Solution Gate EC 29.2022

Aditya Vikram Singh*

Question 29 Let H(X) denote the entropy of a discrete random variable X taking K possible distinct real values. Which of the following statements is/are necessarily true?

(A) $H(X) \leq \log_2 K$ bits

(B) $H(X) \leq H(2X)$

(C) $H(X) \leq H(X^2)$

(D) $H(X) \leq H(2^X)$

Solution:

Random independent variable	value of R.V	Description
X	$X \in (x_1, x_2..., x_K)$	Value of the discrete variable X

1) For Option(A) we will find the $H(X)max$
   We know that :

$$\sum_{k=0}^{K} p_X(k) = 1 \tag{1}$$

$$H(X) = \sum_{k=0}^{K} p_X(k) \log_2 \frac{1}{p_X(k)} \tag{2}$$

   Now,we use lagranges multiplier to find the maximum entropy subject to the lagranges multiplier constant λ and $p_X(k)$

$$L\left(p_X(k), \lambda\right) = -\sum_{k=0}^{K} p_X(k) \log_2 p_X(k) + \lambda \left(\sum_{k=0}^{k} p_X(k) - 1 \right) \tag{3}$$

$$\frac{\partial L}{\partial p_X(k)} = -\log_2 p_X(k) - 1 + \lambda \tag{4}$$

   Now, we take the derivative of L with respect to each $p_X(k)$ equal to zero for $H(X)max$

$$\lambda = \log_2 \frac{2}{k} \tag{5}$$

$$p_X(k) = 1/K \tag{6}$$

   On solving, we get the value of

$$H(X)max = \log_2 K \tag{7}$$

$$H(X) \leq \log_2 K \tag{8}$$

   Hence, Option(A) is correct
   Let's consider the discrete variable as follows

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

$X \in x_i$	$p_X(k)$
-1	$\frac{1}{4}$
0	$\frac{1}{2}$
1	$\frac{1}{4}$

$$H(X) = \frac{1}{4}\log_2 4 + \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 \tag{9}$$
$$H(X) = 1.5units \tag{10}$$

2) Now $Y = 2X$

$Y \in y_i$	$p_Y(k)$
-2	$\frac{1}{4}$
0	$\frac{1}{2}$
2	$\frac{1}{4}$

$$H(Y) = \sum_{i=0}^{2} p_Y(k)\log_2 \frac{1}{p_Y(k)} \tag{11}$$
$$H(Y) = 1.5units \tag{12}$$
$$H(Y) = H(2X) = H(X) \tag{13}$$

Hence, Option(B) is correct

3) Similarly on substituting $Y = X^2$

$Y \in y_i$	$p_Y(k)$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

$$H(Y) = \sum_{i=0}^{1} p_Y(k)\log_2 \frac{1}{p_Y(k)} \tag{14}$$
$$H(Y) = 1units \tag{15}$$
$$H(Y) = H(X^2) \leq H(X) \tag{16}$$

Hence, Option(C) is incorrect

4) Now for $Y = 2^X$
As the function Y is monotonically increasing, so for every distinct X, we get a distinct Y

$$H(Y) = H(2^X) = H(X) \tag{17}$$

Hence, Option(D) is correct
The ans is (A), (B), (D)