6.5.3.1

EE24BTECH11001 - Aditya Tripathy

Question:
Find the local minimum/maximum of the given function:
$f(x) = x^2$
Solution:
We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$x_{n+1} = x_n - \mu f'(x_n) \tag{0.1}$$
$$f'(x_n) = 2x_n \tag{0.2}$$
$$\to x_{n+1} = x_n - 2\mu x_n \tag{0.3}$$
$$= (1 - 2\mu) x_n \tag{0.4}$$

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1 - 2\mu) X(z) \tag{0.5}$$
$$(z - (1 - 2\mu)) X(z) = zx_0 \tag{0.6}$$
$$X(z) = \frac{zx_0}{z - (1 - 2\mu)} \tag{0.7}$$
$$X(z) = \frac{x_0}{1 - (1 - 2\mu) z^{-1}} \tag{0.8}$$
$$= \sum_{0}^{\infty} (1 - 2\mu)^n z^{-n} \tag{0.9}$$

From the last equation, ROC is

$$|z| > |1 - 2\mu| \tag{0.10}$$
$$\to |1 - 2\mu| > 0 \tag{0.11}$$
$$\to \mu \in \mathcal{R} \setminus \left\{\frac{1}{2}\right\} \tag{0.12}$$

Now, if μ satisfies the previous condition,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{0.13}$$
$$\to \lim_{n\to\infty} \|(1-2\mu)\, x_n\| = 0 \tag{0.14}$$
$$\to \|(1-2\mu)\| \lim_{n\to\infty} \|x_n\| = 0 \tag{0.15}$$
$$\to \lim_{n\to\infty} \|x_n\| = 0 \tag{0.16}$$
$$\to \lim_{n\to\infty} x_n = 0 \tag{0.17}$$

Taking initial guess = 8
step size = 0.01
tolerance(minimum value of gradient) = 1e-5
We get
$x_{min} = 4.910792425174398e-06$
As a fun exercise, the given question can be posed as the following quadratic programming problem:
Find the point lying on the line y = 1, which is nearest to origin.
We can formulate the problem as follows:

$$\min_{\mathbf{x}} \left\| e_2^{\top} \mathbf{x} \right\|^2 \tag{0.18}$$
$$\text{s.t.} \tag{0.19}$$
$$\mathbf{x}^{\top} V \mathbf{x} + 2\mathbf{u}^{\top}\mathbf{x} + f = 0 \tag{0.20}$$
$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.21}$$
$$\mathbf{u} = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \tag{0.22}$$
$$f = 0 \tag{0.23}$$

In the current form, the constraint is non-convex since the constraint defines a set which is not convex, since points on the line joining any 2 points on the curve don't belong to the set. However, if we become lenient and make the constraint

$$\mathbf{x}^{\top} V \mathbf{x} + 2\mathbf{u}^{\top}\mathbf{x} \leq 0 \tag{0.24}$$

the constraint becomes convex. Using cvxpy to solve this convex optimization problem, we get

$$Optimal x : [[0.00000000e+00] \tag{0.25}$$
$$[8.63053066e-11]] \tag{0.26}$$

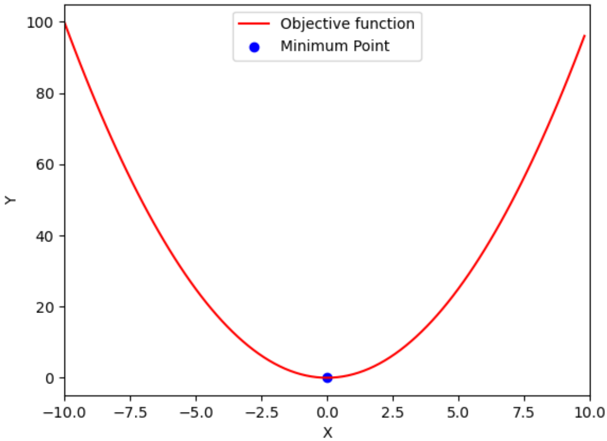

Fig. 0.1: Minimum value of objective function