HYPERBOLIC FUNCTIONS

CIRCULAR FUNCTIONS:

From Euler's formula, we have $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

If $z = x + iy$ is complex number, then $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$

These are called circular function of complex numbers.

cos(x) is a periodic/circular function while cos(hx) is a hyperbolic function

HYPERBOLIC FUNCTIONS:

If x is real or complex, then sine hyperbolic of x is denoted by $sinh\, x$ and is given as, $\sinh x = \frac{e^{x}-e^{-x}}{2}$

and Cosine hyperbolic of x is denoted by $cosh\, x$ and is given as, $\cosh x = \frac{e^{x}+e^{-x}}{2}$

From above expressions, other hyperbolic functions can also be obtained as

$\tan \mathrm{hx} = \frac{\sinh x}{\cosh x} = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$, $\mathrm{cosechx} = \frac{1}{\sinh x} = \frac{2}{e^{x}-e^{-x}}$, $\mathrm{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^{x}+e^{-x}}$, and

$\coth x = \frac{1}{\tanh x} = \frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

TABLE OF VALUES OF HYPERBOLIC FUNCTION:

From the definitions of sinh x, cosh x, tanh x, we can obtain the following values of hyperbolic function.

x	$-\infty$	0	∞
$\sinh x$	$-\infty$	0	∞
$\cosh x$	∞	1	∞
$\tanh x$	-1	0	1

Note: since $\tanh(-\infty) = -1$, $\tanh(0) = 0$, $\tanh(\infty) = 1$

$\therefore |\tanh x| \le 1$

in case of sin the i comes out and in cos the i is absorbed

RELATION BETWEEN CIRCULAR AND HYPERBOLIC FUNCTIONS :

(i)	$\sin ix = i\sinh x$ & $\sinh x = -i\sin ix$	$\sinh ix = i\sin x$ & $\sin x = -i\sinh ix$
(ii)	$\cos ix = \cosh x$	$\cosh ix = \cos x$
(iii)	$\tan ix = i\tanh x$ & $\tanh x = -i\tan ix$	$\tanh ix = i\tan x$ & $\tan x = -i\tanh ix$

in sin^2 and sin^3 & tan^2 and tan^3 the sign changes as we go from circular functions to hyperbolic functions but in cos^2and cos^3 the sign remains the same.

the sign of all the trigo fuctions with degree 1 remain the same when we go from circular functions to hyperbolic functions

similar impact on cosec , sec and cot is seen

the cosec behaves like sin while going from circular functions to hyperbolic functions

the sec behaves like cos while going from circular functions to hyberbolic functions

cot behaves like tan while going from circular functions to hyperbolic functions .

so sign of cosec remains same
but sign of cosec^2 and cosec^3 gets negated while going from circular functions to hyperbolic functions

sign of sec , sec^2 and sec^3 remain the same while going from circular functions to hyperbolic functions

sign of cot remains the same but of cot^2 and cot^3 gets negated while going from circular functions to hyperbolic functions

sign remains the same while going from circular functions to hyperbolic functions :

all trigo functions with degree 1
cos^2
cos^3
sec^2
sec^3

sign gets negated while going from circular functions to hyperbolic functions :

sin^2
sin^3
tan^2
tan^3
cosec^2
cosec^3
cot^2
cot^3

Eg sinx(siny)

in case of two sin being adjacent to each other in a formula
the sign (while going from circular functions to hyperbolic functions) gets negated

<u>since two sin are adjacent to each other , they behave like sin^2 (in terms of sign change while going from circular functions to hyperbolic functions) (*1)</u>

similarly two tan adjacent to each other behave like tan^2 (in terms of sign change while going from circular functions to hyperbolic functions)

FORMULAE ON HYPERBOLIC FUNCTIONS :

	CIRCULAR FUNCTIONS	**HYPERBOLIC FUNCTIONS**
1	$\sin(-x) = -(\sin x)$	$\sinh(-\mathrm{x}) = -\sinh \mathrm{x},$
2	$\cos(-x) = (cosx)$	$\cosh(-\mathrm{x}) = \cosh \mathrm{x}$
3	$e^{ix} = \cos x + i \sin x$	$e^{x} = \cosh x + \sinh x$
4	$e^{-ix} = \cos x - i \sin x$	$e^{-x} = \cosh x - \sinh x$
5	$sin^2x + cos^2x = 1$	$\cosh^2\mathrm{x} - \sinh^2\mathrm{x} = 1$
6	$1 + \tan^2\mathrm{x} = \sec^2\mathrm{x}$	$\mathrm{sech}^2\mathrm{x} + \tanh^2\mathrm{x} = 1$
7	$1 + \cot^2\mathrm{x} = \mathrm{cosec}^2\mathrm{x}$	$\coth^2\mathrm{x} - \mathrm{cosech}^2\mathrm{x} = 1$
8	$\sin 2\mathrm{x} = 2 \sin x \cos x$ $= \frac{2 \tan \mathrm{x}}{1 + \tan^2 \mathrm{x}}$	$\sinh 2\mathrm{x} = 2 \sinh x \cosh x$ $= \frac{2 \tanh \mathrm{x}}{1 - \tanh^2 \mathrm{x}}$
9	$\cos 2\mathrm{x} = \cos^2\mathrm{x} - \sin^2\mathrm{x}$ $= 2\cos^2\mathrm{x} - 1$ $= 1 - 2\sin^2\mathrm{x}$ $= \frac{1 - \tan^2 \mathrm{x}}{1 + \tan^2 \mathrm{x}}$	$\cosh 2\mathrm{x} = \cosh^2\mathrm{x} + \sinh^2\mathrm{x}$ $= 2\cosh^2\mathrm{x} - 1$ $= 1 + 2\sinh^2\mathrm{x}$ $= \frac{1 + \tanh^2 \mathrm{x}}{1 - \tanh^2 \mathrm{x}}$
10	$\tan 2\mathrm{x} = \frac{2 \tan \mathrm{x}}{1 - \tan^2 \mathrm{x}}$	$\tanh 2\mathrm{x} = \frac{2 \tanh \mathrm{x}}{1 + \tanh^2 \mathrm{x}}$
11	$\sin 3\mathrm{x} = 3 \sin \mathrm{x} - 4 \sin^3\mathrm{x}$	$\sinh 3\mathrm{x} = 3 \sinh \mathrm{x} + 4 \sinh^3\mathrm{x}$
12	$\cos 3\mathrm{x} = 4\cos^3\mathrm{x} - 3 \cos \mathrm{x}$	$\cosh 3\mathrm{x} = 4\cosh^3\mathrm{x} - 3 \cosh \mathrm{x}$
13	$\tan 3\mathrm{x} = \frac{3 \tan \mathrm{x} - \tan^3\mathrm{x}}{1 - 3 \tan^2\mathrm{x}}$	$\tanh 3\mathrm{x} = \frac{3 \tanh \mathrm{x} + \tanh^3\mathrm{x}}{1 + 3 \tanh^2\mathrm{x}}$
14	$\sin(x \pm y) = \sin x \cos y \pm \cos x \, sin \, y$	$\sinh(\mathrm{x} \pm \mathrm{y}) = \sinh \mathrm{x} \cosh \mathrm{y} \pm \cosh \mathrm{x} \sinh \mathrm{y}$
*1 **15**	$\cos(x \pm y) = \cos x \cos y \mp \sin x \, sin \, y$	$\cosh(\mathrm{x} \pm \mathrm{y}) = \cosh \mathrm{x} \cosh \mathrm{y} \pm \sinh \mathrm{x} \sinh \mathrm{y}$
16	$\tan(\mathrm{x} \pm \mathrm{y}) = \frac{\tan \mathrm{x} \pm \tan \mathrm{y}}{1 \mp \tan \mathrm{x} \tanh \mathrm{y}}$	$\tanh(\mathrm{x} \pm \mathrm{y}) = \frac{\tanh \mathrm{x} \pm \tanh \mathrm{y}}{1 \pm \tanh \mathrm{x} \tanh \mathrm{y}}$
17	$\cot(\mathrm{x} \pm \mathrm{y}) = \frac{\cot \mathrm{x} \cot \mathrm{y} \mp 1}{\cot \mathrm{y} \pm \cot \mathrm{x}}$	$\coth(\mathrm{x} \pm \mathrm{y}) = \frac{-\coth \mathrm{x} \coth \mathrm{y} \mp 1}{\coth \mathrm{y} \pm \coth \mathrm{x}}$
18	$\sin \mathrm{x} + \sin \mathrm{y} = 2 \sin\left(\frac{\mathrm{x}+\mathrm{y}}{2}\right) \cos\left(\frac{\mathrm{x}-\mathrm{y}}{2}\right)$	$\sinh \mathrm{x} + \sinh \mathrm{y} = 2 \sinh \frac{\mathrm{x}+\mathrm{y}}{2} \cosh \frac{\mathrm{x}-\mathrm{y}}{2}$
19	$\sin \mathrm{x} - \sin \mathrm{y} = 2 \cos\left(\frac{\mathrm{x}+\mathrm{y}}{2}\right) \sin\left(\frac{\mathrm{x}-\mathrm{y}}{2}\right)$	$\sinh \mathrm{x} - \sinh \mathrm{y} = 2 \cosh \frac{\mathrm{x}+\mathrm{y}}{2} \sinh \frac{\mathrm{x}-\mathrm{y}}{2}$

20	$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$	$\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$
21	$\cos x - \cos y$ $= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$	$\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$
22	$2\sin x\cos y = \sin(x+y) + \sin(x-y)$	$2\sinh x\cosh y = \sinh(x+y) + \sinh(x-y)$
23	$2\cos x\sin y = \sin(x+y) - \sin(x-y)$	$2\cosh x\sinh y = \sinh(x+y) - \sinh(x-y)$
24	$2\cos x\cos y = \cos(x+y) + \cos(x-y)$	$2\cosh x\cosh y$ $= \cosh(x+y) + \cosh(x-y)$
25	$2\sin x\sin y = \cos(x-y) - \cos(x+y)$	$2\sinh x\sinh y = \cos h(x+y)$ $- \cos h(x-y)$

-2(sinhx)(sinhy)= cosh(x-y) - cosh(x+y)

PERIOD OF HYPERBOLIC FUNTIONS:

$$\begin{aligned}\sinh(2\pi i + x) &= \sinh(2\pi i)\cosh x + \cosh(2\pi i)\sinh x\\ &= i\sin 2\pi\ \cosh x + \cos 2\pi\ \sinh x\\ &= 0 + \sinh x\\ &= \sinh x\end{aligned}$$

this periodicity is not visible in the 2D plane but is visible in the complex 3D plane

Hence $sinh\ x$ is a periodic function of period 2πi

Similarly we can prove that $cosh\ x$ and $tanh\ x$ are periodic functions of period 2πi and πi .

sinhx and coshx ===> 2πi
tanhx ===> πi

DIFFERENTIATION AND INTRGRATION :

(i) If $y = \sinh x$, $y = \frac{e^x - e^{-x}}{2}$ $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$

If $y = \sinh x$, $\frac{dy}{dx} = \cosh x$

(ii) If $y = cosh\ x$, $y = \frac{e^x + e^{-x}}{2}$, $\quad \therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^x - e^{-x}}{2} = \sinh x$

If $y = cosh\ x$, $\frac{dy}{dx} = \sinh x$

(iii) If $y = \tanh x$, $y = \frac{\sinh x}{\cosh x}$

$$\therefore \frac{dy}{dx} = \frac{\cosh x \cdot \cosh x - \sinh x \cdot \sinh x}{cosh^2 x} = \frac{1}{cosh^2 x} = sech^2 x$$

If $y = \tanh x$, $\frac{dy}{dx} = sech^2 x$

Hence, we get the following three results

$\int \cosh x\ dx = \sinh x$, $\quad \int \sinh x\ dx = \cosh x$, $\quad \int sech^2 x dx = \tanh x$

cos and sec mein negation , baki sab mein as it is

Derivative list :-

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cos x) = -\sin x$ $\qquad$ $\frac{d}{dx}(\cosh x) = \sinh x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ $\qquad$ $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\qquad$ $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ $\qquad$ $\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

In sums Always assume log to the base e i.e ln if not specified , log base 2 or log base 10 etc.... will be explicitly specified

Note:

$16 \sinh^5 7x = 16 (\sinh 7x)^5$

$\sinh^{-1} a = \alpha \cdot \quad a = \sinh \alpha$

refer graphs of hyperbolic functions from video lectures proof for the claim in Q8 part i

$1-\sin^2 x = \cos^2 x$. $\therefore \dfrac{1-\sin x}{\cos x} = \dfrac{\cos x}{1+\sin x}$

$$\frac{1}{\cos x} - \frac{\sin x}{\cos x} = \left(\frac{1+\sin x}{\cos x}\right)^{-1} = \left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)^{-1}$$

$$\sec x - \tan x = (\sec x + \tan x)^{-1}$$

take log both sides.

$$\ln(\sec x - \tan x) = \ln(\sec x + \tan x)^{-1}$$

$\therefore$ we claim that :-

$$\ln(\sec x - \tan x) = -\ln(\sec x + \tan x)$$