Solution Gate EC 29.2022

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Question 29 Let H(X) denote the entropy of a discrete random variable X taking K possible distinct real values. Which of the following statements is/are necessarily true?

(A) $H(X) \leq \log_2 K$ bits

(B) $H(X) \leq H(2X)$

(C) $H(X) \leq H(X^2)$

(D) $H(X) \leq H(2^X)$

Solution: For Option(A) we will find the $H(X)max$

Random independent variable	value of R.V	Description
X	$X \in (x_1, x_2..., x_K)$	Value of the discrete variable X

We know that :

$$\sum_{i=0}^{k} P_X x_i = 1 \tag{1}$$

$$H(X) = \sum_{i=0}^{k} P_X x_i \log_2 \frac{1}{P_X x_i} \tag{2}$$

Now,we use lagranges multiplier to find the maximum entropy subject to the lagranges multiplier constant λ and $P_X x_i$

$$L(P_X x_i, \lambda) = -\sum_{0}^{k} P_X x_i \log_2 P_X x_i + \lambda \left(\sum_{0}^{k} P_X x_i - 1 \right) \tag{3}$$

$$\frac{\partial L}{\partial P_X x_i} = -\log_2 P_X x_i - 1 + \lambda \tag{4}$$

Now, we take the derivative of L with respect to each $P_X x_i$ equal to zero for $H(X)max$

$$\lambda = \log_2 \frac{2}{k} \tag{5}$$

$$P_X x_i = 1/K \tag{6}$$

If all the K distinct values of the variable have the same probability, then Entropy will be maximum On solving, we get the value of

$$H(X)max = \log_2 K \tag{7}$$

but, we know that every value will have a different value so,

$$H(X) \leq \log_2 K \tag{8}$$

Hence, Option(A) is correct Let's consider the discrete variable as follows

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$X \in x_i$	$P_X x_i$
-1	$\frac{1}{4}$
0	$\frac{1}{2}$
1	$\frac{1}{4}$

$$H(X) = \frac{1}{4}\log_2 4 + \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 \tag{9}$$

$$H(X) = 1.5units \tag{10}$$

Now $Y = 2X$

$Y \in y_i$	$P_Y y_i$
-2	$\frac{1}{4}$
0	$\frac{1}{2}$
2	$\frac{1}{4}$

$$H(Y) = \sum_{i=0}^{2} P_Y y_i \log_2 \frac{1}{P_Y y_i} \tag{11}$$

$$H(Y) = 1.5units \tag{12}$$

$$H(Y) = H(2X) = H(X) \tag{13}$$

Hence, Option(B) is correct
Similarly on substituting $Y = X^2$

$Y \in y_i$	$P_Y y_i$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

$$H(Y) = \sum_{i=0}^{1} P_Y y_i \log_2 \frac{1}{P_Y y_i} \tag{14}$$

$$H(Y) = 1units \tag{15}$$

$$H(Y) = H(X^2) \leq H(X) \tag{16}$$

Hence, Option(C) is incorrect
Now for $Y = 2^X$
As the function Y is monotonically increasing, so for every distinct X, we get a distinct Y

$$H(Y) = H(2^X) = H(X) \tag{17}$$

Hence, Option(D) is correct
The ans is (A), (B), (D)