problems involving tan,cos take conjugate both sides and use direct expansion

SEPARATION OF REAL AND IMAGINARY PARTS:

Many a time we are required to separate real and imaginary parts of a given complex function.

For this, we have to use identities of circular and hyperbolic functions.

In problem where we are given $\tan(\alpha + i\beta) = x + i\,y$, we proceed as shown below

Since $\tan(\alpha + i\beta) = x + i\,y$, we get $\tan(\alpha - i\beta) = x - i\,y$.

$\therefore\ \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta).\ \tan(\alpha-i\beta)}$$

$$= \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-x^2-y^2}$$

in sin and cos, direct expansion of sin(x+iy) using sin(A+B) formula works but for tan we use this kind of approach

$\therefore\ 1 - x^2 - y^2 = 2x\cot 2\alpha$ $\qquad \therefore\ x^2 + y^2 + 2x\cot 2\alpha - 1 = 0$

Further, $\tan(2i\beta) = \tan[(\alpha + i\beta) - (\alpha - i\beta)]$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\ \ \tan(\alpha-i\beta)}$$

$$i\tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = \frac{2i\,y}{1+x^2+y^2}$$

$$\therefore\ \tanh 2\beta = \frac{2y}{1+x^2+y^2}$$

$\therefore\ 1 + x^2 + y^2 = 2y\coth 2\beta$ $\qquad$ i.e., $x^2 + y^2 - 2y\coth 2\beta + 1 = 0$

SOME SOLVED EXAMPLES:

1. Separate into real and imaginary parts $tan^{-1}\left(e^{i\,\theta}\right)$

Solution: Let $tan^{-1}e^{i\theta} = x + iy$ $\quad \therefore\ e^{i\theta} = \tan(x + iy)$

$\therefore\ cos\theta + i\sin\theta = \tan(x + iy)$

Similarly, $cos\theta - i\sin\theta = \tan(x - iy)$

Now, $tan\ 2x = \ tan\ [\ (x + iy) +\ (x - iy)]$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{(cos\theta+i\sin\theta)+(cos\theta-i\sin\theta)}{1-(cos\theta+i\sin\theta)(cos\theta-i\sin\theta)} = \frac{2\ cos\theta}{1-(cos^2\theta+sin^2\theta)}$$

we don't use this approach for tan

$\tan(\alpha+i\beta) = x+iy$.

$$\frac{\tan\alpha + \tan i\beta}{1-\tan\alpha\tan i\beta} = \frac{\tan\alpha + i\tanh\beta}{1-i\tan\alpha\tanh\beta}$$

Now * ~~#~~ in num & deno the conjugate of denominator is a very difficult approach.

Thus we use diff approach.

$= \frac{2\, cos\theta}{1-1} = \frac{2\, cos\theta}{0} = \infty$

$\therefore 2x = \frac{\pi}{2} \quad \therefore x = \frac{\pi}{4}$

Also $\tan 2\, iy = \tan[(x + iy) - (x - iy)]$

$= \frac{\tan(x+iy) - \tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$

$= \frac{(cos\theta + i\sin\theta) - (cos\theta - i\sin\theta)}{1+(cos\theta+i\sin\theta)(cos\theta - i\sin\theta)} = \frac{2\, isin\theta}{1+(cos^2\theta+sin^2\theta)} = \frac{2isin\theta}{2}$

$\therefore i\tan h\, 2y = i\sin\theta \qquad \therefore \tan h\, 2y = \sin\theta$

$\therefore 2y = tanh^{-1}\sin\theta \qquad \therefore y = \frac{1}{2}\tan h^{-1}\sin\theta$

2. If $\sin(\alpha - i\,\beta) = x + i\,y$ then prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1$ and $\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = 1$

Solution: $\sin(\alpha - i\,\beta) = x + i\,y$

$\therefore \sin\alpha\cos h\,\beta + i\cos\alpha\sin h\,\beta = x + iy$

Equating real and imaginary parts, we get,

$\sin\alpha\cos h\,\beta = x \;\; and \cos\alpha\sin h\,\beta = y$

$\therefore \frac{x^2}{\cos h^2\beta} + \frac{y^2}{\sin h^2\beta} = sin^2\alpha + cos^2\alpha = 1$ and

$\frac{x^2}{sin^2\alpha} - \frac{y^2}{cos^2\alpha} = \cos h^2\beta - \sin h^2\beta = 1$

3. If $cos(x + i\,y) = \cos\alpha + i\sin\alpha\, , prove\ that$

(i) $\sin\alpha = \pm\, sin^2x = \pm\, \sin h^2 y$ **(ii)** $\cos 2x + \cosh 2y = 2$

Solution: $\cos(x + iy) = \cos\alpha + i\sin\alpha$

$\cos x\cos(iy) - \sin x\sin(iy) = \cos\alpha + i\sin\alpha$

$\cos x\cosh y - i\sin x\sinh y = \cos\alpha + i\sin\alpha$

Equating real and imaginary parts, we get,

$\cos x\cosh y = \cos\alpha$ and $-\sin x\sinh y = \sin\alpha$

(i) Since $\sin^2\alpha + \cos^2\alpha = 1$

$\therefore \sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y = 1$

$\sin^2 x \sinh^2 y + (1-\sin^2 x)(1+\sinh^2 y) = 1$

$\sin^2 x \sinh^2 y + 1 + \sinh^2 y - \sin^2 x - \sin^2 x \sinh^2 y = 1$

$1 + \sinh^2 y - \sin^2 x = 1$

$\sinh^2 y - \sin^2 x = 0$

$\therefore \sinh^2 y = \sin^2 x$ ....(i)

$\therefore \sinh y = \pm \sin x$

$\therefore \sin\alpha = -\sin x \sinh y = -\sin x\,(\pm \sin x) = \pm \sin^2 x$

(ii) $\cos 2x + \cosh 2y = 1 - 2\sin^2 x + 1 + 2\sinh^2 y$

$= 2 - 2\sin^2 x + 2\sin^2 x$ .... from (i)

$= 2$

4. If $x + i\,y = \tan(\pi/6 + i\,\alpha)$, prove that $x^2 + y^2 + 2x/\sqrt{3} = 1$

Solution: We have to separate real part $\pi/6$ and imaginary part α

$\because tan\left(\frac{\pi}{6} + i\alpha\right) = x + iy \qquad \therefore tan\left(\frac{\pi}{6} - i\alpha\right) = x - iy$

$\therefore tan\left[\left(\frac{\pi}{6} + i\alpha\right) + \left(\frac{\pi}{6} - i\alpha\right)\right] = \frac{tan\left(\frac{\pi}{6}+i\alpha\right)+\tan\left(\frac{\pi}{6}-i\alpha\right)}{1-\tan\left(\frac{\pi}{6}+i\alpha\right).\tan\left(\frac{\pi}{6}-i\alpha\right)}$

$\therefore tan\frac{\pi}{3} = \frac{(x+iy)+(x-iy)}{1-(x+iy).(x-iy)}$

$\therefore \sqrt{3} = \frac{2x}{1-x^2-y^2}$

$\therefore 1 - x^2 - y^2 = \frac{2x}{\sqrt{3}}$

$\therefore x^2 + y^2 + \frac{2x}{\sqrt{3}} = 1.$

5. If $x + i\,y = c\cot(u + i\,v)$, show that $\frac{x}{\sin 2u} = -\frac{y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u}$.

Solution: We have $x + iy = c\cot(u+iv) \qquad \therefore\; x - iy = c\cot(u - iv)$

$\therefore 2x = c[\cot(u+iv) + \cot(u-iv)]$

$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)} + \frac{\cos(u-iv)}{\sin(u-iv)}\right]$

$$= c\,\frac{[\cos(u+iv)\sin(u-iv)+\sin(u+iv)\cos(u-iv)]}{\sin(u+iv)\sin(u-iv)}$$

$$\therefore 2x = \frac{c\sin[(u-iv)+(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u-iv-u+iv)]/2}$$

$$\therefore x = \frac{c\sin 2u}{-[\cos 2u-\cos 2\,iv]} = \frac{c\sin 2u}{\cos h\,2v-\cos 2u} \quad \text{...........(1)}$$

Now, $2iy = c[\cot(u+iv)-\cot(u-iv)]$

$$= c\left[\frac{\cos(u+iv)}{\sin(u+iv)}-\frac{\cos(u-iv)}{\sin(u-iv)}\right]$$

$$= c\left[\frac{\cos(u+iv)\sin(u-iv)-\cos(u-iv)\sin(u+iv)}{\sin(u+iv)\sin(u-iv)}\right]$$

$$\therefore 2iy = \frac{c\sin[(u-iv)-(u+iv)]}{-[\cos(u+iv+u-iv)-\cos(u+iv-u+iv)]/2}$$

$$\therefore\ iy = \frac{c\sin(-2iv)}{-[\cos 2u-\cos 2iv]} = -\frac{i\,c\sin h\,2v}{\cos h\,2v-\cos 2u}$$

$$\therefore\ y = \frac{-\,c\sin h\,2v}{\cos h\,2v-\cos 2u} \quad \text{..........(2)}$$

From (1) & (2) $\dfrac{x}{\sin 2u} = -\dfrac{y}{\sinh 2v} = \dfrac{c}{\cosh 2v-\cos 2u}$

6. If $u+i\,v = cosec\left(\frac{\pi}{4}+i\,x\right), prove\ that\ (u^2+v^2)^2 = 2(u^2-v^2)$

Solution: We have $\dfrac{1}{\sin[(\pi/4)+ix]} = u+iv$

$$\therefore \sin\left(\frac{\pi}{4}+ix\right) = \frac{1}{u+iv} = \frac{1}{u+iv}.\frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore \sin\frac{\pi}{4}\cos i\,x+\cos\frac{\pi}{4}\sin ix = \frac{u-iv}{u^2+v^2}$$

$$\frac{1}{\sqrt{2}}\cos h\,x + i\frac{1}{\sqrt{2}}\sin hx = \frac{u-iv}{u^2+v^2}$$

Equating real and imaginary parts $\cos hx = \sqrt{2}.\left(\frac{u}{u^2+v^2}\right)$; $\sin hx = -\sqrt{2}.\left(\frac{v}{u^2+v^2}\right)$

But $cosh^2x - sinh^2x = 1$

$$\therefore 2\left(\frac{u^2}{(u^2+v^2)^2}\right) - 2\left(\frac{v^2}{(u^2+v^2)^2}\right) = 1$$

$$\therefore 2(u^2-v^2) = (u^2+v^2)^2$$

7. If $x+iy=\cos(\alpha+i\,\beta)$ or if $\cos^{-1}(\mathrm{x}+\mathrm{iy})=\alpha+\mathrm{i}\,\beta$ express x and y in terms of α and β.

Hence show that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$\lambda^2-(x^2+y^2+1)\lambda+x^2=0$

Solution: We have $\cos\alpha\cos i\beta-\sin\alpha\sin i\beta=x+iy$

$\therefore \cos\alpha\cos h\,\beta-i\sin\alpha\sin h\beta= x+iy$

Equating real and imaginary parts $\cos\alpha\cos h\,\beta=x\;\; and \;\; \sin\alpha\sin h\beta=-y$

We know that, in terms of the roots, the quadratic equation is given by

$\lambda^2-(sum\ of\ the\ roots)\lambda+(product\ of\ the\ roots)=0$

Hence the equation whose roots are $\cos^2\alpha$ and $\cosh^2\beta$ is

$\lambda^2-(cos^2\alpha+cos^2\beta)\lambda+(cos^2\alpha.cos^2\beta)=0$

This means we have to prove that $x^2+y^2+1=cos^2\alpha+\cos h^2\beta$ and

$x^2=cos^2\alpha\cos h^2\beta$

Now, $x^2+y^2+1=cos^2\alpha\cos h^2\beta+sin^2\alpha\sin h^2\beta+1$

$=cos^2\alpha\cos h^2\beta+(1-cos^2\alpha)(\cos h^2\beta-1)+1$

$=cos^2\alpha\cos h^2\beta+\cos h^2\beta-1-cos^2\alpha\cos h^2\beta+cos^2\alpha+1$

$=cos^2\alpha+\cos h^2\beta=sum\ of\ the\ roots$

And $x^2=cos^2\alpha\cos h^2\beta$ = Product of the roots

Hence the equation whose roots are $cos^2\alpha$, $\cos h^2\beta$ is

$\lambda^2-(x^2+y^2+1)\lambda+x^2=0$

SOME PRACTICE PROBLEMS:

1. Separate into real and imaginary parts.

(i) $\cosh(x+iy)$ **(ii)** $\cos(x+iy)$ **(iii)** $\coth(x+iy)$

(iv) $\mathrm{sech}(x+iy)$ **(v)** $\coth i(x+iy)$ **(vi)** $\tan(x+iy)$

(vii) $\cot(x+iy)$

2. Separate into real and imaginary parts $tan^{-1}(\alpha+i\,\beta)$

3. Separate into real and imaginary parts $sin^{-1}\left(e^{i\,\theta}\right)$

4. If $A + i\,B = C\tan(x + iy)$, prove that $tan2x = \frac{2CA}{C^2 - A^2 - B^2}$

5. If $\cos(\theta + i\,\Phi) = r(\cos\alpha + i\sin\alpha)$, prove that

$$r^2 = \frac{1}{2}[\cosh 2\,\Phi + \cos 2\,\theta] \;\&\; \tan\alpha = -\tan\theta\tanh\Phi$$

6. If $\cos(\alpha + i\,\beta) = x + i\,y$, Prove that $\frac{x^2}{cosh^2\beta} + \frac{y^2}{sinh^2\beta} = 1, \quad \frac{x^2}{cos^2\alpha} - \frac{y^2}{sin^2\alpha} = 1$

7. If $\sinh(a + i\,b) = x + i\,y$, prove that

$$x^2 cosech^2 a + y^2 sech^2 a = 1 \quad and \quad y^2 cosec^2 b - x^2 sec^2 b = 1$$

8. If $\sin(x + iy) = \cos\alpha + i\sin\alpha$, Prove that

(i) $\cosh 2y - \cos 2x = 2$ **(ii)** $y = \frac{1}{2}\,log\frac{\cos(x-\alpha)}{\cos(x+\alpha)}$

(iii) $\sin\alpha = \pm\,cos^2 x = \pm\,sinh^2 y$

9. If $\cosh(\theta + i\,\Phi) = e^{i\,\alpha}$, prove that $sin^2\alpha = sin^4\Phi = sinh^4\theta$

10. If $\cos(u + i\,v) = x + i\,y$ Prove that, $(1+x)^2 + y^2 = (\cosh v + \cos u)^2$ and $(1-x)^2 + y^2 = (\cosh v - \cos u\,)^2$

11. If $tan(\alpha + i\,\beta) = x + i\,y$, prove that

$x^2 + y^2 + 2x\cot 2\alpha = 1, \quad x^2 + y^2 - 2y\coth 2\beta + 1 = 0$

12. If $\tan\left(\frac{\pi}{3} + i\,\alpha\right) = x + i\,y$, prove that, $x^2 + y^2 - \frac{2x}{\sqrt{3}} - 1 = 0$

13. If $cot(\alpha + i\,\beta) = x + i\,y$, prove that

$x^2 + y^2 - 2x\cot 2\alpha = 1, \quad x^2 + y^2 + 2y\coth 2\beta + 1 = 0$

14. If $tanh\left(\alpha + \frac{i\,\pi}{6}\right) = x + i\,y$, prove that, $x^2 + y^2 + \frac{2y}{\sqrt{3}} = 1$

15. If $\coth(\alpha + i\,\pi/8) = x + i\,y$, prove that $x^2 + y^2 + 2y = 1$

16. If $\sinh(x + i\,y) = e^{i\,\pi/3}$, prove that

(i) $3cos^2 y - sin^2 y = 4sin^2 y\,cos^2 y$

(ii) $3sinh^2 x + cosh^2 x = 4sinh^2 x cosh^2 x$

17. If $x + i\,y = 2\cosh\left(\alpha + \frac{i\,\pi}{3}\right)$, prove that $3x^2 - y^2 = 3$

18. If $\cot(u + i\,v) = cosec(x + i\,y)$, prove that $\text{cothy}\sinh 2v = \cot x \sin 2u$

19. Show that $tan\left(\frac{u + i\,v}{2}\right) = \frac{\sin u + i\sinh v}{\cos u + \cosh v}$

20. If $\sin^{-1}(\alpha + i\,\beta) = x + i\,y$,

show that $\sin^2 x$ and $\cos h^2 y$ are the roots of the equation

$\lambda^2 - (\alpha^2 + \beta^2 + 1)\lambda + \alpha^2 = 0$

21. If $(u + iv) = x + iy$, prove that the curves u = constant , v = constant are a family of circles which are mutually orthogonal

ANSWERS

1. **(i)** $\cosh x \cos y + i \sinh x \sin y$

(ii) $cosx \cosh y - i \sin x \sinh y$

(iii) $(\sinh 2x - i \sin 2y)/(\cosh 2x - \cos 2y)$

(iv) $\frac{(2 \cosh x \cos y - 2\, i \sinh x \sin y)}{(\cosh 2x + \cos 2y)}$

(v) $(-\sinh 2y - i \sin 2x)/(\cosh 2y - \cos 2x)$

(vi) $(\sin 2x + i \sinh 2y)/(\cos 2x + \cosh 2y)$

(vii) $(\sin 2x - i \sinh 2y)/(\cosh 2y - \cos 2x)$

2. $tan^{-1}[2\alpha/(1 - \alpha^2 - \beta^2)]$, $\frac{1}{2} tanh^{-1}[2\,\beta/(1 + \alpha^2 + \beta^2)]$.

3. $cos^{-1}\sqrt{\sin\theta} + i\; sinh^{-1}\sqrt{\sin\theta}$