Linear Algebra (20MA412)

CO1 :- Unit - 1 :- Linear Equations

2)

(i) System of Linear Equations
(ii) Row Reduction and Echelon form
(iii) Vector Equations
(iv) Matrix Equation
(v) Solution sets of Linear Systems
(vi) Linear Independence

⇒ **Linear Algebra:-**

It is the study of vectors and linear functions

⇒ **Linear Equations:-**

Any equation is said to be linear which has the highest degree 1. i.e., no variable in a linear Equation has an exponent more than 1.

Standard form :-

$$ax + b = 0 \;/\; Ax + B = 0$$

$$ax + by = c \;/\; Ax + By = c$$

⇒ System of Linear Equations:-

A System of Linear Equation (or) a linear system is a collection of one (or) more linear equations involving the same variables. say the variables are $x_1, x_2 \ldots x_n$

Let 'm' number of linear equations with 'n' unknown variables $(x_1, x_2 \ldots x_n)$ is a collection of 'm' equations in the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_2 + \ldots + a_{1n}x_n = d_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = d_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = d_m$$

where, $a_{ij} \neq 0$ are called coeffecients,

$x_1, x_2, \ldots x_n$ are called variables

$d_1, d_2, \ldots d_m$ are called constants.

⇒ Matrix form of System of Linear Equation:-

The above system of linear equations can be put into matrix form as, $Ax = B$

where, $A = \begin{bmatrix} a_{11}, a_{12} \ldots & a_{1n} \\ a_{21}, a_{22} \ldots & a_{2n} \\ \vdots & \\ a_{m1}, a_{m2} \ldots & a_{mn} \end{bmatrix}$ } ~~Row~~ Coefficient matrix

$$X = \left.\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right\}$$ Column Matrix called as Variable Matrix

$$B = \left.\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}\right\}$$ Constant Matrix

⇒ Augmented Matrix :-

A matrix formed by appending to 'A' an extra Column consisting of constant Matrix. is called Augmented Matrix . Denoted as $[A : B]$

$$[A : B] = \begin{bmatrix} a_{11}, a_{12} & \cdots & a_{1n} & : & d_1 \\ a_{21} \; a_{22} & \cdots & a_{2n} & : & d_2 \\ \vdots & & \vdots & & \vdots \\ a_{m1} \; a_{m2} & \cdots & a_{mn} & : & d_m \end{bmatrix}$$

⇒ Solution of System of Linear Equations :-

* The system of Linear equation is said to be consistent if it has atleast one Solution.

* A linear System may not have a solution at all, in this case it is called Inconsistent.

⇒ Examples :-

① Solve the following system of Equation :-

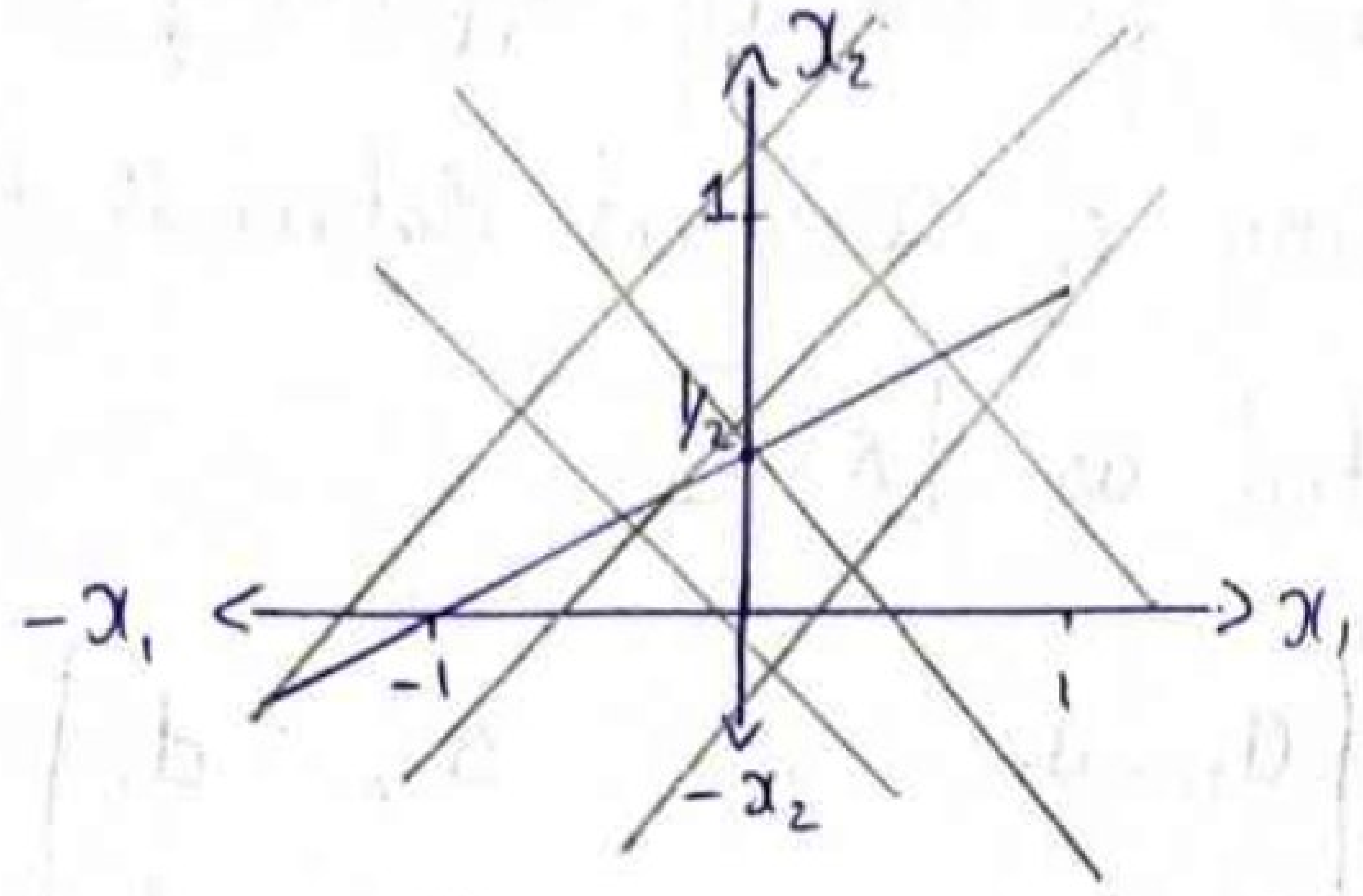
(i) $x_1 - 2x_2 = -1 \Rightarrow l_1$

$-x_1 + 3x_2 = 3 \Rightarrow l_2$

For l_1 :-

Put, $x_1 = 0 \Rightarrow x_2 = 0.5$

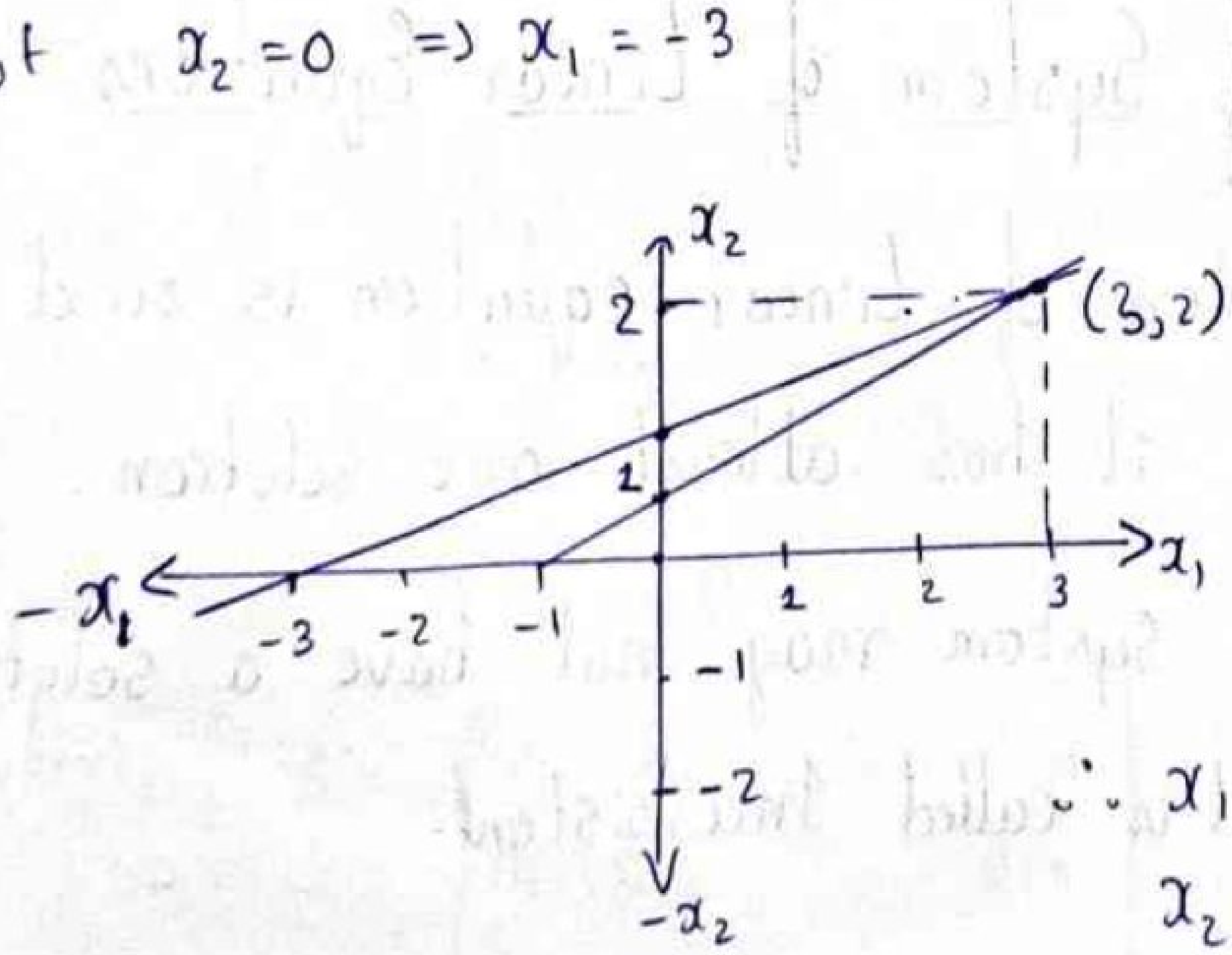
Put, $x_2 = 0 \Rightarrow x_1 = -1$



For l_2 :-

Put $x_1 = 0 \Rightarrow x_2 = 1$

Put $x_2 = 0 \Rightarrow x_1 = -3$



$\therefore x_1 = 3$

$x_2 = 2$

② Solve $x_1 - 2x_2 = -1$ and $-x_1 + 2x_2 = 3$

③ Solve $x_1 - 2x_2 = -1$ and $-x_1 + 2x_2 = 1$

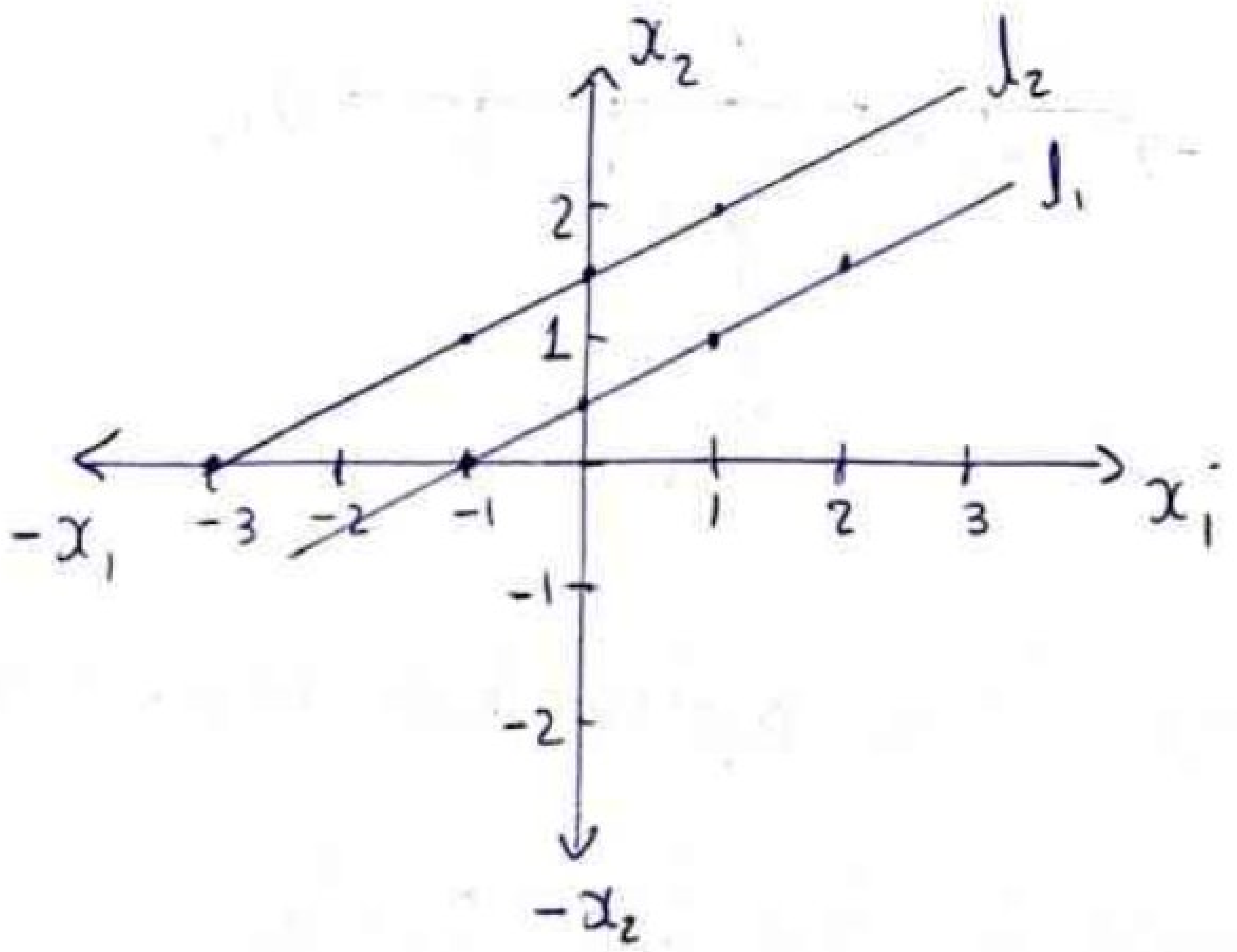
② $\underset{\downarrow \; l_1}{x_1 - 2x_2 = -1}$ & $\underset{\downarrow \; l_2}{-x_1 + 2x_2 = 3}$

For l_1

x_1	0	-1	1	2
x_2	$1/2$	0	1	$3/2$

For l_2

x_1	0	-3	1	2	-1
x_2	$3/2$	0	2	$+5/2$	1



* The given linear system has no solutions. Because the two lines are parallel.

* The linear system is inconsistent

③ $x_1 - 2x_2 = -1 \longrightarrow l_1$

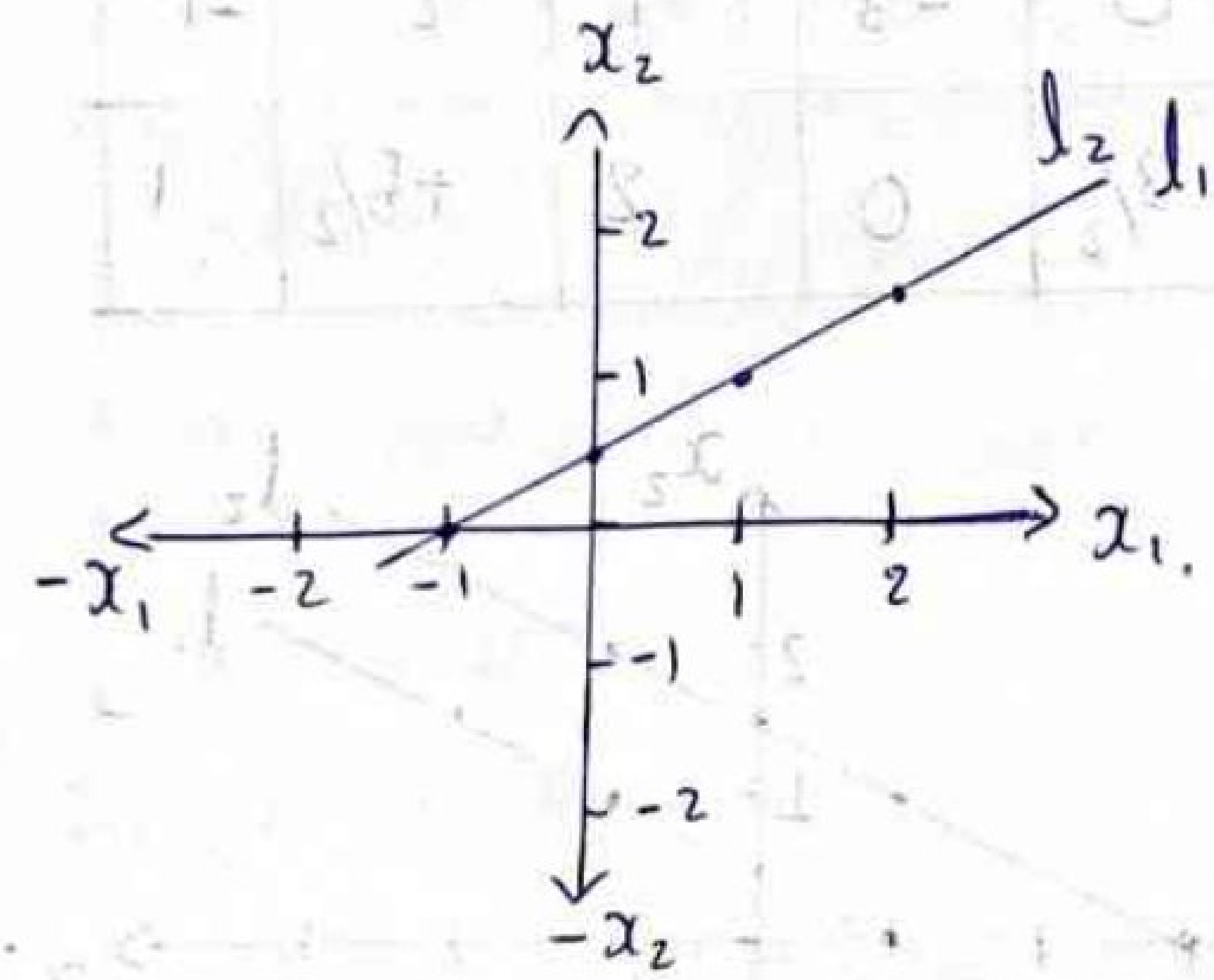
$-x_1 + 2x_2 = 1 \longrightarrow l_2$

For l_1,

x_1	0	-1	1	2
x_2	$1/2$	0	1	$3/2$

For l_2,

x_1	0	-1	1	2
x_2	$1/2$	0	1	$3/2$



* The given linear system has infinitely many solution because the two lines overlap.

⇒ Solution of system of Linear Equations:-

① Solve a linear system of equation :- $x_1 - 2x_2 + x_3 = 0$, $-4x_1 + 5x_2 + 9x_3 = -9$, $2x_2 - 8x_3 = 8$

The given equations are,

$$x_1 - 2x_2 + x_3 = 0 \rightarrow \text{(i)}$$

$$2x_2 - 8x_3 = 8 \rightarrow \text{(ii)}$$

$$-4x_1 + 5x_2 + 9x_3 = -9 \rightarrow \text{(iii)}$$

⇒ Solving by Matrix :-

The augmented matrix is given as,

$$[A : B] = \begin{bmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 2 & -8 & : & 8 \\ -4 & 5 & 9 & : & -9 \end{bmatrix} \begin{array}{l} \\ \\ \rightarrow R_3 \end{array}$$

$$R_3 \rightarrow R_3 + 4R_1$$

$$[A : B] = \begin{bmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 2 & -8 & : & 8 \\ 0 & -3 & 13 & : & -9 \end{bmatrix}$$

$$R_3 \rightarrow 2R_3 + 3R_2$$

$$[A : B] = \begin{bmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 2 & -8 & : & 8 \\ 0 & 0 & 2 & : & 6 \end{bmatrix}$$

$$R_2 \rightarrow R_2/2$$

$$[A:B] = \begin{bmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -4 & : & 4 \\ 0 & 0 & 2 & : & 6 \end{bmatrix}$$

$$R_3 \rightarrow R_3/2$$

$$[A:B] = \begin{bmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & -4 & : & 4 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 4R_3$

$$[A:B] = \begin{bmatrix} 1 & -2 & 1 & : & 0 \\ 0 & 1 & 0 & : & 16 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

⇒ Solving by simplification :-

$$x_1 - 2x_2 + x_3 = 0 \longrightarrow (1)$$

$$2x_2 - 8x_3 = 8 \longrightarrow (2)$$

$$-4x_1 + 5x_2 + 9x_3 = -9 \rightarrow (3)$$

Multiply (1) by 4

$$4x_1 - 8x_2 + 4x_3 = 0 \rightarrow (4)$$

Add (4) & (3)

$$\begin{array}{r} 4x_1 - 8x_2 + 4x_3 = 0 \\ -4x_1 + 5x_2 + 9x_3 = -9 \\ \hline 0 - 3x_2 + 13x_3 = -9 \\ \hline \end{array}$$

∴ Eqn (3) becomes $-3x_2 + 13x_3 = -9 \rightarrow (3)$

Multiply ③ by 2 & Multiply ② by 3

$$\cancel{6x_2} - 24x_3 = 24 \rightarrow ②$$
$$\cancel{-6x_2} + 26x_3 = -18 \rightarrow ③$$
$$2x_3 = 6$$
$$x_3 = 3$$

The new equations obtained are,

$$x_1 - 2x_2 + x_3 = 0 \rightarrow ①$$
$$2x_2 - 8x_3 = 8 \rightarrow ②$$
$$\underline{x_3 = 3} \rightarrow ③$$

Substitute x_3 in ②

$$2x_2 - 8(3) = 8$$
$$2x_2 = 8+24$$
$$\underline{x_2 = 16}$$

Substitute x_2, x_3 in ①

$$x_1 - 2(16) + 3 = 0$$
$$x_1 = 32-3$$
$$\underline{x_1 = 29}$$

② Solve the system of equations :-

$$x_1 - 3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$R_2 \to -2R_1 + R_2$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

$R_3 \to R_2 - 2R_3$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & 5 & -5 \end{array}\right]$$

From the above matrix, the equations are

$$x_1 - 3x_3 = 8 \to ①$$
$$2x_2 + 15x_3 = -9 \to ②$$
$$5x_3 = -5$$
$$\therefore \quad x_3 = -1 \to ③$$

Sub x_3 in ①

$$x_1 - 3(-1) = 8$$
$$x_1 = 8 - 3$$
$$x_1 = 5$$

Sub x_3 in ②

$$2x_2 + 15(-1) = -9$$
$$x_2 = \frac{-9+15}{2}$$
$$x_2 = 3$$

$\therefore\ x_1 = 5,\ x_2 = 3,\ x_3 = -1$

⇒ Row Elimentary Operations / Row Reduction

(i) Row Replacement :-

Replace one row by the sum of itself and a multiple of another two row

$$R_j \longrightarrow R_j + KR_i \quad , \quad K \neq 0$$

(or)

$$R_j \longrightarrow R_j + R_i/K \quad , \quad K \neq 0$$

(ii) Row Interchange :-

Interchanging any two rows

$$R_j \longleftrightarrow R_i$$

(iii) Row Scaling :-

Multiply all entries in a row by a non zero constant.

$$R_j \longrightarrow KR_j$$

(or) $\quad , \ K \neq 0$

$$R_j \rightarrow R_j/K$$

⇒ Row Reduction and Echelon Form :-

Any matrix 'A' is said to be in Echelon form if it has the following 3 properties.

(i) All non-zero rows are above any zero rows

(ii) Each leading entry of a row is in a column to the right of leading entry of the row above it.

(iii) All entries in a column below a leading entry (Pivot) must be zero

Examples :-

$$\begin{bmatrix} \square & * & * & * \\ 0 & \square & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note :- No need to write Augmented matrix while writing Echelon form

→ **Pivot Position :-**

A Pivot position in a matrix 'A' is a location in 'A' that corresponds to a leading '1' in the reduced Echelon form.

→ **Pivot Column :-**

It is a Column in 'A' that contains a pivot position.

→ **Row Reduced Echelon form :-**

If a matrix in Echelon form satisfies the following additional conditions, then it is called as Row-Reduced Echelon form :-

1. The leading entry in each non-zero row is '1'.
2. Each leading '1' is the only non zero entry in the column.

Ex :-

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note :-

* The first non-zero element in each row is Pivot element

① Row reduce the matrix 'A' to Echelon form and locate the Pivot Columns of 'A'.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Interchanging R_1 & R_4 $\Rightarrow$ $R_1 \leftrightarrow R_4$

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Replacing rows $\Rightarrow$ $R_2 \to R_1 + R_2$, $R_3 \to 2R_1 + R_3$

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Scaling rows $\Rightarrow$ $R_2 \to R_2/2$

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Replacing rows, $R_3 \to R_3 - 5R_2$, $R_4 \to R_4 + 3R_2$

$$A = \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Replacing → Interchanging rows $\Rightarrow R_3 \leftrightarrow R_4$

$$\begin{array}{c} \begin{array}{ccccc} \uparrow C_1 & \uparrow C_2 & & \uparrow C_4 & \end{array} \\ A = \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Columns C_1, C_2, C_4 are Pivot Columns

$x_1, x_2, x_4 \to$ Basic variables

$x_3, x_5 \to$ Free variables

→ **Basic Variable** :-

A variable is a basic variable if it corresponds to a Pivot Column

→ **Free Variable** :-

A variable that does not correspond to a Pivot Column.

Note:-

Consider, a System of linear equation whose Echelon form is

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

⇒ if there are 'p' equations with 'q' unknowns

(i) if $p = q$, then there are no free variables
All the variables are called Basic

(ii) if $q > p$, then, there will be '$q - p$' number of free variable

So, in the above Echelon form $p = 3$, $q = 5$

$\therefore$ $5 - 3 = 2$ Free variables are there.

→ Rank of Matrix $\rho(A)$:-

The rank of any given matrix 'A' is the number of non-zero rows in the Echelon form after using the elementary row transformation. It is denoted as $\rho(A) = r$

⇒ Consistency and Inconsistency :-

(i) $\rho[A] = \rho[A:B] \Rightarrow$ Consistent

a) Unique Solution $\Rightarrow r = n$ ($n \to$ no of variables)

b) Infinite Solutions $\Rightarrow r < n$ ($n \to$ no of variables)

(ii) $\rho[A] \neq \rho[A:B] \Rightarrow$ Inconsistent $(r \neq n)$

Q) Test the consistency and solve the system of Linear equation. Also, identify the basic and free variables.

$$x_2 - 4x_3 = 8 \longrightarrow (i)$$

$$2x_1 - 3x_2 + 2x_3 = 1 \longrightarrow (ii)$$

$$5x_1 - 8x_2 + 7x_3 = 1 \longrightarrow (iii)$$

The matrix form of given system is $Ax = B$

$$\begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

The augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc:c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{2} & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right] \quad R_3 \to R_3 - \frac{5}{2}R_1$$

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right] \quad R_3 \to R_3 + \frac{1}{2}R_2$$

$$[A:B] = \begin{bmatrix} \boxed{2} & -3 & 2 & : & 1 \\ 0 & \boxed{1} & -4 & : & 8 \\ 0 & 0 & 0 & : & \frac{5}{2} \end{bmatrix}$$

Here, Rank of 'A', $\rho(A) = 2$

Rank of $[A:B]$, $\rho[A:B] = 3$

Here, $\rho(A) \neq \rho[A:B]$

$\therefore$ The System is inconsistent.

② Solve the system and test the consistency :

$x_1 - 3x_3 = 8 \quad 2x_1 + 2x_2 + 9x_3 = 7 \quad x_2 + 5x_3 = -2$

Sol^n:- The matrix form is

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix}$$

Augmented Matrix,

$$[A:B] = \begin{bmatrix} \boxed{1} & 0 & -3 & : & 8 \\ 2 & 2 & 9 & : & 7 \\ 0 & 1 & 5 & : & -2 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$[A:B] = \begin{bmatrix} \boxed{1} & 0 & -3 & : & 8 \\ 0 & \boxed{2} & 15 & : & -9 \\ 0 & 1 & 5 & : & -2 \end{bmatrix} \quad R_3 \to R_3 - \frac{R_2}{2}$$

$$[A:B] = \begin{bmatrix} 1 & 0 & -3 & : & 8 \\ 0 & 2 & 15 & : & -9 \\ 0 & 0 & -5/2 & : & 5/2 \end{bmatrix}$$

$$\rho(A) = 3 \qquad \rho[A:B] = 3$$

Here, $\rho(A) = \rho[A:B] = 3$

$\therefore$ The system is consistent.

Here, we have a unique solution, because

Rank of $A = 3 = \underline{\text{No}}$ of variables

The equations from matrix are,

$$x_1 - 3x_3 = 8$$

$$2x_2 + 15x_3 = -9$$

$$-5/2\, x_3 = 5/2$$

$$\therefore x_3 = -1$$

By back substitution

$$-x_1 - 3(-1) = 8 \qquad\qquad 2x_2 + 15(-1) = -9$$

$$x_1 = 5 \qquad\qquad x_2 = \frac{-9+15}{2}$$

$$x_2 = 3$$

$\therefore$ Solution is $x_1 = 5,\ x_2 = 3,\ x_3 = -1$

③ Test the consistency and solve the system:

$4x - 2y + 6z = 8 \quad x + y - 3z = -1 \quad 15x - 3y + 9z = 21$

The matrix form is,

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

Augmented matrix

$$[A:B] = \begin{bmatrix} 4 & -2 & 6 & : & 8 \\ 1 & 1 & -3 & : & -1 \\ 15 & -3 & 9 & : & 21 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$[A:B] = \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 4 & -2 & 6 & : & 8 \\ 15 & -3 & 9 & : & 21 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 15R_1 \end{matrix}$$

$$[A:B] = \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -6 & 18 & : & 12 \\ 0 & -18 & 54 & : & 36 \end{bmatrix} \quad R_3 \to R_3 - 3R_2$$

$$= \begin{bmatrix} 1 & 1 & -3 & : & -1 \\ 0 & -6 & 18 & : & 12 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\rho(A) = 2 \qquad \rho[A:B] = 2$

Since, $\rho(A) = \rho[A:B]$

The system is consistent

But, since $r = 2 < n = 3$

The system has infinite Solutions.

→ Discuss the nature of following system for different values of λ and μ.

$$2x + 3y + 5z = 9$$
$$7x + 3y - 2z = 8$$
$$3x + 3y + \lambda z = \mu$$

The corresponding augmented matrix is

$$[A:B] = \begin{bmatrix} \boxed{2} & 3 & 5 & : & 9 \\ 7 & 3 & -2 & : & 8 \\ 3 & 3 & \lambda & : & \mu \end{bmatrix} \quad \begin{matrix} R_2 \to \frac{7}{2}R_1 - R_2 \\ R_3 \to 3/2\, R_1 - R_3 \end{matrix}$$

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & 15/2 & 39/2 & : & \frac{47}{2} \\ 0 & 1/2 & \frac{15}{2} - \lambda & : & \frac{27}{2} - \mu \end{bmatrix} \quad \begin{matrix} R_3 \to 2R_3 \\ R_2 \to 2R_2 \end{matrix}$$

$$[A:B] = \begin{bmatrix} 2 & 3 & 5 & : & 9 \\ 0 & 15 & 39 & : & 47 \\ 0 & 1 & 15-2\lambda & : & 27-2\mu \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{2} & 3 & 5 & 9 \\ 0 & \boxed{15} & 39 & 47 \\ 0 & 3 & 15-2\lambda & 27-2\mu \end{array}\right] R_3 \to \frac{R_2}{5} - R_3$$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & \frac{10\lambda-36}{5} & \frac{10\mu-358}{5} \end{array}\right] R_3 \to R_3 \times 5$$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & 3 & 5 & 9 \\ 0 & 15 & 39 & 47 \\ 0 & 0 & 10\lambda-36 & 10\mu-88 \end{array}\right]$$

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{2} & 3 & 5 & 9 \\ 0 & \boxed{15} & 39 & 47 \\ 0 & 0 & 10\lambda-36 & 10\mu-88 \end{array}\right]$$

Case ①:- Consistent

The system will have infinite solution when,

$r < n(3)$

i.e $r = 2$

For $r = 2$, $10\lambda - 36 = 0$ & $10\mu - 88 = 0$

$\lambda = \frac{18}{5}$ $\mu = \frac{44}{5}$

~~Case~~

The System will have unique solution when,

$r = n(3)$

i.e $r = 3$

For $r = 3$, $10\lambda - 36 \neq 0$ & $10\mu - 88 = 0$ (or) $10\mu - 88 \neq 0$

The system will not have any solution when

$$\rho(A) \neq \rho(A:B)$$

$$\therefore \quad 10\lambda - 36 = 0 \text{ and } 10\lambda - 88 \neq 0$$

$\Rightarrow$ General Solution of a system of linear equation :-

The general solution of a system is the solution that gives an explicit description of all solution.

(1) Find the general solution of linear system whose augmented matrix has been reduced to :

$$[A:B] = \begin{bmatrix} \boxed{1} & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & \boxed{2} & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix} \quad R_2 \to R_2/2$$

Row Echelon form [REF]

$$[A:B] = \begin{bmatrix} \boxed{1} & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & \boxed{1} & -4 & -1/2 & 3/2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix} \quad R_1 \to R_1 - 2R_2$$

$$[A:B] = \begin{bmatrix} \boxed{1} & 6 & 0 & +3 & -1 & -7 \\ 0 & 0 & \boxed{1} & -4 & -1/2 & +3/2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_3/2 \\ R_1 \to R_1 + R_3 \end{matrix}$$

$$[A:B] = \begin{bmatrix} \boxed{1} & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$

Pivot Positions :- 1, 3, 5

$\therefore$ Basic variables = x_1, x_3, x_5

Free variables = x_2, x_4

The General Solution is,

$$x_1 + 6x_2 + 3x_4 = 0$$

$$x_3 - 4x_4 = 5$$

$$x_5 = 7$$

$$\Rightarrow \quad x_1 = -6x_2 - 3x_4$$

$$x_3 = 5 + 4x_4$$

$$x_5 = 7$$

$\left.\begin{matrix} x_2 = \text{free} \\ x_4 = \text{free} \end{matrix}\right\}$ They can be given any arbitrary value

(2) Find the general solution of the system :-

$$x_1 - 2x_2 - x_3 + 3x_4 = 0$$

$$-2x_1 + 4x_2 + 5x_3 - 5x_4 = 3$$

$$3x_1 - 6x_2 - 6x_3 + 8x_4 = 2$$

The augmented Matrix is given by

$$[A : B] = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$[A : B] = \begin{bmatrix} \boxed{1} & -2 & -1 & 3 & 0 \\ 0 & 0 & \boxed{3} & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix} \quad R_3 \to R_2 + R_3$$

$$[A : B] = \begin{bmatrix} \boxed{1} & -2 & -1 & 3 & 0 \\ 0 & 0 & \boxed{3} & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad R_2 \to R_2/3$$

Basic variables $= x_1, x_3$

Free variables $= x_2, \cancel{x_4}$ $\quad x_4 \to$ cannot be decided

$$[A : B] = \begin{bmatrix} \boxed{1} & -2 & -1 & 3 & 0 \\ 0 & 0 & \boxed{1} & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$[A : B] = \begin{bmatrix} \boxed{1} & -2 & 0 & 10/3 & 1 \\ 0 & 0 & \boxed{1} & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

The General Solution, is not possible because the system is inconsistent

$\because \rho[A] \neq \rho[A : B]$

The presence of free variable is irrelevant.

(3) In the following, determine the value of 'h' such that the matrix is augmented matrix of a consistent linear equation. The augmented matrix.

$$[A:B] = \begin{bmatrix} \boxed{1} & h & : & 4 \\ 3 & 6 & : & 8 \end{bmatrix} \quad R_2 \to R_2 - 3R_1$$

$$[A:B] = \begin{bmatrix} \boxed{1} & h & : & 4 \\ 0 & \boxed{6-3h} & : & -4 \end{bmatrix}$$

Here, in the above Echelon form

$$\rho[A] = \rho[A:B]$$

since, the system is consistent

$$\therefore \quad 6-3h \neq 0$$

$$\therefore \quad \underline{h \neq 2}$$

(4) For what values of h and K is the following system consistent

$$2x_1 - x_2 = h$$

$$-6x_1 + 3x_2 = K$$

The Augmented Matrix is

$$[A:B] = \begin{bmatrix} 2 & -1 & : & h \\ -6 & 3 & & K \end{bmatrix} \quad R_2 \to R_2 + 3R_1$$

$$[A:B] = \begin{bmatrix} 2 & -1 & : & h \\ 0 & 0 & : & K+3h \end{bmatrix}$$

Here,

$$\rho[A] = \rho[A:B]$$

Since, System is consistent

$$K + 3h = 0$$

$$\underline{K = -3h}$$

$$\underline{h = -\frac{K}{3}}$$

→ **Vector Equation :-**

A vector equation is an equation involving 'n' number of vectors, more formally it can be defined as an equation involving a linear combination of vectors with possibly unknown coefficient.

Upon solving it gives a vector in return.

⇒ Vectors in R^2 :-

A matrix with only one column is called a Column Vector (or) simply a vector.

$$u = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \quad v = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

* The set of all vectors with two entries is denoted by R^2 i.e $R^2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \mid a_1, a_2 \in R \right\}$ where, $R \to$ Real numbers that appear as entries in the vector and the exponent '2' indicates that the vectors each contain two entries.

Note:-
* Two vectors in R^2 are equal if and only if their corresponding entries are equal.

$$v = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \neq u = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

* We say that vectors in R^2 are ordered pairs of real numbers.

$$v = \begin{bmatrix} 4 \\ 7 \end{bmatrix} = (4, 7)$$

→ Sum of two vectors :-

Given two vectors v, u ~~and~~ in R^2, their sum is a vector $v + u$ obtained by adding corresponding entries of v and u.

$$v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad u = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$v + u = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

→ Scalar Multiplication :-

Given a vector u, and a scalar (real number) 'c' then the scalar multiple of 'u' by 'c' is a vector

$$u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$c = 8$$

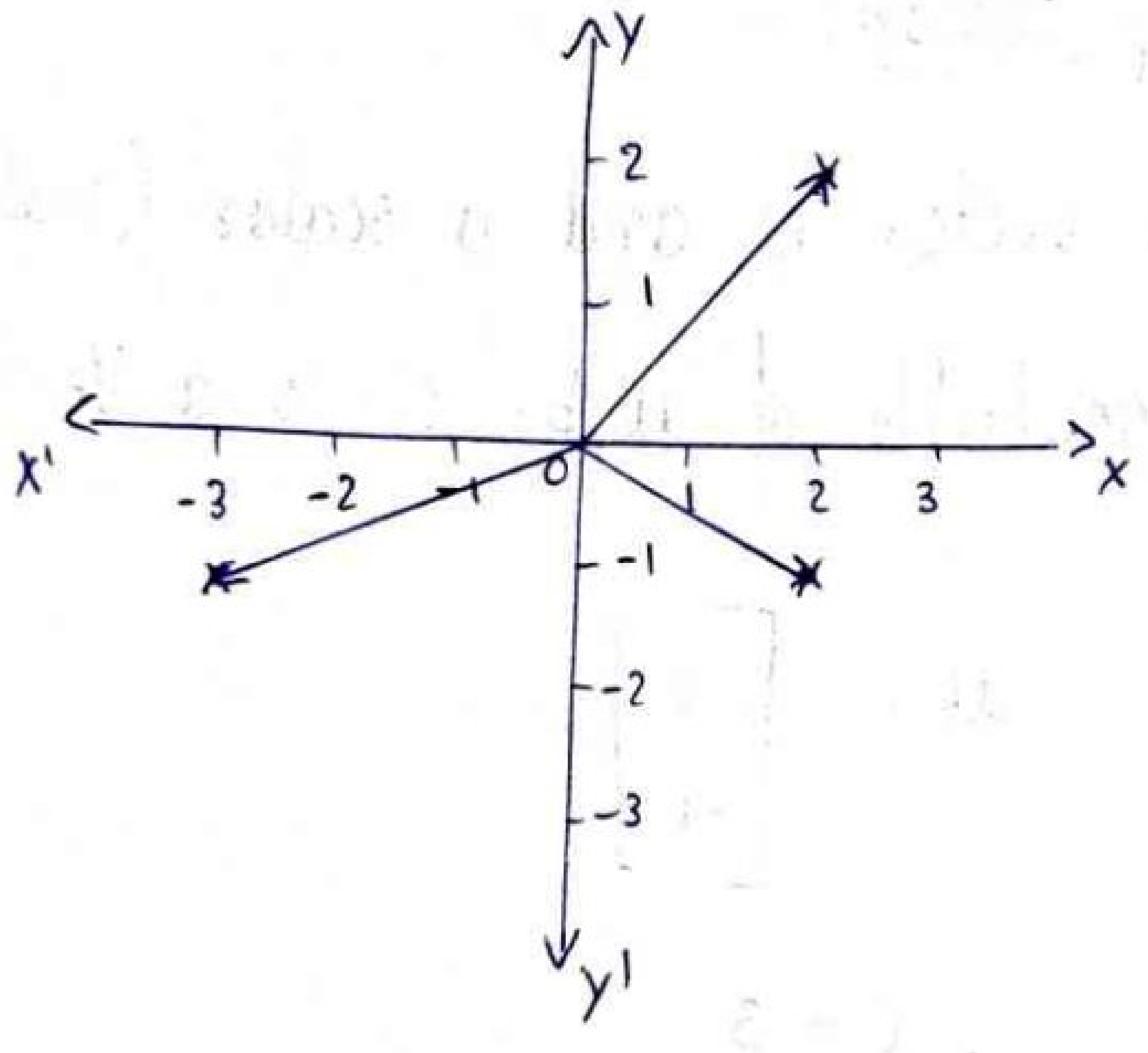
$$c \cdot u = \begin{bmatrix} 16 \\ -8 \end{bmatrix}$$

⇒ Geometric Descriptions of $\mathbb{R}^2$:-

① Vectors as points :-

Consider, a rectangular coordinate system in a Plane since each point in the plane is determined by an ordered pair of number. We can identify a geometric point (a, b) with a column vector $\begin{bmatrix} a \\ b \end{bmatrix}$ i.e

$$u = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad w = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$



② Vectors as Arrows :-

A line segment with arrow is drawn to the Vector Point from origin.

⇒ Parallelogram Rule of Vectors:-

If u and v in R^2 are represented as points in the plane $u+v$ corresponds to the fourth vertex of Parallelogram whose other vertices are $(0,0)$, u, v

→ Vector in R^3 :-

Vectors in R^3 are ~~are~~ 3 by 1 column matrix with three entries i.e

$$R^3 = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \ a, b, c \in R \right\}$$

Ex:- If $u = (2, 3, 4)$ and $2u = (4, 6, 8)$ Draw the 3-dimensional figure.

⇒ Vectors in R^n :-

If 'n' is a positive integer, R^n denotes the collection of all 'n' real numbers (or) ordered tuples usually denoted as $n \times 1$ column matrix.

$$R^n = \left\{ \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \middle| \, u_i \in R, \, \underset{1 \le i \le n}{i \in N} \right\}$$

→ Algebraic Properties of R^n :-

For all u, v, w in R^n and all scalars c & d :-

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. $u + 0 = 0 + u = u$
4. $u + (-u) = -u + u = 0$
5. $c(u + v) = cu + cv$
6. $(c + d)u = cu + du$
7. $c(du) = (cd)u$
8. $1 \cdot u = u$

⇒ Linear Combinations :-

Given vectors $v_1, v_2, \ldots v_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector 'y' defined by

$$y = c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots + c_p v_p$$

is called Linear Combination of $v_1, v_2, \ldots v_p$ with weights $c_1, c_2, \ldots c_p$

⇒ Span :-

* If the vectors $v_1, v_2, \ldots, v_p$ in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, v_2, \ldots, v_p$ is denoted by span of $v_1, v_2, \ldots v_p$

$$\text{Span}\{v_1, v_2, \ldots v_p\}$$

* If $S = \{v_1, v_2, \ldots v_p\}$ is a set of vectors in a vector space V, then the span of the set S is the linear combination of vectors in S

$$\text{Span}(S) = \{c_1 v_1 + c_2 v_2 + \ldots + c_p v_p \mid c_i \in \mathbb{R}, 1 \le i \le n\}$$

* Every vector In a given vector space can be written as a linear combination of vectors in a given set 'S', then 'S' is called a spanning set of the vector space.

⇒ Remarks :-

(i) Span $\{v_1, v_2, \dots v_n\}$ contains infinitely many vectors

(ii) Zero vector is in span$\{v_1, v_2, \dots, v_n\}$, since the zero vector can be expressed as a liner combination of $v_1, v_2, \dots v_n$

(iii) Any vector $b \in$ span$\{v_1, v_2, \dots v_p\}$, this implies that the vector equation $x_1v_1 + x_2v_2 + \dots + x_pv_p = b$ has a solution

(or)

A linear system with augmented matrix,
$[A : b] = \{v_1, v_2 \dots v_n : b\}$ has a solution

(1) Ex:- If possible, express $u = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ as a linear combination of vectors $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

The linear vector equation of u is given as,

$$u = c_1v_1 + c_2v_2$$

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ 2c_1 \end{bmatrix} + \begin{bmatrix} -c_2 \\ 4c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} c_1 - c_2 \\ 2c_1 + 4c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$\therefore \; c_1 - c_2 = -3 \qquad 2c_1 + 4c_2 = 7$

$c_1 = c_2 - 3 \qquad 2(c_2 - 3) + 4c_2 = 7$

$c_1 = \frac{13}{6} - 3 \qquad 2c_2 + 4c_2 - 6 = 7$

$6c_2 = 13$

$c_1 = -\frac{5}{6} \qquad c_2 = \frac{13}{6}$

Augmented matrix

$$\left[\begin{array}{cc:c} 1 & -1 & -3 \\ 2 & 4 & 7 \end{array}\right] R_2 \Rightarrow R_2/2$$

$$\left[\begin{array}{cc:c} 1 & -1 & -3 \\ 1 & 2 & 7/2 \end{array}\right] R_2 \Rightarrow R_2 - R_1$$

$$\left[\begin{array}{cc:c} 1 & -1 & -3 \\ 0 & 3 & \frac{13}{2} \end{array}\right] \Rightarrow \quad 3c_2 = \frac{13}{2} \quad c_2 = \frac{13}{6} \qquad c_1 - c_2 = -3 \quad \therefore c_1 = -\frac{5}{6}$$

$\therefore$ u can be expressed as

$$u = -\frac{5}{6} v_1 + \frac{13}{6} v_2$$

$$\begin{bmatrix} -3 \\ 7 \end{bmatrix} = -\frac{5}{6}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{13}{6}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} -5/6 \\ -10/6 \end{bmatrix} + \begin{bmatrix} -13/6 \\ +\frac{52}{6} \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$$

$\therefore$ LHS = RHS

$\therefore$ u can be expressed as linear ~~equation~~ combination of v_1 & v_2

② Let, $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine

wether b can be generated as a linear combination of a_1 & a_2

Determine wether weights x_1, x_2 exists such $x_1 a_1 + x_2 a_2 = b$

Soln:- The linear combination of b is written as

To prove:- $b \in \text{span}\{a_1, a_2\}$ $b = x_1 a_1 + x_2 a_2$

$$\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

The augmented matrix is given as,

$$\left[\begin{array}{cc:c} \boxed{1} & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \begin{array}{l} R_3 \to R_3 + 5R_1 \\ R_2 \to R_2 + 2R_1 \end{array}$$

$$\left[\begin{array}{cc:c} \boxed{1} & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right] R_2 \to \frac{R_2}{9},\ R_3 \to \frac{R_3}{16}$$

$$\left[\begin{array}{cc:c} \boxed{1} & 2 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 1 & 2 \end{array}\right] R_3 \to R_3 - R_2$$

$$\left[\begin{array}{cc:c} \boxed{1} & 2 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rho[A] = \rho[A:B]$$

$\therefore$ The system is consistent

Since, $n = r = 2$

The system has unique Solution

From the Echelon form

$$x_1 + 2x_2 = 7 \qquad x_2 = 2$$

By back substitution

$$x_1 + 4 = 7$$

$$x_1 = 3$$

$\therefore \quad b \in \text{span}\{a_1, a_2\}$

(3) For what values of 'h' will 'y' be in span of v_1, v_2, v_3 if $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$, $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$

For $y \in \text{span}\{v_1, v_2, v_3\}$

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = y$$

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 + 5x_2 - 3x_3 \\ -x_1 - 4x_2 + x_3 \\ -2x_1 - 7x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + 2R_1 \\ R_2 \to R_2 + R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 5 & -3 & -4 \\ 0 & \boxed{1} & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right] \quad R_3 \to R_3 - 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right]$$

$\therefore$ From the above Echelon form for system to be consistent

$$h - 5 = 0$$

$$\therefore h = 5$$

(4) Let, $a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ $a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$ $b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$, then span$\{a_1, a_2\}$ is a plane through the origin is b in that plane.

For, $b \in$ span$\{a_1, a_2\}$

Linear Combination is given as

$$x_1 a_1 + x_2 a_2 = b$$

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 5x_2 \\ -2x_1 - 13x_2 \\ -3x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix}$$

The augmented matrix is

$$[A:B] = \begin{bmatrix} \boxed{1} & 5 & : & -3 \\ -2 & -13 & : & 5 \\ 3 & -3 & : & 1 \end{bmatrix} \begin{matrix} \\ \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$= \begin{bmatrix} \boxed{1} & 5 & : & -3 \\ 0 & \boxed{-3} & : & 2 \\ 0 & -15 & : & 10 \end{bmatrix} R_3 \to R_3 - 6R_2$$

$$= \begin{bmatrix} \boxed{1} & 5 & : & -3 \\ 0 & \boxed{-3} & : & 2 \\ 0 & 0 & : & -2 \end{bmatrix}$$

From the above Row Echelon form

$$\rho[A] \neq \rho[A:B]$$

$\therefore$ The system of equation is inconsistent
i.e system has no solution

$\therefore$ 'b' $\notin$ span$\{a_1, a_2\}$

→ Matrix Equation :-

If 'A' is a $m \times n$ matrix with columns $a_1, a_2, \dots a_n$ and 'x' is in R^n then the product of A and x is denoted by Ax is the linear combination of the Column of 'A' using corresponding entries in 'x' as weights.

$$A = [a_1 \ a_2 \ a_3 \dots a_n] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\therefore \ Ax = a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n$$

① Given the system of equation, Write the vector equation and matrix equation.

$$x_1 + 2x_2 - x_3 = 4$$
$$-5x_2 + 3x_3 = 1$$

Vector Equation

$$\begin{array}{c} a_1 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} x_1 + \begin{array}{c} a_2 \\ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \end{array} x_2 + \begin{array}{c} a_3 \\ \begin{bmatrix} -1 \\ 3 \end{bmatrix} \end{array} x_3 = \begin{array}{c} b \\ \begin{bmatrix} 4 \\ 1 \end{bmatrix} \end{array}$$

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

Matrix Equation:-

$$Ax = b$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

⇒ Solution of Non-Homogeneous System:-

When A non homogeneous linear system has many solutions, then the general solution can be written in Parametric Vector form as one vector plus an arbitrary linear combination of vectors.

i.e Parametric vector form of the solution is written as

$$X = P + [a_1 x_1 + a_2 x_2 + a_3 x_3]$$

① Write all the solutions of $Ax = B$ where $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$

$$b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

The augmented matrix is,

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{3} & +5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} \boxed{3} & 5 & -4 & 7 \\ 0 & \boxed{3} & 0 & 6 \\ 0 & -9 & 0 & -18 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/3 \\ R_3 \to R_3/9 \end{array}$$

$$= \left[\begin{array}{ccc:c} \boxed{3} & 5 & -4 & 7 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & -1 & 0 & -2 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$= \left[\begin{array}{ccc:c} \boxed{3} & 5 & -4 & 7 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - 5R_2$$

$$= \left[\begin{array}{ccc:c} \boxed{3} & 0 & -4 & -3 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1/3$$

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{1} & 0 & -4/3 & -1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the above, Row Reduced Echelon form

Basic Variables :- x_1 & x_2

Free variable : x_3

$\therefore$ General Solution is written as

$$x_1 - \frac{4}{3}x_3 = -1 \Rightarrow x_1 = \frac{4}{3}x_3 - 1$$

$$x_2 = 2 \qquad x_2 = 2$$

$$x_3 = \text{free}$$

General Sol^n

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix}$$

Parametric Vector form :-

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\text{Vector}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\text{Arbitrary linear Combinations.}}$$

$$X = p + x_3 v \quad \text{where, } v \in R$$

~~Linear~~ Independence ~~and Independence~~

⇒ Solution set of Homogeneous and Non Homogeneous System :-

→ Solution Set of Linear System :-

* A system of linear equation is said to be homogeneous if it can be written in form of $Ax = 0$ where A is a $m \times n$ matrix and '0' is a zero-vector in R^n. Such a System $Ax = 0$ always has one solution, $x = 0$ (zero vector in R^n). This zero solution is usually called the Trivial Solution.

* The Homogeneous equation is said to have a non trivial Solution iff the system has atleast one free variable

⇒ Linear Dependence and Independence :-

* A set of vector $\{v_1, v_2, v_3 \dots v_p\}$ in R^n, the set said to be linearly Independent if the vector equation $x_1v_1 + x_2v_2 + \dots x_pv_p$ ① has only the Trivial Solution i.e these weights $x_1 = 0, x_2 = 0, \dots x_p = 0$.

* If the vector set $\{v_1, v_2, v_3 \dots, v_p\}$ in R^n is said to be Linearly Dependent if the vector equation ① has a non trivial Solution i.e it has atleast one $x_i \neq 0$

* The column of matrix 'A' are linearly independent iff the equation $Ax = 0$ has a trivial solution

* If there are two vectors in the vector set $\{v_1, v_2\}$ is said to be linearly independent if neither of vector are multiple of others

* It is linearly dependent if atleast one of the vector is a multiple of other.

⇒ Characteristation of Linearly Dependent Set

A set of vector $S = \{v_1, v_2, \ldots v_p\}$ if two or more vectors are linear dependent iff atleast one of the vector in S is a linear combination of the other

① Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

(i) Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent

(ii) If possible find, the linear dependence relation among the vectors v_1, v_2, v_3.

① Let the vector equation be,

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 4x_2 + 2x_3 \\ 2x_1 + 5x_2 + x_3 \\ 3x_1 + 6x_2 + 0x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix is given as.

$$[A:0] = \left[\begin{array}{ccc:c} \boxed{1} & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$[A:0] = \left[\begin{array}{ccc:c} \boxed{1} & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \quad R_3 \to -\frac{R_3}{6}, \; R_2 \to -\frac{R_2}{03}$$

$$[A:0] = \left[\begin{array}{ccc:c} \boxed{1} & 4 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$[A:0] = \left[\begin{array}{ccc:c} \boxed{1} & 4 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Basic Variable :- x_1, x_2

Free Variable :- x_3

The system has a free varaibe, So it has a non-trivial solution.

Hence, The system of vector set is linearly dependent

$$[A : 0] = \left[\begin{array}{ccc:c} \boxed{1} & 4 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - 4R_2$$

$$[A : 0] = \left[\begin{array}{ccc:c} \boxed{1} & 0 & -2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \text{Row reduced Echelon form}$$

The General Solution is

$$x_1 - 2x_3 = 0$$

$$\Rightarrow x_1 = 2x_3$$

$$x_2 + x_3 = 0$$

$$\Rightarrow x_2 = -x_3$$

$$x_3 = \text{free}$$

Choose $x_3 = 1$

$$x_1 = 2 \quad x_2 = -1 \quad x_3 = 1$$

$\therefore$ The linear dependence relation of $\{v_1, v_2, v_3\}$

$$2v_1 - v_2 + v_3 = 0$$

H.W

(2) Let, $v_1 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix}$

For what values of h

(i) v_3 in span of $\{v_1, v_2\}$

(ii) $\{v_1, v_2, v_3\}$ are linearly dependent.