MA4340: Probability Theory in Finance

GARCH Model: An Analysis of Financial Volatility

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***Abstract*—Financial markets exhibit notable periods of volatility clustering, where high fluctuations tend to be followed by high volatility and low fluctuations by calm periods. Traditional linear models that assume constant variance are inadequate to describe such dynamics. To address this, the Autoregressive Conditional Heteroskedasticity (ARCH) model and its extension, the Generalized ARCH (GARCH) model, have become essential tools in financial econometrics. This project explores multiple GARCH variants—including EGARCH, IGARCH, GJR-GARCH, and GARCH-in-Mean—to analyze and forecast market volatility on both synthetic and real stock data. By estimating these models using rigorous statistical methods in Python and comparing their performance through diagnostic tests, RMSE metrics, and variance forecast evaluations, we provide a comprehensive assessment of their strengths and limitations. Our findings reveal that while no single model uniformly outperforms across all assets, GJR-GARCH exhibit superior performance for certain stocks, suggesting that incorporating asymmetry and risk premium effects is vital for accurate volatility estimation.**

I. Introduction

A. Background

Financial markets are characterized by periods of varying volatility, with clusters of high and low fluctuations occurring over time. Traditional linear models often assume constant variance, which fails to capture this dynamic nature. To address this, Robert Engle introduced the Autoregressive Conditional Heteroskedasticity (ARCH) model in 1982, which was later extended by Tim Bollerslev in 1986 to the Generalized ARCH (GARCH) model. The GARCH model accounts for time-varying volatility by incorporating past variances into current estimations, making it a valuable tool for modeling financial time series data.

B. Topic of Interest

The GARCH model has become a cornerstone in financial econometrics for analyzing and forecasting market volatility. It effectively captures the phenomenon of volatility clustering, where large changes in asset prices are likely to be followed by more large changes. Various extensions of the GARCH model, such as EGARCH, FIGARCH, GJR-GARCH and GARCH-M, have been developed to address specific characteristics of financial data, including asymmetries in volatility responses.

C. Aim of the Project

This project aims to delve into the GARCH model's theoretical foundations and practical applications in financial volatility analysis. The objectives include:

- Understanding the mathematical underpinnings of the GARCH model.
- Analyzing financial time series data to identify patterns of volatility clustering.
- Implementing the GARCH model using Python to estimate parameters.
- Forecasting future market volatility and evaluating the model's predictive performance.

By achieving these goals, the project seeks to enhance the understanding of financial market dynamics and improve the accuracy of volatility forecasting.

II. Theory Review

A. Terms Used

- **Price**: The market value of a financial asset at a given point in time.
- **Returns**: The percentage change in price of an asset over time, often used to measure performance. The returns is what is to be modeled in the following sections. Given below is the formulation of the return used in the subsequent sections:

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$$

  where:
  - r_t is the log return at time t,
  - P_t is the adjusted closing price at time t,

- P_{t-1} is the adjusted closing price at the previous time step $t-1$.
- **Weak Stationarity**: A time series $\{y_t\}$ is weakly stationary if the following conditions hold:
  - **Constant mean:**
  $$\mathbb{E}[y_t] = \mu, \quad \text{for all } t.$$
  - **Constant variance:**
  $$\text{Var}(y_t) = \sigma^2, \quad \text{for all } t.$$
  - **Constant autocovariance:**
  $$\text{Cov}(y_t, y_{t+h}) = \gamma(h), \quad \text{for all } t \text{ and } h.$$

  where μ is the mean, σ^2 is the variance, and $\gamma(h)$ is the autocovariance function, which depends only on the lag h, not on time t.
  We generally assume that the returns are weakly stationary, to be able to consider time series techniques, and use tools like the auto-correlation function
- **Autocorrelation Function**: Autocorrelation Function (ACF) shows the correlation of a series with its past values. It is given by:
$$\rho_h = \frac{\text{Cov}(r_t, r_{t-h})}{\text{Var}(r_t)} = \frac{\gamma_h}{\gamma_0} = \frac{\text{Cov}(r_t, r_{t-h})}{\sqrt{\text{Var}(r_t)\text{Var}(r_{t-h})}}$$
  where:
  - ρ_h is the autocorrelation at lag h,
  - γ_h is the autocovariance at lag h,
  - γ_0 is the variance of r_t.
- **Partial ACF**: Partial ACF (PACF) shows the correlation after removing the effects of earlier lags.
- **White Noise**: White noise is a series of uncorrelated random variables with constant mean and variance. The models utilize white noise for the model formulation. Practically, if the ACF values are almost close to 0, the series may be considered white noise

B. AR(p) model

The autoregressive (AR) model uses the following formulation for the model:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + a_t, \tag{1}$$

where ϕ are the coefficients, a is white noise, and r represents the returns.

This model is used to capture autoregressive relationships in the r_t series

C. MA(q) model

The moving average (MA) model uses the following formulation for the model:

$$r_t = c_0 + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, \tag{2}$$

where θ are the coefficients, a_t is white noise, and r_t represents the returns.

This formulation has been derived using an infinite order AR model, where the coefficients decay exponentially (refer Analysis of Financial Time Series book for further details)

D. ARMA(p, q) model

The autoregressive moving average (ARMA) model combines both AR and MA components and is given by the formulation:

$$r_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} + a_t - \sum_{i=1}^{q} \theta_i a_{t-i}, \tag{3}$$

where ϕ and θ are the AR and MA coefficients respectively, a_t is white noise, and r_t represents the returns.

E. Volatility

The volatility captures the fluctuation of the returns. The GARCH model aims to model this. The following are benefits of modeling volatility:

- Volatility is crucial for options pricing, especially in models like Black-Scholes.
- It helps assess the risk associated with a financial position.
- Understanding volatility enables improved forecasting of future returns.

The following are some commonly observed properties of volatility:

- Forms clusters of high and low variances
- Evolves in a continuous manner (jumps are rare)
- Varies in a fixed range and doesn't diverge to infinity
- Leverage effect: Volatility responds differently to price increases and price drops (This effect has been considered in GARCH model variations)

Since the above properties have been observed, we look to capture these relationships in the form of mathematical models.

F. Conditional Heteroskedasticity Modeling

Conditional Heteroskedastic models aim to predict the conditional variance along with the conditional mean:

$$\mu_t = \mathbb{E}(r_t|\mathcal{F}_{t-1}) \tag{4}$$

$$\sigma_t^2 = \text{Var}(r_t|\mathcal{F}_{t-1}) = \mathbb{E}[(r_t - \mu_t)^2|\mathcal{F}_{t-1}] \tag{5}$$

It has been observed empirically that generally the ACF of returns is close to zero, but the absolute/squared returns have a significant ACF. This shows that the returns are uncorrelated and dependent (see Figure 1)

Since the ACF of returns is almost 0, we may model the mean using a simple time series model (AR/MA/ARMA) with a lower order. This is the **mean equation**. The difference between the predicted mean and the actual return gives the **residuals**

$$r_t = \mu_t + a_t \tag{6}$$

where μ_t is modeled using a simple time series model

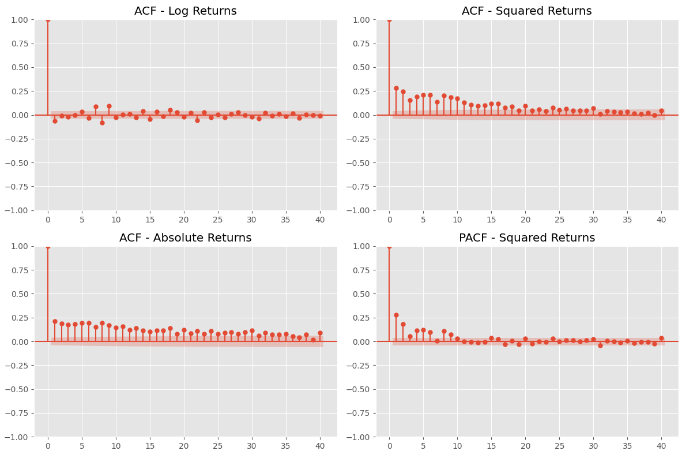

Fig. 1. ACF Plots for Apple Stock Returns

Hence Conditional Heteroskedastic Modeling deals with modeling the following.

$$\sigma_t^2 = \text{Var}(r_t \mid \mathcal{F}_{t-1}) = \text{Var}(a_t \mid \mathcal{F}_{t-1}) \quad (7)$$

Modeling above gives the **volatility equation**.

G. *Model Building Steps*

The following steps have been used to model conditional heteroskedastic models:

- Specify a mean equation.
- Use the residuals of the mean equation to test for ARCH effects.
- Use a volatility model if the ARCH effects are significant, and perform a joint estimation of the mean and volatility equations.
- Check and refine the model.

H. *ARCH effect testing*

Ljung-Box statistics test has been used to test if there is enough dependency between the residuals to consider an ARCH model. It considers the ACF of the a_t^2 series, with the null hypothesis being that the ACF values are 0. If p-value of this test is close to zero, it implies there is significant correlation in the a_t^2 series.

I. *ARCH(m) model*

The Autoregressive Conditional Heteroskedasticity (ARCH) model captures time-varying volatility in a time series and is defined by the equations:

$$a_t = \sigma_t \cdot \varepsilon_t, \quad (8)$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2, \quad (9)$$

where ε_t is white noise with mean 0 and variance 1, a_t is the residual, and σ_t^2 represents the conditional variance. The parameters $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i = 1, \ldots, m$ for positive variance.

The above formulation has been defined similarly to considering an AR model on the a_t^2 series. The intuition is that high a_t values may cause a higher variance in the future.

J. *GARCH(m, s) model*

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model extends the ARCH model by including lagged conditional variances. It is defined by the following equations:

$$a_t = \sigma_t \cdot \varepsilon_t, \quad (10)$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2, \quad (11)$$

where ε_t is white noise with mean 0 and variance 1, a_t is the residual, and σ_t^2 is the conditional variance.

The parameters must satisfy: $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, and $\sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1$.

The intuition behind the above is that the σ_t^2 is modeled as a function of both past squared residuals and past conditional variances, similar to applying an ARMA model to the a_t^2 series.

Parameter Estimation: The model parameters are estimated using the L-BFGS-B optimization algorithm. This has been implemented using the "arch" Python library.

Order Selection: The order of the GARCH model has been chosen based on the AIC scores. The AIC score is a function of the log-likelihood of the estimated parameters and the number of parameters in the model. It quantifies the tradeoff between model fit and complexity. The data has been fit for different order values, and the AIC and BIC scores are evaluated. Lower values of AIC and BIC implies there is a good tradeoff between the likelihood and the complexity of the model

$$\text{AIC} = -2 \log L + 2k, \qquad \text{BIC} = -2 \log L + k \log n \quad (12)$$

where L is the maximized value of the likelihood function, k is the number of estimated parameters, and n is the number of observations.

III. GARCH MODELS

A. *GARCH-M model*

In finance, the return of a security may depend on its volatility. To model such phenomenon, we may use the GARCH-M model, or Garch *in the mean*.

A simple GARCH(1, 1)-M model can be written as

$$r_t = \mu + c\sigma_t^2 + a_t, a_t = \sigma_t \epsilon_t,$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_1^2$$

with μ and c being constants. Here we see that as opposed to a regular GARCH model, we now see that the return r_t

depends on c, called the risk premium parameter. A positive c indicates that the return is positively related to the volatility and a negative c otherwise. Other formulations have also been used, such as $c\ln(\sigma_t^2)$ or $c\sigma_t$ replacing the $c\sigma_t^2$ term above.

Note that now, there are serial correlations between returns. Here, it is introduced by the volatility σ_t^2.

B. GARCH-E model

A weakness of GARCH models is that they use squared returns, therefore they treat positive and negative returns similarly. However, this is untrue in the financial market. In truth, negative returns generally leads to more volatility, since the market becomes more unstable. To model this, the E-GARCH model was proposed, to allow for asymmetric effects between positive and negative returns. E-GARCH considers the weighted innovation

$$g(\epsilon_t) = \theta\epsilon_t + \gamma(|\epsilon_t| - E(|\epsilon_t|))$$

where θ and γ are real constants

We can clearly see that this is asymmetric, since if $\epsilon_t > 0$, then $g = (\theta + \gamma)\epsilon_t - E(|\epsilon_t|)$ and $g = (\theta - \gamma)\epsilon_t - E(|\epsilon_t|)$ otherwise.

- An E-GARCH(m, s) model can be written as

  $$a_t = \sigma_t\epsilon_t, ln(\sigma_t^2) = \alpha_0 + \frac{1 + \beta_1 B + \dots \beta_{s-1}B^{s-1}}{1 - \alpha_1 B - \dots \alpha_m B^m} g(\epsilon_{t-1})$$

  , where B is a backshift operator, i.e, $Bg(\epsilon_t) = g(\epsilon_{t-1})$
- An alternate form for the E-GARCH(m, s) model is

  $$\ln(\sigma_t^2) = \alpha_0 + \sum_{i=1}^{s} \alpha_i \frac{|a_{t-i}| + \gamma_i a_{t-i}}{\sigma_{t-i}} + \sum_{j-1}^{m} \beta_j \ln(\sigma_{t-j}^2)$$

  Here the effects of a_t or the shock can be seen more directly. A positive a_{t-i} contributes $\alpha_i(1 + \gamma_t)|\epsilon_{t-i}|$ to the log volatility, whereas a negative a_{t-i} contributes $\alpha_i(1-\gamma_t)|\epsilon_{t-i}|$. In applications, we would expect γ_t to be negative, since the volatility would decrease with negative a.

C. GARCH-I model

IGARCH (Integrated GARCH) was created to model persistent volatility, where the effects of a shock to volatility never completely die out. It's used when volatility seems to have "infinite memory". IGARCH models are unit-root GARCH models. As mentioned, a key feature of IGARCH models is that the impact of past squared shocks $\eta_{t-i} = a_{t-i}^2 - \sigma_{t-i}^2$ for $i > 0$ on a_t^2 is persistent.

A simple IGARCH(1,1) model can be written as

$$a_t = \sigma_t\varepsilon_t, \quad \sigma_t^2 = \alpha_0 + \beta_1\sigma_{t-1}^2 + (1 - \beta_1)a_{t-1}^2,$$

where $1 > \beta_1 > 0$.

The case of $\alpha_0 = 0$ is of particular interest in studying the IGARCH(1,1) model. In this case, the volatility forecasts are simply $\sigma_h^2(1)$ for all forecast horizons. This special IGARCH(1,1) model is the volatility model used in *RiskMetrics*, which is an approach for calculating value at risk. The model is also an exponential smoothing model for the $\{a_t^2\}$ series.

To see this, we rewrite the model as:

$$\sigma_t^2 = (1 - \beta_1)a_{t-1}^2 + \beta_1\sigma_{t-1}^2,$$
$$= (1 - \beta_1)a_{t-1}^2 + \beta_1\left[(1 - \beta_1)a_{t-2}^2 + \beta_1\sigma_{t-2}^2\right],$$
$$= (1 - \beta_1)a_{t-1}^2 + (1 - \beta_1)\beta_1 a_{t-2}^2 + \beta_1^2\sigma_{t-2}^2.$$

By repeated substitutions, we have:

$$\sigma_t^2 = (1 - \beta_1)\left(a_{t-1}^2 + \beta_1 a_{t-2}^2 + \beta_1^2 a_{t-3}^2 + \cdots\right),$$

which is the well-known exponential smoothing formulation with β_1 being the discounting factor. Exponential smoothing methods can thus be used to estimate such an IGARCH(1,1) model. For our analysis we are utilising FI-GARCH which is a generalised version of the IGARCH model that allows partial differencing.

D. GARCH-GJR model

GJR-GARCH / Threshold GARCH (TGARCH) was created to model the asymmetric effect of shocks — especially the fact that negative returns (bad news) tend to cause higher volatility than positive returns (good news) of the same size. This is known as the leverage effect.A GJR-GARCH(m, s) model assumes the form:

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s}(\alpha_i + \gamma_i N_{t-i})a_{t-i}^2 + \sum_{j=1}^{m}\beta_j\sigma_{t-j}^2 \qquad (3.33)$$

where N_{t-i} is an indicator for negative a_{t-i}, that is,

$$N_{t-i} = \begin{cases} 1 & \text{if } a_{t-i} < 0, \\ 0 & \text{if } a_{t-i} \geq 0, \end{cases}$$

and α_i, γ_i, and β_j are non-negative parameters satisfying conditions similar to those of GARCH models.

From the model, it is seen that a positive a_{t-i} contributes $\alpha_i a_{t-i}^2$ to σ_t^2, whereas a negative a_{t-i} has a larger impact $(\alpha_i + \gamma_i)a_{t-i}^2$, assuming $\gamma_i > 0$.

The model uses zero as its threshold to separate the impacts of past shocks. Other threshold values can also be used.

IV. Analysis of GARCH Variants on Synthetic Data

We generated and analyzed seven synthetic datasets that capture different volatility structures. These include:

1) Data generated from a standard GARCH
2) High-low-high volatility pattern
3) Square wave
4) Data sampled from a t-distribution
5) Normal noise added to a low-variance process
6) Data from an EGARCH process
7) Geometric Brownian Motion (GBM)

(Please refer to code for returns generated from this data)

TABLE I
LAGRANGE MULTIPLIER TEST p-VALUES FOR THE DIFFERENT SYNTHETIC DATASETS

Dataset	p-value
GARCH generated data	0.0000
High Low High	0.0000
Square Wave	1.0000
t-distribution	0.9831
Added Normal	0.0000
EGARCH generated data	0.3020
GBM	0.4931

From the above table, we see that these synthetic datasets capture different levels of the ARCH effect.

Each of these datasets were first modeled using an AR(1) model. We then applied the following GARCH model variations: GARCH, EGARCH, FIGARCH, GJRGARCH, and GARCH-M.

For each of the above combinations, we display:

- the predicted and the realized variance plot
- the forecasted returns with a red region highlighting the area that the return will end up with a confidence interval of 95%

Observations from Results:

- The GARCH model is able to capture regions of high and low volatilities as seen from results of dataset 2. This shows that the GARCH model is able to achieve the intended purpose of detecting clusters of high and low variances (Refer Figure 2)

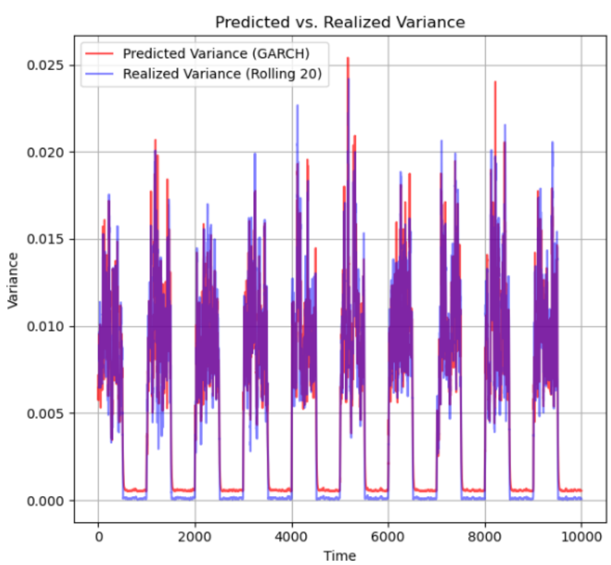

Fig. 2. Predicted vs Realized Variance for Dataset 2 (High-Low Volatilities)

- Dataset 4, which has a poor ARCH effect (high LM p-value), is not well modeled using the different garch models (see Figure 3, Figure 4). These poor results are present in other datasets that have high LM p-value
- FI-GARCH assumes gradual memory decay and cannot adapt to rapid volatility changes. It remembers the past low volatility conditions when a high volatility period

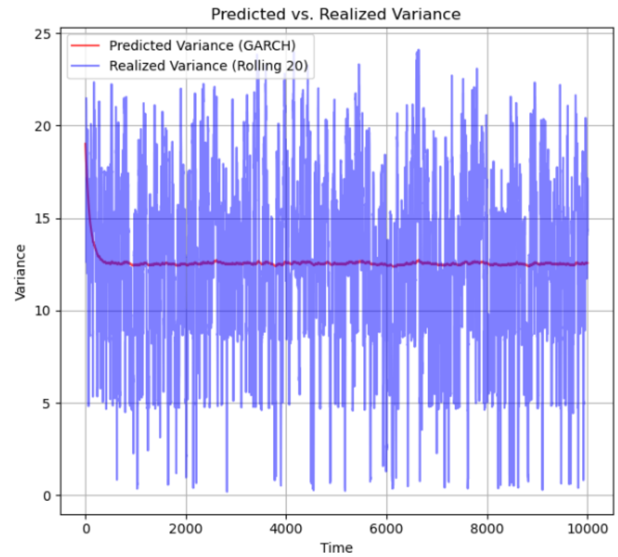

Fig. 3. Predicted vs Realized Variance for Dataset 4 with GARCH

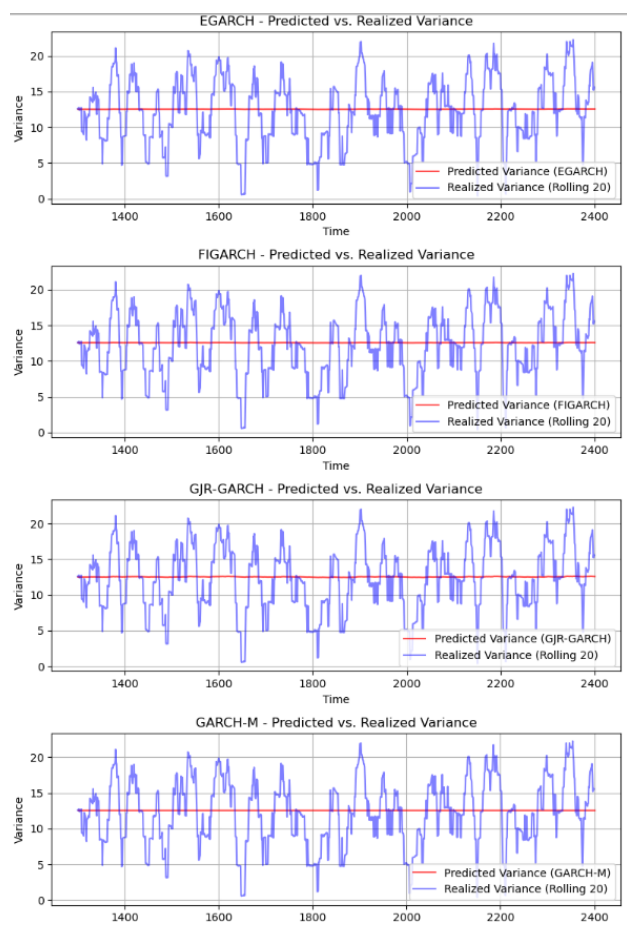

Fig. 4. Predicted vs Realized Variance for Dataset 4 with different GARCH models

begins and underestimates the volatility for a while. Similarly, when a high volatility period begins after a low volatility phase, it overestimates the volatility for while.
- If we look at the performance of E-GARCH of the square-wave returns , we see that the predictions explode to a very large value. This is because the returns that EGARCH expects has to be low enough (between ± 0.1) but our returns are pretty high ± 1.0 .

V. Analysis of GARCH Variants on Real Stock Data

We analyzed and compared the predictive performance of various GARCH-type models on stock return volatility across several companies, including Reliance Industries. Our approach involved the following steps:

1) We collected historical daily adjusted closing prices for selected stocks (Apple, Microsoft, Reliance, etc.) over a 10-year period (2015–2025) using the `yfinance` Python library.
2) We computed daily log returns from the price data using the formula:
$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$$
where P_t is the adjusted closing price at time t.
3) To ensure the validity of our time series modeling, we used the Augmented Dickey-Fuller (ADF) test to check the stationarity of the return series.
4) We performed an initial analysis by calculating the 21-day rolling standard deviation to visualize time-varying volatility. We also plotted the autocorrelation and partial autocorrelation (ACF and PACF) of both raw returns and residual transformations to examine temporal dependencies and signs of conditional heteroskedasticity.
5) We fitted an AR(1) model to the return series to account for short-term autocorrelation. The model takes the form:
$$r_t = \alpha + \phi r_{t-1} + \epsilon_t$$
We then analyzed the residuals ϵ_t to detect the presence of volatility clustering.
6) We used the Ljung-Box test on the squared residuals to formally test for ARCH effects, indicating the suitability of GARCH-type models.
7) For the standard GARCH model, we varied the order parameters p and q from 1 to 2 and selected the optimal configuration based on the lowest AIC and BIC values. We also fitted a range of GARCH variants, including:
   - EGARCH, which accounts for asymmetry in the impact of shocks on volatility.
   - FIGARCH, to capture long-memory effects in volatility.
   - GJR-GARCH, which includes a leverage term to better model asymmetries.
   - GARCH-M, which incorporates volatility directly into the mean equation.
8) To evaluate and compare the predictive performance of these models, we computed the Root Mean Squared Error (RMSE) between the predicted conditional volatility (standard deviation) from each model and the realized volatility estimated from the data (e.g., via rolling standard deviation).

TABLE II
RMSE Comparison of GARCH-type Models Across Different Stocks

Stock	GARCH	EGARCH	FIGARCH	GJR-GARCH	GARCH-M
Apple	0.0038	0.0045	0.0049	0.0036	**0.0035**
Microsoft	0.0039	0.0050	0.0053	**0.0037**	0.0054
HDFC Bank	0.0031	0.0033	0.0039	**0.0025**	0.0045
Reliance	0.0043	**0.0038**	0.0040	0.0048	0.0044
TCS	**0.0028**	0.0038	0.0040	0.0048	0.0044

Inferences: The RMSE values across GARCH-type models for different stocks yield the following insights:

1) **Apple:** GARCH-M achieves the best RMSE (0.0035), slightly outperforming GJR-GARCH (0.0036).
2) **Microsoft:** GJR-GARCH yields the lowest RMSE (0.0037); GARCH-M performs the worst (0.0054).
3) **HDFC Bank:** GJR-GARCH achieves the lowest RMSE (0.0025), clearly outperforming other models.
4) **Reliance:** EGARCH shows the best RMSE (0.0038), suggesting its ability to capture asymmetric volatility effects.
5) **TCS:** The basic GARCH model performs best with the lowest RMSE (0.0028).
6) GJR-GARCH is a frequent top performer, with the lowest RMSE in two out of five cases (Microsoft and HDFC Bank). Other models like GARCH-M and EGARCH each excel for one stock.
7) The effectiveness of each variant appears asset-specific, depending on the volatility characteristics of the underlying stock.

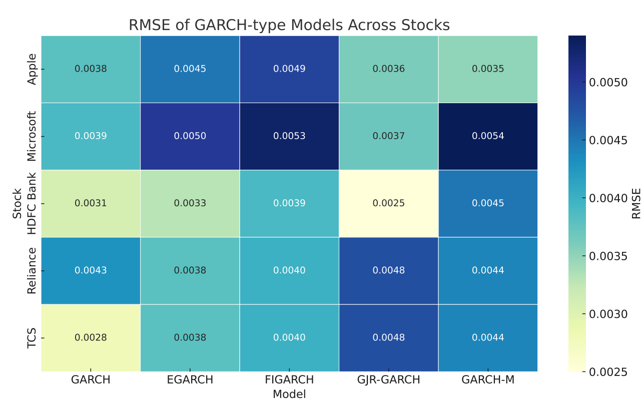

Fig. 5. Heatmap of RMSE values for GARCH models — darker colors indicate higher RMSE (worse accuracy).

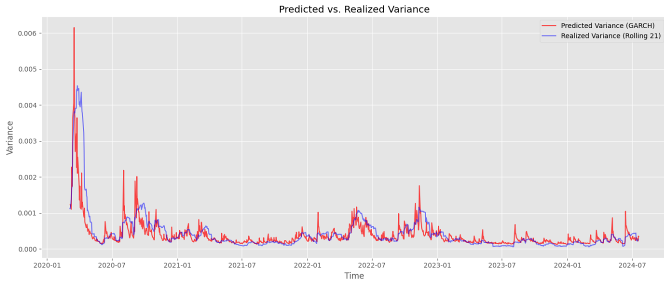

Fig. 6. Predicted vs Realised Variance for the GARCH model.

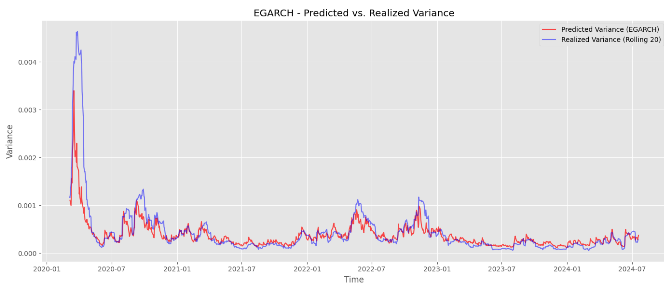

Fig. 7. Predicted vs Realised Variance for the EGARCH model.

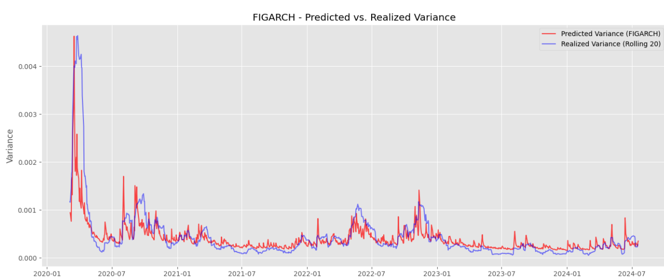

Fig. 8. Predicted vs Realised Variance for the FIGARCH model.

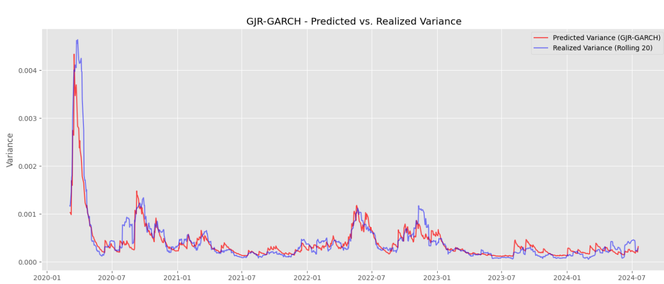

Fig. 9. Predicted vs Realised Variance for the GJR-GARCH model.

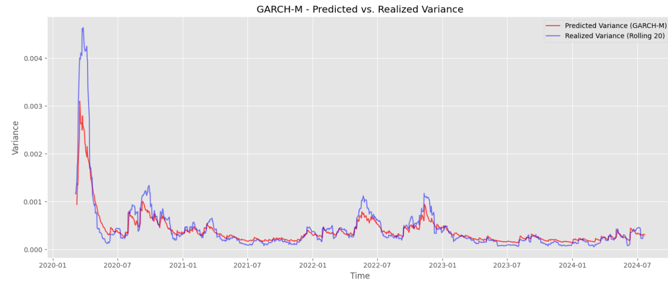

Fig. 10. Predicted vs Realised Variance for the GARCH-M model.

VI. References

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