11.16.3.2

EE24BTECH11001 - Aditya Tripathy

Question:
A coin is tossed twice, what is the probability that atleast one tail occurs?
Solution:
The sample space is

$$\Omega = \{HH, HT, TH, TT\} \tag{0.1}$$

Assuming equally likely outcomes,

$$\Pr(\omega \in \Omega) = \frac{1}{4} \tag{0.2}$$

Define a discrete random variable X = number of tails in the sequence. Probability Mass Function $\Pr_X(x)$ is given by:

$$\Pr_X(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases} \tag{0.3}$$

The CDF (Cumulative Distribution Function) is given by:

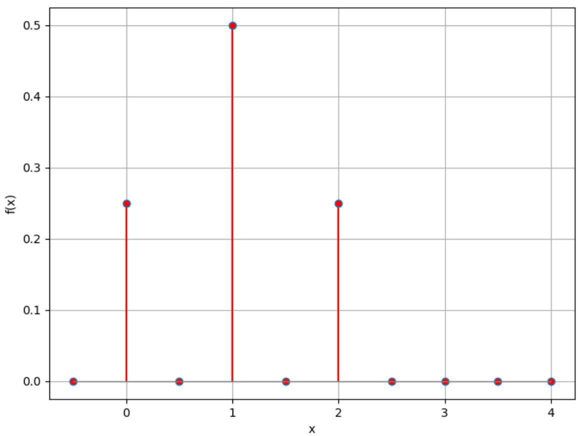

Fig. 0.1: Probability Mass Function

$$F_X(x) = \Pr_X(X \le x) = \begin{cases} 0 & x < 0 \\ \frac{1}{4} & 0 \le x < 1 \\ \frac{3}{4} & 1 \le x < 2 \\ 1 & x >= 2 \end{cases} \tag{0.4}$$

$$\Pr(X \ge 1) = 1 - \Pr(X < 1) \tag{0.5}$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \tag{0.6}$$

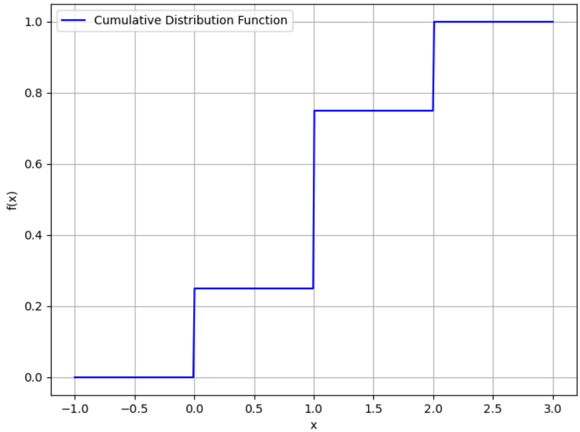

Fig. 0.2: Cumulative Distribution Function

Simulation:
To run a simulation we need to generate random numbers with uniform probability, which is done as shown below(Algorithm taken from OpenSSL's random_uniform.c):

1) Generate 32 bits of entropy using /dev/urandom.
2) Treat this as a fixed point number in the range [0, 1)
3) Scale this to desired range using fixed point multiplication and treat as 64bit number(upper 32 bits integer and rest as fractional part)
4) Return the integer part of the fixed point numbers

The following shows how the relative frequency reaches true probability with increasing number of trials of the event.
Approach 2 :
In this approach, we treat our random variable as the sum of outcomes of two bernoulli random variables

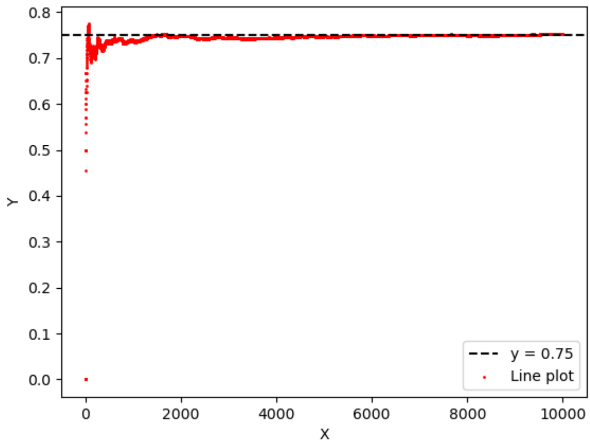

Fig. 4.1: Relative Frequency tends to True Probability

$$X = X_1 + X_2 \tag{4.1}$$

where,

$$X_i = \begin{cases} 0 & \text{Outcome is Heads} \\ 1 & \text{Outcome is Tails} \end{cases} \tag{4.2}$$

$$\Pr_X(n) = \begin{cases} p = \frac{1}{2} & n = 0 \\ 1 - p = \frac{1}{2} & n = 1 \end{cases} \tag{4.3}$$

Our random variable is a sum of a two bernoulli random variables

$$Y = X_1 + X_2 \tag{4.4}$$

Using properties of Z transform of PMF,

$$M_Y(z) = M_{X_1}(z) M_{X_2}(z) \tag{4.5}$$

$$M_{X_1}(z) = \sum_{n=-\infty}^{\infty} \Pr_{X_1}(n) z^{-n} = p + (1-p) z^{-1} \tag{4.6}$$

$$M_{X_2}(z) = \sum_{n=-\infty}^{\infty} \Pr_{X_2}(n) z^{-n} = p + (1-p) z^{-1} \tag{4.7}$$

$$M_Y(z) = \left(p + (1-p) z^{-1}\right)^2 \tag{4.8}$$

$$= \sum_{n=-\infty}^{\infty} {}^2C_k p^{2-k} (1-p)^k z^{-k} \tag{4.9}$$

$$P_Y(n) = {}^2C_k p^{2-k} (1-p)^k \tag{4.10}$$

The blue plot is the $n = 2$ case and red plot is the $n = 10$ case.

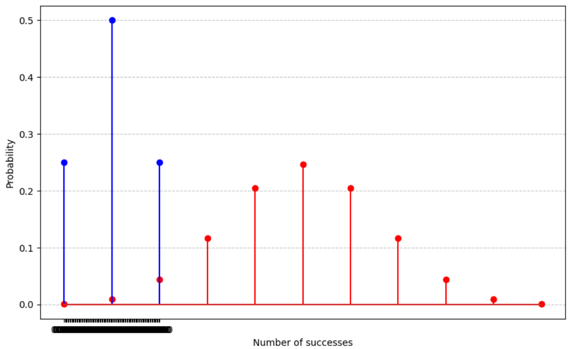

Fig. 4.2: Generating binomial distribution from bernoulli