INVERSE HYPERBOLIC FUNCTIONS

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If $x = \sinh \mathrm{u}$ then $\mathrm{u} = \sinh^{-1} \mathrm{x}$ is called sine hyperbolic inverse of x, where x is real.
Similarly we can define $\cosh^{-1}\mathrm{x}, \tanh^{-1}\mathrm{x}, \coth^{-1}\mathrm{x}, \mathrm{sech}^{-1}\mathrm{x}, \mathrm{cosech}^{-1}\mathrm{x}$.

Theorem: If x is real.

(i) $\sinh^{-1}\mathrm{x} = \log\ (\mathrm{x} + \sqrt{\mathrm{x}^2 + 1})$

(ii) $\cosh^{-1}\mathrm{x} = \log\ (\mathrm{x} + \sqrt{\mathrm{x}^2 - 1})$

(iii) $\tanh^{-1}\mathrm{x} = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof is also important and can come in exam

Proof :- (i) let $\sinh^{-1}(x) = y$

$$\therefore \sinh y = x$$

$$\frac{e^y - e^{-y}}{2} = x$$

$$e^y - e^{-y} = 2x$$

$$e^{2y} - 2x e^y - 1 = 0 \quad (\text{multi by } e^y)$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2+1}\right)$$

Now $x - \sqrt{x^2+1} < 0$ $\qquad x < \sqrt{x^2+1}$

$\therefore \log\left(x - \sqrt{x^2+1}\right)$ is not defined

$$\therefore y = \log\left(x + \sqrt{x^2+1}\right)$$

$$\therefore \sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$$

(ii) Let $\cosh^{-1} x = y$

$$\therefore \cosh y = x$$

$$\therefore \frac{e^y + e^{-y}}{2} = x$$

$$\therefore e^y + e^{-y} = 2x$$

multi by e^y

$$e^{2y} - 2x e^y + 1 = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$\therefore e^y = x \pm \sqrt{x^2-1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2-1}\right) \quad \text{—— (1)}$$

Let $y = \log\left(x - \sqrt{x^2-1}\right)$ —— (2)

$$e^y = x - \sqrt{x^2-1}$$

$$\therefore e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$\therefore e^{-y} = \frac{x+\sqrt{x^2-1}}{(x)^2-(\sqrt{x^2-1})^2} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1}$$

$$\therefore e^{-y} = x+\sqrt{x^2-1}$$

$$\therefore -y = \log(x+\sqrt{x^2-1})$$

$$\therefore y = -\log(x+\sqrt{x^2-1}) \quad \text{--- } (3)$$

from (2) & (3) $\Rightarrow$ $\log(x-\sqrt{x^2-1}) = -\log(x+\sqrt{x^2-1})$

Subst in (1) $\therefore y = \pm \log(x+\sqrt{x^2-1})$

$$\therefore \cosh^{-1} x = \pm \log(x+\sqrt{x^2-1})$$

$$\therefore x = \cosh(\pm \log(x+\sqrt{x^2-1}))$$

$$[\text{but } \cosh(-z) = \cosh z]$$

$$x = \cosh(\log(x+\sqrt{x^2-1}))$$

$$\therefore \cosh^{-1} x = \log(x+\sqrt{x^2-1})$$

(iii) TPT. $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Soln :- Let $\tanh^{-1}(x) = y$

$$\therefore x = \tanh y$$

$$\frac{x}{1} = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

Using componendo - dividendo

$$\frac{1+x}{1-x} = \frac{(e^{y} + e^{-y}) + (e^{y} - e^{-y})}{(e^{y} + e^{-y}) - (e^{y} - e^{-y})}$$

$$\frac{1+x}{1-x} = \frac{2e^{y}}{2e^{-y}} = e^{2y}$$

$$\therefore e^{2y} = \frac{1+x}{1-x}$$

$$\therefore 2y = \log\left(\frac{1+x}{1-x}\right)$$

$$\therefore y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh log\, \sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log\sqrt{5/3} + \tanh \log\sqrt{7} = 1$

Soln :-

method 1.

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\tanh(\log\sqrt{x}) = \frac{e^{\log\sqrt{x}} - e^{-\log\sqrt{x}}}{e^{\log\sqrt{x}} + e^{-\log\sqrt{x}}}$$

method 2.

Let $\tanh(\log\sqrt{x}) = a$

$$\therefore \log\sqrt{x} = \tanh^{-1}(a)$$

$$\log\sqrt{x} = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right)$$

$$\tanh(\log\sqrt{x}) = \frac{e^{\log\sqrt{x}} - e^{-\log\sqrt{x}}}{e^{\log\sqrt{x}} + e^{-\log\sqrt{x}}}$$

$$= \frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$\tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

$$\log\sqrt{x} = \log\sqrt{\frac{1+a}{1-a}}$$

$$\sqrt{x} = \sqrt{\frac{1+a}{1-a}}$$

$$x = \frac{1+a}{1-a}$$

$$\frac{x-1}{x+1} = \frac{1+a-1+a}{1+a+1-a} = a$$

$$\therefore \tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

TPT $\tanh\left(\log\sqrt{\frac{5}{3}}\right) + \tanh(\log\sqrt{7}) = 1.$

$$\tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

$$\tanh\left(\log\sqrt{\frac{5}{3}}\right) = \frac{\frac{5}{3}-1}{\frac{5}{3}+1} = \frac{2}{8} = \frac{1}{4}$$

$$\tanh(\log\sqrt{7}) = \frac{7-1}{7+1} = \frac{6}{8} = \frac{3}{4}$$

$$\therefore \tanh\left(\log\sqrt{\frac{5}{3}}\right) + \tanh(\log\sqrt{7}) = \frac{1}{4} + \frac{3}{4} = 1$$

2. (i) Prove that $\cosh^{-1}\sqrt{1+x^2} = \sinh^{-1}x$

(ii) Prove that $\tanh^{-1}x = \sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

(iv) Prove that $\coth^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}\log\left(\frac{x+a}{x-a}\right)$ (H.W.) (do similar to $\tanh^{-1}(x)$)

(v) Prove that $\operatorname{sech}^{-1}(\sin\theta) = \log\cot\frac{\theta}{2}$

Soln:- (i) Let $\cosh^{-1}\sqrt{1+x^2} = y$

$$\therefore \sqrt{1+x^2} = \cosh y$$

$$\therefore 1+x^2 = \cosh^2 y$$

$$\therefore x^2 = \cosh^2 y - 1$$

$$\therefore x^2 = \sinh^2 y$$

$$\therefore x = \sinh y$$

$$\therefore \sinh^{-1} x = y$$

$$\therefore \cosh^{-1} \sqrt{1+x^2} = \sinh^{-1} x$$

(ii) Tpt: $\tanh^{-1} x = \sinh^{-1} \frac{x}{\sqrt{1-x^2}}$

<u>soln</u>:- Let $\tanh^{-1} x = y$

$$\therefore x = \tanh y$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}} = \frac{\tanh y}{\operatorname{sech} y}$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \sinh y$$

$$\therefore y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\therefore \tanh^{-1} x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii) Tpt: $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$ (H.W.)

Let $\cosh^{-1}\left(\sqrt{1+x^2}\right) = y$

$\sqrt{1+x^2} = \cosh y$

(4) TPT: $\operatorname{sech}^{-1}(\sin\theta) = \log \cot \frac{\theta}{2}$

Let $\operatorname{sech}^{-1}(\sin\theta) = y$

$\sin\theta = \operatorname{sech} y$

$$\sin\theta = \frac{2}{e^y + e^{-y}} = \frac{2e^y}{e^{2y}+1}$$

$$(\sin\theta)\, e^{2y} - 2e^y + \sin\theta = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(\sin\theta)(\sin\theta)}}{2\sin\theta}$$

$$\therefore e^y = \frac{2 \pm \sqrt{4 - 4\sin^2\theta}}{2\sin\theta}$$

$$\therefore e^y = \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$e^y = \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2}$$

$$\therefore y = \log \cot\frac{\theta}{2}$$

3. Separate into real and imaginary parts $cos^{-1}e^{i\theta}$ ***or*** $cos^{-1}(\cos\theta + i\sin\theta)$

Soln:- Let $\cos^{-1}(\cos\theta + i\sin\theta) = x+iy$

$$\therefore \cos(x+iy) = \cos\theta + i\sin\theta$$

$$\cos x \cos iy - \sin x \sin iy = \cos\theta + i\sin\theta$$

$$\cos x \cosh y - i\sin x \sinh y = \cos\theta + i\sin\theta$$

$(\cos iy = \cosh y, \ \sin iy = i\sinh y)$

$$\therefore \cos\theta = \cos x \cosh y \ \& \ \sin\theta = -\sin x \sinh y \quad \text{—— (1)}$$

Now. $\cosh^2 y - \sinh^2 y = 1$

$$\left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{-\sin\theta}{\sin x}\right)^2 = 1$$

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\frac{\sin^2 x - \sin^2 x \sin^2\theta - \sin^2\theta + \sin^2 x \sin^2\theta}{\sin^2 x - \sin^4 x} = 1$$

$$\sin^2 x - \sin^2\theta = \sin^2 x - \sin^4 x$$

$$\therefore \sin^4 x = \sin^2\theta$$

$$\therefore \sin^2 x = \sin\theta$$

$$\sin x = \sqrt{\sin\theta} \quad \text{—— (A)}$$

$$\therefore \boxed{x = \sin^{-1}\left(\sqrt{\sin\theta}\right)} \quad \text{—— (2)}$$

from (1) $\quad \sin\theta = -\sin x \sinh y$

$$\sin\theta = -\sqrt{\sin\theta}\,\sinh y$$

$$\sinh y = -\sqrt{\sin\theta}$$

$$\therefore y = \sinh^{-1}\left(-\sqrt{\sin\theta}\right)$$

Now $\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

$$\therefore y = \log\left[-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right]$$

$$\therefore y = \log\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$

$$\therefore \cos^{-1}(\cos\theta + i\sin\theta) = x + iy$$

$$= \sin^{-1}\left(\sqrt{\sin\theta}\right) + i\log\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$