Unit-5 :- Orthogonality & least Squares

→ Gram Schmidt Process :-

This process is used to convert non-orthogonal vectors to orthogonal vectors using the following procedure.

Theorem :- Given, a basis $\{x_1, x_2 \dots x_p\}$ for a subspace 'w' of $\mathbb{R}^n$ defined

$$V_1 = x_1$$

$$V_2 = x_2 - \frac{\langle x_2 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1$$

$$V_3 = x_3 - \frac{\langle x_3 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 - \frac{\langle x_3 \cdot V_2 \rangle}{\langle V_2 \cdot V_2 \rangle} \cdot V_2$$

$$\vdots$$

$$V_p = x_p - \frac{\langle x_p \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 - \frac{\langle x_p \cdot V_2 \rangle}{\langle V_2 \cdot V_2 \rangle} \cdot V_2 \dots - \frac{\langle x_p \cdot V_{p-1} \rangle}{\langle V_{p-1} \cdot V_{p-1} \rangle} \cdot V_{p-1}$$

The obtained set of vectors $\{V_1, V_2 \dots V_p\}$ are orthogonal vectors. i.e it is an orthogonal basis for 'w'

In addition,

$$\text{span}\{v_1, v_2 \dots v_p\} = \text{span}\{x_1, x_2 \dots x_p\}$$

① Construct an orthogonal basis for 'W' where is

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ by Gram Shmidt Process.

Sol^n:-

Given ,- $W = [x_1, x_2, x_3]$

$$\Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x_1 \cdot x_2 = [1\ 1\ 1\ 1] \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3 \neq 0$$

$$x_1 \cdot x_3 = [1\ 1\ 1\ 1] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$$

$$x_2 \cdot x_3 = [0\ 1\ 1\ 1] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$$

$\therefore \{x_1, x_2, x_3\}$ are not orthogonal to each other

$\therefore$ By Gram Shmidt Process,

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$V_2 = x_2 - \frac{\langle x_1 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \; V_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$V_3 = x_3 - \frac{\langle x_3 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 - \frac{\langle x_3 \cdot V_2 \rangle}{\langle V_2 \cdot V_2 \rangle} \cdot V_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}{\begin{bmatrix} -3/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}} \cdot V_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{4}}\begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 - \frac{1}{2} + \frac{1}{2} \\ 0 - \frac{1}{2} - \frac{1}{6} \\ 1 - \frac{1}{2} - \frac{1}{6} \\ 1 - \frac{1}{2} - \frac{1}{6} \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 - 2/3 \\ 1 - 2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\therefore\ v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

<u>Verification</u>:-

Check, $v_1 \cdot v_2 = 0 = v_2 \cdot v_3 = v_1 \cdot v_3$

→ Orthogonal basis :-

It is constructed from an orthogonal basis $v_1, v_2, \ldots v_p$ by normalizing all v_K for $K = 1, 2, \ldots p$.

$$u_k = \frac{v_K}{\|v_K\|}$$

① Let, $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$, Construct an orthonormal basis for A

Soln :-

Let $A = [x_1 \; x_2 \; x_3]$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix}.$$

$$\langle x_1 \cdot x_2 \rangle = [1 \; 1 \; 1 \; 1] \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = 3 + 1 + 1 + 3 = 8 \neq 0$$

$$\langle x_2 \cdot x_3 \rangle = [3 \; 1 \; 1 \; 3] \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} = 15 + 0 + 2 + 9 = 26 \neq 0$$

$$\langle x_1 \cdot x_3 \rangle = [1 \; 1 \; 1 \; 1] \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} = 5 + 0 + 2 + 3 = 10 \neq 0$$

∴ $\{x_1, x_2, x_3\}$ are not orthogonal to each other.

Apply Gram Smidth Process

$$V_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$V_2 = x_2 - \frac{\langle x_2 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 3 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \cdot V_1$$

$$V_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 1-2 \\ 1-2 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$V_3 = x_3 - \frac{\langle x_3 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} V_1 - \frac{\langle x_3 \cdot V_2 \rangle}{\langle V_2 \cdot V_2 \rangle} \cdot V_2$$

$$V_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{10}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}$$

Thus, $\{u_1, u_2, u_3\}$ form the Orthonormal basis for A

⇒ Least Squares Solution :- $(Ax = b)$

→ Theorem-1 :- Let A be an $m \times n$ matrix and b be any vector in R^m. The least squared solution of $Ax=b$ are the solutions of the matrix equation, $A^TAx = A^Tb$

→ Theorem-2 :- Let A be an $m \times n$ matrix and b be any vector in R^m. The following are equivalent

1. $Ax=b$ has a unique least square solution
2. The columns of A are Linearly Independent.
3. A^TA is invertible

(*) In other words a least square solution solves the equation $Ax=b$ as closely as possible in sense that the sum of squares of difference $b-Ax$ is minimized.

(1) Find the least square solution of the inconsistent system $Ax=b$ where, $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

WKT, the normal equation of $Ax=b$ is given by

$$(A^T A x) = A^T b$$

$$A^T A x = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 16+1 & 1 \\ 1 & 4+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$LHS = A^T A x = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$RHS = A^T b$$

$$= \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Now, by equating LHS and the RHS

$$A^T A x = A^T b$$

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Method-1:-

The Augmented Matrix is,

$$\left[\begin{array}{cc:c} 17 & 1 & 19 \\ 1 & 5 & 11 \end{array}\right] \quad R_2 \to R_2 - \frac{R_1}{17}$$

$$\left[\begin{array}{cc:c} 17 & 1 & 19 \\ 0 & \frac{84}{17} & \frac{168}{17} \end{array}\right] \quad R_2 \to R_2 \times \frac{17}{84}$$

$$\left[\begin{array}{cc:c} 17 & 1 & 19 \\ 0 & 1 & 2 \end{array}\right]$$

$\therefore \quad 17x_1 + x_2 = 19 \Rightarrow 17x_1 = 17 \Rightarrow x_1 = 1$

$x_2 = 2$

$$\therefore \quad \hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

MATHS

Method-2 :-

$$\hat{x} = (A^T A)^{-1} (A^T b)$$

$$(A^T A)^{-1} = \frac{1}{|A^T A|} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$|A^T A| = 85 - 1 = 84$$

$$(A^T A)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 5/84 & -1/84 \\ -1/84 & 17/84 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} \frac{95 - 11}{84} \\ \frac{-19 + 187}{84} \end{bmatrix}$$

$$\therefore \hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

② Find the least square Solution :- $(Ax = b)$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

Sol^n :-

The normal equation is,

$$A^T A x = A^T b$$

$$LHS = A^T A x = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$RHS = A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$\therefore \quad A^T A x = A^T b$$

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

Augmented matrix :-

$$\sim \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{array}\right] \quad \begin{array}{l} R_4 \to R_4 - \frac{R_1}{3} \\ R_2 \to R_2 - \frac{R_1}{3} \\ R_3 \to R_3 - \frac{R_1}{3} \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 0 & 4/3 & -2/3 & -2/3 & -16/3 \\ 0 & -2/3 & 4/3 & -2/3 & 2/3 \\ 0 & -2/3 & -2/3 & 4/3 & 14/3 \end{array}\right] \quad \begin{array}{l} R_2 \to \frac{3}{2} R_2 \\ R_3 \to \frac{3}{2} R_3 \\ R_4 \to \frac{3}{2} R_4 \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 0 & 2 & -1 & -1 & -8 \\ 0 & -2 & 4 & -2 & 2 \\ 0 & -2 & -2 & 4 & 14 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + R_2 \\ R_4 \to R_4 + R_2 \end{array}$$

$$\sim \left[\begin{array}{cccc:c} 6 & 2 & 2 & 2 & 4 \\ 0 & 2 & 1 & 1 & -8 \\ 0 & 0 & 3 & -3 & -6 \\ 0 & 0 & -3 & 3 & 6 \end{array}\right] \quad R_4 \to R_4 + R_3$$

$$\left[\begin{array}{cccc:c} \boxed{6} & 2 & 2 & 2 & 4 \\ 0 & \boxed{2} & -1 & -1 & -8 \\ 0 & 0 & \boxed{+3} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$\left[\begin{array}{cccc:c} \boxed{6} & 0 & 3 & 3 & 12 \\ 0 & \boxed{2} & -1 & -1 & -8 \\ 0 & 0 & \boxed{3} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + \frac{R_3}{3} \\ R_1 \to R_1 - R_3 \end{array}$$

$$\left[\begin{array}{cccc:c} \boxed{6} & 0 & 0 & 6 & 18 \\ 0 & \boxed{2} & 0 & -2 & -10 \\ 0 & 0 & \boxed{3} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/6 \\ R_2 \to R_2/2 \\ R_3 \to R_3/3 \end{array}$$

Row reduced Echelon form:-

$$\sim \left[\begin{array}{cccc:c} \boxed{1} & 0 & 0 & 1 & 3 \\ 0 & \boxed{1} & 0 & -1 & -5 \\ 0 & 0 & \boxed{1} & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variable :- x_1, x_2, x_3

free variable :- x_4

General Sol^n :-

$$x_1 + x_4 = 3 \qquad x_1 = -x_4 + 3$$

$$x_2 - x_4 = -5 \qquad x_2 = x_4 - 5$$

$$x_3 - x_4 = -2 \qquad x_3 = x_4 - 2$$

$$x_4 = \text{free} \qquad x_4 = \text{free}$$

$$\therefore \hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} -x_4 + 3 \\ x_4 - 5 \\ x_4 - 2 \\ x_4 \end{bmatrix}$$

$$\hat{x} = x_4 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix}$$

Let us choose, $x_4 = 0$

$$\therefore \hat{x} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix}$$

$\Rightarrow$ QR Factorization:-

If A is a $m \times n$ matrix, then A can be factorized (or) decomposed as, $A = QR$ where Q is an $m \times n$ matrix, whose columns form an 'orthonormal basis' for Columns of A and R is a $m \times n$ Upper Triangular Invertible matrix with positive entries.

$$\therefore A = QR$$

$$\Rightarrow Q^T A = Q^T Q R$$

$$Q^T A = IR$$

$$Q^T A = R$$

<u>Note</u>:- Steps to solve QR factorization

① Let, Col A = span $\{x_1, x_2, x_3\}$

② Find an orthonormal vectors

Let $u_1 = \frac{v_1}{\|v_1\|}$, $u_2 = \frac{v_2}{\|v_2\|}$, $u_3 = \frac{v_3}{\|v_3\|}$

③ Use the Orthonormal vectors as column of Q

Let $Q = [u_1 \;\; u_2 \;\; u_3]$

④ Find $Q^T A = R$

Q.① : Find the QR factorization of matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

<u>Sol^n</u> :-

Let, $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$ $x_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix}$

$$\langle x_1 \cdot x_2 \rangle = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = 3+1+1+3 = 8$$

$$\langle x_2 \cdot x_3 \rangle = \begin{bmatrix} 3 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} = 15+2+9 = 26$$

$$\langle x_1 \cdot x_3 \rangle = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} = 5+2+3 = 10$$

Since, $\langle x_1 \cdot x_2 \rangle \neq 0$, $\langle x_2 \cdot x_3 \rangle \neq 0$, $\langle x_1 \cdot x_3 \rangle \neq 0$

$\therefore$ x_1, x_2, x_3 are not orthogonal

To convert into Orthogonal, we use Gram Smidt Process

Let v_1, v_2, v_3 be 3 vectors.

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{\langle x_2 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 3 & 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 3-2 \\ 1-2 \\ 1-2 \\ 3-2 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$V_3 = x_3 - \frac{\langle x_3 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 - \frac{\langle x_3 \cdot V_2 \rangle}{\langle x_3 \cdot V_2 \rangle} \cdot V_2$$

$$= \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{[5\ 0\ 2\ 3]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{[1\ 1\ 1\ 1]\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{[5\ 0\ 2\ 3]\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}{[1\ -1\ -1\ 1]\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{10}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 - 5/2 - 3/2 \\ 0 - 5/2 + 3/2 \\ 2 - 5/2 + 3/2 \\ 3 - 5/2 - 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\cancel{V_3 = \begin{bmatrix} 3/2 \\ -1/2 \\ 3/2 \\ -1/2 \end{bmatrix}}$$

$$\therefore\ V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\langle V_1 \cdot V_2 \rangle = \langle V_2 \cdot V_3 \rangle = \langle V_1 \cdot V_3 \rangle = 0$$

$\therefore$ The vectors $\{V_1, V_2, V_3\}$ are Orthogonal.

Now, to find Q, we need orthonormal basis

Let $Q = [u_1 \; u_2 \; u_3]$ form the Orthormal basis

$$\therefore \quad u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{[1\;1\;1\;1]\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{4}}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\+1/2\\-1/2\end{bmatrix}$$

$$\therefore \quad Q = \begin{bmatrix}1/2 & 1/2 & 1/2\\ 1/2 & -1/2 & -1/2\\ 1/2 & -1/2 & 1/2\\ 1/2 & 1/2 & -1/2\end{bmatrix}$$

Now, to find R,

$$R = Q^T A$$

$$\therefore \quad R = \begin{bmatrix}1/2 & 1/2 & 1/2 & 1/2\\ 1/2 & -1/2 & -1/2 & 1/2\\ 1/2 & -1/2 & 1/2 & -1/2\end{bmatrix}\begin{bmatrix}1 & 3 & 5\\ 1 & 1 & 0\\ 1 & 1 & 2\\ 1 & 3 & 3\end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \quad A = QR$$

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

⇒ <u>Least Square Solution by QR factorization</u> :-

Given a $m \times n$ matrix 'A' with linearly independent columns,. Let, $A = QR$ be a QR factorization of 'A', then for each 'B' in $\mathbb{R}^m$, the equation $Ax = b$ has a unique Least Square solution given by

$$\hat{x} = R^{-1} Q^T b \quad (\text{where } R \text{ is } 2 \times 2)$$

(or)

$$\cancel{R} \; RX = Q^T b$$

① Find the least square Solution of $Ax = b$ where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

WKT, From previous problem

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \text{ and } R = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & a \end{bmatrix}$$

To find the Least Square Solution we solve the equation

$$RX = Q^T b$$

$$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & +1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

∴ Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 5 & 6 \\ 0 & 2 & 3 & -6 \\ 0 & 0 & 3 & 4 \end{array}\right]$$

$\therefore \quad 2x_1 + 4x_2 + 5x_3 = 6$

$2x_2 + 3x_3 = -6$

$2x_3 = 4$

$\therefore x_3 = 2$

By back substitution

$2x_2 + 6 = -6$

$x_2 = \frac{-12}{2} = -6$

$2x_1 + 4(-6) + 10 = 6$

$x_1 = \frac{24 - 4}{2}$

$x_1 = 10$

$\therefore$ Least Square Solution,

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$$

② Find the least Square Solution by QR factorization

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$