

9.1.2

EE24BTECH11001 - Aditya Tripathy

Question:

Plot a solution to the following differential equation:

$$y' + 5y = 0$$

Solution:

The aim is to find the difference equation using the trapezoidal law using the following initial conditions, $x_0 = 0, y_0 = 1$

$$y' = -5y \quad (0.1)$$

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} -5y dx \quad (0.2)$$

$$(0.3)$$

Using the trapezoidal rule,

$$J = \int_a^b f(x) dx \quad (0.4)$$

$$\approx h \left(\frac{1}{2} f(x) + f(x_1) + f(x_2) \cdots + f(x_{n-1}) + \frac{1}{2} f(b) \right) \quad (0.5)$$

$$\frac{y_{n+1} - y_n}{1} \left(\frac{1}{2} + \frac{1}{2} \right) = -5 \frac{x_{n+1} - x_n}{1} \left(\frac{y_n}{2} + \frac{y_{n+1}}{2} \right) \quad (0.6)$$

$$\rightarrow y_{n+1} = \frac{(2 - 5h) y_n}{5h + 2} \quad (0.7)$$

Another way we can arrive at the difference equation is by using the Bilinear transform, Applying Laplace Transform to both sides of the differential equation,

$$sY - sy_0 + 5Y = 0 \quad (0.8)$$

$$Y(s) = \frac{sy_0}{s + 5} \quad (0.9)$$

$$(0.10)$$

Apply Bilinear Transform, with $T = h$

$$s = \frac{2}{T} \frac{1 + z^{-1}}{1 - z^{-1}} \quad (0.11)$$

$$Y(z) = \frac{2y_0(1 + z^{-1})}{2(1 + z^{-1}) + 5h(1 - z^{-1})} \quad (0.12)$$

$$Y(z) = \frac{2y_0(1 + z^{-1})}{(2 + 5h) - (5h - 2)z^{-1}} \quad (0.13)$$

$$\alpha = \frac{5h - 2}{5h + 2} \quad (0.14)$$

$$Y = \frac{2y_0}{5h + 2} \left(\frac{1}{1 - \alpha z^{-1}} + \frac{z^{-1}}{1 - \alpha z^{-1}} \right) \quad (0.15)$$

Applying inverse Z-transform,

$$y_n = \frac{2y_0}{5h + 2} (\alpha^n u[n] + \alpha^{n-1} u[n - 1]) \quad (0.16)$$

$$(0.17)$$

To check how close the approximate graph is to the actual solution, we will solve the original equation using a Laplace Transform method:

Let $\mathcal{L}(y) = Y$

$$(sY - y_0) + 5Y = 0 \quad (0.18)$$

$$(s + 5)Y = y_0 \quad (0.19)$$

$$Y = \frac{1}{s + 5} \quad (0.20)$$

$$(0.21)$$

Substituting $y_0 = 1$,

$$Y = \frac{1}{s + 5} \quad (0.22)$$

$$(0.23)$$

Now, we take the inverse laplace transform to get a solution,

$$\mathcal{L}^{-1} \left(\frac{1}{s + 5} \right) = e^{-5x} u(x) \quad (0.24)$$

For the following approximate graph, I chose $h = 0.01$ and $h = 0.1$.

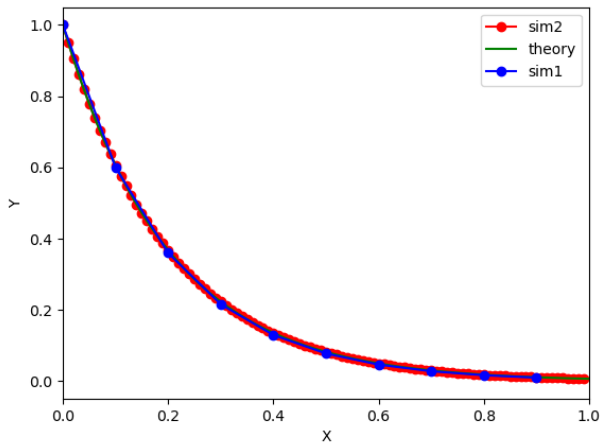


Fig. 0.1: Approximate solution of the DE