

Solution Gate EC 29.2022

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Question 29 Let $H(X)$ denote the entropy of a discrete random variable X taking K possible distinct real values. Which of the following statements is/are necessarily true?

(A) $H(X) \leq \log_2 K$ bits

(B) $H(X) \leq H(2X)$

(C) $H(X) \leq H(X^2)$

(D) $H(X) \leq H(2^X)$

Solution:

Random independent variable	value of R.V	Description
X	$X \in (x_1, x_2, \dots, x_K)$	Value of the discrete variable X

1) For Option(A) we will find

We know that :

$$\begin{aligned} \max_{p_X(k)} \quad & H(x) \\ \text{s.t.} \quad & \sum_{k=0}^K p_X(k) = 1 \end{aligned}$$

\Rightarrow

$$\begin{aligned} \max_{p_X(k)} \quad & - \sum_{k=0}^K p_X(k) \log_2 p_X(k) \\ \text{s.t.} \quad & \sum_{k=0}^K p_X(k) = 1 \end{aligned}$$

Now, we use lagranges multiplier to find the maximum entropy subject to the lagranges multiplier constant λ and $p_X(k)$

$$L(p_X(k), \lambda) = - \sum_{k=0}^K p_X(k) \log_2 p_X(k) + \lambda \left(\sum_{k=0}^K p_X(k) - 1 \right) \quad (1)$$

$$\frac{\partial L}{\partial p_X(k)} = -\log_2 p_X(k) - 1 + \lambda \quad (2)$$

Now, we take the derivative of L with respect to each $p_X(k)$ equal to zero for $H(X)_{max}$

$$\lambda = \log_2 \frac{2}{k} \quad (3)$$

$$p_X(k) = 1/K \quad (4)$$

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On solving, we get the value of

$$H(X)_{max} = \log_2 K \quad (5)$$

$$H(X) \leq \log_2 K \quad (6)$$

Hence, Option(A) is correct

2) Let's consider the discrete variable as follows

$X \in x_i$	$p_X(k)$
-1	$\frac{1}{4}$
0	$\frac{1}{2}$
1	$\frac{1}{4}$

$$H(X) = \frac{1}{4} \log_2 4 + \frac{1}{2} \log_2 2 + \frac{1}{4} \log_2 4 \quad (7)$$

$$H(X) = 1.5 \text{ units} \quad (8)$$

Now $Y = 2X$

$Y \in y_i$	$p_Y(k)$
-2	$\frac{1}{4}$
0	$\frac{1}{2}$
2	$\frac{1}{4}$

$$H(Y) = \sum_{i=0}^2 p_Y(k) \log_2 \frac{1}{p_Y(k)} \quad (9)$$

$$H(Y) = 1.5 \text{ units} \quad (10)$$

$$H(Y) = H(2X) = H(X) \quad (11)$$

Hence, Option(B) is correct

3) Similarly on substituting $Y = X^2$

$Y \in y_i$	$p_Y(k)$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

$$H(Y) = \sum_{i=0}^1 p_Y(k) \log_2 \frac{1}{p_Y(k)} \quad (12)$$

$$H(Y) = 1 \text{ units} \quad (13)$$

$$H(Y) = H(X^2) \leq H(X) \quad (14)$$

Hence, Option(C) is incorrect

4) Now for $Y = 2^X$

As the function Y is monotonically increasing, so for every distinct X, we get a distinct Y

$$H(Y) = H(2^X) = H(X) \quad (15)$$

Hence, Option(D) is correct

The ans is (A), (B), (D)

These options are correct for the particular example.