# Computer Graphics I

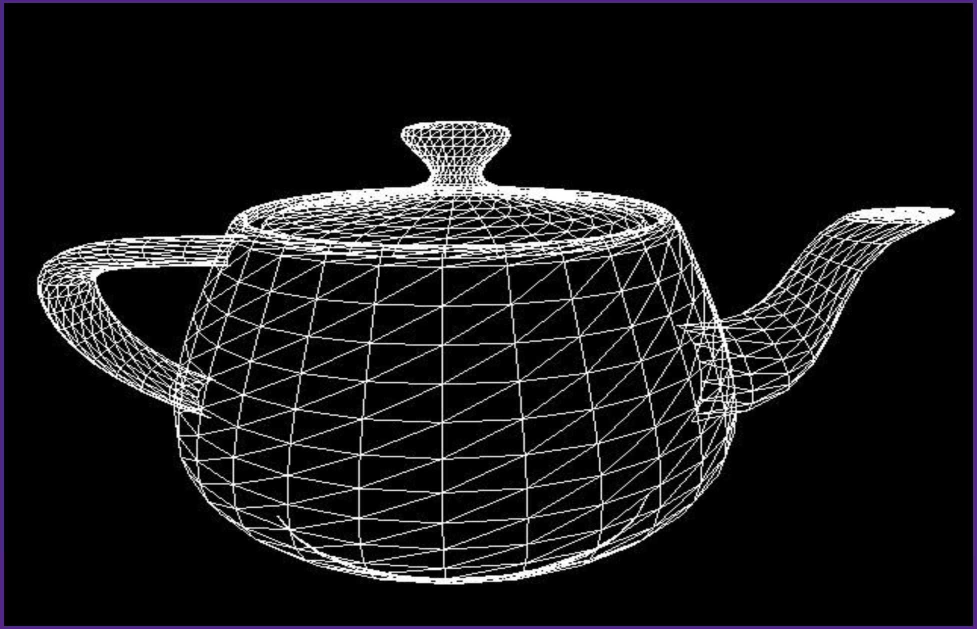
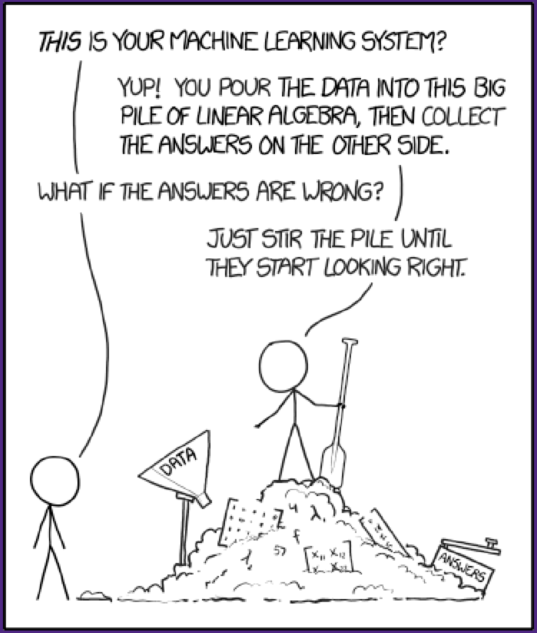


*Illustration 1:*
*A wire-mesh model of a once famous teapot*

# Review of Linear Algebra




- Useful for much more than graphics!

- Many applied scientists in stats, ML, graphics end up wishing they had spent more time with math foundations early on in studies.

# Review of Linear Algebra

- We're going to start with a review of linear algebra, vectors, and matrices. This mathematical basis for computer graphics, its implementation, and its manipulation is very important. As we explore topics within computer graphics, we will see new, varying, and specific applications of vectors and matrices. In this first section we'll just review the basics.

- Linear algebra is so important in computer graphics mainly because of their ability to describe transformations. We'll revisit transformations in the 2D and 3D cases independently in further sections.

- With this section we should re-familiarize ourselves with:
  - Coordinate Systems
  - Points, vectors, directions
  - Dot product, cross product.
  - Matrices and their arithmetic

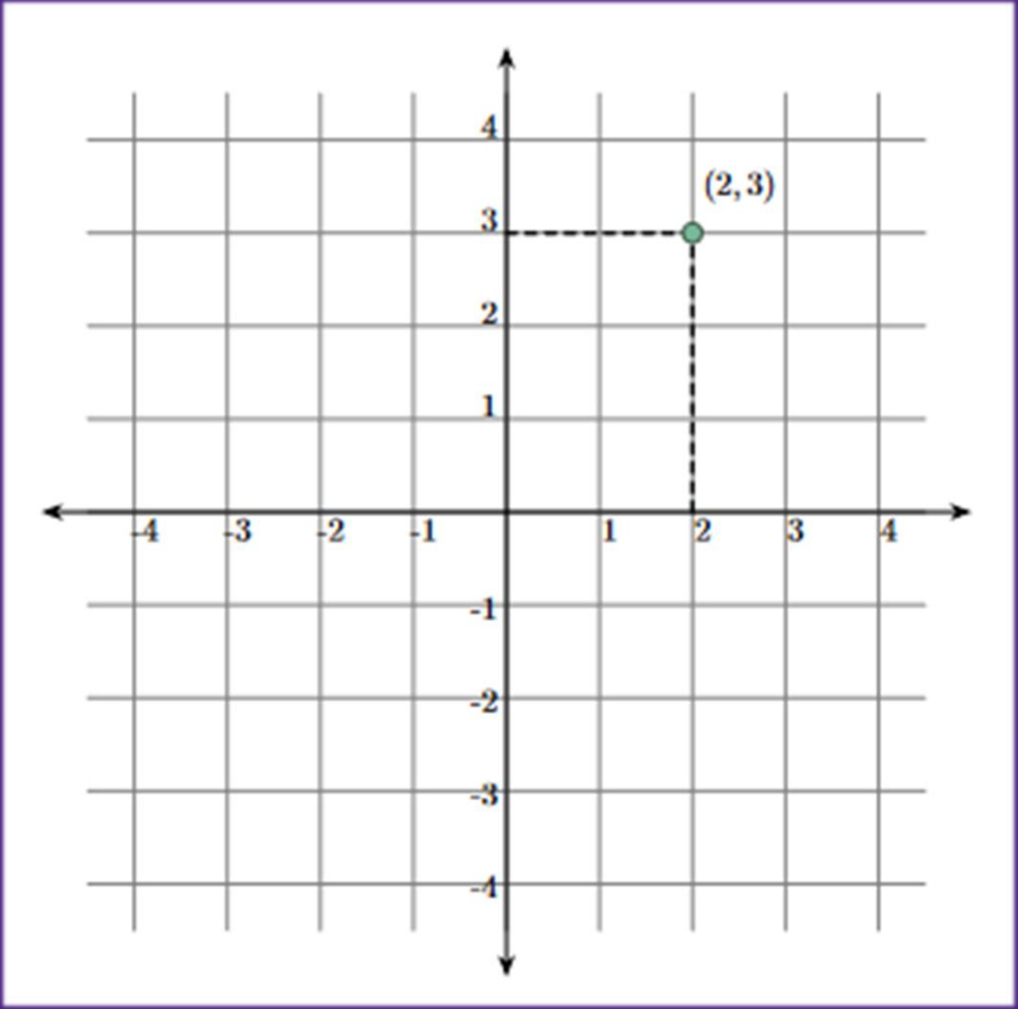
# Coordinate System

- A coordinate system, in general, is a geometrical system where tuples of numbers are used to uniquely describe the positions of points, lines, surfaces, and other geometrical structures in some space. Each number in the tuples is a coordinate.

- The real number line is really just a one-dimensional coordinate system. Each point in that one-dimensional space is determined by a single real number. For example, (3.14).

- We should be very familiar with the Cartesian plane or two-dimensional Cartesian coordinate system. In this system, each point on the plane is given by a pair of real numbers (x, y).

# Coordinate System



# Coordinate System

- Formally, this coordinate system is defined by an origin and two orthonormal basis vectors. In the two-dimensional case, those orthonormal basis vectors can be thought of just two perpendicular lines. The coordinates (x, y) represents the signed distance x to the first line, and the signed distance y to the second line.

- Like the real number line, you still need a starting place, a place for 0, an origin. In most coordinate systems the origin is given by the point where all coordinates are 0s.

- Another way of thinking about coordinate systems is based much more explicitly on vectors.

- We'll return to explore that further later.

# Points and Vectors in Space

- A point in *n* dimensional space represents a position in space and is given by a tuple $p=( p_1, p_2,\ldots, p_n )^T$ where $p_i$ are scalars.
- The position of a point is relative to a coordinate system with an origin given by $O=(0,0,\ldots, 0)^T$ and unit axes:
  - $\vec{u}_1=(1,0,\ldots, 0)^T$, $\vec{u}_2=(0,1,0,\ldots, 0)^T$, … , $\vec{u}_n=(0,0,\ldots, 1)^T$
- Hence, a 3D point is written as $p=(x , y ,z)^T$, and a 2D point, as $p=(x , y)^T$.

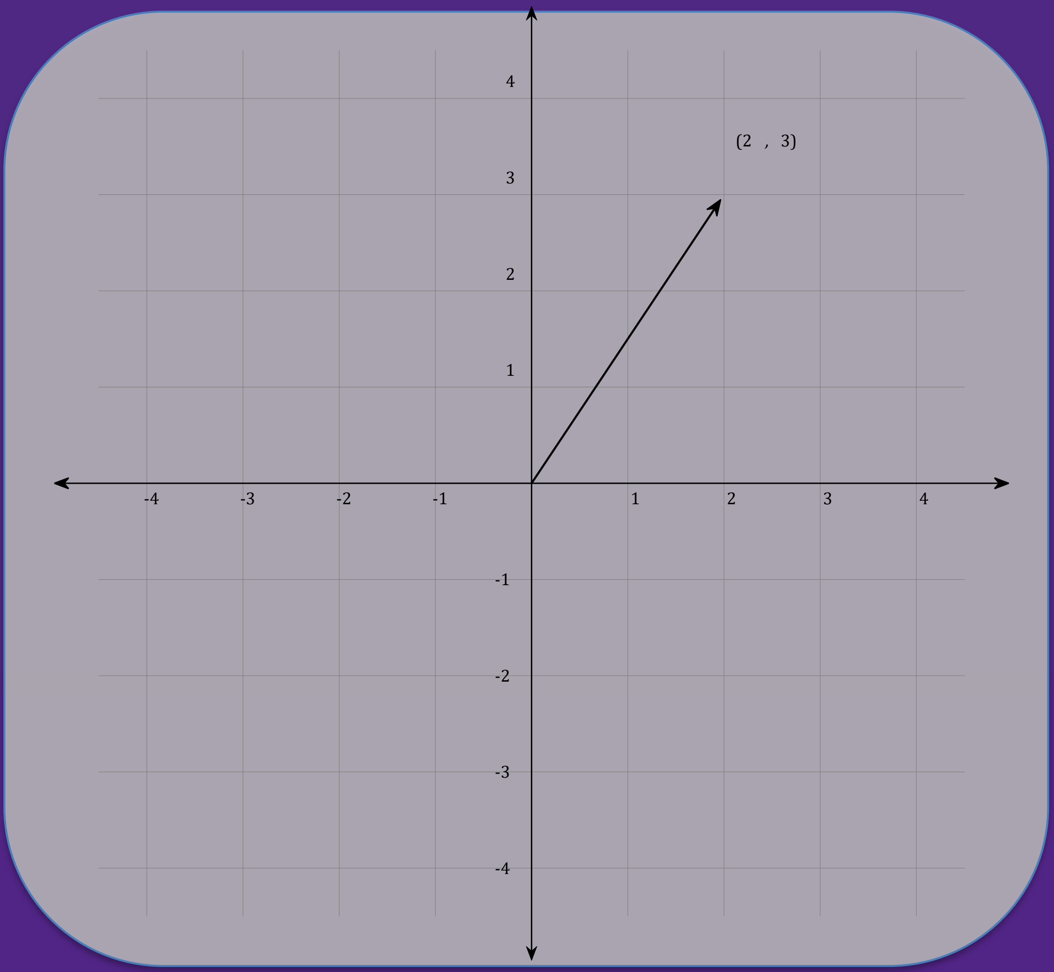
# Vector and Unit Vector Notation

- I'll be using the general format of $^{\rightarrow}$ to the left of the assigned vector, to keep it separate from subscripts and exponents

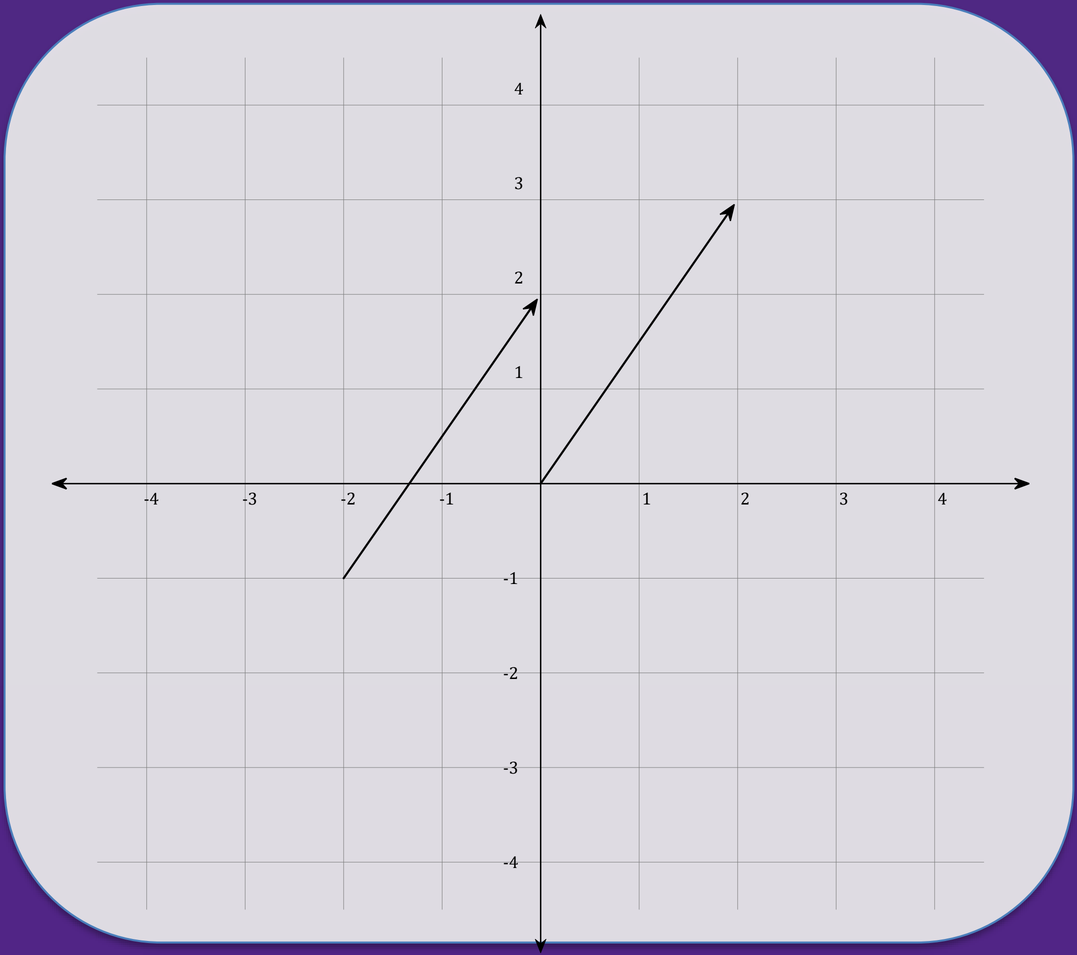
- Similarly, ^ will be to the left.

- Ex: a $^{\rightarrow}$v is referring to a vector v and a scalar constant a
- $^{\rightarrow}$a $^{\rightarrow}$v is referring to vectors a and v
- $^{\rightarrow}$va would be a vector v and a scalar constant a
- ^v will be referring to the unit vector of v

# Vector and Unit Vector Notation



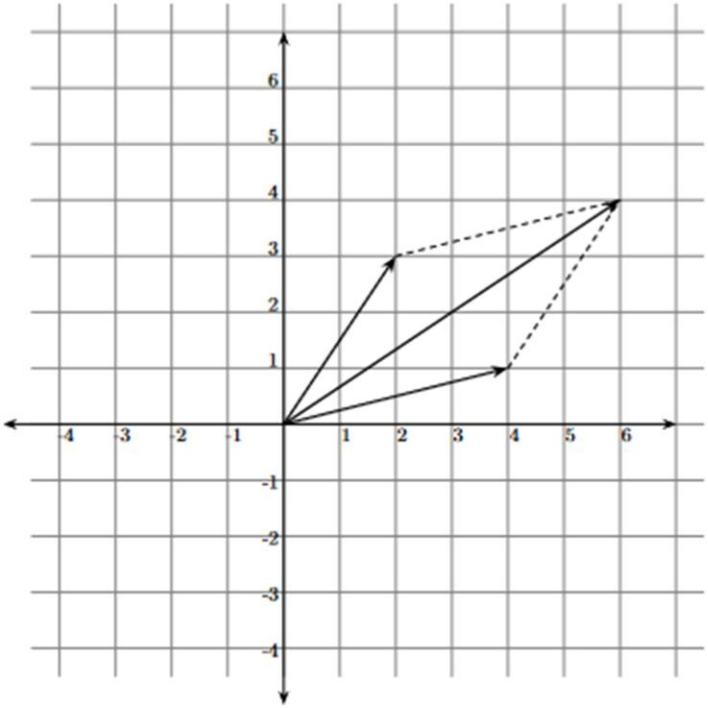
# Equivalent Vectors, Different Points

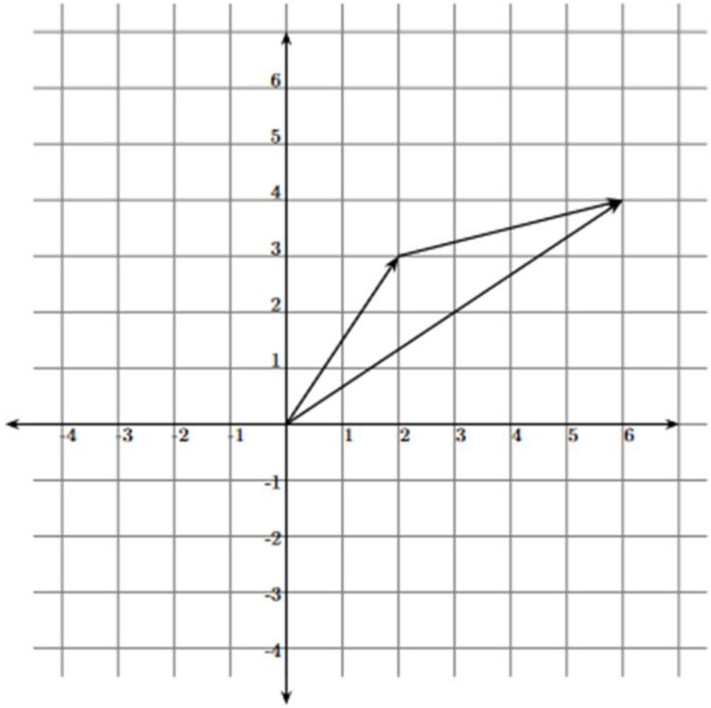


# Points and Vectors in Space

- A vector in $n$ dimensional space represents a direction and is given by a tuple $\vec{v}=(v_{1,}\, v_{2,}\ldots,v_n)^T$ where $v_i$ are scalars.
- A vector is the result of the subtraction of two points.
- For example the vector $\vec{v}(1,2)^T$ is the result of the subtraction of the two points $\vec{v}(1,2)= \boldsymbol{p}(1,2)^T - 0^T$ .
- The resulting vector represents the direction and distance between the points. Thus we can write, for any two points $\boldsymbol{p}_{1,}\boldsymbol{p}_2$ : $\vec{v} = \boldsymbol{p}_2 - \boldsymbol{p}_1$ .
- It follows directly that $\boldsymbol{p}_1+\vec{v} = \boldsymbol{p}_2$ .
- Adding a vector to a point results in a point.

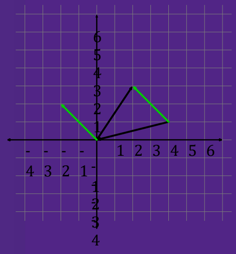
# Vector Addition




For $\vec{u} = (2, 3)$, $\vec{v} = (4, 1)$:

# Vector Subtraction



- For $\vec{u} = (2, 3), \vec{v} = (4, 1)$:
- $\vec{u} - \vec{v} = (2 - 4, 3 - 1) = (-2, 2)$

Now, the geometric interpretation of vector subtraction is a little different. We travel/displace backwards by the subtracting vector, and then travel/displace along the subtracted vector.

This makes more sense if we think about subtraction as first multiplying by $-1$ and then doing vector addition.

$\vec{u} - \vec{v} = \vec{u} + (-1\vec{v})$

# Scalar Multiplication

Scalar multiplication is defined as $a\vec{v}=(a\,v_1, a\,v_2,\ldots,av_n)$ and has the following properties:

(Side note: -ve scalar multiplication changes the direction by exactly 180 degrees)

1. Scalar associative: $(ab)\vec{v}=a(b\vec{v})$
2. Scalar distributive: $(a+b)\vec{v}=a\vec{v}+b\vec{v}$
3. Vector distributive: $a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$

# Linear Combination

- Typically, points are not ‘added’ since they describe a spot in relation to an origin.
- However, a linear combination of points can be used to indicate a space.
  - –For example, a vector space over a field
  - –Represented by:

$$p = \sum_{i=1}^{n} a_i p_i$$

# Affine Combination of Points

- When the following is true:

$$\sum_{i=1}^{n} a_i = 1$$

- We can say this is an ‘affine’ combination of points, where $p_i$ are elements (vectors) of a vector space over a field *K*, and the coefficients $a_i$ are elements of *K*.

# Affine Space

- An affine space is a geometric structure that generalizes some Euclidean space properties, but leaves it independent of the concepts of distance and measurement of angles

- There is no distinguished point that serves as an origin
  - No vector is uniquely distinguished as the origin (No zero vector)
  - No vector has a fixed origin, no vector can be uniquely assigned to a point
  - Instead, there are ‘translation’ or ‘displacement’ vectors, between two points of the space.

# Affine Space

- An affine space is like a "playground" where points and directions (vectors) exist, but there’s no fixed starting point (origin). Imagine a blank piece of paper: you can draw points and connect them with arrows, but there's no special dot that you have to call "home base.“

Key points about affine spaces:

1. **No Origin**: There's no single, special point that's the starting place (no zero vector).
2. **Vectors Connect Points**: You can only talk about the "difference" or "displacement" between points, not their position relative to a fixed origin.
3. **Generalizes Euclidean Geometry**: It's like regular geometry but without needing distances or angles.

- Think of it as focusing on relationships (like how far apart two points are) rather than absolute positions.

# Vector Length

- The length of a vector is known as its magnitude and is defined as:

$$||\vec{v}|| = \sqrt{\sum_{i=1}^{n} v_i^2}$$

- A unit vector is a vector of length equal to one.

- Any vector can be made a unit vector by:

$$\hat{v} = \frac{1}{||\vec{v}||}\vec{v}$$

# Unit Vector & Direction

- When a vector is a unit vector it can also be called *a* **direction**.
- Why?
- Because there is **exactly one** unit vector in any direction. Remember, vectors only care about direction and length. If we fix the length to be exactly 1, then there is exactly one vector in each possible direction.
- Now, you might wonder why we haven't precisely defined direction.
- There's been no angles or north/south/east/west. This is intentional.
- Those kinds of directions only make sense when we are *relative* to some fixed position. To be north, it must be north *of something*. If you are in plane 10, 000 feet above the north pole, which direction is north?

# Unit Vector & Direction

- Instead, we let directions be the vectors themselves. remember that there is exactly one unit vector in each direction. Thus, two vectors point in the *same direction* if their normalized vectors are equal.

- (2, 4) and (1, 2) point in the same direction.

- In addition to two vectors pointing in the same direction, two other classifications are important: parallel and perpendicular vectors. We'll see those soon in the next two sections.

# Vector Dot Product

- Dot product between two vectors is defined as:

$$\vec{v} \cdot \vec{u} = \sum_{i=1}^{n} v_i u_i$$

- It has the following properties:

- Vector length: $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
- Scalar associative: $(a\vec{u}) \cdot (b\vec{v}) = ab(\vec{u} \cdot \vec{v})$
- Commutative $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- Addition distributive $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$
- $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$ , where θ is the angle between the vectors

# Vector Dot Product

- The last property in particular is helpful for checking information about the vectors. Consider the case where we normalize both vectors to unit length so we have $\hat{u} \cdot \hat{v} = \cos\theta$ , so $\theta = \cos^{-1}(\hat{u} \cdot \hat{v})$
- Additionally:
  - $\hat{u} \cdot \hat{v} = 0$ implies $\theta = 90°$
  - $\hat{u} \cdot \hat{v} > 0$ implies $\theta < 90°$
  - $\hat{u} \cdot \hat{v} < 0$ implies $\theta > 90°$

# Orthonormal Bases

So now we know what orthogonal means. Looking back, we can now make a little sense of how coordinate systems are defined. They defined using an origin and an orthonormal basis.

Orthonormal vectors are unit vectors which are orthogonal.

The Cartesian plane is defined with the origin at (0,0) and two orthonormal vectors denoted $\hat{i}$ and $\hat{j}$:

$$\hat{i} = (1, 0) \quad \hat{j} = (0, 1)$$

# Orthonormal Bases

These are exactly the unique normalized vectors pointing in the directions of the x-axis, and y-axis, respectively. It is not hard to show that these two vectors are perpendicular using their dot product.

In three dimensions, the Cartesian coordinate system has three basis vectors. We re-use $\hat{i}$, $\hat{j}$, and add $\hat{k}$.

$$\hat{i} = (1, 0, 0) \qquad \hat{j} = (0, 1, 0) \qquad \hat{k} = (0, 0, 1)$$

# Orthonormal Bases

- Again, one can use dot products to show that these vectors are perpendicular to the other two.

- When vectors are mutually orthogonal, they form an **orthogonal basis**. When normalized vectors are mutually orthogonal, they form an orthonormal basis.

- In n dimensions we need n perpendicular vectors to form a orthogonal basis. You might recall this from linear algebra.

- But, an n-dimensional space can be described by many different bases. This will become important later when we start dealing with 3D transformations and multiple coordinate systems at the same time.

# 3D Cross Product

- Cross Product exists for a limited range of dimensions (3,7) where an input of two vectors produces a vector in the same space. (The why is better saved for a math class)

- We compute it as:

$$\vec{u}\mathrm{x}\vec{v} = \left( \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

# 3D Cross Product

- The cross product has the following properties:
  - Nilpotent: $\vec{v} \times \vec{v} = \vec{0} = (0,0,0)$
  - Scalar associative: $(a\vec{u}) \times (b\vec{v}) = (ab)(\vec{u} \times \vec{v})$
  - Anti-symmetric: $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
  - Addition distributive: $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
  - Dot-cross associative: $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
  - $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2 \theta$ where $\theta$ is the angle between vectors

# 3D Cross Product

- The geometric interpretation of the cross product is very useful for a number of processes in computer graphics.

- Suppose $\vec{u}$ and $\vec{v}$ are not parallel vectors.

- Then the vector $\vec{w}=\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$ .

- In particular, if both $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) and of unit length then, with $\vec{w}=\vec{u}\times\vec{v}$ , we have that the vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ form an orthonormal basis for 3D space.
  - Most common orthonomal basis for $R^3$ is $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$

- For any non parallel vectors $\vec{u}$ and $\vec{v}$ , the magnitude of vector $\vec{w}=\vec{u}\times\vec{v}$ represents the area of the parallelogram subtended by vectors $\vec{u}$ and $\vec{v}$

# 3D Cross Product

- Cross Product exists for a limited range of dimensions (3,7) where an input of two vectors produces a vector in the same space. (The why is better saved for a math class)
- We compute it as:

$$\vec{u}\mathrm{x}\vec{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right)$$

where

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

# 3D Cross Product

- The cross product has the following properties:

- Nilpotent: $\vec{v} \times \vec{v} = \vec{0} = (0,0,0)$
- Scalar associative: $(a\vec{u}) \times (b\vec{v}) = (ab)(\vec{u} \times \vec{v})$
- Anti-symmetric: $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
- Addition distributive: $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
- Dot-cross associative: $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- $\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 \sin^2 \theta$ where $\theta$ is the angle between vectors

# 3D Cross Product Example

•Imagine you have a triangular surface in a 3D environment, defined by three points A,B,A,B, and CC. To properly light this surface, you need to compute its normal vector, which is perpendicular to the surface.

- Define two edges of the triangle:
  - $\vec{u} = B - A$
  - $\vec{v} = C - A$

•Use the 3D cross product to find the normal vector:

- $\vec{n} = \vec{u} \times \vec{v}$

•Normalize the result:

- Convert $\vec{n}$ to a unit vector (divide by its magnitude) so it can be used for lighting calculations.

# 3D Cross Product Example

- Why is this important?
  - The normal vector helps the graphics engine determine how light interacts with the surface (e.g., reflection, refraction, shadows). This process is part of the Phong lighting model or other shading techniques.
- Practical Application:
  - In a 3D game or simulation, every frame requires these calculations to create realistic lighting effects, making the cross product a fundamental tool in computer graphics.

# Handedness in Coordinate Systems

•A coordinate system is also endowed with a "handedness" which gives us a cheap trick to determine the direction of cross products (and later, of rotations).

•In a right-hand coordinate system we use our right hand. For $\vec{u} \times \vec{v}$, put your fingers in the direction of $\vec{u}$, with your palm facing toward $\vec{v}$ (in the direction where the angle between $\vec{u}$ and $\vec{v}$ is less than 180°). The thumb of your right hand points in the direction of their cross product.

•If you repeat this procedure with $\vec{v} \times \vec{u}$, you'll see that you have to flip your palm in the opposite direction, and thus your thumb flips around 180°.

# Handedness in Coordinate Systems

- In a left-hand coordinate system, you do the exact same as above, but with the left hand.
- This handedness also tells us about the orientation of that the coordinate system. In a righthand coordinate system, we find that k^ = i^ × k^.

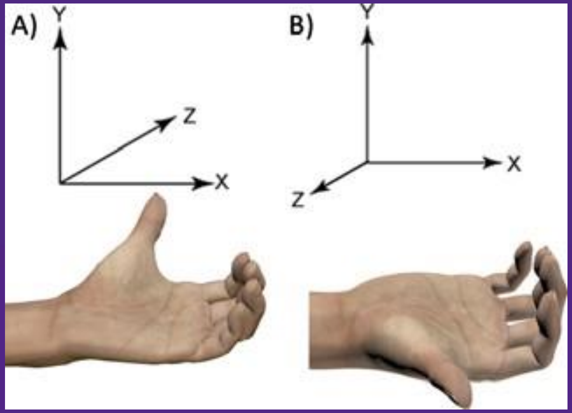



Figure 5: A left-hand coordinate system and a right-hand coordinate system.

# Matrices

- We’ll revisit matrices a lot in 3D graphics when we talk about affine transforms, and OpenGL. OpenGL loves matrices.
- For now, let’s just recall the basics of matrices and dealing with them.
- An m by n matrix is a rectangular array of entries organized into m rows and n columns. An m by n matrix can be denoted as $A_{m \times n}$
- We give special names to matrices with certain dimensions.

# Matrices

1. When $m = 1$ we have a **row vector**. Below is a row vector with 7 entries.

$$\begin{bmatrix} 4 & 1 & 4 & 2 & 3 & 7 & 8 \end{bmatrix}$$

2. When $n = 1$ we have a **column vector**. Below is a column vector with 4 entries.

$$\begin{bmatrix} 5 \\ -3 \\ 4 \\ -11 \end{bmatrix}$$

3. When $m = n$, we have a **square matrix**. We say a square matrix is *of order $n$* for a matrix with dimension $n$ by $n$. A square matrix of order 3 is shown below.

$$\begin{bmatrix} 9 & -1 & 16 \\ 3 & -5 & 12 \\ 11 & 4 & -8 \end{bmatrix}$$

# Matrices

1. When $m = 1$ we have a **row vector**. Below is a row vector with 7 entries.

$$\begin{bmatrix} 4 & 1 & 4 & 2 & 3 & 7 & 8 \end{bmatrix}$$

2. When $n = 1$ we have a **column vector**. Below is a column vector with 4 entries.

$$\begin{bmatrix} 5 \\ -3 \\ 4 \\ -11 \end{bmatrix}$$

3. When $m = n$, we have a **square matrix**. We say a square matrix is *of order $n$* for a matrix with dimension $n$ by $n$. A square matrix of order 3 is shown below.

$$\begin{bmatrix} 9 & -1 & 16 \\ 3 & -5 & 12 \\ 11 & 4 & -8 \end{bmatrix}$$

# Matrices

1. When $m = 1$ we have a **row vector**. Below is a row vector with 7 entries.

$$\begin{bmatrix} 4 & 1 & 4 & 2 & 3 & 7 & 8 \end{bmatrix}$$

2. When $n = 1$ we have a **column vector**. Below is a column vector with 4 entries.

$$\begin{bmatrix} 5 \\ -3 \\ 4 \\ -11 \end{bmatrix}$$

3. When $m = n$, we have a **square matrix**. We say a square matrix is *of order $n$* for a matrix with dimension $n$ by $n$. A square matrix of order 3 is shown below.

$$\begin{bmatrix} 9 & -1 & 16 \\ 3 & -5 & 12 \\ 11 & 4 & -8 \end{bmatrix}$$

# Matrices

1. When $m = 1$ we have a **row vector**. Below is a row vector with 7 entries.

$$\begin{bmatrix} 4 & 1 & 4 & 2 & 3 & 7 & 8 \end{bmatrix}$$

2. When $n = 1$ we have a **column vector**. Below is a column vector with 4 entries.

$$\begin{bmatrix} 5 \\ -3 \\ 4 \\ -11 \end{bmatrix}$$

3. When $m = n$, we have a **square matrix**. We say a square matrix is *of order $n$* for a matrix with dimension $n$ by $n$. A square matrix of order 3 is shown below.

$$\begin{bmatrix} 9 & -1 & 16 \\ 3 & -5 & 12 \\ 11 & 4 & -8 \end{bmatrix}$$

# Matrices

Scalar multiplication is a simple operation which multiplies a single number against each entry of a matrix to produce another matrix of the same dimensions.

Say $A = (a_{i,j})$ is an $m$ by $n$ integer matrix. Given some other integer $c$, $cA$ is another $m$ by $n$ matrix where:

$$cA = (c \cdot a_{i,j}) = c \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

$$= \begin{bmatrix} c \cdot a_{1,1} & c \cdot a_{1,2} & \cdots & c \cdot a_{1,n} \\ c \cdot a_{2,1} & c \cdot a_{2,2} & \cdots & c \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m,1} & c \cdot a_{m,2} & \cdots & c \cdot a_{m,n} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 5 & 12 \\ -1 & -7 & 4 \end{bmatrix} \qquad 4A = \begin{bmatrix} 12 & 20 & 48 \\ -4 & -28 & 16 \end{bmatrix}$$

# Matrices

- Two matrices can be added or subtracted when they have the same dimensions. For two $m$ by $n$ matrices, $A = (a_{i,j})$ and $B = (b_{i,j})$, their sum $A + B$ is equal to $(a_{i,j} + b_{i,j})$.

- That is, each entry of $A$ is added to the corresponding entry in $B$ in the same position.

  - Thanks to scalar multiplication, we can define subtraction via addition –
    $$A – B = A + (-1)B$$

- To add or subtract matrices they must be of the same dimensions. Every entry in the first matrix must have a corresponding entry in the second matrix to be added to.

# Matrices

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,n} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,n} \end{bmatrix} =$$

$$\begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,n} + b_{1,n} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,n} + b_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} + b_{m,1} & a_{m,2} + b_{m,2} & \cdots & a_{m,n} + b_{m,n} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 8 \\ 11 & 2 & 24 \\ 12 & 4 & 1 \end{bmatrix} + \begin{bmatrix} -9 & 8 & 6 \\ 0 & 15 & 2 \\ 3 & 14 & 0 \end{bmatrix} = \begin{bmatrix} -8 & 4 & 14 \\ 11 & 17 & 26 \\ 15 & 18 & 1 \end{bmatrix}$$

# Matrices

- Matrix multiplication is much more involved than addition. First, we must consider under which conditions two matrices can be multiples.
- Matrix multiplication between two matrices is only defined when the number of columns in the left-hand matrix equals the number of rows in the right-hand matrix.
- The result of the multiplication is another matrix whose number of rows equals the left-hand matrix's and whose number of columns equals the right-hand matrix's.
  - $A_{m \times n} \cdot B_{n \times p} = C_{m \times p}$

# Matrices

- In this notation, the "inner" dimensions must be the same, and the "outer" dimensions give the dimensions of the product. In this case, $n = n$ are the inner dimensions, and $m$, $p$ are the outer dimensions.

- But how do we define the entries of the product $C = (c_{i,j})$?

- Each entry of the matrix product $c_{i,j}$ is an *inner product* of the $i$th row of $A$ and the $j$th column of $B$.

# Matrices

$$c_{i,j} = \begin{bmatrix} a_{i,1} & a_{i,2} & a_{i,3} & \cdots & a_{i,n} \end{bmatrix} \cdot \begin{bmatrix} b_{1,j} \\ b_{2,j} \\ b_{3,j} \\ \vdots \\ b_{n,j} \end{bmatrix}$$

$$= (a_{i,1} \cdot b_{1,j} + a_{i,2} \cdot b_{2,j} + \cdots + a_{i,n} \cdot b_{n,j})$$

$$= \sum_{k=1}^{n} a_{i,k} \cdot b_{k,j}$$

Therefore, the matrix multiplication $A_{m\times n} \cdot B_{n\times p} = C_{m\times p}$ is actually $m \times p$ individual inner products.

$$\begin{bmatrix} 0 & 4 & 1 & 0 \\ \boxed{6 \quad 5 \quad 1 \quad 8} \\ 5 & 2 & 7 & 9 \\ 0 & 2 & 4 & 7 \end{bmatrix} \times \begin{bmatrix} 3 & 6 & 0 & 3 \\ 7 & 5 & 7 & 1 \\ 3 & 9 & 2 & 9 \\ 6 & 7 & 8 & 4 \end{bmatrix} = \begin{bmatrix} c_{1,1} & c_{2,1} & c_{3,1} & c_{4,1} \\ c_{1,2} & \boxed{126} & c_{3,2} & c_{4,2} \\ c_{1,3} & c_{2,3} & c_{3,3} & c_{4,3} \\ c_{1,4} & c_{2,4} & c_{3,4} & c_{4,4} \end{bmatrix}$$

$$c_{2,2} = (6 \cdot 6) + (5 \cdot 5) + (1 \cdot 9) + (8 \cdot 7) = 126$$

# Matrices

$$\begin{bmatrix} -9 & 6 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}_{3\times 2} \times \begin{bmatrix} 1 & -4 & 5 \\ 7 & 2 & 2 \end{bmatrix}_{2\times 3}$$

$$= \begin{bmatrix} (-9\cdot 1)+(6\cdot 7) & (-9\cdot -4)+(6\cdot 2) & (-9\cdot 5)+(6\cdot 2) \\ (0\cdot 1)+(2\cdot 7) & (0\cdot -4)+(2\cdot 2) & (0\cdot 5)+(2\cdot 2) \\ (3\cdot 1)+(0\cdot 7) & (3\cdot -4)+(0\cdot 2) & (3\cdot 5)+(0\cdot 2) \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} 33 & 48 & -33 \\ 14 & 4 & 4 \\ 3 & -12 & 15 \end{bmatrix}_{3\times 3}$$

Square matrices are particularly important in matrix multiplication. Why? Because matrices must have compatible dimensions to be multiplied, and their product, in general, has different dimensions. However, for square matrices of order $n$, their product is also a square matrix of order $n$.

# Matrices

•Square matrices are particularly important in matrix multiplication. Why? Because matrices must have compatible dimensions to be multiplied, and their product, in general, has different dimensions. However, for square matrices of order $n$, their product is also a square matrix of order $n$.

•The consequence of this is that the multiplication of square matrices is *associative*. For square matrices $A$, $B$, $C$, we have:

•$ABC = A(BC) = (AB)C$

•However, note that matrix multiplication is still not commutative in general.

•$ABC \neq CBA$

•This will become very very important when we talk about 3D transformations.

•With that, you know everything you need to know about linear algebra for graphics. The world is your oyster. Check your knowledge now by completing this week's problem set.

# Matrices