Solution Gate EC 29.2022

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Question 29 Let H(X) denote the entropy of a discrete random variable X taking K possible distinct real values. Which of the following statements is/are necessarily true?

(A) $H(X) \leq \log_2 K$ bits

(B) $H(X) \leq H(2X)$

(C) $H(X) \leq H(X^2)$

(D) $H(X) \leq H(2^X)$

Solution:

Random independent variable	value of R.V	Description
X	$X \in (x_1, x_2..., x_K)$	Value of the discrete variable X

1) For Option(A) we will find
   We know that :

$$\max_{p_X(k)} \quad H(x)$$
$$\text{s.t.} \quad \sum_{k=0}^{K} p_X(k) = 1$$

$\implies$

$$\max_{p_X(k)} \quad -\sum_{k=0}^{K} p_X(k) \log_2 p_X(k)$$
$$\text{s.t.} \quad \sum_{k=0}^{K} p_X(k) = 1$$

Now,we use lagranges multiplier to find the maximum entropy subject to the lagranges multiplier constant λ and $p_X(k)$

$$L\left(p_X(k), \lambda\right) = -\sum_{k=0}^{K} p_X(k) \log_2 p_X(k) + \lambda\left(\sum_{k=0}^{k} p_X(k) - 1\right) \tag{1}$$

$$\frac{\partial L}{\partial p_X(k)} = -\log_2 p_X(k) - 1 + \lambda \tag{2}$$

Now, we take the derivative of L with respect to each $p_X(k)$ equal to zero for $H(X)max$

$$\lambda = \log_2 \frac{2}{k} \tag{3}$$

$$p_X(k) = 1/K \tag{4}$$

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On solving, we get the value of

$$H(X)_{max} = \log_2 K \tag{5}$$

$$H(X) \leq \log_2 K \tag{6}$$

Hence, Option(A) is correct

2) Consider

$$Y = 2X \tag{7}$$

let m denote the entries for Y

$$m = 2k \tag{8}$$

$$\begin{aligned} p_Y(m) &= p_{2X}(2k) && (9)\\ &= P(2X = 2k) && (10)\\ &= P(X = k) && (11)\\ p_Y(m) &= p_X(k) && (12) \end{aligned}$$

Entropy for Y is

$$\begin{aligned} H(Y) = H(2X) &= -\sum_{m=0}^{M} p_Y(m)\log_2 p_Y(m) && (13)\\ &= -\sum_{k=0}^{K} p_X(k)\log_2 p_X(k) && (14)\\ H(2X) &= H(X) && (15) \end{aligned}$$

Hence, Option(B) is correct

3) Similarly on substituting $Y = X^2$

$Y \in y_i$	$p_Y(k)$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

$$\begin{aligned} H(Y) &= \sum_{i=0}^{1} p_Y(k)\log_2 \frac{1}{p_Y(k)} && (16)\\ H(Y) &= 1 units && (17)\\ H(Y) &= H(X^2) \leq H(X) && (18) \end{aligned}$$

Hence, Option(C) is incorrect

4)

$$Y = 2^X \tag{19}$$

let m denote the entries for Y

$$m = 2^k \tag{20}$$

$$p_Y(m) = p_{2^X}(2^k) \tag{21}$$
$$= P(2^X = 2^k) \tag{22}$$
$$\tag{23}$$

Taking $\log_2$ both sides

$$= P(X = k) \tag{24}$$
$$p_Y(m) = p_X(k) \tag{25}$$

Entropy for Y is

$$H(Y) = H(2^X) = -\sum_{m=0}^{M} p_Y(m) \log_2 p_Y(m) \tag{26}$$
$$= -\sum_{k=0}^{K} p_X(k) \log_2 p_X(k) \tag{27}$$
$$H(2^X) = H(X) \tag{28}$$

Hence, Option(D) is correct
The ans is (A), (B), (D)