

10.4.1.2

EE24BTECH11001 - Aditya Tripathy

Question:

Find the roots of the equation $x^2 - 2x = (-2)(3 - x)$

Solution:

On simplification, the equation to solve is,

$$x^2 - 4x + 6 = 0 \quad (0.1)$$

The motive now is to solve the equation at hand using a matrix based approach.

Recall that for a matrix A of order n , the characteristic equation is given by,

$$\det(A - \lambda I) = a_n \lambda^n + a_{n-1} \lambda^{n-1} \cdots + a_0 = 0 \quad (0.2)$$

Recall that the solutions to the characteristic polynomial are the eigenvalues of the the matrix A . So if we can somehow construct the corresponding matrix from the characteristic polynomial, our job will be finished, since we can find the eigen values using the QR algorithm.

Companion matrix: A matrix is said to be the companion of a polynomial $f(x)$ if $\det(A - \lambda I) = 0 \implies f(x) = 0$.

For,

$$f(x) = c_0 + c_1 x \cdots + x^n \quad (0.3)$$

The companion matrix is,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \quad (0.4)$$

For the equation at hand, the companion matrix is,

$$\begin{pmatrix} 0 & -6 \\ 1 & 4 \end{pmatrix} \quad (0.5)$$

Using the QR algorithm we can now solve for the eigenvalues and thus the solutions for the given equation.

Understanding QR algorithm,

The basic idea is to use similarity transforms

$$B = P^{-1}AP \quad (0.6)$$

to get the Schur decomposition,

$$A = QUQ^{-1} \quad (0.7)$$

for some unitary matrix Q (so that the inverse Q^{-1} is also the conjugate transpose Q^* of Q), and some upper triangular matrix U . The diagonal entries of the U matrix are the eigenvalues.

The steps to solve QR quickly :

- 1) Convert original matrix to an intermediate form (hessenberg form)
- 2) Apply givens rotation
- 3) Diagonal elements are eigenvalues

Step 1:

Since a 2×2 matrix is always in Hessenberg form, we need not proceed with step 1.

The general method for converting to upper hessenberg is :

Definition: A matrix of the form

$$P = I - 2\mathbf{u}\mathbf{u}^* \quad (3.1)$$

$$(3.2)$$

is called a Householder reflector.

$$P\mathbf{x} = \mathbf{x} - \mathbf{u}(2\mathbf{u}^*\mathbf{x}) \quad (3.3)$$

$$(3.4)$$

A task that we repeatedly want to carry out with Householder reflectors is to transform a vector \mathbf{x} on a multiple of e_1

$$P\mathbf{x} = \mathbf{x} - \mathbf{u}(2\mathbf{u}^*\mathbf{x}) = \alpha e_1 \quad (3.5)$$

$$(3.6)$$

Since P is unitary, we must have $\alpha = \rho\|\mathbf{x}\|$, where $\rho \in \mathbb{C}$ has absolute value one. Therefore,

$$\mathbf{u} = \frac{\mathbf{x} - \rho\|\mathbf{x}\|e_1}{\|\mathbf{x} - \rho\|\mathbf{x}\|e_1\|} \quad (3.7)$$

We can freely choose ρ provided that $|\rho| = 1$. Let $x_1 = |x_1|e^{iu_k^*\phi}$. To avoid numerical cancellation we set $\rho = -e^{i\phi}$. In the real case, one commonly sets $\rho = -\text{sign}(x_1)$. If $x_1 = 0$ we can set ρ in any way.

To visualize

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & I_4 - 2u_1u_1^* \end{bmatrix} \quad (3.8)$$

$$A = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_1^*} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} \xrightarrow{*P_1} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} \quad (3.9)$$

$$P_1^* A P_1 = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} \xrightarrow{P_2^*/*P_2} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix} \xrightarrow{P_3^*/*P_3} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \\ = P_3 P_2 P_1 A P_1 P_2 P_3 \quad (3.10)$$

Step 2: Constructing Givens Rotation Matrix,

$$c_k = \frac{0}{\sqrt{0^2 + 1^2}} = 0 \quad (3.11)$$

$$s_k = \frac{1}{\sqrt{0^2 + 1^2}} = 1 \quad (3.12)$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -6 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 6 & 0 \end{pmatrix} \quad (3.13)$$

Givens rotations are used to zero specific elements of a vector or matrix by rotating the vector in the plane of two coordinates. A Givens rotation matrix is defined as:

$$G(i, j, \theta) = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -\bar{s} & \dots & \bar{c} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{pmatrix} \quad (3.14)$$

where

$$c_k = \frac{\bar{x}_i}{\sqrt{|x_i|^2 + |x_j|^2}} \quad (3.15)$$

$$s_k = \frac{\bar{x}_j}{\sqrt{|x_i|^2 + |x_j|^2}} \quad (3.16)$$

As eq 3.12 shows, QR algorithm has failed since we did not get a upper triangular block, since the eigenvalues are complex and the entries of the matrix are real.

Sometimes the QR algorithm doesn't cleanly converge to upper triangular but has certain 2x2 blocks called Jordan blocks. The rest of the diagonal entries are the real eigenvalues

and the complex conjugate pair of eigenvalues is given by manually solving for the Jordan Blocks.

$$\begin{bmatrix} a_0 & & & & & & \\ & a_1 & & & & & \\ & & a_2 & & & & \\ & & & a_3 & a_4 & & \\ & & & a_5 & a_6 & & \end{bmatrix} \quad (3.17)$$

On running the code to solve for eigenvalues we get:

Upper Hessenberg Matrix

```
0.0000000000 +0.0000000000i -6.0000000000 +0.0000000000i
1.0000000000 +0.0000000000i 4.0000000000 +0.0000000000i
eigenvalue 1: 2.000000 + 1.414214i
eigenvalue 2: 2.000000 + -1.414214i
```