(1) Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$. Diagonalize matrix A, if possible

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= 3-\lambda[(3-\lambda)^2 - 1] - 1[(3-\lambda) - 1] + 1[1-(3-\lambda)]$$

$$= 3-\lambda(9+\lambda^2 - 6\lambda - 1) - 2 + \lambda - 2 + \lambda$$

$$= 24 + 3\lambda^2 - 18\lambda - 8\lambda - \lambda^3 + 6\lambda^2 - 4 + 2\lambda$$

$$= -\lambda^3 + 9\lambda^2 - 24\lambda + 20$$

The Characteristic Polynomial is

$$|A - \lambda I| = 0$$

$$-\lambda^3 + 9\lambda^2 - 24\lambda + 20 = 0$$

$\therefore$ By solving above equation

$$\lambda = 5,\ 2,\ +2$$

For, $\lambda = 5$,

$$A - \lambda I = A - 5I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Augmented Matrix:-

$$[A - \lambda I : 0] = \left[\begin{array}{ccc:c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + \frac{R_1}{2} \\ R_2 \to R_2 + \frac{R_1}{2} \end{array}$$

$$= \left[\begin{array}{ccc:c} \boxed{-2} & 1 & 1 & 0 \\ 0 & \boxed{-\frac{3}{2}} & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\sim \left[\begin{array}{ccc:c} \boxed{-2} & 1 & 1 & 0 \\ 0 & \boxed{-3/2} & 3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1/-2 \\ R_2 \to R_2 \times -\frac{2}{3} \end{array}$$

$$\sim \left[\begin{array}{ccc:c} \boxed{+1} & -1/2 & -1/2 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 + \frac{R_2}{2}$$

$$\sim \left[\begin{array}{ccc:c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \quad x_1, x_2 \rightarrow$ Basic $\quad x_3 \rightarrow$ Free

$$\therefore \quad \begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_3 &= \text{free} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= x_3 \\ x_2 &= x_3 \\ x_3 &= \text{free} \end{aligned}$$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \|v_1\| = \sqrt{3}$$

$$u_1 = \frac{v_1}{\|v_1\|} \Rightarrow u_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Now, for $\lambda = 2$

$$A - \lambda I = A - 2I = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Augmented Matrix.

$$A - 2I = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Basic $\to x_1$ Free $\to x_2, x_3$

General Solution, $x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$

$x_2 = \text{free}$ $\qquad x_2 = \text{free}$

$x_3 = \text{free}$ $\qquad x_3 = \text{free}$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \|V_2\| = \sqrt{2} \quad \|V_3\| = \sqrt{2}$$

$$U_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad U_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$P = [u_1 \; u_2 \; u_3]$

$$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then, P orthogonally diagonalizes A and

$$D = PAP^{T} = PAP^{-1}$$

② Orthogonally diagonalize the matrix $A = \begin{matrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{matrix}$

$$|A - \lambda I| = \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix}$$

$$= (6-\lambda)\left[(6-\lambda)(5-\lambda) - 1\right] - (-2)\left[-10 + 2\lambda - 1\right] - 1\left(2 + 6 - \lambda\right)$$

$$= (6-\lambda)\left[30 - 6\lambda - 5\lambda + \lambda^2 - 1\right] + 2\left[-11 + 2\lambda\right] - (+8 - \lambda)$$

$$= (6-\lambda)(\lambda^2 - 11\lambda + 29) - 22 + 4\lambda + 4 + \lambda$$

$$= 6\lambda^2 - 66\lambda + 174 - \lambda^3 + 11\lambda^2 - 29\lambda - 22 - 8 + 5\lambda$$

$$= -\lambda^3 + 17\lambda^2 - 90\lambda + 144$$

$\therefore$ The Characteristic Polynomial is,

$$|A - \lambda I| = 0$$

$$\lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$$

$\therefore$ By Solving above equation

$$\lambda = 8, 6, 3$$

For $\lambda = 8$,

$$A - 8I = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix} - \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1/2 \end{matrix}$$

$$= \begin{bmatrix} -2 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & -5/2 \end{bmatrix} \quad \begin{matrix} R_3 \to -2/5\, R_3 \\ R_1 \to R_1 \times \left(-\frac{1}{2}\right) \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & +1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 \to R_1 - \frac{R_2}{2}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Basic $\to x_1, x_3$ $\qquad$ Free $\to x_2$

$\therefore$ $x_1 + x_2 = 0$ $\qquad$ $x_2 =$ free $\qquad$ $x_3 = 0$

$x_1 = -x_2$

$$\therefore \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \quad V_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \|V_1\| = \sqrt{2}$$

$$u_1 = \frac{V_1}{\|V_1\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

For, $\lambda = 6$,

$$A - 6I = \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

Augmented Matrix,

$$A - 6I : 0 \quad \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \qquad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} -1 & -1 & -1 \\ -2 & 0 & -1 \\ 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & -1 \\ 0 & -2 & -1 \end{bmatrix} \qquad R_2 \rightarrow R_2 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 \rightarrow R_1 - \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

The General Solution,

$$x_1 + \frac{x_3}{2} = 0 \qquad x_1 = -\frac{x_3}{2}$$

$$x_2 + \frac{x_3}{2} = 0 \qquad x_1 = -\frac{x_3}{2}$$

$$x_3 = \text{free} \qquad x_3 = \text{free}$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3/2 \\ -x_3/2 \\ +x_3 \end{bmatrix}$$

$$\therefore x = x_3 \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

Choose $x_3 = 2$

$$\therefore x = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore V_a = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \qquad \|V_2\| = \sqrt{6}$$

$$u_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Augmented Matrix,

$$A - 3I : 0 = \left[\begin{array}{ccc:c} 3 & -2 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc:c} -1 & -1 & 2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ -2 & 3 & -1 & 0 \\ 3 & -2 & -1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2 \quad \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to \frac{R_2}{5}$$

$$A - 3I : 0 = \left[\begin{array}{ccc:c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1, x_2 \to$ Basic $\quad x_3 =$ free

The General Solution is,

$$x_1 - x_3 = 0 \qquad x_1 = x_3$$

$$x_2 - x_3 = 0 \qquad x_2 = x_3 \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

$$x_3 = \text{free} \qquad x_3 = \text{free}$$

$$x = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \therefore V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \|V_3\| = \sqrt{3}$$

$$\therefore u_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$P = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

This matrix P diagonalizes A,

$\therefore \; A = PDP^{-1}$

③ Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ if possible

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\left[(5-\lambda)(1-\lambda)+9\right] - 3\left((-3)(1-\lambda)+9\right) + 3\left(-9-3(-5-\lambda)\right)$$

$$= (1-\lambda)\left(-5+4\lambda+\lambda^2+9\right] - 3(-3+3\lambda+9) + 3(-9+15+3\lambda)$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) - \cancel{9\lambda} - \cancel{15} + \cancel{15} + \cancel{9\lambda} \cancel{+ \lambda^2 + 4\lambda + 4}$$

$$= \lambda^2 - 4\lambda + 4 - \lambda^3 - 4\lambda^2 - 4\lambda$$

$$= -\lambda^3 - 3\lambda^2 + 4$$

Characteristic Polynomial, $A - \lambda I = 0 \Rightarrow \lambda^3 + 3\lambda^2 - 4 = 0$

$\therefore \; \lambda = 1, -2, -2$

For $\lambda = 1$, form

$$A - \lambda I = A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

The Augmented Matrix

$$A - I : 0 \quad \left[\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -3 & -6 & -3 & 0 \\ 3 & 3 & 0 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$A - I : 0 \quad \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$$\left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \quad R_1 \to \frac{R_1}{3},\ R_2 \to \frac{R_2}{-3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 - R_2$$

$$(A - I : 0) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Basic :- x_1, x_2

Free :- x_3

$$x_1 - x_3 = 0 \Rightarrow x_1 = x_3$$

$$x_1 + x_3 = 0 \Rightarrow x_2 = -x_3$$

The General Solution is,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Choosing, $x_3 = 1$, we get $v_1 = \begin{vmatrix} 1 \\ -1 \\ 1 \end{vmatrix}$ $\quad \|v_1\| = \sqrt{3}$

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

For $\lambda = -2$

$$A - \lambda I = A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix}$$

Augmented Matrix

$$(A + 2I) = \left[\begin{array}{ccc:c} 3 & 3 & 3 & 0 \\ -3 & -3 & -3 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\therefore \quad A+2I = \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 \rightarrow$ Basic, $x_3, x_2 \rightarrow$ free

$$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -x_2 - x_3$$

General Solution,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \ \& \ \|v_2\| = \sqrt{2} \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \ \|v_3\| = \sqrt{2}$$

$$u_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

$$\therefore \quad P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -1/\sqrt{2} & -1 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

A is diagnolizable $\therefore$ $A = PDP^{-1}$

④ Make a change of variable $x = Py$, that transforms the Quadratic form $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$ into quadratic form with no cross product.

Given, $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$

$$A = \begin{array}{cc} & \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x \end{array} & \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix} \end{array}$$

The matrix of Quadratic form is

$$A = \begin{bmatrix} 1 & -4 \\ -4 & -5 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & -4 \\ -4 & -5-\lambda \end{bmatrix}$$

$$= (1-\lambda)(-5-\lambda) + (-16)$$

$$= \lambda^2 + 4\lambda - 21$$

The characteristic Polynomial is

$$A - \lambda I = 0 \Rightarrow \lambda^2 + 4\lambda - 21 = 0$$

$$\therefore \lambda^2 + 3\lambda + 7\lambda - 21 = 0$$

$$\lambda(\lambda \mp 3) + 7(\lambda - 3) = 0$$

$$(\lambda - 3)(\lambda + 7) = 0$$

$$\lambda = 3, -7$$

For $\lambda = 3$

$$A - \lambda I = A - 3I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix}$$

Augmented Matrix

$$A - 3I : 0 = \begin{bmatrix} -2 & -4 & : & 0 \\ -4 & -8 & : & 0 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$= \begin{bmatrix} -2 & -4 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \quad R_1 \to \frac{R_2}{-2} \Rightarrow \begin{bmatrix} 1 & 2 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

$\therefore$ Basic $\to x_1$, Free $\to x_2$

$$x_1 + 2x_2 = 0 \qquad x_1 = -2x_2$$

The General solution, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Corresponding Eigen vector, $v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda = -7$,

$$A - \lambda I = A + 7I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}$$

Augmented Matrix

$$A + 7I : 0 = \begin{bmatrix} 8 & -4 & : & 0 \\ -4 & 2 & : & 0 \end{bmatrix} \quad R_2 \to R_2 + \frac{R_1}{2} \sim \begin{bmatrix} 8 & -4 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \quad R_1 \to \frac{R_1}{8}$$

$$A+7I:0 \quad \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Basic $\rightarrow x_1$
Free $\rightarrow x_2$

$\therefore \quad x_1 - \frac{x_2}{2} = 0 \qquad x_1 = \frac{x_2}{2}$

General Solution, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2/2 \\ x_2 \end{bmatrix} \quad x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

choose $x_2 = 2 \quad \therefore \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Compute, $v_1 \cdot v_2 = (-2)(1) + (1)(2) = -2+2 = 0$

$\therefore \{v_1, v_2\}$ is an orthogonal set

Normalize the vectors v_1 and v_2

$\|v_1\| = \sqrt{2^2+1^2} = \sqrt{5} \quad \|v_2\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\therefore \quad u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \qquad u_2 = \frac{v_2}{\|v_2\|} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

Now, $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \qquad P = [u_1 \; u_2] \quad \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

Then, $A = PDP^{-1} \Rightarrow D = P^{-1}AP = P^TAP$

The desired change of variable is $x = Py$ and the new Quadratic form is

$$x^TAx = (Py)^TA(Py)$$
$$= y^TP^TAPy$$
$$= y^T(P^TAP)y$$
$$= y^TDy$$
$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 3y_1 \\ -7y_2 \end{bmatrix}$$
$$= 3y_1^2 - 7y_2^2$$

(5) Make a change of variable $x = Py$, that transforms the Quadratic form $9x_1^2 - 8x_1x_2 + 3x_2^2$ into a Quadtratic form with no cross product term. Give P and new Quadratic form.

Given, $Q(x) = 9x_1^2 - 8x_1x_2 + 3x_2^2$

$$A = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_1 & 9 & -4 \\ x_2 & -4 & 3 \end{array}$$

The Matrix of Quodtratic form is

$$A = \begin{bmatrix} 9 & -4 \\ -4 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 9-\lambda & -4 \\ -4 & 3-\lambda \end{bmatrix}$$

$$= (9-\lambda)(3-\lambda) - 16$$

$$= 27 - 12\lambda + \lambda^2 - 16$$

$$= \lambda^2 - 12\lambda + 11$$

The characteristic Polynomial is, $A - \lambda I = 0$

$$\lambda^2 - 12\lambda + 11 = 0$$

$$\lambda^2 - \lambda - 11\lambda + 11 = 0$$

$$\lambda(\lambda - 1) - 11(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda - 11) = 0$$

$$\therefore \lambda = 1, 11$$

For, $\lambda = 11$, form

$$A - \lambda I = A - 11I = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix}$$

The augmented matrix $\Rightarrow (A - 11I : 0) = \begin{bmatrix} -2 & -4 & : & 0 \\ -4 & -8 & : & 0 \end{bmatrix}$ $R_1 \to \frac{R_1}{-2}$

$R_2 \to \frac{R_2}{-4}$

$$A - 1I : 0 = \begin{bmatrix} 1 & 2 & : & 0 \\ 1 & 2 & : & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$= \begin{bmatrix} 1 & 2 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \quad \begin{matrix} \text{Basic} \to x_1 \\ \text{Free} \to x_2 \end{matrix}$$

$$x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

$\therefore$ General Solution is, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} \quad x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$$\therefore \quad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$A - \lambda I = A - I = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}$$

The augmented Matrix $\Rightarrow A - I : 0 \begin{bmatrix} 8 & -4 & : & 0 \\ -4 & 2 & : & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1/8 \\ R_2 \to R_2/-4 \end{matrix}$

$$A - I : 0 \Rightarrow \begin{bmatrix} 1 & -1/2 & : & 0 \\ 1 & -1/2 & : & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$= \begin{bmatrix} 1 & -1/2 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

Basic $\to x_1$ Free $\Rightarrow x_2$

$$x_1 - \frac{x_2}{2} = 0 \Rightarrow x_1 = \frac{x_2}{2}$$

The General Solution, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_2}{2} \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

choose, $x_2 = 2$ $\therefore$ $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Compute: $v_1 \cdot v_2 = -2 \times 1 + 1 \times 2 = 0$

Now To normalize v_1 and v_2

$$||v_1|| = \sqrt{-2^2 + 1^2} = \sqrt{5}$$

$$||v_2|| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\therefore\ u_1 = \frac{v_1}{||v_1||} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \qquad u_2 = \frac{v_2}{||v_2||} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\therefore\ D = \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = [u_1\ u_2] \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Then, $A = PDP^{-1}$

$$D = P^{-1}AP = P^{T}AP \quad (\because P^T = P^{-1})$$

The desired change of variable is $x = Py$, and the new Quadratic form is.

$$Q(x) = x^T A x = (Py)^T A (Py)$$

$$= y^T P^T A P y$$

$$= y^T (P^T A P) y$$

$$= y^T D y$$

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 11 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 11y_1 \\ y_2 \end{bmatrix}$$

$$= 11y_1^2 + y_2$$

⑥ Define $T: P_3 \to R^4$ by $T(P) = \begin{bmatrix} P(-2) \\ P(3) \\ P(1) \\ P(0) \end{bmatrix}$

(i) Show that T is a linear Transformation

(ii) Find the matrix for T relative to the basis $\{1, t, t^2, t^3\}$ for P_3 and standard basis for R^4

(i) Let p and q be polynomials in P_3 and let c be any scalar, Then

$$T(p+q) = \begin{bmatrix} (p+q)(-2) \\ (p+q)(+3) \\ (p+q)(1) \\ (p+q)(0) \end{bmatrix} = \begin{bmatrix} p(-2) + q(-2) \\ p(3) + q(3) \\ p(1) + q(1) \\ p(0) + q(0) \end{bmatrix}$$

$$= \begin{bmatrix} p(-2) \\ p(3) \\ p(1) \\ p(0) \end{bmatrix} + \begin{bmatrix} q(-2) \\ q(3) \\ q(1) \\ q(3) \end{bmatrix}$$

$$\Rightarrow T(p+q) = T(p) + T(q)$$

$$T(c \cdot p) = \begin{bmatrix} (c \cdot p)(-2) \\ (c \cdot p)(3) \\ (c \cdot p)(1) \\ (c \cdot p)(0) \end{bmatrix} = \begin{bmatrix} c \cdot p(-2) \\ c \cdot p(3) \\ c \cdot p(1) \\ c \cdot p(0) \end{bmatrix} = c \begin{bmatrix} p(-2) \\ p(3) \\ p(1) \\ p(0) \end{bmatrix}$$

$$\therefore T(c \cdot p) = c \cdot T(p)$$

$\Rightarrow$ T is a Linear Transformation.

(ii) Let $B = \{1, t, t^2, t^3\}$ and $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ be standard basis of R^3.

$$[T(b_1)]_{\mathcal{E}} = T(b_1) = T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|T(b_2)|_{\mathcal{E}} = T(b_2) = T(t) = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

$$|T(b_3)|_{\mathcal{E}} = T(b_3) = T(t^2) = \begin{bmatrix} 4 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

$$|T(b_4)|_{\mathcal{E}} = T(b_4) = T(t^3) = \begin{bmatrix} -8 \\ 27 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Matrix T relative to B and $\mathcal{E}$ is

$$\left[[T(b_1)]_{\mathcal{E}} \; [T(b_2)]_{\mathcal{E}} \; [T(b_3)]_{\mathcal{E}} \; [T(b_4)]_{\mathcal{E}}\right] = \begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 3 & 9 & 27 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

(7) Show that 7 is an Eigen Value of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding Eigen vectors

Given, $\lambda = 7$

Consider, $(A - \lambda I)x = 0$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = 0$$

$$\left(\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = 0$$

Augmented Martrix

$$(A - \cancel{7}I : 0) = \begin{bmatrix} -6 & 6 & : & 0 \\ 5 & -5 & : & 0 \end{bmatrix} \quad \begin{matrix} R_1 \to \frac{-R_1}{6} \\ R_2 \to \frac{R_2}{5} \end{matrix} \sim \begin{bmatrix} +1 & -1 & : & 0 \\ 1 & -1 & : & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

$$= \begin{bmatrix} 1 & -1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix}$$

The System has trivial solution, 7 is an Eigen Value of A

$$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$$

General Solution, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to 7

⑧ Find Eigen value and corresponding Eigen vectors.

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

Consider,

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 4 \\ 3 & 2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(2-\lambda)(-1-\lambda)+1\right] + 1\left(3(-1-\lambda) - (-1)(2)\right)$$
$$+ 4\left(3 - (2-\lambda)(2)\right)$$

$$= (1-\lambda)(\lambda^2 - \lambda - 2 + 1) + (-3 - 3\lambda + 2) + 4(3 - 4 + 2\lambda)$$

$$= (1-\lambda)(\lambda^2 - \lambda - 1) + (-3\lambda - 1) - 4 + 8\lambda$$

$$= \lambda^2 - \lambda - 1 - \lambda^3 + \lambda^2 + \lambda - 3\lambda - 1 - 4 + 8\lambda$$

$$|A - \lambda I| = -\lambda^3 + 2\lambda^2 + 5\lambda - 6$$

$$|A - \lambda I| = 0 \Rightarrow \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$$

On solving above equation,

$$\lambda = -2, 3, 1$$

$$x_1 + x_3 = 0 \qquad \Rightarrow \qquad x_1 = -x_3$$

$$x_2 - x_3 = 0 \qquad\qquad x_2 = x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \qquad \therefore\ v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \|v_1\| = \sqrt{3}$$

$$\therefore\ u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Consider, for $\lambda = 3$

$$A - 3I = \begin{bmatrix} 1-3 & -1 & 4 \\ 3 & 2-3 & -1 \\ 2 & 1 & -1-3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix}$$

Augmented Matrix,

$$(A - 3I : 0) = \left[\begin{array}{ccc:c} -2 & -1 & 4 & 0 \\ 3 & -1 & -1 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 + R_1 \times \frac{3}{2} \\ R_3 \to R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} -2 & -1 & 4 & 0 \\ 0 & -5/2 & +5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad R_2 \to R_2 \times \frac{-2}{5}$$

For, $\lambda = -2$

$$A + 2I = \begin{bmatrix} 3 & -1 & 4 \\ 3 & 4 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

Augmented Matrix

$$A+2I:0 \quad \left[\begin{array}{ccc:c} 3 & -1 & 4 & 0 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - \frac{2}{3} R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 3 & -1 & 4 & 0 \\ 0 & 5 & -5 & 0 \\ 0 & \frac{5}{3} & -\frac{5}{3} & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/5 \\ R_3 \to R_3 \times \frac{3}{5} \end{array} \quad \left[\begin{array}{ccc:c} 3 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$(A+2I:0) = \left[\begin{array}{ccc:c} 3 & -1 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$$= \left[\begin{array}{ccc:c} 3 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore x_1, x_2 \to$ Basic and $x_3 \Rightarrow$ Free

$$\begin{bmatrix} -2 & -1 & 4 & : & 0 \\ 0 & 1 & -2 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$\begin{bmatrix} -2 & 0 & 2 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \quad R_1 \to -R_1/2$$

$$A - 2I : 0 \qquad \begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & 1 & -2 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$x_1, x_2 \to$ Basic

$x_3 \to$ Free

$$x_1 - x_3 = 0 \quad \Rightarrow \quad x_1 = x_3$$

$$x_2 - 2x_3 = 0 \qquad\quad x_2 = 2x_3$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \therefore V_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \qquad \|V_2\| = \sqrt{6}$$

$$u_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

For $\lambda = 1$

$$A - I = \begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$$

Augmented Matrix,

$$(A-I:0) = \left[\begin{array}{ccc:c} 0 & -1 & 4 & 0 \\ 3 & 1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{array}\right] \underset{R_1 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc:c} 2 & 1 & -2 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1 \times \frac{3}{2}$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & -2 & 0 \\ 0 & -1/2 & 2 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc:c} 2 & 1 & -2 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_1 \to R_1 + R_2}{\sim} \left[\begin{array}{ccc:c} 2 & 0 & 2 & 0 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 \times (-1) \\ R_1 \to R_{1/2} \end{array}$$

$$(A-I:0) = \left[\begin{array}{ccc:c} 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1, x_2 \to$ Basic

Free $\to x_3$

$x_1 + x_3 = 0 \quad x_2 - 4x_3 = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \quad \therefore V_3 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \quad \|V_3\| = \sqrt{18}$$

$$\therefore U_3 = \frac{V_3}{\|V_3\|} = \begin{bmatrix} -1/3\sqrt{2} \\ 4/3\sqrt{2} \\ 1/3\sqrt{2} \end{bmatrix}$$

$\therefore U_3$ is Eigen vector corresponding to $\lambda = 1$

9. Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Find the orthogonal projection of y onto u. Then write y as the sum of 2 orthogonal vectors, one in span $\{u\}$ and one orthogonal to u.

Orthogonal Projection of y onto u,

$$\hat{y} = \frac{\langle y \cdot u \rangle}{\langle u \cdot u \rangle} \cdot u = \frac{\begin{bmatrix} 7 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \frac{28+12}{16+4} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Rightarrow \hat{y} = \begin{matrix} 8 \\ 4 \end{matrix}$$

Orthogonal vector in span $\{u\} = z$

$$z = y - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} \Rightarrow z = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Consider, $z + \hat{y} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$$= \begin{bmatrix} 7 \\ 6 \end{bmatrix} = y$$

$\therefore$ $y = z + \hat{y}$ and $z, \hat{y}$ are orthogonal

(10) Let, $u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $x = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

Show that $\{u_1, u_2, u_3\}$ is an orthogonal basis for R^3. Then express x as as a linear combination of u_1, u_2 & u_3

Consider,

$$u_1 \cdot u_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} = -1 + 0 + 1 = 0$$

$$u_2 \cdot u_3 = \begin{bmatrix} -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ +1 \\ -2 \end{bmatrix} = -2 + 4 - 2 = 0$$

$$u_3 \cdot u_1 = \begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2 - 2 = 0$$

Since, $\langle u_1 \cdot u_2 \rangle = \langle u_2 \cdot u_3 \rangle = \langle u_1 \cdot u_3 \rangle = 0$

$\Rightarrow \{u_1, u_2, u_3\}$ is an Orthogonal basis of R^3

Consider, $x = c_1 u_1 + c_2 u_2 + c_3 u_3$

$$c_1 = \frac{x \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 8 & -4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} = \frac{8 + 0 - 3}{1 + 0 + 1} = \frac{5}{2} = c_1$$

$$c_2 = \frac{x \cdot u_2}{u_2 \cdot u_2} = \frac{\begin{bmatrix} 8 & -4 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}} = \frac{-8 - 16 - 3}{1 + 16 + 1} = \frac{-3}{2} = c_2$$

$$c_3 = \frac{x \cdot u_3}{u_3 \cdot u_3} = \frac{\begin{bmatrix} 8 & -4 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}}{\begin{bmatrix} 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}} = \frac{18}{9} = 2 = c_3$$

$\therefore \quad x = \frac{5}{2} u_1 + \left(-\frac{3}{2}\right) u_2 + 2 u_3$

$$\begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 5/2 + 3/2 + 4 \\ 0 - \frac{3}{2} \times 4 + 4 \\ \frac{5}{2} - \frac{3}{2} - 4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$$

(11) Let, $A = (1, -2, 2, 0)$, Find a unit vector is in the some direction as V.

$$V = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \qquad \|V\| = \langle V^T . V \rangle = \sqrt{\begin{bmatrix} 1 & -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}}$$

$$= \sqrt{1^2 + 4 + 4 + 0} = \sqrt{9} \Rightarrow \|V\| = 3$$

Unit vector along V, is given as $u = \frac{V}{\|V\|}$

$$= \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \\ 0 \end{bmatrix}$$

(12) Construct an Orthogonal basis for W using Gram Smidt Process

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Let, $W = [x_1 \quad x_2 \quad x_3]$

$$\therefore \; x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x_1 \cdot x_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 3 \neq 0$$

$$x_2 \cdot x_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$$

$$x_1 \cdot x_3 = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 2 \neq 0$$

By Gram Smidt Process

$$V_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{\langle x_2 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$v_3 = x_3 - \frac{\langle x_3 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1 - \frac{\langle x_3 \cdot v_2 \rangle}{\langle v_2 \cdot v_2 \rangle} \cdot v_2$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}{\left(-3/4^2 + 1/4^2 + 1/4^2 + 1/4^2\right)} \cdot \begin{bmatrix} -3/4 \\ +1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 \cdot v_3 = 0$

$\therefore \{v_1, v_2, v_3\}$ is an Orthogonal basis

(13) Find the least square solution of the inconsistent system $Ax = b$ for $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

Normal Equation,

$$A^T A x = A^T b$$

$$A^T A x = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Consider,

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 8+0+11 \\ 0+0+11 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

The least square solution gives,

$$\hat{x} = (A^T A)^{-1} (A^T b)$$

$$(A^T A)^{-1} = \frac{1}{|A^T A|} \operatorname{adj} (A^T A)$$

$$= \frac{1}{85-1} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 95 - 11 \\ -19 + 187 \end{bmatrix}$$

$$= \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Least Square Solution, $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(14) Use QR factorization $A = QR$ to find the least squares Solution of $Ax = b$. Given,

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

Take, $A = [x_1 \ \ x_2]$

$$\therefore \ x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 4 \end{bmatrix}$$

$$x_1 \cdot x_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ -1 \\ 4 \end{bmatrix} = -1+4-1+4 = 6$$

$$x_1 \cdot x_2 \neq 0$$

$\therefore$ x_1 & x_2 are not Orthogonal

By Using Gram Smidth Process,

Let, $v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 4 \end{bmatrix} - \frac{\begin{bmatrix} -1 & 4 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -1 \\ 4 \\ -1 \\ 4 \end{bmatrix} - \frac{6}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \\ -5/2 \\ 5/2 \end{bmatrix}$$

$$\|V_1\| = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$$

$$U_1 = \frac{V_1}{\|V_1\|} \Rightarrow U_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\|V_2\| = \sqrt{\frac{25}{4} + \frac{25}{4} + \frac{25}{4} + \frac{25}{4}} = \sqrt{\frac{25}{4} \times 4} = 5$$

$$\therefore U_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$Q = U_1 \quad U_2$$

$$\therefore Q = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$R = Q^T A$$

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 4 \\ 1 & -1 \\ 1 & 4 \end{bmatrix}$$

$$R = \begin{bmatrix} 1/2 + 1/2 + 1/2 + 1/2 & -1/2 + 2 - 1/2 + 2 \\ -\cancel{1/2} + \cancel{1/2} - \cancel{1/2} + \cancel{1/2} & 1/2 + 2 + 1/2 + 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$$

To obtain the least square solution, we need to solve

$$R\hat{x} = Q^T b$$

$$Q^T b = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & +1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 6 \\ 5 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 + 3 + 5/2 + 7/2 \\ 1/2 + 3 - 5/2 + 7/2 \end{bmatrix}$$

$$Q^T b = \begin{bmatrix} \frac{17}{2} \\ 9/2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 17/2 \\ 9/2 \end{bmatrix}$$

From the above equation

$$2x_1 + 3x_2 = 17/2$$

$$5x_2 = 9/2 \Rightarrow x_2 = 9/10$$

By Back Substitution $\Rightarrow 2x_1 + \frac{27}{10} = \frac{17}{2}$

$$x_1 = \cdot 29/10$$

$\therefore$ Least Square Solution.

$$\hat{x} = \begin{bmatrix} \cdot 29/10 \\ 9/10 \end{bmatrix}$$

(15) Find the Least Square Solution of $Ax = b$ by QR factorization

where, $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$

Consider,

$$A = [x_1 \quad x_2 \quad x_3]$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \quad x_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

By Gram Smidt Process

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \frac{\langle x_2 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} \cdot v_1$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 3 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 3-2 \\ 1-2 \\ 1-2 \\ 3-2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$V_3 = x_3 - \frac{\langle x_3 \cdot V_1 \rangle}{\langle V_1 \cdot V_1 \rangle} \cdot V_1 - \frac{\langle x_3 \cdot V_2 \rangle}{\langle V_2 \cdot V_2 \rangle} \cdot V_2$$

$$= \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{\begin{bmatrix} 5 & 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 5 & 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{10}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{6}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$V_3 = \begin{bmatrix} 5 - 5/2 - 3/2 \\ 0 - 5/2 + 3/2 \\ 2 - 5/2 + 3/2 \\ 3 - 5/2 - 3/2 \end{bmatrix} \qquad V_3 = \begin{bmatrix} 1 \\ -1 \\ +1 \\ -1 \end{bmatrix}$$

$$u_1 = \frac{V_1}{\|V_1\|} \Rightarrow \|V_1\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

$$u_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \qquad |||^{ly} \quad \|V_2\| = 2 \qquad \|V_3\| = 2$$

$$\therefore u_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ +1/2 \end{bmatrix} \qquad U_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$Q = [u_1 \quad u_2 \quad u_3]$$

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A = QR$$

The least square solution is obtained by solving

$$Rx = Q^T b$$

$$Q^T b = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

$$Q^T b = \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{ccc:c} 2 & 4 & 5 & 6 \\ 0 & 2 & 3 & -6 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$\therefore$ $2x_1 + 4x_2 + 5x_3 = 6$

$2x_2 + 3x_3 = -6$

$2x_3 = 4$

$\Rightarrow \underline{\underline{x_3 = 2}}$

By back substitution

$2x_2 + 6 = -6$

$\underline{\underline{x_2 = -6}}$

$2x_1 - 24 + 10 = 6 \Rightarrow x_1 = \frac{20}{2} \Rightarrow \underline{\underline{x_1 = 10}}$

$\therefore$ The Least square Solution is,

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$$