

LOGARITHMS OF COMPLEX NUMBERS:

Let $z = x + iy$ and also let $x = r \cos \theta$, $y = r \sin \theta$ so that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos \theta + i \sin \theta)) = \log(r.e^{i\theta})$ take : tan inverse zero as pi
 $= \log r + \log e^{i\theta} = \log r + i\theta$

$$\therefore \log(x + iy) = \log r + i\theta$$

$$\therefore \log(x + iy) = \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \frac{y}{x} \quad \dots\dots\dots (1)$$

This is called **principal value** of $\log(x + iy)$

The general value of $\log(x + iy)$ is denoted by $\text{Log}(x + iy)$ and is given by

$$\therefore \text{Log}(x + iy) = 2n\pi i + \log(x + iy)$$

$$\therefore \text{Log}(x + iy) = 2n\pi i + \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \frac{y}{x}$$

$$\text{Log}(x + iy) = \frac{1}{2} \log(x^2 + y^2) + i(2n\pi + \tan^{-1} \frac{y}{x}) \quad \dots\dots\dots (2)$$

Caution: $\theta = \tan^{-1} y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r \cos \theta$, $y = r \sin \theta$, $-\pi \leq \theta \leq \pi$.

SOME SOLVED EXAMPLES:

1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$

Solution: Since $\log(x + iy) = \frac{1}{2} \log(x^2 + y^2) + i \tan^{-1} \frac{y}{x}$

Putting $x = -3$, $y = 0$

we have $\log(-3) = \frac{1}{2} \log(9) + i \tan^{-1} \left(\frac{0}{-3} \right) = \frac{1}{2} \log 3^2 + i\pi = \log 3 + i\pi$

$$\log_2(-3) = \frac{\log_e(-3)}{\log_e 2} = \frac{\log 3 + i\pi}{\log 2}$$

2. Find the general value of $\text{Log}(1 + i) + \text{Log}(1 - i)$

Solution: $\log(1 + i) = \frac{1}{2} \log 2 + i \frac{\pi}{4} = \log \sqrt{2} + i \frac{\pi}{4}$

$$\therefore \text{Log}(1 + i) = \log \sqrt{2} + i \left(2n\pi + \frac{\pi}{4} \right) \quad (\text{General value})$$

Changing the sign of i ,

$$\text{Log}(1 - i) = \log \sqrt{2} - i \left(2n\pi + \frac{\pi}{4} \right)$$

By addition, we get $\text{Log}(1 + i) + \text{Log}(1 - i) = 2 \log \sqrt{2} = 2 \cdot \frac{1}{2} \log 2 = \log 2$

3. Prove that $\log(1 + e^{2i\theta}) = \log(2 \cos \theta) + i\theta$

Solution: $\log(1 + e^{2i\theta}) = \log(1 + \cos 2\theta + i \sin 2\theta)$
 $= \log(2 \cos^2 \theta + i 2 \sin \theta \cos \theta)$

$$\begin{aligned}
&= \log(2 \cos \theta (\cos \theta + i \sin \theta)) \\
&= \log(2 \cos \theta \cdot e^{i\theta}) \\
&= \log(2 \cos \theta) + \log(e^{i\theta}) \\
&= \log(2 \cos \theta) + i\theta
\end{aligned}$$

4. Find the value of $\log [\sin(x + i y)]$

Solution: We have, $\sin(x + i y) = \sin x \cos hy + i \cos x \sin hy$

$$\therefore \log \sin(x + i y) = \frac{1}{2} \log(\sin^2 x \cos^2 hy + \cos^2 x \sin^2 hy) + i \tan^{-1} \left(\frac{\cos x \sin hy}{\sin x \cos hy} \right)$$

$$\text{Now, } \sin^2 x \cos^2 hy + \cos^2 x \sin^2 hy = (1 - \cos^2 x) \cos^2 hy + \cos^2 x (\cos^2 hy - 1)$$

$$\begin{aligned}
&= \cos^2 hy - \cos^2 x \\
&= \left(\frac{1 + \cos 2y}{2} \right) - \left(\frac{1 + \cos 2x}{2} \right) \\
&= \frac{1}{2} (\cos 2y - \cos 2x)
\end{aligned}$$

$$\therefore \log \sin(x + i y) = \frac{1}{2} \log \left(\frac{\cos 2y - \cos 2x}{2} \right) + i \tan^{-1}(\cot x \tan hy)$$

5. Show that $\tan \left[i \log \left(\frac{a - ib}{a + ib} \right) \right] = \frac{2ab}{a^2 - b^2}$

Solution: We have $\log(a - bi) = \frac{1}{2} \log(a^2 + b^2) - i \tan^{-1} \frac{b}{a}$

$$\text{And } \log(a + bi) = \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1} \frac{b}{a}$$

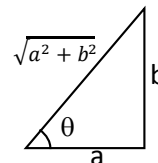
$$\therefore \log \left(\frac{a - bi}{a + bi} \right) = \log(a - bi) - \log(a + bi) = -2i \tan^{-1} \frac{b}{a}$$

$$\therefore i \log \left(\frac{a - bi}{a + bi} \right) = -2i^2 \tan^{-1} \frac{b}{a} = 2 \tan^{-1} \frac{b}{a}$$

$$\therefore \tan \left\{ i \log \left(\frac{a - bi}{a + bi} \right) \right\} = \tan \left(2 \tan^{-1} \frac{b}{a} \right)$$

$$\therefore \tan \left\{ i \log \left(\frac{a - bi}{a + bi} \right) \right\} = \tan 2\theta \quad \text{where } \tan^{-1} \frac{b}{a} = \theta$$

$$= \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2(b/a)}{1 - (b^2/a^2)} = \frac{2ab}{a^2 - b^2}$$



6. Prove that $\cos \left[i \log \left(\frac{a - ib}{a + ib} \right) \right] = \frac{a^2 - b^2}{a^2 + b^2}$

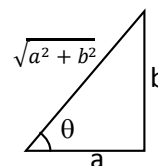
Solution: We have $\log(a - bi) = \frac{1}{2} \log(a^2 + b^2) - i \tan^{-1} \frac{b}{a}$

$$\text{And } \log(a + bi) = \frac{1}{2} \log(a^2 + b^2) + i \tan^{-1} \frac{b}{a}$$

$$\therefore \log \left(\frac{a - bi}{a + bi} \right) = \log(a - bi) - \log(a + bi) = -2i \tan^{-1} \frac{b}{a}$$

$$\therefore i \log \left(\frac{a - bi}{a + bi} \right) = -2i^2 \tan^{-1} \frac{b}{a} = 2 \tan^{-1} \frac{b}{a}$$

$$\cos \left[i \log \left(\frac{a - ib}{a + ib} \right) \right] = \cos \left(2 \tan^{-1} \frac{b}{a} \right)$$



$$\cos \left[i \log \left(\frac{a-ib}{a+ib} \right) \right] = \cos 2\theta \quad \text{where } \tan^{-1} \frac{b}{a} = \theta$$

$$= \cos^2 \theta - \sin^2 \theta = \frac{a^2}{a^2+b^2} - \frac{b^2}{a^2+b^2} = \frac{a^2-b^2}{a^2+b^2}$$

7. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Solution: We have $\sqrt{i} = i^{1/2} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{1/2} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}}$

$$\text{Also } \sqrt{i} = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{1/2} = \left(e^{i\pi/2} \right)^{1/2} = e^{i\pi/4}$$

$$\therefore (\sqrt{i})^{\sqrt{i}} = \left\{ e^{i\pi/4} \right\}^{\left(\frac{1}{\sqrt{2}} + i \cdot \frac{1}{\sqrt{2}} \right)} = e^{i\pi/4\sqrt{2} - \pi/4\sqrt{2}} = e^{-\pi/4\sqrt{2}} \cdot e^{i\pi/4\sqrt{2}}$$

$$= e^{-\pi/4\sqrt{2}} \left(\cos \frac{\pi}{4\sqrt{2}} + i \sin \frac{\pi}{4\sqrt{2}} \right)$$

$$\therefore \text{Real Part} = e^{-\pi/4\sqrt{2}} \cos \left(\frac{\pi}{4\sqrt{2}} \right) \quad \& \quad \text{Imaginary Part} = e^{-\pi/4\sqrt{2}} \sin \left(\frac{\pi}{4\sqrt{2}} \right)$$

8. Find the principal value of $(1+i)^{1-i}$

Solution: $z = (1+i)^{1-i}$

$$\therefore \log z = (1-i) \log(1+i)$$

$$\therefore \log z = (1-i) [\log \sqrt{1+1} + i \tan^{-1} 1]$$

$$= (1-i) \left[\frac{1}{2} \log 2 + i \cdot \frac{\pi}{4} \right]$$

$$= \left(\frac{1}{2} \log 2 + \frac{\pi}{4} \right) + i \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right) = x + iy \text{ say}$$

$$\therefore z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

$$= e^{\left(\frac{1}{2} \log 2 + \frac{\pi}{4} \right)} \left[\cos \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right) + i \sin \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right) \right]$$

$$= \sqrt{2} e^{\pi/4} \left[\cos \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right) + i \sin \left(\frac{\pi}{4} - \frac{1}{2} \log 2 \right) \right] \quad \because e^{\frac{1}{2} \log 2} = e^{\log \sqrt{2}} = \sqrt{2}$$

9. Prove that the general value of $(1+i \tan \alpha)^{-i}$ is $e^{2m\pi+\alpha} [\cos(\log \cos \alpha) + i \sin(\log \cos \alpha)]$

Solution: Let $1+i \tan \alpha = r e^{i\theta}$

$$\therefore r^2 = 1 + \tan^2 \alpha = \sec^2 \alpha \quad \therefore r = \sec \alpha$$

$$\text{And } \theta = \tan^{-1} \left(\frac{\tan \alpha}{1} \right) = \tan^{-1}(\tan \alpha) = \alpha$$

$$\text{Now, } \log(1+i \tan \alpha) = \log(r e^{i\theta}) = \log r + (2m\pi + \theta)i$$

$$= \log \sec \alpha + (2m\pi + \alpha)i$$

$$\therefore 1+i \tan \alpha = e^{[\log \sec \alpha + (2m\pi + \alpha)i]}$$

$$\therefore (1+i \tan \alpha)^{-i} = e^{-i[\log \sec \alpha + (2m\pi + \alpha)i]}$$

$$= e^{2m\pi + \alpha} \cdot e^{-i \log \sec \alpha}$$

$$= e^{2m\pi + \alpha} \cdot e^{i(\log \cos \alpha)}$$

$$= e^{2m\pi + \alpha} \cdot [\cos(\log \cos \alpha) + i \sin(\log \cos \alpha)]$$

10. Considering only principal value, if $(1 + i \tan \alpha)^{1+i \tan \beta}$ is real, prove that its value is $(\sec \alpha)^{\sec^2 \beta}$

Solution: Let $z = (1 + i \tan \alpha)^{1+i \tan \beta}$

Taking logarithms of both sides,

$$\begin{aligned}\log z &= (1 + i \tan \beta) \log(1 + i \tan \alpha) \\ &= (1 + i \tan \beta) \left[\frac{1}{2} \log(1 + \tan^2 \alpha) + i \tan^{-1} \tan \alpha \right] \\ &= (1 + i \tan \beta) [\log \sec \alpha + i \alpha]\end{aligned}$$

$$\therefore \log z = (\log \sec \alpha - \alpha \tan \beta) + i(\alpha + \tan \beta \log \sec \alpha) = x + iy \text{ say}$$

$$\text{Where } x = \log \sec \alpha - \alpha \tan \beta \text{ and } y = \alpha + \tan \beta \log \sec \alpha \dots\dots\dots(i)$$

$$\text{Now, } z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

Since by data z is real

$$\therefore e^x \sin y = 0 \quad \therefore y = 0 \quad \therefore \cos y = 1$$

$$\therefore z = e^x \cos y = e^x = e^{\log \sec \alpha - \alpha \tan \beta} \text{ from (i)}$$

$$\therefore z = e^{\log \sec \alpha} \cdot e^{-\alpha \tan \beta} = \sec \alpha \cdot e^{-\alpha \tan \beta} \dots\dots\dots(ii)$$

$$\text{But since } y = 0, \text{ from (i) } \alpha + \tan \beta \log \sec \alpha = 0$$

$$\therefore -\alpha = \tan \beta \log \sec \alpha$$

$$\therefore -\alpha \tan \beta = \tan^2 \beta \cdot \log \sec \alpha = \log (\sec \alpha)^{\tan^2 \beta}$$

$$\therefore e^{-\alpha \tan \beta} = (\sec \alpha)^{\tan^2 \beta}$$

$$\therefore \text{from (ii) } z = \sec \alpha \cdot (\sec \alpha)^{\tan^2 \beta} = (\sec \alpha)^{(1+\tan^2 \beta)} = (\sec \alpha)^{\sec^2 \beta}$$

11. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\beta$, find α and β

Solution: $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\beta$,

$$\text{Taking logarithms of both sides, } \log \left(\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} \right) = \log (\alpha + i\beta)$$

$$\log(\alpha + i\beta) = (x + iy) \log(a + ib) - (x - iy) \log(a - ib)$$

$$\log(\alpha + i\beta) = (x + iy) \left[\frac{1}{2} \log(a^2 + b^2) + i \tan^{-1} \left(\frac{b}{a} \right) \right] - (x - iy) \left[\frac{1}{2} \log(a^2 + b^2) - i \tan^{-1} \left(\frac{b}{a} \right) \right]$$

$$\log(\alpha + i\beta) = 2i \left[x \tan^{-1} \frac{b}{a} + \frac{y}{2} \log(a^2 + b^2) \right]$$

$$= 2ik \text{ say} \quad \text{where } k = \left[x \tan^{-1} \frac{b}{a} + \frac{y}{2} \log(a^2 + b^2) \right]$$

$$\therefore (\alpha + i\beta) = e^{2ik} = \cos 2k + i \sin 2k$$

$$\therefore \alpha = \cos 2k, \beta = \sin 2k \quad \text{where } k = \left[x \tan^{-1} \frac{b}{a} + \frac{y}{2} \log(a^2 + b^2) \right]$$

5. Prove that the real part of the principal value of $(1+i)^{\log i}$ is $e^{-\pi^2/8} \cos\left(\frac{\pi}{4} \log 2\right)$
6. If $\frac{(1+i)^{x+iy}}{(1-i)^{x-iy}} = \alpha + i\beta$, find α and β
7. If $\sqrt[n]{1}^{\infty} = \alpha + i\beta$, prove that $\alpha^2 + \beta^2 = e^{-\pi\beta/2}$
8. Find all the root of the equation $\tan h z + 2 = 0$
9. If $\tan[\log(x + i y)] = a + i b$ Prove that $\tan[\log(x^2 + y^2)] = \frac{2a}{1-a^2-b^2}$ when $a^2 + b^2 \neq 1$.

ANSWERS

1. (i) $e^{-(2n+\frac{1}{2})\pi}$ (ii) $e^{\pi/2}(\cos \pi/2 + i \sin \pi/2)$
 (iii) $i e^{-\pi/2}(\cos \pi/2 + i \sin \pi/2)$
6. $\alpha = \cos k, \beta = \sin k$ where $k = \left(x\frac{\pi}{2} + y \log 2\right)$ 8. $-\frac{1}{2} \log 3 + i\left(n + \frac{1}{2}\right)\pi$