9.1.2

EE24BTECH11001 - Aditya Tripathy

Question:
Plot a solution to the following differential equation:
$y' + 5y = 0$
Solution:
The aim is to find the difference equation using the trapezoidal law using the following initial conditions, $x_0 = 0, y_0 = 1$

$$y' = -5y \tag{0.1}$$
$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} -5y\,dx \tag{0.2}$$
$$\tag{0.3}$$

Using the trapezoidal rule,

$$J = \int_a^b f(x)\,dx \tag{0.4}$$
$$\approx h\left(\frac{1}{2}f(x) + f(x_1) + f(x_2)\cdots + f(x_{n-1}) + \frac{1}{2}f(b)\right) \tag{0.5}$$
$$\frac{y_{n+1} - y_n}{1}\left(\frac{1}{2} + \frac{1}{2}\right) = -5\frac{x_{n+1} - x_n}{1}\left(\frac{y_n}{2} + \frac{y_{n+1}}{2}\right) \tag{0.6}$$
$$\rightarrow y_{n+1} = \frac{(2-5h)\,y_n}{5h+2} \tag{0.7}$$

Another way we can arrive at the differnce equation is by using the Bilinear transform, Applying Laplace Transform to both sides of the differential equation,

$$sY - sy_0 + 5Y = 0 \tag{0.8}$$
$$Y(s) = \frac{sy_0}{s+5} \tag{0.9}$$
$$\tag{0.10}$$

Apply Bilinear Transform, with T = h

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{0.11}$$

$$Y(z) = \frac{2y_0\left(1-z^{-1}\right)}{2\left(1-z^{-1}\right)+5h\left(1+z^{-1}\right)} \tag{0.12}$$

$$Y(z) = \frac{2y_0\left(1-z^{-1}\right)}{(2+5h)+(5h-2)\,z^{-1}} \tag{0.13}$$

$$\alpha = -\frac{5h-2}{5h+2} \tag{0.14}$$

$$Y = \frac{2y_0}{5h+2}\left(\frac{1}{1-\alpha z^{-1}} - \frac{z^{-1}}{1-\alpha z^{-1}}\right) \tag{0.15}$$

$$\left(1-\alpha z^{-1}\right)Y = \frac{2y_0}{5h+2}\left(1-z^{-1}\right) \tag{0.16}$$

Applying inverse Z-transform,

$$y_n - \alpha y_{n-1} = \frac{2y_0}{5h+2}\left(\delta[n] - \delta[n-1]\right) \tag{0.17}$$

$$\tag{0.18}$$

To check how close the approximate graph is to the actual solution, we will solve the original equation using a Laplace Transform method:
Let $\mathcal{L}(y) = Y$

$$(sY - y_0) + 5Y = 0 \tag{0.19}$$

$$(s+5)\,Y = y_0 \tag{0.20}$$

$$Y = \frac{1}{s+5} \tag{0.21}$$

$$\tag{0.22}$$

Substituting $y_0 = 1$,

$$Y = \frac{1}{s+5} \tag{0.23}$$

$$\tag{0.24}$$

Now, we take the inverse laplace transform to get a solution,

$$\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) = e^{-5x}u(x) \tag{0.25}$$

For the following approximate graph, I chose $h = 0.01$ and $h = 0.1$.

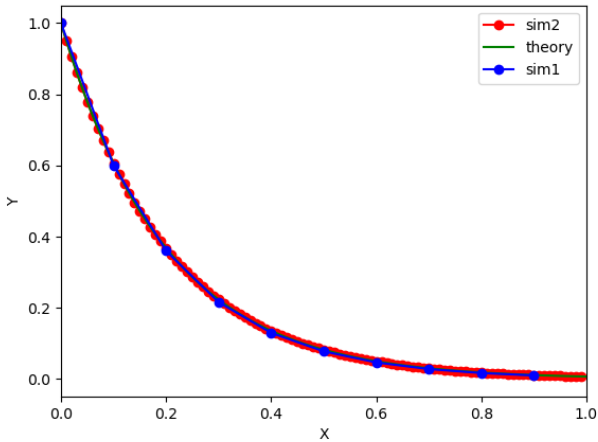

Fig. 0.1: Approximate solution of the DE