⇒ Matrix Algebra :-

→ Introduction to Linear Transformation :-

A transformation (or) function (or) mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns to each 'x' in $\mathbb{R}^n$ to a vector $T(x)$ in $\mathbb{R}^m$, the set $\mathbb{R}^n$ is called the domain and $\mathbb{R}^m$ is the codomain of linear Transformation.

The notation $T: \mathbb{R}^n \to \mathbb{R}^m$ indicates the domain of T is $\mathbb{R}^n$ and codomain is $\mathbb{R}^m$

For every 'x' in $\mathbb{R}^n$, the vector $T(x)$ in $\mathbb{R}^m$ is called the image of 'x', ~~the transform~~ under the transformation 'T'

The set of all images $T(x)$ is called as the range of the Transformation.

→ Matrix Transformation :-

For each 'x' in $\mathbb{R}^n$, $T(x)$ is computed as Ax where A is a $m \times n$ matrix sometimes we donote the Matrix Transformation as,

$$x \longmapsto Ax \quad (x \text{ bow into } Ax)$$

* Observe that the domain of T is $\mathbb{R}^n$ where 'A' has 'n' number of columns and codomain of T is $\mathbb{R}^m$ where the each column of matrix 'A' has 'm' number of entries.

① Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ +5 \end{bmatrix}$ and

define a Transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$, then

(i) Find the image of 'u' under the Transformation 'T'

(ii) Find an 'x' in $\mathbb{R}^2$ whose image under 'T' is 'b'

(iii) Is there more than one 'x' whose image under 'T' is 'b'

(iv) Determine if 'c' is the range of Transformation 'T'

Soln:- Let, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$T(x) = Ax$$

$$= \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \Bigg[$$

(i) Image of 'u' under 'T' is $T(u)$

$$T(u) = Au$$

$$= \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$T(u) = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

(ii) $T(x) = b$

$$Ax = b$$

$$\begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

$$[A:b] = \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$= \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{array}\right] \begin{array}{l} R_2 \to R_2/7 \\ R_3 \to R_3/2 \end{array}$$

$$= \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{array}\right] R_3 \to R_3 - R_2$$

$$[A : b] = \begin{bmatrix} 1 & -3 & : & 3 \\ 0 & 2 & : & -1 \\ 0 & 0 & : & 0 \end{bmatrix}$$

$\therefore \quad x_2 = -\frac{1}{2}$

$$x_1 - 3x_2 = 3$$
$$x_1 - 3\left(-\frac{1}{2}\right) = 3$$
$$x_1 = 3 - \frac{3}{2}$$
$$x_1 = \frac{3}{2}$$

$$\therefore \quad x = \begin{bmatrix} +3/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$$

$\therefore$ x is a Vector in R^2 whose image under 'T' is 'b'

(iii) Since, $\rho[A] = \rho[A : B]$

and $n = r = 2$

The Equation $Ax = b$ (or $T(x) = b$) has a unique solution

$\therefore$ There is only one 'x' for which image under 'T' is 'b'

(iv) To find if the vector 'c' is in the range of 'T' i.e to find (or) solve $T(x) = c$

i.e $Ax = c$

$$\begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$[A : C] = \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_1 + R_3 \end{array}$$

$$= \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2/7 \\ R_3 \to R_3/4 \end{array}$$

$$= \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{array}\right] \quad R_3 \to R_3 - \frac{R_2}{2}$$

$$= \left[\begin{array}{cc:c} 1 & -3 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 3/2 \end{array}\right]$$

Here, $\rho[A] \neq \rho[A : C]$

$\therefore$ The system is inconsistent

$\Rightarrow$ $T(x) = C$ does not have any solution

Hence, 'C' is not in the range of Transformation 'T'

(2) In the following Transformation defined as $T(x) = Ax$

Find the vector 'a' whose image under transformation T is 'b'

and determine wether 'x' is unique.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} ; \quad b = \begin{bmatrix} -1 \\ +7 \\ 3 \end{bmatrix}$$

To find 'x' such that

$$T(x) = b$$

$$Ax = b$$

$$\begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + 0x_2 - 2x_3 \\ -2x_1 + x_2 + 6x_3 \\ 3x_1 - 2x_2 - 5x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 7 \\ 3 \end{bmatrix}$$

$$[A : b] = \left[\begin{array}{ccc:c} \boxed{1} & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & 3 \end{array}\right] \quad R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - 3R_1$$

$$= \left[\begin{array}{ccc:c} \boxed{1} & 0 & -2 & -1 \\ 0 & \boxed{1} & 2 & 5 \\ 0 & -2 & 1 & 6 \end{array}\right] \quad R_3 \to R_3 + 2R_2$$

$$[A : b] = \left[\begin{array}{ccc:c} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 16 \end{array}\right]$$

$$x_1 - 2x_3 = -1$$
$$x_2 + 2x_3 = 5$$
$$5x_3 = 16 \Rightarrow x_3 = \frac{16}{5}$$

By back substitution

$$x_2 + 2\left(\frac{16}{5}\right) = 5 \qquad x_1 - 2\left(\frac{16}{5}\right) = -1$$

$$x_2 = 5 - \frac{32}{5} \qquad x_1 = \frac{-5+32}{5}$$

$$x_2 = -\frac{7}{5} \qquad x_1 = \frac{27}{5}$$

$\therefore$ $x_1 = 27/5$ $\quad x_2 = -7/5$ $\quad x_3 = 16/5$

$$x = \begin{bmatrix} 27/5 \\ -7/5 \\ 16/5 \end{bmatrix}$$

Since, in row echelon form

$$\rho[A] = \rho[A:b]$$

and there are no free variables

The solution to the system is unique

$\Rightarrow$ The solution 'x' is unique.

③ In the following, find all 'x' in R^n that are mapped into zero vector by Transformation $x \mapsto Ax$ for the given matrix 'A'.

(i) $$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

To find 'x' such that. $T(x)=0$ defined as $\mathbb{R}^4 \to \mathbb{R}^3$

$$Ax = 0$$

$$\therefore \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Augmented matrix

$$[A:b] = \left[\begin{array}{cccc:c} \boxed{1} & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{array}\right] R_3 \to R_3 - 2R_1$$

$$= \left[\begin{array}{cccc:c} \boxed{1} & -4 & 7 & -5 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{array}\right] R_3 \to R_3 - 2R_1$$

$$= \left[\begin{array}{cccc:c} \boxed{1} & -4 & 7 & -5 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables :- x_1, x_2

Free variables :- x_3, x_4

$$x_1 - 4x_2 + 7x_3 - 5x_4 = 0$$

$$\Rightarrow x_1 - 4x_2 = 5x_4 - 7x_3$$

$$x_2 + 0x_1 - 4x_3 + 3x_4 = 0$$

$$x_2 = 4x_3 - 3x_4$$

$$x_1 - 4(4x_3 - 3x_4) = 5x_4 - 7x_3$$

$$x_1 - 16x_3 + 12x_4 = 5x_4 - 7x_3$$

$$x_1 = 9x_3 - 7x_4$$

$\therefore$ There are infinitely many solution of x in R^n that
are mapped to '0' vec
because there are free variables.

⇒ Onto Function :-

A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each 'y' in $\mathbb{R}^m$ is the image of atleast one vector x in $\mathbb{R}^n$

⇒ One to One Function :-

The linear Transformation $T: \mathbb{R}^m \rightarrow \mathbb{R}^m$ if each 'b' in $\mathbb{R}^m$ atmost one x in $\mathbb{R}^n$.

Remark :-

* Let 'T' : $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the linear Transformation then 'T' is one-to-one iff the equation $T(x) = 0$ has only Trivial Solution

* Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear Transformation and let 'A' be the standard matrix of 'T', then 'T' maps $\mathbb{R}^n$ ~~onto~~ $\mathbb{R}^m$, the column of matrix A which span $\mathbb{R}^m$

① Let, T be the linear Transformation whose standard matrix $A = \begin{bmatrix} 2 & 1 & -3 \\ -6 & 4 & 0 \\ 2 & -5 & -1 \\ 0 & -2 & 3 \end{bmatrix}$.

(i) decide if T is one-one

(ii) decide if T is onto

(i) for T to be one-one, the $T(x) = 0$ must be Trivial.

$$\therefore \begin{bmatrix} 2 & 1 & -3 \\ -6 & 4 & 0 \\ 2 & -5 & -1 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[A:b] = \begin{bmatrix} \boxed{2} & 1 & -3 & : & 0 \\ -6 & 4 & 0 & : & 0 \\ 2 & -5 & -1 & : & 0 \\ 0 & -2 & 3 & : & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$-\frac{54}{7} + 2$

$$= \begin{bmatrix} \boxed{2} & 1 & -3 & : & 0 \\ 0 & \boxed{7} & -9 & : & 0 \\ 0 & -6 & +2 & : & 0 \\ 0 & -2 & 3 & : & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 + \frac{6R_2}{7} \\ R_4 \to R_4 + \frac{2R_2}{7} \end{matrix}$$

$$= \begin{bmatrix} \boxed{2} & 1 & -3 & : & 0 \\ 0 & \boxed{7} & -9 & : & 0 \\ 0 & 0 & -40/7 & : & 0 \\ 0 & 0 & 3/7 & : & 0 \end{bmatrix} \begin{matrix} R_3 \to R_3 \times 7 \\ R_4 \to R_4 \times 7 \end{matrix}$$

$$[A:0] = \left[\begin{array}{ccc:c} \boxed{2} & 1 & -3 & 0 \\ 0 & \boxed{7} & -9 & 0 \\ 0 & 0 & \boxed{-40} & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \quad R_4 \to \frac{R_3}{40} \times 3 + R_4$$

$$= \left[\begin{array}{ccc:c} \boxed{2} & 1 & -3 & 0 \\ 0 & \boxed{7} & -9 & 0 \\ 0 & 0 & \boxed{-40} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore$ The Echelon form has no free variables

$\Rightarrow$ The system $\underset{Ax=0}{,}$ has a Trivial solution

$\therefore$ Transformation 'T' is one-one

(ii) The Total number of Rows in Echelon form i.e each row does not have a pivot

$\therefore$ The columns of matrix 'A' do not span $\mathbb{R}^4$

$\because$ Here, the 4^{th} row does not have a pivot position.

$\Rightarrow$ $\mathbb{R}^3$ not onto $\mathbb{R}^4$

② Let 'T' be linear Transformation whose ~~Transf~~. standard matrix is $A = \begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix}$

(i) Decide if 'T' is one-one

(ii) Decide if it is onto

$$A = \begin{bmatrix} 7 & 5 & 4 & -9 \\ 10 & 6 & 16 & -4 \\ 12 & 8 & 12 & 7 \\ -8 & -6 & -2 & 5 \end{bmatrix} \quad R_2 \to R_2 - \frac{10R_1}{7}, \; R_3 \to R_3 - \frac{12R_1}{7}$$

$$R_4 \to R_4 + \frac{8R_1}{7}$$

$$A = \begin{bmatrix} \boxed{7} & 5 & 4 & -9 \\ 0 & -8/7 & 72/7 & 62/7 \\ 0 & -4/7 & 36/7 & 157/7 \\ 0 & -2/7 & 18/7 & -37/7 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 \times 7 \\ R_3 \to R_3 \times 7 \\ R_4 \to R_4 \times 7 \end{array}$$

$$A = \begin{bmatrix} \boxed{7} & 5 & 4 & -9 \\ 0 & -8 & 72 & 62 \\ 0 & -4 & 36 & 157 \\ 0 & -2 & 18 & -37 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - \frac{R_2}{2} \\ R_4 \to R_4 - \frac{R_2}{4} \end{array}$$

$$A = \begin{bmatrix} \boxed{7} & 5 & 4 & -9 \\ 0 & \boxed{-8} & 72 & 62 \\ 0 & 0 & 0 & \boxed{126} \\ 0 & 0 & 0 & -\frac{105}{2} \end{bmatrix} \quad R_4 \to R_4 + \frac{105 R_3}{126}$$

$$A = \begin{bmatrix} \boxed{7} & 5 & 4 & -9 \\ 0 & \boxed{-8} & 72 & 62 \\ 0 & 0 & 0 & \boxed{126} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Here, x_3 is a free variable

$\Rightarrow$ ~~A(x) = 0~~ $Ax = 0$ has a non Trivial solution

$\Rightarrow$ $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is not one-one

(ii) The 4th row in the Row Echelon form does not does not have a pivot element

$\therefore$ The columns of 'A' do not span $\mathbb{R}^3$

$\Rightarrow$ T is not mapped from $\mathbb{R}^2$ onto $\mathbb{R}^3$

③ Let $T: \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation,

$$T(x_1, x_2, x_3, x_4) = (0, x_1 + x_2, x_2 + x_3, x_3 + x_4)$$

ⓐ Determine if it is

(i) one to one

(ii) onto

Given, $T(x) = Ax$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Solving $T(x) = Ax$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} R_3 \leftrightarrow R_4$$

$$= \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) Here, x_4 is the free variable

$\therefore$ The system has non-trivial solution

$\Rightarrow$ The Transformation $T: \mathbb{R}^4 \to \mathbb{R}^4$ is not one to one

(ii) The 4^{th} row in Echelon form does not have a pivot

$\therefore$ Columns of A do not span $\mathbb{R}^4$

Properties:-

A Transformation (or) mapping 'T': $U \to V$ is said to be linear (i) $T(u+v) = T(u) + T(v)$ for $\forall\, u, v \in U$

(ii) $T(cu) = c \cdot T(u)$, $\forall\, u \in U$ & c is any scalar

Equivalently:-

T is said to be linear if

(i) $T(0) = 0$

(ii) $T(cu+dv) = c \cdot T(u) + d \cdot T(v)$ $\forall\, u, v \in U$

and c, d are scalars

① Show that the transformation 'T' defined by

$$T(x_1, x_2) = (2x_1 - 3x_2,\ x_1 + 4,\ 5x_2)$$

Clearly the mapping is not linear,

Since, $T(0,0) = (0, 4, 0)$ is not a zero vector.

② Show that the mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_1 - 3x_2 - 2x_3)$$

Sol^n:- From the first property, it is clear that $T(0) = 0$

To check 2^{nd} property, let us consider

$u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be any vectors in $\mathbb{R}^3$

and let 'c' & 'd' be any '2' scalars.

Let $cu + dv = c(u_1, u_2, u_3) + d(v_1, v_2, v_3)$

$$= (cu_1 + dv_1,\ cu_2 + dv_2 + cu_3 + dv_3)$$

$\therefore$ LHS $= T(cu + dv)$

$$= T(cu_1 + dv_1,\ cu_2 + dv_2 + cu_3 + dv_3)$$

$$\text{LHS} = \begin{bmatrix} 3(cu_1 + dv_1) - 2(cu_2 + dv_2) + cu_3 + dv_3 \\ cu_1 + dv_1 - 3(cu_2 + dv_2) - 2(cu_3 + dv_3) \end{bmatrix}$$

$$RHS = c \cdot T(u) + d \cdot T(v)$$

$$= c \cdot T(u_1, u_2, u_3) + d \cdot T(v_1, v_2, v_3)$$

$$\cancel{\neq c \begin{bmatrix} 3x_1 - 2x_2 + x_3 \\ x_1 + 3x_2 - 2x_3 \end{bmatrix} + d \Big[}$$

$$= c \begin{bmatrix} 3u_1 - 2u_2 + u_3 \\ u_1 - 3u_2 - 2u_3 \end{bmatrix} + d \begin{bmatrix} 3v_1 - 2v_2 + v_3 \\ v_1 - 3v_2 - 2v_3 \end{bmatrix}$$

$$= \begin{bmatrix} 3cu_1 - 2cu_2 + cu_3 + 3dv_1 - 2dv_2 + dv_3 \\ cu_1 - 3cu_2 - 2cu_3 + dv_1 - 3dv_2 - 2dv_3 \end{bmatrix}$$

$$RHS = \begin{bmatrix} 3(cu_1 + dv_1) - 2(cu_2 + dv_2) + (cu_3 + dv_3) \\ (cu_1 + dv_1) - 3(cu_2 + dv_2) - 2(cu_3 + dv_3) \end{bmatrix}$$

$\therefore$ LHS = RHS

$$\Rightarrow T(cu + dv) = c \cdot T(u) + d \cdot T(v)$$

$\therefore$ The Transformation is linear//

⇒ Matrix Algebra:-

Properties of Matrix Multiplications

Let A be $m \times n$ matrix and let B and C have sizes for which the dindicated sums and products are defined.

(i) $A(BC) = (AB)C$ [Associative law]

(ii) $A(B+C) = AB + AC$ [Distributive Law]

(iii) $(B+C)A = BA + CA$ [Right Distributive Law]

(iv) $r(AB) = (rA)B = A(rB)$ for any scalar r

(v) $I_m A = A = A \cdot I_m$

⇒ Transpose of a Matrix:-

Given $m \times n$ matrix A, the transpose of the matrix is the $n \times m$ matrix denoted by A^T whose columns are formed from the corresponding rows of matrix A.

Ex:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Let A and B be matrices whose sizes are appropriate for the following sums and products.

(i) $(A^T)^T = A$ (ii) $(A+B)^T = A^T + B^T$ (iii) For any sclar r $(rA)^T = rA^T$

(iv) $(AB)^T = B^T A^T$

⇒ Inverse of a Matrix :-

* An $n \times n$ matrix A is said to be inverted if there is an $n \times n$ matrix 'C' such that $CA = I$ and $AC = I$ where $I = I_n$ an $n \times n$ Identity matrix. In this case the matrix C is the inverse of the matrix A. Infact the matrix C is uniquely determined by matrix A. Because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$. This unique inverse is denoted by A^{-1} so that $A^{-1}A = AA^{-1} = I$

* A matrix that is not invertible is sometimes called a singular matrix and an invertible matrix is called a non singular matrix

Note :- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $ad - bc \neq 0 \Rightarrow A$ is invertible

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A is not invertible

* If A is invertible $n \times n$ matrix, then for each B, in R^n the equation $Ax = b$ has unique solution $x = A^{-1}b$

* If A is invertible matrix then A^{-1} is also invertible and $(A^{-1}$

* If A and B are $n \times n$ invertible matrices then the product AB is also invertible and inverse of AB is product of inverses of A B is reverse order $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

* If A is an invertible matrix then its transpose A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

⇒ Elementary Matrices :-

An elementary matrix is one that is obtained by performing a single Elementary row operation on any Indentity matrix.

Ex:- $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

⇒ Theorem :-

An $n \times n$ matrix A is invertible iff A is row equivalent to I_n and in this case any sequence of elementary row operations that reduces matrix A to I_n. Also transforms I_n to A^{-1}.

Ex:- If $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ then $[A : I_n] = \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$

→ Row reducing :-

$$[A : I_n] \sim \left[\begin{array}{cc:cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc:cc} 1 & 0 & 4 & -3 \\ 0 & 1 & -1 & 1 \end{array}\right] = [I_2 : A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$$

⇒ Inverse of a Matrix :-

① Find the inverse of a matrix if it exists

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

$$|A| = 0(0+9) - 1(8-12) + 2(-3-0)$$

$$|A| = 0 + 4 - 6$$

$$|A| = -2$$

Since, $|A| \neq 0$

$\therefore$ A^{-1} exists

By Invertible Matrix Theorem,

Augmented matrix

$$[A : I_3] \sim \left[\begin{array}{ccc:ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] R_3 \rightarrow R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc:ccc} \boxed{1} & 0 & 3 & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \quad R_3 \to R_3 + 3R_3$$

$$\sim \left[\begin{array}{ccc:ccc} \boxed{1} & 0 & 3 & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{2} & 3 & -4 & 1 \end{array}\right] \quad R_3 \to R_3/2$$

$$\sim \left[\begin{array}{ccc:ccc} \boxed{1} & 0 & 3 & 0 & 1 & 0 \\ 0 & \boxed{1} & 2 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & \frac{1}{2} \end{array}\right]$$

$$\therefore [A : I_3] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\qquad\qquad\qquad I_3 \qquad\qquad\qquad A^{-1}$$

From above Row reduced Echelon form

$$A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

(a) Find the inverse of matrix A,

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

The augmented matrix is,

$$[A : I_3] = \left[\begin{array}{ccc:ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$|A| = \left[1(25+24) + (-2)(-5+30) - 1(4-25) \right]$$

$$= -49+70-21$$

$$|A| = 0$$

A^{-1} is not possible

$$[A : I_3] = \left[\begin{array}{ccc:ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$= \left[\begin{array}{ccc:ccc} \boxed{1} & -2 & -1 & 1 & 0 & 0 \\ 0 & \boxed{3} & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right] R_3 \to R_3 - 2R_2$$

$$= \left[\begin{array}{ccc:ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right]$$

In the Row echelon form, all elements of 'A' is last row is '0'

Hence, application of any other Row Elementary operation does not transform the matrix

$\therefore$ The Inverse of Matrix 'A' does not exist

⇒ Partitioned Matrices :-

Using a system of Horizontal and vertical lines, we can partition a matrix 'A' into sub matrices called Blocks (or) Cells of the matrix 'A'

Clearly a given matrix may be divided into blocks in different ways.

(i)

$$A = \left[\begin{array}{cc|cc|cc} 1 & 3 & 5 & -1 & 2 & 3 \\ 1 & 3 & 2 & 0 & 0 & 0 \\ \hline 2 & 7 & 2 & 1 & 2 & 3 \\ 1 & -1 & 2 & 3 & 2 & 1 \end{array}\right]$$

(ii)

$$A = \left[\begin{array}{c|ccc|cc} 1 & 0 & 0 & 0 & 2 & 3 \\ 2 & 0 & 0 & 0 & 5 & 7 \\ \hline 3 & 2 & 1 & 3 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 & 0 \end{array}\right]$$

(1) A matrix of the form A, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is said to be Block Upper Triangular. Assume that A_{11} is $p \times p$ and matrix A_{22} is $q \times q$, if A is invertible find formula for A^{-1}.

Solⁿ :- Given, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$

Denote, A^{-1} by matrix B and partition the matrix 'B'.

$$A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Let, $AB = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$

There exists a matrix B such that $AB = BA = I$

$$\begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{22}B_{21} & A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

$$A_{11}B_{11} + A_{12}B_{21} = I_p \rightarrow (1)$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \rightarrow (2)$$

$$A_{22}B_{21} = 0 \rightarrow (3) \Rightarrow B_{21} = 0$$

$$A_{22}B_{22} = I_q \rightarrow (4) \Rightarrow B_{22} = A_{22}^{-1}$$

Put $B_{21} = 0$ in (1)

$$A_{11}B_{11} = I_p$$

$$\therefore B_{11} = A_{11}^{-1}$$

Put B_{22} in (2)

$$A_{11}B_{12} + A_{12}(A_{22}^{-1}) = 0$$

$$B_{12} = -A_{12}A_{22}^{-1}A_{11}^{-1}$$

$$\therefore B = A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{12}A_{22}^{-1}A_{11}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$$

② Find the inverse of a matrix without using Row reduction

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

$$A = \left[\begin{array}{cc|ccc} 1 & 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 5 & 6 \end{array}\right]$$

Let, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$$A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \Rightarrow A_{11}^{-1} = \frac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

$$A_{12} = 0 \Rightarrow A_{12}^{-1} = 0$$

$$A_{21} = 0 \Rightarrow A_{21}^{-1} = 0$$

$$A_{22} = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 7 & 8 \\ 0 & 5 & 6 \end{array}\right] \Rightarrow \text{Let } A_{22} = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$$

$$P = [2] \qquad P^{-1} = \left[\tfrac{1}{2}\right]$$

$$Q = 0 \Rightarrow Q^{-1} = 0$$

$$R = 0 \Rightarrow R^{-1} = 0$$

$$S = \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} \Rightarrow S^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 3 & -4 \\ -5/2 & 7/2 \end{bmatrix}$$

$$\therefore A_{22}^{-1} = \begin{bmatrix} P^{-1} & Q^{-1} \\ R^{-1} & S^{-1} \end{bmatrix}$$

$$A_{22}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 3 & -4 \\ 0 & -5/2 & 7/2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} A_{11}^{-1} & A_{12}^{-1} \\ A_{21}^{-1} & A_{22}^{-1} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & +1/2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & -5/2 & 7/2 \end{bmatrix}$$

LU- Factorization :-

① Let Find LU Decomposition of the following matrix :-

(i) $A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 4 & -3 \\ -2 & 8 & 5 \\ 3 & 4 & 7 \end{bmatrix} \quad R_2 \to R_2 + 2R_1 , \; R_3 \to R_3 - 3R_1$$

$$= \begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & -8 & 16 \end{bmatrix} \quad R_3 \to R_3 + \frac{R_2}{2}$$

$$= \begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 31/2 \end{bmatrix}$$

$$\therefore \quad U = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 16 & -1 \\ 0 & 0 & 31/2 \end{bmatrix} \quad \text{(Upper Triangular Matrix)}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1/2 & 1 \end{bmatrix} \quad \text{(Unit Lower Triangular Matrix)}$$

** => Solving System of Linear equation Using LU Factorization

(i) Factorize A into LU

(ii) $Ax = b$

$(LU)x = b$

$L(Ux) = b$

Put $Ux = y$

$Ly = b$ => Solve

(iii) Then solve $Ux = y$ to get x

(1) Solve the system of linear equation $Ax = b$ using LU Factorization where,

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix}$$

Now, Solving $Ly = b$, where, $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$[L : b] = \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 2 \\ 3 & 0 & 1 & : & 5 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$= \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & -1 \end{bmatrix}$$

$\therefore$ From the above matrix

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Now Solving, $Ux = y$

$$\begin{bmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$[U:y] = \left[\begin{array}{ccc:c} 2 & 3 & 3 & 2 \\ 0 & 5 & 7 & 2 \\ 0 & 0 & -1 & -1 \end{array}\right] \quad R_3 \rightarrow -R_3$$

$$= \left[\begin{array}{ccc:c} 2 & 3 & 3 & 2 \\ 0 & 5 & 7 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \rightarrow R_2 - 7R_3 \\ R_1 \rightarrow R_1 - 3R_3 \end{array}$$

$$= \left[\begin{array}{ccc:c} 2 & 3 & 0 & -1 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_2 \rightarrow \frac{R_2}{5}$$

$$= \left[\begin{array}{ccc:c} 2 & 3 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1 \rightarrow R_1 - 3R_2$$

$$= \left[\begin{array}{ccc:c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1 \rightarrow \frac{R_1}{2}$$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

2) Solve $Ax = b$ where $A = \begin{bmatrix} 3 & 7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$ by LU Factors.

$$A \sim \begin{bmatrix} 3 & 7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$A \sim \begin{bmatrix} 3 & 7 & -2 \\ 0 & 12 & -1 \\ 0 & -18 & 4 \end{bmatrix} R_3 \to R_3 + \frac{18}{12} R_2$$

$$A \sim \begin{bmatrix} 3 & 7 & -2 \\ 0 & 12 & -1 \\ 0 & 0 & 5/2 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -\frac{3}{2} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 & 7 & -2 \\ 0 & 12 & -1 \\ 0 & 0 & 5/2 \end{bmatrix}$$

Solving $Ly = b$, Augmented matrix is

$$[L : b] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 5 \\ 2 & -\frac{3}{2} & 1 & 2 \end{array}\right] \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$[L:b] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & -\frac{3}{2} & 1 & 4 \end{array}\right] \quad R_3 \to R_3 + \frac{3}{2} R_2$$

$$= \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

$$\therefore \; y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 10 \end{bmatrix}$$

Now Solving $Ux = y$

$$[U:y] = \left[\begin{array}{ccc:c} 3 & 7 & -2 & -1 \\ 0 & 12 & -1 & 4 \\ 0 & 0 & 5/2 & 10 \end{array}\right]$$

By Back Substitution,

$$\frac{\cancel{5}}{2} x_3 = \overset{2}{\cancel{10}}$$

$$\Rightarrow x_3 = 4$$

$$12x_2 - x_3 = 4$$

$$x_2 = \frac{8}{12} = \frac{2}{3}$$

$$\therefore \; x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/9 \\ 2/3 \\ 4 \end{bmatrix}$$

$$3x_1 + 7x_2 - 2x_3 = -1$$

$$3x_1 = -1 + 8 - \frac{14}{3}$$

$$x_1 = \frac{-3 + 24 - 14}{9} = \frac{7}{9}$$