Probability

Aditya Tripathy
Dept. of Electrical Engg.,
IIT Hyderabad.

January 29, 2025

Outline

Table Of Contents

Problem

Problem Statement

A coin is tossed twice, what is the probability that atleast one tail occurs?

Defining the random variables

Formulate in terms of bernoulli

Let Y be the random variable representing the number of tails. Y can be represented as the sum of two bernoulli random variables, X_1, X_2,

$$Y = X_1 + X_2 \tag{3.1}$$

The Bernoulli R.V is defined as,

$$X_i = \begin{cases} 0 & \text{Outcome is Heads} \\ 1 & \text{Outcome is Tails} \end{cases} \tag{3.2}$$

PMF of bernoulli

The PMF of Bernoulli R.V is given by,

$$p_X(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \end{cases} \tag{3.3}$$

Z-transform to compute binomial PMF

Using properties of Z transform of PMF on eq. (0.1),

$$M_Y(z) = M_{X_1}(z)\, M_{X_2}(z) \tag{3.4}$$

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(k)\, z^{-k} = p + (1-p)\, z^{-1} \tag{3.5}$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(k)\, z^{-k} = p + (1-p)\, z^{-1} \tag{3.6}$$

$$M_Y(z) = \left(p + (1-p)\, z^{-1}\right)^2 \tag{3.7}$$

$$= \sum_{k=-\infty}^{\infty} {}^2C_k p^{2-k} (1-p)^k z^{-k} \tag{3.8}$$

$$p_Y(n) = {}^2C_n p^{2-n} (1-p)^n \tag{3.9}$$

Final PMF of Binomial distribution

Substituting $p = \frac{1}{2}$,

$$p_Y(n) = {}^{2}C_n \left(\frac{1}{2}\right)^2 \tag{3.10}$$

CDF of binomial distribution

CDF as a sum of PMF

Using eq. (0.9) the CDF (Cumulative Distribution Function) is given by:

$$F_X(n) = \sum_{k=-\infty}^{n} {}^2C_k \left(\frac{1}{2}\right)^2 =$$

$$\begin{cases} 0 & x < 0 \\ {}^2C_0 \left(\frac{1}{2}\right)^2 = \frac{1}{4} & 0 \le x < 1 \\ {}^2C_1 \left(\frac{1}{2}\right)^2 + {}^2C_0 \left(\frac{1}{2}\right)^2 = \frac{3}{4} & 1 \le x < 2 \\ {}^2C_2 \left(\frac{1}{2}\right)^2 + {}^2C_1 \left(\frac{1}{2}\right)^2 + {}^2C_0 \left(\frac{1}{2}\right)^2 = 1 & x >= 2 \end{cases} \tag{4.1}$$

Back to the problem

$$\Pr(X \geq 1) = 1 - F_X(1) \tag{4.2}$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \tag{4.3}$$

Simulating random bits with known bias

Random bits

To run a simulation we need to generate random numbers with uniform probability, which is done as shown below(Algorithm taken from OpenSSL's random_uniform.c):

1. Generate 1byte(8 bits) of entropy using OpenSSL/rand.h.
2. Scale down this number in the range [0, 255] to [0, 1]by dividing by 255.
3. Return 0 if the scaled down number is less than p and return 1 otherwise.

PMF of random variable

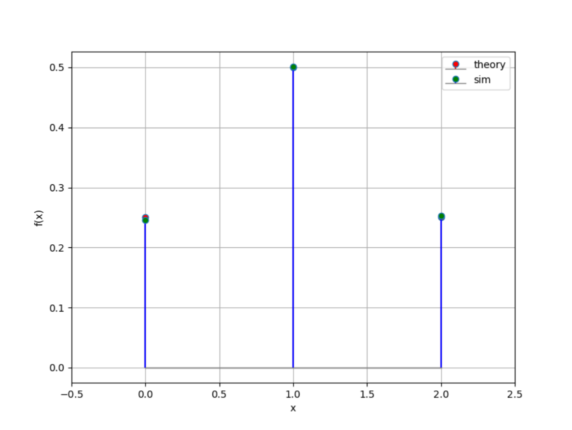

Figure 1: Probability Mass Function

CDF of random variable

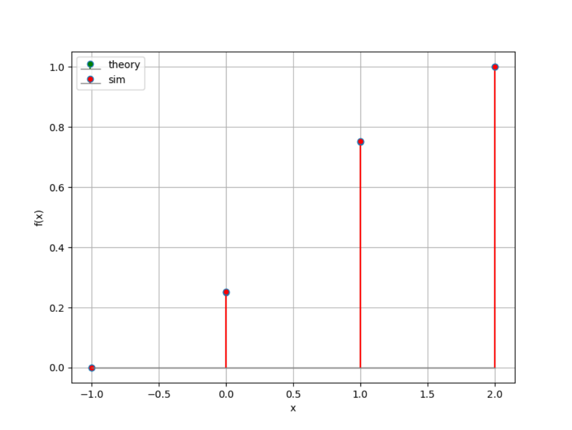

Figure 2: Cumulative Distribution Function

Generating Binomial distribution for higher n

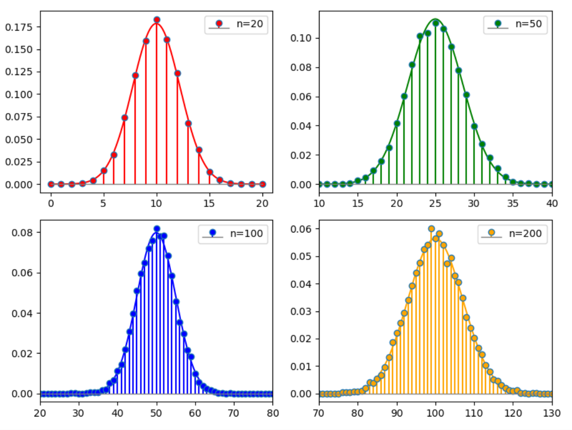

Figure 3: Generating binomial distribution from bernoulli