Solution to DES using Laplace Transforms

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Outline

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Problem Statement

Plot the solution to $y' + 2y = \sin x$

Solution

Euler's Method

To plot a curve in the solution family, we take the initial condition to be

$x_0 = 0, y_0 = 1$

Using Euler's Method, we represent the the differential equation in the following difference equations:

$$x_{n+1} = x_n + h \tag{3.1}$$

$$y_{n+1} - y_n + 2hy_n = h \sin x_n \tag{3.2}$$

$$\rightarrow y_{n+1} = (1 - 2h)\, y_n + h \sin x_n \tag{3.3}$$

Now we can iteratively generate points which lie close to the graph.

Laplace Transform

Let $\mathcal{L}(y) = Y$

$$(sY - y_0) + 2Y = \mathcal{L}(\sin x) \tag{3.4}$$

$$\mathcal{L}(\sin x) = \int_0^\infty e^{-sx} \sin x = \frac{1}{s^2+1} \tag{3.5}$$

$$(s+2)\,Y = y_0 + \frac{1}{s^2+1} \tag{3.6}$$

$$Y = \frac{y_0}{s+2} + \frac{1}{(s^2+1)(s+2)} \tag{3.7}$$

$$\tag{3.8}$$

Using method of partial fractions,

$$\frac{1}{\left(s^2+1\right)s+2} = \frac{a}{s+2} + \frac{bs+c}{s^2+1} \tag{3.9}$$

$$\tag{3.10}$$

On solving we get,

$$a = \frac{1}{5} \tag{3.11}$$

$$b = \frac{-1}{5} \tag{3.12}$$

$$c = \frac{2}{5} \tag{3.13}$$

$$\tag{3.14}$$

Substituting $y_0 = 1$,

$$Y = \frac{1}{s+2} + \frac{0.2}{s+2} + \frac{-0.2s}{s^2+1} + \frac{0.4}{s^2+1} \tag{3.15}$$

$$\rightarrow Y = \frac{1.2}{s+2} + \frac{-0.2s}{s^2+1} + \frac{0.4}{s^2+1} \tag{3.16}$$

$$\tag{3.17}$$

Now, we take the inverse laplace transform to get a solution,

$$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2x}u\left(x\right) \tag{3.18}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin xu\left(x\right) \tag{3.19}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos xu\left(x\right) \tag{3.20}$$

Therefore the final solution to the differential equation is,

$$y(x) = \left(1.2e^{-2x} - 0.2\cos x + 0.4\sin x\right) u(x) \tag{3.21}$$

Graph

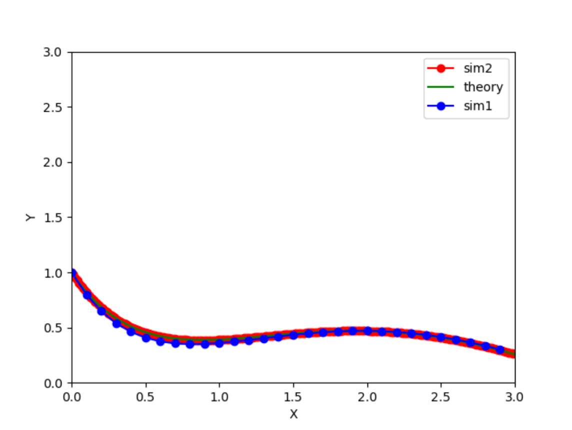

Figure 1: equilateral triangle of side 5cm