

DE MOIVRE'S THEOREM

8. If $a = \cos 2\alpha + i \sin 2\alpha$, $b = \cos 2\beta + i \sin 2\beta$, $c = \cos 2\gamma + i \sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2 \cos(\alpha + \beta - \gamma)$

Solution:
$$\frac{ab}{c} = \frac{(\cos 2\alpha + i \sin 2\alpha)(\cos 2\beta + i \sin 2\beta)}{(\cos 2\gamma + i \sin 2\gamma)}$$

$$= \cos(2\alpha + 2\beta - 2\gamma) + i \sin(2\alpha + 2\beta - 2\gamma)$$

$$= \cos 2(\alpha + \beta - \gamma) + i \sin 2(\alpha + \beta - \gamma)$$

$$\sqrt{\frac{ab}{c}} = [\cos 2(\alpha + \beta - \gamma) + i \sin 2(\alpha + \beta - \gamma)]^{1/2}$$

$$= \cos(\alpha + \beta - \gamma) + i \sin(\alpha + \beta - \gamma)$$

Similarly, $\sqrt{\frac{c}{ab}} = \cos(\alpha + \beta - \gamma) - i \sin(\alpha + \beta - \gamma)$

By addition we get $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2 \cos(\alpha + \beta - \gamma)$

9. If $x - \frac{1}{x} = 2i \sin \theta$, $y - \frac{1}{y} = 2i \sin \Phi$, $z - \frac{1}{z} = 2i \sin \psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2 \cos(\theta + \Phi + \psi)$

(ii) $\frac{m\sqrt{x}}{n\sqrt{y}} + \frac{n\sqrt{y}}{m\sqrt{x}} = 2 \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Solution: Since $x - \frac{1}{x} = 2i \sin \theta \therefore x^2 - 2ix \sin \theta - 1 = 0$

Solving the quadratic for x , we get,

$$x = \frac{2i \sin \theta \pm \sqrt{4i^2 \sin^2 \theta - 4(1)(-1)}}{2(1)} = i \sin \theta \pm \sqrt{1 - \sin^2 \theta} = i \sin \theta \pm \cos \theta$$

consider $x = \cos \theta + i \sin \theta$

Similarly, $y = \cos \Phi + i \sin \Phi$, $z = \cos \psi + i \sin \psi$

(i) $xyz = (\cos \theta + i \sin \theta)(\cos \Phi + i \sin \Phi)(\cos \psi + i \sin \psi)$

$$= \cos(\theta + \Phi + \psi) + i \sin(\theta + \Phi + \psi)$$

$$\therefore \frac{1}{xyz} = \cos(\theta + \Phi + \psi) - i \sin(\theta + \Phi + \psi)$$

$$\text{Adding we get } xyz + \frac{1}{xyz} = 2 \cos(\theta + \Phi + \psi)$$

$$(ii) \frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \frac{(\cos \theta + i \sin \theta)^{1/m}}{(\cos \Phi + i \sin \Phi)^{1/n}} = \frac{\left(\cos \frac{\theta}{m} + i \sin \frac{\theta}{m}\right)}{\left(\cos \frac{\Phi}{n} + i \sin \frac{\Phi}{n}\right)} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) + i \sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$$

$$\text{Similarly, } \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right) - i \sin\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$$

$$\text{Adding we get } \frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2 \cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$$

Imp **10.** If $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = \sin \alpha + 2 \sin \beta + 3 \sin \gamma = 0$, Prove that $\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma = 18 \sin(\alpha + \beta + \gamma)$.

Solution: We have $\cos \alpha + 2 \cos \beta + 3 \cos \gamma = \sin \alpha + 2 \sin \beta + 3 \sin \gamma = 0$

$$\therefore (\cos \alpha + 2 \cos \beta + 3 \cos \gamma) + i (\sin \alpha + 2 \sin \beta + 3 \sin \gamma) = 0$$

$$\therefore (\cos \alpha + i \sin \alpha) + 2(\cos \beta + i \sin \beta) + 3(\cos \gamma + i \sin \gamma) = 0$$

$$\text{Let } x = \cos \alpha + i \sin \alpha, y = 2(\cos \beta + i \sin \beta), z = 3(\cos \gamma + i \sin \gamma)$$

$$\therefore x + y + z = 0$$

$$\therefore (x + y + z)^3 = 0$$

$$\therefore x^3 + y^3 + z^3 + 3(x + y + z)(xy + yz + zx) - 3xyz = 0$$

$$\therefore x^3 + y^3 + z^3 = 3xyz$$

$$\begin{aligned} \therefore (\cos \alpha + i \sin \alpha)^3 + 2^3(\cos \beta + i \sin \beta)^3 + 3^3(\cos \gamma + i \sin \gamma)^3 \\ = 3(\cos \alpha + i \sin \alpha) \cdot 2 \cdot (\cos \beta + i \sin \beta) \cdot 3 \cdot (\cos \gamma + i \sin \gamma) \end{aligned}$$

\therefore By De Moivre's Theorem,

$$\begin{aligned} (\cos 3\alpha + i \sin 3\alpha) + 8(\cos 3\beta + i \sin 3\beta) + 27(\cos 3\gamma + i \sin 3\gamma) \\ = 18[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)] \end{aligned}$$

$$\begin{aligned} (\cos 3\alpha + 8 \cos 3\beta + 27 \cos 3\gamma) + i(\sin 3\alpha + 8 \sin 3\beta + 27 \sin 3\gamma) \\ = 18[\cos(\alpha + \beta + \gamma) + i \sin(\alpha + \beta + \gamma)] \end{aligned}$$

Equating imaginary parts, we get the required result.

11. If $x_r = \cos \frac{\pi}{3^r} + i \sin \frac{\pi}{3^r}$, prove that (i) $x_1 x_2 x_3 \dots \text{ad. inf.} = i$

(ii) $x_0 x_1 x_2 \dots \text{ad. inf.} = -i$

Solution: We have $x_r = \cos \frac{\pi}{3^r} + i \sin \frac{\pi}{3^r}$

Putting $r = 0, 1, 2, 3 \dots \dots \dots$ we get $x_0 = \cos \frac{\pi}{3^0} + i \sin \frac{\pi}{3^0} = \cos \pi + i \sin \pi = -1$

$$x_1 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}, \quad x_2 = \cos \frac{\pi}{3^2} + i \sin \frac{\pi}{3^2} \dots \dots \dots \text{and so on}$$

$$x_1 x_2 x_3 \dots \dots \dots$$

$$= \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) \left(\cos \frac{\pi}{3^2} + i \sin \frac{\pi}{3^2} \right) \left(\cos \frac{\pi}{3^3} + i \sin \frac{\pi}{3^3} \right) \dots \dots \dots$$

$$= \cos \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \dots \dots \right) \pi + i \sin \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \dots \dots \right) \pi$$

$$\text{But } \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots \dots \dots \infty = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$$

$$x_1 x_2 x_3 \dots \dots \dots = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = 0 + i(1) = i$$

$$\text{Also } x_0 x_1 x_2 x_3 \dots \dots \dots = x_0(i) = (-1)(i) = -i$$

12. If $(\cos \theta + i \sin \theta)(\cos 3\theta + i \sin 3\theta) \dots \dots [\cos(2n-1)\theta + i \sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2r\pi}{n^2}$

Solution:

$$\text{L.H.S} = (\cos \theta + i \sin \theta)(\cos 3\theta + i \sin 3\theta) \dots \dots [\cos(2n-1)\theta + i \sin(2n-1)\theta]$$

$$= \cos[1 + 3 + \dots + (2n-1)]\theta + i \sin[1 + 3 + \dots + (2n-1)]\theta$$

But $1 + 3 + \dots + (2n-1)$ is an A.P. with first term 1, the number of terms n and common difference 2.

$$\therefore \text{The Sum, } S_n = \frac{n}{2}[2a + (n-1).d] = \frac{n}{2}[2 + (n-1).2] = n^2$$

$$\therefore \text{L.H.S} = \cos(n^2\theta) + i \sin(n^2\theta)$$

$$\text{R.H.S} = 1 = \cos 2r\pi + i \sin 2r\pi \quad \text{where } r = 0, 1, 2 \dots \dots$$

$$\text{Equating the two sides, we get } n^2\theta = 2r\pi \quad \therefore \theta = \frac{2r\pi}{n^2}$$

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know

13. By using De Moivre's Theorem show that

$$\sin \alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{\sin \alpha/2}$$

Solution: $\frac{1-z^6}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5 \dots\dots\dots(i)$

Let $z = \cos \alpha + i \sin \alpha$, then by De Moivre's theorem, $z^n = \cos n\alpha + i \sin n\alpha$

$$\begin{aligned} \therefore 1 + z + z^2 + z^3 + z^4 + z^5 &= 1 + (\cos \alpha + i \sin \alpha) + (\cos 2\alpha + i \sin 2\alpha) \\ &\quad + (\cos 3\alpha + i \sin 3\alpha) + (\cos 4\alpha + i \sin 4\alpha) + (\cos 5\alpha + i \sin 5\alpha) \\ &= (1 + \cos \alpha + \cos 2\alpha + \cos 3\alpha + \cos 4\alpha + \cos 5\alpha) \\ &\quad + i (\sin \alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha + \sin 5\alpha) \dots\dots\dots(ii) \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{1-z^6}{1-z} &= \frac{1-(\cos \alpha + i \sin \alpha)^6}{1-(\cos \alpha + i \sin \alpha)} = \frac{1-\cos 6\alpha - i \sin 6\alpha}{1-\cos \alpha - i \sin \alpha} = \frac{2\sin^2 3\alpha - 2i \sin 3\alpha \cos 3\alpha}{2\sin^2(\alpha/2) - 2i \sin(\alpha/2) \cos(\alpha/2)} \\ &= \frac{\sin 3\alpha (\sin 3\alpha - i \cos 3\alpha) [\sin(\alpha/2) + i \cos(\alpha/2)]}{\sin(\alpha/2) [\sin(\alpha/2) - i \cos(\alpha/2)] [\sin(\alpha/2) + i \cos(\alpha/2)]} \\ &= \frac{\sin 3\alpha (\sin 3\alpha - i \cos 3\alpha) [\sin(\alpha/2) - i \cos(\alpha/2)]}{\sin(\alpha/2) [\sin^2(\alpha/2) + \cos^2(\alpha/2)]} \\ &= \frac{\sin 3\alpha}{\sin(\alpha/2)} (\sin 3\alpha - i \cos 3\alpha) [\sin(\alpha/2) - i \cos(\alpha/2)] \\ &= \frac{\sin 3\alpha}{\sin(\alpha/2)} \left[\cos\left(\frac{\pi}{2} - 3\alpha\right) - i \sin\left(\frac{\pi}{2} - 3\alpha\right) \right] \times \left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \right] \\ &= \frac{\sin 3\alpha}{\sin(\alpha/2)} \left[\cos\left(-\frac{\pi}{2} + 3\alpha\right) + i \sin\left(-\frac{\pi}{2} + 3\alpha\right) \right] \times \left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) \right] \\ \therefore \frac{1-z^6}{1-z} &= \frac{\sin 3\alpha}{\sin(\alpha/2)} \left[\cos\left(3\alpha - \frac{\alpha}{2}\right) + i \sin\left(3\alpha - \frac{\alpha}{2}\right) \right] \\ &= \frac{\sin 3\alpha}{\sin(\alpha/2)} \left[\cos\left(\frac{5\alpha}{2}\right) + i \sin\left(\frac{5\alpha}{2}\right) \right] \dots\dots\dots(iii) \end{aligned}$$

Using (i) equating real parts, from (ii) and (iii), we get

$$1 + \cos \alpha + \cos 2\alpha + \cdots + \cos 5\alpha = \frac{\sin 3\alpha \cos(5\alpha/2)}{\sin(\alpha/2)}$$

And equating imaginary parts, we get

$$\sin \alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha \sin(5\alpha/2)}{\sin(\alpha/2)}$$

medium = grey

challenging = yellow

SOME PRACTICE PROBLEMS:

1. Simplify

(i)
$$\frac{(\cos 2\theta - i \sin 2\theta)^5 (\cos 3\theta + i \sin 3\theta)^6}{(\cos 4\theta + i \sin 4\theta)^7 (\cos \theta - i \sin \theta)^8}$$

(ii)
$$\frac{(\cos 2\theta + i \sin 2\theta)^3 (\cos 3\theta - i \sin 3\theta)^2}{(\cos 4\theta + i \sin 4\theta)^5 (\cos 5\theta - i \sin 5\theta)^4}$$

2. Prove that

(i)
$$\frac{(1+i)^8 (1-i\sqrt{3})^3}{(1-i)^6 (1+i\sqrt{3})^9} = \frac{i}{32}$$

(ii)
$$\frac{(1+i\sqrt{3})^9 (1-i)^4}{(\sqrt{3}+i)^{12} (1+i)^4} = -\frac{1}{8}$$

3. Find the modulus and the principal value of the argument of $\frac{(1+i\sqrt{3})^{17}}{(\sqrt{3}-i)^{15}}$ 4. Express $(1 + 7i)(2 - i)^{-2}$ in the form of $r(\cos \theta + i \sin \theta)$ and prove that the second power is a negative imaginary number and the fourth power is a negative real number.5. If $x_n + iy_n = (1 + i\sqrt{3})^n$, prove that $x_{n-1}y_n - x_n y_{n-1} = 4^{n-1}\sqrt{3}$.6. Simplify $(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n$ 7. Prove that $\frac{1+\sin \theta+i \cos \theta}{1+\sin \theta-i \cos \theta} = \sin \theta + i \cos \theta$ Hence deduct that

$$\left(1 + \sin \frac{\pi}{5} + i \cos \frac{\pi}{5}\right)^5 + i \left(1 + \sin \frac{\pi}{5} - i \cos \frac{\pi}{5}\right)^5 = 0.$$

8. If $z = \frac{1}{2} + i \frac{\sqrt{3}}{2}$ and \bar{z} is the conjugate of z find the value of $(z)^{15} + (\bar{z})^{15}$.9. Prove that, if n is a positive integer, then

(i)
$$(a + ib)^{m/n} + (a - ib)^{m/n} = 2(\sqrt{a^2 + b^2})^{m/n} \cos \left(\frac{m}{n} \tan^{-1} \frac{b}{a}\right)$$

(ii)
$$(\sqrt{3} + i)^{120} + (\sqrt{3} - i)^{120} = 2^{121}$$

10. If n is a positive integer, prove that $(1 + i)^n + (1 - i)^n = 2 \cdot 2^{n/2} \cos n \pi / 4$ Hence, deduce that $(1 + i)^{10} + (1 - i)^{10} = 0$ Doubt 11. Prove that $\left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n$ is equal to -1 if $n = 3k \pm 1$ and 2 if $n = 3k$ where k is an integer.12. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$,prove that $\alpha^n + \beta^n = 2^{n+1} \cos(n\pi/3)$.

(i) Deduce that $\alpha^{15} + \beta^{15} = -2^{16}$ (ii) Deduce that $\alpha^6 + \beta^6 = 128$

13. If α, β are the roots of the equation $z^2 \sin^2 \theta - z \sin 2\theta + 1 = 0$, prove that

$$\alpha^n + \beta^n = 2 \cos n \theta \operatorname{cosec}^n \theta$$

14. If $a = \cos 3\alpha + i \sin 3\alpha, b = \cos 3\beta + i \sin 3\beta, c = \cos 3\gamma + i \sin 3\gamma$,

prove that $\sqrt[3]{\frac{ab}{c}} + \sqrt[3]{\frac{c}{ab}} = 2 \cos(\alpha + \beta - \gamma)$

15. If $x + \frac{1}{x} = 2 \cos \theta, y + \frac{1}{y} = 2 \cos \phi, z + \frac{1}{z} = 2 \cos \psi$, prove that

$$(i) \quad xyz + \frac{1}{xyz} = 2 \cos(\theta + \phi + \psi) \quad (ii) \quad \sqrt{xyz} + \frac{1}{\sqrt{xyz}} = 2 \cos\left(\frac{\theta + \phi + \psi}{2}\right)$$

$$(iii) \quad \frac{x^m}{y^n} + \frac{y^n}{x^m} = 2 \cos(m\theta - n\phi) \quad (iv) \quad \frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2 \cos\left(\frac{\theta}{m} - \frac{\phi}{n}\right)$$

16. If $a = \cos \alpha + i \sin \alpha, b = \cos \beta + i \sin \beta, c = \cos \gamma + i \sin \gamma$, prove

$$\text{that } \frac{(b+c)(c+a)(a+b)}{abc} = 8 \cos \frac{(\alpha-\beta)}{2} \cos \frac{(\beta-\gamma)}{2} \cos \frac{(\gamma-\alpha)}{2}.$$

Doubt 17. If a, b, c are three complex numbers such that $a + b + c = 0$, prove that

$$(i) \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0 \quad \text{and} \quad (ii) \quad a^2 + b^2 + c^2 = 0$$

18. If $\cos \alpha + \cos \beta + \cos \gamma = 0$ and $\sin \alpha + \sin \beta + \sin \gamma = 0$, Prove that

$$(i) \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma = 0, \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 0.$$

$$(ii) \quad \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{3}{2}$$

$$(iii) \quad \cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha) = 0.$$

$$(iv) \quad \sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha) = 0.$$

$$(v) \quad \cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3 \cos(\alpha + \beta + \gamma)$$

$$(vi) \quad \sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3 \sin(\alpha + \beta + \gamma)$$

Remaining to do :
18 and 19

19. If $a \cos \alpha + b \cos \beta + c \cos \gamma = a \sin \alpha + b \sin \beta + c \sin \gamma = 0$, Prove that

$$a^3 \cos 3\alpha + b^3 \cos 3\beta + c^3 \cos 3\gamma = 3abc \cos(\alpha + \beta + \gamma) \text{ and}$$

$$a^3 \sin 3\alpha + b^3 \sin 3\beta + c^3 \sin 3\gamma = 3abc \sin(\alpha + \beta + \gamma)$$

20. If $x_r = \cos\left(\frac{2}{3}\right)^r \pi + i \sin\left(\frac{2}{3}\right)^r \pi$, prove that

$$(i) \quad x_1 x_2 x_3 \dots \infty = 1,$$

$$(ii) \quad x_0 x_1 x_2 \dots \infty = -1$$

ANSWERS

1. (i) $\cos 12\theta - i \sin 12\theta$ (ii) 1

3. $4, \frac{\pi}{6}$

4. $z = -1 + i = \sqrt{2}(\cos 3\pi/4 + i \sin 3\pi/4)$

$$z^4 = (\sqrt{2})^4 \cos(3\pi + i \sin 3\pi) = -4$$

6. $2^{n+1} \cos^n \left(\frac{\theta}{2}\right) \cos \left(\frac{n\theta}{2}\right)$

8. -2