Probability Assignment 3 (12.13.5.7)

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Question

In an examination, 20 questions of true-false type are asked. Suppose a student tosses a fair coin to determine his answer to each question. If the coin falls heads, he answers 'true'; if it fails, he answers 'false'. Find the probability that he answers at least 12 questions correctly.

Solution

VARIABLE	VALUE	DESCRIPTION
n	20	Number of questions in the examination
p	0.5	Probability of coin showing heads
X	-	Number of questions answered correctly

TABLE 1

X follows a binomial distribution,

$$X = Bin(n, p) \tag{1}$$
$$= Bin(20, 0.5) \tag{2}$$

The mean of X,

$$\mu = n \times p \tag{3}$$
$$= 10 \tag{4}$$

The Variance of X,

$$\sigma^2 = n \times p \times (1 - p) \tag{5}$$
$$= 5 \tag{6}$$

1) **Gaussian**
   Applying the normal approximation to X for large values of n,

$$\Pr(X \geq a) = Q(a) = \int_{a-0.5}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(k-\mu)^2}{2\sigma^2}}\, dk \tag{7}$$
$$Q(12) = \int_{11.5}^{\infty} \frac{1}{\sqrt{10\pi}} e^{-\frac{(k-10)^2}{10}}\, dk \tag{8}$$
$$\tag{9}$$

   On computation,

$$Q(12) \approx 0.2512 \tag{10}$$

2) **Binomial**

Since X follows binomial distribution,

$$\Pr(X = k) = {}^{n}C_{k} p^{k} q^{n-k} \tag{11}$$

$$\Pr(X \geq k) = \sum_{t=k}^{20} {}^{n}C_{t} p^{t} q^{n-t} \tag{12}$$

for $k = 12$,

$$\Pr(X \geq 12) = \sum_{t=12}^{20} {}^{20}C_{t} (0.5)^{t} (0.5)^{20-t} \tag{13}$$

$$= 0.2517 \tag{14}$$

3) **Comparison**
The approximation holds since the value of p = 0.5 and n is large enough. The approximation has a relative error of 0.2%.