

6.5.3.1

EE24BTECH11001 - Aditya Tripathy

Question:

Find the local minimum/maximum of the given function:

$$f(x) = x^2$$

Solution:

We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $x^2 > 0$, we expect to find a local minimum.

$$x_{n+1} = x_n - \mu f'(x_n) \quad (0.1)$$

$$f'(x_n) = 2x_n \quad (0.2)$$

$$\rightarrow x_{n+1} = x_n - 2\mu x_n \quad (0.3)$$

$$= (1 - 2\mu) x_n \quad (0.4)$$

Applying unilateral Z-transform,

$$zX(z) - zx_0 = (1 - 2\mu) X(z) \quad (0.5)$$

$$(z - (1 - 2\mu)) X(z) = zx_0 \quad (0.6)$$

$$X(z) = \frac{zx_0}{z - (1 - 2\mu)} \quad (0.7)$$

$$X(z) = \frac{x_0}{1 - (1 - 2\mu) z^{-1}} \quad (0.8)$$

$$= \sum_{n=0}^{\infty} (1 - 2\mu)^n z^{-n} \quad (0.9)$$

From the last equation, ROC is

$$|z| > |1 - 2\mu| \quad (0.10)$$

$$\rightarrow |1 - 2\mu| > 0 \quad (0.11)$$

$$\rightarrow \mu \in \mathcal{R} \setminus \left\{ \frac{1}{2} \right\} \quad (0.12)$$

Now, if μ satisfies the previous condition,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \quad (0.13)$$

$$\rightarrow \lim_{n \rightarrow \infty} \|(1 - 2\mu) x_n\| = 0 \quad (0.14)$$

$$\rightarrow \|(1 - 2\mu)\| \lim_{n \rightarrow \infty} \|x_n\| = 0 \quad (0.15)$$

$$\rightarrow \lim_{n \rightarrow \infty} \|x_n\| = 0 \quad (0.16)$$

$$\rightarrow \lim_{n \rightarrow \infty} x_n = 0 \quad (0.17)$$

Taking initial guess = 8

step size = 0.01

tolerance(minimum value of gradient) = 1e-5

We get

$x_{min} = 4.910792425174398e - 06$

As a fun exercise, the given question can be posed as the following quadratic programming problem:

Find the point lying on the line $y = 1$, which is nearest to origin.

We can formulate the problem as follows:

$$\min_{\mathbf{x}} \|e_2^\top \mathbf{x}\|^2 \quad (0.18)$$

$$\text{s.t.} \quad (0.19)$$

$$\mathbf{x}^\top V \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (0.20)$$

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (0.21)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -0.5 \end{pmatrix} \quad (0.22)$$

$$f = 0 \quad (0.23)$$

Using `scipy.minimize` to solve the convex optimization problem, we get

Optimal value: 0.0

Optimal x: [0. 0.]

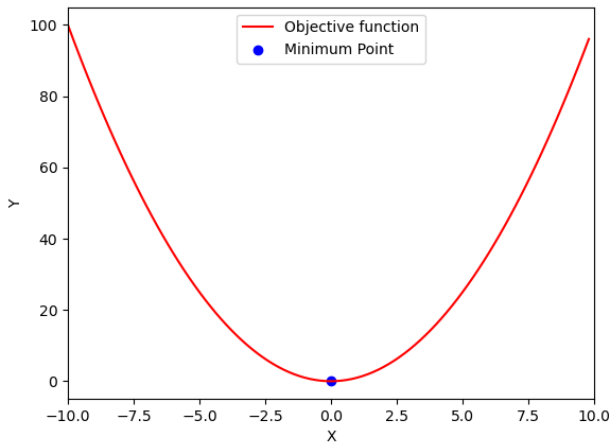


Fig. 0.1: Minimum value of objective function