HYPERBOLIC FUNCTIONS

SOME SOLVED EXAMPLES:

1. If $\tanh x = \frac{1}{2}, find \sinh 2x\ and \cosh 2x$ -

Solution: $\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{2}$

$\therefore \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2} \quad \therefore 2\,e^{2x} - 2 = e^{2x} + 1 \quad \therefore\ e^{2x} = 3$

Now, $sin\ h2x = \frac{e^{2x} - e^{-2x}}{2} = \frac{3-(1/3)}{2} = \frac{4}{3}$ Now, $\cos h2x = \frac{e^{2x}+e^{-2x}}{2} = \frac{3+(1/3)}{2} = \frac{5}{3}$

2. Solve the equation $7cosh\ x + 8sinh\ x = 1$ for real values of x.

Solution: $7cosh\ x + 8sinh\ x = 1$

Putting the values of $coshx$ and $sin\ hx$, we get

$\therefore 7\left(\frac{e^x+e^{-x}}{2}\right) + 8\left(\frac{e^x-e^{-x}}{2}\right) = 1$

$\therefore 7e^x + 7e^{-x} + 8e^x - 8e^{-x} = 2 \qquad \therefore 15e^x - e^{-x} = 2$

$\therefore 15e^{2x} - 2e^x - 1 = 0$ Solving it as a quadratic equation in e^x,

$e^x = \frac{2\pm\sqrt{4-4(15)(-1)}}{2(15)} = \frac{2\pm 8}{30} = \frac{1}{3}\ \ or - \frac{1}{5}$

$\therefore x = \log\left(\frac{1}{3}\right)\ \ or\ x = \log\left(-\frac{1}{5}\right)$

Since x is real, $x = log\left(\frac{1}{3}\right) = -\log 3$

3. If $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x$ then prove that $x = a\sqrt{1+b^2} + b\sqrt{1+a^2}$

Solution: Let $\sin h^{-1}\ a = \ \alpha,\ \sin h^{-1}\ b = \ \beta\ \ and\ \ \sin h^{-1}\ x = \ \gamma$

We are given $sinh^{-1}a + sinh^{-1}b = sinh^{-1}x \qquad \therefore \alpha + \beta = \gamma$

$\therefore \sinh(\alpha+\beta) = \sinh\gamma$

$\therefore \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta = \sinh\gamma$ ....(A)

But $\sinh\alpha$ = a, $\sinh\beta = b,\ \sinh\gamma = x$

$\therefore \cos h\ \alpha = \sqrt{1+\sin h^2\alpha} = \sqrt{1+a^2}$ and $\cos h\ \beta = \sqrt{1+\sin h^2\beta} = \sqrt{1+b^2}$

Putting this values in (A), we get $a\sqrt{1+b^2} + b\sqrt{1+a^2} = x$

4. Prove that $16\ sinh^5 x = \sinh 5x - 5\sinh 3x + 10\sinh x$

Solution: LHS$= 16\sin^5 x \qquad = 16\left(\frac{e^x-e^{-x}}{2}\right)^5$

$= \frac{16}{32}\left(e^{5x} - 5e^{4x}e^{-x} + 10e^{3x}e^{-2x} - 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} - e^{-5x}\right)$

$= \frac{1}{2}\left(e^{5x} - 5e^{3x} + 10e^{x} - 10e^{-x} + 5e^{-3x} - e^{-5x}\right)$

using binomial expansion

$$= \left(\frac{e^{5x}-e^{-5x}}{2}\right) - 5\left(\frac{e^{3x}+e^{-3x}}{2}\right) + 10\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$$= \sinh 5x - 5\sinh 3x + 10\sinh x \quad = \text{RHS}$$

5. Prove that $16cosh^5 x = \cosh 5x + 5\cosh 3x + 10\cosh x$

Solution: $l.h.s = 16cosh^5 x \qquad = 16\left(\frac{e^x+e^{-x}}{2}\right)^5$ [By Binomial Theorem]

$$= \frac{16}{32}[e^{5x} + 5e^{4x}\, e^{-x} + 10e^{3x}e^{-2x} + 10e^{2x}e^{-3x} + 5e^{x}e^{-4x} + e^{-5x}]$$

$$= \frac{(e^{5x}+e^{-5x})}{2} + 5\frac{(e^{3x}+e^{-3x})}{2} + 10\frac{(e^{x}+e^{-x})}{2}$$

$$= \cos h\, 5\, x + 5\cos h\, 3\, x + 10\cos h\, x = r\,.h.s$$

6. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1-cosh^2 x}}} = cosh^2 x$

Solution: $l.h.s = \frac{1}{1-\frac{1}{1-\frac{1}{-sinh^2 x}}} = \frac{1}{1-\frac{1}{1+cosec\, h^2 x}} = \frac{1}{1-\frac{1}{\cot h^2 x}} = \frac{1}{1-\tan h^2 x}$

$$= \frac{1}{1-\frac{sinh^2 x}{\cos h^2 x}} = \frac{\cos h^2 x}{\cos h^2 x - \sin h^2 x} = \cos h^2 x$$

7. If $u = \log tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, Prove that

(i) $\cosh u = \sec\theta$ **(ii)** $\sinh u = \tan\theta$ **(iii)** $\tanh u = \sin\theta$

(iv) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

Solution: **(i)** $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

$$\therefore e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$\therefore e^{-u} = \frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$\therefore \cosh u = \frac{e^u+e^{-u}}{2}$$

$$= \frac{1}{2}\left[\frac{(1+2\tan\theta/2+\tan^2\theta/2)+(1-2\tan\theta/2+\tan^2\theta/2)}{1-\tan^2\theta/2}\right]$$

$$= \frac{1}{2}\left(\frac{2+2\tan^2\theta/2}{1-\tan^2\theta/2}\right) \qquad = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2} = \frac{1}{\cos\theta} = \sec\theta$$

(ii) $\sinh u = \sqrt{\cosh^2 u - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

(iii) $\tanh u = \frac{\sinh u}{\cosh u} = \frac{\tan\theta}{\sec\theta} = \frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta$

(iv) $\tan h\left(\frac{u}{2}\right) = \frac{\sin h(u/2)}{\cos h(u/2)} \quad = \frac{2\sin h(u/2).\cos h(u/2)}{2\cos h(u/2)\cos h(u/2)} \quad = \frac{\sin hu}{1+\cos hu} = \frac{\tan\theta}{1+\sec\theta}$

(By (i) and (ii))

$$\therefore \tan h\left(\frac{u}{2}\right) = \frac{sin\theta/cos\theta}{(\cos\theta+1)/\cos\theta} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)} = \frac{sin(\theta/2)}{\cos(\theta/2)} = \tan\frac{\theta}{2}$$

8. If $\cosh x = \sec\theta$, Prove that

(i) $x = \log(\sec\theta + \tan\theta)$ **(ii)** $\theta = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$ **(iii)** $\tanh\frac{x}{2} = tan\frac{\theta}{2}$

Solution: **(i)** $\cosh x = \sec\theta$

$\therefore \frac{e^x+e^{-x}}{2} = \sec\theta$ By definition $\cos hx = \frac{e^x+e^{-x}}{2}$

$\therefore e^x - 2\sec\theta + e^{-x} = 0$ $\therefore (e^x)^2 - 2e^x sec\theta + 1 = 0$

Solving the quadratic in e^x,

$e^x = sec\theta \pm \sqrt{sec^2\theta - 1} = \sec\theta \pm \tan\theta$

$\therefore x = \log(\sec\theta \pm \tan\theta) = \pm log(\sec\theta + \tan\theta)$

(we can prove that $log(\sec\theta - \tan\theta) = -log(\sec\theta + \tan\theta)$)

(ii) Let $tan^{-1}e^{-x} = \alpha$ $\therefore e^{-x} = \tan\alpha$ $\therefore e^x = \cot\alpha$

Now, by data $\sec\theta = \cos hx = \frac{e^x+e^{-x}}{2} = \frac{cot\alpha+tan\alpha}{2}$

$$2\ sec\theta = cot\alpha + tan\alpha = \frac{cos\alpha}{\sin\alpha} + \frac{\sin\alpha}{cos\alpha} = \frac{2}{\sin 2\alpha}$$

$$\therefore cos\theta = \sin 2\alpha = \cos\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\therefore \theta = \frac{\pi}{2} - 2\alpha = \frac{\pi}{2} - 2tan^{-1}(e^{-x})$$

(iii) $\tan h\frac{x}{2} = \frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}} = \frac{e^x-1}{e^x+1} = \frac{sec\theta+tan\theta-1}{sec\theta+tan\theta+1} = \frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$

$$= \frac{(1-cos\theta)+\sin\theta}{(1+cos\theta)+\sin\theta} = \frac{2sin^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)}{2cos^2(\theta/2)+2\sin(\theta/2)\cos(\theta/2)} = \frac{\sin(\theta/2)}{cos(\theta/2)} = tan\frac{\theta}{2}$$

SOME PRACTICE PROBLEMS

1. If $\tanh x = 2/3$, find the value of x and then $cosh\ 2x$.

2. Solve the equation for real values of x, $17\cosh x + 18\sinh x = 1$.

3. If $6\sinh x + 2\ cosh\ x + 7 = 0$, find $tanh\ x$.

4. If $cosh^{-1}a + cosh^{-1}b = cosh^{-1}x$, then prove that

$a\sqrt{b^2-1} + b\sqrt{a^2-1} = \sqrt{x^2-1}$.

5. If $cosh^6 x = a\cosh 6x + b\cosh 4x + c\cosh 2x + d$,

Prove that $25a - 5b + 3c - 4d = 0$

6. Prove that $cosh^7 x = \frac{1}{64}[\cosh 7x + 7\cosh 5x + 21\cosh 3x + 35\cosh x]$

7. If $\cos\alpha\cosh\beta = x/2,\quad \sin\alpha\sinh\beta = y/2$, show that

(i) $\sec(\alpha - i\,\beta) + \sec(\alpha + i\,\beta) = \frac{4x}{x^2+y^2}$

(ii) $\sec(\alpha - i\,\beta) - sec(\alpha + i\,\beta) = \frac{-4iy}{x^2+y^2}$

8. Prove that $cosech\, x + \coth x = \coth\frac{x}{2}$

9. Prove that $(\cosh x + sinh\, x)^n = \cosh nx + \sinh nx$

10. Prove that $\left(\frac{\cosh x+\sinh x}{\cosh x-\sinh x}\right)^n = \cosh 2nx + \sinh 2nx$

11. If $\log\tan x = y$, prove that $\cosh ny = \frac{1}{2}[tan^n x + cot^n x]$ and

$\sinh(n+1)y + \sinh(n-1)\,y = 2\sinh ny\ \ cosec\ 2x$

12. Prove that $\frac{1}{1-\frac{1}{1-\frac{1}{1+sinh^2 x}}} = -sinh^2 x$

13. If $\cosh u = \sec\theta\, , prove\ that$

(i) $\sinh u = \tan\theta$ **(ii)** $\tanh u = \sin\theta$ **(iii)** $u = \log\left[tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$

ANSWERS

1. $\frac{1}{2}\log 5\ ,\ \frac{13}{5}$ **2.** $x = -\log 5$ **3.** $\frac{3}{5}\, , \frac{-15}{17}$