⇒ Unit - 4 :-

⇒ Inner Product, Length and Orthognality :-

(*) If u & v are any two vectors, then the number $u^T \cdot v$ is called ~~Inner~~ inner product of 'u' and 'v' and it is written as $u \cdot v$ where $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

then Inner product of u & v is given by

$$u^T v = [u_1 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots + u_n v_n$$

(1) Compute $u \cdot v$ and $v \cdot u$ when $u = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

$$u \cdot v = u^T v = [2 \ -5 \ 1] \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$$

$$= 6 - 10 - 3 = -7$$

$$v \cdot u = v^T u = [3 \ 2 \ -3] \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$

$$= 6 - 10 - 3 = -7$$

$\Rightarrow u \cdot v = v \cdot u$

Theorem :- Let u, v, w be vectors in $\mathbb{R}^n$ ~~then~~ and let 'c' be any scalar, then

(i) $u \cdot v = v \cdot u$

(ii) $(u + v) \cdot w = u \cdot w + v \cdot w$

(iii) $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$

(iv) $u \cdot v \geq 0$ ~~iff~~, $u \cdot u = 0$ iff $u = 0$

→ Length of the Vector / Norm of Vector :-

The length of the vector (Norm) is a non-negative scalar, denoted as $\|v\|$ defined as,

$$\|v\| = \sqrt{v \cdot v}$$

For any given scalar 'c', the length of cv is

$$\|c \cdot v\| = |c| \, \|v\|$$

→ Unit vector :-

A vector whose length is one is called a unit vector.

Normalizing a Vector

A non-zero vector 'v' is divided by its length we obtain a unit vector ~~vector since, the length of~~ denote that unit vector as 'u'

i.e, $u = \dfrac{v}{\|v\|}$

This process of ~~normali~~ obtaining 'u' from 'v' is called Normalizing 'v'.

We say that 'u' is in the same direction of 'v'

① Let $v = (1, -2, 2, 0)$, Find a unit vector 'u' in the same direction as 'v'.

$$\|v\| = \sqrt{v \cdot v}$$

$$= \sqrt{v^T \cdot v}$$

$$= \sqrt{\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 & 0 \end{bmatrix}}$$

$$= \sqrt{(1 + 4 + 4 + 0)}$$

$$= \sqrt{9}$$

$$\therefore \|v\| = 3$$

$\therefore$ Unit vector,

$$u = \frac{v}{\|v\|}$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore u = (1/3, -2/3, 2/3, 0)$$ //

To prove 'u' is a unit vector,

$$\|u\| = \sqrt{u \cdot u}$$

$$= \sqrt{\begin{bmatrix} 1/3 & -2/3 & 2/3 & 0 \end{bmatrix}\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}}$$

$$= \sqrt{(1/9 + 4/9 + 4/9 + 0)}$$

$$= \sqrt{\frac{9}{9}}$$

$$\|u\| = 1$$ //

$\therefore$ u is a unit vector in direction same as 'v'.

(2) Let 'W' be the subspace of $\mathbb{R}^2$ spanned by $x = (2/3, 1)$

Find a unit vector z such that it is a basis for 'W'

'W' consists of all multiples of 'x'

$\therefore$ Any non-zero vector in 'W' is a basis for 'W'

Scaling the given vector 'x'

$\therefore$ Multiply 'x' by 3

$$y = (2, 3)$$

$\therefore$ Normalizing vector 'y'

$$\|y\| = \sqrt{[2, 3]\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$$

$$= \sqrt{(4+9)}$$

$$= \sqrt{13} = \sqrt{13}$$

$\therefore$ Unit vector,

$$z = \frac{y}{\|y\|}$$

$$= \frac{(2, 3)}{\sqrt{13}}$$

$$z = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right) = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}\right)$$

Verifying 'z' as Unit vector,

$$\|z\| = \sqrt{\begin{bmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} 2/\sqrt{13} \\ 3/\sqrt{13} \end{bmatrix}}$$

$$= \cancel{\sqrt{\tfrac{4}{13} + \tfrac{9}{13}}} = \sqrt{\frac{4}{13} + \frac{9}{13}}$$

$$\|z\| = 1$$

$\therefore$ z is a unit vector whichis basis of 'w'

⇒ <u>Distance</u> <u>in</u> <u>$\mathbb{R}^n$</u> :-

If a & b are real numbers, the distance on the number line b/w a & b is the number $|a-b|$

If u & v are two vectors in $\mathbb{R}^n$, the distance between u & v is written as dis (u,v) is the length of the vector $u-v$ i.e

$$\text{dis}(u,v) = \|u-v\|$$

① Compute the distance between the vectors $u = (7, 1)$, $v = (3, 2)$.

$$dis(u, v) = \|u - v\|$$

$$u - v = \begin{bmatrix} 7 - 3 \\ -2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$\therefore dis(u, v) = \|u - v\|$$

$$= \sqrt{4^2 + (-1)^2}$$

$$= \sqrt{17}$$ //

② If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ then find $dis(u, v)$

$$u - v = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{bmatrix}$$

$$\therefore dis(u, v) = \|u - v\|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$$

⇒ Orthogonal Vectors :-

Two vectors u & v in $\mathbb{R}^n$ are orthogonal if

$u \cdot v = 0$

The zero vector is Orthogonal to every vector in $\mathbb{R}^n$

Since, $0^T \cdot v = 0 \quad \forall\, v \in \mathbb{R}^n$

→ Orthogonal Compliments :- ($W^\perp$)

If a vector z is orthogonal to every vector in a subspace 'W' of $\mathbb{R}^n$, then z is said to be orthogonal to 'W', the set of all vectors 'z' that are all orthogonal to 'W' is called the Orthogonal Compliment of 'W'.

→ Theorem :- ~~If a vector z is orth~~

Let 'A' be a $m \times n$ matrix, the orthogonal compliment of Row space of A is the Null space of A and orthogonal compliment of col.A is equal to Null space of A^T

i.e $(\text{Row}A)^\perp = \text{Null } A \qquad (\text{Col } A)^\perp = \text{Nul } A^T$

① Let, $A = (-2, 1)$, $B = (-3, 1)$, $C = (4/3, -1, 2/3)$

$d = (5, 6, -1)$

(i) Compute $\frac{a \cdot b}{a \cdot a}$ and $\frac{a \cdot b}{a \cdot a} \cdot a$

(ii) Find unit vector 'u' in the direction of 'c'

(iii) Show that 'd' is orthogonal to 'c'

(i)

$$\frac{a \cdot b}{a \cdot a} = \frac{\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix}} = \frac{6+1}{4+1} = \frac{7}{5}$$

$$\frac{a \cdot b}{a \cdot a} \cdot a = \frac{7}{5} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -14/5 \\ 7/5 \end{bmatrix}$$

(ii)

$$u = \frac{c}{\|c\|}$$

$$= \frac{(4/3, -1, 2/3)}{\sqrt{\frac{16}{9} + 1 + \frac{4}{9}}}$$

$$= \frac{(4/3, -1, 2/3)}{\frac{\sqrt{29}}{3}}$$

$$= \frac{3}{\sqrt{29}} \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{29} \\ -3/\sqrt{29} \\ 2/\sqrt{29} \end{bmatrix}$$

(iii) $$c \cdot d = \begin{bmatrix} 4/3 & -1 & 2/3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$$

$$= \frac{20}{3} - 6 - \frac{2}{3}$$

$$= \frac{20 - 18 - 2}{3}$$

$$c \cdot d = 0$$

Since, $c \cdot d = 0$, c & d are orthogonal

→ Orthogonal Sets :-

A set of vectors $\{u_1, u_2 \ldots u_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal.

i.e $u_i \cdot u_j = 0$ for $\forall\, i \neq j$

→ Orthogonal Basis :-

An orthogonal basis for a subspace 'w' of $\mathbb{R}^n$ is a basis for 'w' that is also an orthogonal set.

→ Theorem 1 :- Let $S = \{u_1, u_2, \ldots u_p\}$ is an orthogonal set of a non-zero vectors in R^n, then S is linearly independent and hence is a basis for subspace spanned by 'S'.

→ Theorem 2 :- Let the set $\{u_1, u_2, \ldots u_p\}$ be an orthogonal basis for a subspace 'W' of $\mathbb{R}^n$ for each 'y' in 'W', the weights is the linear combination.

i.e $y = c_1 u_1 + c_2 u_2 + \ldots + c_p u_p$

where, $c_i = \dfrac{y \cdot u_j}{u_j \cdot u_j}$, for $j = 1, 2, \ldots p$

① Show that the vectors $S = \{u_1, u_2, u_3\}$ where, $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ +2 \\ 1 \end{bmatrix}$

$u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ is an orthogonal set. The set $\{u_1, u_2, u_3\}$ is an orthogonal basis of $\mathbb{R}^3$, then express $y = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$

as a linear combination of vectors in S.

To prove $S = \{u_1, u_2, u_3\}$ is an orthogonal set

We need to prove,

$$u_1 \cdot u_2 = u_2 \cdot u_3 = u_1 \cdot u_3 = 0$$

$$u_1 \cdot u_2 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3+2+1 \end{bmatrix} = 0$$

$$u_2 \cdot u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{1}{2} - 4 + \frac{7}{2} = 0$$

$$u_1 \cdot u_3 = \begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} = \frac{-3}{2} - 2 + \frac{7}{2} = 0$$

$\therefore$ $u_1 \cdot u_2 = u_2 \cdot u_3 = u_1 \cdot u_3 = 0$

$\therefore$ u_1, u_2, u_3 are orthogonal to each other

$\therefore$ $S = \{u_1, u_2, u_3\}$ is an Orthogonal Set.

To express y as,

$$y = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}} = \frac{18+1-8}{9+1+1} = \frac{11}{11} = 1$$

$$C_2 = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{\begin{bmatrix} 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}} = \frac{-6+2-8}{1+4+1} = \frac{-12}{6} = -2$$

$$C_2 = -2$$

$$C_3 = \frac{y \cdot u_3}{u_3 \cdot u_3} = \frac{\begin{bmatrix} 6 & 1 & -8 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}}{\begin{bmatrix} -1/2 & -2 & 7/2 \end{bmatrix} \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}} = \frac{-3-2-28}{\frac{1}{4}+4+\frac{49}{4}} = \frac{-33 \times 4}{1+16+49}$$

$$= \frac{-132}{66}$$

$$C_3 = -2$$

$$\therefore \quad y = 1 \cdot u_1 - 2u_2 - 2u_3$$

$$\begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 & +2 & +1 \\ 1 & -4 & +4 \\ 1 & -2 & -7 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$$

⇒ Orthogonal Projections :-

⊗ Let 'y' and 'u' be two vectors in a vector space, then orthogonal projection of 'y' on u is denoted by $\hat{y}$ and which is defined by,

$$\hat{y} = \frac{\langle y \cdot u \rangle}{\langle u \cdot u \rangle} \cdot u$$

⊗ The orthogonal projection of 'u' on 'y' is denoted as $\hat{u}$ is defined as,

$$\hat{u} = \frac{\langle u \cdot y \rangle}{\langle y \cdot y \rangle} \cdot y$$

⊗ The component of 'y' orthogonal to 'u' is given by

$$y - \hat{y} = y - \frac{\langle y \cdot u \rangle}{\langle u \cdot u \rangle} \cdot u$$

⊗ The component of 'u' orthogonal to 'y' is defined as

$$u - \hat{u} = u - \frac{\langle u \cdot y \rangle}{\langle y \cdot y \rangle} \cdot y$$

① Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. ① Find the component of 'y' orthogonal to 'u' and also component of 'u' orthogonal to 'y'.

② If $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, Compute the orthogonal projection of 'y' on 'u' and 'u' on 'y'.

①
$$y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \frac{\begin{bmatrix} 7 & 6 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \frac{28 + 12}{16 + 4} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 6 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$y - \hat{y} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$u - \hat{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix}}{\begin{bmatrix} 7 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix}} \cdot \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \frac{28 + 12}{49 + 36} \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$u - \hat{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \frac{40}{85} \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$= \begin{matrix} 4 - \frac{56}{17} \\ 2 - \frac{48}{17} \end{matrix}$$

$$u - \hat{u} = \begin{bmatrix} \frac{12}{17} \\ \frac{-14}{17} \end{bmatrix}$$

2

$$\hat{y} = \frac{y \cdot u}{u \cdot u} \cdot u$$

$$= \frac{\begin{bmatrix} 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \frac{14+6}{4+1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$\hat{u} = \frac{u \cdot y}{y \cdot y} \cdot y = \frac{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix}}{\begin{bmatrix} 7 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix}} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \frac{20}{85} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{14}{17} \\ \frac{12}{17} \end{bmatrix}$$

⇒ Orthogonal Decomposition Theorem :-

Let 'w' be a subspace of $\mathbb{R}^n$, then each 'y' in $\mathbb{R}^n$ can be written uniquely in the form $y = \hat{y} + z$ where, $\hat{y}$ is in 'w' and z is in Orthogonal compliment to 'w'.

If $\{u_1, u_2 \ldots u_p\}$ is any Orthogonal basis of 'w', then $\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2 + \ldots \frac{y \cdot u_p}{u_p \cdot u_p} \cdot u_p$

and $z = y - \hat{y}$

① Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ & $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, observe that $\{u_1, u_2\}$ is orthogonal basis for $w = \text{span}\{u_1, u_2)$

Write 'y' as sum of vector in 'w' and a vector orthogonal to 'w'

Sol^n:- To write 'y' as a sum of a vector in 'w'

i.e. $y = \hat{y} + z$

where,

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2$$

$$\hat{y} = \frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 & 5 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}} \cdot u_1 + \frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}} \cdot u_2$$

$$\hat{y} = \frac{2+10-3}{4+25+1} \cdot u_1 + \frac{-2+2+3}{4+1+1} \cdot u_2$$

$$= \frac{9}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{6}{10} - 1 \\ \frac{3}{2} + \frac{1}{2} \\ -\frac{3}{10} + \frac{1}{2} \end{bmatrix}$$

$$\therefore \quad \hat{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

$$\therefore \quad z = y - \hat{y}$$

$$z = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$

This proves that $z = y - \hat{y}$ is orthogonal compliment of 'w'

Since, the vectors u_1 & u_2 are both orthogonal to z

i.e

$$u_1 \cdot z = \begin{bmatrix} 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix} = 0$$

$$u_2 \cdot z = \begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix} = 0$$

$\therefore$ The vectors in 'w' are all orthogonal to 'z'

$\therefore$ The desired decomposition of 'y' is $y = \hat{y} + z$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} //$$

⇒ Properties of Orthogonal Projection :-

① If, y is in $W = \text{span}\{u_1, u_2, u_3 \dots u_p\}$ then projection of y on W i.e

$$\text{proj}_W y = \hat{y}$$

② The Best approximation Theorem :-

Let 'W' be a subspace of $\mathbb{R}^n$ and let 'y' be any vector in $\mathbb{R}^n$ and $\hat{y}$ the orthogonal projection of y on 'W', then $\hat{y}$ is the closest point in 'W' to 'y'

i.e

$$\|y - \hat{y}\| < \|y - v\| \quad \forall \; v \text{ in 'W' distinct from } \hat{y}$$

① If $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Then find the closest path in 'W' and $W = \text{span}\{u_1, u_2\}$

Soln :- WKT, by Best Approximation Theorem, the closest path in 'W' to 'y' is $\hat{y}$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2$$

$$\hat{y} = \frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}}{\begin{bmatrix} 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}} \cdot u_1 + \frac{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}} \cdot u_2$$

$$= \frac{3}{10} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$$

<u>Note</u> :- The distance from a point y in $\mathbb{R}^n$ to a subspace 'w' is defined as distance of y to the nearest point in 'w'.

i.e distance y to w $= \|y - \hat{y}\|$

① Find the distance from y to w where, $W = \text{span}\{u_1, u_2\}$

where, $u_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}$

WKT, distance from y to w

$$= \|y - \hat{y}\|$$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} \cdot u_2$$

$$= \frac{\begin{bmatrix} -1 & -5 & 10 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}} u_1 + \frac{\begin{bmatrix} -1 & -5 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}} \cdot u_2$$

$$= \frac{-5+10+10}{25+4+1} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \frac{-1-10-10}{4+1+1} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} - \frac{\cancel{21}^{\,7}}{\cancel{6}_{\,2}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} - \frac{7}{2} \\ -1 - 7 \\ 1/2 + 7/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \\ 4 \end{bmatrix}$$