + \omega_0^2 x = \alpha x^2 + \beta x^3 + \gamma x^4$..... (anharmonic oscillator) ...(3)

(ii) $\frac{d^2x}{dt^2} - \in(1-x^2)\frac{dx}{dt} + x = 0 \quad (0 < \in << 1)$ (Van der Pol equation) ...(4)

etc. Eq. (4) is a very important equation in the theory of vacuum tube oscillators.

The equation of a simple pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \qquad ...(5a)$$

or $$\frac{d^2\theta}{dt^2} = -K\theta + K_1q^3 - K_2\theta^5 + ... \left(\text{because } \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\right) \qquad ...(5b)$$

is an example of nonlinear differential equation (an harmonic motion). When θ is small, the terms in higher power of θ can be neglected and we obtain what is known as linear approximation.

16.3. PHASE TRAJECTORIES-SINGULAR POINTS (*TOPOLOGICAL METHODS)*

If the nonlinear differential equation is of the second order and does not contain the independent variable t explicitly, a good amount of information, regarding the properties of the solution, may be obtained by a geometrical procedure without solving the equation itself. Dynamical systems, whose equations of motion do not contain the time t explicitly, are known as **autonomous systems.** Large amplitude oscillations of simple pendulum and the system represented by Van der Pol's equation are the examples of autonomous systems.

The equations of motion of second order automomous systems generally possess the following form:

$$\frac{d^2x}{dt^2} + F(\dot{x}) + f(x) = 0 \qquad ...(6)$$

where $F(x)$ is any function of velocity $\dot{x}(= dx/dt)$ and $f(x)$ that of displacement x.

If we introduce the velocity $y = \dot{x}$ as another dependent variable, the differential equation (6) may be written in the form of first order equations :

$$\frac{dx}{dt} = y \qquad ...(7)$$

and $$\frac{dy}{dt} = -[F(y) + f(x)] \qquad ...(8)$$

For a more general system, we may write

$$\frac{dx}{dt} = P(x,y) \qquad ...(9)$$

$$\frac{dy}{dt} = Q(x,y) \qquad ...(10)$$

where P and Q are functions of x and y. Equations (7) and (8) are the special cases of eqs. (9) and (10) respectively.

In order to investigate the qualitative features of the solutions of (6), some definitions are given below.

Phase Plane : The quantities x and y can be considered as cartesian coordinates in x-y plane. This plane is called phase plane.

Phase Trajectory : If we divide eq. (7) by eq. (8), we obtain

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=-\frac{F(y)+f(x)}{y} \qquad ...(11)$$

From eq. (11) we may obtain an equation of a definite curve in the phase plane (x, y). This curve is called the **phase trajectory** or **phase path** corresponding to the differential equation (6). Further the phase trajectory of the system, defined by eqs. (9) and (10), is given by

$$\frac{dy}{dx}=\frac{Q(x,y)}{P(x,y)}, \qquad P(x,y)\neq 0 \qquad ...(12)$$

By integrating eqs. (9) and (10), we can get

$$x=x(t) \text{ and } y=y(t) \qquad ...(13)$$

These are called **parametric equations of the trajectory.**

One may note the similarity between this and the concepts, given in Sec. 7·7. Here the canonical variables q and p are replaced by x and y respectively.

Singular Point : A point (x_0, y_0) of the phase plane for which

$$P(x_0,y_0)=Q(x_0,y_0)=0 \qquad ...(14)$$

is called a ***singular point.***

Ordinary Points : All the points of the phase plane, for which the functions P and Q do not vanish simultaneously, are called ***ordinary points.***

Representative Point : We may consider the derivatives (9) and (10) as the x and y components of the velocity of a point in the phase plane. This point is called the ***representative point*** of the system. The x and y components of the velocity of the representative point are given by

$$v_x=\frac{dx}{dt}=P(x,y) \text{ and } v_y=\frac{dy}{dt}=Q(x,y) \qquad ...(15)$$

Obviously at a singular point $v_x=v_y=0$. Therefore, a singular point represents a position of equilibrium of the system with zero velocity.

In order to illustrate, we discuss below some phase trajectories and singular points of the differential equations for some linear and nonlinear systems.

16.4. PHASE TRAJECTORIES OF LINEAR SYSTEMS

(1) Linear Harmonic Oscillator. The differential equation of a linear harmonic oscillator having mass m and spring force constant k is given by

$$m\frac{d^2x}{dt^2}+kx=0 \text{ or } \frac{d^2x}{dt^2}+\omega_0^2x=0 \qquad ...(16)$$

where $\omega_0=\sqrt{k/m}$.

Let $\frac{dx}{dt}=y$. Then eq. (16) is equivalent to

$$\frac{dy}{dt}=-\omega_0^2x \qquad ...(17)$$

and

$$\frac{dx}{dt}=y \qquad ...(18)$$

In fact, these are coupled first order differential equations.

From eqs. (17) and (18), we have

$$\frac{dy}{dx} = -\frac{kx}{my} \text{ or } m\,ydy + k\,xdx = 0$$

Integrating, we get

$$m\frac{y^2}{2} + \frac{kx^2}{2} = C \qquad \text{...(19)}$$

where C is a constant, representing the total energy E of the oscillator, *i.e.*,

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 = E \qquad \text{...(20)}$$

This can be written in the form

$$\frac{x^2}{2E/k} + \frac{y^2}{2E/m} = 1 \qquad \text{...(21)}$$

If we put $2E/k = a^2$ and $2E/m = b^2$, then eq. (21) represents an ellipse, given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{...(22)}$$

whose centre is at the origin (0,0) with semimajor axis $a = \sqrt{2E/k}$ and semiminor axis $b = \sqrt{2E/m}$ (Fig. 16.1).

The phase trajectory of the differential equations (17) and (18) [or eq. (16)] is a definite ellipse for a definite amount of total energy E. Different values of the total energy E fix correspondingly different ellipses. From the equation $y = dx/dt$, we see that the representative point P moves along the phase trajectory in a *clockwise direction*.

The **Parametric equations** of the ellipse (22) are obtained by integrating eqs. (17) and (18) with respect to time t as follows :

$$x = x_0 \cos\omega_0 t + \frac{y_0}{\omega}\sin\omega_0 t \qquad \text{...(23a)}$$

and

$$y = -\omega_0 x_0 \sin\omega_0 t + y_0 \cos\omega_0 t \qquad \text{...(23b)}$$

where $\omega_0 = \sqrt{k/m}$ and at $t = 0$, $x = x_0$, $y = y_0$.

These equations can be obtained as follows :

$$\frac{dy}{dt} = -\omega_0{}^2 x,\; y = \frac{dx}{dt}$$

Let the solution be $x = a\sin(\omega_0 t + \phi)$

Therefore, $y = a\omega_0 \cos(\omega_0 t + \phi)$

At $t = 0$, $x = x_0$, $y = y_0$ and hence

$$x_0 = a\sin\phi,\; y = a\omega_0\cos\phi$$

Therefore, $x = a\sin\omega_0 t\cos\phi + a\cos\omega_0 t\sin\phi$

or $$x = x_0\cos\omega_0 t + \frac{y}{\omega_0}\sin\omega_0 t$$

Hence $$y = -\omega x_0 \sin\omega t + y\cos\omega t$$

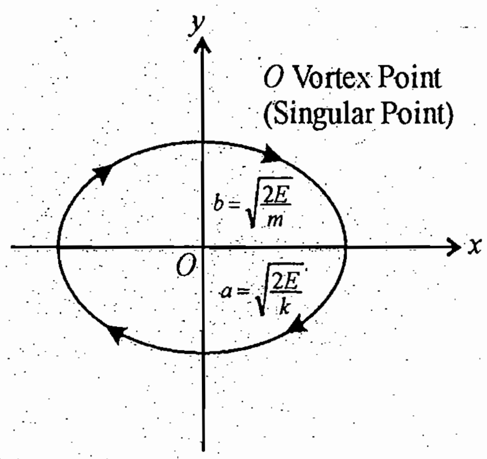


Fig. 16.1. Phase trajectory of linear harmonic oscillator

Obviously the periodicity of the solution (23) tells us that the representative point completes one revolution of the ellipse in time

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}} \qquad \text{...(24)*}$$

From eqs. (17) and (18) for the point (0,0)

$$\frac{dx}{dt} = P(0{,}0) = 0 \text{ and } \frac{dy}{dt} = Q(0{,}0) = 0 \qquad \text{...(25)}$$

Therefore the centre of the ellipse $x = 0, y = 0$ is a *singular point* of the differential eqs. (17) and (18). The possible phase trajectories are ellipses which enclose this point. A singular point of this type enclosed by the phase trajectories and approached by none is called a **vortex point.** This vortex point represents a position of stable equilibrium of the system.

(2) Aperiodic Motion. Let a mass m be repulsed from the origin by a force proportional to its distance x from the origin O (Fig. 16·2), *i.e.*,

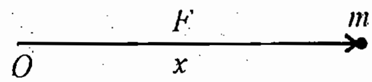


Fig. 16.2. Force proportional to x

$$F \propto x \text{ or } F = kx$$

The equation of motion is

$$m\frac{d^2x}{dt^2} = x \text{ or } \frac{d^2x}{dt^2} = \alpha^2 x \qquad \text{...(26)}$$

where $\alpha = \sqrt{k/m}$.

Eq. (26) can be represented as

$$\int \frac{dx}{y} = \int \frac{dx}{\sqrt{\frac{2}{ml^2}[E - V(x)]}}$$

$$\frac{dy}{dt} = \alpha^2 x \qquad \text{...(27)}$$

and $$\frac{dx}{dt} = y \qquad \text{...(28)}$$

Hence $$\frac{dy}{dx} = \frac{\alpha^2 x}{y} \text{ or } y\,dy - \alpha^2 x dx = 0$$

Integrating it, we get

$$y^2 - \alpha^2 x^2 = C \qquad \text{...(29)}$$

where C is a constant. Eq. (29) represents a hyperbola. For different values of C, eq. (29) gives a family of hyperbolas in the phase plane. If at time $t = 0$, $x = x_0$ and $y = y_0$, the solution of the differential equation gives.

---

* From the equation $y = \frac{dx}{dt}$, we have $dt = \frac{dx}{y}$

Integrating for one revolution $T = \int \frac{dx}{y} = 4\int_0^l \frac{dx}{\left(\frac{2E}{m} - \omega_0^2 x^2\right)} = \frac{2\pi}{\omega_0}$

because from eq. (20) $y = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2}$.

$$x = x_0 \cosh \alpha t + \frac{y_0}{\alpha} \sinh \alpha t \qquad \text{...(30a)}$$

$$y = \alpha x_0 \sinh \alpha t + y_0 \cosh \alpha t \qquad \text{...(30b)}$$

These are the parametric equations of the phase trajectories, given by eqs. (30). There are two asymptotes $y = \pm \alpha x$ to the family of hyperbolas. In the Fig 16·3, origin O is a singular point of a special type, known as a **saddle point.** There are two special trajectories which pass through the saddle point (for $C = 0$). The motion of the representative point along the trajectories which approach the saddle point is asymptotic with the time t. The trajectories in the neighbourhood of a saddle point (O) represent the possible motions that occur in the neighbourhood of a point of unstable equilibrium.

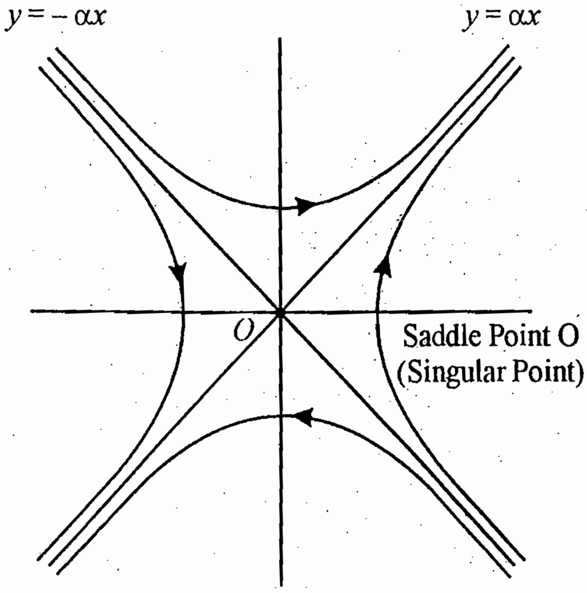


Fig. 16.3. Phase trajectory of aperiodic motion

(3) Damped Harmonic Oscillator and Overdamped Motion. When dissipation, proportional to the speed, is present, the equation of harmonic oscillator with damping can be written as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0 \qquad \text{...(31)}$$

or equivalently,

$$\frac{dy}{dt} = -2by - \omega_0^2 x \qquad \text{...(32)}$$

and

$$\frac{dx}{dt} = y \qquad \text{...(33)}$$

Remembering that by the substitution of the trial solution $x = Ae^{\alpha t}$ in eq. (31), we obtain a quadratic equation in α whose roots are

$$\alpha = -b \pm \sqrt{b^2 - \omega_0^2} \qquad \text{...(34)}$$

The solution, therefore, depends on the nature of the quantity $\sqrt{b^2 - \omega_0^2}$. If $b^2 < \omega_0^2$, the roots are complex and the solution of equation (31) represents the damped harmonic oscillations.

From eqs. (32) and (33), we obtain

$$\frac{dy}{dx} = \frac{-2by - \omega_0^2 x}{y} \qquad \text{...(35)}$$

If we put $\omega = \sqrt{\omega_0^2 - b^2}$, the general solution of eqn. (31), or equivalently eqs. (32) and (33), is

$$x = Ae^{-by} \cos(\omega t + \phi) \qquad \text{...(36)}$$

or

$$y = \dot{x} = -Ae^{-bt}[b\cos(\omega t + \phi) + \omega \sin(\omega t + \phi)] \qquad \text{...(37)}$$

where A and ϕ are constants.

Eqs. (36) and (37) are the parametric equations of phase trajectories representing a family of spirals and one of the spirals is shown in Fig 16·4. The origin O is a singular point of a type called a **focal point.**

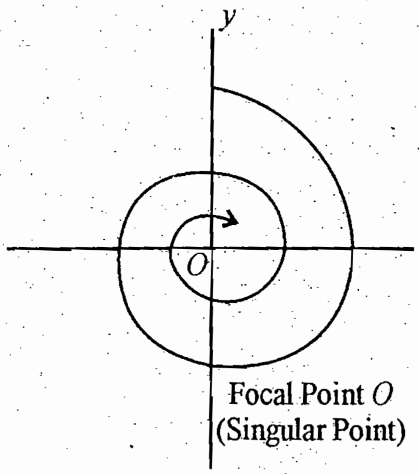


Fig. 16.4. Phase trajectory of an underdamped oscillator

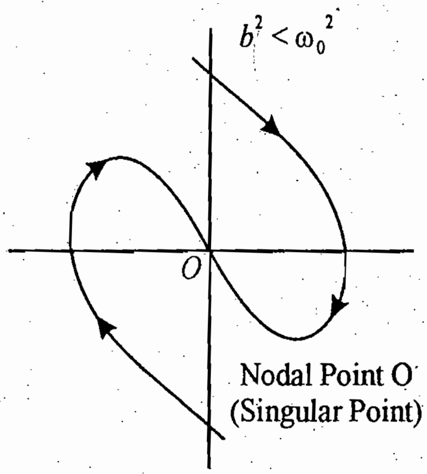


Fig. 16.5. Phase trajectory of an overdamped oscillator

The representative point moves along a spiral in the phase plane and approaches the focal point at the origin O in an asymplotic manner with no definite direction. The focal point O is a point of *stable equilibrium.*

In case, the damping is heavy, *i.e.,* $b^2 > \omega_0^2$ the motion is no longer oscillatory. If we put $\sqrt{b^2 - \omega_0^2} = \beta$, then the solutions are of the form

$$x = Ae^{-bt} \cos h(bt + \phi) \qquad ...(38a)$$

and $$y = \dot{x} = Ae^{-bt}[\beta \sin h(\beta t + \phi) - b \cos h(\beta t + \phi)]$$

where A and ϕ are constants.

Eqs. (38) are the parametric equations, representing the phase trajectories of a overdamped case. One of the phase trajectories is shown in Fig 16·5. The origin is a singular point of the type, called a **nodal point**. We find that for each phase trajectory (curve), the representative point moves toward the nodal point with a *definite direction.*

We summarise below the four basic types of singular points :

Table : Singular Points

S.N.	Name	Type of Motion	Approach	Stability
1.	Vortex point	Oscillatory	None	Stable
2.	Saddle point	Aperiodic	Along asymptotes	Unstable
3.	Focal point	Damped oscillatory	With no definite direction	Stable
4.	Nodal point	Nonoscillatory	With a definite direction	Stable

16.5. PHASE TRAJECTORIES OF NONLINEAR SYSTEMS

We may effectively study the motion of nonlinear conserative and nonconservative systems with specific examples by topological methods.

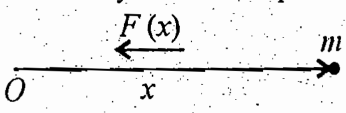


Fig. 16.6. Nonlinear force F(x)

(1) Nonlinear Conservative Systems

As an example of such a system, we consider the motion of a mass m.

attracted towards a fixed point O, the origin, by a nonlinear restoring force $F(x)$ at the displacement x (Fig 16·6). The equation of motion of the system is

$$m\frac{d^2x}{dt^2} = -F(x) \qquad ...(39)$$

This is equivalent to the first order differential equations, given by

$$\frac{dy}{dt} = -\frac{F(x)}{m} = -f(x) \quad \text{(say)} \qquad ...(40)$$

and
$$\frac{dx}{dt} = y \qquad ...(41)$$

Dividing (40) by (41), we get the following equation for the phase trajectory of the motion :

$$\frac{dy}{dx} = -\frac{F(x)}{my} \qquad ...(42)$$

From eq. (42), we obtain

$$mydy = -F(x)dx$$

Integrating it, we get

$$\frac{1}{2}my^2 + \int_0^x F(x)dx = C \qquad ...(43)$$

where C is a constant of integration and the mass m is at $x = 0$, when $t = 0$,

This constant C can be evaluated, if we put the condition at $t = 0$, $x = 0$ and $y = y_0$ in eq. (43), *i.e.*,

$$\frac{1}{2}my_0^2 = C$$

Therefore, eq. (43) takes the form

$$\frac{1}{2}my^2 + \int_0^x F(x)dx = \frac{1}{2}my_0^2 \qquad ...(44)$$

In eq. (44), the quantity $\frac{1}{2}my^2$ is the kinetic energy of the system and

$$\int_0^x F(x)dx = V(x) \qquad ...(45)$$

is the potential energy of the system. Hence, eq. (44) can be written as

$$\frac{1}{2}my^2 + V(x) = \frac{1}{2}my_0^2 = E \qquad ...(46)$$

where E is the total energy of the system.

From eq. (46), we obtain the expression for the velocity of the system as

$$y = \sqrt{\frac{2}{m}[E - V(x)]} \qquad ...(47)$$

We interpret this equation topologically by taking for a special case, where

$$F(x) = x(x+a) \quad \text{...(48)}$$

Hence the equation of motion of the system is

$$\frac{d^2x}{dt^2} = -\frac{x(x+a)}{m} \quad \text{...(49)}$$

which is equivalent to

$$\frac{dy}{dt} = -\frac{x(x+a)}{m} \text{ and } \frac{dx}{dt} = y \quad \text{...(50)}$$

So that $$\frac{dy}{dx} = -\frac{x(x+a)}{my} \quad \text{...(51)}$$

Also $$\frac{1}{2}my^2 + V(x) = E \quad \text{...(52)}$$

where $$V(x) = \int_0^x F(x)dx = \int_0^x x(x+a)dx \quad \text{...(53)}$$

The phase trajectories for the equation (49) is obtained from eq. (52) as

$$y = \sqrt{\frac{2}{m}[E-V(x)]} \quad \text{...(54)}$$

For the force of the form (48), we have drawn the curves for $F(x)$ versus x, energy diagram $V(x)$ versus

(a) F – x graph

(b) Energy diagram : V – x graph

(c) Phase trajectory : y – x graph

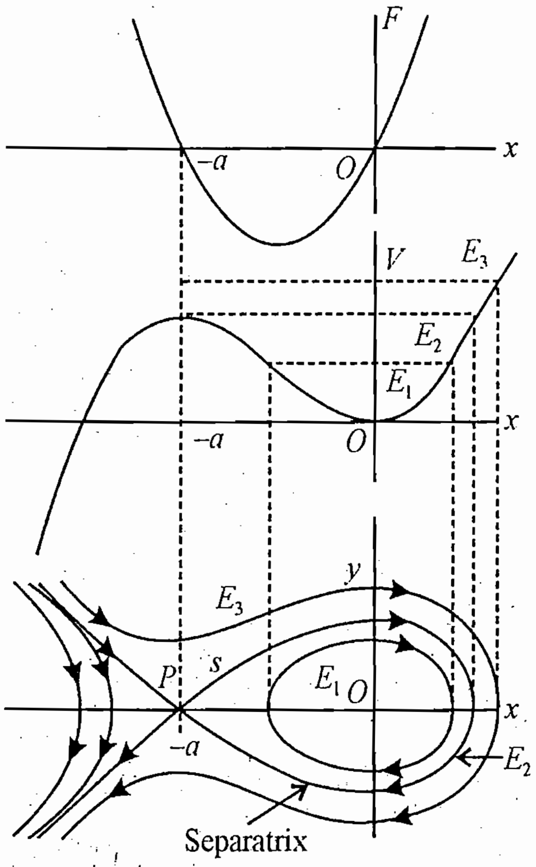


Fig. 16.7. Phase trajectory corresponding to equation $F(x) = x(x + a)$

x and phase trajectories for three values E_1, E_2 and E_3 of the total energy. O is the vortex point, P is the saddale point and s is the separatrix (Fig 16.7).

Trajectory of the Equation of a Simple Pendulum. The restoring couple on a simple pendulum at angular displacement θ is given by

$$I \frac{d^2\theta}{dt^2} = -mgl \sin\theta \qquad \text{...(55)}$$

Since $I = ml^2$, we get from eq. (55)

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \text{ or } \frac{d^2\theta}{dt^2} = -\omega_0^2 \sin\theta \qquad \text{...(56)}$$

where $\omega_0^2 = \sqrt{g/l}$. This is the *equation of motion of the simple pendulum.*

Now, writing $\theta = x$, eq. (56) can be written as

$$\frac{dy}{dt} = -\omega_0^2 \sin x \qquad \text{...(57)}$$

and $$\frac{dx}{dt} = y \qquad \text{...(58)}$$

From eqs. (57) and (58), we get

$$\frac{dy}{dx} = -\frac{\omega_0^2 \sin x}{y} \qquad \text{...(59)}$$

From eq. (59), we obtain

$$y dy = -\omega_0^2 \sin x \, dx$$

Integrating it, we get

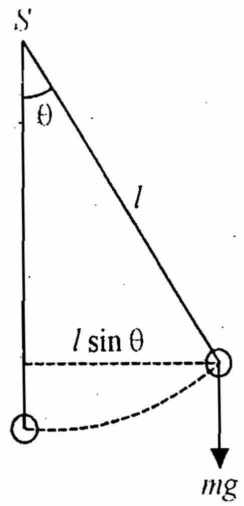


Fig. 16.8. Simple pendulum

$$\frac{y^2}{2} + \int_0^x \omega_0^2 \sin x \, dx = C \text{ or } \frac{y^2}{2} + \omega_0^2(1-\cos x) = C$$

where C is a constant.

Multiply by ml^2, we obtain

$$\frac{ml^2 y^2}{2} + mgl(1-\cos x) = E \qquad \text{...(60)}$$

(Kinetic energy + Potential energy = Total energy)

where the constant E is the total energy for the conservative system.

The potential energy $V(x) = mgl(1-\cos x) = 2mgl\sin^2(x/2)$ ...(61)

From (60) and (61)

$$y = \sqrt{\frac{2}{ml^2}[E - V(x)]} \text{ or } y = \sqrt{\frac{2}{ml^2}[E - 2mgl\sin^2\frac{x}{2}]} \qquad \text{...(62)}$$

In order to study, the topology of the equation, we plot the energy diagram [$V(x)$ against x] by using eq. (61) and phase trajectory [y against x] by using eq. (62) in Fig 16.9.

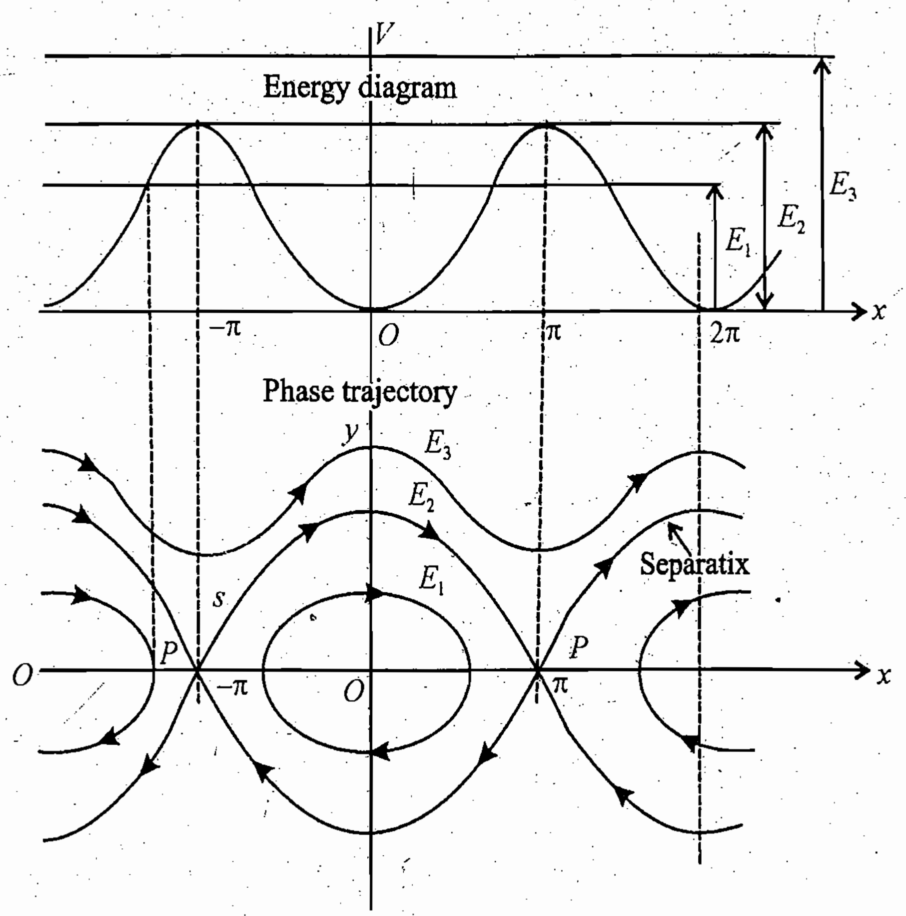


Fig. 16.9. Phase trajectories of simple pendulum

In Fig 16·9 (*a*), the quantity $V(x) = 2mgl\sin^2(x/2)$ has been plotted against x. We have drawn three horizontal lines on the energy diagram corresponding to the three distinct values of total energy E_1, E_2 and E_3, where $E_1 < E_2 < E_3$.

In Fig 16·9 (*b*), we have drawn three different phase trajectories corresponding to the three values of energies E_1, E_2 and E_3 by making use of eq. (62). We discuss below the three cases of phase trajectories for the nonlinear differential equation of the simple pendulum.

Case (i) $E_1 < 2mgl$ **:** This represents the phase trajectory for E_1 of a oscillating pendulum, whose energy (E_1) is less than the energy necessary ($2mgl$) to go over the top.

Case (ii), $E_2 = 2mgl$ **:** The phase trajectory in this case is the one in which the system has just enough total energy to take it to the top at $x = \pi$ and it is called **separatrix** because it separates the motion of one type (oscillatory) from those of entirely different type (nonoscillatory). The point P of the phase trajectory is a **saddle point,** where $V(x)$ is maximum ($2mgl$). This point represents **a point of unstable equilibrium,** corresponding to the inverted position of the pendulum. The point P has completely different properties, when compared to the point O. The point O is a **vortex point,** corresponding to a position of stable equilibrium, the bottom position of the pendulum.

Case (iii), $E_3 > 2mgl$ **:** When the pendulum has total energy E_3, which is greater than the critical energy E_2 to take it over the top then the phase trajectory is a curve that continues with x and the motion is no longer oscillatory. In such a case, the pendulum revolves about its point of support continuously with

maximum angular velocity at $x = 0, 2\pi, 4\pi$ ....*etc.* (at bottom) and **minimum angular velocity** at $x = \pi, 3\pi, 5\pi$ .... *etc.* (at top).

Period of nonlinear oscillations, when the phase trajectory is a closed path, can be obtained by using eq. (58) and (62), *i.e.*,

$$T = \int \frac{dx}{y} = \int \frac{dx}{\sqrt{\frac{2}{ml^2}[E - V(x)]}} \qquad ...(63)$$

Remember that for simple pendulum $x = \theta$, the angular displacement. Hence from (60) at maximum angular displacement $\theta_0, y = \dot{\theta} = 0$,

$$mgl(1-\cos\theta_0) = E$$

Also $$V(x) = mgl(1-\cos x) = mgl(1-\cos\theta)$$

Therefore, $$T = 2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2}{ml^2}[mgl(1-\cos\theta_0) - mgl(1-\cos\theta)]}}$$

because the period is twice the time taken by the pendulum to oscillate from $\theta = -\theta_0$ to $\theta = +\theta_0$.

Now, $$T = 2\int_{-\theta_0}^{\theta_0} \frac{d\theta}{\sqrt{2\frac{g}{l}(\cos\theta - \cos\theta_0)}} = \frac{2}{\sqrt{g/l}} \int_{-\theta_0}^{\theta} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}} \qquad ...(64)$$

$$= \frac{1}{\omega_0} \int_{-\theta_0}^{\theta} \frac{d\theta}{\sqrt{[\sin^2(\theta_0/2) - \sin^2(\theta/2)]}}$$

Let $k = \sin\theta_0/2$ and $\sin\theta/2 = k\sin\alpha$

So that $$d\theta = \frac{2k\cos\alpha\, d\alpha}{\cos(\theta/2)} = \frac{2k\cos\alpha\, d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

Hence $$T \frac{2}{\omega_0} \int_0^{\pi/2} \frac{2k\cos\alpha\, d\alpha}{\sqrt{k^2 - k^2\sin^2\alpha}\sqrt{1-k^2\sin^2\alpha}}$$

or $$T = \frac{4}{\omega_0} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \qquad ...(65)$$

The integral $$\int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = K(k) \qquad ...(66)$$

is called the **elliptical integral of first kind**. This $K(k)$ is given by

$$K(k) = \frac{\pi}{2}\left[1 + (1/2)^2 k^2 + (1\cdot 3/2\cdot 4)^2 k^4 + ...\right]$$

Thus $$T = \frac{4}{\omega_0}K(k) = \frac{2\pi}{\omega_0}\left[1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{3}{8}\right)^2 k^4 + \ldots\ldots\right]$$

But $k = \sin(\theta_0/2)$ and $\theta_0 =$ angular amplitude, hence

$$T = \frac{2\pi}{\omega_0}\left[1+\frac{1}{4}\sin^2\frac{\theta_0}{2}+\frac{9}{64}\sin^4\frac{\theta_0}{2}+\ldots\ldots\right] \qquad \ldots(67)$$

For $\theta_0 = 2°$ and $\theta_0 = 60°$,

$$T_{(2°)} = 1\cdot000076\ T_0 \text{ and } T_{(60°)} = 1\cdot07T_0 \qquad \ldots(68)$$

where $T_0 = 2\pi/\omega_0$. Thus **the period of a pendulum depends on amplitude.**

Further for relatively small amplitude, $k = \sin\theta_0/2 = \theta_0/2$ and neglecting higher order terms more than k^2, we have

$$T = \frac{2\pi}{\omega_0}\left[1+\frac{\theta_0^{\,2}}{16}\right] \text{ or } T = T_0\left[1+\frac{\theta_0^{\,2}}{16}\right] \qquad \ldots(69)$$

From the above discussion we see the power of the topological aspects of the phase trajectories to provide general qualitative information regarding the behaviour of autonomous systems.

(2) Non-linear Non-conservative Systems

As an example of non-linear non-conservative systems, we write the **Van Der Pol Equation**, represented by

$$\frac{d^2x}{dt^2} - \epsilon(1-x^2)\frac{dx}{dt} + x = 0 \qquad \ldots(70)$$

The parametric equation of the phase trajectories of eqn. (70) are

$$\frac{dy}{dt} = \epsilon(1-x^2)y - x \qquad \ldots(71a)$$

and $$\frac{dx}{dt} = y \qquad \ldots(72b)$$

The equation of the phase trajectory is

$$\frac{dy}{dx} = \epsilon(1-x^2) - \frac{x}{y} \qquad \ldots(72)$$

This equation can be plotted for different values of the parameter ϵ. When $\epsilon = 0$, the phase path is a circle, because

$$\frac{dy}{dx} = -\frac{x}{y} \text{ or } xdx + ydy = 0 \text{ or } x^2+y^2 = a^2 \qquad \ldots(73)$$

In such a case the motion will be simple harmonic.

When $\epsilon = 0\cdot1$, the phase trajectory is sightly different from a circle. If we increase the value of ϵ, the phase trajectories differ much from circles. The phase trajectory of Van der Pol's equation for small values

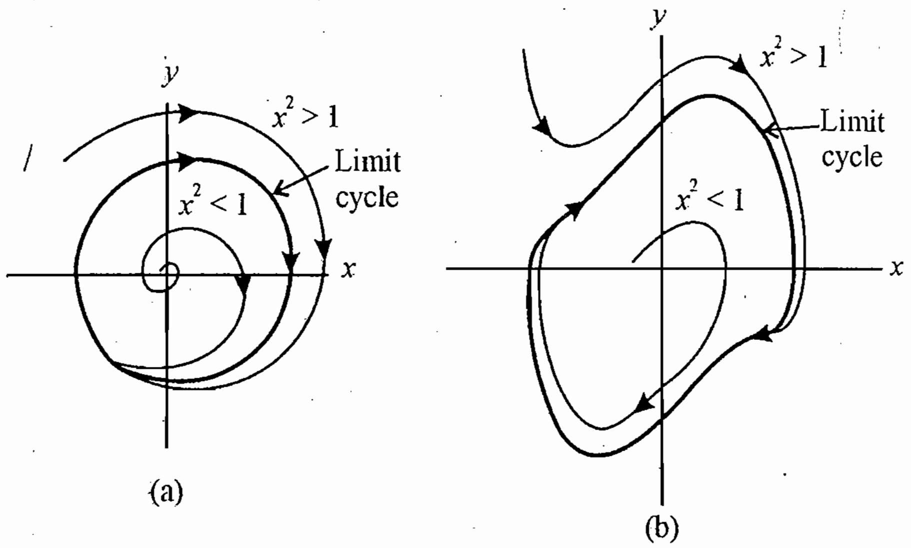


Fig. 16.10. Limit cycles of the Van der Pol equation : (a) approximately circle for small damping coefficient $\in$ and (b) distorted curve for high damping coefficient.

of $\in$ (say $\in = 0{\cdot}2$) is shown in Fig 16·10(a). If $x^2 > 1$, the damping is positive and the motion spirals inward toward a stable path in phase space. This stable path is called the **limit cycle.** For $x^2 < 1$, the damping is negative and the motion spirals inward toward the limit cycle [Fig 16.10(a)]. The final state of motion is stabilized as the damping vanishes for $x = 1$ and the system moves on a closed path approximately circular.

When $\in$ is large enough, the damping term becomes comparable in magnitude to the other terms in eq. (70). The trajectories are still drawn toward the limit cycle but the cycle becomes sufficiently distorted from circular shape [Fig 16.10(b)]. Also, due to strong damping, the frequency relative to that of said simple harmonic motion, decreases and the oscillatons become distorted. For large damping, the shape of the trajectory is very much distorted from circular shape.

16.6. LIMIT CYCLES – ATTRACTORS

Let us consider a system in which the initial condition starts the motion on a trajectory which does not lie on a stable path. However the motion envolves toward a **fixed point** in phase space or toward a stable orbit in phase space. Such a stable path in phase space is called a **limit cycle.** A fixed point (as mentioned) and a limit cycle are the examples of **attractors.**

An attractor is a set of points in phase space to which the solution of an equation evolves long after the transients have died out.

For example, the equilibrium position of a pendulum at rest is a fixed point attractor. If the pendulum is oscillating under small dragging force, the amplitude of successive oscillations will decrease and finally the pendulum will come to a stop at its equilibrium position. It is then said that the motion is drawn to the attractor. In case the motion is overdamped, the pendulum acquires the rest position without any oscillation. In both cases, the motion of the pendulum is such that it reaches the attractor.

In order to discuss more about a limit cycle, consider the equation of motion of an oscillator, when dissipation and excitation forces dependent on the instantaneous amplitude and velocity are present :

$$m\frac{d^2x}{dt^2} + b(x)\frac{dx}{dt} + f(x) = 0 \qquad ...(74)$$

This is equivalent to

$$\frac{dy}{dt} = -\frac{b(x)y + f(x)}{m} \qquad \text{...(75a)}$$

and
$$\frac{dx}{dt} = y \qquad \text{...(75b)}$$

Dividing (75*a*) by (75*b*) , we get

$$\frac{dy}{dx} = -\frac{b(x)y + f(x)}{my} \quad \text{or} \quad mydy + f(x)dx = -b(x)ydx$$

Integrate

$$\frac{1}{2}my^2 + \int f(x)dx = -\int b(x)y^2 dt \qquad \text{...(77)}$$

Let the solution of eq. (74) be

$$x = a\cos(\omega t + \phi) \qquad \text{...(78)}$$

where ω and ϕ are slowly varying functions of x. If in the term $b(x)$ the dissipative and exciting terms are separable, then the integration of the right hand side of eq. (77) over a cycle gives

$$\int \left(\frac{dx}{dt}\right)^2 b(x)dt = E_1 - E_2 \qquad \text{...(79)}$$

where E_1 is the energy dissipated by the system over a cycle and E_2 is the energy supplied to the system over the cycle.

If the dissipation is linear, *i.e.*, $[b(x)\dot{x}] = 2bx$ (say) where b is constant, then over one cycle

$$E_1 = \int_0^\tau 2by^2 dt = \int_0^\tau 2b\left(\frac{dx}{dt}\right)^2 dt = \int_0^\tau 2b.a^2\omega^2 \sin^2(\omega t + \phi)dt = ba^2\omega^2\tau = 2ba^2\omega\pi \quad (\because \omega = 2\pi/\tau) \qquad \text{...(80)}$$

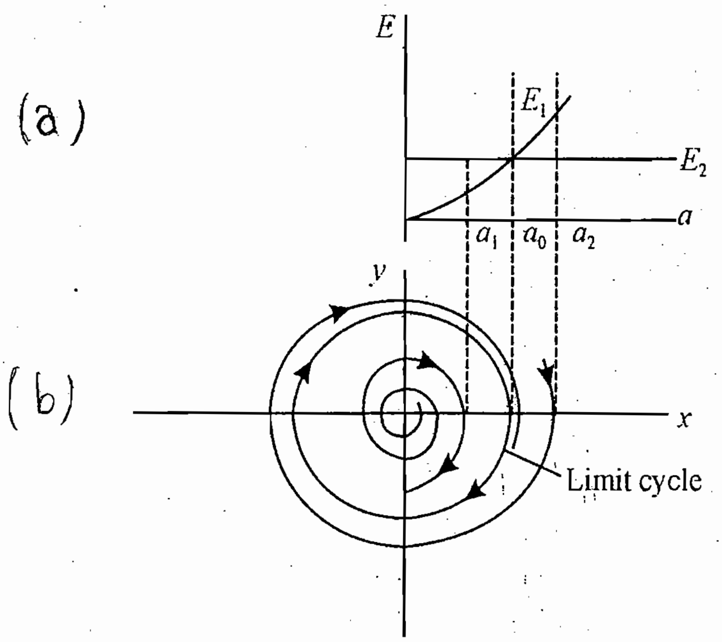


Fig. 16.11. Phase trajectories of the differential equation (74) for $E_1 < E_2$, $E_1 = E_2$ and $E_1 > E_2$.

In Fig. 16·11(a) E_1, the energy dissipated over one cycle, is plotted against the amplitude a. If the supplied energy (E_2) over a cycle is asssumed to be independent of the amplitude, then E_2 will represent a straight line parallel to amplitude axis. When $a = a_0$, the energy dissipated by the system is compensated entirely by the energy supplied to the system. Consequently, the system behaves like an ideal oscillator whose trajectory in the phase plane is a **closed curve** [Fig 16·11(b)]. When $a_1 < a_0$, the system gets a net amount of energy per cycle and therefore the phase trajectory is a spiral coming out of the origin. For a point $a_2 > a_0$, there is a net loss of energy over a cycle by the system. Therefore, the phase trajectory is again a spiral, but now the spiral tends to moove in. The closed curve in between the two spirals is an example of a **limit cycle.**

Poincare showed that a system of equations of the form

$$\frac{dx}{dt} = P(x,y) \text{ and } \frac{dy}{dt} = Q(x,y)$$

of which eq. (75) [or (40) and (41)] is a special case, may have solutions which give isolated closed phase trajectories. He called such phase trajectories as **limit cycles.** If all nearby phase trajectories approach a limit cycle as t increases, it is said to be a **stable limit cycle.** (*e.g.*, in Fig 16·11). A stable limit cycle is an example of an **attractor,** which attracts all the trajectories in the neighbourhood. A stable limit cycle represents a stable stationary oscillation of the given system similar to a singular point which represents a stable equilibrium. One important application of the theory of limit cycles is in relation to the self-sustained oscillations which develop in an electron tube oscillator which is an example of damped nonlinear oscillatory system.

This is to be mentioned that if all the trajectories move away from the limit cycle, it is said to be an **unstable limit cycle.**

Next to the limit cycle which represents a singly-periodic motion, we have a biperiodic torus. This is a drough-nut attractor in the phase space, which is at least three dimensional. We give below a brief description about N-torus.

16·7. N-TORUS

For an harmonic oscillator,

$$\frac{1}{2}my^2 + \frac{1}{2}kx^2 = E \text{, a constant or, } \frac{p'^2}{2m} + \frac{1}{2}m\omega^2 x^2 = E$$

where $p' = my$, $y = dx/dt$ and $\sqrt{k/m} = \omega$.

Writing $\frac{p'}{\sqrt{2m}} = p$ and $\left(\frac{1}{2}m\omega^2\right)^{1/2} x = q$, we obtain

$$p^2 + q^2 = E \qquad \text{...(81)}$$

This represents a circle in $p-q$ phase plane [Fig. 16·12(a)]. Thus we can represent a harmonic oscillator by a uniform circular motion in phase space.

For a system of two uncoupled oscillators

$$p_1^2 + q_1^2 + p_2^2 + q_2^2 = E \qquad \text{...(82)}$$

For convenience, we consider ω_2 for second oscillator much greater than ω_1 for the first oscillator *i.e.*, $\omega_2 >> \omega_1$. We consider the motion of low frequency oscillator ω_1 proceeding on a circle of large radius in the $p_1 - q_1$ plane and then plot the phase trajectory of the high frequency oscillator ω_2 along a small circle in $p_2 - q_2$ plane drawn perpendicular to the circle of ω_1 and centered on its circuference [Fig. 16·12(*b*)]. Combined motion of the two oscillators in the total phase space is a spiraling of the representative point along the surface of a torus, as shown in Fig 16·12(*b*). In case, the frequency ω_2 is a multiple of ω_1, *i.e.*,

$$\frac{\omega_2}{\omega_1} = N \qquad ...(83)$$

where N is an integer, then the trajectories will close on itself and the same pattern will be repeated after a period $T_1 = 2\pi / \omega_1$. In general, when the frequencies are commensurate, *i.e.*, N in eq. (83) is rational number like $3/4$, then the orbit will of course be closed, but it will trace out more than one path around the circle in $p_1 - q_1$ plane before closing onto itself. However if the frequencies are in commensurate, *i.e.* N in eq. (83) is an irrational number, then the phase trajectory will not be closed one and it will gradually cover the surface of the torus, but it will never pass through exactly the same point twice. In the course of the time. The trajectory will pass arbitrary close to every point on the surface. We call such a trajectory as a dense periodic orbit. This orbit is bounded, lying on a surface, but it is not closed. Note that this motion is confined to four dimensional space.

We can generalize this approach more than two harmonic oscillators. For three such oscillators with frequencies ω_1, ω_2 and ω_3, the motion will be confined to a 3-dimensional surface, which we call a 3-torus in the 6-dimensional $p_1, p_2, p_3, q, q_2, q_3$ phase space. In case of N oscillators, we will have an N-torus in a $2N$-dimensional phase space. However it is difficult to visualize the N-tori when N is greater than two.

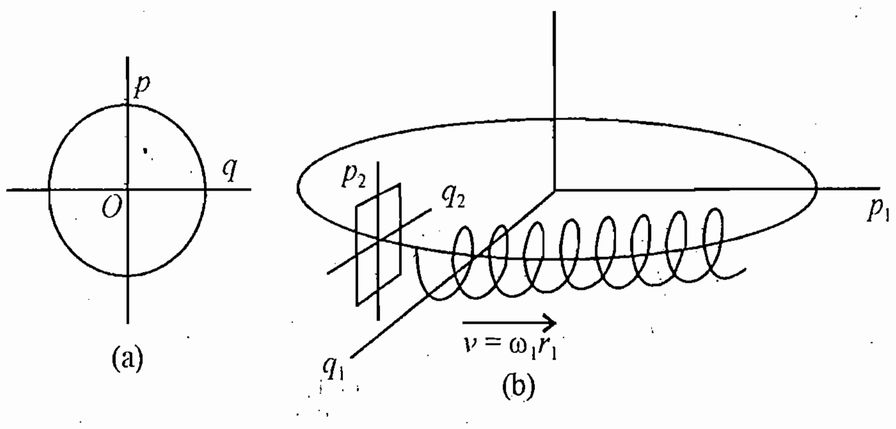


Fig. 16.12. (a) Phase trajectory of reduced harmonic oscillator in p-q plane (b) Circular trajectories for low ω_1 harmonic oscillator in the horizontal p_1- q_1 plane and for high ω_2 (>21) harmonic oscillator in the vertical plane; resultant spiraling trajectory is 2-torus.

We have mentioned above three distinct types of attractors : (*i*) fixed point (*ii*) limit cycle, and (*iii*) torus. When the motion is chaotic, the attractor is said to be chaotic or strange attractor. These four types of attractors in a dynamical system with three degrees of freedom are shown in Fig. 16.13.

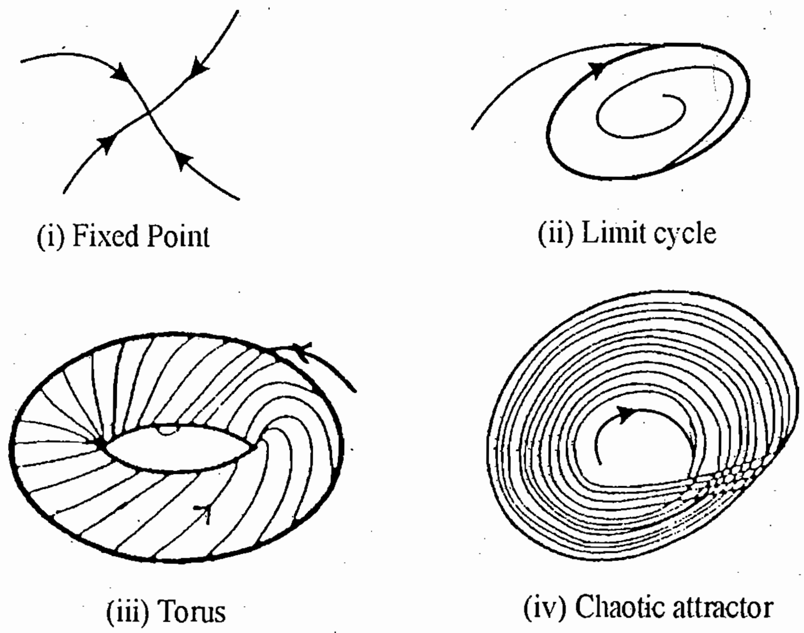


Fig. 16.13. Different types of attractors

16.8. CHAOS

By chaos we mean an irregular complex motion whose long run behaviour in detail is not predictable. The long term bahaviour in a nonlinear system may become senstive to the initial conditions. A small change in the initial conditions may completely change the future path of the system and hence the resulting moton is not predictable. Here the system is classical and hence its motion is governed by the laws of classical mechanics. Thus the motion is completely **deterministic**, but its long term behaviour is chaotic. This is why the phenomenon of chaos is called **deterministic chaos.**

It is to be understood that the chaotic motion is not completely random. In a random sequence, the successive terms are governed by a difinite probability distribution but are not completely determined. In a chaotic sequence, each successive term is uniquily determined by the preceding term and to that extent it is fully deterministic. However, the chaotic sequence is completely a periodic and the long term behaviour of the chaotic system is not predictable. Further in chaos, specific solutions change exponentially in response to small changes in the inital conditions. In fact, chaos is a type of motion that lies between the regular deterministic trajectories (obtained from the solutions of integrable equations) and a state of noise or stochastic behaviour, having characteristics of complete randomness. Example of random motion is the brownian motion of a speck (particle) of pollen, which moves zigzag in water due to innumerable collisions with unobserved water molecules. The unpredictability in the system results from the statistical complexity of the system. However the case of deterministic chaos is different. A small system with small number of degrees of freedom may become chaotic. For example, a system of two coupled oscillators can show chaos if it is excited beyond a threshold of energy value. In deterministic chaos, it is this complex and seemingly unpredicatable behaviour of relatively simple systems, where watchng of their own evolution continuously is the shortest and the most efficient procedure to determine their future course.

16.9. LOGISTIC MAP

Several concepts related with chaos and the phenomenon of chaos itself were introduced following the nonlinear or iteration (logistic) equation, given by

$$x_{n+1} = \mu x_n(1-x_n) = f_\mu(x_n) \qquad ...(83)$$

where μ is the control parameter, which lies in the range $1<\mu \leq 4$ *and x is a variable resticted to the region $0 \leq x \leq 1$ or $x \in [0,1]$. The function $f_\mu(x)$ is zero at the end points of the unit interval [0,1], *i.e.*, $f_\mu(0) = 0 = f_\mu(1)$. The maximum of the function $f_\mu(x)$ is obtained from the condition

$$f'_\mu(x_m) = \mu - 2\mu x_m = 0 \text{ or } x_m = 1/2 \quad ...(84)$$

So that $$f_\mu(1/2) = \mu/4 \quad ...(85)$$

In eq. (83) ; x_n is the value of x after n iterations. Such difference equations are known as **logistic equations** or **logistic maps.** In fact, eq. (83) is representative of many dynamical systems in biology, chemistry, physics and even economics. This equation illustrates most of the characteristics of chaos. Its solutions show regularities as well as chaotic behaviour.

The logistic mapping was introduced first in 1845 by P. F. Verhulst to model the biological population by the differential equation $dx/dt = \mu x(1-x)$. The logistic map is the discrete version of this continuous differential equation for the population growth of animals (whose generations do not overlap) subject to the limited resources, *i.e*,. logistics. This is why eq. (83) is called **logistic map.** Here x_n denotes the population density in the n^{th} year. The linear term simulates the birth rate and the nonlinear term the death rate of the animals in a constant environment controlled by the parameter μ. For low population density, the reproduction rate is positive (for $\mu > 1$) and the population multiplies. But when it grows too large, the resource limitation is felt and the population declines. It may also lead to oscillations or even chaotic fluctuations, depending on the logistic.

The logistic equation gives the evolution of a variable x as function $f_\mu(x_n)$ of discretized time n. Hence, we start with an initial condition x_0 (first value of x_n for $n = 0$) in the interval [0,1] and put it on the right hand side of eq. (83) so that we get x_1 as the output. The process is repeated (iterated) by taking x_1 as the new input on the right hand side of eq. (83) and the result is the output x_2. Further iterations give successively $x_3, x_4 ...... x_n, x_{n+1}$ outputs and the plotting of the function x_{n+1} against x_n is done as follows :

Plotting of the Logistic Map—We take x_{n+1} (output) as the ordinate against x_n (input) as the abscissa. If we start off with an initial value x_0 between 0 and 1, the output also lies in the same unit interval. Hence the phase space is a finite one in two dimensions and in fact it is unit square.

Now we plot the function $f(x)$ against x for $\mu = 2$, which is a **parabola**, standing on the x_n-axis, giving us the ordinate x_{n+1} for the abscissa x_n. Then we draw the **diagonal** between (0,0) to (1, 1). Let us take $x_0 = 0 \cdot 1$ as the initial value, then from $x_{n+1} = \mu x_n(1-x_n)$, $x_1 = \mu x_0(1-x_0) = 2 \times 0 \cdot 1(1-0 \cdot 1) = 0 \cdot 18$

For $x = x_0 = 0 \cdot 1$ on abscissa we move vertically to meet the parabola at $x_1 = 0 \cdot 18$ and a horizontal line from $(x_0, x_1) = (0 \cdot 1, 0 \cdot 18)$ intersects the diagonal at $(x_1, x_1) = (0 \cdot 18, 0 \cdot 18)$. The vertical line through (x_1, x_1) meets the parabola at x_2, given by $x_2 = \mu x_1(1-x_1) = 2 \times 0.18 \times (1 - 0 \cdot 18) = 0 \cdot 2952$ and a horizontal line

* The function $f_\mu(x) = \mu x(1-x)$ represents a parabola and its maximum value becomes more than 1, when $\mu > 4$. Also if $\mu \leq 1$, the sequence $x_1, x_2 .....$ tends to zero. Hence the values of μ in the range $1 < \mu \leq 4$ are considered.

from (x_1, x_2) meets the diagonal at (x_2, x_2) (Fig. 16.14). The process is repeated for $x_3, x_4.....$ values on abscissa and a zig-zag sequence is obtained. The sequence ends at **a fixed point** or **attractor** *P.*

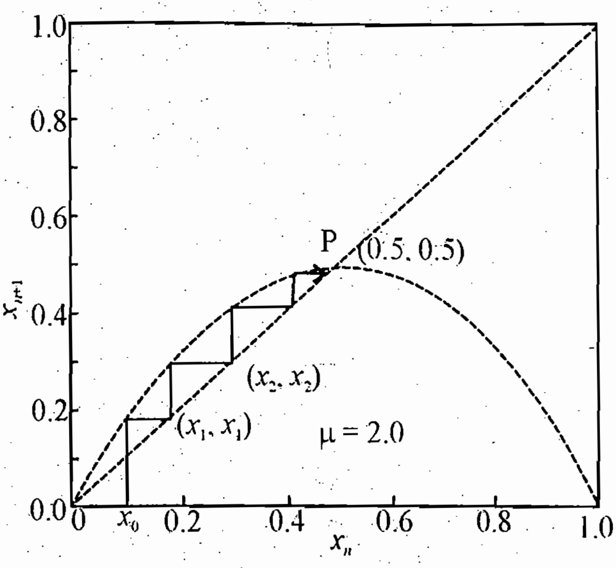


Fig. 16.14. Plotting of logistic map for $\mu = 2.0$

A fixed point (attractor) is defined by the condition

$$f_\mu(x) = x \qquad ...(86)$$

and is the point where the parabola of $f_\mu(x)$ **intersects the line** $x_{n+1} = x_n$ (or $y = x$). Thus for a fixed point

$$x_{n+1} = x_n \qquad (\text{say } x_n = x^*) \qquad ...(87)$$

So that $x^* = \mu x^*(1 - x^*)$ or $x^*[1 - \mu(1 - x^*)] = 0$

Therefore, $x^* = 0 = x^*_0$ (say) and $x^* = 1 - \dfrac{1}{\mu} = x^*_1$ (say) ...(88)

Thus these are the two fixed points for μ in the region $1 < \mu \leq 3$.

The derivatives at these points are given by

$$f'_\mu\,(x^*_0) = \mu \text{ and } f'(x_1^*)' = 2 - \mu \qquad ...(89)$$

Since $\mu > 1$, the first point $x_0^* = 0$ is always unstable. The second point $x_1^* = 1 - 1/\mu$ is stable (point attractor) for $1 < \mu \leq 3$.

For stability, we require $\left| f'_\mu\,(x^*) \right| < 1$ **at the fixed point.** ...(90)

Further, for a stable fixed point, the points near to it are moved even closer to the fixed point, while for an unstable fixed point, nearby points move away as time progresses.

For $\mu = 2$, the fixed point attractor is at x^*_1 = 1 – 1/2 = 1/2.

The manner in which the sequence ends on the fixed point depends on the slope of the curve $f_\mu(x) = \mu x(1 - x)$ at that point. When the slope $f'_\mu\,(x^*)$ lies between 1 and 0 corresponding to $1 < \mu < 2$ [from eq. (89)], the sequence converges to the fixed point while remaining on the same side of the diagonal $x_{n+1} = x_n$ ($y = x$) [Fig. 16·15]. If the slope $f'_\mu\,(x_1^*)$ lies between 0 and -1 corresponding to $2 < \mu < 3$, the sequence

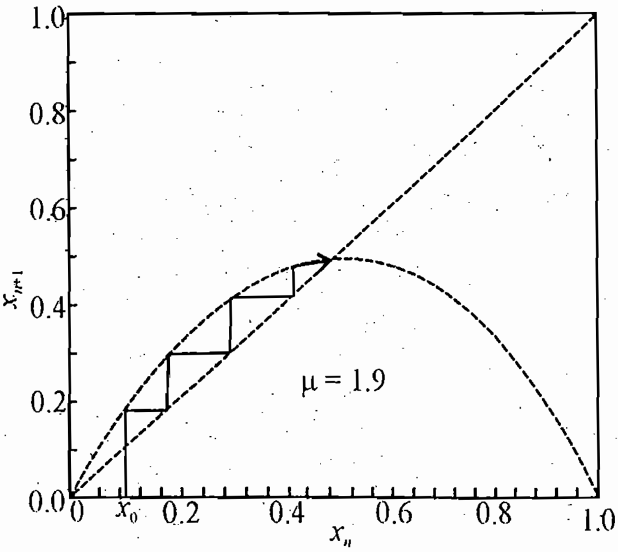


Fig. 16.15. Logistic map for $\mu = 1.9$

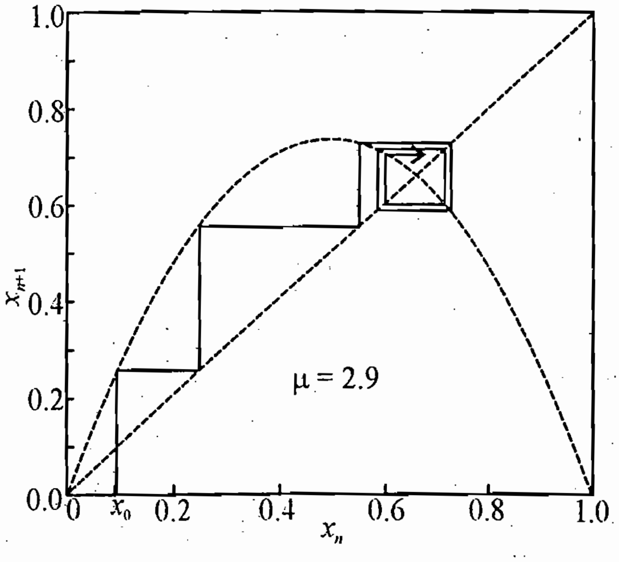


Fig. 16.16. Logistic map for μ = 2.9

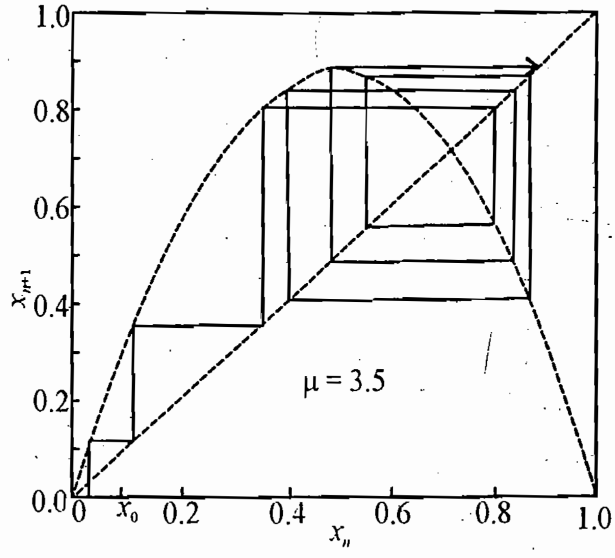


Fig. 16.17. Logistic map for μ = 3.5

converges in an oscillatory way [Fig. 16.16]. Now, if the slope $f_{\mu}'(x_1^*) < -1$ correspondng to $\mu > 3$, the sequence diverges away from the fixed point [Fig 16.17] and the fixed point becomes unstable because $|f'(x_1^*)| > 1$.

Bifurcations—Study of $f_{\mu}^k(x_n)$ versus x_n graph. We have seen above that for $\mu > 3$, both the points $x_0^* = 0$ and $x_1^* = 1 - 1/\mu$, are unstable. For example, when $\mu = 3 \cdot 3$, the fixed points are $x_1^* = 0$ and $x_1^* = 0 \cdot 6969$. Now there are no fixed points which attract the sequence or orbit for any value of x_0 in between 0 and 1. In fact, the sequence settles for any value of x_0 (say 0·2, 0·5, 0·95 etc.) into a pattern of alternating points $x_2^*(1) = 0 \cdot 4794$ and $x_2^*(2) = 0 \cdot 8236$. We have drawn in Fig 16·18 the sequence for $x_0 = 0 \cdot 2$. The values of $x_1, x_2, x_3, x_4, \ldots, x_9, x_{10}, x_{11}, x_{12}, \ldots$ are 0·5280, 0·8224, 0·4820, 0·8239, ... , 0·4794, 0·8236, 0·4794, 0·8236, ... The figure shows typical behaviour of the sequence converging to a period-2 and the stable attractor constitutes a pair of ponts $x_2^*(1)$ and $x_2^*(2)$. The sequence or orbit is attracted to $x_1^*(1)$ every two iterates and to $x_2^*(2)$ at alternate iterates. Hence for $\mu = 3 \cdot 3$, **a stable limit cycle of period-2 has evolved.** This cycle of period-2 of the sequence is called **period-2 orbit** or a **2-cycle** and we say that there has been period doubling bifurcation. This situation occurs for $3 < \mu < 3 \cdot 45$.

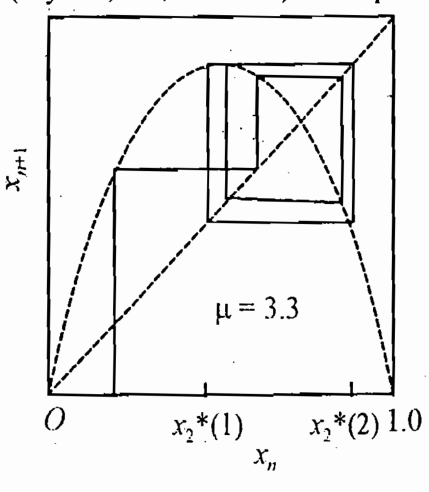


Fig. 16.18. Logistic map for μ = 3.3 with period -2 attractor

We observe that

$$f_{\mu}[x_2^*(2)] = x_2^*(1) \text{ and } f_{\mu}[x_2^*(1)] = x_2^*(2)$$

or
$$f_{\mu}^2[x_2^*(1)] = x_2^*(1) \text{ and } f_{\mu}^2[x_2^*(2)] = x_2^*(2) \quad \ldots(91)$$

Thus the two fixed points of the stable attractor in the range $3 < \mu < 3 \cdot 45$ can be located by solving the equation :

$$f_{\mu}^2(x_2^*) = x_2^* \quad \ldots(92)$$

In fact, the period doubling can also be studied by inspecting the interated mapping

$$x_{n+2} = \mu x_{n+1}(1-x_{n+1}) = f_\mu(x_{n+1})$$

or $$x_{n+2} = f_\mu^{\,2}(x_n) = \mu^2 x_n(1-x_n)(1-\mu x_n + \mu x_n^{\,2}) \qquad ...(93)$$

where $$\begin{aligned} f_\mu(x_{n+1}) &= f_\mu f_\mu(x_n) = f_\mu^2(x_n) = f_\mu[\mu x_n(1-x_n)] \\ &= \mu\mu x_n(1-x_n)[1-\mu x_n(1-x_n)] \\ &= \mu^2 x_n(1-x_n)(1-\mu x_n + \mu x_n^{\,2}) \end{aligned}$$

We study the behaviour of $f_\mu^{\,2}(x)$ versus x plot. $f_\mu^{\,2}(x)$ is a polynomial of fourth order with zeros at $x=0$ and $x=1$. Fig. 16·19 shows the plots for various values of the parameter μ. For $\mu<3$, the graph of $f_\mu^{\,2}(x)$ intersects the line $x_{n+2}=x_n$ ($y=x$) at two points x_0 and x_1 similar to $f_\mu(x)$ does, given in eq. (88). Since the function $f_\mu^{\,2}(x)$ is of fourth order, eq. (93) gives four intersection points with the straight line ($x_{n+2}=x_n$). This happens for $\mu>3$ and for this value of μ, the solution of $f_\mu^{\,2}(x_2{*}) = x_2{*}$ gives the intersection points as

$$x^* = 0, x^* = 1-\frac{1}{\mu},\ x^* = \frac{(\mu+1)}{2\mu} \mp \frac{1}{2\mu}\sqrt{(\mu+1)^2 - 4(\mu+1)} \qquad ...(94)$$

For $\mu>3$, the stability condition $\left(\left|f_\mu^{\,2}(x)\right|<1\right)$ gives the points $x_0{*}=0$ and $x_1{*}=\left(1-\frac{1}{\mu}\right)$ as unstable and $x^* = A \mp B$ points are the two stable points x_2* (1) and x_2* (2), as mentioned in eq. (91). Of course , for $\mu = 3{\cdot}3$ last equation of (94) gives x_2*(1) = 0.4794 and x_2* (2) = 0.8236, as obtained above by numerical calculations.

If the parameter μ is further increased, these two fixed points also become unstable at the critical value

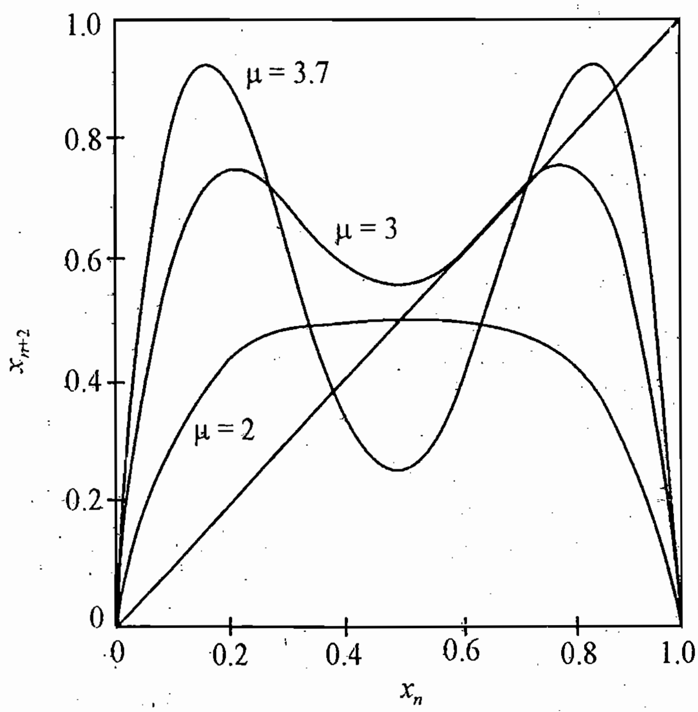


Fig. 16.19. Plot of x_{n+2} [or $f_m^{\,2}(x)$] versus x_n for different values of the parameter μ.

$\mu = 1+\sqrt{6} = 3 \cdot 4495$, and for this value and ahead, one unstable fixed point gives way to two stable fixed points so that $f_4(x_4^*) = x_4^*$ has period 4. *i.e*,. the new attractor consists of a set of 2^2 points and we have 2^2 -cycle. Now as we increase μ, the cycle of period 4 becomes unstable and we get 2^3-cycle at a certain (critical) value. In general, at the k^{th} critical value of the parameter μ (say μ_k), we have the k^{th} period doubling bifurcations† to period 2^k orbit (sequence) or a 2^k-cycle.

Thus at $\mu_1 = 3, \mu_2 = 3 \cdot 4495$, $\mu_3, \mu_4, \ldots$ etc., we get a full cascade of period doublings. In the interval $\mu_k < \mu < \mu_{k+1}$, there exists a stable limit cycle of period 2^k. However, as we increase the value of the parameter μ, its successive critical values come closer and closer, the sequence of bifurcations continues with larger periods until for $k \to \infty$, $\mu_\infty = 3 \cdot 5699\ldots$, where an infinite number of bifurcations occur, the period becomes infinite. The motion has now become aperiodic and it never quite repeats itself. The bands of fixed points x^* start to form a continuum (shown by dark vertical line in Fig 16·20). This is the point where **chaos** starts. When μ approaches 4, x^* spans the unit interval [0,1].

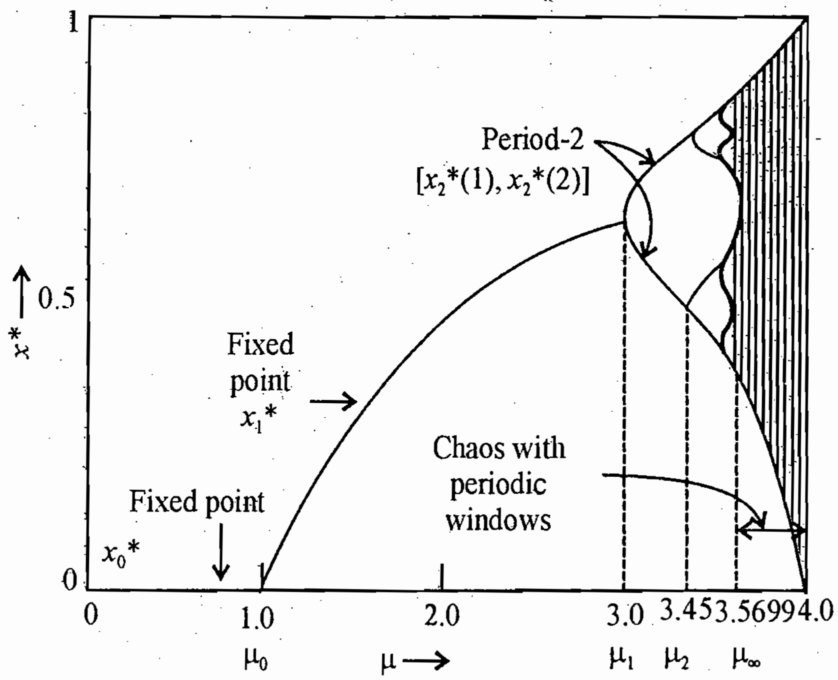


Fig. 16.20. Attractor diagram of the logistic mapping

We find that as the controlling parameter μ is changed, then for $\mu = \mu_1 = 3$, one unstable point (apart from the trivial point $x^* = 0$) is obtained for the sequence x_{n+1} versus x_n. **When one unstable fixed point gives rise to two stable fixed points (or period 2-orbit or 2-cycle), the process is called bifurcation.** The point at which bifurcation takes place is called **a critical** (or **threshold) point.** The bifurcation for $\mu = 3$, where the period doubling occurs, is called a **pitchfork bifurcation** because of its special shape. As the control parameter μ is varied, at successive critical points $\mu_2, \mu_3, \ldots$, each branch of fixed points bifurcates again and the sequence of bifurcations continuous with $2^2 - , 2^4 - \ldots$ cycle. Ultimately at μ_∞, we get infinite number of bifurcations and the chaos starts. The period doubling bifurcation route to chaos is ilustrated in fig. 16·20 through a scheme, known as the **bifurcation diagram,** where we plot the attractor

† With increased period doublings, it becomes impossible to obtain analytical solutions. The iterations are better done on a programmable calculator or personal computer.

set (x^*) against the control parameter (μ). In the diagram, we see the pitchfork bifurcations, cascading into chaos at $\mu_\infty = 3{\cdot}5699......$.

Universal Constants. We have seen above that successive critical points (μ_n), at which the bifurcations occur, get closer and closer as $n \to \infty$. The ratio of the successive spacings as $n \to \infty$ acquires a constant value, given by

$$\lim_{n\to\infty} \frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n} = 3{\cdot}6692\,......,\ \text{say } \delta \qquad ...(95)$$

This constant value $\delta = 3{\cdot}6692$ is called **Feigenbaum number,** which was first recognized by Feigenbaum in the study of period doubling in the logistic map. In fact, this dimensionless number δ is universal for the route to chaos via period doublings for all maps with a quadratic maximum similar to the logistic map. This is why the Feigenbaum number is known as **universal constant.**

There is also a constant, related with the ratio of spacings for successive attractors (x^*) at the successive critical values μ_n as $n \to \infty$. With reference to pitchfork bifurcations, it is really the ratio of the successive openings of the successive generations of forks at $x = 0{\cdot}5$, given by (Fig 16·21)

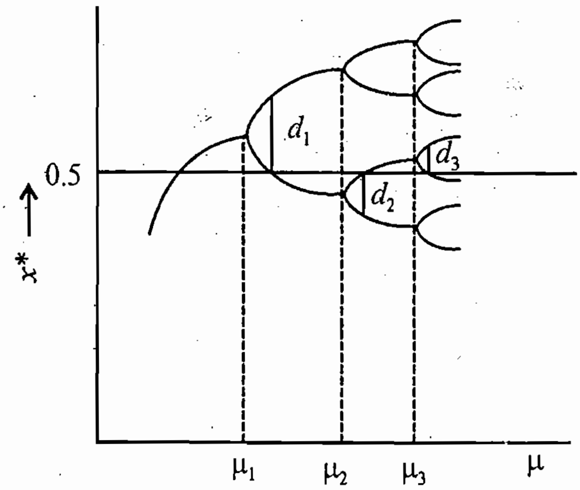


Fig. 16.21. Spacings for successive attractors at the successive values of μ.

$$\lim_{n\to\infty} \frac{|d_n|}{|d_{n+1}|} = 2{\cdot}5029......,\ \text{say}\ \alpha \qquad ...(96)$$

This number α is *another Feigenbaum universal constant.*

Beyond μ_∞. This is interesting what happens beyond $\mu_\infty = 3{\cdot}5699......$ At $\mu = \mu_\infty$, the period becomes infinite. The motion has now become aperiodic and it never quite repeats itself. For $\mu > \mu_\infty$, this is the domain of chaos, characterized by irregular and seemingly random trajectories. A chaotic trajectory of the logistic map is shown for $\mu = 3{\cdot}9$ in fig 16·22.

First the bands of fixed points x^* are formed and then for greater values of μ, these bands begin to form continuum which spans the interval [0,1] as $\mu \to 4$.

In the chaotic region, there are, however, **windows** of periodic solutions (or nonchaotic character). Particularly the region of period-3 occurs above $\mu = 3{\cdot}82$ (Fig. 16.23).

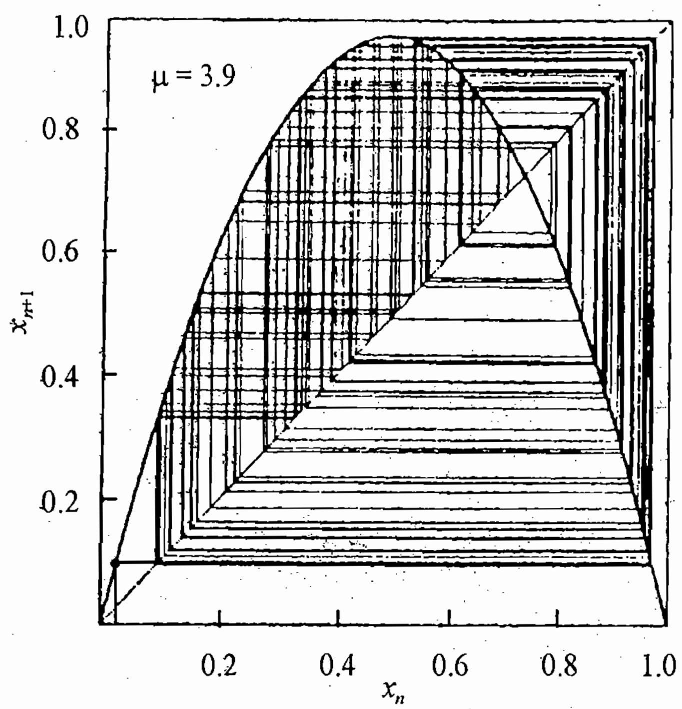


Fig. 16.22. A chatotic trajectory of the logistic map for $\mu = 3.9$

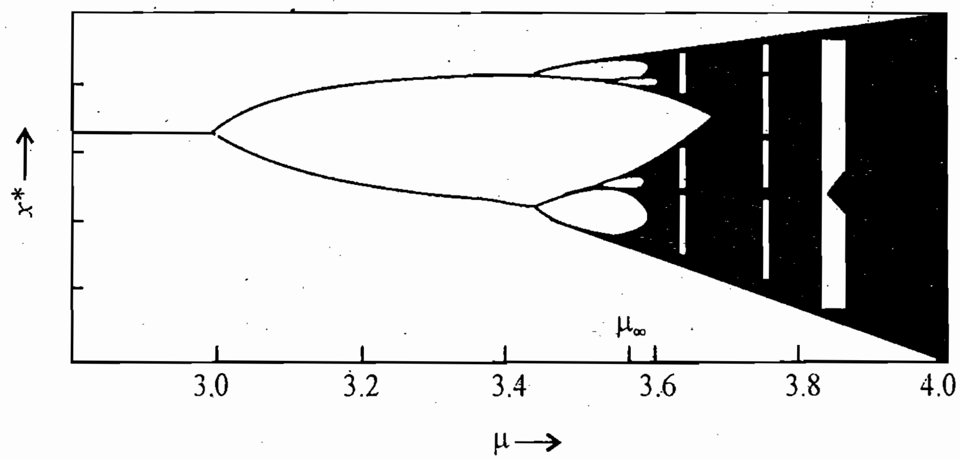


Fig. 16.23. Attractor beyound μ∞.

Ex. 1. *Show that* $x^* = 1$ *is a nontrivial fixed point of the map*

$$x_{n+1} = x_n \exp[\gamma(1-x_n)]$$

having slope $1-\gamma$. *Further show that the equilibrium is stable for* $0<\gamma<2$.

Solution. For fixed point or attractor,

$$x^* \; exp. \; [\gamma(1 - x^*)] = x^*$$

whence, exp. $[\gamma (l - x^*)] = 1$ or $x^* = 1$

The attractor is stable, if

$$\left|\frac{df(x^*)}{dx}\right| < 1, \text{ where } f(x) = x\exp[\gamma(1-x)]$$

Now, $$\frac{df}{dx} = \exp[\gamma(1-x)] - \gamma x \exp[\gamma(1-x)]$$

Hence $$\frac{df(x^*)}{dx} = 1-\gamma \quad \text{(slope)}$$

For stability $|1-\gamma| < 1$ or $1-\gamma<1$ and $-1+\gamma<1$

or $\gamma > 0$ and $\gamma < 2$ or $0 < \gamma < 2$.

16.10. STRANGE ATTRACTOR

For $3<\mu<3{\cdot}45$, the two stable fixed points of $f_\mu^{\,2}(x)$ curve constitute **attractor** of period-2. At $\mu \ge 3{\cdot}45$, these two fixed points of $f_\mu^{\,2}(x)$ curve become unstable, giving rise to four stable fixed points. These four points constitute an **attractor** of $f_\mu^{\,4}(x)$ curve of period-4. On increasing the value of μ, even these four points become unstable and undergo bifurcations. The period doubling continues upto $\mu_\infty = 3{\cdot}5699$, where the period becomes infinite and we say that attractor has infinite set of points. Now, the logistic map has become **chaotic** and the related attractor is said to be **chaotic** or **strange attractor.** In fact, this is an example of strange attractor. The important distinction of the strange attractor when compared to stable attractor, is that the dynamics on them is chaotic and is very sensitive to initial conditions. Further, these attractors have fractal dimensions.

In case of chaotic trajectories, the motion wanders around an extensive and perhaps irregularly shaped region of phase space so that it appears to be random but it is tempered by the constraints. This path or region where the meandering takes place is a strange attractor. This attractor is called strange because of its fractal geometry. The chaotic trajectory roams around, back and forth through this attractor region, but it does not pass through the same point twice. A chaotic orbit, that exists or revisits all regions of the available phase space, is identified with a strange attractor. A fixed point or limit cycle is a localized attractor, but the strange attractor is associated with a very extended region of the phase space and hence the name qualifies.

16.11. SENSITIVITY TO INITIAL CONDITIONS AND PARAMETERS – LYAPUNOV EXPONENT

Sensitivity to initial conditions means that if two trajectories which started off initially at two closely spaced points in phase space, they diverge out far apart. Thus if the map $x_{n+1} = f(x_n)$ is started with two infinitely closeby points, then the separation between them grows exponentially (Fig 16.24). After time n or n iterations, the exponential divergence grows by a factor of $e^{\lambda n}$, where $\lambda > 0$ is called **Lyapunov exponent.** This exponent is a measure of how fast the divergence develops. For example, two nearby points remain close together as they move along in a liquid flow; however after the starting of tubulence the same two points on average keep moving farther and farther apart.

For one dimensional mapping $x_{n+1} = f(x_n)$, two trajectories, starting at the neghbouring points x_0 and $x_0 + \in$ with $\in << 1$, may diverge out after n iterations to a separation d_n, given by

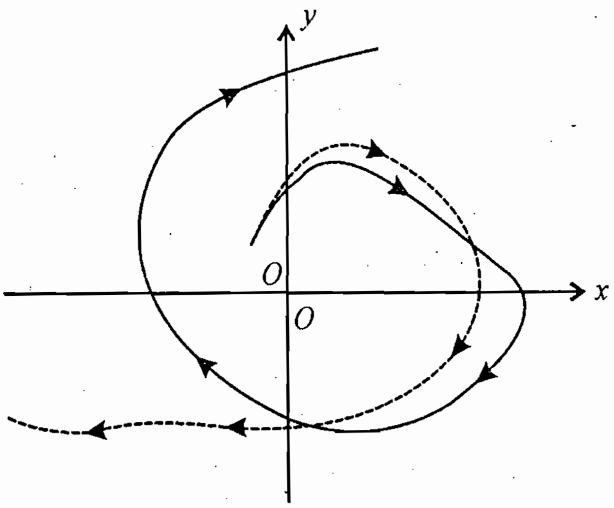


Fig. 16.24. Sensitivity to initial conditions : divergence of two initially neighbouring trajectories.

$$d_n = \left| f^n(x_0 + \in) - f^n(x_0) \right|$$

where $f^n(x_0)$ stands for the n^{th} iterate of x_0. From experience with chaotic behaviour, this distance is expected to increase exponentially with $n \to \infty$, *i.e.*,

$$\frac{d_n}{\in} = \frac{\left| f^n(x_0 + \in) - f^n(x_0) \right|}{\in} = e^{\lambda n}$$

Hence $$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{f^n(x_0 + \in) - f^n(x_0)}{\in} \right) \text{ or } \lambda = \lim_{n\to\infty} \frac{1}{n} \ln\left| \frac{df^n(x_0)}{dx} \right| \quad \text{...(97)}$$

Using the chain rule of differentiation for $\dfrac{df^n(x)}{dx}$, we get

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \ln\left| \frac{df(x_i)}{dx} \right| \quad \text{...(98)}$$

where x_i is the i^{th} iterate of x_0 and λ is calculated at the point x_0.

The Lyapunov exponent λ is a quantitative measure of chaos. A one- dimensional function, which is similar to logistic map, has chaotic cycles $(x_0, x_1, ...)$ for the parameter μ, if the average Lyapunov exponent is positive $(\lambda > 0)$ for that value of μ. Such an initial point x_0 is called a strange or chaotic attractor. For cycles of finite period, λ is negative $(\lambda < 0)$.

16.12. POINCARE SECTIONS

We feel difficulty in visualizing a trajectory in a higher than two dimensional phase space. Suppose (x_1, x_2, x_3) represent a three dimensional state space. We choose a section plane say $x_1 - x_2$ and mark the points $P_0, P_1, P_2 \ldots\ldots$ at which the trajectory crosses this plane successively in the same sense, say downward along the negative x_3-axis. (Fig. 16.25). This set of points constitutes what is known as **Poincare section**. For deterministic motion, there must be a rule or mapping which relates the successive points, *i.e.,* the rule may have the form

$$P_{n+1} = f(P_n) \qquad \ldots(99)$$

In fact this equation represents the coordinates of the points P_{n+1} in terms of those of the earlier point P_n through the function *f*. This is called the **Poincare map** or the first return map. The time periods between the successive returns may not be equal. We obtain here the reduction of the three-dimensional continuous flow to a lower two-dimensional discrete map.

It may also happen that the points *P*'s may lie on a smooth curve, resulting in dimensional reduction. In consequence, the nonlinear three dimensional differential equations are replaced by nonlinear two-dimensional difference equations which can be handled relatively easily. If the phase flow is periodic as in a limit cycle attractor, the successive points $P_0, P_1, P_2, \ldots\ldots$ will come together to a single point. In case, if the trajectory lies on a biperiodic torus, the Poincare section will be a finite set of points or a quasi-continuous curve, which depends on whether the two frequencies of the motion are mutually commensurate (with rational ratio) or incommensurate (with irrational ratio). However, if the dynamics is chaotic, the Poincare section will be a splatter of points over an area, where the successive points of intersection lie erratically.

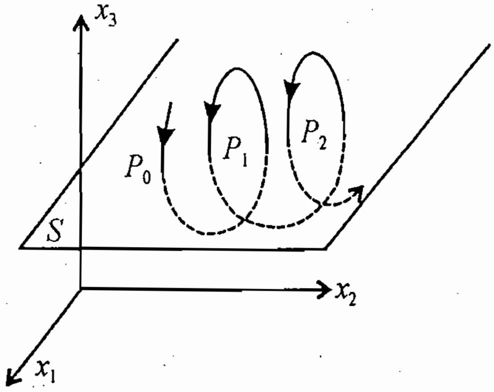


Fig. 16.25. Poincare section for 3-diamensional phase flow.

In general, for an *N*-dimensional phase trajectory, we take a section with (*N*–1) dimensional hypersurface, locally perpendicular to the trajectory. The dimension is reduced by one and thus the procedure is simplified.

Alternatively we construct the Poincare surface of section and obtain the successive intersections of the trajectory with the surface as discussed above, and then, consider the sequence for just one variable, say x_n. From this sequence, one can plot x_{n+1} against x_n – the first return map. It is possible to reconstruct the underlying attractor from such a map. Thus, for example, if this plot gives a curve with a single hump and a smooth maximum, the system is expected to possess the universal features of the logistic map.

16.13. DRIVEN DAMPED HARMONIC OSCILLATOR

In order to exhibit chaos, a system, having first order equations, must be nonlinear and have at least three variables. An example of a simple system, whose motion becomes chaotic, is the driven damped harmonic oscillator, namely periodically driven pendulum. Let θ be the displacement at an instant *t*. Then the equation of motion of the system is

$$\frac{d^2\theta}{dt^2}+b\frac{d\theta}{dt}+\omega_0{}^2\sin\theta = f\cos pt$$

where f is the amplitude of the periodic driving force with frequency p, b is damping constant and ω_0 damping free oscillation frequency. This nonlinear second order differential equation can be written in a system of three coupled first order differential equations as

$$\frac{d\omega}{dt} = -b\omega - \sin\theta + f\cos\phi$$

$$\frac{d\theta}{dt} = \omega \qquad \text{...(101)}$$

$$\frac{d\phi}{dt} = p$$

where $\phi = pt$ is the phase of driving term and we have taken $\omega_0 = 1$.

Depending on the values of the parameter f, p and b, the driven oscillator exihibits many distinct types of motion. Here, we plan to investigate the dependence of motion on the value of the amplitude (force strength) f for constant values of the frequency $p = 2/3$ and $b = 0{\cdot}5$. This f plays the role of the control parameter. The system of differential equations is integrated numerically for various values of the parameter f. The initial solutions, related to transients, will be ignored. Here we reduce three-dimensional representation to two-dimensional phase space diagram. Poincare section and the trajectory is plotted in $\omega - \theta$ plane.

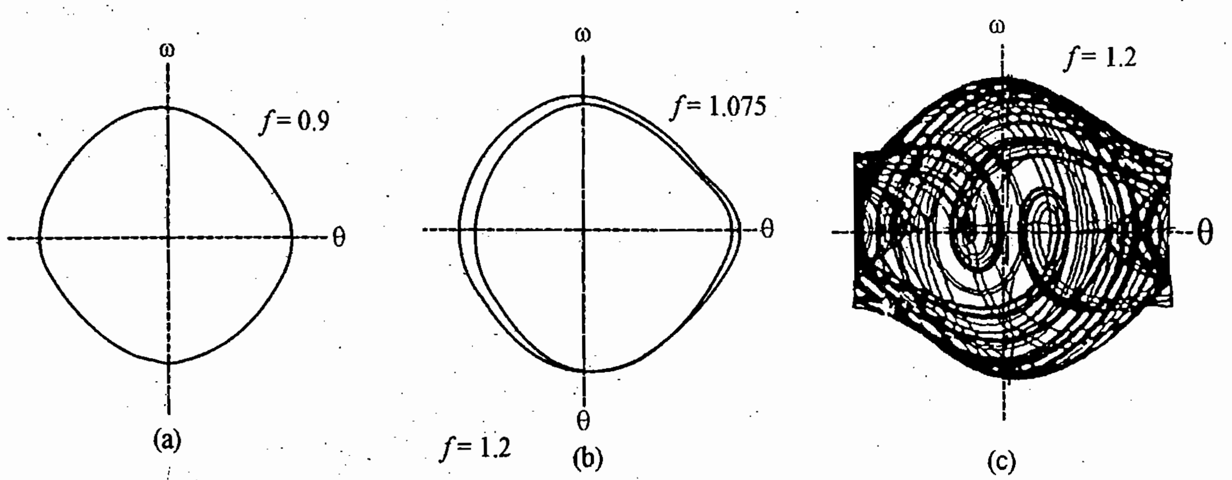


Fig. 16.26. Phase trajectories of a periodically driven damped pendulum for different values of the parameter f (p = 2/3 and b = 0.5 fixed)

For smaller value of the parameter, say f = 0.9, the pendulum performs approximately harmonic librations and the phase path is approximately an ellipse [Fig. 16.26(a)]. With increasing the value of the control parameter, a **bifurcation arises at about** f = 1.07 with a **period doubling** as shown for f = 1.075 [Fig. 16.26(b)], where two slightly different vibrational tracks are alternating. The bifurcatons are associated with normal or nonchaotic behaviour. After further period doublings in the range 1.15 to 1.30, chaotic solutions are obtained. For f = 1.2, the trajectory fluctuates in an erratic way between librations and rotations in both dimensions and the corresponding path densely covers a domain in phase space [Fig 16.26(c)].A survey of the behaviour of the system is shown in the attractor diagram in which the bifurcations are indicated and the chaos is shown by shaded region [Fig. 16.27]. For comparison, two trajectories $\theta(t)$ for $f = 1{\cdot}12$ (regular) and $f = 1{\cdot}2$ (chaotic) are shown in Fig 16·28.

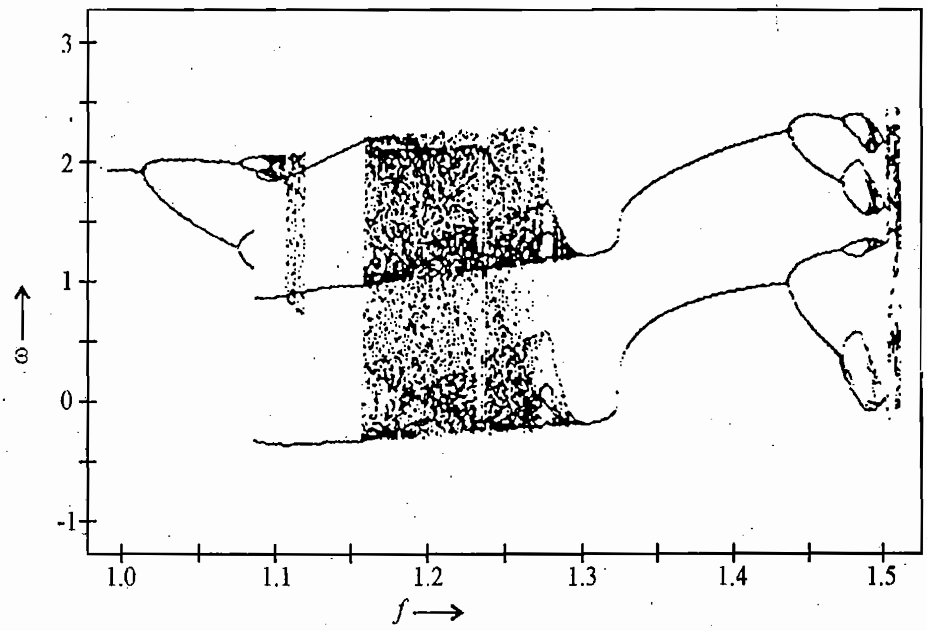


Fig. 16.27. Bifurcation diagram for driven damped harmonic oscillator, showing regions of regular and chaotic character.

For increasing f values, more than 1·3, the chaos do not occur and rotating periodic solutions and also then period doublings are obtained. After a bifurcation cascade, there occurs again a range with chaotic region, e.g., at f = 1·5 (Fig 16·27).

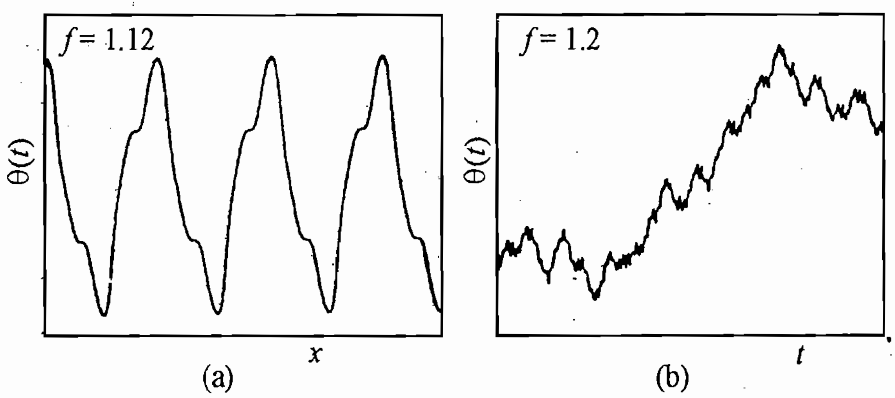


Fig. 16.28. Two trajectories θ(*t*) for (a) *f* = 1.12, a periodic motion and (b) *f* = 1.2, a chaotic motion

16.14. FRACTALS

A fractal is geometrical object or set with nonintegral dimensions which exhibits the property of self similarity. The shape of a fractal is intricate and the name is due to its noninteger dimension. Intuitively a fractal is a set that is self- similar under magnification, the dimension is typically not integral. We call a set of attracting points with noninteger dimension as a *strange attractor.*

Cantor set is an example of a fractal which can be constructed as follows. Consider a straight line segment, remove its middle third part to get two equal line segments. Now remove the middle third part of the later line segments to obtain a total of four line segments, and so on (Fig. 16·29). If we continue this process of removing middle third of successively smaller line segments indefinitely, we obtain an infinite number of points (of length zero). This is called

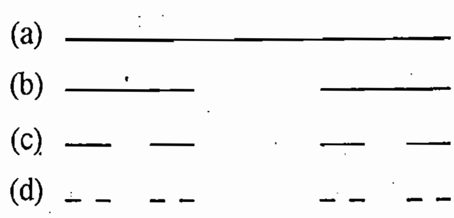


Fig. 16.29. Cantor set

a **Cantor set**. This set at different stages of its generation is self similar in the sense that magnifications of the set at later stages of generation have the same form as the set itself at earlier stages of formation. Now, we want to discuss the dimensionality of the cantor set.

First we discuss the dimensionality D in the ordinary Cartesian or Euclidian space. In one dimension, consider a line segment of length l_0, which is divided into a large number of equal small lengths, each of magnitude $l << l_0$. In two dimensions we have a square of side l_0, which is subdivided into many small squares, each of side $l << l_0$. In case of three dimensions, similar subdivision is made of a cube of side l_0. In each case, we represent the total number of subdivisions by $N(l)$, given by

$$N(l) = \left(\frac{l_0}{l}\right)^D \qquad ...(102)$$

where $D = 1, 2, 3$ is the dimensionality for the three cases.

From (102), we obtain

$$\log N(l) = D \log \frac{l_0}{l}$$

So that

$$D = \frac{\log N(l)}{\log(l_0 / l)} \qquad ...(103)$$

If we consider l_0 to be unity, *i.e.*, $l_0 = 1$, then

$$D = \frac{\log N(l)}{\log(1/l)} \qquad ...(104)$$

This formula is applicable for dimension D for systematic division of ordinary space in any number of dimensions. This is also applicable to the subdivisions of space which are characteristics of fractals. In the later case, the dimensionality, determined by the equation (104) is called the **fractal dimension** D_f and it has nonintegral value.

In case of cantor set, a line length unity is subdivided. This is one dimensional with Euclidean dimensionality $D_e = 1$. Since after an infinite number of splittings, the small lines dimnish to points, having dimensionality of zero and the topological dimensionality of the cantor set is said to be $D_t = 0$*. At the n^{th} level of subdivision, the line segments have length l with their number $N(l)$, given by

$$l = \frac{1}{3^n} \quad \text{and} \quad N(l) = 2^n \qquad ...(105)$$

The **fractal dimension** D_f is defined by

$$D_f = \frac{\log N(l)}{\log(1/l)} \qquad ...(106)$$

This D_f is also called **Hausdorff dimension.**

From (105), $\log N(l) = n \log 2$ and $\log (1/l) = n \log 3$

Hence

$$D_f = \frac{\log 2}{\log 3} = 0{\cdot}63 \qquad ...(107)$$

---

* For a single point, $N(l) = 1$ or $\log N(l) = 0$ or $D = 0$. For a differentiable curve of unit length, $N(l) = 1/l$, $D = 1$.

In the discussion ahead, we shall use the following symbols :

D_e = Initial Eucliclean dimension,

D_t = Final limiting Euclidean (called topological) dimension.

and D_f = non-integer dimension, characteristic of fractals and strange attractors.

The fractal dimension D_f is always in between the two limiting values D_e and D_t, *i.e.*,

$$D_e > D_f > D_t \quad \text{...(108)}$$

In case of cantor set

$$1 > 0 \cdot 63 > 0 \quad \text{...(109)}$$

which is in accordance with relation (108).

In the general case of a fractal object in a D_e dimensional Euclidean space, the **fractal dimensionality** D_f is defined by covering the region, occupied by the object, by D_e-dimensional cubes of side l (or spheres of radius l) by the expression.

$$D_f = \lim_{l \to 0} \frac{\log N(l)}{\log(1/l)} \quad \text{...(110)}$$

which is also called the **box (cube) counting dimension** or **capacity dimension** of the set. When $D_e = 3$, the box is 3-dimensional cube of side l, for $D_e = 2$, it is 2-dimensional square of side l and for $D_e = 1$, it is a line segment of length l.

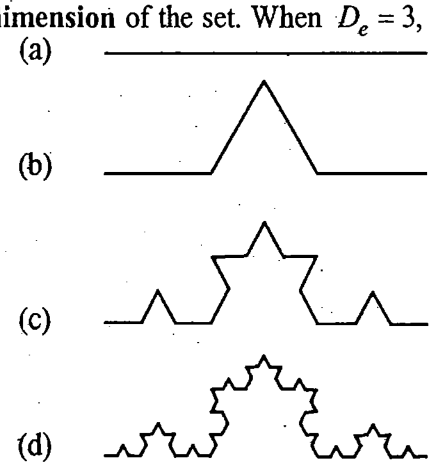


Fig. 16.30. Koch curve

Evaluation of fractal dimensions for some cases : (1) Koch curve. In order to construct the Koch curve, we start with a line segment of unit length [Fig. 16.30 (*a*)] and remove the middle one third. Then we replace it with two segments of length one-third and the two segments form an equilateral triangle [Fig 16.30(*b*)]. The total length of the curve is now (4/3). This process is repeated with each of the segment, so that the total length of all segments is $(4/3)^2$ [fig. 10.30(*c*)]. If we continue to repeat this process with each segment of the successive figures obtained, we get what is known as **Koch curve.**

After n steps, the total number of the segments is

$$N(r_n) = 4^n$$

and the length of each segment is

$$r_n = \left(\frac{1}{3}\right)^n$$

Hence the **fractal dimension of the Koch curve** is

$$D_f = \lim_{r_n \to 0} \frac{\log N(r_n)}{\log(1/r_n)} = \frac{\log 4}{\log 3} = 1 \cdot 26 \cdots \quad \text{...(111)}$$

This number 1.26..... is more than 1 (the dimension of line segment) and less than 2 (the dimension of surface). Since the Koch curve is produced by iteration of the first step, it is correctly self similar.

(2) Siepinski gasket. The fundamental element of the Siepinski gasket is a two dimensional area, namely an equilateral triangle. The iteration rule is as follows : We subdivide each triangle by 4 congruent parts and remove the central triangle (Fig 16.31). After iterations, the resulting object is something between area and curve. It has the property of self similarity.

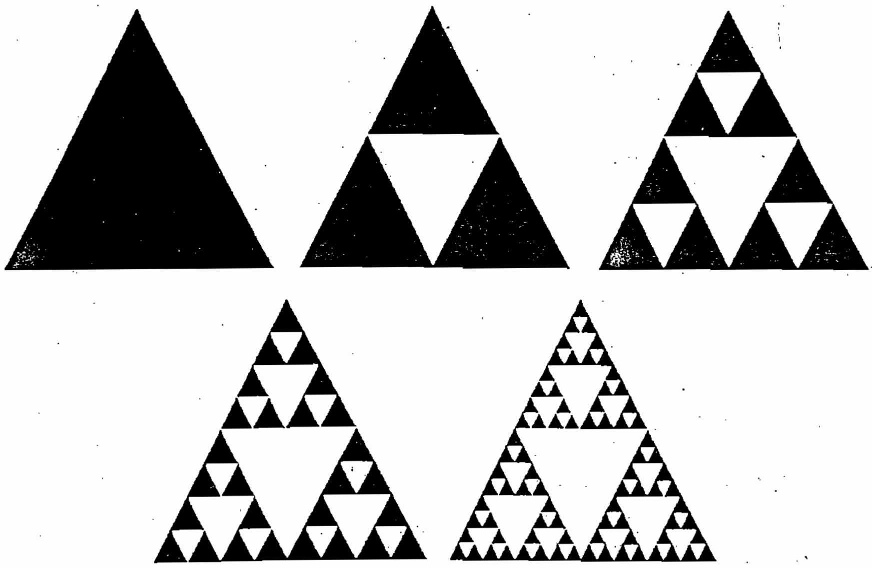


Fig. 16.31. Siepinski gasket

At the n^{th} level of subdivision, the traingle has the side $l = 1/2^n$ or $1/l = 2^n$ and the number $N(l) = 3^n$. Hence the fractal dimension, as $l \to 0$, is

$$D_f = \frac{\log 3}{\log 2} = 1 \cdot 58496 \qquad ...(112)$$

Here $D_e = 2$ and $D_t = 1$ and D_f satisfies $1 < 1 \cdot 58496 < 2$.

(3) Logistic map. The strange attractor associated with the logistic map can be obtained by plotting on a line the values of x^*, where the bifurcations take place. The resulting set is somewhat similar to the cantor set, but not strictly self-simular (Fig 16.32). At μ_∞ (onset of chaos) the box counting dimension of the logistic map has been found by Grassberger (1981) to be 0·538. This is in between the topological dimension $D_t = 0$ corresponding to individual points x^* and the Euclidean dimension $D_e = 1$ corresponding to the range of $x, 0 \le x \le 1$, being a line segment.

Other examples of fractals are the formation of ice crystals, river and vascular networks. It is interesting to mention that the fractal dimension of the coastline of England is about 1·2 and that of the distribution of the stars in the sky is about 1·23.

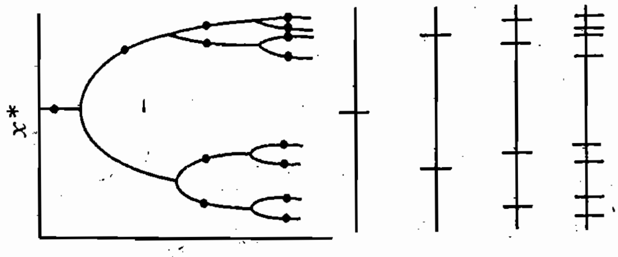


Fig. 16.32. Bifurcations in logistic map, somewhat similar to cantor set

16·15 INTEGRABLE HAMILTONIAN AND INVARIANT TORI

A convenient way to represent the periodic motion is to transform the Hamiltonian to action-angle variables. The new momentum, known as the action variable $J = \oint p dq$, is a constant of motion and a new conjugate coordinate $w = vt + \beta$, depends linearly upon time. In case of harmonic oscillator (conservative system) $J = 2\pi E / \omega$ $(\omega = \sqrt{k/m})$ and $w = \frac{\omega}{2\pi} t + \beta$. For such a system, we can draw a circle with J = constant (as radius) and $0 \le w \le 2\pi$ as the angle. In general suppose that the conservative system admits a canonical transformation from the old (q_k, p_k) $(k = 1,2,3,.....)$ to the new (w_k, J_k) coordinate-momenta system such that the transformed Hamilloan depends only on the N momenta $J_1, J_2, ..., J_N$, *i.e.*,

$$H(q_k, p_k) = H'(J_k) = \alpha_1$$

Writing the equations of motion for action-angle variables as

$$\dot{J}_k = -\frac{\partial H'(J_k)}{\partial w_k} = 0, \text{ hence } J_k = \text{ constant} \quad ...(113)$$

$$\dot{w}_k = \frac{\partial H'(J_k)}{\partial J_k} = \nu_k(J_k), \text{ giving } w_k = \nu_k(J_k)t + \beta_k \quad ...(114)$$

Thus the set of N momenta $J_1, J_2, ...J_N$ are the N constants of motion. Such a system is called **integrable system.** In other words, a Hamiltonian system of N degrees of freedom is said to be integrable, if there are N constants of motion.

In case of an integrable Hamillonian system with the action-angle variables (J_k and w_k) with k = 1, 2, ...N, we may view each pair (J_k, w_k) as polar coordinates describing a circle with J_k = constant, as the radius and $0 \leq w_k \leq 2\pi$ as the angle. Now N such independent and simultaneous circular motions together describe a trajectory of the representative phase point, lying on an N-torus in the 2N-dimensional phase space. Thus for N = 1, the torus collapses in the form of a circle in the 2-dimensional phase space. For N = 2, the 2-torus is a droughnut-like surface embedded in a 4-dimensional phase space. Here, ν_1 and ν_2 are respectively the circular frequencies of motion round the θ-circle and ϕ-circle respectively. Similarly we have for N-torus.

Thus the trajectories of an integrable Hamiltonion system are confined to N-tori, each of which is labelled by the N constants of motion which are N actions or momenta. We call them as **invariant tori.** If the N frequencies ν_k are commensurate *i.e.*, rationally related, then the trajectory will close exactly on itself. This is a kind of closed Lissajous figure. Here, we have a periodic or more correctly multiperiodic orbit. These invariant tori are called **rational tori.** In case, if the frequencies are incommensurate, *i.e.*, not rationally rélated, the trajectory will wind eternally around the torus, never closing exactly, never self- intersecting but passing arbitrarily close to any point on the torus. This is a kind of endless Lissajous figure and we call these invariant tori as the **irrational tori.** In this case, the motion is said to be conditionally periodic or quasi-periodic. In Fig. 16.33, invariant tori for N = 2 has been shown with energy as one of the two constants of motion. The 2-torus can be visualized as embedded in a 3-dimensional subspace- a surface of constant energy.

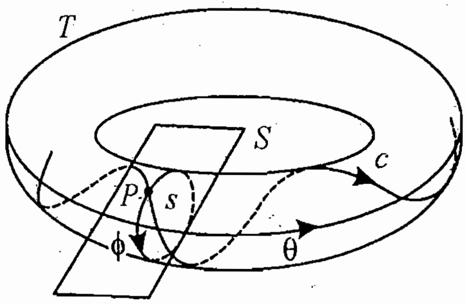


Fig. 16.33. Invariant 2-torus, showing surface of section S: the trajectory e generates poincare section s.

A conservative or Hamiltonian system can be chaotic, provided the system is nonintegrable. Integrable Hamiltonian systems have no chaos.

16.16 KAM THEOREM

We consider a Hamiltonion H in which the main interaction arises from the integrable Hamiltonion H_0 for which the solution can be obtained and an additional interaction due to small ΔH, *i.e.*,

$$H = H_0 + \Delta H$$

Pertubation theory can provide us a solution, when ΔH is small in comparison to H_0. However the question arises as to whelter the perturbed solution is stable, and whether or not the orbits will remain close to the unperturbed orbits over long periods of time. Of course the large perturbation can disturb the regular motion. In this context, a theorem, namely Kolmogorov-Arnold-Moser (KAM) theorem provides the conditions for the breakdown of the regularity. The theorem states as follows :

If the bounded motion of an integrable Hamiltonian H_0 is disturbed by a small pertubation, ΔH, so that the total Hamiltonian becomes nonintegrable and if the following two conditions are satisfied :

(1) the perturbation ΔH is small and

(2) the frequencies ω_k of H_0 are incommensurate.

then the motion remains confined to an N-torus, except for a negligible set of initial conditions that result in a wandering trajectory hither and thither on the energy surface.

Therefore, the pertubed orbits are stable, only slightly changed in shape and lie in the same region in which unpertubed orbits lie. However, the sentence 'except for a negligible set of initial conditions' means that the initial conditions are possible for which the theorem is not valid. When KAM theorem does not hold, chaos can occur.

Questions

1. What are nonlinear differential equations ? Show that the equation of a simple pendulum represents a nonlinear equation. What is linear approximation ?
2. What do you understand by a phase trajectory ? What are singular points ? Discuss the phase trajectories of a linear harmonic oscillator.
3. Show that the phase trajectories of the system whose equation of motion is given by

$$\frac{d^2x}{dt^2} = \alpha^2 x, \quad \frac{dx}{dt} = y$$

are hyperbola with asymptotes $y = \pm \alpha x$.

4. Explain the meaning of the following in relation to phase trajectories :
(*i*) Vortex point, (*ii*) Saddle point, (*iii*) Focal point, (*iv*) Nodal point.
5. The equation of a damped harmonic oscillator is

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = 0$$

Discuss the phase trajectories for $b^2 < \omega_0^2$ and $b^2 > \omega_0^2$.

6. Discuss the phase trajectories of a nonlinear conservative system by considering the motion of a mass, attracted towards a fixed point by a nonlinear restoring force $F(x)$. If $F(x) = k\sin x$, plot the phase trajectories. What is separatrix ?
7. Discuss the phase trajectories of a simple pendulum, whose equation of motion is given by

$$\frac{d^2x}{dt^2} + \omega_0^2 \sin x = 0$$

where $\omega_0 = \sqrt{g/l}$. Plot the energy diagram and phase trajectories.

8. Show that the oscillations of a simple pendulum ore nonlinear with time period

$$T = \frac{4}{\omega_0} K(\sin\theta_0/2)$$

where $K(\sin\theta_0/2)$ is the elliptical integral and θ_0 the angular amplitude. Show that for relatively small amplitude

$$T = T_0\left[1+\frac{\theta_0^2}{16}\right]$$

where $T_0 = 2\pi/\omega_0$.

9. The Van der Pol's equation is given by

$$\frac{d^2x}{dt^2} - \in(1-x^2)\frac{dx}{dt} + x = 0$$

Write the parametric equations of the system. Discuss the phase trajectories for the different values of the parameter $\in$. What is a limit cycle ?

10. What do you understand by a limit cycle ? What is an attractor ? Discuss the concepts involved by taking a specific example.
11. What do you understand by chaos ? Explain the phenomenon by giving examples.
12. What is logistic map ? What is its importance ? How will you plot a logistic map ? How do you locate the position of an attractor on the map ?
13. With reference to the standard quadratic map $x_{n+1} = \lambda x_n(1-x_n)$, discuss fixed points, stability of fixed points and periodic attractors. Explain how bifurcations lead to chaos, when λ. exceeds 3·57. **(Mumbai 2001)**
14. What do you understand by pitchfork bifurcation ? How does this depend on control parameter ? Draw a graph between the attractors and the control parameter ? When does chaos start ?
15. What is period doubling and Feigenbaum universal constants in relation to the logistic map ? Explain with the help of diagrams, if necessary. What happens when the control parameter (μ) has the value more than μ_∞.
16. By taking the example of driven damped harmonic oscillator, explain the phenomenon of chaos.
17. Write notes on the following :

    (*i*) Strange attractor, (*ii*) Lyapunov exponent, (*iii*) Poincare Sections,
    (*iv*) Invariant tori, (*v*) KAM Theorem
18. (*a*) What are fractals ? Explain the meaning of fractal dimension by taking an example.

    (*b*) Calculate the fractal dimension for Koch curve and Sieipinski gasket.
19. What is meant by a fractal dimension of an object ? Give an example to illustrate how a chaotic system exhibits fractal geometry ? **(Mumbai 2001, 2002)**

Problems

1. For a system

$$\frac{dy}{dt} = -\omega^2 x \text{ and } \frac{dx}{dt} = y,$$

show that the phase trajectories for the above equations are ellipses. If we take $y/\omega = p$ and $x = q$, then show that the trajectories are circles in $p-q$ plane.

2. For a nonlinear system, the differential equations are

$$\frac{dx}{dt} = x - y^2 \text{ and } \frac{dy}{dt} = x^2 - y$$

Find the real singular points and equation of trajectories.

Ans. (0,0), (1,1), $y^3 - 3xy + x^3 = C$, a constant.

3. Show that in case of the following equations

$$\frac{dx}{dt} = 2x - 2y \text{ and } \frac{dy}{dt} = 2x - 4y$$

the origin (0,0) is a saddle point.

4. Consider a set of following equations :

$$\frac{dx}{dt} = -y + x(x^2 + y^2 - 1)$$

and
$$\frac{dy}{dt} = x + y(x^2 + y^2 - 1)$$

Plot the phase space trajectories and show that there exists an unstable limit cycle.

5. For a system, the equations

$$\frac{dx}{dt} = y, \frac{dy}{dt} = -\lambda^2 x + \in (1 - x^2) y$$

where $\in$ is a positive parameter, are equivalent to the Van der Pol equation :

$$\frac{d^2x}{dt^2} - \in (1 - x^2)\frac{dx}{dt} + \lambda^2 x = 0$$

Solve numerically and show that a stable limit cycle exists. What happens when $\in$ is negative ?

6. Show that the system, given by

$$y_{n+1} = 1 - Ay_n{}^2$$

with the conditions $-1 < y < 1$ and $0 < A \leq 2$ can be transformed to the logistic equation

$$x_{n+1} = \mu x_n (1 - x_n)$$

when $y = ax + b$ is substituted. Determine A, a and b in terms of the control parameter μ.

Ans. :
$$A = \frac{\mu(\mu - 2)}{4}, \quad a = -\frac{4}{2 - \mu}, \quad b = \frac{2}{2 - \mu}$$

7. Show that the second bifurcation for the logistic map that leads to cycles of period 4 is obtained for $\mu = 1 + \sqrt{6}$.

Consider the map

$$x_{n+1} = f(x_n)$$

where
$$f(x) = a + bx \text{ for } x < 1$$
$$f(x) = a' + b'x \text{ for } x > 1$$

with $b > 0$ and $b' < 0$. Show that its Lyapunov exponent is positive for $b > 1, b' < -1$. Plot a few interations in the (x_{n+1}, x_n) plane.

Objective Type Questions

1. In chaos,

(*a*) the phase trajectories are regular shaped. (*b*) motion is completely random.
(*c*) in between (*a*) and (*b*) (*d*) the phase trajectories are elliptical.

Ans. (*c*).

2. The equation for a simple pendulum

$$\frac{d^2x}{dt^2} + \omega_0^2 \sin x = 0 \text{ is}$$

(*a*) linear (*b*) nonlinear
(*c*) neither linear nor nonlinear (*d*) nonlinear, when $\sin x = x$.
Ans. (*b*)

3. Saddle point
(*a*) is stable (*b*) is unstable
(*c*) corresponds to oscillatory motion (*d*) corresponds to damped oscillattory motion.
Ans. (*b*)

4. Van der Pol equation

$$\frac{d^2x}{dt^2} - \in(1 - x^2)\frac{dx}{dt} + x = 0$$

is (with nonzero $\in$) an example of
(*a*) linear conservative system (*b*) nonlinear conservative system
(*c*) nonlinear nonconservative system (*d*) all of the above.
Ans. (*c*).

5. Stable limit cycle in phase space is an example of
(*a*) attractor (*b*) strange attractor
(*c*) neither attractor nor strange attractor (*d*) both attractor and strange attractor.
Ans. (*a*).

6. For different values of the control parameter, the logistic map can have an attractor with
(*a*) one fixed point (*b*) two fixed points
(*c*) four fixed points (*d*) all of the above.
Ans. (*d*).

7. Beyond the value of the control parameter μ_∞
(*a*) logistic map becomes chaotic, (*b*) the attractor is strange,
(*c*) there are windows of periodic solutions (*d*) all of the above alternatives are incorrect.
Ans. (*a*), (*b*), (*c*).

Short Type Questions

1. What is phase trajectory ?
2. Show that the phase trajectory for a linear harmonic oscillator is an ellipse.
3. What is singular point ?
4. Discuss the phase trajectory for the force equation $F = kx$, where k is a positive constant.
5. What are focal point and nodal point ?
6. Show that the equation of a simple pendulum represents a nonlinear equation.
7. Show that the period of nonlinear oscillations of a simple pendulum is $T = T_0\left[1 + \frac{\theta_0^2}{16}\right]$,

   where $T_0 = 2\pi\sqrt{\frac{l}{g}}$ and θ_0 = amplitude of oscillation.
8. Discuss phase trajectories of Van der Pol equation for different values of damping constant.
9. What do you understand by limit cycles and attractors ?

10. What are different types of attractors ?
11. What is chaos ? Give an example.
12. What is logistic map ?
13. How will you plot a logistic map ?
14. What is the condition of stability ?
15. What do you understand by bifurcations ?
16. Plot the attractor diagram of the logistic mapping.
17. What are universal constants ?
18. What is strange attractor ?
19. Discuss the sensitivity to initial conditions. What is Lyapunov exponent ?
20. What is Poincare section ?
21. What are fractals ?
22. Evaluate the fractal dimension for Koch curve.
23. Fill in the blanks :

    (*i*) The focal point is a point of ....equilibrium.

    (*ii*) The fractal dimension for Cantor set is........

    Ans. : (*i*) stable (*ii*) 0.63.

Ab
Ac
Ac
Ac
Ac
An
An
Ap
Are
Att
Atv
Bifi
Bili
Bra
Brac
Bod
Calc
Canc
Canc
Cent
Cent
Cent
Chac
Conj
Com
Conf
Cont
Cons
Cons

Index

Aberration of light from stars 353
Accelerated frames 320
Acceleration in rotating frame 324
Action 3, 51, 154, 207
Action and angle variables 207, 210
Angle variable 208
Angular velocity 280, 282
Aperiodic motion 442
Areal velocity 18, 107
Attroctors 451
Atwood machine 16, 43
Bifurcations 458
Bilinear invariant condition 170
Brackets 179
 – Poisson's 179
 – Lagrange's 182
Brachristochrone problem 143
Body angle 278
 – coordinate system 274
Calculus of variation 198
Canonical momentum 75
Canonical transformations 163
Central force 18, 47, 108
Centre of mass 10
 – frame of reference 11
Centrifugal force 322
Chaos 455, 460
Conjugate momentum 75
Compound pendulum 44, 89
Configuration space 28
Contact transformations 163
Conservation laws 6
Conservation theorem 77
 – of linear momentum 6, 55, 77
 – of momentum at relativistic speeds 421
 – of angular momentum 6, 56, 78
 – of energy 6, 80, 81
Conservative force 7
Constraints 29
 – holonomic 29
 – nonholonomic 30
 – rheonomous 30
 – scleronomous 30
Coordinate systems 2, 27
 – cartesian 2, 27
 – cylindrical 27
 – spherical 28
Coriolis force 325
Coupled oscillators (two) 235
 – pendulums 246
 – many 256
Covariance of Maxwell's field equations in terms of four vectors 426
 – in terms of electromagnetic field tensor 430
Covariance of physical laws 337
Covariant formulation of electrodynamics 420
Covariant formulation of Lagrangian and Hamiltonian 407
Cyclic coordinates 76, 83
D'alembert's principle 36
De Broglie relation 217
Degrees of freedom 28
 δ-variation 153
 Δ - variation 153
Differential equation for an orbit 108
Differential scattering cross-section 126
Dispersion curve 261
Dispersion relation 261
Double pendulum 250
Dynamics of a rigid body 273
Eigenvalue equation 242
Eigenvectors 243
Einstein's mass-energy relation 371
Eikonel 216

Electromagnetic field tensor 427
Equation of continuity 422
Equilibrium of right-angled level 375
Ether hypothesis 341
Euler-Lagrange's equations 138, 139
Euler's angles 276
Euler's equations of motion for a rigid body 289
Fermat's principle 157, 218
Fictitious force 320
First integrals 77
Force 2
– in relativistic mechanics 374
Force-free motion of a symmetrical top 295
Foucault's pendulum 372
Four dimensional formulation 327
Four space 389
Four vector 396
– acceleration 399
– application 403
– current 424
– force 400
– momentum 399
– potential 425
– scalar product 397
– velocity 398
Frame of reference 1
– inertial 4
– noninertial 320
Galilean invariance hypothesis 336
Galilean transformations 334
Galilean law of addition of velocities 335
Gauge transformation 422
Generalized-coordinates 29, 34
–force 38
– momentum 75
– potential 49
– velocities 38
Generating functions 164
Geometrical interpretation of Lorentz transformations 411
Geometrical representation of simultaneity 414
– length contradiction 415
– time dilation 415
Gravitational mass 5
Gyroscope 307
Hamilton-Jacobi equation 197
– time-independent 202
– geometrical optics and wave mechanics 214
Hamilton's equations 91
– canonical equations 92
– from modified Hamilton's principle 147
Hamiltonian 82
– in relative mechanics 380
Hamilton's characteristic function 201, 202
Hamilton's principal function 52, 198
Hamilton's principle 51, 140
– extended 148
– from D'Alembert's principle 146
– modified 147
Ignorable coordinates 76
Impact parameter 127
Inertia 2
– ellipsoid 292
– moment of 283
– tensor 282
Inertial frames 4
Inertial mass 2
Infinitesimal contact transformations 173
Integral invariance of Poincare 171
Invariable plane 293
Invariant tori 469
Inverse square law of force 109
Jacobi's indentity 187
KAM theorem 470
Kepler's laws 110
Kepler's problem–solution by Hamilton-Jacobi method 204
– in action-angle variables 211
Kinetic energy 7
– in generalized coordinates 41
Lagrange brackets 182
Lagrange's equations 40, 41
– for L-C circuit 46
– from D'Alembert's principle 38
– from variational principle for non-conservative system 148
– from Hamilton's principle 51

– in presence of non-conservative forces 47
– of motion for nonholonomic systems 149
– for a charged particle moving in an electromagnetic field 49
Lagrange's method of undetermined multipliers 149
Lagrangian 40
– dynamics 27
– in relativistic mechanics 380
Lagendre transformations 163
Length contraction 345
Light cone 396
Ligt like interval 395
Limit cycles 451
Line of nodes 278
Liouville's theorem 190
Logistic map 455
Lorentz condition 423
Lorentz force on charged particle 432
Lorentz force in covariant form 433
Lorentz transformations 342, 391
– of force 375
– of electric and magnetic fields 429
Lyapunov exponent 463
Maxwells field equations 421
– in terms of electromagnetic potentials **A** and ϕ 423
– covariance 426
Mechanics of a particle 6
Mechanics of a system of particle 9
Michelson-Morley experiment 339
Minkowski diagrams 411
Minkowski space 389
Modes 232
– normal 232
Modified Hamilton's principle 147
Moment of Inertia 283
Motion under central force 107
N-couples masses 256
Newton's equation from Lagrange's equations 41
Newton's laws of motion 2
Noninertial frames 320
Nonlinear differential equations 439
Nonlinear systems 444
Normal coordinates 238, 245
– frequency 232, 237
– modes 238, 245
– mode frequency 245
N-torus 453
Nutational angle 278
Phase integral 208
Phase trajectories 439
Phase space 171, 189
Phase velocity 215
Poincare sections 464
Poisson brackets 179
– and quantum mechanics 219
– fundamental 181
Positronium 105
Potential energy 7
– and equilibrium 232
– curve 233
Principal axes 284
Principal moments of inertia 284
Precessional angle 277
Principle of least action 154
– Jacobi's form 157
Principle of relativity 337
Principle of virtual work 35
Rayleigh's dissipation function 48
Reduced mass 104
Reduction of two-body problem to one-body problem 103
Redudant coordinates 29
Relativistic Doppler's sffect 355
Relativistic energy 370
Relativistic Hamiltonian of a charged particle 382
Relativistic kinetic energy 370
Relativistic Lagrangian of a charged particle 382
Relativistic law of addition of velocities 349
Rigid body 273
Rotating frames 323
Routherford scattering cross-section 127
Routhian 94
Scattering angle 127
– cross-section 126
– in a central force field 125
– in a repulsive force field 128

Schrodinger equation 217, 218
Simple pendulum 42 , 90
Simultaneity 345
Singular points 439
Small oscillations 232, 240
Sommerfield-Wilson rule of quantization 219
Space-time continuum 389
Space-time intervals 392
Space and time 1
Special theory of relativity 334, 342
Spherical pendulum 63
Stability of orbit under central force 114
Stable, unstable and neutral equilibrium 233
Strange attractor 462
Superfluous coordinates 29
Symmetrical top 295
 – heavy 298
 – sleeping 306
Theorem of parallel axes 288
Time dilation 346
Torque-free motion of a rigid body 291
Transformations of four vector potential 425
Transformation of force 336, 375
Triatomic molecule (vibrations of linear) 252
Twin paradox 348
Two-body central force problem 103
Variation of mass with velocity 367, 369
Variational principles 138
Velocity dependent potential 49
Vibrations of continuous string 266
Virial theorem 124
 – of Clausius 125
Virtual work 35
Work 7
Work-energy theorem 7
World line 392
World point 392
World region 395
World space 385