11.16.3.2

EE24BTECH11001 - Aditya Tripathy

Question:
A coin is tossed twice, what is the probability that atleast one tail occurs?
Solution :
Let Y be the random variable representing the number of tails. Y can the represented the sum of two bernoulli random variables, X_1, X_2,

$$Y = X_1 + X_2 \tag{0.1}$$

The Bernoulli R.V is defined as,

$$X_i = \begin{cases} 0 & \text{Outcome is Heads} \\ 1 & \text{Outcome is Tails} \end{cases} \tag{0.2}$$

The PMF of Bernoulli R.V is given by,

$$p_X(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \end{cases} \tag{0.3}$$

Using properties of Z transform of PMF on eq. (0.1),

$$M_Y(z) = M_{X_1}(z) M_{X_2}(z) \tag{0.4}$$

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(k) z^{-k} = p + (1-p) z^{-1} \tag{0.5}$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(k) z^{-k} = p + (1-p) z^{-1} \tag{0.6}$$

$$M_Y(z) = \left(p + (1-p) z^{-1}\right)^2 \tag{0.7}$$

$$= \sum_{k=-\infty}^{\infty} {}^2C_k p^{2-k} (1-p)^k z^{-k} \tag{0.8}$$

$$p_Y(n) = {}^2C_n p^{2-n} (1-p)^n \tag{0.9}$$

Substituting $p = \frac{1}{2}$,

$$p_Y(n) = {}^2C_n \left(\frac{1}{2}\right)^2 \tag{0.10}$$

Using eq. (0.9) the CDF (Cumulative Distribution Function) is given by:

$$F_X(n) = \sum_{k=-\infty}^{n} {}^2C_k \left(\frac{1}{2}\right)^2 = \begin{cases} 0 & x < 0 \\ {}^2C_0 \left(\frac{1}{2}\right)^2 = \frac{1}{4} & 0 \le x < 1 \\ {}^2C_1 \left(\frac{1}{2}\right)^2 + {}^2C_0 \left(\frac{1}{2}\right)^2 = \frac{3}{4} & 1 \le x < 2 \\ {}^2C_2 \left(\frac{1}{2}\right)^2 + {}^2C_1 \left(\frac{1}{2}\right)^2 + {}^2C_0 \left(\frac{1}{2}\right)^2 = 1 & x >= 2 \end{cases} \tag{0.11}$$

$$p(X \ge 1) = 1 - p(X < 1) \tag{0.12}$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \tag{0.13}$$

Simulation:
To run a simulation we need to generate random numbers with uniform probability, which is done as shown below(Algorithm taken from OpenSSL's random_uniform.c):

1) Generate 32 bits of entropy using /dev/urandom.
2) Treat this as a fixed point number in the range [0, 1)
3) Scale this to desired range using fixed point multiplication and treat as 64bit number(upper 32 bits integer and rest as fractional part)
4) Return the integer part of the fixed point numbers

The following shows how the relative frequency reaches true probability with increasing number of trials of the event. The blue plot is the $n = 2$ case and red plot is the $n = 10$ case.

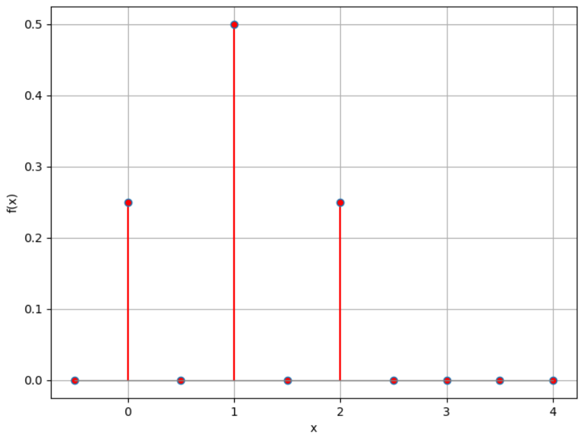

Fig. 4.1: Probability Mass Function

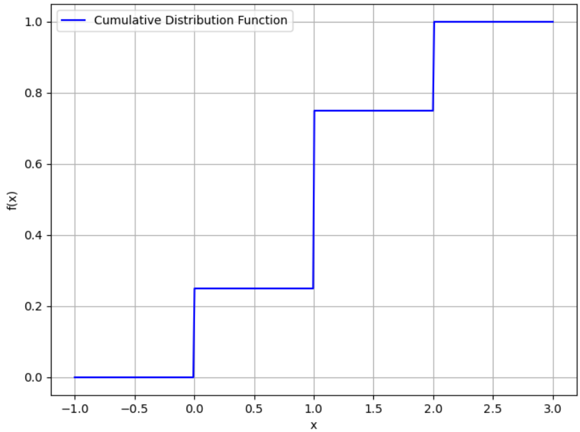

Fig. 4.2: Cumulative Distribution Function

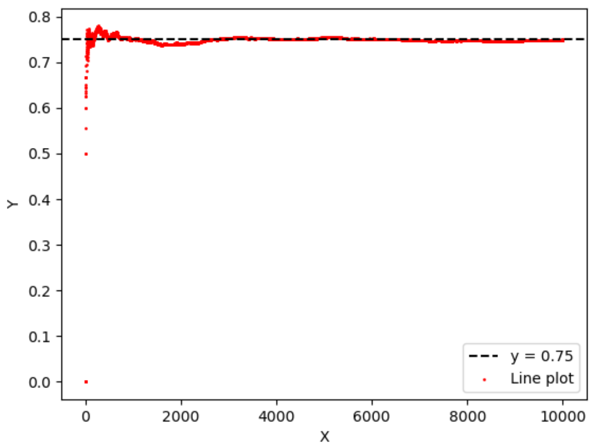

Fig. 4.3: Relative Frequency tends to True Probability

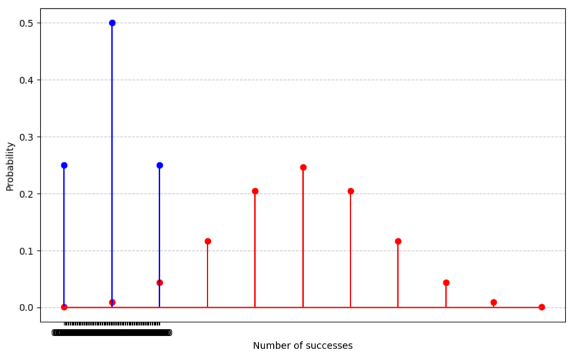

Fig. 4.4: Generating binomial distribution from bernoulli