Optimization

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January 11, 2025

Outline

Table Of Contents

Problem

Problem Statement

Find the local minimum/maximum of the given function:
$f(x) = x^2$

Gradient Descent

Getting the Difference Equation

We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$\begin{aligned} x_{n+1} &= x_n - \mu f'(x_n) && (3.1) \\ f'(x_n) &= 2x_n && (3.2) \\ \rightarrow x_{n+1} &= x_n - 2\mu x_n && (3.3) \\ &= (1-2\mu)\, x_n && (3.4) \end{aligned}$$

Addressing ROC

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1-2\mu)\,X(z) \tag{3.5}$$

$$(z-(1-2\mu))\,X(z) = zx_0 \tag{3.6}$$

$$X(z) = \frac{zx_0}{z-(1-2\mu)} \tag{3.7}$$

$$X(z) = \frac{x_0}{1-(1-2\mu)\,z^{-1}} \tag{3.8}$$

$$= \sum_{0}^{\infty} (1-2\mu)^n\, z^{-n} \tag{3.9}$$

ROC

From the last equation, ROC is

$$|z| > |1 - 2\mu| \tag{3.10}$$

$$\rightarrow |1 - 2\mu| > 0 \tag{3.11}$$

$$\rightarrow \mu \in \mathcal{R} \setminus \left\{ \frac{1}{2} \right\} \tag{3.12}$$

Finding point of convergence

Now, if μ satisfies the previous condition,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{3.13}$$

$$\to \lim_{n\to\infty} \|(-2\mu)\, x_n\| = 0 \tag{3.14}$$

$$\to \|(-2\mu)\| \lim_{n\to\infty} \|x_n\| = 0 \tag{3.15}$$

$$\to \lim_{n\to\infty} \|x_n\| = 0 \tag{3.16}$$

$$\to \lim_{n\to\infty} x_n = 0 \tag{3.17}$$

Solution

Taking initial guess = 8
step size = 0.01
tolerance(minimum value of gradient) = 1e-5
We get
$x_{min} = 4.910792425174398e - 06$

Quadratic Programming

Reformulating

As a fun exercise, the given question can be posed as the following quadratic programming problem:
Find the point lying on the line y = 1, which is nearest to origin.
We can formulate the problem as follows:

$$\min_{\mathbf{x}} \|e_2^\top \mathbf{x}\|^2 \tag{4.1}$$

$$\text{s.t.} \tag{4.2}$$

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{4.3}$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \tag{4.5}$$

$$f = 0 \tag{4.6}$$

(4.7)

Relaxation of Constraint

In the current form, the constraint is non-convex since the constraint defines a set which is not convex, since points on the line joining any 2 points on the curve don't belong to the set. However, if we become lenient and make the constraint

$$\mathbf{x}^\top V\mathbf{x} + 2\mathbf{u}^\top\mathbf{x} \leq 0 \tag{4.8}$$

the constraint becomes convex. Using cvxpy to solve this convex optimization problem, we get

$$\begin{aligned} Optimal x : [[0.00000000e+00] \quad (4.9) \\ [8.63053066e-11]] \quad (4.10) \end{aligned}$$

Graph

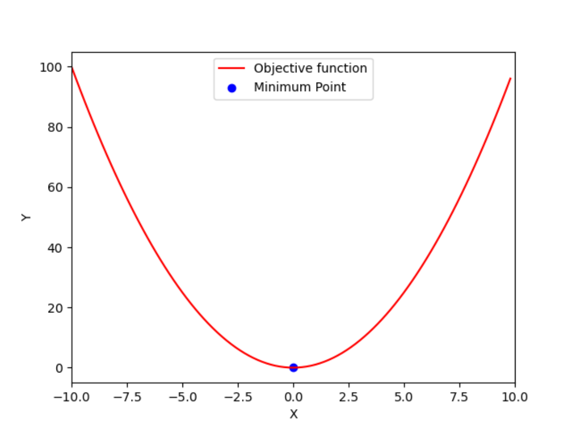

Figure 1: Minimum Value of Objective function