Applications of De-Moivre's Theorem

ROOTS OF ALGEBRAIC EQUATIONS:

De Moivre's theorem can be used to find the roots of an algebraic equation.

General values of $\cos\theta = \cos(2k\pi + \theta)$ and $\sin\theta = \sin(2k\pi + \theta)$ where k is an integer.

To solve the equation of the type $z^n = \cos\ \theta + i\sin\ \theta$, we apply De Moivre's theorem

$$z = (\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n}$$

This shows that $\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the n roots of $z^n = \cos\theta + i\sin\theta$.

The other roots are obtain by expressing the number in the general form

$$z = \{\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)\}^{\frac{1}{n}} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

Taking k = 0, 1, 2,……………,(n – 1). We get n roots of the equation.

Note: (i) Complex roots always occur in conjugate pair if coefficients of different powers of x including constant terms in the equation are real.

(ii) Continued products mean products of all the roots of the equation.

SOME SOLVED EXAMPLES:

1. If ω is a cube root of unity, prove that $(1-\omega)^6 = -27$

Solution: Consider $x^3 = 1 \quad \therefore x = 1^{1/3}$

$$\therefore x = (\cos 0 + i\sin 0)^{1/3} = (\cos 2\,k\,\pi + i\sin 2\,k\,\pi)^{1/3} = \cos\frac{2\,k\,\pi}{3} + i\sin\frac{2\,k\,\pi}{3}$$

Putting $k = 0, 1, 2$, the cube roots of unity are

$x_0 = 1,\ x_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$ (say)

And $x_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]^2 = \omega^2$

$$\text{Now, } 1 + \omega + \omega^2 = 1 + \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - 1 = 0$$

$\therefore 1 + \omega^2 = -\omega$

Now, $(1-\omega)^6 = [(1-\omega)^2]^3 = (1 - 2\omega + \omega^2)^3$

$$= (-\omega - 2\omega)^3 = (-3\omega)^3 - 27\omega^3 = -27$$

For determining the values of K:

Always check the root, not the power

$$Z^{m/n} = (\cos\theta + i\sin\theta)$$

$$z = (\cos\theta + i\sin\theta)^{n/m} = \left[\cos(2K\pi + \theta) + i\sin(2K\pi + \theta)\right]^{n/m}$$

$$= \cos(2K\pi + \theta)\frac{n}{m} + i\sin(2K\pi + \theta)\frac{n}{m}$$

$$K = 0, 1, 2 \ldots (m-1)$$

n = power m = root

doubt for usage of different method

2. Find all the values of $\sqrt[3]{(1+i)/\sqrt{2}} + \sqrt[3]{(1-i)/\sqrt{2}}$

Solution: $\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^{1/3}$

$$= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3} \qquad = \left[\cos\left(2k\pi + \frac{\pi}{4}\right) + i\sin\left(2k\pi + \frac{\pi}{4}\right)\right]^{1/3}$$

$$= \left[\cos\left((8k+1)\frac{\pi}{4}\right) + i\sin\left((8k+1)\frac{\pi}{4}\right)\right]^{1/3}$$

$$\sqrt[3]{\frac{(1+i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) + i\sin\left((8k+1)\frac{\pi}{12}\right)$$

Similarly, $\sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = \cos\left((8k+1)\frac{\pi}{12}\right) - i\sin\left((8k+1)\frac{\pi}{12}\right)$

$$\therefore \sqrt[3]{\frac{(1+i)}{\sqrt{2}}} + \sqrt[3]{\frac{(1-i)}{\sqrt{2}}} = 2\cos\left((8k+1)\frac{\pi}{12}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$2\cos\frac{\pi}{12}$, $2\cos\frac{9\pi}{12}$, $2\cos\frac{17\pi}{12}$ i.e., $2\cos\frac{r\pi}{12}$ where $r = 1, 9, 17$

imp **3.** Find the cube roots of $(1 - cos\theta - i\ sin\theta)$.

Solution: $(1-\cos\theta - i\sin\theta)^{1/3} = \left[2\sin^2\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right]^{1/3}$

$$= \left[2\sin\left(\frac{\theta}{2}\right)\left(2\sin\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\right)\right]^{1/3} = \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) - i\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(2k\pi - \left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right) + i\sin\left(2k\pi - \left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)\right]^{1/3}$$

$$= \left(2\sin\left(\frac{\theta}{2}\right)\right)^{1/3}\left[\cos\left(\frac{(4k-1)+\theta}{6}\right) + i\sin\left(\frac{(4k-1)+\theta}{6}\right)\right]$$

Putting $k = 0, 1, 2$ we get the three roots

pi

4. Find the continued product of all the value of $(-i)^{2/3}$

Solution: $(-i)^{2/3} = \left(0 + i(-1)\right)^{2/3} = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{2/3}$

$$= \left[\cos\left(2k\pi + \frac{\pi}{2}\right) - i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{2/3}$$

$$= \cos\left((4k+1)\frac{\pi}{3}\right) - i\sin\left((4k+1)\frac{\pi}{3}\right)$$

Putting $k = 0, 1, 2$ we get the three roots as

$$\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right), \left(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right), \left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

∴ Continued product

$$= \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\left(\cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}\right)\left(\cos\frac{9\pi}{3} - i\sin\frac{9\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + \frac{5\pi}{3} + \frac{9\pi}{3}\right) - i\sin\left(\frac{\pi}{3} + \frac{5\pi}{3} + \frac{9\pi}{3}\right)$$

$$= \cos 5\pi - i\sin 5\pi \qquad = -1 - i(0) \quad = -1$$

5. Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ and show that their continued product is 1.

Solution: $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}} = \left\{\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^3\right\}^{1/4}$

$$= (\cos\pi + i\sin\pi)^{1/4} \qquad = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4}$$

$$= \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}$$

Putting $k = 0,1,2,3$ we get the four roots as,

$$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right), \left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right), \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$$

$$\therefore \left(\cos\frac{r\pi}{4} + i\sin\frac{r\pi}{4}\right) \text{ where } r = 1,3,5,7$$

The required product $= \cos\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right)$

$$= \cos 4\pi + i\sin 4\pi = 1.$$

6. SOLVE: $x^7 + x^4 + x^3 + 1 = 0$

Solution: $x^7 + x^4 + x^3 + 1 = 0 \quad \therefore x^4(x^3 + 1) + (x^3 + 1) = 0$

$$\therefore (x^3 + 1)(x^4 + 1) = 0 \quad \therefore x^3 = -1, x^4 = -1$$

Consider $x^3 = -1$

$$\therefore x = (-1 + i0)^{1/3} = (\cos\pi + i\sin\pi)^{1/3} = [\cos(2k+1)\pi - i\sin(2k+1)\pi]^{1/3}$$

$$= \cos(2k+1)\frac{\pi}{3} + i\sin(2k+1)\frac{\pi}{3}$$

Putting $k = 0, 1, 2$ we get the three roots

Similarly from $x^4 = -1$ we get the remaining four roots as

$$x = \cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4} \qquad \text{where } k = 0, 1, 2, 3$$

7. SOLVE: $x^4 + x^3 + x^2 + x + 1 = 0$

Solution: $x^4 + x^3 + x^2 + x + 1 = 0$

Multiplying the given equation by $x - 1$, we get $(x-1)(x^4 + x^3 + x^2 + x + 1) = 0$

$$\therefore \text{ We have } x^5 - 1 = 0 \quad \therefore x^5 = 1 = \cos 0 + i\sin 0$$

$$\therefore x^5 = \cos(2k\pi) + i\sin(2k\pi)$$

$$\therefore x = (\cos 2k\pi + i\sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$$

learning from Q5:

always put the power inside using demoivres before assuming the general value of theta i.e before making 2Kpi+theta

in Q5 , 3 is the power and 4 is the root
1st put 3 inside using demovires and then achieve a (cosθ+isinθ)^(1/n) format and then assume general value of θ using 2kpi+θ and then put the 1/n inside using the demoivres theorum

$$x^5 - 1 = (x-1)\left(x^4 + x^3 + x^2 + x^1 + x^0\right)$$

$$x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + x^{n-3} + ... + x^0\right)$$

$$x^6 + 1 = \left(x^2 + 1\right)\left(x^4 - x^2 + x^0\right)$$

Putting $k = 0, 1, 2, 3, 4$, we get the roots of the equation.

$x_0 = \cos 0 + i \sin 0 = 1,$

$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \quad x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$

$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \quad x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

It is clear that 1 is the roots of $x - 1 = 0$

and the remaining roots are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$

i.e., $cos\frac{2\pi}{5} \pm i\, sin\frac{2\pi}{5}, \quad cos\frac{4\pi}{5} \pm i\, sin\frac{4\pi}{5}$

8. SOLVE: $x^4 - x^2 + 1 = 0$

Solution: $x^4 - x^2 + 1 = 0$

Multiplying the given equation by $(x^2 + 1)$, we get, $(x^2 + 1)(x^4 - x^2 + 1) = 0$

$\therefore (x^2)^3 + (1)^3 = 0 \qquad \therefore x^6 + 1 = 0 \qquad \therefore x^6 = -1$

$\therefore x = (-1 + 0i)^{1/6} = (\cos\pi + i\sin\pi)^{1/6}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/6} = \cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots of equation

$x_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} \qquad x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

$x_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} \qquad x_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

$x_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = 0 + i(-1) = -i \qquad x_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$

It is clear that i and $-i$ are the roots of $x^2 + 1 = 0$ and the remaining roots x_0, x_2, x_3, x_5 are roots of $x^4 - x^2 + 1 = 0$

9. Find the roots common to $x^4 + 1 = 0$ *and* $x^6 - i = 0$.

Solution: Consider $x^4 + 1 = 0 \qquad \therefore x^4 = -1$

$x = (-1 + i0)^{1/4} = (\cos\pi + i\sin\pi)^{1/4} = [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{1/4}$

$x = \cos\left((2k+1)\frac{\pi}{4}\right) + i\sin\left((2k+1)\frac{\pi}{4}\right)$

Putting $k = 0, 1, 2, 3$ we get the three roots as

$x_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = 1 \qquad x_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} \qquad x_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Now consider, $x^6 - i = 0 \qquad \therefore x^6 = i$

$x = (0 + 1i)^{1/6} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/6} = \left[\cos\left(2k\pi + \frac{\pi}{2}\right) + i\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{1/6}$

$= \cos\left((4k+1)\frac{\pi}{12}\right) + i\sin\left((4k+1)\frac{\pi}{12}\right)$

Putting $k = 0, 1, 2, 3, 4, 5$ we get the six roots as

$x_0 = \cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ $\quad x_1 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$x_2 = \cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$x_3 = \cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}$

$x_4 = \cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}$

$x_5 = \cos\frac{21\pi}{12} + i\sin\frac{21\pi}{12} = -\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$\therefore$ common roots are $\pm\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

10. If $(1+x)^6 + x^6 = 0$

show that $x = -\frac{1}{2} - \frac{i}{2}cot\frac{\theta}{2}$ *where* $\theta = (2n+1)\,\pi/6\,, n = 0,1,2,3,4,5.$

Solution: $(1+x)^6 + x^6 = 0$ $\quad \therefore \frac{(1+x)^6}{x^6} = -1$

$\frac{1+x}{x} = (-1)^{1/6} = (\cos\pi + i\sin\pi)^{1/6} = [\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/6}$

$= \cos\left((2k+1)\frac{\pi}{6}\right) + i\sin\left((2k+1)\frac{\pi}{6}\right)$

$\frac{x+1-x}{x} = \cos\theta + i\sin\theta - 1$

$\frac{1}{x} = (\cos\theta - 1) + i\sin\theta$

$x = \frac{1}{(\cos\theta-1)+i\sin\theta} \times \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)-i\sin\theta} = \frac{(\cos\theta-1)-i\sin\theta}{(\cos\theta-1)^2+\sin^2\theta} = \frac{(\cos\theta-1)-i\sin\theta}{2(1-\cos\theta)}$

$= \frac{-2\sin^2(\theta/2) - i2\sin(\theta/2)\cos(\theta/2)}{2(2\sin^2(\theta/2))}$

$= -\frac{1}{2} - \frac{i}{2}\cot\left(\frac{\theta}{2}\right)$ $\quad$ where $\theta = (2k+1)\frac{\pi}{6}$

11. If one root of $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0\ is\ 1+i$, find all other roots.

Solution: The given equation is $x^4 - 6x^3 + 15x^2 - 18x + 10 = 0$

Since one of the root is $1+i$

$\therefore$ other root must be $1-i$ (since roots always occurs as complex conjugate pairs)

$\therefore x = 1 \pm i$ are the two roots

$\therefore x - 1 = \pm i$ $\quad \therefore (x-1)^2 = (\pm i)^2$ $\quad \therefore x^2 - 2x + 1 = -1$

$\therefore x^2 - 2x + 2 = 0$

Now we want to find other two remaining roots for that we divide

$x^4 - 6x^3 + 15x^2 - 18x + 10$ by $x^2 - 2x + 2$ and we obtain

$\therefore x^4 - 6x^3 + 15x^2 - 18x + 10 = (x^2 - 2x + 2)(x^2 - 4x + 5)$

$\therefore$ the remaining two roots are the roots of equation $x^2 - 4x + 5 = 0$

$\therefore x = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(5)}}{2(1)} = \frac{4\pm\sqrt{16-20}}{2} = \frac{4\pm 2i}{2} = 2 \pm i$

$\therefore$ The required remaining roots of given equation are $1 - i,\ 2 \pm i$

12. If $\alpha, \alpha^2, \alpha^3, \alpha^4$, are the roots of $x^5 - 1 = 0$, find them & show that $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5.$

Solution: We have $x^5 = 1 = \cos O + i \sin O$ $\therefore x^5 = cos(2k\pi) + i\sin(2k\pi)$

$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$

Putting $k = 0, 1, 2, 3, 4$, we get the five roots as

$x_0 = \cos 0 + i \sin 0 = 1,$ $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5},$

$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5},$ $x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5},$ $x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5},$

Putting $x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha$, we see that $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$

$\therefore$ the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4$, and hence

$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = \frac{x^5-1}{x-1}$

$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$

Putting $x = 1$, we get $(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^5) = 5$

13. Solve the equation $z^4 = i(z-1)^4$ and show that the real part of all the roots is $1/2$.

Solution: We have $z^4 = i(z-1)^4$

$\therefore \left(\frac{z}{z-1}\right)^4 = i = cos\frac{\pi}{2} + i\ sin\frac{\pi}{2} = cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$

$\therefore \frac{z}{z-1} = \left[cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)\right]^{1/4}$

$= cos(4n+1)\frac{\pi}{8} + i\sin(4n+1)\frac{\pi}{8}$

$\therefore \frac{z}{z-1} = \cos\theta + i\sin\theta$ where $\theta = (4n+1)\frac{\pi}{8}$

$\therefore \frac{z}{z-1-z} = \frac{z}{-1} = \frac{\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta}$ Simplifying as in the above example, we get

$\therefore \frac{z}{-1} = \frac{-\sin(\theta/2) + i\cos(\theta/2)}{2\sin(\theta/2)}$

$\therefore -z = -\frac{1}{2} + \frac{i}{2}\ cot\ \frac{\theta}{2}$ $\therefore z = \frac{1}{2} - \frac{i}{2}\ cot\ \frac{\theta}{2},$ $where\ \theta = (4n+1)\frac{\pi}{8}$

For, $n = 0, 1, 2$, we get three roots, All these roots have the real part $1/2$

14. If ω is a 7th root of unity, prove that

$S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n} = 7$ if n is a multiple of 7 and is equal to zero otherwise.

Solution: We have $x = 1^{\frac{1}{7}} = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{7}}$

$$= \cos\frac{2n\pi}{7} + i \sin\frac{2n\pi}{7}, \quad where \;\; n = 0, 1, 2, 3, 4, 5, 6$$

Let $\omega = \cos\frac{2\pi}{7} + i \sin\frac{2\pi}{7}$

$\therefore \omega^7 = \left(\cos\frac{2\pi}{7} + i \sin\frac{2\pi}{7}\right)^7 = \cos 2\pi + i \sin 2\pi = 1 \quad \therefore \omega^{7n} = 1^n = 1$

If n is not a multiple of 7, $\therefore \omega^n \neq 1$

Now, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \ldots\ldots\ldots + \omega^{6n} = \frac{1-\omega^{7n}}{1-\omega^n}$ sum of 7 terms of G.P

$$= \frac{1-1}{1-\omega^n} = \frac{0}{1-\omega^n} = 0$$

If n is a multiple of 7, say $n = 7k$

Then, $S = 1 + \omega^n + \omega^{2n} + \omega^{3n} + \omega^{4n} + \omega^{5n} + \omega^{6n}$

$$= 1 + (\omega^7)^k + (\omega^7)^{2k} + (\omega^7)^{3k} + (\omega^7)^{4k} + (\omega^7)^{5k} + (\omega^7)^{6k}$$

$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7$$

15. Prove that $\sqrt{1 + sec(\theta/2)} = \left(1 + e^{i\theta}\right)^{-1/2} + \left(1 + e^{-i\theta}\right)^{-1/2}$

Solution: We have to show that $\sqrt{1 + sec(\theta/2)} = \frac{1}{\sqrt{1+e^{i\theta}}} + \frac{1}{\sqrt{1+e^{-i\theta}}}$

Squaring both sides, we get, $1 + sec\frac{\theta}{2} = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

We shall prove this result

Now, $r.h.s = \frac{1}{1+e^{i\theta}} + \frac{1}{1+e^{-i\theta}} + \frac{2}{\sqrt{(1+e^{i\theta})(1+e^{-i\theta})}}$

$$= \frac{1}{1+e^{i\theta}} + \frac{e^{i\theta}}{1+e^{i\theta}} + \frac{2}{\sqrt{1+e^{-i\theta}+e^{i\theta}+1}}$$

$$= 1 + \frac{2}{\sqrt{2+(e^{i\theta}+e^{-i\theta})}} = 1 + \frac{2}{\sqrt{2+2\cos\theta}}$$

$$= 1 + \frac{2}{\sqrt{2(1+\cos\theta)}} = 1 + \frac{2}{\sqrt{4cos^2(\theta/2)}}$$

$$= 1 + \frac{2}{2cos(\theta/2)} = 1 + sec\frac{\theta}{2} = l.h.s$$

Refer to the class notes that you made during the lecture to see examples of multiplying with something to simplify the expression

SOME PRACTICE PROBLEMS

1. Find the cube roots of unity. If ω is a complex cube root of unity prove that

(i) $1+\omega+\omega^2=0$ **(ii)** $\frac{1}{1+2\omega}+\frac{1}{2+\omega}-\frac{1}{1+\omega}=0$

2. Prove that the n nth roots of unity are in geometric progression.

how **3.** Show that the sum of the n nth roots of unity is zero.

4. Prove that the product of n nth roots of unity is $(-1)^{n-1}$

5. Find all the values of the following :

(i) $(-1)^{1/5}$ **(ii)** $(-i)^{1/3}$ **(ix)** $\left(1-i\sqrt{3}\right)^{1/4}$

6. Find the continued product of all the values of $\left(\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{3/4}$

7. Find all the value of $(1+i)^{2/3}$ and find the continued product of these values.

8. Solve the equations

(i) $x^9+8x^6+x^3+8=0$ **(ii)** $x^4-x^3+x^2-x+1=0$

(iii) $(x+1)^8+x^8=0$

9. If $(x+1)^6=x^6, show\ that\ x=-\frac{1}{2}-i\ cot\frac{\theta}{2}\ where\ \theta=\frac{2k\pi}{6}, k=0,1,2,3,4,5.$

10. Show that the roots of $(x+1)^7=(x-1)^7$ are given by $\pm i\ cot\frac{r\pi}{7}, r=1,2,3.$

11. If $\alpha,\alpha^2,\alpha^3,\dots\alpha^6$ are the roots of $x^7-1=0$, find them and prove that

$(1-\alpha)(1-\alpha^2)\dots\dots\dots\dots(1-\alpha^6)=7.$

12. Prove that $x^5-1=(x-1)\left(x^2+2x\cos\frac{\pi}{5}+1\right)\left(x^2+2x\cos\frac{3\pi}{5}+1\right)=0.$

13. Solve the equation $z^n=(z+1)^n$ and show that the real part of all the roots is $-1/2$.

14. If $a=e^{i\,2\pi/7}$ and $b=a+a^2+a^4,\ c=a^3+a^5+a^6$**.** then prove that b & c are roots of quadratic equation $x^2+x+2=0$.

15. Prove that **(i)** $\sqrt{1-cosce(\theta/2)}=\left(1-e^{i\theta}\right)^{-1/2}-\left(1-e^{-i\theta}\right)^{-1/2}$

(iv) $\sqrt{1-sce(\theta/2)}=\left(1+e^{i\theta}\right)^{-1/2}-\left(1+e^{-i\theta}\right)^{-1/2}$

16. If $1+2i$ is a root of the equation $x^4-3x^3+8x^2-7x+5=0$, find all the other roots.

Answers:

5. **(i)** $-1,\ cos\frac{\pi}{5}\pm i\sin\frac{\pi}{5},\ \cos\frac{3\pi}{5}\pm i\sin\frac{3\pi}{5}$ **(ii)** $i\ ,\ \pm\frac{\sqrt{3}}{2}-\frac{i}{2}$

(iii) $2^{1/4}\left[cos\frac{(6k+5)\pi}{12}+i\ sin\frac{(6k+5)\pi}{12}\right]\ where\ k=0,1,2,3.$

6. 1

7. $2^{1/3}\left(cos\frac{8\pi k+\pi}{6}+i\sin\frac{8\pi k+\pi}{6}\right), k=0,1,2\ ,\ product=2i$

8. **(i)** $\cos(2k+1)\,\pi/6+i\sin(2k+1)\,\pi/6\,, k=0,1,2,3,4,5$

and $2[\cos(k+1)\,\pi/3+i\sin(2k+1)\,\pi/3]\ ,\ k=0,1,2$

(ii) $\cos(2k+1)\,\pi/5+i\sin(2k+1)\,\pi/5\,, k=0,1,2,3,4$

(iii) $x=1/[cos(2k+1)\,\pi/8+i\ sin(2k+1)\,\pi/8\ -1]\quad here\ k=0,1,2,3,4,5,6,7$

16. $1-2i,\quad \left(1\pm i\sqrt{3}\right)/2$