Numerical Solutions to Equations

Aditya Tripathy
Dept. of Electrical Engg.,
IIT Hyderabad.

January 16, 2025

Outline

Table Of Contents

Problem

Problem Statement

Solve the equation $x^2 - 2x = -2(3 - x)$ or $x^2 - 4x + 6 = 0$

Characteristic Polynomial

Characteristic Polynomial

Recall that for a matrix A of order n, the characteristic equation is given by,

$$det\left(A-\lambda I\right)=a_n\lambda^n+a_{n-1}\lambda^{n-1}\cdots+a_0=0 \tag{3.1}$$

Recall that the solutions to the characteristic polynomial are the eigenvalues of the the matrix A. So if we can somehow construct the corresponding matrix from the characteristic polynomial, our job will be finished, since we can find the eigen values using the QR alogorithm.

Companion Matrix

A matrix is said to be the companion of a polynomial $f(x)$ if $det(A - \lambda I) = 0 \implies f(x) = 0$.
For,

$$f(x) = c_0 + c_1 x \cdots + x^n \tag{3.2}$$

The companion matrix is,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{3.3}$$

Our Companion Matrix

$$\begin{pmatrix} 0 & -6 \\ 1 & 4 \end{pmatrix} \tag{3.4}$$

Using the QR algorithm we can now solve for the eigenvalues and thus the solutions for the given equation.

Understanding QR Algorithm

Similarity Transform

Two matrices $A, B \in \mathbb{C}^{n \times n}$ are said to be similar if there exists an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that

$$B = P^{-1}AP \tag{4.1}$$

Main Idea

The idea is to apply similarity transforms strategically to convert given matrix to Upper triangular matrix, because diagonal entries are the eigenvalues of the original matrix.

Intermediate Steps

The steps to solve QR quickly :

1. Convert original matrix to an intermediate form (hessenberg form) (4.2)
2. Apply givens rotation (4.3)
3. Diagonal elements are eigenvalues (4.4)

(4.5)

Conversion To Hessenberg Form

Householder Reflections

A matrix of the form

$$P = I - 2\mathbf{u}\mathbf{u}^* \tag{5.1}$$

$$\tag{5.2}$$

is called a Householder reflector. A task that we repeatedly want to carry out with Householder reflectors is to transform a vector $\mathbf{x}$ on a multiple of e_1

$$P\mathbf{x} = \mathbf{x} - \mathbf{u}(2\mathbf{u}^*\mathbf{x}) = \alpha e_1 \tag{5.3}$$

$$\tag{5.4}$$

Since P is unitary, we must have $\alpha = \rho||\mathbf{x}||$, where $\rho \in \mathbb{C}$ has absolute value one. Therefore,

$$\mathbf{u} = \frac{\mathbf{x} - \rho||\mathbf{x}||e_1}{||\mathbf{x} - \rho||\mathbf{x}||e_1||} \tag{5.5}$$

Householder Reflections

We can freely choose ρ provided that $|\rho| = 1$. Let $x_1 = |x_1| e^{i {u_k}^* \phi}$.To avoid numerical cancellation we set $\rho = -e^{i\phi}$

Visualizing The process

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & I_4 - 2u_1 u_1^* \end{bmatrix} \tag{5.6}$$

$$P_1^* A P_1 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} \xrightarrow[P_2 * / * P_2]{} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix} \tag{5.7}$$

Returning to problem

Since a 2x2 matrix is always upper hessenber, we need not do the above steps

Step 2: Apply Givens Rotation

Givens rotations are used to zero specific elements of a vector or matrix by rotating the vector in the plane of two coordinates. A Givens rotation matrix is defined as:

$$G(i,j,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix},$$

where

$$c_k = \frac{\overline{x_i}}{\sqrt{|x_i|^2 + |x_j|^2}}, s_k = \frac{\overline{x_j}}{\sqrt{|x_i|^2 + |x_j|^2}} \tag{5.8}$$

Step 2: Apply Givens Rotation

$$c_k = \frac{A_{11}}{\sqrt{A_{11}^2 + A_{12}^2}} = \frac{0}{\sqrt{0^2 + 1^2}} = 0 \tag{5.9}$$

$$s_k = \frac{A_{12}}{\sqrt{A_{11}^2 + A_{12}^2}} = \frac{0}{\sqrt{0^2 + 1^2}} = 1 \tag{5.10}$$

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.11}$$

$$G^\top A G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -6 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.12}$$

$$= \begin{pmatrix} 4 & -1 \\ 6 & 0 \end{pmatrix} \tag{5.13}$$

Failure?

The QR algorithm has failed to converge to an upper triangular matrix, which is to be expected. A matrix with real entries but complex eigenvalues, cannot be transformed by any sort of matrix multipliation of conjugation.

Jordan Blocks

$$\begin{bmatrix} a_0 & & & & \\ & a_1 & & & \\ & & a_2 & & \\ & & & a_3 & a_4 \\ & & & a_5 & a_6 \end{bmatrix} \tag{5.14}$$

Sometimes the QR algorithm ends like this. The propper diagonal elements are the true eigenvalues, and the eigenvalues of the 2×2 block is the complex conjugate pair of eigenvalues

Code Output

$$\text{Upper Hessenberg Matrix} \tag{5.15}$$

$$0.000000000 + 0.000000000\text{i} \text{ -}6.000000000 + 0.000000000\text{i} \tag{5.16}$$

$$1.000000000 + 0.000000000\text{i } 4.000000000 + 0.000000000\text{i} \tag{5.17}$$

$$\text{eigenvalue 1: } 2.000000 + 1.414214\text{i} \tag{5.18}$$

$$\text{eigenvalue 2: } 2.000000 + \text{-}1.414214\text{i} \tag{5.19}$$

$$\tag{5.20}$$