11.16.3.2

EE24BTECH11001 - Aditya Tripathy

Question:
A coin is tossed twice, what is the probability that atleast one tail occurs?
Solution :
Let Y be the random variable representing the number of tails. Y can be represented as the sum of two bernoulli random variables, X_1, X_2,

$$Y = X_1 + X_2 \tag{0.1}$$

The Bernoulli R.V is defined as,

$$X_i = \begin{cases} 0 & \text{Outcome is Heads} \\ 1 & \text{Outcome is Tails} \end{cases} \tag{0.2}$$

The PMF of Bernoulli R.V is given by,

$$p_X(n) = \begin{cases} p & n = 0 \\ 1 - p & n = 1 \end{cases} \tag{0.3}$$

Using properties of Z transform of PMF on eq. (0.1),

$$M_Y(z) = M_{X_1}(z) M_{X_2}(z) \tag{0.4}$$

$$M_{X_1}(z) = \sum_{k=-\infty}^{\infty} p_{X_1}(k) z^{-k} = p + (1 - p) z^{-1} \tag{0.5}$$

$$M_{X_2}(z) = \sum_{k=-\infty}^{\infty} p_{X_2}(k) z^{-k} = p + (1 - p) z^{-1} \tag{0.6}$$

$$M_Y(z) = \left(p + (1 - p) z^{-1}\right)^2 \tag{0.7}$$

$$= \sum_{k=-\infty}^{\infty} {}^2C_k p^{2-k} (1 - p)^k z^{-k} \tag{0.8}$$

$$p_Y(n) = {}^2C_n p^{2-n} (1 - p)^n \tag{0.9}$$

Substituting $p = \frac{1}{2}$,

$$p_Y(n) = {}^2C_n \left(\frac{1}{2}\right)^2 \tag{0.10}$$

Using eq. (0.9) the CDF (Cumulative Distribution Function) is given by:

$$F_X(n) = \sum_{k=-\infty}^{n} {}^2C_k \left(\frac{1}{2}\right)^2 = \begin{cases} 0 & x < 0 \\ {}^2C_0\left(\frac{1}{2}\right)^2 = \frac{1}{4} & 0 \leq x < 1 \\ {}^2C_1\left(\frac{1}{2}\right)^2 + {}^2C_0\left(\frac{1}{2}\right)^2 = \frac{3}{4} & 1 \leq x < 2 \\ {}^2C_2\left(\frac{1}{2}\right)^2 + {}^2C_1\left(\frac{1}{2}\right)^2 + {}^2C_0\left(\frac{1}{2}\right)^2 = 1 & x >= 2 \end{cases} \tag{0.11}$$

$$p(X \geq 1) = 1 - p(X < 1) \tag{0.12}$$

$$= 1 - \frac{1}{4} = \frac{3}{4} \tag{0.13}$$

Simulation:
To run a simulation we need to generate random numbers with uniform probability, which is done as shown below(Algorithm taken from OpenSSL's random_uniform.c):

1) Generate 1byte(8 bits) of entropy using OpenSSL/rand.h.
2) Scale down this number in the range [0, 255] to [0, 1]by dividing by 255.
3) Return 0 if the scaled down number is less than p and return 1 otherwise.

The following shows how the relative frequency reaches true probability with increasing number of trials of the event. The blue plot is the $n = 2$ case and red plot is the $n = 10$ case.

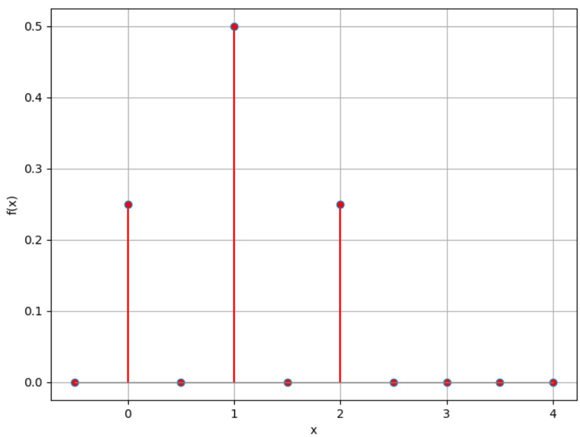

Fig. 3.1: Probability Mass Function

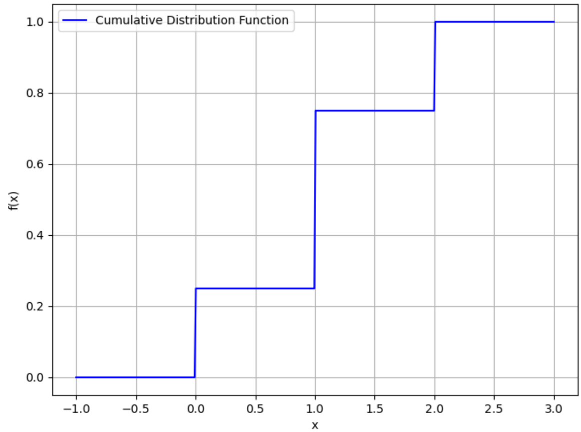

Fig. 3.2: Cumulative Distribution Function

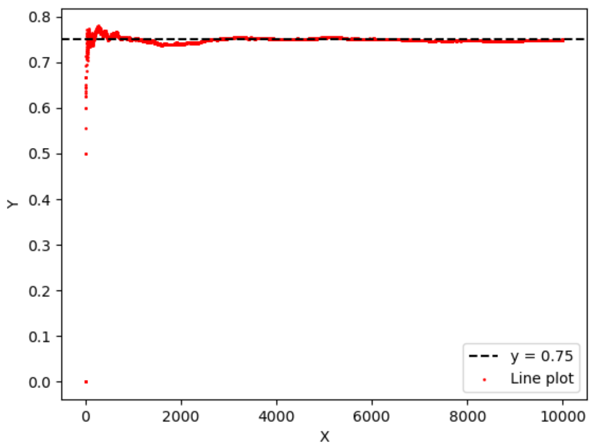

Fig. 3.3: Relative Frequency tends to True Probability

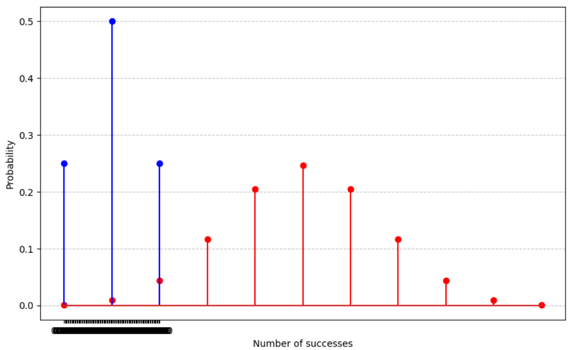

Fig. 3.4: Generating binomial distribution from bernoulli