DE MOIVRE'S THEOREM

Wednesday, October 13, 2021 1:11 PM

DE MOIVRE'S THEOREM:

Statement : For any rational number n the value or one of the values of $\boxed{(\cos\theta + i\sin\theta)^n = \cos\, n\,\theta + i\,\sin n\,\theta}$

1. If $z = cos\,\theta + i\sin\theta$ then

$\frac{1}{z} = z^{-1} = (\cos\theta + \,i\,sin\,\theta)^{-1} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - \,i\sin\theta$

i.e. $\frac{1}{z} = \cos\theta - \,i\sin\theta$

2. $(\cos\theta - i\sin\theta)^n = \cos\, n\,\theta - i\,\sin n\,\theta$

For, $(\cos\theta - i\sin\theta)^n = \{\cos\,(-\theta) + i\,\sin\,(-\theta)\}^n$

$= \cos(-n\theta) + isin(-n\theta).$

$= \cos n\,\theta - i\sin\,n\,\theta$

$(\cos\theta - i\sin\theta)^n$

$[\cos(-\theta) + i\sin(-\theta)]^n$

$\cos(-n\theta) + i\sin(-n\theta)$

$\cos(n\theta) - i\sin n\theta.$

Note : Note carefully that ,

(1) $(\sin\theta + i\cos\theta)^n \neq \sin n\,\theta + i\cos n\,\theta$

But $(\sin\theta + i\cos\theta)^n = [cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)]^n$

$= \cos\, n\left(\frac{\pi}{2} - \theta\right) + i\sin n\,\left(\frac{\pi}{2} - \theta\right)$

(2) $(\cos\,\,\theta + \mathrm{i}\sin\square\,)^n \neq \cos\, n\,\theta + \,i\sin n\,.$

$(\cos\theta + i\sin\phi)^n \neq \cos n\theta + i\sin n\phi$

SOME SOLVED EXAMPLES:

1. Simplify $\frac{(\cos 2\theta - i\sin 2\theta)^7(\cos 3\theta + i\sin 3\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{12}(\cos 5\theta - i\sin 5\theta)^7}$

$\cos 2\theta - i\sin 2\theta = (\cos\theta + i\sin\theta)^{-2} = e^{-i2\theta}$

$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3 = e^{i3\theta}$

$\cos 5\theta - i\sin 5\theta = (\cos\theta + i\sin\theta)^{-5} = e^{-i5\theta}$

Given expression $= \frac{(e^{-i2\theta})^7(e^{i3\theta})^5}{(e^{i3\theta})^{12}(e^{-i5\theta})^7}$

$= \frac{e^{-i14\theta} \times e^{i15\theta}}{e^{i36\theta} \times e^{-i35\theta}} = \frac{e^{i\theta}}{e^{i\theta}}$

$= 1$

2.

$(1+i)^8\left(\sqrt{3} - i\right)^4$ 1

$re^{i\theta}$

2.

Prove that $\frac{(1+i)^8\left(\sqrt{3}-i\right)^4}{(1-i)^4\left(\sqrt{3}+i\right)^8} = -\frac{1}{4}$

$re^{i\theta}$

for $1+i$, $r = \sqrt{2}$ $\theta = \pi/4 \Rightarrow 1+i = \sqrt{2}e^{i\pi/4}$

$1-i$ $\quad r = \sqrt{2}$ $\theta = -\pi/4 \Rightarrow 1-i = \sqrt{2}e^{-i\pi/4}$

$\sqrt{3}-i$ $\quad r = 2$ $\theta = -\pi/6 \Rightarrow \sqrt{3}-i = 2e^{-i\pi/6}$

$\sqrt{3}+i$ $\quad r = 2$ $\theta = \frac{\pi}{6} \Rightarrow \sqrt{3}+i = 2e^{i\pi/6}$

Given expression $= \dfrac{\left(\sqrt{2}e^{i\pi/4}\right)^8 \left(2e^{-i\pi/6}\right)^4}{\left(\sqrt{2}e^{-i\pi/4}\right)^4 \left(2e^{i\pi/6}\right)^8}$

$$= \frac{2^8 e^{i2\pi} \cdot e^{-i2\pi/3}}{2^{10} e^{-i\pi} e^{i4\pi/3}} = \frac{1}{4} e^{i\left(2\pi - \frac{2\pi}{3} + \pi - \frac{4\pi}{3}\right)}$$

$$= \frac{1}{4} e^{i\pi}$$

$$= \frac{1}{4}(\cos\pi + i\sin\pi)$$

$$= -\frac{1}{4} \qquad \left(\begin{array}{l}\cos\pi = -1 \\ \sin\pi = 0\end{array}\right)$$

3.

Find the modulus and the principal value of the argument of $\frac{\left(1+i\sqrt{3}\right)^{16}}{\left(\sqrt{3}-i\right)^{17}}$

Soln: $1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\pi/3}$

$\sqrt{3}-i = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2e^{-i\pi/6}$

given expression $= \dfrac{\left(2e^{i\pi/3}\right)^{16}}{\left(2e^{-i\pi/6}\right)^{17}} = \frac{1}{2} e^{i\left(\frac{16\pi}{3} + \frac{17\pi}{6}\right)}$

$$= \frac{1}{2} e^{i\left(\frac{49\pi}{6}\right)} = \frac{1}{2}\left[\cos\frac{49\pi}{6} + i\sin\frac{49\pi}{6}\right]$$

$$= \frac{1}{2}\left[\cos\left(8\pi + \frac{\pi}{6}\right) + i\sin\left(8\pi + \frac{\pi}{6}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

Hence the modulus is $\frac{1}{2}$ and principal value of the argument is $\frac{\pi}{6}$.

4. Simplify $\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n$

Soln :- $1 = \sin^2\alpha + \cos^2\alpha = \sin^2\alpha - i^2\cos^2\alpha$

$$= (\sin\alpha + i\cos\alpha)(\sin\alpha - i\cos\alpha)$$

$$1+\sin\alpha+i\cos\alpha = (\sin\alpha+i\cos\alpha)(\sin\alpha-i\cos\alpha) + (\sin\alpha+i\cos\alpha)$$

$$= (\sin\alpha+i\cos\alpha)[\sin\alpha - i\cos\alpha + 1]$$

$$1+\sin\alpha+i\cos\alpha = (\sin\alpha+i\cos\alpha)(1+\sin\alpha-i\cos\alpha)$$

$$\therefore \frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha} = \sin\alpha + i\cos\alpha$$

$$\left(\frac{1+\sin\alpha+i\cos\alpha}{1+\sin\alpha-i\cos\alpha}\right)^n = (\sin\alpha+i\cos\alpha)^n$$

$$= \left[\cos\left(\frac{\pi}{2}-\alpha\right) + i\sin\left(\frac{\pi}{2}-\alpha\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2}-\alpha\right) + i\sin n\left(\frac{\pi}{2}-\alpha\right)$$

5. If $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ *and* $\bar{z}$ is the conjugate of z prove that $(z)^{10} + (\bar{z})^{10} = 0$.

Soln: $z = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$\bar{z} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$

$$z^{10} + (\bar{z})^{10} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10}$$

$$= \cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2} + \cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}$$

$$= 2\cos\frac{5\pi}{2}$$

$\cos\frac{n\pi}{2} = 0$ if n is odd.

$$= 2(0)$$

$$= 0.$$

Q.5 **(ii)** $\left(1+i\sqrt{3}\right)^n + \left(1-i\sqrt{3}\right)^n = 2^{n+1}cos(n\pi/3).$

$1+i\sqrt{3} = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\frac{\pi}{3}\right]$

$(1+i\sqrt{3})^n = 2^n\left[\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)\right]$

$(1-i\sqrt{3})^n = 2^n\left[\cos\left(\frac{n\pi}{3}\right) - i\sin\left(\frac{n\pi}{3}\right)\right]$

$$LHS = 2^n\left\{\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right) + \cos\frac{n\pi}{3} - i\sin\left(\frac{n\pi}{3}\right)\right\}$$

$$= 2^n\left\{2\cos\left(\frac{n\pi}{3}\right)\right\}$$

$$= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)$$

$$= RHS.$$

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2\,2^{n/2}\cos n\pi/4$. Hence, deduce

6. If α, β are the roots of the equation $x^2 - 2x + 2 = 0$, prove that $\alpha^n + \beta^n = 2.2^{n/2} \cos n\pi/4$, Hence, deduce that $\alpha^8 + \beta^8 = 32$

Soln :- $x^2 - 2x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= 1 \pm i$$

$\therefore \alpha = 1+i \quad , \beta = 1-i$

$\alpha = 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\beta = 1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$\alpha^n + \beta^n = (\sqrt{2})^n \left\{\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right\}$

$= (\sqrt{2})^n \left\{2\cos\frac{n\pi}{4}\right\}$

$= 2\cdot(\sqrt{2})^n \cos\frac{n\pi}{4}$.

$= 2\cdot(2)^{n/2} \cos\frac{n\pi}{4}$.

$=$ RHS.

put $n=8$

$\alpha^8 + \beta^8 = 2\cdot(2)^{8/2} \cos\frac{8\pi}{4} = 2\cdot(2)^4 \cos 2\pi$

$= 2^5 = 32$.

7. If α, β are the roots of the equation $x^2 - 2\sqrt{3}x + 4 = 0$, Prove that $\alpha^3 + \beta^3 = 0$ and $\alpha^3 - \beta^3 = 16\,i$ (HW)

8. If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$, prove that $\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = 2\cos(\alpha+\beta-\gamma)$

Sol^n :- $a = e^{i2\alpha}$, $b = e^{i2\beta}$, $c = e^{i2\gamma}$

$$\therefore \frac{ab}{c} = \frac{e^{i2\alpha}\cdot e^{i2\beta}}{e^{i2\gamma}} = e^{i2(\alpha+\beta-\gamma)}$$

Similarly $\frac{c}{ab} = e^{-i2(\alpha+\beta-\gamma)}$

$$\sqrt{\frac{ab}{c}} + \sqrt{\frac{c}{ab}} = \sqrt{e^{i2(\alpha+\beta-\gamma)}} + \sqrt{e^{-i2(\alpha+\beta-\gamma)}}$$

$$= e^{i(\alpha+\beta-\gamma)} + e^{-i(\alpha+\beta-\gamma)}$$

$$= \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma) + \cos(\alpha+\beta-\gamma) - i\sin(\alpha+\beta-\gamma)$$

$$= 2\cos(\alpha+\beta-\gamma)$$

= RHS.

9. If $x - \frac{1}{x} = 2i\sin\theta$, $y - \frac{1}{y} = 2i\sin\Phi$, $z - \frac{1}{z} = 2i\sin\psi$, prove that

(i) $xyz + \frac{1}{xyz} = 2\cos(\theta+\Phi+\psi)$ **(ii)** $\frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}} = 2\cos\left(\frac{\theta}{m} - \frac{\Phi}{n}\right)$

Sol^n :- Given $x - \frac{1}{x} = 2i\sin\theta$

$$x^2 - 1 = 2i\sin\theta\, x$$

$$x^2 - 2i\sin\theta\, x - 1 = 0$$

This is a quadratic in x

we solve this to find x $\rightarrow$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{\quad}$$

$$2a$$

$$x = \frac{2i\sin\theta \pm \sqrt{-4\sin^2\theta + 4}}{2}$$

$$= \frac{2i\sin\theta \pm 2\sqrt{1-\sin^2\theta}}{2}$$

$$= i\sin\theta \pm \cos\theta$$

let $x = \cos\theta + i\sin\theta \quad = e^{i\theta}$

Similarly, $y = \cos\phi + i\sin\phi \quad = e^{i\phi}$

$z = \cos\psi + i\sin\psi \quad = e^{i\psi}$

(i) LHS $= xyz + \frac{1}{xyz}$

$$= \left(e^{i\theta}.e^{i\phi}.e^{i\psi}\right) + \frac{1}{e^{i\theta}.e^{i\phi}.e^{i\psi}}$$

$$= e^{i(\theta+\phi+\psi)} + \frac{1}{e^{i(\theta+\phi+\psi)}}$$

$$= e^{i(\theta+\phi+\psi)} + e^{-i(\theta+\phi+\psi)}$$

$$= \cos(\theta+\phi+\psi) + i\sin(\theta+\phi+\psi)$$
$$+ \cos(\theta+\phi+\psi) - i\sin(\theta+\phi+\psi)$$

$$= 2\cos(\theta+\phi+\psi)$$

$=$ RHS.

(ii) LHS $= \frac{\sqrt[m]{x}}{\sqrt[n]{y}} + \frac{\sqrt[n]{y}}{\sqrt[m]{x}}$

$$= \frac{(e^{i\theta})^{\frac{1}{m}}}{(e^{i\phi})^{\frac{1}{n}}} + \frac{(e^{i\phi})^{\frac{1}{n}}}{(e^{i\theta})^{\frac{1}{m}}}$$

$$= \frac{(e^{i\theta})}{(e^{i\phi})^{\frac{1}{n}}} + \frac{(e^{\ \ })}{(e^{i\theta})^{\frac{1}{m}}}$$

$$= \frac{e^{i(\frac{\theta}{m})}}{e^{i(\phi/n)}} + \frac{e^{i(\phi/n)}}{e^{i(\theta/m)}}$$

$$= e^{i(\frac{\theta}{m}-\frac{\phi}{n})} + e^{i(\frac{\phi}{n}-\frac{\theta}{m})}$$

$$= e^{i(\frac{\theta}{m}-\frac{\phi}{n})} + e^{-i(\frac{\theta}{m}-\frac{\phi}{n})}$$

$$= \cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right) + i\sin\left(\frac{\theta}{m}-\frac{\phi}{n}\right) + \cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right) - i\sin\left(\frac{\theta}{m}-\frac{\phi}{n}\right)$$

$$= 2\cos\left(\frac{\theta}{m}-\frac{\phi}{n}\right)$$

$= RHS.$

10. If $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$, Prove that $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha+\beta+\gamma)$. VIMP

Soln :- $\cos\alpha + 2\cos\beta + 3\cos\gamma = \sin\alpha + 2\sin\beta + 3\sin\gamma = 0$

$\therefore (\cos\alpha + 2\cos\beta + 3\cos\gamma) + i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$

$\therefore (\cos\alpha + i\sin\alpha) + 2(\cos\beta + i\sin\beta) + 3(\cos\gamma + i\sin\gamma) = 0$

Let $x = \cos\alpha + i\sin\alpha$, $y = 2(\cos\beta + i\sin\beta)$

$z = 3(\cos\gamma + i\sin\gamma)$

$\therefore \quad x+y+z=0$

$(x+y+z)^3 = 0$

$\Rightarrow x^3+y^3+z^3 + 3\underline{(x+y+z)}(xy+yz+zx) - 3xyz = 0$

$$\Rightarrow x^3+y^3+z^3+3(x+y+z)(xy+yz+zx)-3xyz=0$$

$$\therefore \quad x^3+y^3+z^3-3xyz=0$$

$$\therefore \quad x^3+y^3+z^3=3xyz$$

$$(\cos\alpha+i\sin\alpha)^3+[2(\cos\beta+i\sin\beta)]^3+[3(\cos\gamma+i\sin\gamma)]^3$$

$$=3(\cos\alpha+i\sin\alpha)[2(\cos\beta+i\sin\beta)][3(\cos\gamma+i\sin\gamma)]$$

$$(\cos 3\alpha+i\sin 3\alpha)+8(\cos 3\beta+i\sin 3\beta)+27(\cos 3\gamma+i\sin 3\gamma)$$

$$=18[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$$

$$\therefore (\cos 3\alpha+8\cos 3\beta+27\cos 3\gamma)+i(\sin 3\alpha+8\sin 3\beta+27\sin 3\gamma)$$

$$=18\cos(\alpha+\beta+\gamma)+i18\sin(\alpha+\beta+\gamma)$$

comparing the imaginary parts we get the answer.

11.

If $x_r = cos\frac{\pi}{3^r} + isin\frac{\pi}{3^r}$, prove that **(i)** $x_1x_2x_3 \dots ad.inf. = i$ **(ii)** $x_0x_1x_2 \dots ad.inf. = -i$

Soln :- $x_r = \cos\frac{\pi}{3^r}+i\sin\frac{\pi}{3^r}$

$$\therefore x_0 = \cos\frac{\pi}{3^0}+i\sin\frac{\pi}{3^0} = \cos\pi+i\sin\pi = -1$$

$$x_1 = \cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$x_2 = \cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2}$$ and so on

Now (i) $x_1x_2x_3 \dots \dots$ ad inf

$$=\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3}+i\sin\frac{\pi}{3^3}\right)\dots$$

$$=\cos\left(\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\dots\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\dots\right)$$

Now $\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\cdots$ is nothing but infinite sum of a G.P. with $a=\frac{\pi}{3}$, $r=\frac{1}{3}$

$$\therefore S_\infty = \frac{a}{1-r} = \frac{\pi/3}{1-1/3} = \frac{\pi}{2}$$

$$\therefore LHS = \cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right) = 0+i(1) = i$$

(ii) $x_0 x_1 x_2 \cdots \text{ad inf} = x_0(x_1 x_2 \cdots \text{ad inf})$

$$= (-1)(i)$$

$$= -i$$

12. If $(cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)\ldots\ldots[cos(2n-1)\theta + i\sin(2n-1)\theta] = 1$ then show that the general value of θ is $\frac{2r\pi}{n^2}$

Sol^n :- $(\cos\theta+i\sin\theta)(\cos 3\theta+i\sin 3\theta)\cdots[\cos(2n-1)\theta+i\sin(2n-1)\theta]=1$

$$\therefore \cos(\theta+3\theta+\cdots+(2n-1)\theta)+i\sin(\theta+3\theta+\cdots+(2n-1)\theta)=1$$

$$\cos(1+3+\cdots(2n-1))\theta+i\sin(1+3+\cdots(2n-1))\theta=1$$

but $1+3+\cdots(2n-1)$ is sum of n terms of an A.P. with $a=1$, $d=2$

$$\therefore S_n = \frac{n}{2}[2a+(n-1)d] = \frac{n}{2}[2+(n-1)2] = n^2$$

$$\therefore \cos n^2\theta + i\sin n^2\theta = 1$$

$$\therefore \cos n^2\theta + i\sin n^2\theta = \cos 0 + i\sin 0 \rightarrow \text{Principal value}$$

$$= \cos 2r\pi + i\sin 2r\pi \rightarrow \text{general value}$$

$$\Rightarrow n^2\theta = 2r\pi$$

$$\Rightarrow \theta = \frac{2r\pi}{n^2}$$

Hence proved

13. By using De Moivre's Theorem show that $\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha\,\sin(5\alpha/2)}{sin\alpha/2}$

Sol^n :- $1+z+z^2+z^3+z^4+z^5 = 1-z^6$

$1-z$ ——— (i)

$LHS = 1+z+z^2+z^3+z^4+z^5$

Let $z = \cos\alpha + i\sin\alpha$ then $z^n = \cos n\alpha + i\sin n\alpha$

$LHS = 1 + (\cos\alpha + i\sin\alpha) + (\cos 2\alpha + i\sin 2\alpha) + \cdots + (\cos 5\alpha + i\sin 5\alpha)$

$= (1+\cos\alpha + \cos 2\alpha + \cdots + \cos 5\alpha) + i(\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha)$ ——— (ii)

RHS :- $\dfrac{1-z^6}{1-z} = \dfrac{1-(\cos 6\alpha + i\sin 6\alpha)}{1-(\cos\alpha + i\sin\alpha)}$

$= \dfrac{(1-\cos 6\alpha) - i\sin 6\alpha}{(1-\cos\alpha) - i\sin\alpha}$

$= \dfrac{2\sin^2 3\alpha - i\,2\sin 3\alpha\cos 3\alpha}{2\sin^2\alpha/2 - i\,2\sin\alpha/2\,\cos\alpha/2}$

$= \dfrac{2\sin 3\alpha\,(\sin 3\alpha - i\cos 3\alpha)}{2\sin\alpha/2\,(\sin\alpha/2 - i\cos\alpha/2)}$

$= \dfrac{\sin 3\alpha\,[\sin 3\alpha - i\cos 3\alpha][\sin\alpha/2 + i\cos\alpha/2]}{\sin\alpha/2\,[\sin\alpha/2 - i\cos\alpha/2][\sin\alpha/2 + i\cos\alpha/2]}$

$= \dfrac{\sin 3\alpha}{\sin\alpha/2}\;\dfrac{[\sin 3\alpha - i\cos 3\alpha][\sin\alpha/2 + i\cos\alpha/2]}{\sin^2\alpha/2 + \cos^2\alpha/2}$

$= \dfrac{\sin 3\alpha}{\sin\alpha/2}\left[\cos\left(\frac{\pi}{2} - 3\alpha\right) - i\sin\left(\frac{\pi}{2} - 3\alpha\right)\right]\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$

$= \dfrac{\sin 3\alpha}{\sin\alpha/2}\left[\cos\left(3\alpha - \frac{\pi}{2}\right) + i\sin\left(3\alpha - \frac{\pi}{2}\right)\right]\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$

$= \dfrac{\sin 3\alpha}{}\left[\cos\left(3\alpha - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(3\alpha - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$

$$= \frac{\sin 3\alpha}{\sin \alpha/2}\left[\cos\left(3\alpha - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\alpha}{2}\right) + i \sin\left(3\alpha - \frac{\pi}{2} + \frac{\pi}{2} - \frac{\alpha}{2}\right)\right]$$

$$= \frac{\sin 3\alpha}{\sin \alpha/2}\left[\cos\left(\frac{5\alpha}{2}\right) + i \sin\left(\frac{5\alpha}{2}\right)\right] \quad \text{——— (iii)}$$

using (i), (ii) & (iii)

$$\sin\alpha + \sin 2\alpha + \cdots + \sin 5\alpha = \frac{\sin 3\alpha}{\sin \alpha/2} \cdot \sin\left(\frac{5\alpha}{2}\right)$$

Applications of De-Moivre's Theorem

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ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.
General values of $\cos\theta = \cos(2k\pi+\theta)$ and $\sin\theta = \sin(2k\pi+\theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\theta + i\sin\theta$, we apply De Moivre's theorem

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

This shows that $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos\theta + i\sin\theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)\}^{\frac{1}{n}} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,..............,(n – 1). We get n roots of the equation.

n values of K.

$$z^4 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z^4 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$z = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/4}$$

$$= \cos\frac{\pi}{16} + i\sin\frac{\pi}{16}$$

$$z = \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/4}$$

$k = 0, 1, 2, 3$

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

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$$z^n = a + ib = r(\cos\theta + i\sin\theta)$$

$$z = [r(\cos\theta + i\sin\theta)]^{1/n} = r^{1/n}[\cos\theta + i\sin\theta]^{1/n}$$

$$z = r^{1/n}[\cos(2k\pi+\theta) + i\sin(2k\pi+\theta)]^{1/n}$$

$$= r^{1/n}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right] \quad \text{where } k = 0, 1, \cdots, n-1$$

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Soln :- Let $z = (1)^{1/3} = [1 + i0]^{1/3} = [\cos 0 + i\sin 0]^{1/3}$

$$z = [\cos(2k\pi) + i\sin(2k\pi)]^{1/3}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right) \quad k = 0, 1, 2$$

$k = 0$, $z_0 = \cos 0 + i\sin 0 = 1$

$k = 1$, $z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$k = 2$, $z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \omega^2$

$\therefore$ 1, ω and ω^2 are the cube roots of unity

$$1 + \omega + \omega^2 = 1 + \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0.$$

$\therefore \; 1+\omega^2 = -\omega$.

Now $(1-\omega)^6 = \left[(1-\omega)^2\right]^3 = (1-2\omega+\omega^2)^3$

$= (-\omega-2\omega)^3$

$= (-3\omega)^3$

$= -27\omega^3 = -27(1) = -27$

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

$\frac{1}{n} = \frac{1}{3}$ $\boxed{n=3}$

Soln :- Let $z = \sqrt[3]{(1+i)/\sqrt{2}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$= \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{1/3}$

$= \left[\cos\left(2k\pi+\frac{\pi}{4}\right) + i\sin\left(2k\pi+\frac{\pi}{4}\right)\right]^{1/3}$

$= \left[\cos\left(\frac{8k+1}{4}\right)\pi + i\sin\left(\frac{8k+1}{4}\right)\pi\right]^{1/3}$

$= \cos\left(\frac{8k+1}{12}\right)\pi + i\sin\left(\frac{8k+1}{12}\right)\pi$

where $k = 0, 1, 2$

Similarly $\sqrt[3]{(1-i)/\sqrt{2}} = \cos\left(\frac{8k+1}{12}\right)\pi - i\sin\left(\frac{8k+1}{12}\right)\pi$

$k = 0, 1, 2$

$\therefore \sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}} = 2\cos\left(\frac{8k+1}{12}\right)\pi$ where $k = 0, 1, 2$

$= 2\cos\frac{\pi}{12}, \; 2\cos\frac{9\pi}{12}, \; 2\cos\frac{17\pi}{12}$

3. Find the cube roots of $(1 - \cos\theta - i\sin\theta)$. $\boxed{n=3}$

Soln :- $(1-\cos\theta - i\sin\theta)^{1/3}$

$= \left[2\sin^2\theta/2 - 2i\sin\theta/2\cos\theta/2\right]^{1/3}$

$(\ldots)^{1/3}\left[\sin\frac{\theta}{\ldots} - i\cos\frac{\theta}{\ldots}\right]^{1/3}$

$$= \left(2\sin\theta/2\right)^{1/3}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]^{1/3}$$

$$= \left(2\sin\theta/2\right)^{1/3}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\theta/2\right)^{1/3}\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{2}\right) + i\sin\left(\frac{\theta}{2}-\frac{\pi}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\theta/2\right)^{1/3}\left[\cos\left(2k\pi+\frac{\theta}{2}-\frac{\pi}{2}\right) + i\sin\left(2k\pi+\frac{\theta}{2}-\frac{\pi}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\theta/2\right)^{1/3}\left[\cos\left(\frac{(4k-1)\pi+\theta}{2}\right) + i\sin\left(\frac{(4k-1)\pi+\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\theta/2\right)^{1/3}\left[\cos\left(\frac{(4k-1)\pi+\theta}{6}\right) + i\sin\left(\frac{(4k-1)\pi+\theta}{6}\right)\right]$$

where $k = 0, 1, 2$

4. Find the continued product of all the value of $(-i)^{2/3}$

Sol^n:- continued product = product of all the roots.

$$z = (-i)^{2/3} = \left[(-i)^2\right]^{1/3}$$

$$= (-1)^{1/3}$$

$$= \left[\cos\pi + i\sin\pi\right]^{1/3}$$

$$= \left[\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)\right]^{1/3}$$

$$= \cos\left(\frac{2k+1}{3}\right)\pi + i\sin\left(\frac{2k+1}{3}\right)\pi$$

$k = 0, 1, 2$

$$z_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$$

$$z_1 = \cos\pi + i\sin\pi$$

$$z_2 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$\therefore$ The continued product

$$z = (-i)^{2/3}$$

$$= \left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]^{2/3}$$

$$= \left[\cos\left(2k\pi+\frac{3\pi}{2}\right) + i\sin\left(2k\pi+\frac{3\pi}{2}\right)\right]^{2/3}$$

$$= \left[\cos\left(\frac{4k+3}{2}\right)\pi + i\sin\left(\frac{4k+3}{2}\right)\pi\right]^{2/3}$$

$$= \cos\left(\frac{4k+3}{3}\right)\pi + i\sin\left(\frac{4k+3}{3}\right)\pi$$

$k = 0, 1, 2$

$$z_0 = \cos\pi + i\sin\pi$$

$$z_1 = \cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}$$

$$z_2 = \cos\frac{11\pi}{3} + i\sin\frac{11\pi}{3}$$

product

3 → $z_2 = \cos\frac{11\pi}{3} + i\sin\frac{11\pi}{3}$

$\therefore$ The continued product

$= z_0 \cdot z_1 \cdot z_2$

$= \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(\cos\pi + i\sin\pi)\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$= \cos\left(\frac{\pi}{3} + \pi + \frac{5\pi}{3}\right) + i\sin\left(\frac{\pi}{3} + \pi + \frac{5\pi}{3}\right)$

$= \cos(3\pi) + i\sin(3\pi)$

$= -1 + i(0)$

$= -1$

product

$\cos\left(\pi + \frac{7\pi}{3} + \frac{11\pi}{3}\right) + i\sin\left(\pi + \frac{7\pi}{3} + \frac{11\pi}{3}\right)$

$\cos(7\pi) + i\sin(7\pi)$

$(-1) + i(0)$

$= -1$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4} = \left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]^{3/4}$$

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

no. of roots = deg of eqns.

Soln :-

$$x^7 + x^4 + x^3 + 1 = 0$$
$$x^4(x^3+1) + 1(x^3+1) = 0$$
$$(x^3+1)(x^4+1) = 0$$

$x^3 + 1 = 0$

$x^3 = -1 = \cos\pi + i\sin\pi$

$= \cos(2k+1)\pi + i\sin(2k+1)\pi$

$x = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/3}$

$= \cos\frac{(2k+1)\pi}{3} + i\sin\frac{(2k+1)\pi}{3}$

where $k = 0, 1, 2$

$x_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$x_1 = \cos\pi + i\sin\pi = -1$

$x_2 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

$x^4 + 1 = 0$

$x^4 = -1 = \cos\pi + i\sin\pi$

$= \cos(2k+1)\pi + i\sin(2k+1)\pi$

$x = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}$

$= \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}$

where $k = 0, 1, 2, 3$

$x_3 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$x_4 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_5 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$x_6 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

we multiply given eqn by $(x-1)$

$$\boxed{x^5 - 1 = 0.}$$

$$x^5 = 1 = \cos 0 + i\sin 0$$

$$= \cos 2k\pi + i\sin 2k\pi$$

$$x = (\cos 2k\pi + i\sin 2k\pi)^{1/5}$$

$$x = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

where $k = 0,1,2,3,4$

$x_0 = \cos 0 + i\sin 0 = 1$

$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

$x_0 = 1$ is the root of $(x-1) = 0$ and the remaining are roots of given eqn.

H.W.

Note :- $x^4 - x^3 + x^2 - x + 1 = 0.$

we multiply by $x+1$.

we get $x^5 + 1 = 0. \Rightarrow x^5 = -1 \rightarrow$ find roots.

8. SOLVE: $x^4 - x^2 + 1 = 0$

we multiply by (x^2+1)

we get $x^6 + 1 = 0 \Rightarrow x^6 = -1$

$$\Rightarrow x^6 = \cos\pi + i\sin\pi$$

$$\therefore x^6 = \cos(2k+1)\pi + i\sin(2k+1)\pi$$

$$\therefore x = \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}$$

$k = 0,1,2,3,4,5$

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$

$x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

It is clear that i and $-i$ are the roots of $x^2+1=0$ and the remaining roots x_0, x_2, x_3, x_5 are the roots of $x^4 - x^2 + 1 = 0$.

H.W

Note :- $x^4 + x^2 + 1 = 0$

we multiply by $x^2 - 1$

$\Rightarrow x^6 - 1 = 0 \quad \Rightarrow x^6 = 1$

9. Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$.

$x^4 + 1 = 0$

$x^4 = -1$

$x = \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}$

$k = 0, 1, 2, 3$

$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

$x^6 - i = 0$

$x^6 = i = \cos(2k+\frac{1}{2})\pi + i\sin(2k+\frac{1}{2})\pi$

$x = \cos\frac{(4k+1)\pi}{12} + i\sin\frac{(4k+1)\pi}{12}$

$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$

$x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}$

$x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12}$

$\therefore$ The common roots are $\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ and $\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$.

10. If $(1+x)^6 + x^6 = 0$ show that $x = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$ where $\theta = (2k+1)\pi/6$, $k = 0,1,2,3,4,5$.

Soln :- $(1+x)^6 + x^6 = 0$

$\left(\frac{1+x}{x}\right)^6 + 1 = 0$

$\therefore \left(\frac{1+x}{x}\right)^6 = -1 = \cos\pi + i\sin\pi = \cos(2k+1)\pi + i\sin(2k+1)\pi$

$$\frac{1+x}{x} = \cos\frac{(2k+1)\pi}{6} + i\sin\frac{(2k+1)\pi}{6}$$

$k = 0, 1, 2, 3, 4, 5$

Let $\theta = \frac{(2k+1)\pi}{6}$

$\therefore \frac{1+x}{x} = \cos\theta + i\sin\theta$

$\therefore 1 + \frac{1}{x} = \cos\theta + i\sin\theta$

$$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$$

$$x = \frac{1}{(\cos\theta - 1) + i\sin\theta}$$

$$= \frac{1}{(\cos\theta-1) + i\sin\theta} \times \frac{(\cos\theta-1) - i\sin\theta}{(\cos\theta-1) - i\sin\theta}$$

$$= \frac{(\cos\theta-1) - i\sin\theta}{(\cos\theta-1)^2 + \sin^2\theta} = \frac{(\cos\theta-1) - i\sin\theta}{2(1-\cos\theta)}$$

$$= -\frac{1}{2} - i\frac{\sin\theta}{2(1-\cos\theta)}$$

$$= -\frac{1}{2} - i\frac{2\sin\theta/2\cos\theta/2}{2(2\sin^2\theta/2)}$$

$x = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$ where $\theta = \left(\frac{2k+1}{6}\right)\pi$

$k = 0, 1, 2, 3, 4, 5$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$ *is* $1 + i$, find all other roots.

Soln:- since $1+i$ is one of the roots,

$1-i$ is also a roots (complex roots always occur in pairs)

$\therefore x = 1 \pm i$ are two roots

$x - 1 = \pm i$

$(x-1)^2 = (\pm i)^2$

$$x^2 - 2x + 1 = -1$$
$$x^2 - 2x + 2 = 0$$

Now we want to find the remaining roots, for that we divide $x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 2x + 2$ and we get

$$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = (x^2 - 2x + 2)(x^2 - 4x + 5)$$

$\therefore$ The remaining roots are the roots of the equation $x^2 - 4x + 5 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$\therefore$ The required remaining roots are $1 - i,\ 2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$.

Soln:- $x^5 - 1 = 0 \Rightarrow x^5 = 1 \Rightarrow x = (1)^{1/5}$

$$x = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{5}}$$

$$= \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5} \qquad k = 0, 1, 2, 3, 4$$

$\therefore x_0 = \cos 0 + i \sin 0 = 1$

$x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$

$x_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$

$x_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$

$x_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$

taking $x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \alpha$

$\therefore x_2 = \alpha^2, \quad x_3 = \alpha^3, \quad x_4 = \alpha^4$

$\therefore 1, \alpha, \alpha^2, \alpha^3$ & α^4 are roots of $x^5 - 1 = 0$

$$\therefore x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\frac{x^5 - 1}{x - 1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore x^4 + x^3 + x^2 + x + 1 = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

put $x = 1$

$$\therefore \quad 5 = (1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)$$

Hence proved.

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$. (H.w.)

Soln:- $\frac{z^4}{(z-1)^4} = i$

$$\left(\frac{z}{z-1}\right)^4 = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

14. If ω is a 7th root of unity, prove that $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$ if n is a multiple of 7 and is equal to zero otherwise.

Soln:- 7th roots of unity, $x = (1)^{1/7}$

$$= \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$$

where $k = 0, 1, 2, 3, 4, 5, 6$

$\therefore$ let $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

$$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i\sin 2\pi = 1.$$

$\therefore \boxed{\omega^7 = 1}$ $\quad\therefore \boxed{\omega^{7n} = 1}$ for any n

if n is not a multiple of 7 then $\underline{\underline{\omega^n \neq 1}}$.

Case (i) If n is a multiple of 7

let $n = 7k$

$$\therefore S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \cdots + \omega^{6n}$$

$$= 1 + \omega^{7k} + (\omega^{7k})^2 + (\omega^{7k})^3 + \cdots + (\omega^{7k})^6$$

but $\omega^{7k} = 1$

$$\therefore S = 1+1+1+1+1+1+1 = 7$$

Case (ii) If n is not a multiple of 7. $\Rightarrow \underline{\underline{\omega^n \neq 1}}$.

$$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \cdots + \omega^{6n}$$

This is sum of a G.P. where $a = 1, r = \omega^n$

$$\therefore S = \frac{1-\omega^{7n}}{1-\omega^n} = \frac{1-1}{1-\omega^n} = 0$$

$$\boxed{\omega^{7n} = 1}$$

15. Prove that $\sqrt{1+sec(\theta/2)} = (1+e^{i\theta})^{-1/2} + (1+e^{-i\theta})^{-1/2}$

Soln:- To show that

$$\sqrt{1+\sec\left(\frac{\theta}{2}\right)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$$

Squaring on both sides

$$1+\sec\left(\frac{\theta}{2}\right) = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$$

we will prove this.

$$RHS = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{i\theta}+e^{-i\theta}+1}}$$

$$= \frac{1+e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{2+e^{i\theta}+e^{-i\theta}}}$$

$$= 1 + \frac{2}{\sqrt{2+2\cos\theta}} \qquad \left(\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}\right)$$

$$= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}}$$

$$= 1 + \frac{2}{\sqrt{2\left(2\cos^2\frac{\theta}{2}\right)}} = 1 + \frac{2}{2\cos\frac{\theta}{2}} = 1+\sec\left(\frac{\theta}{2}\right)$$

HYPERBOLIC FUNCTIONS

Tuesday, October 26, 2021 1:30 PM

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore\ \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\ x$ and is given as, $\sinh x = \frac{e^{x}-e^{-x}}{2}$ and

Cosine hyperbolic of x is denoted by $cosh\ x$ and is given as, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as

$\tan \mathrm{hx} = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$, $\quad \mathrm{cosechx} = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}$, $\quad \mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}$, and

$\coth x = \frac{1}{\tanh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinhx, cos x, tanhx, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1, \tanh(0) = 0, \tanh(\infty) = 1$

$\therefore |\tanh x| \le 1$

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	HYPERBOLIC FUNCTIONS
1	$\sin(-x) = -(\sin x)$	$\sinh(-x) = -\sinh x,$
2	$\cos(-x) = (cosx)$	$\cosh(-x) = \cosh x$
3	$e^{ix} = \cos x + i\sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i\sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2 x - \sinh^2 x = 1$
6	$1 + \tan^2 x = \sec^2 x$	$\mathrm{sech}^2 x + \tanh^2 x = 1$
7	$1 + \cot^2 x = \mathrm{cosec}^2 x$	$\coth^2 x - \mathrm{cosech}^2 x = 1$
8	$\sin 2x = 2\sin x\cos x$ $= \frac{2\tan x}{1+\tan^2 x}$	$\sinh 2x = 2\sinh x\cosh x$ $= \frac{2\tanh x}{1-\tanh^2 x}$
9	$\cos 2x = \cos^2 x - \sin^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$

	$= 2\cos^2 x - 1$ $= 1 - 2\sin^2 x$ $= \frac{1-\tan^2 x}{1+\tan^2 x}$	$= 2\cosh^2 x - 1$ $= 1 + 2\sinh^2 x$ $= \frac{1+\tanh^2 x}{1-\tanh^2 x}$
10	$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$	$\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$
11	$\sin 3x = 3\sin x - 4\sin^3 x$	$\sinh 3x = 3\sinh x + 4\sinh^3 x$
12	$\cos 3x = 4\cos^3 x - 3\cos x$	$\cosh 3x = 4\cosh^3 x - 3\cosh x$
13	$\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$	$\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$
14	$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
15	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$	$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
16	$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tanh y}$	$\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$
17	$\cot(x \pm y) = \dfrac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$	$\coth(x \pm y) = \dfrac{-\coth x \coth y \mp 1}{\coth y \pm \coth x}$
18	$\sin x + \sin y = 2\sin\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$	$\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$
19	$\sin x - \sin y = 2\cos\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$	$\sinh x - \sinh y = 2\cosh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$
20	$\cos x + \cos y$ $= 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}$
21	$\cos x - \cos y$ $= -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}$
22	$2\sin x \cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x \cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x \sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x \sinh y = \sinh(x+y) - \sinh(x-y)$
24	$2\cos x \cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$
25	$2\sin x \sin y = \cos(x-y) - \cos(x+y)$	$2\sinh x \sinh y = \cos h(x+y) - \cos h(x-y)$

PERIOD OF HYPERBOLIC FUNTIONS:

$\sinh(2\pi i + x) = \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x$

$= i\sin 2\pi \cosh x + \cos 2\pi \sinh x$

$= 0 + \sinh x \qquad = \sinh x$

Hence $\sinh x$ is a periodic function of period $2\pi i$

Similarly we can prove that $\cosh x$ and $\tanh x$ are periodic functions of period 2πi and πi .

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$,

$$y = \frac{e^x - e^{-x}}{2}$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

If $y = \sinh x, \frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\ x$,

$$y = \frac{e^x + e^{-x}}{2},$$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$$

If $y = cosh\ x, \frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$,

$$y = \frac{\sinh x}{\cosh x}$$

$$\therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$$

If $y = \tanh x, \frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$$\int \cosh x\, dx = \sinh x\,, \quad \int \sinh x\, dx = \cosh x\,, \quad \int sech^2 x dx = \tanh x$$

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SOME SOLVED EXAMPLES:

Sol: $\tanh x = \frac{1}{2}$

1. If $\tanh x = \frac{1}{2}, find\ \sinh 2x\ and\ \cosh 2x$

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$$

$$2e^x - 2e^{-x} = e^x + e^{-x}$$

$$e^x - 3e^{-x} = 0$$

$$e^{2x} - 3 = 0 \Rightarrow e^{2x} = 3 \Rightarrow e^{-2x} = \frac{1}{3}$$

$$\therefore \sinh 2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{8}{6} = \frac{4}{3}$$

$$\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6} = \frac{5}{3}$$

Alternatively

$$\sinh 2x = \frac{2\tanh x}{1 - \tanh^2 x} = \frac{2 \cdot \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

$$\cosh 2x = \frac{1 + \tanh^2 x}{} \quad 1 + \left(\frac{1}{2}\right)^2 \quad \frac{5}{4} \quad = \frac{5}{}$$

$$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x} = \frac{1+\left(\frac{1}{2}\right)^2}{1-\left(\frac{1}{2}\right)^2} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}.$$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Soln:- $7\cosh x + 8\sinh x = 1$

$$7\left(\frac{e^x+e^{-x}}{2}\right) + 8\left(\frac{e^x-e^{-x}}{2}\right) = 1$$

$$7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2$$

$$\downarrow$$

$$15e^{2x} - 2e^x - 1 = 0$$

This is a quadratic eqn is e^x

$$15y^2 - 2y - 1 = 0$$

$$\therefore y = e^x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(15)(-1)}}{2(15)}$$

$$\Rightarrow e^x = \frac{1}{3} \text{ or } -\frac{1}{5}$$

$$\Rightarrow x = \log\left(\frac{1}{3}\right) \text{ or } x = \log\left(-\frac{1}{5}\right)$$

Since x is real, $x = \log\left(\frac{1}{3}\right) = -\log 3$.

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Soln:- Let $\sinh^{-1} a = \alpha$

$\sinh^{-1} b = \beta$ and $\sinh^{-1} x = y$

$\therefore \alpha + \beta = y$

$$\sinh(\alpha+\beta) = \sinh y$$

$$\left(\sinh(A+B) = \sinh A \cosh B + \cosh A \sinh B\right)$$

$$\therefore \sinh\alpha \cosh\beta + \cosh\alpha \sinh\beta = \sinh y$$

but $\sinh\alpha = a$

$\sinh\beta = b$

$\sinh y = x$

also $\cosh\beta = \sqrt{1+\sinh^2\beta} = \sqrt{1+b^2}$

Similarly $\cosh\alpha = \sqrt{1+\sinh^2\alpha} = \sqrt{1+a2}$

$$\therefore a\sqrt{1+b^2} + b\sqrt{1+a2} = x$$

4. Prove that $16\, sinh^5 x = \sinh 5x - 5\sinh 3x + 10 \sinh x$

Soln :- $LHS = 16\sinh^5 x = 16(\sinh x)^5$

$$= 16\left[\frac{e^x - e^{-x}}{2}\right]^5 = \frac{16}{2^5}\left(e^x - e^{-x}\right)^5$$

$$\left[(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_n b^n\right]$$

$$LHS = \frac{16}{2^5}\Big[(e^x)^5 - 5(e^x)^4(e^{-x}) + 10(e^x)^3(e^{-x})^2 - 10(e^x)^2(e^{-x})^3 + 5(e^x)(e^{-x})^4 - (e^{-x})^5\Big]$$

$$= \frac{16}{2^5}\left[e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x}\right]$$

$$= \frac{16}{2^5}\left[e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x}\right]$$

$$= \frac{16}{2^5}\left[\left(e^{5x} - e^{-5x}\right) - 5\left(e^{3x} - e^{-3x}\right) + 10\left(e^{x} - e^{-x}\right)\right]$$

$$= \left(\frac{e^{5x} - e^{-5x}}{2}\right) - 5\left(\frac{e^{3x} - e^{-3x}}{2}\right) + 10\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= \sinh 5x - 5\sinh 3x + 10\sinh x$$

5. Prove that $16cosh^5x = \cosh 5x + 5\cosh 3x + 10\cosh x$ (HW)

6. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1-cosh^2x}}} = cosh^2x$

Soln:- $$\frac{1}{1-\frac{1}{1-\frac{1}{1-\cosh^2 x}}} = \frac{1}{1-\frac{1}{1-\frac{1}{-\sinh^2 x}}}$$

$$\left(1-\cosh^2 x = -\sinh^2 x\right)$$

$$= \frac{1}{1-\frac{1}{1+\operatorname{cosech}^2 x}}$$

$$\left(1+\operatorname{cosech}^2 x = 1+\frac{1}{\sinh^2 x} = \frac{\sinh^2 x+1}{\sinh^2 x} = \frac{\cosh^2 x}{\sinh^2 x} = \coth^2 x\right)$$

$$= \frac{1}{1-\frac{1}{\coth^2 x}}$$

$$= \frac{1}{1-\tanh^2 x} = \frac{1}{1-\frac{\sinh^2 x}{\cosh^2 x}} = \frac{\cosh^2 x}{\cosh^2 x - \sinh^2 x}$$

$$= \cosh^2 x \qquad (\text{as } \cosh^2 x - \sinh^2 x = 1)$$

7. If $u = \log tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$ **(iv)** $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Soln:- $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$$\Rightarrow e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\tan\left(\frac{\pi}{4}\right) + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\theta}{2}\right)}$$

$$\therefore e^u = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$$

$$\therefore e^{-u} = \frac{1 - \tan\theta/2}{1 + \tan\theta/2}$$

(i) $$\cosh u = \frac{e^u + e^{-u}}{2} = \frac{1}{2}\left[\frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} + \frac{1 - \tan\frac{\theta}{2}}{1 + \tan\frac{\theta}{2}}\right]$$

$$= \frac{1}{2}\left[\frac{\left(1 + \tan\frac{\theta}{2}\right)^2 + \left(1 - \tan\frac{\theta}{2}\right)^2}{\left(1 - \tan\frac{\theta}{2}\right)\left(1 + \tan\frac{\theta}{2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1 + 2\tan\frac{\theta}{2} + \tan^2\frac{\theta}{2} + 1 - 2\tan\frac{\theta}{2} + \tan^2\frac{\theta}{2}}{1 - \tan^2\frac{\theta}{2}}\right]$$

$$= \frac{1 + \tan^2\frac{\theta}{2}}{} = \frac{1}{} \qquad \left[\cos 2\theta = \frac{1 - \tan^2\theta}{}\right.$$

$$= \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{1}{\cos\theta} \qquad \left[\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right.$$

$$\therefore \cosh u = \sec\theta$$

(ii) $\sinh u = \sqrt{\cosh^2 u - 1}$ $\quad (\cosh^2 u - \sinh^2 u = 1)$

$$= \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$$

(iii) $\tanh u = \dfrac{\sinh u}{\cosh u} = \dfrac{\tan\theta}{\sec\theta} = \sin\theta$

(iv) $\tanh\dfrac{u}{2} = \dfrac{\sinh\frac{u}{2}}{\cosh\frac{u}{2}} = \dfrac{2\sinh\frac{u}{2}\cosh\frac{u}{2}}{2\cosh^2\frac{u}{2}}$

(multiplied by $2\cosh\frac{u}{2}$ in N & D)

$$= \frac{\sinh u}{1+\cosh u} = \frac{\tan\theta}{1+\sec\theta} \qquad \text{(using (i) \& (ii))}$$

$$= \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}$$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Soln :- $\cosh x = \sec\theta$

$e^{x} + e^{-x}$ ... $\Rightarrow e^{x} + e^{-x} = 2\sec\theta$

$\therefore \cosh x = \sec\theta$

$$\frac{e^{x}+e^{-x}}{2} = \sec\theta \Rightarrow e^{x}+e^{-x} = 2\sec\theta$$

multiply by e^{x}

$$e^{2x} - 2\sec\theta\, e^{x} + 1 = 0$$

This is a quadratic eqn in e^{x}

$$e^{x} = \frac{-(-2\sec\theta) \pm \sqrt{(-2\sec\theta)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{2\sec\theta \pm \sqrt{4\sec^2\theta - 4}}{2}$$

$$= \sec\theta \pm \sqrt{\sec^2\theta - 1}$$

$$e^{x} = \sec\theta \pm \tan\theta$$

$$\therefore x = \log(\sec\theta \pm \tan\theta)$$

$$= \pm \log(\sec\theta + \tan\theta)$$

(ie. we claim that $\log(\sec\theta - \tan\theta) = -\log(\sec\theta + \tan\theta)$)

$$\therefore x = \log(\sec\theta + \tan\theta)$$

(ii) TPT $\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$

Let $\tan^{-1}(e^{-x}) = \alpha$

$$\therefore e^{-x} = \tan\alpha$$

$$\therefore e^{x} = \cot\alpha$$

by given data $\cosh x = \sec\theta$

$$\therefore \frac{e^{x}+e^{-x}}{2} = \sec\theta$$

$$\therefore \frac{\tan\alpha+\cot\alpha}{2} = \sec\theta$$

$$\frac{1}{2}\left[\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\right] = \sec\theta$$

$$\frac{1}{2}\left[\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}\right] = \sec\theta$$

$$\therefore \frac{1}{\sin 2\alpha} = \sec\theta$$

$$\therefore \cos\theta = \sin 2\alpha$$

$$\cos\theta = \cos\left(\frac{\pi}{2}-2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha$$

$$\theta = \frac{\pi}{2} - 2\tan^{-1}(e^{-x})$$

(iii) $\tanh\left(\frac{x}{2}\right) = \tan\frac{\theta}{2}$

$$\tanh\left(\frac{x}{2}\right) = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{e^{x}-1}{e^{x}+1}$$

$$= \frac{\sec\theta+\tan\theta-1}{\sec\theta+\tan\theta+1}$$

$\sec\theta + \tan\theta + 1$

$$= \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta} = \frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$$

$$= \frac{2\sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{2\sin\theta/2}{2\cos\theta/2}\left[\frac{\sin\theta/2+\cos\theta/2}{\sin\theta/2+\cos\theta/2}\right]$$

$$= \tan\frac{\theta}{2}.$$

SEPARATION OF REAL AND IMAGINARY PARTS

Thursday, October 28, 2021 10:21 AM

Many a time we are required to separate real and imaginary parts of a given complex function. For this, we have to use identities of circular and hyperbolic functions.

sin Z

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$$\therefore \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$$
$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\tan(\alpha-i\beta)}$$
$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$
$$\therefore 1 - x^2 - y^2 = 2x\cot 2\alpha$$
$$\therefore x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$
$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \tan(\alpha-i\beta)}$$
$$i\tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$
$$\therefore \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$
$$\therefore 1 + x^2 + y^2 = 2y\coth 2\beta \qquad \text{i.e.,}\ x^2 + y^2 - 2y\coth 2\beta + 1 = 0$$

SOME SOLVED EXAMPLES:

1. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = 1$ and $\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = 1$

Soln

$$\sin(\alpha - i\beta) = x + iy$$
$$\sin\alpha\cos i\beta - \cos\alpha\sin i\beta = x + iy$$
$$\sin\alpha\cosh\beta - i\cos\alpha\sinh\beta = x + iy$$

$\left[\begin{array}{l}\cos i\beta = \cosh\beta \\ \sin i\beta = i\sinh\beta\end{array}\right]$

$$\therefore x = \sin\alpha\cosh\beta \quad \& \quad y = -\cos\alpha\sinh\beta \quad \text{--- (1)}$$

(i) $$\text{LHS} = \frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta}$$
$$= \frac{\sin^2\alpha\cosh^2\beta}{\cosh^2\beta} + \frac{\cos^2\alpha\sinh^2\beta}{\sinh^2\beta}$$
$$= \sin^2\alpha + \cos^2\alpha = 1. = \text{RHS}$$

(ii) $LHS = \dfrac{x^2}{\sin^2\alpha} - \dfrac{y^2}{\cos^2\alpha}$

$$= \frac{\sin^2\alpha \cosh^2\beta}{\sin^2\alpha} - \frac{\cos^2\alpha \sinh^2\beta}{\cos^2\alpha}$$

$$= \cosh^2\beta - \sinh^2\beta = 1 = RHS.$$

2. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Soln :- $x + iy = \tan\left(\frac{\pi}{6} + i\alpha\right)$

$\therefore\ x - iy = \tan\left(\frac{\pi}{6} - i\alpha\right)$

$$\tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{\tan\left(\frac{\pi}{6} + i\alpha\right) + \tan\left(\frac{\pi}{6} - i\alpha\right)}{1 - \tan\left(\frac{\pi}{6} + i\alpha\right)\tan\left(\frac{\pi}{6} - i\alpha\right)}$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{(x+iy) + (x-iy)}{1 - (x+iy)(x-iy)}$$

$$\sqrt{3} = \frac{2x}{1 - x^2 - y^2}$$

$$\Rightarrow 1 - x^2 - y^2 = \frac{2}{\sqrt{3}}x$$

$$\Rightarrow x^2 + y^2 + \frac{2}{\sqrt{3}}x = 1$$

3. Separate into real and imaginary parts $tan^{-1}\left(e^{i\,\theta}\right)$

Soln :- Let $\tan^{-1}(e^{i\theta}) = x + iy$

$\therefore\ \tan(x+iy) = e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore \tan(x-iy) = \cos\theta - i\sin\theta$$

Now $\tan(2x) = \tan\left[(x+iy)+(x-iy)\right]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$\tan(2x) = \frac{(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)}{1-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$= \frac{2\cos\theta}{1-(\cos^2\theta+\sin^2\theta)} = \frac{2\cos\theta}{1-1}$$

$$\therefore \tan 2x = \infty$$

$$\therefore 2x = \frac{\pi}{2} \Rightarrow \boxed{x = \frac{\pi}{4}}$$

Now $\tan(2iy) = \tan\left[(x+iy)-(x-iy)\right]$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$i\tanh 2y = \frac{(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)}{1+(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)}$$

$$= \frac{2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)} = \frac{2i\sin\theta}{2}$$

$$= \frac{2i\sin\theta}{1+(\cos^2\theta+\sin^2\theta)} = \frac{}{2}$$

$$\therefore \tanh 2y = \sin\theta$$

$$2y = \tanh^{-1}(\sin\theta)$$

$$\therefore y = \frac{1}{2}\tanh^{-1}(\sin\theta)$$

$$\therefore \tan^{-1}(e^{i\theta}) = x+iy = \frac{\pi}{4} + i\frac{1}{2}\tanh^{-1}(\sin\theta)$$

4. If $cos(x+i\,y) = \cos\alpha + i\sin\alpha\,, prove\ that$

(i) $\sin\alpha = \pm\, sin^2 x = \pm\, \sin h^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Soln $\cos(x+iy) = \cos\alpha + i\sin\alpha$

$$\cos x\cos iy - \sin x\sin iy = \cos\alpha + i\sin\alpha$$

$$\cos x\cosh y - i\sin x\sinh y = \cos\alpha + i\sin\alpha$$

$$[\cos iy = \cosh y,\ \sin iy = i\sinh y]$$

$$\therefore \cos\alpha = \cos x\cosh y \ \ \& \ \ \sin\alpha = -\sin x\sinh y \quad (1)$$

Now $\cos^2\alpha + \sin^2\alpha = 1$

$$\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y = 1$$

$$(1-\sin^2 x)(1+\sinh^2 y) + \sin^2 x\sinh^2 y = 1$$

$$1+\sinh^2 y - \sin^2 x - \sin^2 x\sinh^2 y + \sin^2 x\sinh^2 y = 1$$

$$\Rightarrow \sinh^2 y - \sin^2 x = 0$$

$\Rightarrow \sinh^2 y - \sin^2 x = 0$

$\Rightarrow \sin^2 x = \sinh^2 y$ ——— (2)

$\Rightarrow \sin x = \pm \sinh y$

from (1) $\Rightarrow$ $\sin\alpha = -\sin x \sinh y$

$= -\sin x(\pm \sin x) = \pm \sin^2 x$

or $\sin\alpha = -\sin x \sinh y$

$= -(\pm \sinh y)\sinh y = \pm \sinh^2 y$

(ii) TPT $\cos 2x + \cosh 2y = 2$

LHS :- $\cos 2x + \cosh 2y$

$= 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2(\sin^2 x - \sinh^2 y)$

but $\sin^2 x = \sinh^2 y$ (from (2))

$= 2$

$=$ RHS.

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$.

Soln:- We have $x + iy = c\cot(u+iv)$

$\therefore x - iy = c\cot(u-iv)$

$2x = c\left[\cot(u+iv) + \cot(u-iv)\right]$

$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$

$$2x = c\left[\frac{\sin(u-iv)\cos(u+iv)+\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$= c\left[\frac{\sin[(u-iv)+(u+iv)]}{\frac{1}{2}[\cos(u+iv-u+iv)-\cos(u+iv+u-iv)]}\right]$$

$$2\sin A\sin B = \cos(A-B)-\cos(A+B)$$

$$2x = \frac{2c\sin 2u}{\cos(2iv)-\cos 2u}$$

$$\therefore \frac{x}{\sin 2u} = \frac{c}{\cosh 2v-\cos 2u}$$

Now :- $2iy = c[\cot(u+iv)-\cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)}-\frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

Complete this as H.W.

6. If $u+iv = cosec\left(\frac{\pi}{4}+ix\right)$, prove that $(u^2+v^2)^2 = 2(u^2-v^2)$

Soln :- $cosec\left(\frac{\pi}{4}+ix\right) = u+iv$

$$\frac{1}{\quad} = u+iv$$

$\sin\left(\frac{\pi}{4}+ix\right)$

$$\sin\left(\frac{\pi}{4}+ix\right) = \frac{1}{u+iv} \times \frac{u-iv}{u-iv}$$

$$\sin\frac{\pi}{4}\cos ix + \cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$$

$\left[\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},\ \cos ix = \cosh x,\ \sin ix = i\sinh x\right]$

$$\frac{\cosh x}{\sqrt{2}} + i\,\frac{\sinh x}{\sqrt{2}} = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$$

comparing real & imaginary parts

$$\cosh x = \frac{\sqrt{2}u}{u^2+v^2}, \quad \sinh x = \frac{-\sqrt{2}v}{u^2+v^2}$$

Now :- $\cosh^2 x - \sinh^2 x = 1$

$$\frac{2u^2}{(u^2+v^2)^2} - \frac{2v^2}{(u^2+v^2)^2} = 1$$

$$\therefore 2(u^2-v^2) = (u^2+v^2)^2$$

7. If $x + iy = \cos(\alpha + i\beta)$ or if $\cos^{-1}(x+iy) = \alpha + i\beta$ express x and y in terms of α and β. Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation $\lambda^2 - (x^2+y^2+1)\lambda + x^2 = 0$

Soln :- $x+iy = \cos(\alpha+i\beta)$

$x+iy = \cos\alpha\cos i\beta - \sin\alpha\sin i\beta$

$= \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

$x+iy = \cos\alpha\cosh\beta - \sin\alpha\sinh\beta$

$x+iy = \cos\alpha\cosh\beta - i\sin\alpha\sinh\beta$

$\therefore x = \cos\alpha\cosh\beta \quad \& \quad y = -\sin\alpha\sinh\beta$ — (1)

we know that, in terms of roots, the quadratic equation is given by

$$\lambda^2 - (\text{sum of the roots})\lambda + (\text{product of roots}) = 0$$

Hence, to show that $\cos^2\alpha$ and $\cosh^2\beta$ are roots of $\lambda^2 - (x^2+y^2+1)\lambda + x^2 = 0$, it is enough to show that

$$x^2+y^2+1 = \cos^2\alpha + \cosh^2\beta \quad \text{— (2)}$$

and $$x^2 = \cos^2\alpha\cosh^2\beta \quad \text{— (3)}$$

from (1) $x = \cos\alpha\cosh\beta$

$$\therefore x^2 = \cos^2\alpha\cosh^2\beta$$

That proves (3)

Now $x^2+y^2+1 = \cos^2\alpha\cosh^2\beta + \sin^2\alpha\sinh^2\beta + 1$

$= \cos^2\alpha\cosh^2\beta + (1-\cos^2\alpha)(\cosh^2\beta - 1) + 1$

$= \cos^2\alpha\cosh^2\beta + \cosh^2\beta - 1 - \cos^2\alpha\cosh^2\beta$
$+ \cos^2\alpha + 1$

$x^2+y^2+1 = \cos^2\alpha + \cosh^2\beta$

That proves (2)

$\therefore \cos^2\alpha$ & $\cosh^2\beta$ are the roots of given quadratic equation.

INVERSE HYPERBOLIC FUNCTIONS

Friday, October 29, 2021 1:19 PM

If $x = \sinh u$ then $u = \sinh^{-1} x$ is called sine hyperbolic inverse of x, where x is real.
Similarly we can define $\cosh^{-1}x, \tanh^{-1}x, \coth^{-1}x, \mathrm{sech}^{-1}x, \mathrm{cosech}^{-1}x$.

Theorem: If x is real.

(i) $\sinh^{-1}x = \log\,(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1}x = \log\,(x + \sqrt{x^2-1})$

(iii) $\tanh^{-1}x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Proof :- (i) let $\sinh^{-1}(x) = y$

$$\therefore \sinh y = x$$

$$\frac{e^y - e^{-y}}{2} = x$$

$$e^y - e^{-y} = 2x$$

$$e^{2y} - 2x e^y - 1 = 0 \quad (\text{multi by } e^y)$$

This is a quadratic in e^y

$$\therefore \quad e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2+1}\right)$$

Now $x - \sqrt{x^2+1} < 0$ $\qquad x < \sqrt{x^2+1}$

$\therefore \log\left(x-\sqrt{x^2+1}\right)$ is not defined

$$\therefore y = \log\left(x+\sqrt{x^2+1}\right)$$

$$\therefore \sinh^{-1} x = \log\left(x+\sqrt{x^2+1}\right)$$

(ii) Let $\cosh^{-1} x = y$

$$\therefore \cosh y = x$$

$$\therefore \frac{e^y + e^{-y}}{2} = x$$

$$\therefore e^y + e^{-y} = 2x$$

multi by e^y

$$e^{2y} - 2xe^y + 1 = 0$$

This is a quadratic in e^y

$$\therefore e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(1)}}{2(1)}$$

$$\therefore e^y = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\therefore e^y = x \pm \sqrt{x^2-1}$$

$$\therefore y = \log\left(x \pm \sqrt{x^2-1}\right) \quad \text{—— (1)}$$

Let $y = \log\left(x - \sqrt{x^2-1}\right)$ —— (2)

$$e^y = x - \sqrt{x^2-1}$$

$\therefore e^{-y} = \dfrac{1}{x - \sqrt{x^2-1}}$

$$\therefore e^{-y} = \frac{1}{x-\sqrt{x^2-1}} \times \frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$\therefore e^{-y} = \frac{x+\sqrt{x^2-1}}{(x)^2 - \left(\sqrt{x^2-1}\right)^2} = \frac{x+\sqrt{x^2-1}}{x^2-x^2+1}$$

$$\therefore e^{-y} = x+\sqrt{x^2-1}$$

$$\therefore -y = \log\left(x+\sqrt{x^2-1}\right)$$

$$\therefore y = -\log\left(x+\sqrt{x^2-1}\right) \quad \text{--- (3)}$$

from (2) & (3) $\Rightarrow \log\left(x-\sqrt{x^2-1}\right) = -\log\left(x+\sqrt{x^2-1}\right)$

Subst in (1) $\therefore y = \pm \log\left(x+\sqrt{x^2-1}\right)$

$$\therefore \cosh^{-1} x = \pm \log\left(x+\sqrt{x^2-1}\right)$$

$$\therefore x = \cosh\left(\pm \log\left(x+\sqrt{x^2-1}\right)\right)$$

[but $\cosh(-z) = \cosh z$]

$$x = \cosh\left(\log\left(x+\sqrt{x^2-1}\right)\right)$$

$$\therefore \cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$$

(iii) TPT: $\tanh^{-1}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Soln :- Let $\tanh^{-1}(x) = y$

$$\therefore x = \tanh y$$

$$\frac{x}{1} = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$$

Using componendo - dividendo

$$\frac{1+x}{1-x} = \frac{(e^{y} + e^{-y}) + (e^{y} - e^{-y})}{(e^{y} + e^{-y}) - (e^{y} - e^{-y})}$$

$$\frac{1+x}{1-x} = \frac{2e^{y}}{2e^{-y}} = e^{2y}$$

$$\therefore e^{2y} = \frac{1+x}{1-x}$$

$$\therefore 2y = \log\left(\frac{1+x}{1-x}\right)$$

$$\therefore y = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

SOME SOLVED EXAMPLES:

1. Prove that $\tanh \log\sqrt{x} = \frac{x-1}{x+1}$ Hence deduce that $\tanh \log\sqrt{5/3} + \tanh \log\sqrt{7} = 1$

Soln :-

method 1.

$$\tanh x = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\tanh(\log\sqrt{x}) = \frac{e^{\log\sqrt{x}} - e^{-\log\sqrt{x}}}{e^{\log\sqrt{x}} + e^{-\log\sqrt{x}}}$$

method 2.

Let $\tanh(\log\sqrt{x}) = a$

$$\therefore \log\sqrt{x} = \tanh^{-1}(a)$$

$$\log\sqrt{x} = \frac{1}{2}\log\left(\frac{1+a}{1-a}\right)$$

$$\tanh(\log\sqrt{x}) = \frac{e^{\log\sqrt{x}} - e^{-\log\sqrt{x}}}{e^{\log\sqrt{x}} + e^{-\log\sqrt{x}}}$$

$$= \frac{\sqrt{x} - \frac{1}{\sqrt{x}}}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$\tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

$$\log\sqrt{x} = \log\sqrt{\frac{1+a}{1-a}}$$

$$\sqrt{x} = \sqrt{\frac{1+a}{1-a}}$$

$$x = \frac{1+a}{1-a}$$

$$\frac{x-1}{x+1} = \frac{1+a-1+a}{1+a+1-a} = a$$

$$\therefore \tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

TPT $\tanh\left(\log\sqrt{\frac{5}{3}}\right) + \tanh(\log\sqrt{7}) = 1.$

$$\tanh(\log\sqrt{x}) = \frac{x-1}{x+1}$$

$$\tanh\left(\log\sqrt{\frac{5}{3}}\right) = \frac{\frac{5}{3}-1}{\frac{5}{3}+1} = \frac{2}{8} = \frac{1}{4}$$

$$\tanh(\log\sqrt{7}) = \frac{7-1}{7+1} = \frac{6}{8} = \frac{3}{4}$$

$$\therefore \tanh\left(\log\sqrt{\frac{5}{3}}\right) + \tanh(\log\sqrt{7}) = \frac{1}{4} + \frac{3}{4} = 1$$

2. (i) Prove that $cosh^{-1}\sqrt{1+x^2} = sinh^{-1}x$

(ii) Prove that $tanh^{-1}x = sinh^{-1}\frac{x}{\sqrt{1-x^2}}$

(iii) Prove that $cosh^{-1}\left(\sqrt{1+x^2}\right) = tanh^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$

✓**(iv)** Prove that $cot\,h^{-1}\left(\frac{x}{a}\right) = \frac{1}{2}log\left(\frac{x+a}{x-a}\right)$ (H.W.) (do similar to $\tanh^{-1}(x)$)

(v) Prove that $sech^{-1}(\sin\theta) = \log cot\frac{\theta}{2}$

Soln:- (i) Let $\cosh^{-1}\sqrt{1+x^2} = y$

$\therefore \sqrt{1+x^2} = \cosh y$

$\therefore 1+x^2 = \cosh^2 y$

$\therefore x^2 = \cosh^2 y - 1$

$\therefore x^2 = \sinh^2 y$

$\therefore x = \sinh y$

$\therefore \sinh^{-1} x = y$

$\therefore \cosh^{-1} \sqrt{1+x^2} = \sinh^{-1} x$

(ii) Tpt: $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{1-x^2}}$

Soln:- Let $\tanh^{-1} x = y$

$\therefore x = \tanh y$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \frac{\tanh y}{\sqrt{1-\tanh^2 y}} = \frac{\tanh y}{\sqrt{\operatorname{sech}^2 y}} = \frac{\tanh y}{\operatorname{sech} y}$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \sinh y$$

$$\therefore y = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\therefore \tanh^{-1} x = \sinh^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

(iii) Tpt: $\cosh^{-1}\left(\sqrt{1+x^2}\right) = \tanh^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$ (H.W.)

Let $\cosh^{-1}\left(\sqrt{1+x^2}\right) = y$

$\sqrt{1+x^2} = \cosh y$

(4) TPT: $\operatorname{sech}^{-1}(\sin\theta) = \log \cot\frac{\theta}{2}$

Let $\operatorname{sech}^{-1}(\sin\theta) = y$

$$\sin\theta = \operatorname{sech} y$$

$$\sin\theta = \frac{2}{e^{y}+e^{-y}} = \frac{2e^{y}}{e^{2y}+1}$$

$$(\sin\theta)\, e^{2y} - 2e^{y} + \sin\theta = 0$$

This is a quadratic in e^{y}

$$\therefore e^{y} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(\sin\theta)(\sin\theta)}}{2\sin\theta}$$

$$\therefore e^{y} = \frac{2 \pm \sqrt{4-4\sin^2\theta}}{2\sin\theta}$$

$$\therefore e^{y} = \frac{1 \pm \sqrt{1-\sin^2\theta}}{\sin\theta} = \frac{1 \pm \cos\theta}{\sin\theta}$$

$$e^{y} = \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \cot\frac{\theta}{2}$$

$$\therefore y = \log\cot\frac{\theta}{2}$$

v) $\operatorname{sech}^{-1}(\sin\theta) = \log\left(\cot\frac{\theta}{2}\right)$

let $\operatorname{sech}^{-1}(\sin\theta) = y$.

$$\operatorname{sech} y = \sin\theta .$$

$$\cosh y = \operatorname{cosec}\theta .$$

$$y = \cosh^{-1}(\operatorname{cosec}\theta)$$

$$y = \cancel{\log\left(x + \sqrt{x^2 - 1}\right)}$$

$$\log\left(\operatorname{cosec}\theta + \sqrt{\operatorname{cosec}^2\theta - 1}\right)$$

$$\log\left(\operatorname{cosec}\theta + \cot\theta\right)$$

$$\log\left(\frac{1 + \cos\theta}{\sin\theta}\right).$$

$$\log\left(\frac{1}{\tan\left(\frac{\theta}{2}\right)}\right)$$

$$\log\left(\cot\left(\frac{\theta}{2}\right)\right)$$

3. Separate into real and imaginary parts $cos^{-1}e^{i\theta}$ ***or*** $cos^{-1}(\cos\theta + i\sin\theta)$

Soln :- let $\cos^{-1}(\cos\theta + i\sin\theta) = x+iy$

$$\therefore \cos(x+iy) = \cos\theta + i\sin\theta$$

$$\cos x \cos iy - \sin x \sin iy = \cos\theta + i\sin\theta$$

$$\cos x \cosh y - i\sin x \sinh y = \cos\theta + i\sin\theta$$

$$(\cos iy = \cosh y, \ \sin iy = i\sinh y)$$

$$\therefore \cos\theta = \cos x\cosh y \ \& \ \sin\theta = -\sin x\sinh y \quad \text{—— (1)}$$

Now. $\cosh^2 y - \sinh^2 y = 1$ this step is done in order to eliminate y

$$\left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{-\sin\theta}{\sin x}\right)^2 = 1$$

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\frac{1-\sin^2\theta}{1-\sin^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\frac{\sin^2 x - \sin^2 x\sin^2\theta - \sin^2\theta + \sin^2 x\sin^2\theta}{\sin^2 x - \sin^4 x} = 1$$

$$\sin^2 x - \sin^2\theta = \sin^2 x - \sin^4 x$$

$$\therefore \sin^4 x = \sin^2\theta$$

$$\therefore \sin^2 x = \sin\theta$$

$$\sin x = \sqrt{\sin\theta} \quad \text{—— (A)}$$

$$\therefore \boxed{x = \sin^{-1}\left(\sqrt{\sin\theta}\right)} \quad \text{—— } (2)$$

from (1) $\quad \sin\theta = -\sin x \sinh y$

$$\sin\theta = -\sqrt{\sin\theta}\,\sinh y$$

$$\sinh y = -\sqrt{\sin\theta}$$

$$\therefore y = \sinh^{-1}\left(-\sqrt{\sin\theta}\right)$$

Now $\sinh^{-1}(x) = \log\left(x + \sqrt{x^2+1}\right)$

$$\therefore y = \log\left[-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}\right]$$

$$\therefore y = \log\left(\sqrt{\sin\theta + 1} - \sqrt{\sin\theta}\right)$$

$$\therefore \cos^{-1}(\cos\theta + i\sin\theta) = x + iy$$

$$= \sin^{-1}\left(\sqrt{\sin\theta}\right) + i\log\left(\sqrt{\sin\theta+1} - \sqrt{\sin\theta}\right)$$

4. Separate into real and imaginary parts $sinh^{-1}(i\,x)$

Soln :- Let $\sinh^{-1}(ix) = \alpha + i\beta$

$$\therefore ix = \sinh(\alpha + i\beta)$$

$$= \sinh\alpha \cosh(i\beta) + \cosh\alpha \sinh(i\beta)$$

$\cosh(i\beta) = \cos\beta$

$\sinh(i\beta) = i\sin\beta$

$\cosh(i\beta) = \cos\beta$

$\sinh(i\beta) = i\sin\beta$

$ix = \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$

comparing real & imaginary parts

$\Rightarrow \sinh\alpha\cos\beta = 0$ & $\cosh\alpha\sin\beta = x$

Case-I $\sinh\alpha\cos\beta = 0$

$\Rightarrow \sinh\alpha = 0$

$\therefore \alpha = 0$

Now $\cosh\alpha\sin\beta = x$

$\therefore (1)\sin\beta = x$ $(\cosh(0) = 1)$

$\Rightarrow \beta = \sin^{-1}(x)$

$\sinh^{-1}(ix) = \alpha + i\beta$

$= i\sin^{-1}(x)$

Case-II $\sinh\alpha\cos\beta = 0$

or $\cos\beta = 0$

or $\beta = \frac{\pi}{2}$

$\cosh\alpha\sin\beta = x$

$\Rightarrow \cosh\alpha = x$ $(\sin\frac{\pi}{2} = 1)$

$\Rightarrow \alpha = \cosh^{-1}(x)$

$= \log(x + \sqrt{x^2 - 1})$

$\sinh^{-1}(ix)$

$= \log(x + \sqrt{x^2-1}) + i\frac{\pi}{2}$

5. If $\tan z = \frac{i}{2}(1-i)$, prove that $z = \frac{1}{2}tan^{-1}2 + \cancel{\frac{i}{4}log\left(\frac{1}{5}\right)}$ $\frac{i}{4}\log(5)$

Soln :- $\tan z = \frac{i}{2}(1-i) = \frac{i}{2} - \frac{i^2}{2} = \frac{1}{2} + \frac{i}{2}$

let $z = x + iy$

$$\tan(x+iy) = \frac{1}{2} + \frac{i}{2}$$

$$\Rightarrow \tan(x-iy) = \frac{1}{2} - \frac{i}{2}$$

$$\tan(2x) = \tan[(x+iy) + (x-iy)]$$

$$= \frac{\tan(x+iy) + \tan(x-iy)}{1 - \tan(x+iy)\tan(x-iy)}$$

$$= \frac{\left(\frac{1}{2}+\frac{i}{2}\right) + \left(\frac{1}{2}-\frac{i}{2}\right)}{1-\left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right)} = \frac{1}{1-\left(\frac{1}{4}+\frac{1}{4}\right)}$$

$$\therefore \tan 2x = 2$$

$$\Rightarrow x = \frac{1}{2}\tan^{-1}(2)$$

Also, $\tan(2iy) = \tan[(x+iy) - (x-iy)]$

$$= \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$\tan(2iy) = \frac{\left(\frac{1}{2}+\frac{i}{2}\right) - \left(\frac{1}{2}-\frac{i}{2}\right)}{1+\left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{1}{2}-\frac{i}{2}\right)} = \frac{i}{1+\left(\frac{1}{2}\right)}$$

$(\tan(ix) = i\tanh x)$

$$\therefore \quad i\tanh(2y) = \frac{2i}{3}$$

$$\therefore \quad \tanh(2y) = \frac{2}{3} \Rightarrow 2y = \tanh^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \quad 2y = \frac{1}{2}\log\left(\frac{1+2/3}{1-2/3}\right) = \frac{1}{2}\log 5$$

$$\therefore \quad y = \frac{1}{4}\log 5$$

$$\therefore \quad z = x+iy = \frac{1}{2}\tan^{-1}(2) + \frac{i}{4}\log 5.$$

6. Show that $tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \frac{i}{2}log\frac{x}{a}$

Soln:- let $\tan^{-1}\left[i\left(\frac{x-a}{x+a}\right)\right] = \theta$

$$\therefore \quad i\left(\frac{x-a}{x+a}\right) = \tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$$

$$\left(\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$

$$\therefore \quad \frac{x-a}{x+a} = \frac{e^{i\theta} - e^{-i\theta}}{i^2(e^{i\theta} + e^{-i\theta})}$$

$$e^{-i\theta} - e^{i\theta} \qquad (\text{using } i^2 = -1)$$

$$\frac{x-a}{x+a} = \frac{e^{-i\theta} - e^{i\theta}}{e^{i\theta} + e^{-i\theta}} \quad (\text{using } i^2 = -1)$$

using componendo dividendo

$$\frac{(x-a)+(x+a)}{(x-a)-(x+a)} = \frac{(e^{-i\theta} - e^{i\theta}) + (e^{i\theta} + e^{-i\theta})}{(e^{-i\theta} - e^{i\theta}) - (e^{i\theta} + e^{-i\theta})}$$

$$\frac{2x}{-2a} = \frac{2e^{-i\theta}}{-2e^{i\theta}}$$

$$\frac{x}{a} = e^{-2i\theta}$$

$$\therefore \; -2i\theta = \log\left(\frac{x}{a}\right)$$

$$\therefore \; \theta = \frac{-1}{2i} \log\left(\frac{x}{a}\right)$$

$$\theta = \frac{-i}{2i^2} \log\left(\frac{x}{a}\right)$$

$$\theta = \frac{i}{2} \log\left(\frac{x}{a}\right)$$

LOGARITHMS OF COMPLEX NUMBERS

Tuesday, November 9, 2021 2:44 PM

Let $z = x + iy$ and also let $x = r\cos\theta, y = r\sin\theta$ so that $r = \sqrt{x^2+y^2}$ and $\theta = tan^{-1}(y/x)$.

Hence, $\log z = \log(r(\cos\theta + i\sin\theta)) = \log(r.e^{i\theta})$

$= \log r + \log e^{i\theta} = \log r + i\theta$

$\therefore \log(x+iy) = \log r + i\theta$

$\therefore \log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$ .... (1)

This is called **principal value** of log (x + iy)

The general value of log (x + iy) is denoted by Log (x + iy) and is given by

$\therefore \mathrm{Log}\,(x+iy) = 2n\pi i + \log(x+iy)$

$\therefore \mathrm{Log}\,(x+iy) = 2n\pi i + \frac{1}{2}\log(x^2+y^2) + i\,tan^{-1}\frac{y}{x}$

$\mathrm{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i(2n\pi + tan^{-1}\frac{y}{x})$ .... (2)

Caution: $\theta = tan^{-1} y/x$ only when x and y are both positive.

In any other case θ is to be determined from $x = r\cos\theta, y = r\sin\theta, -\pi \le \theta \le \pi$. -pi<theta<=pi

SOME SOLVED EXAMPLES:

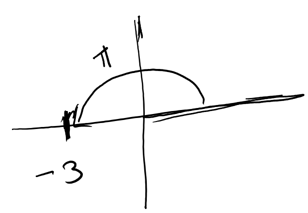
1. Considering the principal value only prove that $\log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$

Soln:- $\log_2(-3) = \frac{\log(-3)}{\log(2)}$

$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$\log(-3) = \frac{1}{2}\log(9+0) + i\tan^{-1}\left(\frac{0}{-3}\right)$

$= \frac{1}{2}\log(9) + i(\pi)$



$\log(-3) = \log 3 + i\pi$

$\therefore \log_2(-3) = \frac{\log 3 + i\pi}{\log 2}$

2. Find the general value of $\mathrm{Log}(1+i) + \mathrm{Log}\,(1-i)$

$\mathrm{Log}\,(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left(\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right)$

$$\operatorname{Log}(1+i) = \frac{1}{2}\log(1^2+1^2) + i\left[\tan^{-1}\left(\frac{1}{1}\right) + 2n\pi\right]$$

$$\operatorname{Log}(1+i) = \frac{1}{2}\log(2) + i\left(\frac{\pi}{4} + 2n\pi\right)$$

$$\operatorname{Log}(1-i) = \frac{1}{2}\log(2) - i\left(\frac{\pi}{4} + 2n\pi\right)$$

$$\operatorname{Log}(1+i) + \operatorname{Log}(1-i) = \log(2)$$

3. Prove that $\log\left(1 + e^{2i\theta}\right) = \log(2\cos\theta) + i\theta$

Soln :-

$$\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$$

$$= \log(2\cos^2\theta + i\,2\sin\theta\cos\theta)$$

$$= \log[2\cos\theta(\cos\theta + i\sin\theta)]$$

$$= \log(2\cos\theta) + \log(\cos\theta + i\sin\theta)$$

$$= \log(2\cos\theta) + \log(e^{i\theta})$$

$$= \log(2\cos\theta) + i\theta$$

Another method :

$$\log(\underbrace{1+\cos 2\theta}_{x} + i\underbrace{\sin 2\theta}_{y}) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

4. Find the value of $\log\left[\sin(x+iy)\right]$

Soln :- $\log[\sin(x+iy)]$

$$= \log[\sin x \cos iy + \cos x \sin iy]$$

$\cos iy = \cosh y$ & $\sin iy = i \sinh y$

$$= \log[\sin x \cosh y + i \cos x \sinh y]$$

$$= \log(a+ib)$$

$$= \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$$

$$= \frac{1}{2}\log\left[\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y\right] + i\tan^{-1}\left(\frac{\cos x \sinh y}{\sin x \cosh y}\right)$$

Now $\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$

$$= (1-\cos^2 x)\cosh^2 y + \cos^2 x(\cosh^2 y - 1)$$

$$= \cosh^2 y - \cos^2 x \cosh^2 y + \cos^2 x \cosh^2 y - \cos^2 x$$

$$= \cosh^2 y - \cos^2 x$$

$$\therefore \log[\sin(x+iy)] = \frac{1}{2}\log(\cosh^2 y - \cos^2 x) + i\tan^{-1}(\cot x \tanh y)$$

5. Show that $\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2ab}{a^2-b^2}$

Soln:- $\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}\left(\frac{b}{a}\right)$

$$\log(a-ib) = \frac{1}{2}\log(a^2+b^2) - i\tan^{-1}\left(\frac{b}{a}\right)$$

$$\log\left(\frac{a-ib}{a+ib}\right) = \log(a-ib) - \log(a+ib)$$

$$\log\left(\frac{a-ib}{a+ib}\right) = \log(a-ib) - \log(a+ib)$$

$$\log\left(\frac{a-ib}{a+ib}\right) = -2i\tan^{-1}\left(\frac{b}{a}\right)$$

$$i\log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}\left(\frac{b}{a}\right)$$

$$\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \tan\left[2\tan^{-1}\left(\frac{b}{a}\right)\right]$$

Let $\tan^{-1}\left(\frac{b}{a}\right) = \theta \Rightarrow \tan\theta = \frac{b}{a}$

$$= \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\tan\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{2b/a}{1-(b/a)^2} = \frac{2ab}{a^2-b^2}$$

6. Prove that $\cos\left[i\log\left(\frac{a-ib}{a+ib}\right)\right] = \frac{a^2-b^2}{a^2+b^2}$ H.W.

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7. Separate into real and imaginary parts $\sqrt{i}^{\sqrt{i}}$

Soln :- we have $\sqrt{i} = (i)^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Also, $\sqrt{i} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = \left(e^{i\pi/2}\right)^{1/2} = e^{i\pi/4}$

$$\sqrt{i}^{\sqrt{i}} = \left(e^{i\pi/4}\right)^{\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}} = e^{i\frac{\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}}$$

$$= e^{-\frac{\pi}{4\sqrt{2}}}\left[e^{i\frac{\pi}{4\sqrt{2}}}\right]$$

$-\pi$

$$\sqrt{i}^{\sqrt{i}} = e^{\overline{4\sqrt{2}}}\left[\cos\left(\frac{\pi}{4\sqrt{2}}\right) + i\sin\left(\frac{\pi}{4\sqrt{2}}\right)\right]$$

Real part $= e^{-\frac{\pi}{4\sqrt{2}}}\cos\left(\frac{\pi}{4\sqrt{2}}\right)$

Imaginary part $= e^{-\pi/4\sqrt{2}}\sin\left(\frac{\pi}{4\sqrt{2}}\right)$

8. Find the principal value of $(1+i)^{1-i}$

Soln :- Let $z = (1+i)^{1-i}$

Taking log on both sides

$$\log z = (1-i)\log(1+i)$$

$$\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$\log z = (1-i)\left[\frac{1}{2}\log(1^2+1^2) + i\tan^{-1}\left(\frac{1}{1}\right)\right]$$

$$= (1-i)\left[\frac{1}{2}\log 2 + i\frac{\pi}{4}\right]$$

$$\log z = \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) = x+iy \text{ (say)}$$

$$\therefore z = e^{x+iy} = e^x \cdot e^{iy}$$

$$= e^x\left[\cos y + i\sin y\right]$$

$$(1+i)^{(1-i)} = z = e^{\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right)}\left[\cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i\sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right)\right]$$

9. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is $e^{2m\pi+\alpha}[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)]$

Soln :- Let $z = (1+i\tan\alpha)^{-i}$

Taking Logarithm on both sides

Taking Logarithm on both sides

$$\operatorname{Log} z = (-i) \operatorname{Log}(1+i\tan\alpha)$$

$$\operatorname{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\left[2m\pi + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\operatorname{Log} z = (-i)\left[\frac{1}{2}\log(1^2+\tan^2\alpha) + i\left[2m\pi + \tan^{-1}\left(\frac{\tan\alpha}{1}\right)\right]\right]$$

$$= (-i)\left[\frac{1}{2}\log(1+\tan^2\alpha) + i(2m\pi+\alpha)\right]$$

$$= (-i)\left[\frac{1}{2}\log(\sec^2\alpha) + i(2m\pi+\alpha)\right]$$

$$= (-i)\left[\log(\sec\alpha) + i(2m\pi+\alpha)\right]$$

$$= (2m\pi+\alpha) - i\log(\sec\alpha)$$

$$\operatorname{Log} z = (2m\pi+\alpha) + i\log(\cos\alpha) = x+iy \ (\text{say})$$

$$z = e^{x+iy} = e^x \cdot e^{iy} = e^x\left[\cos y + i\sin y\right]$$

$$= e^{(2m\pi+\alpha)}\left[\cos(\log\cos\alpha) + i\sin(\log\cos\alpha)\right]$$

10. Considering only principal value, if $(1+i\tan\alpha)^{1+i\tan\beta}$ is real, prove that its value is $(\sec\alpha)^{sec^2\beta}$

Soln:- let $z = (1+i\tan\alpha)^{1+i\tan\beta}$

$$\log z = (1+i\tan\beta)\log(1+i\tan\alpha)$$

$$= (1+i\tan\beta)\left[\frac{1}{2}\log(1+\tan^2\alpha) + i\tan^{-1}\left(\frac{\tan\alpha}{1}\right)\right]$$

$$= (1+i\tan\beta)\left[\log(\sec\alpha)+i\alpha\right]$$

$$\log Z = \left(\log(\sec\alpha) - \alpha\tan\beta\right) + i\left(\alpha + \tan\beta\log(\sec\alpha)\right)$$

$$= x+iy \text{ (say)}$$

where $x = \log(\sec\alpha) - \alpha\tan\beta$, $y = \alpha + \tan\beta\log(\sec\alpha)$ — ①

$$Z = e^{x+iy} = e^{x}\cdot e^{iy} = e^{x}\left[\cos y + i\sin y\right]$$

$$Z = e^{x}\cos y + ie^{x}\sin y$$

Now, it is given that Z is real

$$\Rightarrow e^{x}\sin y = 0 \Rightarrow \sin y = 0$$

$$\Rightarrow y = 0$$

$$\Rightarrow \alpha + \tan\beta\log(\sec\alpha) = 0 \quad \text{—— ②}$$

$$\therefore Z = e^{x}\cos y = e^{x}\cos(0) = e^{x}$$

$$\therefore Z = e^{\log(\sec\alpha) - \alpha\tan\beta}$$

$$Z = e^{\log(\sec\alpha)}\cdot e^{-\alpha\tan\beta} = (\sec\alpha)\, e^{-\alpha\tan\beta} \quad \text{—— ③}$$

from ② $\Rightarrow \alpha + \tan\beta\log(\sec\alpha) = 0$

$$\Rightarrow -\alpha = \tan\beta\log(\sec\alpha)$$

$$\Rightarrow -\alpha\tan\beta = \tan^2\beta\log(\sec\alpha)$$

$$\Rightarrow -\alpha\tan\beta = \log(\sec\alpha)^{\tan^2\beta}$$

$$-\alpha\tan\beta \qquad \tan^2\beta$$

$$\Rightarrow \quad -\alpha\tan\beta = \log(\sec\alpha)$$

$$\Rightarrow \quad e^{-\alpha\tan\beta} = (\sec\alpha)^{\tan^2\beta}$$

Sub. in ③ $\quad Z = (\sec\alpha)\, e^{-\alpha\tan\beta}$

$$= (\sec\alpha)(\sec\alpha)^{\tan^2\beta}$$

$$= (\sec\alpha)^{1+\tan^2\beta}$$

$$Z = (\sec\alpha)^{\sec^2\beta}$$

11. If $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\,\beta$, find α and β

Soln: $\frac{(a+ib)^{x+iy}}{(a-ib)^{x-iy}} = \alpha + i\beta$

Taking log on both sides

$$\log(\alpha+i\beta) = (x+iy)\log(a+ib) - (x-iy)\log(a-ib)$$

H.W.

12. If $i^{\alpha+i\beta} = \alpha + i\beta$ $\left(\text{or } i^{i^{i\cdots\cdots\infty}} = \alpha + i\beta\right)$, prove that $\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$ Where n is any positive integer

$$(\alpha+i\beta) = i^{(\alpha+i\beta)}$$

$$\log(\alpha+i\beta) = (\alpha+i\beta)\log(i)$$

$$= (\alpha+i\beta)\left[\frac{1}{2}\log(0^2+1^2) + i\left(\tan^{-1}\left(\frac{1}{0}\right)+2n\pi\right)\right]$$

$$= (\alpha+i\beta)\left[\frac{1}{2}\log(1) + i\left(\frac{\pi}{2}+2n\pi\right)\right]$$

$$= (\alpha+i\beta)\left[i\left(\frac{\pi}{2}+2n\pi\right)\right]$$

$$\log(\alpha+i\beta) = -\left(\frac{\pi}{2}+2n\pi\right)\beta + i\left(\frac{\pi}{2}+2n\pi\right)\alpha$$

$$(\alpha+i\beta) = e^{-\left(\frac{\pi}{2}+2n\pi\right)\beta} \cdot e^{i\left(\frac{\pi}{2}+2n\pi\right)\alpha}$$

$$\alpha+i\beta = e^{-\left(\frac{\pi}{2}+2n\pi\right)\beta}\left[\cos\left(\frac{\pi}{2}+2n\pi\right)\alpha + i\sin\left(\frac{\pi}{2}+2n\pi\right)\alpha\right]$$

$$\Rightarrow \alpha = e^{-\left(\frac{\pi}{2}+2n\pi\right)\beta}\cos\left(\frac{\pi}{2}+2n\pi\right)\alpha$$

$$\beta = e^{-\left(\frac{\pi}{2}+2n\pi\right)\beta}\sin\left(\frac{\pi}{2}+2n\pi\right)\alpha$$

squaring & adding

$$\alpha^2+\beta^2 = e^{-2\left(\frac{\pi}{2}+2n\pi\right)\beta}\left[\cos^2\left(\frac{\pi}{2}+2n\pi\right)\alpha + \sin^2\left(\frac{\pi}{2}+2n\pi\right)\alpha\right]$$

$$= e^{-(\pi+4n\pi)\beta}$$

$$\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$$

13. Prove that $\log tan\left(\frac{\pi}{4}+i\frac{x}{2}\right) = i\, tan^{-1}(\sinh x)$.

Sol^n:- $\log\tan\left(\frac{\pi}{4}+i\frac{x}{2}\right)$

$$= \log\left[\frac{\tan\frac{\pi}{4}+\tan\left(i\frac{x}{2}\right)}{1-\tan\frac{\pi}{4}\tan\left(i\frac{x}{2}\right)}\right]$$

$$= \log\left[\frac{1+\tan\left(i\frac{x}{2}\right)}{1-\tan\left(i\frac{x}{2}\right)}\right]$$

$$= \log\left[\frac{1+\tan(i\frac{x}{2})}{1-\tan(i\frac{x}{2})}\right]$$

$$\tan(i\alpha) = i\tanh(\alpha)$$

$$= \log\left[\frac{1+i\tanh(\frac{x}{2})}{1-i\tanh(\frac{x}{2})}\right]$$

$$= \log\left[1+i\tanh\left(\frac{x}{2}\right)\right] - \log\left[1-i\tanh\left(\frac{x}{2}\right)\right]$$

$$= \frac{1}{2}\log\left[1^2+\tanh^2\left(\frac{x}{2}\right)\right] + i\tan^{-1}\tanh\left(\frac{x}{2}\right)$$

$$- \left[\frac{1}{2}\log\left(1^2+\tanh^2\left(\frac{x}{2}\right)\right) - i\tan^{-1}\tanh\left(\frac{x}{2}\right)\right]$$

$$= 2i\tan^{-1}\tanh\left(\frac{x}{2}\right) \qquad \left(2\tan^{-1}\alpha = \tan^{-1}\left(\frac{2\alpha}{1-\alpha^2}\right)\right)$$

$$= i\tan^{-1}\left[\frac{2\tanh(\frac{x}{2})}{1-\tanh^2(\frac{x}{2})}\right]$$

$$= i\tan^{-1}(\sinh x)$$

<u>circuits</u>

$i \longrightarrow$ current

$\alpha + i\omega$

$\alpha + j\omega$