# Unit - 3 :- Vector Spaces

**Vector Space :-** It is a non-empty set 'v' of objects called vectors on which are defined two operations called addition and multiplication by scalars subject to the 10 axioms as follows :-

The axioms must hold for all vectors $u, v$ & $w$ in the vector space 'v' and for all scalar 'c' and 'd'

→ **Addition :-**

① $u + v$ is a vector in 'V' [Closure under addition]

② $u + v = v + u$ [Commutative under addition]

③ $(u + v) + w = u + (v + w)$ [Associative under addition]

④ There is a zero-vector '0' in 'V' such that,

$$u + 0 = u \quad \text{[Addition Identity]}$$

⑤ For every $u$ in 'V' there is a vector 'v' denoted by $-u$. ~~Then~~ such that $u + (-u) = 0$ [Additive Inverse]

Multiplication :-

(6) $cu$ is in 'V' [Closure under Multiplication]

(7) $c(u+v) = cu + cv$ [Distributive Property of Scalar Multiplicati

(8) $(c+d)u = cu + du$ [Distributive property] Under Scalar Multiplication]

(9) $c(du) = (cd)u$ [Associative Property]

(10) $1\cdot(u) = u$ [Scalar Multiplicative Identity Property]

Problems :-

(1) Prove that $\mathbb{R}^2 = \{(a_1, a_2) \mid a_1 \in \mathbb{R}, a_2 \in \mathbb{R}\}$ is a vector space over $\mathbb{R}$ with addition and Scalar Multiplication.

Let the two ordered pairs be $(a_1, a_2)$ & $(b_1, b_2)$

By Property of Closure under addition

$$(a_1, a_2) + (b_1, b_2) = (a_1+b_1, a_2+b_2)$$

$$\Downarrow$$

It is also in $\mathbb{R}^2$

$\therefore$ The property is proved.

For property of closure under Multiplication

$$c(a_1, a_2) = (ca_1, ca_2)$$

$$\Downarrow$$

It is also in $\mathbb{R}^2$

$\therefore$ The ~~vecto~~ $\mathbb{R}^2$ is a vector space over $\mathbb{R}^2$ with addition and Scalar Multiplication.

Note :-

① Vectors in 3 dimensional space forms a vector space Over $\mathbb{R}$ with respect to addition & Scalar Multiplication.

② Let "$c^n$ ~~and~~ be the set of all ordered 'n' tuple of, then $c^n$ is a vector space over 'c with addition & Scalar Multiplication

② Show that, for $n \geq 0$ the set $P_n$ of polynomial of degree at most 'n' is a vector space.

Let, a Polynomial

$$P(t) = a_0 + a_1 t + a_2 t^2 + \dots a_n t^n$$ be the set of polynomial where the coefficient $a_0, a_1 \dots a_n$ and the variable 't' are real numbers.

It is the $n^{th}$ order Polynomial.

If, $P(t) = a_0 \neq 0$, then the order of Polynomial is 0.

$\Rightarrow$ All the other coefficients are zero.

① Closure under addition :-

Let,
$$P(t) = a_0 + a_1 t + \dots a_n t^n \in P_n$$
$$Q(t) = b_0 + b_1 t + \dots b_n t^n \in P_n$$

$$P(t) + Q(t) = (a_0 + b_0) + (a_1 + b_1)t + \dots (a_n + b_n)t^n$$

$$\therefore P(t) + Q(t) \in P_n$$

② Associative under addition:-

Let,
$$P(t) = a_0 + a_1 t + \dots a_n t^n \in P_n$$
$$q(t) = b_0 + b_1 t + \dots b_n t^n \in P_n$$
$$r(t) = c_0 + c_1 t + \dots c_n t^n \in P_n$$

To Prove,
$$(P(t) + q(t)) + r(t) = P(t) + [q(t) + r(t)]$$

$$LHS \Rightarrow (a_0 + b_0) + (a_1 + b_1)t + \dots (a_n + b_n)t^n + c_0 + c_1 t + \dots c_n t^n$$
$$= (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)t + \dots (a_n + b_n + c_n)t^n$$

$$RHS = a_0 + a_1 t + \dots a_n t^n + (b_0 + c_0) + (b_1 + c_1)t + \dots (b_n + c_n)t^n$$

$$RHS = (a_0 + b_0 + c_0) + (a_1 + b_1 + c_1)t + \dots \dots (a_n + b_n + c_n)t^n$$

$$\therefore \quad LHS = RHS_{//}$$

③ Commutative under addition :-

Let, $P(t), q(t) \in P_n$

To Prove

$$P(t) + q(t) = q(t) + P(t)$$

$$(a_0 + b_0) + (a_1 + b_1)t \dots (a_n + b_n)t^n = (b_0 + a_0) + (b_1 + a_1)t + \dots (b_n + a_n)t^n$$

$$\therefore (a_0 + b_0) + (a_1 + b_1)t + \dots (a_n + b_n)t^n = (a_0 + b_0) + (a_1 + b_1)t + \dots (a_n + b_n)t^n$$

$$\Rightarrow P(t) + q(t) = q(t) + P(t)$$

④ Existence of Additive Identity :-

Let a zero polynomial, $O(t) = 0 + 0t + 0t^2 + \dots + 0t^n \in P_n$

$$P(t) + O(t) = a_0 + a_1 t + \dots a_n t^n + 0 + 0t + 0t^2 + \dots 0t^n$$

$$= (a_0 + 0) + (a_1 + 0)t + \dots (a_n + 0)t^n$$

$$= a_0 + a_1 t + \dots a_n t^n$$

$$= P(t)$$

$$\Rightarrow P(t) + O(t) = P(t)_{//}$$

⑤ Existence of additive Inverse :-

Let $p(t) = a_0 + a_1 t + \dots a_n t^n \in P_n$

and $q(t) = -p(t) = -a_0 - a_1 t + \dots - a_n t^n \in P_n$

Now, $p(t) + q(t) = p(t) + (-p(t))$

$= a_0 + a_1 t + \dots a_n t^n + (-a_0 - a_1 t - \dots a_n t^n)$

$= (a_0 - a_0) + (a_1 - a_1)t + \dots + (a_n - a_n)t^n$

$= 0 + 0t + \dots 0t^n$

$p(t) + q(t) = 0(t) = p(t) + (-p(t))$

$\therefore$ $q(t) = -p(t)$ is the additive Inverse

⑥ Closure under Multiplication :-

Let, $p(t) = a_0 + a_1 t + \dots + a_n t^n \in P_n$

then $c\,p(t) = c\,[a_0 + a_1 t + \dots + a_n t^n]$

$= ca_0 + ca_1 t + \dots + ca_n t^n \in P_n$

Here, $c \cdot p(t) \in P_n$

Since, the above 6 axioms are satisfied by the set of all Polynomials $P_n$ also satisfies 7, 8, 9, 10 which follow property of $\mathbb{R}$

$\therefore$ $P_n$ is a vector space.

③ Let, 'V' be the vector space in the first Quadrant of XY Plane defined as

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0, y \geq 0 \right\}$$

(i) If 'u', 'v' are in 'V', prove 'u+v' is in 'V'

(ii) Find a specific vector $u \in V$ and a specific scalar such that $cu \notin V$

(i) Let, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Then, $u + v = \begin{bmatrix} 1+3 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$u + v \in V$

(ii) Let, $u = \begin{bmatrix} 7 \\ 45 \end{bmatrix}$ and a scalar $c = -2$

Then, $cu = -2 \begin{bmatrix} 7 \\ 45 \end{bmatrix} = \begin{bmatrix} -14 \\ -90 \end{bmatrix}$

$\therefore$ $cu \notin V$

IIIly For all $c < 0$, $cu \notin V$

→ **Subspace** :-

A subspace of a vector space 'V' is a subset 'H' of 'V' that has 3 properties.

(i) The zero vector of 'V' is in 'H'

(ii) 'H' is closed under vector addition i.e for each $u$ & $v$ in 'H', The sum $u + v \in H$

(iii) 'H' is closed under Scalar Multiplication i.e for each $u \in$ 'H' and any scalar 'c', then $cu \in H$

**Note** :-

⇒ **Zero Subspace** :-

The set consisting of only zero vector in a vector space 'V' is a subspace of 'V' called Zero subspace and is denoted as $\{0\}$

① A plane in $R^3$ not passing through the origin is not a subspace of $R^3$ because the plane does not contain zero vector of $R^3$

② Let 'H' be the set of points inside and on the unit circle in the X-Y plane

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x^2+y^2 \leq 1 \right\}$$

Find a specific example with two vectors and a scalar to show that 'H' is not a subspace of $R^2$

Let, $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Let, $c = 2$

$\therefore$ $cu = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \notin H$

$\therefore$ Since, $cu \notin H$,

$\therefore$ 'H' does not satisfy the closure under Scalar Multiplication

$\therefore$ 'H' is not a subspace of $R^2$

③ Show that the set 'H' of all points in $R^2$ of the form $(3s, 5+2s)$ is not ~~not~~ a vector space by showing that it is not closed under scalar multiplication.

Let, $u = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ $c = 2$

Then, $cu = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$

$\therefore$ $3s = 6$ $\quad$ $5 + 2s = 14$

$s = 2$ $\quad$ $s = 4.5$

$\therefore$ $cu \notin H$

Since, $cu \notin H$

H is not a Vector space.

(ii) ~~(3s, 2+5s)~~ $(4s, 2+3s)$

Let, $u = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $c = 2 \Rightarrow cu = \begin{bmatrix} 8 \\ 10 \end{bmatrix}$

$4s = 8$

$s = 2$

$2 + 3s = 10$

$s = \frac{8}{3}$

$\therefore$ $cu \notin H$

④ Given, $v_1$ and $v_2$ in a vector space 'V'. Let $H = \text{span}\{v_1, v_2\}$. Show that 'H' is a subspace of 'V'.

Since, $H = \text{span}\{v_1, v_2\}$

$\therefore H = x_1 v_1 + x_2 v_2$

(i) To prove Zero vector $\in H$

Let, $\vec{0} = 0v_1 + 0v_2$

$\therefore \vec{0} \in H$

(ii) To prove Closure under addition

i.e $\forall\, u_1, u_2 \in H \Rightarrow u_1 + u_2 \in H$

Let, $u_1 = a_1 v_1 + a_2 v_2$

$u_2 = b_1 v_1 + b_2 v_2$

$\therefore u_1 + u_2 = a_1 v_1 + a_2 v_2 + b_1 v_1 + b_2 v_2$

$= (a_1 + b_1) v_1 + (a_2 + b_2) v_2$

$\therefore u_1 + u_2 \in H$

$\therefore$ 'H' is closed under ~~Mult~~ Addition.

(iii) To prove Closure under Scalar multiplication

let, $u = x_1 v_1 + x_2 v_2$ and some scalar 'c'

$\therefore cu = x_1 c v_1 + x_2 c v_2$

$\therefore cu \in H$

$\therefore$ 'H' is closed under Scalar Multiplication

$\therefore$ Since, all 3 properties are proved.

'H' is a subspace of 'V'.

=> <u>Theorem 1</u> :- Let $v_1, v_2 \ldots v_p$ are in a Vector space 'V' then span$\{v_1, v_2 \ldots, v_p\}$ is a subspace of 'V'

① Let 'W' be the set of all vectors of the form

$\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$. Show that 'W' is a subspace of $R^4$.

<u>Sol^n</u> :- Given, $W = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$

$W = s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$

$\therefore$ $W = sv_1 + tv_2$ where,

$v_1 = (1,1,2,0)$  $v_2 = (3,-1,-1,4)$

$\therefore$ $W = span\{v_1, v_2\}$

Since, $v_1, v_2 \in$ '$\mathbb{R}^4$', By Theorem1

$\therefore$ $W$ is a subspace of $\mathbb{R}^4$

(2) Let, $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \ a - 3b - c = 0 \right\}$ Show that '$W$' is a subspace of $\mathbb{R}^3$.

$W = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$,  $a - 3b - c = 0$  $\Rightarrow a = 3b + c$

$a \to$ basic variable
$b, c \to$ free variable.

$\Rightarrow W = \begin{bmatrix} 3b + c \\ b \\ c \end{bmatrix}$

$W = b\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ $W = bv_1 + cv_2 \Rightarrow W \in span\{v_1, v_2\}$

Since, $v_1, v_2 \in \mathbb{R}^3 \Rightarrow W$ is a subspace of $\mathbb{R}^3$.

## ⇒ Null Space :-

Consider, the system of Homogeneous equation

$$x_1 - 3x_2 - 2x_3 = 0$$
$$-5x_1 + 9x_2 + x_3 = 0$$

where, $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

the set of all 'x' that satisfies the above equation is called the solution set of the system.

We call the set of 'x' that satisfy $Ax = 0$ is the null space of matrix A.

Theorem :- The null space of a $m \times n$ matrix 'A' is a subspace of $R^n$ then equivalently the set of all solution to a system $Ax = 0$ of 'm' homogeneous linear equation with 'n' unknowns is a subspace of $R^n$

## ⇒ The explicit description of Null A :-

① Find an explicit description of Null A by listing vectors that span the null space.

(i) $A = \begin{bmatrix} -3 & 6 & \cancel{9}-1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ $R_1 \rightarrow R_1/-3$

$A \sim \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$ $R_2 \rightarrow R_2 - R_1,\ R_3 \rightarrow R_3 - 2R_1$

$A \sim \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 0 & 0 & 5/3 & 10/3 & -10/3 \\ 0 & 0 & 13/3 & 25/3 & -\frac{26}{3} \end{bmatrix}$ $R_3 \rightarrow R_3 \times 3,\ R_2 \rightarrow R_2 \times \frac{3}{5}$

$A \sim \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 13 & 25 & -26 \end{bmatrix}$ $R_3 \rightarrow R_3 - 13R_2$

$A \sim \begin{bmatrix} 1 & -2 & 1/3 & -1/3 & 7/3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Hence, $x_1, x_3, x_5 \rightarrow$ Basic

$x_2, x_4 \rightarrow$ Free

$x_1 - 2x_2 + 4x_4 = 0$ $\quad$ $x_3 - 9x_4 = 0$ $\quad$ $x_5 = 0$

$x_1 = 2x_2 + 4x_4$ $\quad$ $x_3 = 9x_4$

$x_2 = \text{free},\ x_4 = \text{free}$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + 4x_4 \\ x_2 \\ 9x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$X = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X = x_2 v_1 + x_4 v_2$$

$$\therefore X = \text{span}\{v_1, v_2\}$$

=> **The Column Space of a Matrix:-**

The column space of a $m \times n$ matrix 'A' which is written as ColA is the set of all linear combinations of columns of 'A'.

i.e if $A = [a_1, a_2, \dots a_n]$

$$\text{Col } A = \text{span}\{a_1, a_2, \dots a_n\}$$

**Theorem:-** The column space of an $m \times n$ matrix 'A' is the subspace of $R^m$

① Find a matrix 'A' such that $W = \text{col } A$ where, W is defined as $W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix}, a, b \in R \right\}$

Let, $W = \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix}$

$$W = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore W = av_1 + bv_2$

$W = \text{span}\{v_1, v_2\}$

$\therefore$ The matrix is formed by using $v_1, v_2$ as columns.

$$A = [v_1 \quad v_2]$$

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

$$\therefore \; W = \text{Col} A$$

② Find a matrix 'A' such that given set is ColA

$$\text{Col}A = \left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix}, \; r, s, t \text{ are real} \right\}$$

Let,

$$W = \text{Col}A = \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix}$$

$$= s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{Col}A = rv_1 + sv_2 + tv_3$$

$$\text{Col}A = \text{span}\{v_1, v_2, v_3\}$$

$\therefore$ The matrix formed is

$$A = [v_1 \ v_2 \ v_3]$$

$$\cancel{A \neq \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix}} \quad A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$$

$\Rightarrow$ Remarks :-

The column space of an $m \times n$ matrix 'A' is all of $R^n$ iff the equation $Ax = b$ has a solution for each 'b' in $R^m$

$\Rightarrow$ Linearly Independent Sets & Bases :-

(*) An index set of vectors $\{v_1, v_2 \ldots v_p\}$ in a vector space 'V' is said to be linearly independent if the vector equation $c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$ has only the Trivial Solution..

(*) Basis :- A set 'S' $= \{v_1, v_2, \ldots v_p\}$ of vectors is a basis of the vector space 'V' if it satify following properties

(i) Set 'S' is lineary independent

(ii) 'S' spans vectors space 'V'

① Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$, Determine

if the vector set $\{v_1, v_2, v_3\}$ is a basis for $R^3$

Sol^n:

To Prove, $S = \{v_1, v_2, v_3\}$ is linearly independent

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

must have Trivial solution

$$\therefore \quad x_1 \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix} x_2 + x_3 \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3x_1 - 4x_2 - 2x_3 \\ 0x_1 + x_2 + x_3 \\ -6x_1 + 7x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ Augmented matrix

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \quad R_3 \to R_3 + 2R_1$$

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$A = \begin{bmatrix} \boxed{3} & -4 & -2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

From the above Echelon form, it is clear that the system of equation does not have free variable because each column has Pivot Position.

$\therefore$ System of vector equation has Trivial Solution

$\therefore$ $S = \{v_1, v_2, v_3\}$ is linearly independent

Since, all the rows of Echelon form have pivot elements, the matrix spans $R^3$.

$\therefore$ The vector set $\{v_1, v_2, v_3\}$ is basis for $R^3$

② Let, $v_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$, $v_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, $v_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$

Find the basis for subspace 'W' spanned by $\{v_1, v_2, v_3, v_4\}$

Solⁿ

Let a matrix A be, $A = [v_1 \ v_2 \ v_3 \ v_4]$

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$$

Given, W is spanned by $\{v_1, v_2, v_3, v_4\}$

i.e $W = \text{span}\{v_1, v_2, v_3, v_4\}$

$W = \text{Col}A$

ColA is a subspace of $\mathbb{R}^3$

$$A = \begin{bmatrix} \boxed{1} & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \quad R_2 \to R_2 + 3R_1,\ R_3 \to R_3 - 4R_1$$

$$A \sim \begin{bmatrix} \boxed{1} & 6 & 2 & -4 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{bmatrix} \quad R_2 \to R_2/4,\ R_3 \to R_3/5$$

$$A \sim \begin{bmatrix} \boxed{1} & 6 & 2 & -4 \\ 0 & \boxed{5} & 1 & -5 \\ 0 & -5 & -1 & 5 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

$$A \sim \begin{bmatrix} \boxed{1} & 6 & 2 & -4 \\ 0 & \boxed{5} & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, Basic variables $\rightarrow x_1, x_2$

Free variables $\rightarrow x_3, x_4$

$\therefore \quad x_1 + 6x_2 + 2x_3 - 4x_4 = 0$

$5x_2 + x_3 - 5x_{\cancel{3}4} = 0$

$x_1 + 6x_2 = -2x_3 + 4x_4$

$5x_2 = 5x_4 - x_3$

In the above row Echelon form $c_1$ and $c_2$ columns have pivot positions and hence corresponding vectors span $W$.

$\therefore$ $\{V_1, V_2\}$ ~~span~~ ~~W~~ is the basis of $W$

=> <u>Basis</u> for <u>NulA</u> <u>and</u> <u>colA</u> :-

* The vector in the spanning set for the null space of a matrix forms basis of null space of a matrix.
* The pivot columns of matrix 'A' form a basis for colA.
* ~~Row~~

$\Rightarrow$ Row Space :-

* The set of all linear combination of Row vectors is called the row space of matrix A

* If any matrix 'A' is in echelon form then the nonzero rows in A forms a basis.

① Find a basis for Row Space, Column Space and the null space of matrix A.

Soln :- To find basis of Col A & Row A, reduce A to Echelon form

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad R_1 \Leftrightarrow R_2$$

$$A = \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1, \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$A = \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3/2 \\ R_4 \to R_4/4 \end{array}$$

$$A \sim \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 1 & -2 & 1 & -2 \end{bmatrix} \begin{matrix} R_4 \to R_4 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$A \sim \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 5 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$A \sim \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \boxed{-1} & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to -R_3$$

$$A \sim \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \boxed{+1} & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix}$$

$$A \sim \begin{bmatrix} \boxed{1} & 3 & -5 & 0 & 10 \\ 0 & \boxed{1} & -2 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_1 \to R_1 - 3R_2$$

$$A \sim \begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 \\ 0 & \boxed{1} & -2 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the Echelon form, the first 3 rows (non-zero rows) forms the basis for the row space.

Basis for Row A = $\left\{ (1, 3, -5, 1, 5)\ (0, 1, -2, 2, -7),\ (0, 0, 0, -4, 5) \right\}$

In the Echelon form, the columns 1, 2 & 4 have pivots

$\therefore$ These column forms basis of Col A

$\therefore$ Basis for Column A = $\{ v_1, v_2, v_4 \}$

$$= \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

From row reduced Echelon form,

$$x_1 + x_3 + x_5 = 0$$

$$x_2 - 2x_3 + 3x_5 = 0$$

$$x_4 - 5x_5 = 0$$

$$x_3 = \text{free},$$

$$x_5 = \text{free}$$

$\therefore$ General Solution,

$$x_1 = -x_3 - x_5$$

$$x_2 = 2x_3 - 3x_5$$

$$x_3 = \text{free}$$

$$x_4 = 5x_5$$

$$x_5 = \text{free}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix}$$

$$X = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\therefore X = x_3 v_1 + x_5 v_2$$

$$X \in \text{span}\{v_1, v_2\}$$

$$\text{Basis for Nul}A = \left\{ x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

$\Rightarrow$ The contrast between Null A and Col A :-

① A matrix $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ a) if the column space A

is a subspace of $R^k$, then what is k.

b) if Null A is a subspace of $R^k$, then what is k.

a) WKT, the colum space of a $m \times n$ matrix is a subspace of $R^m$

$\therefore K = m = 3$

b) The Null space of a $m \times n$ matrix is a subspace of $\mathbb{R}^n$

$\therefore K = n = 4$

② Let, $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$,

Find a non-zero vector in Col A and a non-zero vector in Null A.

$\underline{Sol^n}$ :-

A non-zero vector in ColA is any column of A.

$\therefore$ Any one of the non-zero column in 'A' is a non-zero vector in colA

$\rightarrow$ <u>Non Zero vector in NullA</u> :-

To find it, we row reduce 'A' to Echelon form,

$$A = \begin{bmatrix} \boxed{2} & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_1,\ R_3 \rightarrow R_3 - \frac{3}{2}R_1$$

$$A \sim \begin{bmatrix} \boxed{2} & 4 & -2 & 1 \\ 0 & \boxed{-1} & 5 & 4 \\ 0 & 1 & -5 & 9/2 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2$$

$$A \sim \begin{bmatrix} \boxed{2} & 4 & -2 & 1 \\ 0 & \boxed{-1} & 5 & 4 \\ 0 & 0 & 0 & \boxed{17/2} \end{bmatrix}$$

Basic variable, :- $x_1, x_2, x_4$

Free variable :- $x_3$

$$\boxed{x_4 = 0}$$

$$-x_2 + 5x_3 + 4x_4 = 0$$

$$\boxed{x_2 = 5x_3 + 0}$$

$$2x_1 + 4x_2 - 2x_3 + x_4 = 0$$

$$2x_1 + 4(5x_3 + 0) - 2x_3 + 0 = 0$$

$$2x_1 + 20x_3 - 2x_3 = 0$$

$$x_1 = -9x_3$$

$\therefore$ $x_1 = -9x_3$

$x_2 = 5x_3$

$x_3 = $ free

$x_4 = 0$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ +5x_3 \\ x_3 \\ 0 \end{bmatrix}$$

$$\Rightarrow X = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

Let $x_3 = K = 1$

$$\therefore X = \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$

X is a non-zero vector in Null A

③ Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ and $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(i) Determine if $u$ is in Null A

(ii) Could '$u$' be in Col A

(iii) Determine if '$v$' is in Col A

(iv) Could '$v$' be in Null A

(i) Solⁿ :- To prove '$u$' is in Nul A, it is enough to prove $Au = 0$

$$Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6-8+2+0 \\ -6+10-7+0 \\ 9-14+8 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ +3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore Au \neq 0$

$\Rightarrow u$ is not in Nul A

(ii) It is not possible for '$u$' to be in Col A because '$u$' has 4 entries.

(iii) Let us check the equation,

$Ax = V$ is consistent (or) not

$$[A : V] = \begin{bmatrix} \boxed{2} & 4 & -2 & 1 & : & 3 \\ -2 & -5 & 7 & 3 & : & -1 \\ 3 & 7 & -8 & 6 & : & 3 \end{bmatrix} R_2 \to R_2 + R_1,\ R_3 \to R_3 - \frac{3}{2}R_1$$

$$= \begin{bmatrix} \boxed{2} & 4 & -2 & 1 & : & 3 \\ 0 & \boxed{-1} & 5 & 4 & : & 2 \\ 0 & 1 & -5 & 9/2 & : & -3/2 \end{bmatrix} R_3 \to R_3 + R_2$$

$$= \begin{bmatrix} \boxed{2} & 4 & -2 & 1 & : & 3 \\ 0 & \boxed{-1} & 5 & 4 & : & 2 \\ 0 & 0 & 0 & \boxed{17/2} & : & 1/2 \end{bmatrix} R_3 \to 2R_3$$

$$= \begin{bmatrix} \boxed{2} & 4 & -2 & 1 & : & 3 \\ 0 & \boxed{-1} & 5 & 4 & : & 2 \\ 0 & 0 & 0 & \boxed{17} & : & 1 \end{bmatrix}$$

From the Echelon form, we can say that

$$\rho(A) = \rho[A : V]$$

$\therefore$ The system is consistent

$\therefore$ V ~~∉~~ is Col A

(iv) 'V' cannot be in Null A because $V \in \mathbb{R}^3$ but Null $A \in \mathbb{R}^4$

## ⇒ Dimensions of Vector Space :-

If the vector space 'V' is spanned by a finite set, then V is said to be finite dimensional vector space. and it is denoted as 'dimV' and is defined as the number of vectors in a basis of a Vector space.

Note :-

(i) The dimension of NullA is the number of free variable of system $AX = 0$

(ii) The dimension of ColA is the number of Pivot Columns of matrix A.

(iii) We know that, RowA is a subspace of $R^n$. Since the rows of A are identified with Columns of $A^T$

$\therefore$ $RowA = (Col A)^T$

(iv) If two matrices A and B are row equivalent, then their row spaces are same.

(v) If 'B' is Echelon form, then non zero rows of B forms basis of rowspace of A and rowspace of B.

① Find the basis and dimension of a ~~matrix A~~ ~~where~~ A row space, column space and the null space of matrix A, where $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad R_2 \to R_2 + \frac{R_1}{2}, \ R_3 \to R_3 + \frac{3}{2}R_1$$
$$R_4 \to R_4 + \frac{R_1}{2}$$

$$A \sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & \frac{1}{2} & -1 & 1 & -7/2 \\ 0 & 7/2 & -7 & 7 & -49/2 \\ 0 & 9/2 & -9 & 5 & -23/2 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 \\ R_3 \to \frac{2R_3}{7} \\ R_4 \to 2R_4 \end{array}$$

$$A \sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 1 & -1 & 1 & -7 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 9R_2 \end{array}$$

$$A \sim \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -8 & 31 \end{bmatrix}$$

$$A \sim \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \boxed{-4} & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of Row A $= \{(1, 3, -5, 1, 5) (0, 1, -2, 2, -7) (0, 0, 0, -4, 20)\}$

Basis of col A $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

RREF,

$$C = \begin{bmatrix} \boxed{1} & 0 & 1 & 0 & 1 \\ 0 & \boxed{1} & -2 & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

General Solution of the system,

$$x_1 + x_3 + x_5 = 0$$
$$x_2 - 2x_3 + 3x_5 = 0$$
$$x_4 - 5x_5 = 0$$

$x_1, x_2, x_4 \rightarrow$ Basic

$x_3, x_5 \Rightarrow$ Free

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ +2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix}$$

$$\therefore X = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$$\therefore X = x_3 V_1 + x_5 V_2$$

Basis for Null A $= \{V_1, V_2\}$

$$= \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Dimension of Row A = Col A = 3

Dimension of Null A = 2

⇒ **Rank of Matrix:-**

The rank of matrix 'A' is the dimension of the vector space generated or spanned by its columns i.e the rank of A is the dimension of col A

→ **The Rank Theorem:-**

The dimensions of column space and the row space of an $m \times n$ matrix 'A' are equal. This common dimension is rank of A which is equal to number of Pivot Position in A and satisfy equation,

$$\text{Rank } A + \dim(\text{Null } A) = n$$

⇒ **Coordinate System:-**

→ **Unique Representation Theorem:-**

Let, $\mathcal{B} = \{b_1, b_2, \ldots b_n\}$ be a basis for vector space 'V'. Then for each '$x$' in 'V', there exists a unique set of scalars $c_1, c_2, c_3 \ldots c_n$ such that,

$$x = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$$

Suppose, $\mathcal{B} = \{b_1, b_2, \dots b_n\}$ is a basis for 'V' and 'X' is in 'V', the coordinates of 'x' related to basis 'B' or B coordinate of x or the weights $c_1, c_2, c_3 \dots c_n$ such that $x = c_1 b_1 + c_2 b_2 + c_3 b_3 + \dots + c_n b_n$

If $c_1, c_2 \dots c_n$ are the $\mathcal{B}$ coordinates of x, then the vector in $\mathbb{R}^n$, then,

$$\text{B coordinate of } x = [x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ is the B-Coordinate

of 'x' or the Coordinate vector 'x'. The mapping, $x \mapsto [x]_B$ is called Coordinate Mapping.

① Consider a basis, $\mathcal{B} = \{b_1, b_2\}$ for $\mathbb{R}^2$ where $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose an 'x' in $\mathbb{R}^2$ has the Coordinate vector $[x]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, Find x;

The B- coordinate of 'x' from the vectors in B such that

$$X = c_1 b_1 + c_2 b_2 \rightarrow (i)$$

$$X = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Also,

$$[x_B]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \quad X = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$X = \begin{bmatrix} -2 + 3 \\ 0 + 6 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 6 \end{bmatrix} //$$

(2) Let, $b_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $b_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $B = \{b_1, b_2\}$

Find the coordinate vector $[x]_B$ of x related to B

The equation is,

$$X = c_1 b_1 + c_2 b_2$$

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Augmented matrix,

$$[C : X] = \left[\begin{array}{cc:c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right] \quad R_2 \to R_2 - \frac{R_1}{2}$$

$$= \left[\begin{array}{cc:c} 2 & -1 & 4 \\ 0 & 3/2 & 3 \end{array}\right] \quad R_2 \to \frac{2R_2}{3}$$

$$= \left[\begin{array}{cc:c} 2 & -1 & 4 \\ 0 & 1 & 2 \end{array}\right]$$

$$\therefore \quad 2c_1 - c_2 = 4$$

$$c_2 = 2$$

By back substitution

$$2c_1 - 2 = 4$$

$$c_1 = 6/2$$

$$\therefore c_1 = 3$$

$$\therefore \quad [x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

⇒ Change of Basis :-

Let $\mathcal{B} = \{b_1, b_2, \dots b_n\}$ and $C = \{c_1, c_2, c_3 \dots c_n\}$ be the basis of vector space 'V', Then there is a unique $n \times n$ matrix 'P' suchat that $\underset{C \leftarrow B}{P}$ such that 'C' coordinate of x is equal to P coordinate of B to C into C

$$[x]_c = \underset{C \leftarrow B}{P} [x]_B$$

The columns of $\underset{C \leftarrow B}{P}$ are C-coordinate vectors of the vectors in the basis $\mathcal{B}$ i.e

$$\underset{C \leftarrow B}{P} = \{[b_1]_c, [b_2]_c, \dots [b_n]_c\}$$

The matrix $\underset{C \leftarrow B}{P}$ is called the change of coordinate matrix from B to C

① Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$

and consider bases for $\mathbb{R}^2$, is given by $B = \{b_1, b_2\}$

and $C = \{c_1, c_2\}$

(i) Find the change of Coordinate matrix from C to B

(ii) Find the change of Coordinate matrix from B to C

(i) Soln :- To find the change of coordinate matrix from C to B, we have

$$\underset{B \leftarrow C}{P} = [b_1 b_2 : c_1 c_2]$$

$$= \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] R_2 \to R_2 + 3R_1$$

$$= \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right] R_2 \to R_2/2$$

$$= \left[\begin{array}{cc:cc} 1 & -2 & -7 & -5 \\ 0 & 1 & 6 & 4 \end{array}\right] R_1 \to R_1 + 2R_2$$

$$= \left[\begin{array}{cc:cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right] \Rightarrow \underset{B \leftarrow C}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

(ii)

$$\underset{C\leftarrow B}{P} = \left[\underset{B\leftarrow C}{P}\right]^{-1}$$

$$= \frac{1}{2}\begin{bmatrix}4 & -3\\ -6 & 5\end{bmatrix}$$

$$\underset{C\leftarrow B}{P} = \begin{bmatrix}2 & -3/2\\ -3 & 5/2\end{bmatrix}$$