# Box Kernel Convolution Analysis

*An analysis on convolution effects with the box kernel*

*by*

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# Contents

# Chapter 1

# Convolving with the rect Function: Running Average

## 1.1 Rectangular Kernel

A rectangular kernel of width $2T$ is defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

For normalization purposes, we often scale the kernel to ensure its integral equals 1:

$$h(t) = \begin{cases} \frac{1}{2T}, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{1.2}$$

## 1.2 Convolution as Running Average

When we convolve a signal $f(t)$ with the normalized rectangular kernel, we get:

$$(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\,d\tau \tag{1.3}$$

$$= \int_{-\infty}^{\infty} f(\tau) \cdot \begin{cases} \frac{1}{2T}, & \text{for } -T \leq t-\tau \leq T \\ 0, & \text{otherwise} \end{cases} d\tau \tag{1.4}$$

The limits of integration are determined by $-T \leq t - \tau \leq T$, which gives us $t - T \leq \tau \leq t + T$. Therefore:

$$(f * h)(t) = \int_{t-T}^{t+T} f(\tau) \cdot \frac{1}{2T} \, d\tau \tag{1.5}$$

$$= \frac{1}{2T} \int_{t-T}^{t+T} f(\tau) \, d\tau \tag{1.6}$$

This is precisely the definition of a running average: at each point $t$, we compute the average value of the function $f$ over the interval $[t-T, t+T]$. The width of this averaging window is $2T$.

## 1.3 Conclusion

Convolution with the rectangular kernel causes a very important smoothing effect on convolution. Computing the convolution smooths out the original function by taking a running average over a short window and hence the reduces the perceived ”spiky-ness” of the function. The removal (or reduction) of such spikiness will be demonstrated in Chapter 3.

# Chapter 2

# Analytical Solution

## 2.1 Introduction

This report analyzes the convolution of various input signals with rectangular kernels. We examine the analytical solutions for sinusoidal inputs and investigate how kernel modifications affect the output.

## 2.2 Theoretical Background

The convolution of two functions $f(t)$ and $h(t)$ is defined as:

$$(f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, d\tau \tag{2.1}$$

For our analysis, we consider a rectangular kernel defined as:

$$h(t) = \begin{cases} 1, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

## 2.3 Convolution with $\sin(\omega t)$

### 2.3.1 Case 1: Symmetric Kernel

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \frac{1}{\omega}\int\limits_{-\infty}^{\infty} h(\tau)\sin(\omega(t-\tau))d\tau \\
&= \frac{1}{\omega}\int\limits_{-T}^{T} 1.\sin(\omega(t-\tau))d\tau \\
&= \frac{1}{\omega}[\cos(\omega(t-\tau))]_{-T}^{T} \\
&= \frac{1}{\omega}(\cos(\omega(t-T)) - \cos(\omega(t+T))) \\
&= \frac{2}{\omega}\sin\omega t\sin\omega T
\end{aligned}
$$

Therefore , output of the system is $y(t) = \frac{2}{\omega}\sin\omega t\sin\omega T$

### 2.3.2 Case 2: Kernel in $[0, T]$

Let us consider input of the given system is x(t) = sin(t) and output be y(t) , then

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \frac{1}{\omega}\int\limits_{-\infty}^{\infty} h(\tau)\sin(\omega(t-\tau))d\tau \\
&= \frac{1}{\omega}\int\limits_{0}^{T} 1.\sin(\omega(t-\tau))d\tau \\
&= \frac{1}{\omega}[\cos(\omega(t-\tau))]_{0}^{T} \\
&= \frac{1}{\omega}(\cos(\omega(t-T)) - \cos(\omega(t-0))) \\
&= \frac{1}{\omega}(\cos(\omega(t-T)) - \cos(\omega t))
\end{aligned}
$$

Therefore , output of the system is $y(t) = \frac{1}{\omega}(\cos(\omega(t-T)) - \cos(\omega t))$

### 2.3.3 Case 3: Shifted Kernel

Let us consider input of the given system is $x(t) = \sin(t)$ and output be $y(t)$ , then

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \frac{1}{\omega} \int_{-\infty}^{\infty} h(\tau) \sin\left(\omega(t-\tau)\right) d\tau \\
&= \frac{1}{\omega} \int_{-T+\tau_0}^{T+\tau_0} 1. \sin\left(\omega(t-\tau)\right) d\tau \\
&= \frac{1}{\omega} [\cos\left(\omega(t-\tau)\right)]_{-T+\tau_0}^{T+\tau_0} \\
&= \frac{1}{\omega} (\cos\left(\omega(t-\tau_0-T)\right) - \cos\left(\omega(t-\tau_0+T)\right)) \\
&= \frac{2}{\omega} \sin \omega(t-\tau_0) \sin \omega T
\end{aligned}
$$

Therefore , output of the system is $y(t) = \frac{2}{\omega} \sin \omega(t-\tau_0) \sin \omega T$
Here , The output signal also got shifted by $\tau_0$ units.

## 2.4 Convolution with $e^{at}$

### 2.4.1 Case 1: Symmetric Kernel

Let us consider input of the given system is x(t) = $e^t$ and output be y(t) , then

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} h(\tau) A e^{a(t-\tau)} u(t-\tau) d\tau \\
&= e^{(at)} \int_{-T}^{min(T,t)} e^{a(-\tau)} d\tau
\end{aligned}
$$

for $T < t$

$$= \frac{e^{(at)}}{a}[-e^{a(-\tau)}]_{-T}^{T}$$
$$= \frac{e^{(at)}}{a}(e^{aT} - e^{-aT})$$

for $T > t$

$$= \frac{e^{(at)}}{a}[-e^{a(-\tau)}]_{-T}^{t}$$
$$= \frac{e^{(at)}}{a}(e^{aT} - e^{-at})$$

Therefore , output of the system is

$$y(t) = \begin{cases} \frac{1}{a}(e^{a(t+t)} - 1) & t < T \\ \frac{e^{(at)}}{a}(e^{aT} - e^{-aT}) & t > T \end{cases}$$

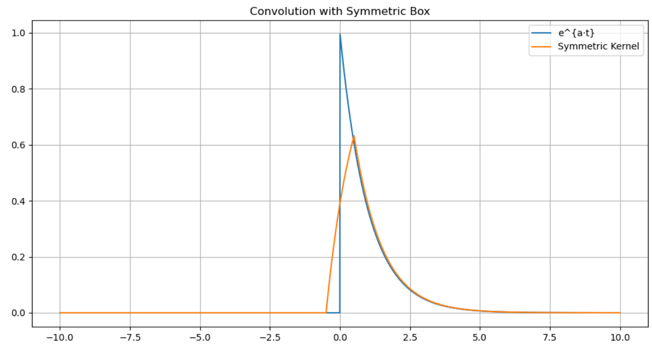



FIGURE 2.1: Convolution with Symmetric Kernel

### 2.4.2 Case 2: Pulse $[0, T]$

Let us consider input of the given system is x(t) = $e^t u(t)$ and output be y(t) , then

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} h(\tau) e^{a(t-\tau)} u(t-\tau) d\tau \\ &= e^{(at)} \int_{0}^{min(t,T)} e^{a(-\tau)} d\tau \end{aligned}$$

for $T < t$

$$\begin{aligned} &= \frac{e^{(at)}}{a} [-e^{a(-\tau)}]_0^T \\ &= \frac{e^{(at)}}{a} (1 - e^{-aT}) \end{aligned}$$

for $T > t$

$$\begin{aligned} &= \frac{e^{(at)}}{a} [-e^{a(-\tau)}]_0^t \\ &= \frac{e^{(at)}}{a} (1 - e^{-at}) \end{aligned}$$

Therefore , output of the system is

$$y(t) = \begin{cases} \frac{1}{a}(e^{(at)} - 1) & t < T \\ \frac{e^{(at)}}{a}(1 - e^{-aT}) & t > T \end{cases}$$

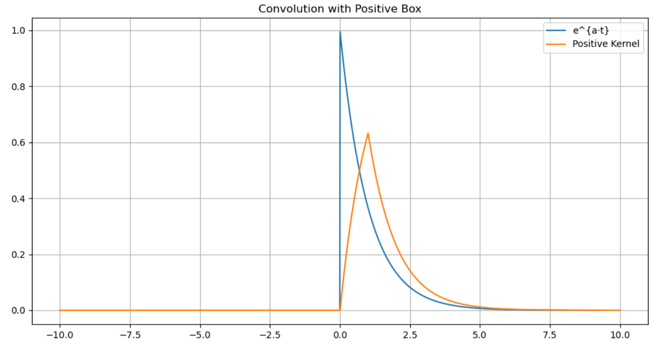



FIGURE 2.2: Convolution with Causal Kernel

### 2.4.3 Case 3: Shifted Kernel

$$
\begin{aligned}
y(t) &= x(t) * h(t) \\
&= \int_{-\infty}^{\infty} h(\tau) e^{a(t-\tau)} u(t-\tau) d\tau \\
&= e^{at} \int_{\tau_0 - T}^{\min(t, \tau_0 + T)} e^{-a\tau} d\tau
\end{aligned}
$$

for $t < \tau_0 - T$

$$y(t) = 0$$

for $\tau_0 - T \leq t < \tau_0 + T$

$$
\begin{aligned}
y(t) &= e^{at} \int_{\tau_0 - T}^{t} e^{-a\tau} d\tau \\
&= \frac{e^{at}}{a} [-e^{-a\tau}]_{\tau_0 - T}^{t} \\
&= \frac{e^{at}}{a} (e^{-a(\tau_0 - T)} - e^{-at}) \\
&= \frac{1}{a} (e^{a(t - \tau_0 + T)} - 1)
\end{aligned}
$$

for $t \geq \tau_0 + T$

$$
\begin{aligned}
y(t) &= e^{at} \int\limits_{\tau_0 - T}^{\tau_0 + T} e^{-a\tau} d\tau \\
&= \frac{e^{at}}{a} [-e^{-a\tau}]_{\tau_0 - T}^{\tau_0 + T} \\
&= \frac{e^{at}}{a} (e^{-a(\tau_0 - T)} - e^{-a(\tau_0 + T)}) \\
&= \frac{e^{a(t-\tau_0)}}{a} (e^{aT} - e^{-aT})
\end{aligned}
$$

Therefore, output of the system is

$$
y(t) = \begin{cases} 0 & t < \tau_0 - T \\ \frac{1}{a}(e^{a(t-\tau_0+T)} - 1) & \tau_0 - T \leq t < \tau_0 + T \\ \frac{e^{a(t-\tau_0)}}{a}(e^{aT} - e^{-aT}) & t \geq \tau_0 + T \end{cases}
$$

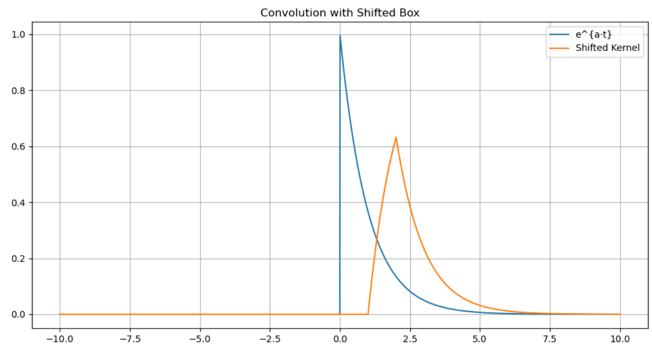



FIGURE 2.3: Convolution with Shifted Kernel

Here , The output signal also got shifted by $\tau_0$ units. **Inference:** Convolution Results in scaling only (except when $\omega T = n\pi$).

## 2.5 Convolution with step function $u(t)$

### 2.5.1 Case 1: Symmetric Pulse $[-T, T]$

**Given:**

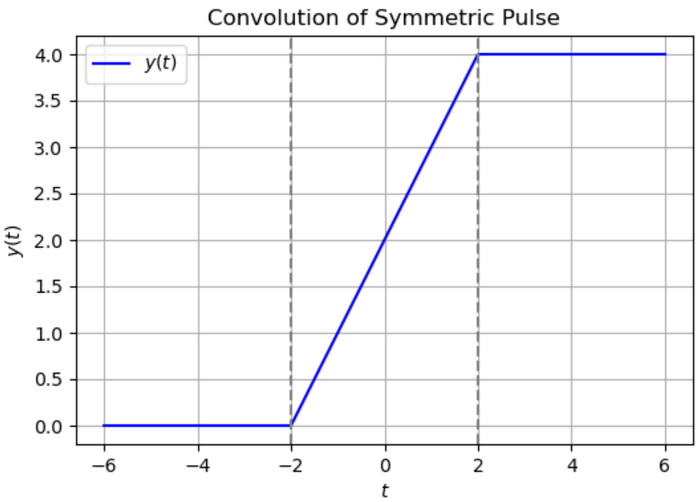
$$h(\tau) = \begin{cases} 1, & -T \leq \tau \leq T \\ 0, & \text{otherwise} \end{cases}$$

Computing the integral:

$$y(t) = \int_{-T}^{\min(t,T)} 1 \, d\tau = \min(t, T) + T$$

**Final Expression:**

$$y(t) = \begin{cases} 0, & t < -T \\ t + T, & -T \leq t < T \\ 2T, & t \geq T \end{cases}$$




### 2.5.2 Case 2: Pulse $[0, T]$

**Given:**

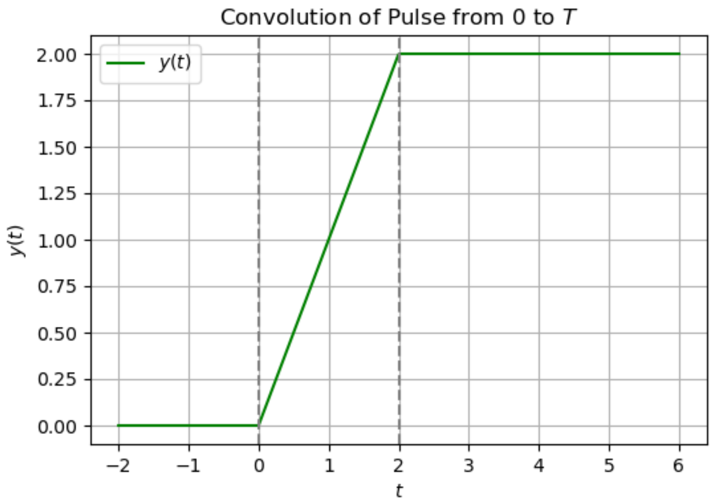
$$h(\tau) = \begin{cases} 1, & 0 \leq \tau \leq T \\ 0, & \text{otherwise} \end{cases}$$

Computing the integral:

$$y(t) = \int_0^{\min(t,T)} 1\, d\tau = \min(t, T)$$

**Final Expression:**

$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < T \\ T, & t \ge T \end{cases}$$




### 2.5.3 Case 3: Shifted Pulse $[-T + \tau_0, T + \tau_0]$

**Given:**

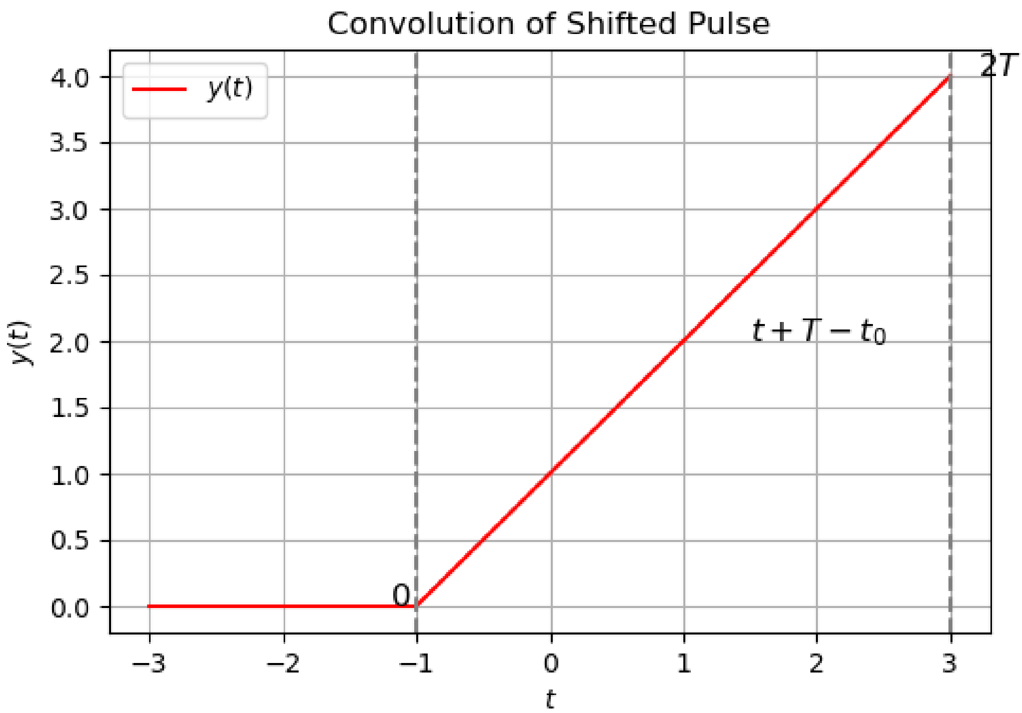
$$h(\tau) = \begin{cases} 1, & -T + \tau_0 \le \tau \le T + \tau_0 \\ 0, & \text{otherwise} \end{cases}$$

Computing the integral:

$$y(t) = \int_{-T+\tau_0}^{\min(t,T+\tau_0)} 1\, d\tau = \min(t, T + \tau_0) + T - \tau_0$$

**Final Expression:**

$$y(t) = \begin{cases} 0, & t < -T + \tau_0 \\ t + T - \tau_0, & -T + \tau_0 \leq t < T + \tau_0 \\ 2T, & t \geq T + \tau_0 \end{cases}$$




## 2.6 Significance in Time-Delayed Systems

The result from the shifted kernel analysis has significant implications for time-delayed systems:

- In signal processing, this allows for precise control of signal delays.
- In physical systems, it represents the causal relationship between input and output.
- In control systems, it models transport delays or dead time.

Time delays are critical in many engineering applications:

- In communication systems, where signals experience propagation delays

- In control systems, where actuator response is not instantaneous
- In digital signal processing, where processing introduces computational delays

The mathematical relationship $(f*h_{\tau_0})(t) = (f*h)(t-\tau_0)$ provides a formal way to analyze and predict the behavior of systems with time delays, which is essential for designing robust control systems and signal processing algorithms.

## 2.7 Conclusion

We have analytically derived the convolution of a sinusoidal input with rectangular kernels. The results show that:

- For sinusoidal inputs, the output remains sinusoidal with the same frequency but with amplitude scaling that depends on both the kernel width and input frequency.
- Modifying the kernel to be half-sided introduces a phase shift in addition to amplitude scaling.
- Shifting the kernel by $\tau_0$ results in a time delay of the output by the same amount.

These findings have important implications for understanding linear time-invariant systems and their response to various inputs, particularly in applications involving signal filtering, audio processing, and control systems.

# Chapter 3

# Applications: Noise Reduction using Moving Average Filter

## 3.1 Noise Reduction via Rectangular Kernel Convolution

Convolution with a rectangular kernel (also known as a box filter) is a widely used technique for noise reduction in signal processing.

When a noisy signal $x(t)$ is convolved with a rectangular kernel, the result is a moving average over the width of the kernel:

$$(y * \mathrm{rect})(t) = \int_{-\infty}^{\infty} x(\tau)\,\mathrm{rect}(t-\tau)\,d\tau \tag{3.1}$$

### 3.1.1 Why Does This Reduce Noise?

1. **Averaging Effect on Zero-Mean Noise:**
   The rectangular kernel computes the average of the signal within its window. If the noise $n(t)$ added to the signal $s(t)$ is zero-mean (i.e., $\mathbb{E}[n(t)] = 0$), then averaging multiple samples tends to cancel out the noise:

$$\mathbb{E}\left[\frac{1}{K}\sum_{i=1}^{K} n_i\right] = \frac{1}{K}\sum_{i=1}^{K}\mathbb{E}[n_i] = 0 \tag{3.2}$$

   where $n_i$ are independent noise samples and $K$ is the number of samples in the kernel window.

2. **Variance Reduction:**

   For zero-mean noise with variance $\sigma^2$, the variance after averaging $K$ samples is reduced to $\sigma^2/K$:

$$\mathrm{Var}\left[\frac{1}{K}\sum_{i=1}^{K} n_i\right] = \frac{\sigma^2}{K} \tag{3.3}$$

   Thus, the wider the kernel, the greater the noise suppression.

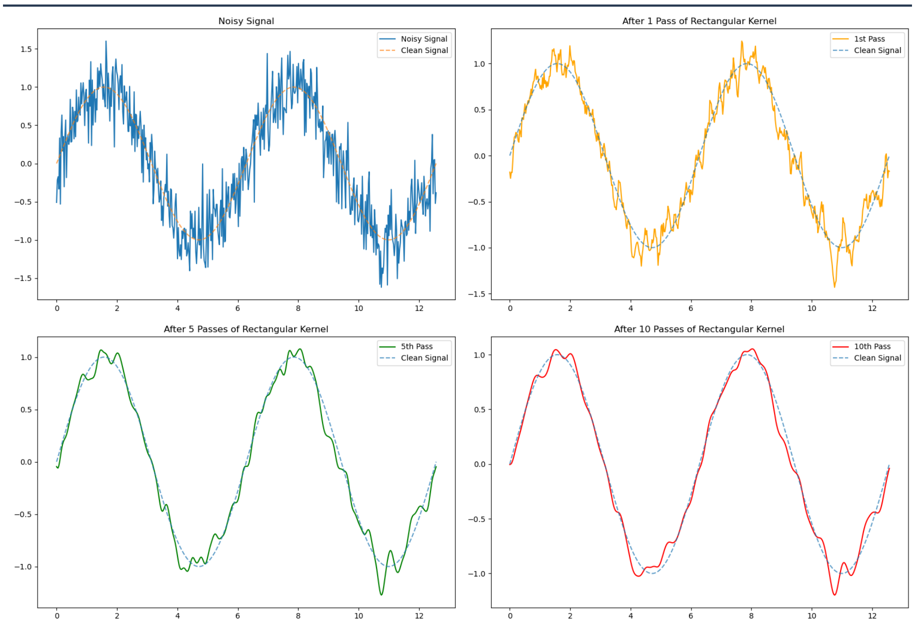



FIGURE 3.1: Noise Reduction with increased number of passes

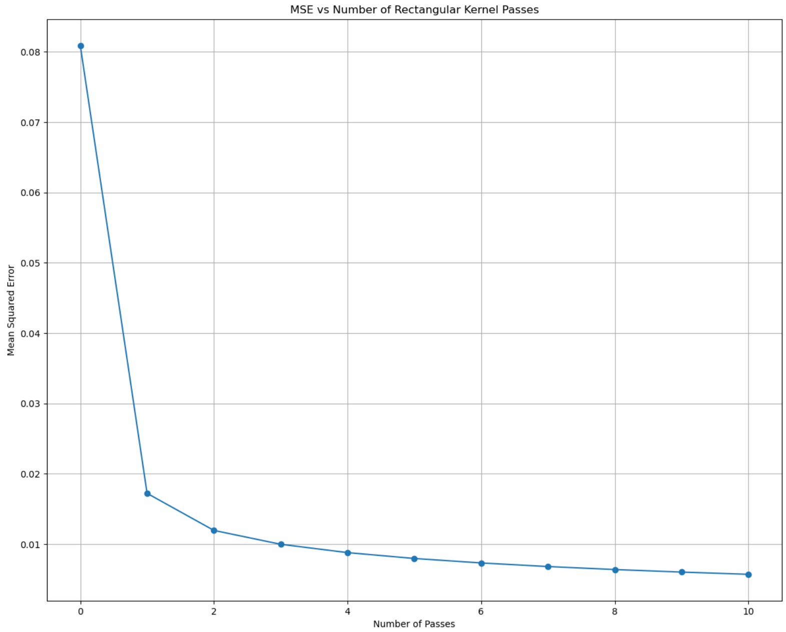



FIGURE 3.2: Decreasing Error with increased passes

### 3.1.2 Side-effects: Frequency Domain Interpretation

Taking the Fourier Transform:

$$Y(f) = X(f) \cdot H(f)$$

The Fourier Transform of the rectangular pulse $h(t)$ is the sinc function:

$$H(f) = T \cdot \operatorname{sinc}(fT) = T \cdot \frac{\sin(\pi f T)}{\pi f T}$$

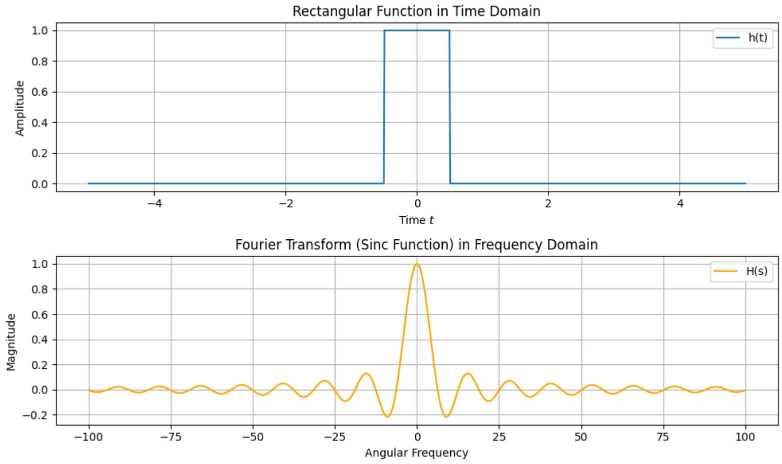



FIGURE 3.3: Fourier Transform of $h(t)$

Since $H(f)$ decays with increasing frequency, multiplication with $X(f)$ attenuates high frequencies—this is the essence of a low-pass filter.

### 3.1.3 Python Demonstration

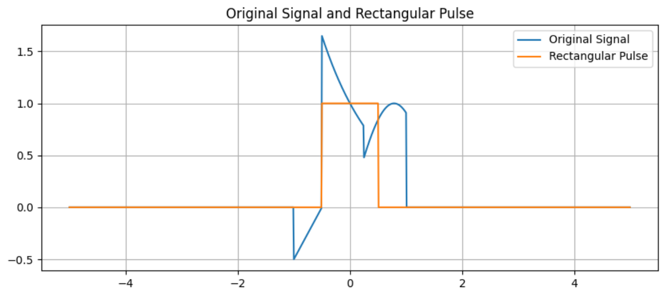



FIGURE 3.4: Original Signal and Rectangular Pulse

On comparison of frequency responses

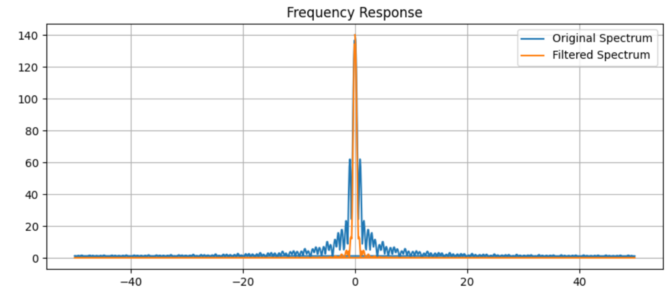



FIGURE 3.5: Magnitude Spectrum Before and After Convolution

## 3.2 Conclusion

With this, we showed that the box kernel can be used to reduce noise from audio signals and also as a crude LPF.

# Chapter 4

# Effect of Repeated Convolutions

## 4.1 Repeated Convolution with the Box Kernel

In this section, we investigate the result of repeatedly convolving a signal with the same box kernel. Repeated convolution involves applying the convolution operation multiple times, where the output of one convolution becomes the input for the next.

### Mathematical Formulation

Let $f(t)$ be a signal, and let $h(t)$ be the box kernel defined as:

$$h(t) = \begin{cases} \frac{1}{2T}, & \text{for } -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

For the first convolution, we have:

$$y_1(t) = (f * h)(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\, d\tau$$

For subsequent convolutions, the result of each convolution is treated as the input for the next:

$$y_n(t) = (y_{n-1} * h)(t)$$

Thus, the $n$-th convolution can be written as:

$$y_n(t) = (f * h^n)(t)$$

where $h^n(t)$ represents the kernel $h(t)$ convolved with itself $n$ times.

### Behavior of Repeated Convolution

Each convolution with the box kernel has the effect of smoothing the signal further, reducing sharp transitions and spreading the signal over a wider interval. With each successive convolution, the result becomes progressively smoother and more spread out. The repeated averaging reduces high-frequency components, resulting in an increasingly gradual response.

As the number of convolutions increases, the signal approaches a smooth, more uniform shape, with less distinction between the original features of the signal.

## 4.2 Gaussian Approximation from Repeated Convolution

In this section, we explore how repeated convolution of a signal with a box kernel leads to a Gaussian distribution. This behavior is a direct consequence of the **Central Limit Theorem (CLT)**.

### Central Limit Theorem (CLT) and Convolution

The CLT states that the sum of a large number of independent random variables, each with the same distribution, approaches a Gaussian distribution as the number of variables increases. In the context of signal processing, this means that repeated convolution with the same kernel can produce a signal that approximates a Gaussian distribution.

The box kernel acts as an averaging filter, and when it is convolved with a signal multiple times, the resulting signal becomes increasingly smooth and more similar to a Gaussian function.

### Gaussian Approximation

As discussed in the previous section, the output of repeated convolutions with the box kernel $h(t)$ can be written as:

$$y_n(t) = (f * h^n)(t)$$

With each convolution, the signal smooths out more, and after many iterations, the result approaches a Gaussian distribution. Specifically, the output $y_n(t)$ converges to a Gaussian function with mean 0 and variance $\sigma^2$, where $\sigma$ depends on the kernel width and the number of convolutions.

$$y_n(t) \to \mathcal{N}(0, \sigma^2)$$

The standard deviation $\sigma$ increases with the number of convolutions, and the width of the Gaussian distribution becomes larger as more convolutions are applied.

### Numerical Illustration

We can observe the convergence to a Gaussian distribution by plotting the result of repeated convolutions. Initially, the signal will have sharp transitions and peaks. After several convolutions, the signal will become smoother and approach the shape of a Gaussian curve.

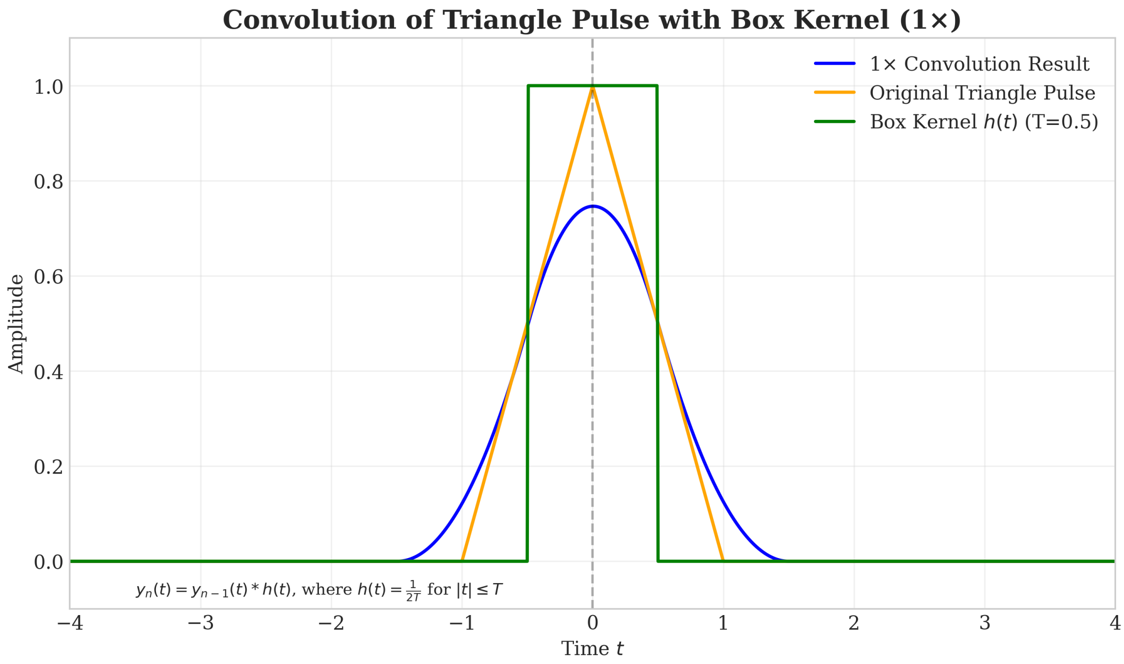



Figure 4.1: Triangle Pulse Convolved Once with Box Kernel

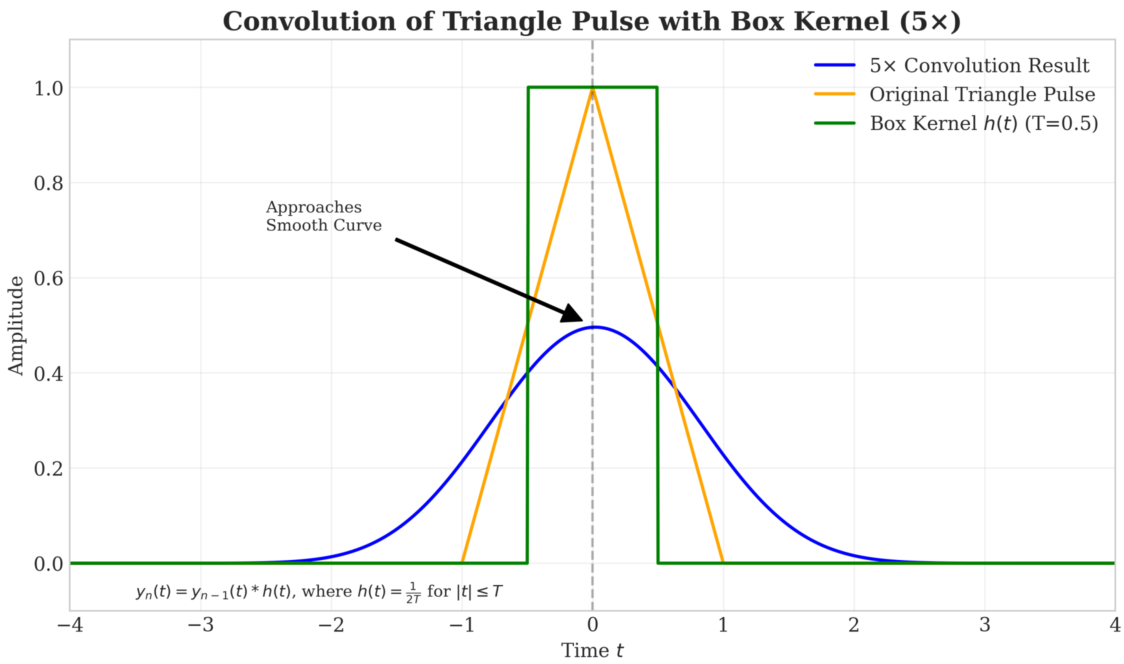



Figure 4.2: Result of Five Consecutive Convolutions with Box Kernel

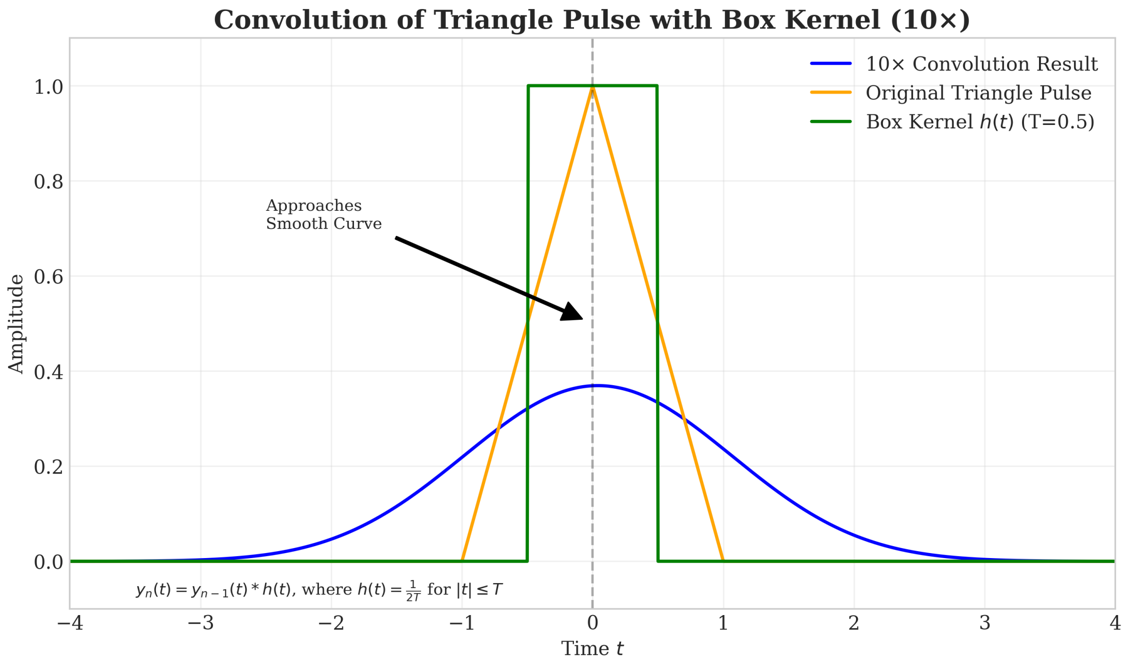



FIGURE 4.3: Result of Ten Consecutive Convolutions with Box Kernel

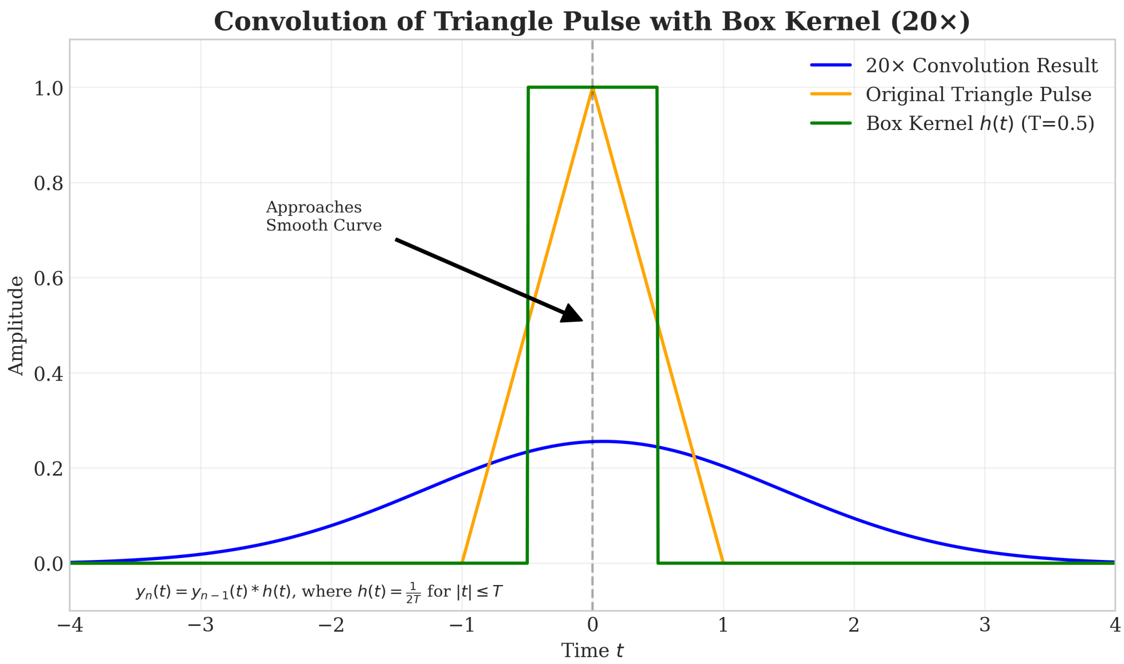



FIGURE 4.4: Result of Twenty Consecutive Convolutions with Box Kernel

## Conclusion

Repeated convolution with a box (rectangular) kernel illustrates a fundamental principle from probability theory: the Central Limit Theorem (CLT). The CLT states that the sum (or average) of a large number of independent and identically distributed (i.i.d.) random variables—regardless of their original distribution—tends toward a Gaussian (normal) distribution as the number of terms increases.

In the context of signal processing, each convolution with a normalized box kernel can be interpreted as a smoothing operation that averages local values. Mathematically, this is analogous to summing independent uniform distributions (since the box kernel is uniform over its support). When the convolution is repeated multiple times, the result approximates the sum of multiple uniform distributions. According to the CLT, the shape of this sum converges to that of a Gaussian function as the number of repetitions increases.

Therefore, the repeated convolution of even a simple step function with a box kernel produces a progressively smoother output that increasingly resembles a Gaussian curve. This behavior is not just theoretical—it has practical importance in signal smoothing, noise reduction, and approximating Gaussian filters using simple kernels.

This convergence highlights the universality of the Gaussian distribution and the power of convolution as a tool in engineering and data analysis.

# Chapter 5

# Supporting Code

This link hosts all the code used for analysis in this report.