

# Optimization

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# Outline

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# Problem

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## Problem Statement

Find the local minimum/maximum of the given function:

$$f(x) = x^2$$

# Gradient Descent

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## Getting the Difference Equation

We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of  $x^2 > 0$ , we expect to find a local minimum.

$$x_{n+1} = x_n - \mu f'(x_n) \quad (3.1)$$

$$f'(x_n) = 2x_n \quad (3.2)$$

$$\rightarrow x_{n+1} = x_n - 2\mu x_n \quad (3.3)$$

$$= (1 - 2\mu) x_n \quad (3.4)$$

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1 - 2\mu)X(z) \quad (3.5)$$

$$(z - (1 - 2\mu))X(z) = zx_0 \quad (3.6)$$

$$X(z) = \frac{zx_0}{z - (1 - 2\mu)} \quad (3.7)$$

$$X(z) = \frac{x_0}{1 - (1 - 2\mu)z^{-1}} \quad (3.8)$$

$$= \sum_{n=0}^{\infty} (1 - 2\mu)^n z^{-n} \quad (3.9)$$



From the last equation, ROC is

$$|z| > |1 - 2\mu| \quad (3.10)$$

$$\rightarrow 1 > |1 - 2\mu| > 0 \quad (3.11)$$

$$\rightarrow \mu \in (0, 1) \setminus \left[ \frac{1}{2} \right] \quad (3.12)$$

Equation 3.11 is due to the following theorem

# Theorem

An LTI system is stable if and only if the ROC of its system function contains the unit circle,  $|z| = 1$

## Finding point of convergence

Now, if  $\mu$  satisfies the previous condition,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \quad (3.13)$$

$$\rightarrow \lim_{n \rightarrow \infty} \|(-2\mu) x_n\| = 0 \quad (3.14)$$

$$\rightarrow \|(-2\mu)\| \lim_{n \rightarrow \infty} \|x_n\| = 0 \quad (3.15)$$

$$\rightarrow \lim_{n \rightarrow \infty} \|x_n\| = 0 \quad (3.16)$$

$$\rightarrow \lim_{n \rightarrow \infty} x_n = 0 \quad (3.17)$$

Taking initial guess = 8

step size = 0.01

tolerance(minimum value of gradient) =  $1e-5$

We get

$$x_{min} = 4.910792425174398e - 06$$

# Quadratic Programming

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# Reformulating

As a fun exercise, the given question can be posed as the following quadratic programming problem:

Find the point lying on the line  $y = 1$ , which is nearest to origin.

We can formulate the problem as follows:

$$\min_{\mathbf{x}} \|\mathbf{e}_2^\top \mathbf{x}\|^2 \quad (4.1)$$

$$\text{s.t.} \quad (4.2)$$

$$\mathbf{x}^\top V \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (4.3)$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (4.4)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \quad (4.5)$$

$$f = 0 \quad (4.6)$$

$$(4.7)$$

## Relaxation of Constraint

In the current form, the constraint is non-convex since the constraint defines a set which is not convex, since points on the line joining any 2 points on the curve don't belong to the set. However, if we become lenient and make the constraint

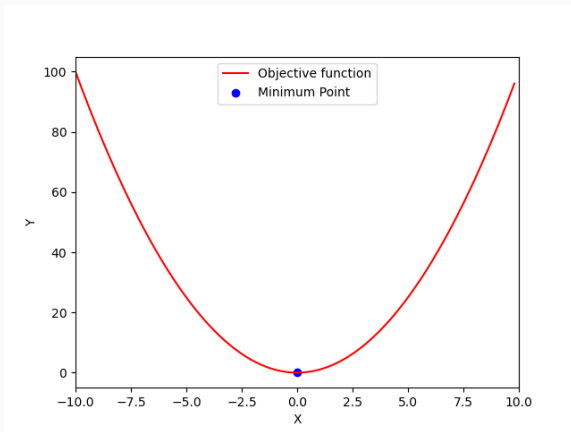
$$\mathbf{x}^\top V \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} \leq 0 \quad (4.8)$$

the constraint becomes convex. Using `cvxpy` to solve this convex optimization problem, we get

$$\text{Optimal } x : [[0.00000000e + 00] \quad (4.9)$$

$$[8.63053066e - 11]] \quad (4.10)$$

# Graph



**Figure 1:** Minimum Value of Objective function