# 9.6.1

EE24BTECH11001 - Aditya Tripathy

**Question**:
Plot a solution to the following differential equation:
$y' + 2y = \sin x$
**Solution:**
To plot a curve in the solution family, we take the initial condition to be
$x_0 = 0, y_0 = 1$
Using Euler's Method, we represent the the differential equation in the following difference equations:

$$x_{n+1} = x_n + h \tag{0.1}$$

$$y_{n+1} - y_n + 2hy_n = h \sin x_n \tag{0.2}$$

$$\rightarrow y_{n+1} = (1 - 2h) y_n + h \sin x_n \tag{0.3}$$

We can get the solution to the difference equation in $y$ using the unilateral Z-transform. Applying the unilateral Z-transform to both sides of equation,(representing transform of $y_n$ with $Y(z)$)

$$Y(z) = \sum_{n=0}^{\infty} y_n z^{-n} \tag{0.4}$$

$$\tag{0.5}$$

Therefore,

$$zY(z) - zy_0 = (1 - 2h) Y(z) + hZ(\{\sin x_n\}) \tag{0.6}$$

$$\rightarrow Y(z)(z - (1 - 2h)) = zy_0 + hZ(\{\sin x_n\}) \tag{0.7}$$

$$Y(z) = \frac{zy_0}{z - (1 - 2h)} + \frac{h}{z - (1 - 2h)} Z(\{\sin x_n\}) \tag{0.8}$$

We know that the Z-transform has the following properties,

1) If the signal $x[n]$ has Z-transform $X(z)$, then Z-transform of $z^{-1}X(z)$
2) If $X_1(z), X_2(z)$ correspond to the Z-transforms of the signals $x_1[n], x_2[n]$ then $X_1(z) X_2(z)$ corresponds to the Z-transform of $x_1[n] \circledast x_2[n]$, where $\circledast$ corresponds to the convolution operator

Applying Inverse Z-transform, we notice
For the signal $x[n] = \alpha^n u(n), (u[n]$ being the unit step signal)

$$X\left(\alpha^{n} u[n]\right)=\frac{1}{1-\alpha z^{-1}}, \text{provided } |z|<|\alpha| \tag{2.1}$$

$$\rightarrow\left(\alpha^{n-1} u[n-1]\right)=\frac{1}{1-\alpha z^{-1}}, \text{provided } |z|>|\alpha| \tag{2.2}$$

On comparison, $\alpha=(1-2h)$, so the solution for $y_n$ is:

$$y_n=y_0(1-5h)^n u[n] \tag{2.3}$$

Substituting $y_0=1$,

$$y_n=(1-2h)^n u[n]+h\sin x_n \circledast (1-2h)^{n-1} u[n-1] \tag{2.4}$$

Now we can iteratively generate points which lie close to the graph.
To check how close the approximate graph is to the actual solution, we will solve the original equation using a Laplace Transform method:
Let $\mathcal{L}(y)=Y$

$$(sY-y_0)+2Y=\mathcal{L}(\sin x) \tag{2.5}$$

$$\mathcal{L}(\sin x)=\int_0^{\infty} e^{-sx}\sin x=\frac{1}{s^2+1} \tag{2.6}$$

$$(s+2)Y=y_0+\frac{1}{s^2+1} \tag{2.7}$$

$$Y=\frac{y_0}{s+2}+\frac{1}{(s^2+1)(s+2)} \tag{2.8}$$

$$\tag{2.9}$$

Using method of partial fractions,

$$\frac{1}{(s^2+1)\,s+2}=\frac{a}{s+2}+\frac{bs+c}{s^2+1} \tag{2.10}$$

$$\tag{2.11}$$

On solving we get,

$$a=\frac{1}{5} \tag{2.12}$$

$$b=\frac{-1}{5} \tag{2.13}$$

$$c=\frac{2}{5} \tag{2.14}$$

$$\tag{2.15}$$

Substituting $y_0=1$,

$$Y = \frac{1}{s+2} + \frac{0.2}{s+2} + \frac{-0.2s}{s^2+1} + \frac{0.4}{s^2+1} \tag{2.16}$$

$$\rightarrow Y = \frac{1.2}{s+2} + \frac{-0.2s}{s^2+1} + \frac{0.4}{s^2+1} \tag{2.17}$$

$$\tag{2.18}$$

Now, we take the inverse laplace transform to get a solution,

$$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2x}u(x) \tag{2.19}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin xu(x) \tag{2.20}$$

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos xu(x) \tag{2.21}$$

Therefore the final solution to the differential equation is,

$$y(x) = \left(1.2e^{-2x} - 0.2\cos x + 0.4\sin x\right)u(x) \tag{2.22}$$

For the following approximate graph, I chose $h = 0.01$ and $h = 0.1$.

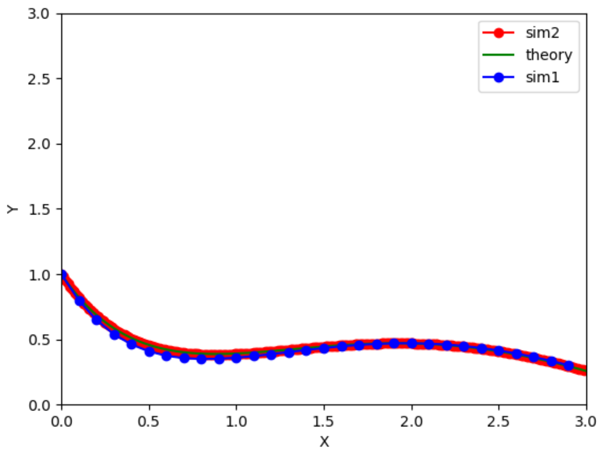


Fig. 2.1: Approximate solution of the DE